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NONLINEAR AND QUANTUM OPTICS

Inversionless Superradiance and the Duffing Model[1]

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Abstract—Superradiance of three-level optical systems with a doublet in the ground state (Λ-scheme) placed in a high-Q cavity is studied theoretically. The conservation laws are obtained, which allow to considerably reduce the dimension of the phase space of the examined model ($\mathbb{R}^{11} \to \mathbb{R}^{5}$). In the particular case of a degenerate doublet, a mapping that makes it possible to reduce the problem of the three-level superradiance to a Duffing oscillator model ($\mathbb{R}^{5} \to \mathbb{R}^{2}$) is found. It is shown the possibility to initiate the superradiance generation even in the case when the population of the upper level is smaller than the total population of the lower doublet, i.e., without population inversion on the whole.

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INTRODUCTION

It is well-known that the necessary condition for the existence of the Dicke superradiance [1] is the presence of the initial population inversion of transition levels [2–13]. In the case of multilevel emitters (in particular, in the case of three-level atoms with the Λ-scheme of operating transitions, which are considered in this work), this restriction is not necessary: the superradiance can occur even when the initial population of the upper level is smaller than the total population of the lower doublet (inversionless superradiance) [14–38]. The essence of the effect is as follows. If one prepares the initial state of the lower doublet as a coherent superposition, transition to which from the upper state is forbidden, then the orthogonal to the initial superposition, transition to which is allowed, appears to be unpopulated. In this case, the transition from the upper level to this superposition state appears to be inverted at an arbitrarily small population of the upper level. The initial coherent state of the doublet can be created by a short low-frequency $\pi/2$-pulse [22–27, 31–36]. Systems of this kind can be realized in crystals, e.g., in a LaF_3 matrix doped with praseodymium ions Pr^{+3} [27–30], the ground state of which has a fine structure. It should be noted that, in systems of two-level emitters, it is frequently difficult to reach population inversion. The presence of an additional level that is close to the ground one opens the opportunity to avoid this problem.

The objective of this work is to theoretically study the nonlinear dynamics of the superradiance of an ensemble of three-level Λ-atoms which are spatially homogeneously and isotropically distributed in a high-Q cyclic cavity. The model of the superradiance that is proposed in this work is conservative (Hamiltonian); i.e., we do not take into account the relaxation of the population and polarization and the dissipation that is related to other (except superradiance) processes, as well as the lateral losses of the superradiance field energy. The time dynamics of the model is considered in terms of the semiclassical approach: the ensemble of three-level emitters is described by equations for the density matrix ρ_{mn} $(m, n = 1, 2, 3)$, while the electromagnetic field is described by the Maxwell equations. The conservation of the system leads to the occurrence of integrals of motion, that considerably reduces the dimension of the phase space of the examined model: $\mathbb{R}^{11} \to \mathbb{R}^{5}$. For the degenerate doublet, mapping is found that makes it possible to reduce the problem of the three-level superradiance to the Duffing oscillator model ($\mathbb{R}^{5} \to \mathbb{R}^{2}$). In this limit, the separatrix solution is investigated, and it is shown that the separatrix represents the point.

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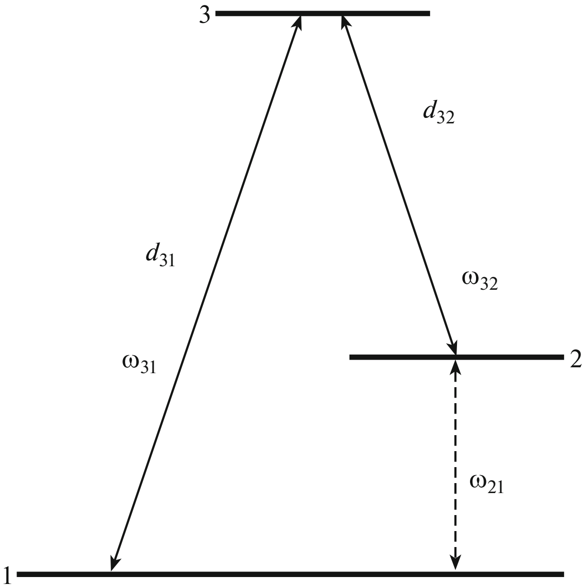

Fig. 1. Energy-level diagram of examined Λ-emitters. The number of the level ($n = 1,2,3$) corresponds to the state of the emitter with energy E_n. Solid and dashed arrows indicate, respectively, the allowed and forbidden transitions between the singled out energy levels of the emitter, with the frequencies of the corresponding transitions being ω_{21}, ω_{31}, and ω_{32} and the transition dipole moments being d_{31} and d_{32} ($d_{21} = 0$).

