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Distributed Event-Triggered Control for Asymptotic Synchronization of Dynamical Networks *

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Abstract

This paper studies synchronization of dynamical networks with event-based communication. Firstly, two estimators are introduced into each node, one to estimate its own state, and the other to estimate the average state of its neighbours. Then, with these two estimators, a distributed event-triggering rule (ETR) with a dwell-time is designed such that the network achieves synchronization asymptotically with no Zeno behaviours. The designed ETR only depends on the information that each node can obtain, and thus can be implemented in a decentralized way.

Key words: distributed event-triggered control, asymptotic synchronization, dynamical networks.

1 Introduction

Synchronization of dynamical networks, and its related problem – consensus of multi-agent systems, have attracted a lot of attention due to their extensive applications in various fields (see Arenas et al. (2008); Olfati-Saber et al. (2007); Ren et al. (2007); Wu (2007) for details). Motivated by the fact that connected nodes in some real-world networks share information over a digital platform, these problems have recently been investigated under the circumstance that nodes communicate to their neighbours only at certain discrete-time instants. To use the limited communication network resources effectively, event-triggered control (ETC) (see Heemels et al. (2012) and reference therein) introduced in networked control systems has been extensively used to synchronize networks. Under such a circumstance,

each node can only get limited information, and the main issue becomes how to use these limited information to design an ETR for each node such that the network achieves synchronization asymptotically and meanwhile to prevent Zeno behaviours that are caused by the continuous/discrete-time hybrid nature of ETC, and undesirable in practice (Tabuada (2007)).

In Dimarogonas and Johansson (2009), distributed ETC was developed to investigate consensus of a multi-agent system. To prevent Zeno behaviour, a decentralized ETR with a time-varying threshold was introduced to achieve consensus in Seyboth et al. (2013). Self-triggered strategies were proposed in De Persis and Frasca (2013) and shown to be robust to skews of the local clocks, delays, and limited precision in the communication. However, all these works focused on dynamical networks with simple node dynamics (single-integrators or double-integrators), which do not appear to extend in a straightforward way to networks with generalized node dynamics. Further, most of these existing results only guarantee bounded synchronization rather than asymptotic synchronization in order to exclude Zeno behaviour (e.g. Demir and Lunze (2012); Zhu et al. (2014)). In view of these issues, we study asymptotic synchronization of networks with generalized linear node dynamics with ETC. Firstly, a new sampling mechanism is used with which

two estimators are introduced into each node. One is used to estimate its own state, and the other is used to estimate the average state of the node's neighbours. A distributed ETR is then designed based on these estimators which guarantees asymptotic synchronization of the network. Moreover, inspired by the method proposed in Tallapragada and Chopra (2014), a dwell-time (Cao and Morse (2010)) is used to exclude Zeno, which can simplify the implementation of the designed ETR.

Our contribution is to propose a control law that for the first time has the three essential and desirable properties: i). the proposed ETR can guarantee asymptotic synchronization with no Zeno behaviours for networks with generalized linear node dynamics, whereas most of the existing results sacrifice synchronization performance and can only get bounded synchronization; ii). by introducing a new sampling mechanism, we reduce the number of estimators needed for each node to two, whereas existing results need $d_i + 1$ estimators (d_i is the degree of the node); iii). by introducing an estimation of synchronization errors between neighbours into the designed ETR, networks with proposed ETC can reduce the number of sampling times significantly.

2 Network Model and Preliminaries

Notation: Denote the set of real numbers, non-negative real numbers, and non-negative integers by $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{Z}^+$; the set of n-dimensional real vectors and $n \times m$ real matrices by $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$. I_n, 1_n and $1_{n \times m}$ are the n-dimensional identity matrix, n-dimensional vector and $n \times m$ matrix with all entries being 1, respectively. $\| \cdot \|$ represents the Euclidean norm for vectors and also the induced norm for matrices. The superscript $\top$ is the transpose of vectors or matrices. $\otimes$ is the Kronecker product of matrices. For a single $\omega : \mathbb{R}^+ \to \mathbb{R}^n$, $\omega(t^-) = \lim_{s \uparrow t} \omega(s)$. Let $\mathcal{G}$ be an undirected graph consisting of a node set $\mathcal{V} = \{1, 2, \ldots, N\}$ and a link set $\mathcal{E} = \{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_M\}$. If there is a link $\bar{e}_k$ between nodes i and j, then we say node j is a neighbour of node i and vice versa. Let $A = (a_{ij}) \in \mathbb{R}^{N \times N}$ be the adjacency matrix of $\mathcal{G}$, where $a_{ii} = 0$ and $a_{ij} = a_{ji} > 0$, $i \neq j$, if node i and node j are neighbours, otherwise $a_{ij} = a_{ji} = 0$. The Laplacian matrix $L = (l_{ij}) \in \mathbb{R}^{N \times N}$ is defined by $l_{ij} = -a_{ij}$, if $j \neq i$ and $l_{ii} = \sum_{j=1}^{N} a_{ij}$.

