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On monodromy in integrable Hamiltonian systems

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Contents

Introduction

One historical line of the present work starts in the 1850s when J. Liouville proved that it is sufficient to find n independent functions in involution to integrate a given Hamiltonian system with n degrees of freedom. In the XIXth century it was also observed that in various examples of integrable Hamiltonian systems the functions in involution give rise to invariant tori and that a neighborhood of any such torus admits *action-angle* coordinates. The precise mathematical result was given much later by Arnol'd in [1, 3] (Arnol'd-Liouville theorem).

The geometric point of view on integrable systems was further developed by the mathematicians Duistermaat, Fomenko, Nekhoroshev, Smale, Weinstein, and by others in the 1970-1980s. Their works laid a foundation to a completely new field in mathematical physics. One major direction in this field was founded by Cushman and Duistermaat. Its history starts with the discovery of a certain 'holonomy' effect, which appears in integrable systems when one tries to construct a set of global action coordinates. Locally, such coordinates exist by the Arnol'd-Liouville theorem, whereas globally this is not necessarily the case. The corresponding obstruction was introduced by Duistermaat in his work [27] in 1980 and it is commonly referred to as *Hamiltonian monodromy* or simply as *monodromy*.

In order to see non-trivial monodromy, it is customary to look at the so-called *energy-momentum map* (also known as the *integral map*) of an integrable system with two degrees of freedom. The energy-momentum map is defined on the phase space of the system by a pair of functions in involution: the Hamiltonian function, which encodes the dynamics of the system, and the momentum, which encodes the symmetry associated to the Hamiltonian. For instance, the energy-momentum map of the spherical pendulum has the form

$$F = (H, J) \colon T^*S^2 \to \mathbb{R}^2,$$

where $H = K + V$ is the Hamiltonian (sum of the kinetic and the potential energy) of the pendulum and the integral J is the component of the angular momentum about the gravitational axis.

The map F defines a fibration of the phase into regular two-dimensional tori and a number of critical fibers, where the differential dF does not have a full rank. The situation is depicted in Fig. 1, where the *bifurcation diagram* of the spherical pendulum, that is, the set of the critical values of F, is shown.

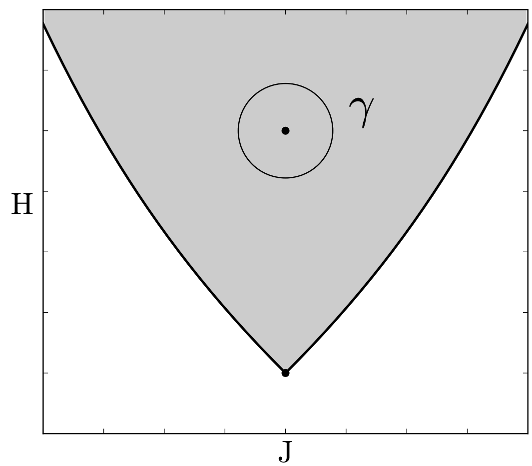

Figure 1: Bifurcation diagram of the spherical pendulum. The image of the energy-momentum map F is shaded gray. The critical values at the boundary of image(F) correspond to Huygens's horizontal periodic solutions. The isolated critical value is the projection of the unstable equilibrium, when the pendulum is at the top of the sphere.

For monodromy, a crucial property of the diagram is the presence of the isolated critical value, which corresponds to the unstable equilibrium when the pendulum is at the top of the sphere. The presence of such a critical value implies that the set R of the regular values of F is not simply connected: the curve γ shown in Fig. 1 cannot be shrunk to a single point within the set R. As a result, the torus fibration $F\colon F^{-1}(\gamma) \to \gamma$ is not necessarily trivial. It turns out that $F\colon F^{-1}(\gamma) \to \gamma$ is a non-trivial torus bundle with the *monodromy matrix*

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{Z}), \qquad (1)$$

which means that the preimage $F^{-1}(\gamma)$ can be obtained from the direct product $\gamma \times T^2$ by re-gluing the fibers using the monodromy matrix M. This result was established by Duistermaat in his work [27] and was shown to be incompatible with the existence of smooth action coordinates along γ.

Hamiltonian monodromy was later found to be non-trivial in various other fundamental integrable problems of physics and mechanics, such as the Lagrange top [22], the hydrogen atom in crossed fields [18], the Jaynes-Cummings model [83] and the two-center problem [70, 99]. The notion of Hamiltonian monodromy was also generalized in several different directions, including the cases of

i. *quantum* [17, 95]

ii. *fractional* [79] and

iii. *scattering* [5, 30] monodromy.