MODEL AND FORMALISM

We consider an ensemble of a three-level atoms with the Λ-scheme of operation transitions (Fig. 1), with these atoms being homogeneously distributed along one of the arms of a high-Q cyclic cavity (Fig. 2). In addition, all vectors (transition dipole moments and polarization of the field) are assumed to be directed identically and perpendicularly to the axis of the system. The evolution of the system then obeys the following (one-dimensional) system of Maxwell–Bloch equations:

$$\begin{gathered}
\dot{\rho}_{11} = i\frac{d_{31}E}{\hbar}(\rho_{31} - \rho_{13}),\\
\dot{\rho}_{22} = i\frac{d_{32}E}{\hbar}(\rho_{32} - \rho_{23}),\\
\dot{\rho}_{33} = -i\frac{d_{31}E}{\hbar}(\rho_{31} - \rho_{13}) - i\frac{d_{32}E}{\hbar}(\rho_{32} - \rho_{23}),\\
\dot{\rho}_{21} = -i\omega_{21}\rho_{21} - i\frac{d_{31}E}{\hbar}\rho_{23} + i\frac{d_{32}E}{\hbar}\rho_{31}, \qquad (1)\\
\dot{\rho}_{31} = -i\omega_{31}\rho_{31} - i\frac{d_{31}E}{\hbar}(\rho_{33} - \rho_{11}) + i\frac{d_{32}E}{\hbar}\rho_{21},\\
\dot{\rho}_{32} = -i\omega_{32}\rho_{32} - i\frac{d_{32}E}{\hbar}(\rho_{33} - \rho_{22}) + i\frac{d_{31}E}{\hbar}\rho_{12},\\
\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E = \frac{4\pi}{c^2}\frac{\partial^2 P}{\partial t^2}.
\end{gathered}$$

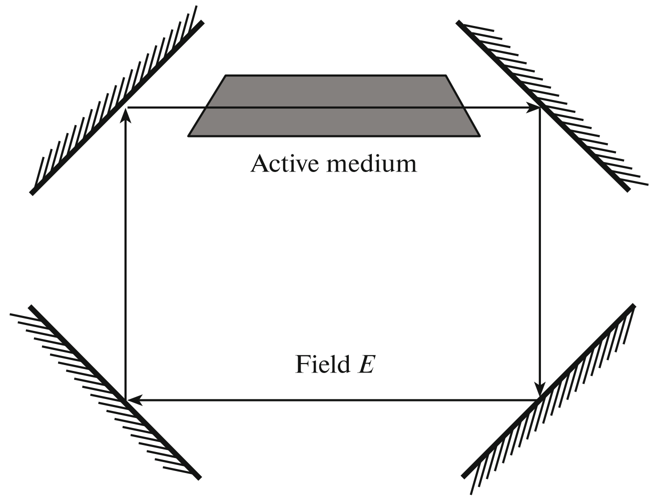

Fig. 2. Scheme of a unidirectional ring cavity. The active medium of Λ-emitters is shown by a gray color.

Here, ρ_{nm} are the elements of the density matrix of the three-level atom at the point with coordinate x at moment of time t ($n,m = 1,2,3$); d_{31} and d_{32} are the dipole moments of the corresponding transitions, which, without loss of generality, can be considered to be real-valued and positive; ω_{31} and ω_{32} are the frequencies of optical transitions between the upper level 3 and the doublet sublevels 1 and 2; ω_{21} is the frequency of the transition between the sublevels of the doublet; $P = N(d_{31}\rho_{31} + d_{32}\rho_{32} + \text{c.c.})$ is the polarization of the medium; N is the concentration of atoms; c is the speed of light in vacuum; and E is the electric field strength. The relaxation of the populations and of the polarization (homogeneous and related to inhomogeneous broadening) is not taken into account: we assume that the superradiance time is considerably shorter than all relaxation times and consider the dynamics of the superradiance on this scale. In addition, we will neglect the decay of the field due to cavity losses. Frequency ω_{21} of the doublet splitting is assumed to be much smaller than frequencies ω_{31} and ω_{32} of the optical transitions. We also assume that the spectrum of the superradiance and the value of doublet splitting ω_{21} do not exceed the spacing between cavity modes; i.e., we restrict ourselves to the single-mode approximation.

We will seek the solution to the system of equations (1) in the form

$$\begin{gathered}
\rho_{31} = \mathcal{R}_{31}e^{-i(\omega t - kx)}, \quad \rho_{32} = \mathcal{R}_{32}e^{-i(\omega t - kx)},\\
E = \mathcal{A}e^{-i(\omega t - kx)} + \text{c.c.},
\end{gathered} \qquad (2)$$

where $k = \omega/c$, while the field amplitude $\mathcal{A}$ and off-diagonal elements $\mathcal{R}_{31}$ and $\mathcal{R}_{32}$ of the density matrix (in what follows, they will be referred to as the high-frequency coherences) are functions that vary slowly on a scale of the optical period $2\pi/\omega$ and the radiation wavelength $\lambda = 2\pi/k$ (the approximation of slowly varying amplitudes). Note that an analogous assumption with respect to low-frequency coherence ρ_{21} (on a scale of $2\pi/\omega$) is not used. It is natural to assume that the passage time L/c (L is the cavity length) is much shorter than characteristic times of the problem; i.e., during one round trip of the light in the cavity, the state of the medium changes insignificantly. In this case, the retardation can be neglected. Then the field at the input into the active medium (by virtue of a high quality factor of the cavity) is equal to the field at the output of it, which also justifies the use of the mean-field approximation. And, finally, let us assume that the $|3\rangle \leftrightarrow |1\rangle$ and $|3\rangle \leftrightarrow |2\rangle$ transition dipole moments are identical ($d_{31} = d_{32} = d$)—the approximation that is not principal for the problem under consideration.