We consider dynamical networks whose state equation is

$$\dot{x}_i(t) = H x_i(t) + B u_i(t), \quad \forall i \in \mathcal{V} \tag{1}$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^\top \in \mathbb{R}^n$ is the state of node i. $H \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^n$, and $u_i \in \mathbb{R}$ are the node dynamics, input matrix, and control input, respectively. Generally, continuous communication between neighbouring nodes is assumed, i.e., $u_i(t) = K \sum_{j=1}^{N} a_{ij}(x_j(t) - x_i(t))$.

This yields the following network

$$\dot{x}_i(t) = H x_i + B K \sum_{j=1}^{N} a_{ij}(x_j(t) - x_i(t)). \tag{2}$$

In this paper, we assume that connections in (1) are realized via discrete communication, i.e., each node only obtains information from its neighbours at certain discrete-time instants. We will present an event-triggered version of network (2), and study how to design an ETR for each node to achieve asymptotic synchronization. We suppose that the topological structure of the network is fixed, undirected and connected. For simplicity, we only consider unweighted networks, i.e., $a_{ij} \in \{0, 1\}$; but the obtained results can be extended to weighted networks directly. We further assume that: there is no time delay for computation and execution, i.e., t_{k_i} represents both the k_ith sampling time and the k_ith time when node i broadcasts updates; and the communication network is under an ideal circumstance, i.e., there are no time delays or data dropouts in communication.

We introduce two estimators $\mathcal{O}_i$ and $\mathcal{O}_{\mathcal{V}_i}$ into each node i, where $\mathcal{O}_i$ is used to estimate its own state, and $\mathcal{O}_{\mathcal{V}_i}$ is used to estimate the average state of its neighbours. We adopt the following control input

$$u_i(t) = K \left(\hat{x}_{\mathcal{V}_i}(t) - l_{ii} \hat{x}_i(t) \right) \tag{3}$$

where $K \in \mathbb{R}^{1 \times n}$ is the control gain to be designed, $\hat{x}_i \in \mathbb{R}^n$ and $\hat{x}_{\mathcal{V}_i} \in \mathbb{R}^n$ are states of $\mathcal{O}_i$ and $\mathcal{O}_{\mathcal{V}_i}$, respectively. The state equations of $\mathcal{O}_i$ and $\mathcal{O}_{\mathcal{V}_i}$ are given by

$$\mathcal{O}_i : \quad \begin{aligned} \dot{\hat{x}}_i(t) &= H \hat{x}_i(t), \ t \in [t_{k_i}, t_{k_i+1}) \\ \hat{x}_i(t) &= x_i(t), \quad t = t_{k_i} \end{aligned} \tag{4}$$

$$\mathcal{O}_{\mathcal{V}_i} : \quad \begin{aligned} \dot{\hat{x}}_{\mathcal{V}_i}(t) &= H \hat{x}_{\mathcal{V}_i}(t), \quad t \in [t_{\bar{k}_i}, t_{\bar{k}_i+1}) \\ \hat{x}_{\mathcal{V}_i}(t) &= \hat{x}_{\mathcal{V}_i}(t^-) - \sum_{j \in \mathcal{J}_i} e_j(t^-), \quad t = t_{\bar{k}_i}. \end{aligned} \tag{5}$$

The increasing time sequences $\{t_{k_i}\}$ and $\{t_{\bar{k}_i}\}$, k_i, $\bar{k}_i \in \mathbb{Z}^+$ represent time instants that node i sends updates to its neighbours and that it receives updates from one or more of its neighbours, respectively. The set $\mathcal{J}_i = \mathcal{J}_i(t_{\bar{k}_i}) = \{j \mid t_{k_j} = t_{\bar{k}_i}, j \in \mathcal{V}_i\}$ is a subset of $\mathcal{V}_i$, from which node i receives updated information at $t = t_{\bar{k}_i}$, and $\mathcal{V}_i = \{j \mid a_{ij} = 1, j \in \mathcal{V}\}$ is the index set of the neighbours for node i. The error vector $e_i(t) = \hat{x}_i(t) - x_i(t)$ represents the deviation between the state of estimator $\mathcal{O}_i$ and its own. The time sequence $\{t_{k_i}\}$ is decided by the following ETR

$$t_{k_i+1} = \inf \left\{ t > t_{k_i} \mid r_i(t, x_i, \hat{x}_i, \hat{x}_{\mathcal{V}_i}) > 0 \right\} \tag{6}$$

where $r_i(\cdot, \cdot, \cdot, \cdot) : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the event-triggering function to be designed. For $t > t_{k_i}$, if $r_i > 0$ at $t = t_{k_i+1}^-$, then node i samples $x_i(t_{k_i+1}^-)$, $\hat{x}_i(t_{k_i+1}^-)$, calculates $e_i(t_{k_i+1}^-)$, sends $e_i(t_{k_i+1}^-)$ to its neighbours,

and reinitialize the estimator $\mathcal{O}_i$ at $t = t_{k_i+1}$ by $x_i(t_{k_i+1})$. In addition, node i will reinitialize the estimator $\mathcal{O}_{\mathcal{V}_i}$ by $\hat{x}_{\mathcal{V}_i}(t_{\bar{k}_i+1}) = \hat{x}_{\mathcal{V}_i}(t^-_{\bar{k}_i+1}) - \sum_{j \in \mathcal{J}_i} e_j(t^-_{\bar{k}_i+1})$ each time when it receives updates from its neighbours.