One important result on Hamiltonian monodromy is the *geometric monodromy theorem* [67, 74, 75, 105], which relates Hamiltonian monodromy to special isolated singularities of the energy-momentum map. The singularities are of the so-called *focus-focus* type and are similar to the unstable equilibrium found in the spherical pendulum. Specifically, the theorem states that the monodromy around a singular focus-focus fiber is always non-trivial and is given by the number of the focus-focus points on this fiber; see Fig. 2. For instance, in the case of the spherical pendulum the critical fiber contains only one focus-focus point. Hence, the off-diagonal entry in Eq. (1) is equal to 1. We note that the geometric monodromy theorem is valid also in the quantum case [95].

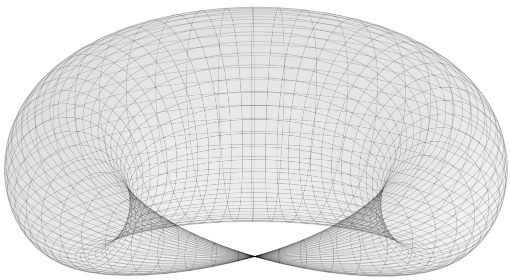

Figure 2: A focus-focus fiber with one singular point (a pinched torus).

There are a few different ways to prove the geometric monodromy theorem. For instance, the theorem can be proven by looking at the variation of the rotation number [96] or by applying methods from symplectic geometry [106]. In Chapter 1, after giving basic definitions in the theory of Hamiltonian systems, we give a new proof of the theorem, which is based on the topological approach proposed by F. Takens in [92]. Specifically, we show how the energy levels $H^{-1}(h)$ and their Euler numbers manifest the non-triviality of monodromy in the focus-focus case.

The approach that we develop in Chapter 1 applies more generally to integrable two degree of freedom systems that are invariant under a circle action. However, this approach does not directly generalize to systems with $n \geq 3$ degrees of freedom since it hinges on the use of the energy levels. In Chapter 2 we demonstrate that the symmetry alone is sufficient for the computation of monodromy. This will allow us to give a unified approach to Hamiltonian monodromy, which is applicable to integrable systems with many degrees of freedom and isolated singularities of a more general type than in the focus-focus case.

In Chapter 3 we consider a more general setting of fractional monodromy and Seifert manifolds. Passing to such manifolds allows us to generalize the obtained results on Hamiltonian monodromy, to generalize the known results on fractional monodromy (see the work [35] and references therein), and to make a connection to Fomenko-Zieschang theory. Our final theorems proven in this chapter can be summarized as follows.

- Fractional monodromy can naturally be defined for closed Seifert manifolds with an orientable base of genus $g > 0$.
- For such a Seifert manifold, the corresponding Euler number and the order of the deck group completely determine fractional monodromy.
- In the case of integrable systems, the Euler number can be computed in terms of the fixed points of the circle action.

We note that the importance of deck groups for fractional monodromy was observed in [35], where it was defined for a different covering. The importance of Seifert manifolds for integrable systems was discovered by Fomenko and Zieschang in the 1980's. In the context of fractional monodromy this was made more explicit by Bolsinov *et al.* in [10]. The question of why and how is the Euler number related to monodromy (Hamiltonian or fractional) is resolved in the present work.

In Chapter 4 we start with the discussion of general scattering theory, mostly following the work of Knauf [61]. Then we show how this theory can be adapted for the context of Liouville integrability. In particular, we propose a general definition of a reference Hamiltonian for scattering and integrable systems and generalize the notion of scattering monodromy [5,30] to this setting. We note that (unlike in the cases of Hamiltonian and fractional monodromy) the Liouville fibrations which are considered here are necessarily non-compact.

In Chapter 5 we apply our methods to the spatial Euler two-center problem. We consider the scattering case of positive energies (the gravitational problem in the case of negative energies was studied in [33, 99]) and show that in this case the problem has non-trivial scattering monodromy of two different types (pure and mixed scattering monodromy) as well as non-trivial Hamiltonian monodromy. Our results show that Hamiltonian and mixed scattering monodromy remain in the limiting case of the Kepler problem and that Hamiltonian monodromy is present also in the spatial free flow.

Several parts of the thesis have previously appeared as journal articles or as preprints. The references are:

Monodromy and Morse theory, preprint, 2018
N. Martynchuk, H.W. Broer and K. Efstathiou
(See Chapters 1 and 2)

Scattering invariants in Euler's two-center problem, preprint, 2018
N. Martynchuk, H.R. Dullin, K. Efstathiou and H. Waalkens
(See Chapters 4 and 5)

Parallel Transport along Seifert Manifolds and Fractional Monodromy
Comm. Math. Phys., 356(2):427-449, doi:10.1007/s00220-017-2988-5, 2017
N. Martynchuk and K. Efstathiou
(See Chapter 3)

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K. Efstathiou and N. Martynchuk
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(See Chapter 4)