By passing in the standard way from the system of equations (1) to a similar system for slowly varying amplitudes, we obtain

$$\dot{\rho}_{11} = \mathscr{E}\mathscr{R}_{31}^* + \mathscr{E}^*\mathscr{R}_{31}, \quad \dot{\rho}_{22} = \mathscr{E}\mathscr{R}_{32}^* + \mathscr{E}^*\mathscr{R}_{32}, \tag{3a}$$

$$\dot{\rho}_{33} = -(\mathscr{E}\mathscr{R}_{31}^* + \mathscr{E}^*\mathscr{R}_{31}) - (\mathscr{E}\mathscr{R}_{32}^* + \mathscr{E}^*\mathscr{R}_{32}), \tag{3b}$$

$$\dot{\rho}_{21} = -i\delta\rho_{21} + \mathscr{E}\mathscr{R}_{32}^* + \mathscr{E}^*\mathscr{R}_{31}, \tag{3c}$$

$$\dot{\mathscr{R}}_{31} = -\frac{i\delta}{2}\mathscr{R}_{31} + \mathscr{E}(\rho_{33} - \rho_{11} - \rho_{21}), \tag{3d}$$

$$\dot{\mathscr{R}}_{32} = \frac{i\delta}{2}\mathscr{R}_{32} + \mathscr{E}(\rho_{33} - \rho_{22} - \rho_{21}^*), \tag{3e}$$

$$\dot{\mathscr{E}} = \mathscr{R}_{31} + \mathscr{R}_{32}. \tag{3f}$$

Here, dots denote the derivatives with respect to the dimensionless time $\tau = t\Omega$, where $\Omega^{-1} = \sqrt{\hbar(2\pi\omega d^2 N)^{-1}}$ is the constant that determines the time scale (Ω^{-1}); $\delta = \omega_{21}/\Omega$ is the dimensionless splitting frequency of the doublet; and $\mathscr{E} = -id\mathscr{A}/(\hbar\Omega)$ is the dimensionless amplitude of the electric field strength. For simplicity, eigenfrequency $\omega = (\omega_{31} + \omega_{32})/2$ of the cavity is considered to be centered between frequencies ω_{31} and ω_{32}.

This system of equations has the following integrals of motion:

$$|\mathscr{E}|^2 + \rho_{33} = \text{const}, \tag{4a}$$

$$\rho_{11} + \rho_{22} + \rho_{33} = 1, \tag{4b}$$

$$\begin{gathered}\rho_{11}^2 + \rho_{22}^2 + \rho_{33}^2 \\ + 2(|\rho_{21}|^2 + |\mathscr{R}_{31}|^2 + |\mathscr{R}_{32}|^2) = \text{const}.\end{gathered} \tag{4c}$$

The last two expressions, (4b) and (4c), represent the normalization conditions for the density matrix and its square, respectively. The interpretation of integral (4a) is not so evident, although it resembles conservation law of the excitation energy. The presence of integrals of motion makes it possible to considerably simplify the analysis of the dynamics of the three-level superradiance.

INITIAL CONDITIONS, SYMMETRY, AND SIMPLIFICATION OF THE MODEL

As compared to the two-level superradiance [11, 12], the scheme with the doublet in the ground state (three-level superradiance) introduces new effects into the response of the system, which are generated by the competition between the transitions $|3\rangle \leftrightarrow |1\rangle$ and $|3\rangle \leftrightarrow |2\rangle$. In connection with this, in order to investigate the kinetics of the three-level superradiance, we will choose the initial conditions such that the interaction process of the parts of the system "cavity + atoms + field" would proceed most efficiently, namely, at any initial population of the upper state and with a minimal delay of the superradiance pulse. For this purpose, let us turn attention to the fact that Eqs. (3d) and (3e) for the high-frequency coherences ($\mathscr{R}_{31}$ and $\mathscr{R}_{32}$) contain terms that are proportional to the low-frequency coherence ρ_{21}. In this case, if $\rho_{21}(0) \neq 0$, the evolution of initial fluctuations of $\mathscr{R}_{31}$ and $\mathscr{R}_{32}$ (their decay or growth) will depend on phase of $\rho_{21}(0)$. At positive values of $\rho_{21}(0)$, these fluctuations will decrease; however, if the values of $\rho_{21}(0)$ are negative, these fluctuations, on the contrary, will increase avalanche-like, thereby initiating the superradiance. We emphasize that this becomes possible at any difference of the populations in channels $3 \leftrightarrow 1$ and $3 \leftrightarrow 2$ and is ensured by the transformation of the low-frequency coherence $\rho_{21}(0)$ into the high-frequency coherences $\mathscr{R}_{31}$ and $\mathscr{R}_{32}$. The latter effect is explicitly reflected in integral of motion (4c). The analysis of the superradiance of this Λ-system is significantly simplified upon passage to a new basis $|3\rangle$, $|+\rangle = (|1\rangle + |2\rangle)/\sqrt{2}$, $|-\rangle = (|1\rangle - |2\rangle)/\sqrt{2}$ [22, 24, 25, 31–33]. In such a case, the elements of the density matrix are transformed in accordance with the following relations:

$$\rho_{++} = \frac{1}{2}(\rho_{11} + \rho_{22} + 2\mathrm{Re}[\rho_{21}]), \tag{5}$$

$$\rho_{--} = \frac{1}{2}(\rho_{11} + \rho_{22} - 2\mathrm{Re}[\rho_{21}]),$$

$$\rho_{+-} = \frac{1}{2}(\rho_{11} - \rho_{22} + \rho_{21} - \rho_{21}^*),$$

$$\mathscr{R}_{3+} = \frac{1}{\sqrt{2}}(\mathscr{R}_{31} + \mathscr{R}_{32}), \quad \mathscr{R}_{3-} = \frac{1}{\sqrt{2}}(\mathscr{R}_{31} - \mathscr{R}_{32}),$$

where ρ_{++} and ρ_{--} are the populations of the active and passive states, respectively; ρ_{+-} is the low-frequency coherence; and $\mathscr{R}_{31}$ and $\mathscr{R}_{32}$ are the high-frequency coherences of the corresponding optical channels.

It can be seen from the expression for the population ρ_{++} of the active state presented in relations (5) that, for the three-level superradiance to take place, the presence of an inversion population in active channel $|3\rangle \leftrightarrow |+\rangle$ is necessary; i.e., at the initial moment of time, inequality $\rho_{33}(0) > \rho_{++}(0)$ should be met. In the ideal case, in which the population of the active state is zero, $\rho_{++}(0) = \rho_{11}(0) + \rho_{22}(0) + 2\mathrm{Re}[\rho_{21}(0)] = 0$, the following conditions should be held:

$$\begin{gathered}\mathrm{Re}[\rho_{21}(0)] = \frac{(\alpha - 1)}{2}, \quad \mathrm{Im}[\rho_{21}(0)] = 0, \\ -2\mathrm{Re}[\rho_{21}(0)] = \rho_{11}(0) = \rho_{22}(0),\end{gathered} \tag{6}$$

where $\rho_{33}(0) = \alpha$ and $0 < \alpha \le 1$. In what follows, we will assume that the lower doublet is prepared in a maximally coherent state if conditions (6) are met at the initial moment of time. We emphasize again that, under these starting conditions, the superradiance can occur at any initial population $\rho_{33}(0)$ of the upper state, even if there is no inversion population on the whole, when the total initial population of the lower doublet exceeds the initial population of the upper level, $\rho_{11}(0) + \rho_{22}(0) > \rho_{33}(0)$.

If the initial electric field strength is zero,

$$\mathrm{Re}[\mathscr{E}(0)] = \mathrm{Im}[\mathscr{E}(0)] = 0, \tag{7}$$

to initiate the superradiance, it suffices to set small seed values of the high-frequency coherences, e.g., as

$$\mathrm{Re}[\mathcal{R}_{31}(0)] = \mathrm{Re}[\mathcal{R}_{32}(0)] = \pm R_0, \tag{8}$$

where, without loss of generality, it is assumed that $\mathrm{Im}[\mathcal{R}_{31}(0)] = \mathrm{Im}[\mathcal{R}_{32}(0)] = 0$, while the value of $\mathcal{R}_0 \ll 1$. We are not interested in fluctuations of the superradiance; therefore, initial values $\mathcal{R}_{31}(0)$ and $\mathcal{R}_{32}(0)$ are specified as determinate parameters, which corresponds to the conditions of the induced superradiance [37, 38].

It is interesting to note that, if the states of the doublet are populated incoherently at the initial moment of time (e.g., $\rho_{21}(0) = 0$), then the presence of population inversion for one of the optical transitions is not a necessary condition to realize the superradiance, which is also the coherent effect; however, we will not consider it in this work.

The system of differential equations (3) with initial conditions (6)–(8) was solved numerically. The following two controlling parameters were varied: the initial population $\rho_{33}(0) = \alpha$ of the upper level and the splitting frequency δ of the doublet. This allowed us to reveal a number of interesting regularities of the time dynamics of the amplitudes of the electric field of the superradiance and elements of the density matrix. Figure 3 presents one of typical examples of these calculations. Here, we can see that the real part of the amplitude of the electric field of the superradiance (Fig. 3.2) reveals a time dynamics ($\mathrm{Re}[\mathscr{E}(\tau)] \ne 0$), whereas its imaginary part (Fig. 3.3) remains undeveloped in time ($\mathrm{Im}[\mathscr{E}(\tau)] = 0$). The real parts of high-frequency coherences $\mathrm{Re}[\mathcal{R}_{31}(\tau)]$ (Fig. 3.5) and $\mathrm{Re}[\mathcal{R}_{32}(\tau)]$ (Fig. 3.12) show a similar behavior, whereas their imaginary parts $\mathrm{Im}[\mathcal{R}_{31}(\tau)]$ (Fig. 3.6) and $\mathrm{Im}[\mathcal{R}_{32}(\tau)]$ (Fig. 3.13) exhibit an antiphase behavior. In accordance with this, the squares of their moduli, $|\mathcal{R}_{31}|^2 = |\mathcal{R}_{32}|^2$, evolve identically (Figs. 3.4 and 3.11). The dynamics of populations $\rho_{11}(\tau)$ (Fig. 3.7) and $\rho_{22}(\tau)$ (Fig. 3.14) are identical and repeat the dynamics for superradiance field intensity $|\mathscr{E}|^2$ (Fig. 3.1). We note that these regularities are a consequence of the initial conditions (6)–(8) and are realized at any physically realistic parameters α and δ. This makes it possible to considerably simplify the mathematical model of the problem under consideration.