We assume the network is well initialized at $t = t_0$, i.e., $\hat{x}_i(t_0) = 0$ and each node samples and sends $e_i(t_0)$ to its neighbours. Therefore, we have $\hat{x}_i(t_0) = x_i(t_0)$, $\hat{x}_{\mathcal{V}_i}(t_0) = \sum_{j \in \mathcal{V}_i} x_j(t_0)$ and $\mathcal{J}_i(t_0) = \mathcal{V}_i$ for all $i \in \mathcal{V}$. Then, the problem is with the given network topology, to determine the time sequence $\{t_{k_i}\}$, $k_i \in \mathbb{Z}^+$ by designing a proper ETR (6) such that network (1) achieves synchronization asymptotically without Zeno behaviours.
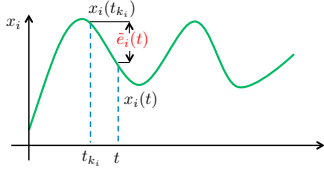

Fig. 1. The block diagram of $\tilde{e}_i(t)$.

Remark 1 *In Liu et al. (2012), $u_i = BK\tilde{z}_i(t)$ and ETR $t_{k_i+1} = \inf\{t \geq t_{k_i} \mid \|\tilde{e}_i(t)\| \geq \rho\|\tilde{z}_i(t)\|\}$ were adopted where $\tilde{e}_i(t) = x_i(t) - x_i(t_{k_i})$ and $\tilde{z}_i(t) = \sum_{j \in \mathcal{V}_i}(x_j(t_{k_j}) - x_i(t_{k_i}))$. It turns out that the ETR with $\tilde{e}_i(t)$ and $\tilde{z}_i(t)$ cannot avoid Zeno behaviours, in particular for networks whose nodes synchronize to a time-varying solution. Suppose indeed the network achieves asymptotic synchronization under the above ETR. As $x_i(t)$ and $x_j(t)$ approach to each other and converge to a time-varying solution, $\tilde{z}_i(t)$ may converge to zero as well. However, $\tilde{e}_i(t)$ will not converge to zero (see Figure 1), and this makes $t_{k_i+1} - t_{k_i}$ close to zero and may lead to Zeno behaviour. This is the reason that we introduce estimators into each nodes. By doing so, $e_i(t)$ will approach zero, which may exclude Zeno behaviours for each node. For each node, $d_i + 1$ estimators were used to achieve bounded synchronization in Demir and Lunze (2012); whereas controller (3) only needs two estimators $\mathcal{O}_i$ and $\mathcal{O}_{\mathcal{V}_i}$. Therefore, the advantage of our control method is clear, in particular for networks with large degrees. Moreover, we will show that our controller (3) with a distributed ETR can achieve asymptotic synchronization with no Zeno behaviours.*

Remark 2 *The state error $e_i(t) = \hat{x}_i(t) - x_i(t)$ (or $e_i(t) = x_i(t_{k_i}) - x_i(t)$ for networks with no estimators) is extensively used to design ETR in the literature (see Seyboth et al. (2013); Tallapragada and Chopra (2014); Zhu et al. (2014) for examples) where each node samples its state and sends the sampled state to its neighbours. In order to reduce the number of estimators, we make each node send the sampled error $e_i(t_{k_i})$ instead of $x_i(t_{k_i})$ to its neighbours who will use this information to update the corresponding estimator $\mathcal{O}_{\mathcal{V}_i}$. The implementation of the this new sampling mechanism needs no more information than that used in the literature. It should be noted that most synchronization algorithms for network*

(2) *with continuous nodes' interactions only use relative state information, and it is very important to study network (7) by also using the relative state information for the design purposes which should be studied in the future.*

To simplify the analysis, we will show that network (1) with controller (3) and estimators (4), (5) is equivalent to the following system where each node maintains an estimator of the state of each of its neighbours.

$$\dot{x}_i(t) = Hx_i(t) - BK\sum_{j=1}^{N} l_{ij}\hat{x}_j(t), \forall i \in \mathcal{V} \quad (7a)$$

$$\mathcal{O}_i : \begin{aligned} \dot{\hat{x}}_i(t) &= H\hat{x}_i(t), \ t \in [t_{k_i}, t_{k_i+1}) \\ \hat{x}_i(t) &= x_i(t), \quad t = t_{k_i}. \end{aligned} \quad (7b)$$

Defining $\bar{z}_i = \sum_{j \in \mathcal{V}_i} \hat{x}_j$ gives $\dot{\bar{z}}_i(t) = \sum_{j \in \mathcal{V}_i} \dot{\hat{x}}_j(t) = H\bar{z}_i(t)$, $t \in [t_{\bar{k}_i}, t_{\bar{k}_i+1})$, which has the same dynamics as $\hat{x}_{\mathcal{V}_i}$ defined in (5). Moreover, at $t = t_{\bar{k}_i}$, we have

$$\begin{aligned} \bar{z}_i(t) &= \sum_{j \in \mathcal{V}_i/\mathcal{J}_i(t)} \hat{x}_j(t^-) + \sum_{j \in \mathcal{J}_i(t)} x_j(t) \\ &= \sum_{j \in \mathcal{V}_i/\mathcal{J}_i(t)} \hat{x}_j(t^-) + \sum_{j \in \mathcal{J}_i(t)} \left(\hat{x}_j(t^-) - e_j(t^-)\right) \\ &= \hat{x}_{\mathcal{V}_i}(t). \end{aligned} \quad (8)$$

Thus, we have $\bar{z}_i(t) = \hat{x}_{\mathcal{V}_i}(t)$ for all $t \geq t_0$. Then, controller (3) becomes

$$u_i = K\left(\bar{z}_i - l_{ii}\hat{x}_i\right) = K\left(\hat{x}_{\mathcal{V}_i} - l_{ii}\hat{x}_i\right). \quad (9)$$

Substituting (9) into (1) gives that network (1) with (3), (4), and (5) is equivalent to (7).

Moreover, let $\hat{z}_i = \sum_{j \in \mathcal{V}_i}(\hat{x}_j - \hat{x}_i)$. We have $\hat{x}_{\mathcal{V}_i} = \bar{z}_i = \hat{z}_i + l_{ii}\hat{x}_i$. Then, ETR (6) can be reformulated as

$$t_{k_i+1} = \inf\{t > t_{k_i} \mid r_i(t, x_i, \hat{x}_i, \hat{z}_i) > 0\}. \quad (10)$$

This paper will use model (7) and ETR (10) for the analysis. But the obtained results can be implemented by using controller (3) with the two estimators $\mathcal{O}_i$, $\mathcal{O}_{\mathcal{V}_i}$ and ETR (6). To finish this section, we give the definition of asymptotic synchronisation based on network (7).

Definition 1 *Let $x(t) = \left(x_1^\top(t), x_2^\top(t), \ldots, x_N^\top(t)\right)^\top \in \mathbb{R}^{nN}$ and $\hat{x}(t) = \left(\hat{x}_1^\top(t), \hat{x}_2^\top(t), \ldots, \hat{x}_N^\top(t)\right)^\top \in \mathbb{R}^{nN}$ be a solution of network (7) with initial condition $x_0 = \left(x_{10}^\top, x_{20}^\top, \ldots, x_{N0}^\top\right)^\top$ and $x_{i0} = x_i(t_0)$. Then, the network is said to achieve synchronization asymptotically, if for every $x_0 \in \mathbb{R}^{nN}$ the following condition is satisfied*

$$\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0, \quad \forall i, j \in \mathcal{V}. \quad (11)$$

Remark 3 *When the communication network is not ideal, model (1) with (3) and $\mathcal{O}_i, \mathcal{O}_{\mathcal{V}_i}$ cannot be simplified*

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to (7). *A more complicated model is needed to describe the network dynamics. Time delays and packet loss will influence the synchronization performance. However, due to the robust property of asymptotic synchronization, bounded synchronization can be guaranteed where the final synchronization error may depend on the time delay magnitude and probability of packet loss. These issues should be studied in the future. Another important problem for this situation is under what conditions the network can still achieve synchronization asymptotically.*

3 Event-Triggered Control

Denote $e(t) = \left(e_1^\top(t), e_2^\top(t), \ldots, e_N^\top(t)\right)^\top$ with $e_i(t) = \hat{x}_i(t) - x_i(t)$. Then, network (7a) can be rewritten by

$$\dot{x} = (I_N \otimes H - L \otimes BK)x - (L \otimes BK)e. \quad (12)$$

Since the topology of the network is undirected and connected, the Laplacian matrix L is irreducible, symmetric, and has only one zero eigenvalue. Further, there exists an orthogonal matrix $\Psi = (\psi_1, \psi_2, \ldots, \psi_N) \in \mathbb{R}^{N \times N}$ with $\psi_i = (\psi_{i1}, \psi_{i2}, \ldots, \psi_{iN})^\top$ and $\Psi^\top \Psi = I_N$ such that $\Psi^\top L \Psi = \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$ where $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N$. Choose $\psi_1 = 1/\sqrt{N}1_N^\top$ for λ_1. Due to the zero row sum property of L, we have $\sum_{j=1}^N \psi_{ij} = 0$ for all $i = 2, 3, \ldots, N$. Defining $\Phi = (\psi_2, \psi_3, \ldots, \psi_N) \in \mathbb{R}^{N \times (N-1)}$ gives

$$\Phi^\top \Phi = I_{N-1}, \quad \Phi\Phi^\top = I_N - 1/N1_{N \times N}. \quad (13)$$

Let $\Lambda_1 = \Phi^\top L \Phi = \mathrm{diag}\{\lambda_2, \lambda_3, \ldots, \lambda_N\}$, $\bar{\Phi} = \Phi \otimes I_n$ and $\bar{\Lambda} = \Lambda_1 \otimes BK = \mathrm{diag}\{\lambda_2 BK, \lambda_3 BK, \ldots, \lambda_N BK\}$. Defining $y = \bar{\Phi}^\top x$ gives

$$\begin{aligned} \dot{y}(t) =& \bar{\Phi}^\top \left((I_N \otimes H)x - (L \otimes BK)(I_{Nn} - \bar{\Phi}\bar{\Phi}^\top \right. \\ & \left. + \bar{\Phi}\bar{\Phi}^\top)(x + e)\right) \\ =& (I_{N-1} \otimes H - \Lambda_1 \otimes BK)y - \bar{\Lambda}\bar{\Phi}^\top e \quad (14) \end{aligned}$$