By introducing the notation

$$\mathrm{Re}[\mathscr{E}] = \epsilon, \quad \mathrm{Im}[\mathscr{E}] = 0, \quad \rho_{21} = \eta + i\chi, \tag{9a}$$

$$\rho_{11} = \rho_{22} = \frac{1 - \rho_{33}}{2}, \quad \rho_{33} = \alpha - \epsilon^2, \tag{9b}$$

$$\begin{gathered}\mathrm{Re}[\mathcal{R}_{31}] = \mathrm{Re}[\mathcal{R}_{32}] = \xi, \\ \mathrm{Im}[\mathcal{R}_{31}] = -\mathrm{Im}[\mathcal{R}_{32}] = \zeta, \\ |\mathcal{R}_{31}|^2 = |\mathcal{R}_{32}|^2 = \xi^2 + \zeta^2,\end{gathered} \tag{9c}$$

we can transform the system of differential equations (3) into the following system:

$$\dot{\epsilon} = 2\xi, \tag{10a}$$

$$\dot{\xi} = \frac{\delta}{2}\zeta + \frac{1}{2}(3\alpha - 1)\epsilon - \frac{3}{2}\epsilon^3 - \epsilon\eta, \tag{10b}$$

$$\dot{\zeta} = -\frac{\delta}{2}\xi - \epsilon\chi, \tag{10c}$$

$$\dot{\eta} = \delta\chi + 2\epsilon\xi, \tag{10d}$$

$$\dot{\chi} = -\delta\eta + 2\epsilon\zeta. \tag{10e}$$

Therefore, the relations (9) implement the reduction of our model from the complex domain to the real one. As a consequence of this, the initial phase space $\mathbb{R}^{11}$ (3) of the model is completely mapped into $\mathbb{R}^5$ (10). In addition, taking into account (4b) and relations (9), the integral of motion (4c) takes the form

$$\begin{gathered}(\epsilon^2 - \gamma)^2 + \frac{4}{3}(\eta^2 + \chi^2 + 2\xi^2 + 2\zeta^2) = \mathrm{const}, \\ \mathrm{const} = \frac{4}{3}(\alpha^2 - \gamma + 2\mathcal{R}_0^2), \quad \gamma = \alpha - \frac{1}{3}.\end{gathered} \tag{11}$$

It is important to note that this law of conservation restricts the domain of existence of phase trajectories of the system and determines a closed hypersurface in