where we use properties $\bar{\Phi}^\top(I_N \otimes H) = (I_{N-1} \otimes H)\bar{\Phi}^\top$ and $(L \otimes BK)(I_{Nn} - \bar{\Phi}\bar{\Phi}^\top) = 0$ for any BK, which are supported by facts $L1_N = 0$ and (13). Denoting $\bar{H} = (I_{N-1} \otimes H) - (\Lambda_1 \otimes BK) = \mathrm{diag}\{H_2, H_3, \ldots, H_N\}$ with $H_i = H - \lambda_i BK$, system (14) can be simplified to

$$\dot{y} = \bar{H}y - \bar{\Lambda}\bar{\Phi}^\top e. \quad (15)$$

By defining $\bar{x} = \frac{1}{N}\sum_{i=1}^N x_i$, we have $\|y\|^2 = x^\top \bar{\Phi}\bar{\Phi}^\top x = \sum_{i=1}^N \|x_i - \bar{x}\|^2$ where the last equality follows from $\Phi^\top \Phi = I_{N-1}$ and $(\bar{\Phi}\bar{\Phi}^\top)^2 = \bar{\Phi}\bar{\Phi}^\top$. Therefore, if $\lim_{t \to \infty} \|y(t)\| = 0$, then $x_i(t) = x_j(t) = \bar{x}(t)$ as $t \to \infty$, i.e., network (7) achieves synchronization asymptotically. This result is summarized in the following lemma.

Lemma 1 *If system* (15) *is asymptotically stable, i.e.,* $\lim_{t \to \infty} \|y(t)\| = 0$, *then network* (7) *achieves synchronization asymptotically.*

It is shown in Trentelman et al. (2013) that a necessary and sufficient condition for asymptotic synchronization of network (2) with continuous interconnections is the existence of positive definite matrices P_i such that

$$H_i^\top P_i + P_i H_i = -2I_n, \ i = 2, 3, \ldots, N. \quad (16)$$

This condition requires all the linear systems with system matrices $H_i = H - \lambda_i BK$, $i = 2, \ldots, N$ are asymptotically stable simultaneously, which is stronger than that (H, B) is stabilizable. From (14), network (7) with ETC can be regarded as network (2) with an external input (or disturbance) $\bar{\Lambda}\bar{\Phi}^\top e$. According to ISS (input-to-state stability) theory, a necessary condition that the linear system (14) is asymptotically stable is that the the corresponding system (also described by (14) but without the term $\bar{\Lambda}\bar{\Phi}^\top e$) is asymptotically stable. Hence, the existence of matrix solutions P_i to Lyapunov equations (16) is also a fundamental requirement for network (7) with ETC to achieve asymptotic synchronization. In this paper, we assume that such matrices P_i exist.

Let $z_i = \sum_{j \in \mathcal{V}_i}(x_j - x_i)$, $\hat{z}_i = \sum_{j \in \mathcal{V}_i}(\hat{x}_j - \hat{x}_i)$, $z = (z_1^\top, z_2^\top, \ldots, z_N^\top)^\top = (-L \otimes I_n)x$, $\hat{z} = (\hat{z}_1^\top, \hat{z}_2^\top, \ldots, \hat{z}_N^\top)^\top = (-L \otimes I_n)\hat{x}$, $\rho = \frac{\delta}{\lambda_N \sqrt{2N(\alpha^2 + \delta^2)}}$, $\rho_1 = \frac{1}{\lambda_2}(\frac{\delta}{\sqrt{2(\alpha^2 + \delta^2)}} + 1)$, $\delta \in (0, 1)$, $\alpha = \max_{i=2,3,\ldots,N} \{\lambda_i \|P_i BK\|\}$, $a = \|H\| + \|\bar{H}\| + \lambda_N \frac{\delta}{\alpha}\|BK\|$, and $b = \lambda_N \|BK\|(1 + \frac{\delta}{\alpha})$. Next, we will give a useful lemma which will be used to prove the main result of the paper.

Lemma 2 *Consider network* (7). *The following two inequalities hold for any* $t \geq t_0$

$$\|\hat{z}\| \leq \lambda_N(\|e\| + \|y\|) \quad (17)$$
$$\lambda_2 \|y\| \leq \lambda_N \|e\| + \|\hat{z}\|. \quad (18)$$

PROOF. Due to $\|(L \otimes I_n)\| = \lambda_N$, we have

$$\|\hat{z}\| = \|(L \otimes I_n)(x + e)\| \leq \|z\| + \lambda_N \|e\| \quad (19)$$

$$\|z\| = \|(L \otimes I_n)(\hat{x} - e)\| \leq \|\hat{z}\| + \lambda_N \|e\|. \quad (20)$$