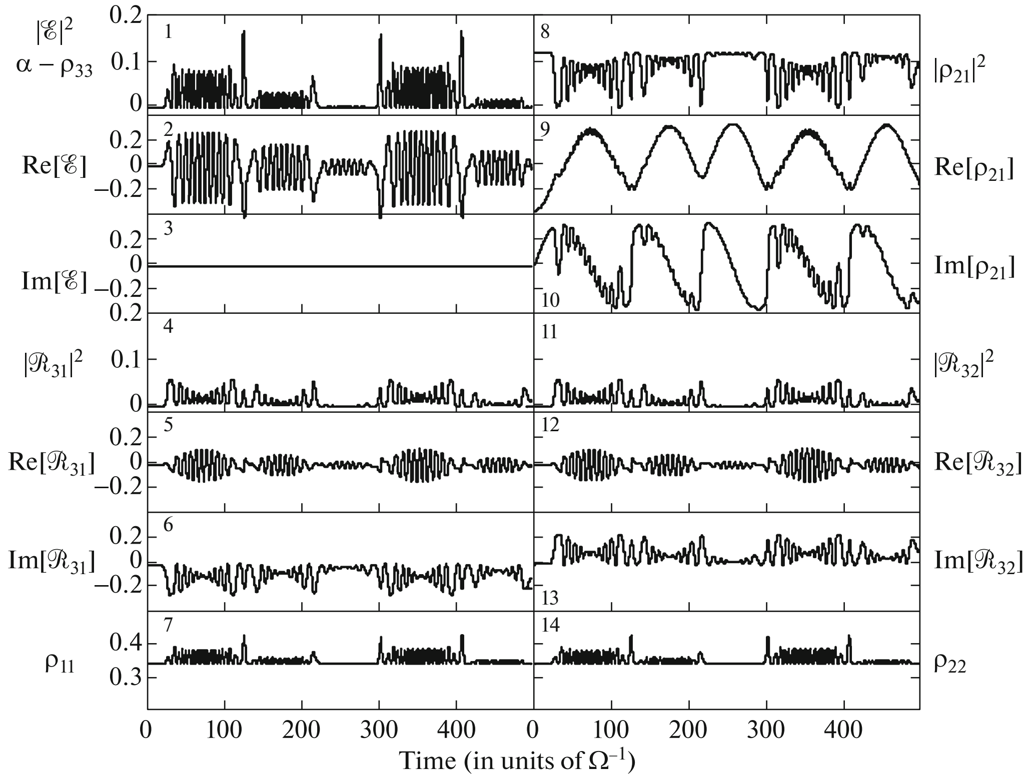

Fig. 3. Dynamics of the superradiance electric field strength and the density matrix elements calculated at the following initial conditions: $\rho_{33}(0) = \alpha = 0.3$, $\rho_{11}(0) = \rho_{22}(0) = 0.35$, $\mathrm{Re}[\rho_{21}] = -0.35$, $\mathrm{Im}[\rho_{21}] = 0$, $\mathrm{Re}[\mathcal{R}_{31}(0)] = \mathrm{Re}[\mathcal{R}_{32}(0)] = 10^{-8}$, $\mathrm{Re}[\mathscr{E}(0)] = \mathrm{Im}[\mathscr{E}(0)] = \mathrm{Im}[\mathcal{R}_{31}(0)] = \mathrm{Im}[\mathcal{R}_{32}(0)] = 0$, and $\delta = 0.05$.

the phase space ($\epsilon, \xi, \zeta, \eta$, and χ) outside of which solutions of the system of equations (10) do not exist at any values of parameters α and δ. This makes it possible to characterize the process of superradiance as a process that is stable in the sense of Lagrange [39]. Topological specific features of the hypersurface (11) depend on the sign of constant γ. At $1 \geq \alpha > \frac{1}{3}$, i.e., at $\gamma > 0$, this is a five-dimensional "dumbbell" with the symmetry axis ϵ. If $\frac{1}{3} \geq \alpha > 0$, i.e., $\gamma < 0$, the hypersurface is a five-dimensional ellipsoid.

DEGENERATE DOUBLET

Let us consider a particular case of a degenerate doublet ($\delta = 0$). In this limit, the system of differential equations (10) is considerably simplified and takes the form

$$\dot{\epsilon} = 2\xi, \quad \dot{\eta} = 2\epsilon\xi, \tag{12a}$$

$$\dot{\chi} = 2\epsilon\zeta, \quad \dot{\zeta} = -\epsilon\chi, \tag{12b}$$

$$\dot{\xi} = \frac{1}{2}(3\alpha - 1)\epsilon - \frac{3}{2}\epsilon^3 - \epsilon\eta. \tag{12c}$$

This system of equations has integrals of motion. First, Eq. (12a) along with the initial conditions $\epsilon(0) = 0$ and $\eta(0) = -(1-\alpha)/2$ yield the first integral of motion:

$$2\eta - \epsilon^2 = \alpha - 1. \tag{13}$$

Second, Eq. (12b) and the initial conditions $\chi(0) = \zeta(0) = 0$ yield the second integral of motion: $\chi^2 + 2\zeta^2 = 0$, which means that functions $\chi(\tau)$ and $\zeta(\tau)$, which are defined in the real domain, remain unchanged and equal to zero within the entire superradiance process: $\chi(\tau) = \zeta(\tau) = 0$, $\tau \geq 0$. Then, expressing functional dependence $\eta(\epsilon)$ from (13) via ϵ^2 and substituting it into (12c), we obtain

$$\dot{\epsilon} = 2\xi, \tag{14a}$$

$$\dot{\xi} = \alpha\epsilon - 2\epsilon^3. \tag{14b}$$

Eliminating variable ξ from (14), we arrive at the following closed equation for the field ϵ

$$\ddot{\epsilon} - 2\alpha\epsilon + 4\epsilon^3 = 0, \quad 1 \geq \alpha > 0, \tag{15}$$

which represents the Duffing equation (Georg Duffing, [40]) for an oscillator with a cubic nonlinearity

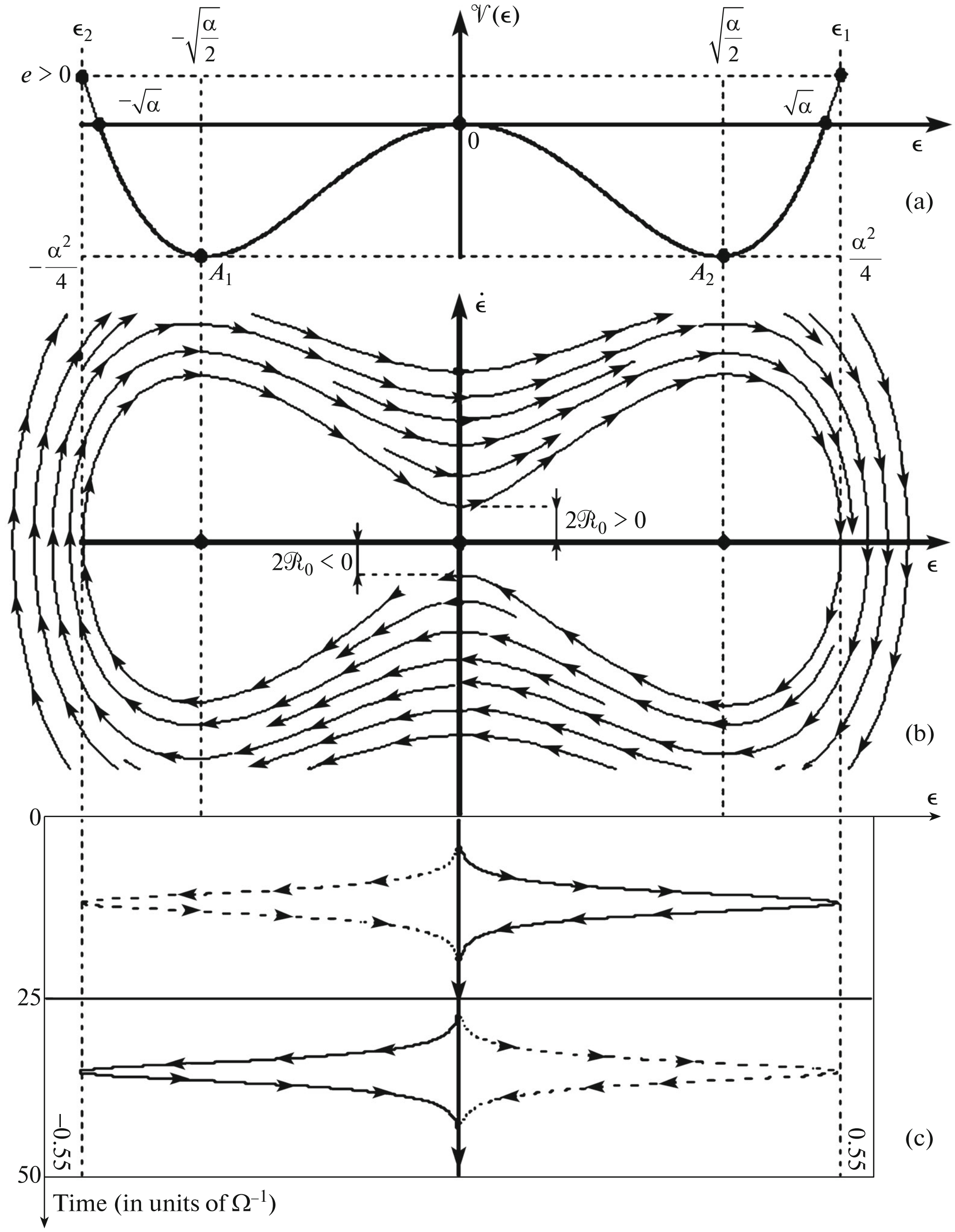

Fig. 4. Case of a degenerate doublet: (a) potential energy curve $\mathcal{V}(\epsilon)$; (b) phase portrait $(\epsilon, \dot{\epsilon})$, in which arrows on phase trajectories show the direction of increasing time; (c) pulse of the superradiance field in relation to the sign of $\mathcal{R}_0$ (solid curve corresponds to the case of $\mathcal{R}_0 > 0$, dashed curve corresponds to the case $\mathcal{R}_0 < 0$, and arrows show the direction of increasing time).

without friction and external driving force. Equation (15) yields the third integral of motion:

$$\frac{1}{2}\dot{\epsilon}^2 + \mathcal{V}(\epsilon) = e, \tag{16}$$

the physical meaning of which is that it corresponds to the total energy of the oscillator (superradiance field), where, taking into account the initial conditions $\epsilon(0) = 0$ and $\xi(0) = \pm R_0 \neq 0$, the value of the total energy is $e = 2\mathcal{R}_0^2 > 0$ ($\mathcal{R}_0 \neq 0$), while the function $\mathcal{V}(\epsilon) = \epsilon^4 - \alpha\epsilon^2$ can be interpreted as a potential energy of the superradiance.

The relation between the signs in front of the linear and nonlinear terms in (15) characterizes the superradiance field as that of a stable oscillation process in double-humped potential $\mathcal{V}(\epsilon)$ with infinite walls, which has three singular points in the phase space $(\epsilon,\dot{\epsilon})$ (Fig. 4a): $A_{1,2}(\pm\sqrt{\alpha/2},-\alpha^2/2)$ are the points of a stable equilibrium of the type of a center (minima of the potential $\mathcal{V}(\epsilon)$); and $O(0,0)$ is the point of an unstable equilibrium of the type of a saddle (maximum of the potential $\mathcal{V}(\epsilon)$). As can be seen from Fig. 4a, the oscillations proceed within the range $\epsilon_2 \le \epsilon \le \epsilon_1$, where $\epsilon_{1,2}$ are the turning points of the oscillator, and are determined by the roots of the equation $\mathcal{V}(\epsilon) = e$; in this case, it can be shown that $\epsilon_1^2 + \epsilon_2^2 = \alpha$.