Let $U = \Phi\Phi^\top$, then for any L, we have $LU = UL$, i.e., L and U are diagonalizable simultaneously. Further, we have $\Psi^\top L \Psi = \Lambda$ and $\Psi^\top U \Psi = \mathrm{diag}\{\lambda_{u1}, \lambda_{u2}, \ldots, \lambda_{uN}\}$, where $\lambda_{u1} = 0$ and $\lambda_{ui} = 1$, $i = 2, 3, \ldots, N$ are eigenvalues of U. Let $\bar{\lambda}_i$, $i = 1, 2, \ldots, N$ be eigenvalues of the matrix $(\lambda_N^2 U^2 - L^2)$. Then with $U^2 = U$, we have $\bar{\lambda}_1 = 0$ and $\bar{\lambda}_i = \lambda_N^2 - \lambda_i^2 \geq 0$, $i = 2, 3, \ldots, N$, which gives $L^2 \leq \lambda_N^2 U^2$. Thus, we have

$$\begin{aligned} \|z\|^2 =& x^\top(L^2 \otimes I_n)x \leq \lambda_N^2 x^\top(U^2 \otimes I_n)x \\ =& \lambda_N^2 \|\bar{\Phi}^\top x\|^2 = \lambda_N^2 \|y\|^2. \end{aligned} \quad (21)$$

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Combining (19) with (21) gives inequality (17). Similar to (21), we have $\|y\|^2 = x^\top(U^2 \otimes I_n)x \leq 1/\lambda_2^2 x^\top(L^2 \otimes I_n)x$ which with (20) gives (18). $\qquad \square$

Theorem 1 *Network (7) achieves synchronization asymptotically under the distributed ETR*

$$t_{k_i+1} = \inf\{t \geq t_{k_i} + \tau^* \mid \|e_i\| > \rho\|\hat{z}_i\|\} \qquad (22)$$

where

$$\tau^* = \frac{1}{a}\ln\left(\frac{a\rho}{b\rho_1} + 1\right) > 0. \qquad (23)$$

Moreover, no Zeno behaviour occurs in the network.

PROOF. Under ETR (22), the existence of $\tau_{k_i} = t_{k_i+1} - t_{k_i} > 0$ is guaranteed by dwell-time τ^*. To show asymptotic synchronization, we claim that the network with (22) satisfies $\|e_i\| \leq \rho\|\hat{z}\|$ for all $t \geq t_0$. This is true at $t = t_0$, as we have $\|e_i(t_0)\| = 0$ and hence $\|e_i(t_0)\| \leq \rho\|\hat{z}(t_0)\|$ for all $i \in \mathcal{V}$. Now, suppose this is not true at some $t > t_0$, and let t^* be the infimum of the times at which there exists a node $i \in \mathcal{V}$ such that $\|e_l(t)\| > \rho\|\hat{z}(t)\|$ (the analysis is the same if this happens in multiple nodes). There holds thus $\|e_i(t)\| \leq \rho\|\hat{z}(t)\|$ for all $i \in \mathcal{V}$ and all $t < t^*$, which gives

$$\|e\|^2 = \sum_{i=1}^N \|e_i\|^2 \leq \frac{\delta^2}{2\lambda_N^2(\alpha^2 + \delta^2)}\|\hat{z}\|^2. \qquad (24)$$

Combining (17) with (24) yields

$$\|e(t)\| \leq \frac{\delta}{\alpha}\|y(t)\|, \quad \forall t \in [t_0, t^*). \qquad (25)$$

Based on (22), we can conclude that $\|e_l(t)\| > \rho\|\hat{z}(t)\|$ can only happen when $t^* \in (t_{k_l}, t_{k_l} + \tau^*]$ with $k_l \geq 1$ since $\|e_l\| \leq \rho\|\hat{z}_l\| \leq \rho\|\hat{z}\|$, $\forall t \in (t_{k_l} + \tau^*, t_{k_l+1})$. Calculating $\frac{d}{dt}\frac{\|e_l\|}{\|y\|}$ for $t \in [t_{k_l}, t^*)$ directly gives

$$\frac{d}{dt}\frac{\|e_l\|}{\|y\|} \leq \left(\|H\| + \|\bar{H}\|\right)\frac{\|e_l\|}{\|y\|} + \frac{\|\bar{\Lambda}\|\|e_l\|\|e\|}{\|y\|^2}$$
$$+ \lambda_N\|BK\|\frac{\|e\|}{\|y\|} + \lambda_N\|BK\| \qquad (26)$$

where we use (17) in Lemma 2 to get (26). Substituting (25) into (26) gives

$$\frac{d}{dt}\frac{\|e_l\|}{\|y\|} \leq a\frac{\|e_l\|}{\|y\|} + b. \qquad (27)$$

Based on the comparison theory (Khalil (2002)), we have $\|e_l(t)\|/\|y(t)\| \leq \phi(t - t_{k_l})$, whenever $\|e_l(t_{k_l})\|/\|y(t_{k_l})\| \leq \phi(t_{k_l})$ where $\phi(t - t_{k_l})$ is the solution of the ordinary differential equation

$$\dot{\phi} = a\phi + b \qquad (28)$$

with the initial condition $\phi(t_{k_l})$. At $t = t_{k_l}$, we have $\|e_l(t_{k_l})\|/\|y(t_{k_l})\| = 0$. Setting $\phi(t - t_{k_l}) = 0$ gives

$$\frac{\|e_l(t)\|}{\|y(t)\|} \leq \phi(t - t_{k_l}), \forall t \in [t_{k_l}, t^*). \qquad (29)$$

Further, combining (18) with (24) gives $\|\hat{z}\| \geq \|y\|/\rho_1$ which with (29) leads to

$$\frac{\|e_l(t)\|}{\|\hat{z}(t)\|} \leq \rho_1\frac{\|e_l(t)\|}{\|y(t)\|} \leq \rho_1\phi(t - t_{k_l}), \quad \forall t \in [t_{k_l}, t^*).$$

Solving (28) shows that it will take $\phi(t - t_{k_l})$ a positive time constant τ^* to change its values from 0 to ρ/ρ_1, so does $\|e_l(t)\|/\|y(t)\|$. Therefore, it requires at least τ^* to make $\|e_l(t)\|$ move from 0 to $\rho\|\hat{z}(t)\|$.