In the general case, $\xi(0) = \mathcal{R}_0 \neq 0$, the value of total energy e (16) is always positive, $2\mathcal{R}_0^2 > 0$. Consequently, the oscillation process of the superradiance always proceeds in the supernonlinear ("episeparatrix") regime (Fig. 4a). In this case, Eq. (16) will have an exact solution. In terms of the elliptic functions, it can be obtained by applying the following substitution:

$$\epsilon = \epsilon_1 \cos[\varphi], \quad \epsilon_1^2 = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + 2\mathcal{R}_0^2}.$$

Then,

$$\varphi = \mathrm{am}[(\phi\tau - \tau_d); m], \quad \epsilon(\tau) = \epsilon_1 \mathrm{cn}[(\phi\tau - \tau_d); m], \quad (17)$$

$$\tau_d = K(m), \quad \phi = \sqrt{2\alpha + \frac{8\mathcal{R}_0^2}{\epsilon_1^2}},$$

$$T = 4K(m), \quad m^2 = 1 - \frac{2\mathcal{R}_0^2}{4\mathcal{R}_0^2 + \alpha\epsilon_1^2},$$

where $\mathrm{am}[\tau'; m]$ is the amplitude of Jacobi functions; $\mathrm{cn}[\tau'; m]$ is the elliptic cosine; $\tau' = \phi\tau - \tau_d$, where τ_d is the delay time of the superradiance pulse (initial stage of superradiance); $K(m)$ is the complete elliptic integral of the first kind; and T is the period of oscillations of the electric field strength.

In this case, the physical picture of the superradiance is rather transparent. From Eq. (14a), we have $\dot{\epsilon}(0) = \pm 2\xi(0) = \pm 2\mathcal{R}_0$. If $\xi(0) = \mathcal{R}_0$, then $\dot{\epsilon}(\tau) = \sqrt{2[e - \mathcal{V}(\epsilon)]}$ and the field of the superradiance will increase ($\epsilon(\tau) > 0$) in the time interval $0 \le \tau < T/4$ with delay τ_d; after that, the field will decrease within the interval of the same length, initiating a superradiance pulse. Upon the reverse motion, the system emits an antiphase pulse, and this process is periodically reproduced, since the system is conservative (Fig. 4c). If $\xi(0) = -\mathcal{R}_0$ and $\dot{\epsilon}(\tau) = -\sqrt{2[e - \mathcal{V}(\epsilon)]}$, the field of the superradiance has a phase shift by $T/2 = \pi$ (Fig. 4b).

It is of interest to consider the particular case of the separatrix solution of Eq. (16), when $\xi(0) = \mathcal{R}_0 = 0$, and then $e = 0$. In this case, if $\xi(0) = 0$, then $\dot{\epsilon}(0) = 0$, $\epsilon(0) = 0$, and $\dot{\xi}(0) = 0$ by the assumption. This means that, in the phase plane $(\epsilon,\dot{\epsilon})$, the separatrix represents the point that is located at the coordinate origin (Fig. 4b). The physical interpretation of this solution is very simple: an oscillator, residing in the position of an unstable equilibrium (point O; Fig. 4a), does not have any initial displacement and any initial velocity. Consequently, the system will reside for an arbitrarily long time in this state, even though it is unstable, and does not generate any superradiance field.

We also note that features of the superradiance dynamics of the degenerate doublet manifest themselves in a more general case when the doublet is split ($\delta \neq 0$). Namely, the superradiance process will be determined by the episeparatrix scenario. In the phase space ($\epsilon, \xi, \zeta, \eta$, and χ), the separatrix will also be the point, the coordinate origin, that corresponds to the unstable position of the oscillator. For the activation of its motion, i.e., for the initiation of the superradiance process, a small seed is always necessary, e.g., $\xi(0) = \mathcal{R}_0$.

CONCLUSIONS

For multilevel systems, in particular, for systems with the Λ-scheme of operating transitions, we showed that, at any population of the upper level, even without the inversion population on the whole, it is possible to initiate the process of generation of a superradiance pulse. Analysis of the new collective basis of the doublet state allowed us to obtain the conservation laws, which made it possible to considerably reduce the dimension of the phase space of the examined model ($\mathbb{R}^{11} \to \mathbb{R}^5$) and to realize convertion of the model from the complex to the real domain. As a consequence of this, in the case of a degenerate doublet ($\delta = 0$), we found mapping that allowed us to reduce the problem of the three-level superradiance to the Duffing oscillator model ($\mathbb{R}^5 \to \mathbb{R}^2$). In this case, it was found that the oscillation process proceeds in the episeparatrix (supernonlinear) regime. For this case, in terms of elliptic functions, we found its analytical solution. The separatrix itself represents the point of an unstable equilibrium and is located at the coordinate origin. At this point, the system can reside for an arbitrarily long time, without generating any superradiance field.

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