Suppose, to obtain a contradiction, that $t^* < t_{k_l} + \tau^*$. In that case, $\|e_l(t)\|/\|y(t)\| \leq \phi(t - t_{k_l}) < \phi(\tau^*) \leq \rho/\rho_1$, for all $t \leq t^*$. By continuity of $\|e_l\|/\|y\|$, this implies the existence of an $\varepsilon > 0$ such that $\|e_l(t)\|/\|y(t)\| < \phi(\tau^*)$ for all $t \leq t^* + \varepsilon$. Therefore, there holds then $\|e_i(t)\| < \rho\|\hat{z}(t)\|$ for all $t < t^* + \varepsilon$, in contradiction with t^* being the infimum of the times at which $\|e_i(t)\| > \rho\|\hat{z}(t)\|$.

Hence if such a t^* exists, there must hold $t^* \geq t_{k_l} + \tau^*$, and $\|e_l(t)\| \leq \rho\|\hat{z}(t)\|$ for all $t \in [t_{k_l}, t_{k_l} + \tau^*)$. Thus, we conclude that $\|e_i\| \leq \rho\|\hat{z}\|$, $\forall t \in [t_{k_i}, t_{k_i} + \tau^*)$. According to ETR (22), we have $\|e_i\| \leq \rho\|\hat{z}\|$, $\forall t \geq t_0$. Therefore, we have (25) hold $\forall t \geq t_0$. This is equivalent to say that (25) holds $\forall t \geq t_0$.

Now, select the Lyapunov function candidate $V = y^\top Py$ with $P = \text{diag}\{P_2, P_3, \ldots, P_N\}$. Then, the derivative of V along system (15) satisfies

$$\dot{V} \leq -2\|y\|^2 + 2\alpha\|y\|\|\bar{\Phi}^\top e\|. \qquad (30)$$

Combining (25) with $\|\bar{\Phi}\| = 1$ yields

$$\|\bar{\Phi}^\top e\| \leq \|\bar{\Phi}^\top\|\|e\| = \|e\| \leq \frac{\delta}{\alpha}\|y\|. \qquad (31)$$

Substituting (31) into (30) gives

$$\dot{V} \leq -2(1 - \delta)\|y\|^2 < 0, \quad \forall\|y\| \neq 0. \qquad (32)$$

Therefore, equilibrium point $y = 0$ of system (15) is asymptotically stable. Based on Lemma 1, the network achieves synchronization asymptotically. $\qquad \square$

Remark 4 *Like most works in synchronization of dynamical network (with/without ETC), in particular for networks with generalized linear node dynamics (e.g. Trentelman et al. (2013); Guinaldo et al. (2013)), one usually needs some global parameters to guarantee asymptotic synchronization as well as to exclude*

Zeno behaviours. These parameters can be estimated by using methods proposed in the related literature (e.g. Franceschelli et al. (2013)), and can be initialized to each node at the beginning. However, how to use local parameters rather than global ones (e.g. how to replace N by using local parameter such as the degree of the node d_i) remains open, and deserves attention.

Remark 5 *Most works in the literature (e.g. Demir and Lunze (2012); Guinaldo et al. (2013); Seyboth et al. (2013); Zhu et al. (2014)) use decentralized ETRs which can be summarized in a compact from as follows (Guinaldo et al. (2013))*

$$t_{k_i+1} = \inf\left\{t \mid \|e_i\| > c_0 + c_1 e^{-\gamma t}\right\} \qquad (33)$$

where $e_i(t) = \hat{x}_i(t) - x_i(t)$ (or $e_i(t) = x_i(t_{k_i}) - x_i(t)$ for networks with no estimators), and $c_0 \geq 0$, $c_1 \geq 0$, $\gamma > 0$ are three design parameters. It is obvious that only bounded synchronization can be achieved under ETR (33) with $c_0 \neq 0$ which is the case extensively studied in the literature, i.e., $c_0 > 0$ and $c_1 = 0$ (see Demir and Lunze (2012); Zhu et al. (2014) for examples). Further, these decentralized ETRs ignore the interactions (or differences) between neighbours, and hence may have conservativeness, in particular when these differences are large. Our new distributed ETR (22) achieves asymptotic synchronization by introducing $\|\hat{z}_i\|$. The term $\|\hat{z}_i\|$ updated by $x_j(t_{k_j})$ estimates synchronization errors between neighbours continuously, and thus provides each node useful information for determining its sampling times. Therefore, the proposed ETR can reduce the sampling times significantly, in particular for cases where $\|\hat{z}_i\|$ is large (see the example in Section 4 for details).

Remark 6 *To simplify notations, this paper only considers the case where u_i is a scalar. However, the obtained results can extend to multiple-input case directly. It is pointed out in Heemels et al. (2013) that the joint design of the controller and event-triggering rule is a hard problem. However, we can select any control gain K to synchronizes the continuous-time network (2), i.e., to stabilize $(H, \lambda_i B)$, $i = 2, \ldots, N$ simultaneously. It can be selected by solving a group of linear matrix inequalities. Further, it is shown in Liu et al. (2013) that a similar distributed ERT as (22) but with exponential term $c_1 e^{-\gamma t}$ can also guarantees asymptotic synchronization. However, this paper replaces the exponential term by dwell time term which can be implemented easily in practice. Such a τ^* gives an upper bound for the designed ETR (22), and therefore, a modified ETR with $0 < \tau_i^* \leq \tau^*$ can also synchronize the network without Zeno behaviours.*

Remark 7 *Instead of monitoring the triggering condition continuously, a periodic ETC method was proposed to stabilize linear systems exponentially in Heemels et al. (2013) where the triggering condition was verified periodically. Similar idea was used to achieve consensus*

of multi-agent systems in Meng and Chen (2013); Xiao et al. (2015). However, bi-directional communication were used, i.e., at each event time, the node needs to send its sampled state to its neighbours and meanwhile also needs its neighbours' newest sampled state to update its control signal; whereas in our paper, the node only needs to send it sampled information to its neighbours but does not need information from its neighbours. In the paper, we don't check the event-triggering condition in the time interval $[t_{k_i}, t_{k_i} + \tau^*)$, but need to check the condition continuously during the period $[t_{k_i} + \tau^*, t_{k_i+1})$. It is of great interest to study asymptotic synchronization by using periodic ETC and one-directional communications.

4 An Example

To show the effectiveness of our method, consider a network with 10 nodes that have parameters as follows

$$H = \begin{pmatrix} 0 & -0.5 \\ 0.5 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad K = \begin{pmatrix} -0.5 & 1 \end{pmatrix}.$$

We adopt the two-nearest-neighbour graph to describe the topology, i.e., $\mathcal{V}_i = \{j \mid j = i-2, i-1, i+1, i+2\}$, $i = 1, 2, \ldots, 10$. If $j \in \mathcal{V}_i$ and $j < 0$, then $j = j + 11$. If $j \in \mathcal{V}_i$ and $j > 0$, then $j = j - 10$. Since the matrix H has two eigenvalues on the imaginary axis of the complex plane, the network will synchronize to a stable time-varying solution determined by the initial condition. By calculating, we get $\alpha = 2.9061$. We select $\delta = 0.9$. Figure 2 gives the simulation results of the network with the distributed ETR (22) (DDT), which shows the effectiveness of the proposed method. In the figure, we only give the sampling time instants in the first 2 seconds for clarity. The theoretical value of τ^* is 0.0013 s. The minimum and maximum sample periods (τ_{min}/τ_{max}) for each node during the simulation time are given in Table 1 which shows that the actual sample periods are much larger than τ^*.

We also compared our method with the decentralized ETR (33) (DET) proposed in Guinaldo et al. (2013). According to Remark 5, only bounded synchronization can be guaranteed with $c_0 \neq 0$ in (33) (Seyboth et al. (2013)). For this case, the advantage of our method is clear. So here, we only compare our method with the case $c_0 = 0$ where asymptotic synchronization under (33) can also be achieved. We select $c_1 = \rho$ and $\gamma = 0.30579$. During the simulation period $(0 - 18\,\text{s})$, the network with DDT samples 3432 times in total, whereas the network with DET samples 212 times more (3644 times in total).

5 Conclusion

This paper has investigated asymptotic synchronization of dynamical networks by using distributed ETC. With
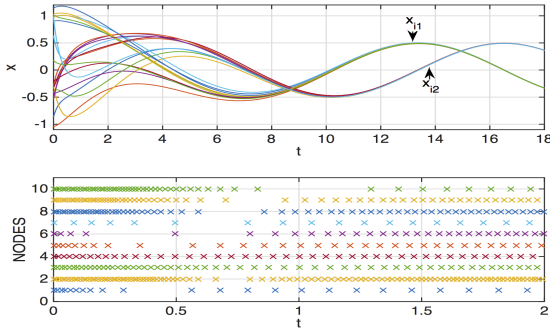
Fig. 2. Simulation for the network with DDT.

Table 1. The minimum/maximum sample period

	Node 1	Node 2	Node 3	Node 4	Node 5
τ_{min}	0.0153	0.0114	0.016	0.0214	0.0188
τ_{max}	0.2651	0.5292	0.6336	0.0817	0.1851
	Node 6	Node 7	Node 8	Node 9	Node 10
τ_{min}	0.0046	0.0688	0.0116	0.0117	0.0117
τ_{max}	0.3584	0.2841	1.4677	0.5347	0.5238

the help of the introduced estimators, a distributed ETR for each node has been explored, which only relies on the state of the node and the states of the introduced estimators. It has been shown that the proposed ETC synchronizes the network asymptotically with no Zeno behaviours. It is worth pointing out that time-delay and data packet dropout are common phenomena which definitely affects the synchronization of a network with event-based communication. Thus, it appears that the synchronization of such networks with imperfect communication is an important issue to pursue further for both theoretical interest and practical consideration.

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