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A modular design of incremental Lyapunov functions for microgrid control with power sharing

C. De Persis and N. Monshizadeh

Abstract—In this paper we contribute a theoretical framework that sheds a new light on the problem of microgrid analysis and control. The starting point is an energy function comprising the kinetic energy associated with the elements that emulate the rotating machinery and terms taking into account the reactive power stored in the lines and dissipated on shunt elements. We then shape this energy function with the addition of an adjustable voltage-dependent term, and construct incremental storage functions satisfying suitable dissipation inequalities. Our choice of the voltage-dependent term depends on the voltage dynamics/controller under investigation. Several microgrids dynamics that have similarities or coincide with dynamics already considered in the literature are captured in our incremental energy analysis framework. The twist with respect to existing results is that our incremental storage functions allow for a large signal analysis of the coupled microgrid obviating the need for simplifying linearization techniques and for the restrictive decoupling assumption in which the frequency dynamics is fully separated from the voltage one. A complete Lyapunov stability analysis of the various systems is carried out along with a discussion on their active and reactive power sharing properties.

I. INTRODUCTION

Microgrids have been envisioned as one of the leading technologies to increase the penetration of renewable energies in the power market. A thorough discussion of the technological, physical and control-theoretic aspects of microgrids is provided in many interesting comprehensive works, including [44], [43], [18], [2], [28].

Power electronics allows inverter in the microgrids to emulate desired dynamic behavior. This is an essential feature since when the microgrid is in grid forming mode, inverters have to inject active and reactive power in order to supply the loads in a shared manner and maintain the desired frequency and voltage values at the nodes. Hence, much work has focused on the design of dynamics for the inverters that achieve these desired properties and this effort has involved both practitioners and theorists, all providing a myriad of solutions, whose performance has been tested mainly numerically and experimentally.

The main obstacle however remains a systematic design of the microgrid controllers that achieve the desired properties in terms of frequency and voltage regulation with power sharing. The difficulty lies in the complex structure of these systems, comprising dynamical models of inverters and loads that are physically interconnected via exchange of active and reactive power. In quasi steady state working conditions,

these quantities are sinusoidal terms depending on the voltage phasor relative phases. As a result, mathematical models of microgrids reduce to high-order oscillators interconnected via sinusoidal coupling. Moreover the coupling weights depend on the voltage magnitudes, which obey extra coupled dynamics. Two additional features complicate the situation: the presence of loads typically leads to differential-algebraic models and the presence of unmeasured loads requires controllers that can deal with such uncertainty.

To deal with the complexity of these dynamical models a common assumption is to decouple frequency and voltage dynamics thus to enable a separate analysis of the two dynamics. Once separated, the two dynamics are simpler to analyze and the presence of algebraic constraints can be investigated. In this case, a common tool to infer stability results is to rely on small signal arguments that focus on a linearized model of the system; see e.g. [34]. Results that deal with the fully coupled system are also available ([29], [41], [24]). In this case, the results mainly concern network-reduced models with primary control, namely stability rather than stabilization of the equilibrium solution. Furthermore, lossy transmission lines can also be studied ([13], [41], [4], [41], [24], and also [9]).

In spite of these many advances, what is still missing is a comprehensive approach to deal with the analysis and control design for microgrids. In this paper we provide a contribution in this direction. The starting point is the energy function associated with the system, a combination of kinetic and potential energy. Relying on an extended notion of incremental dissipativity, a variety of shifted Lyapunov functions whose critical points have desired features are constructed. The construction is inspired by works in the control of networks in the presence of disturbances, which makes use of incremental passivity and internal model controllers ([36], [5], [23]). The Lyapunov functions that we design encompass several microgrid dynamics that have appeared in the literature, including the conventional droop controller ([44], [29]), the quadratic droop controller [34], and the reactive power consensus dynamics ([30]). Our analysis, however, suggests suitable modifications (such as a suitable voltage-dependent weighting of the reactive power consensus dynamics of [30]) and inspires new controllers, such as the so-called reactive current controller. Our approach has two additional distinguishing features: we do not need to assume decoupled dynamics and we perform a large signal analysis.

Our contribution also expands the knowledge on the use of energy functions in the context of microgrids. Although energy functions have played a substantial role to deal with quite accurate models of power systems ([39], [10], [8]), our approach based on the incremental dissipativity notion sheds

a new light into the construction of these energy functions, allow us to cover a wider range of microgrid dynamics, and paves the way for the design of dynamic controllers, following the combination of passivity techniques and internal model principles as in [5]. We refer the reader to e.g. [25], [12] for seminal work on passivity-based control of power networks.

In this paper we focus on network reduced models of microgrids ([29], [41], [24], [35]). These models are typically criticized for not providing an explicit characterization of the loads ([34]). Focusing on network reduced models allows us to reduce the technical complexity of the arguments and to provide an elegant analysis. However, one of the advantages of the use of the energy functions is that they remain effective also with network preserved models ([39]). In fact, a preliminary investigation not reported in this manuscript for the sake of brevity shows that the presented results extend to the case of network preserved models. A full investigation of this case will be reported elsewhere.

The outline of the paper is as follows. In Section II, details on the model under consideration are provided. In Section III the design of incremental energy functions is carried out and incremental dissipativity of various models of microgrids associated with different voltage dynamics/controllers is shown. A few technical conditions on these energy functions are discussed in Section IV, and a decentralized test to check them is also provided. Based on the results of these sections, attractivity of the prescribed synchronous solution and voltage stability is presented in Section V, along with a discussion on power sharing properties of the proposed controllers. A few accessory results on power sharing in the presence of homogeneous transmission lines are also presented. Concluding remarks are provided in the last section.

II. MICROGRID MODEL AND THE SYNCHRONOUS SOLUTION

We consider the following network-reduced model of a microgrid

$$\begin{aligned}\dot{\theta} &= \omega \\ T_P \dot{\omega} &= -(\omega - \omega^*) - K_P(P - P^*) + u_P \\ T_Q \dot{V} &= f(V, Q, u_Q)\end{aligned}\quad (1)$$

where $\theta \in \mathbb{T}^N$ is the vector of voltage angles, $\omega \in \mathbb{R}^N$ is the frequency, $P \in \mathbb{R}^N$ is the active power vector, $Q \in \mathbb{R}^N$ is the reactive power vector, and $V \in \mathbb{R}_{>0}^N$ is the vector of voltage magnitudes. The matrices T_P , T_V , and K_P are diagonal and positive definite. The vectors ω^* and P^* denote the frequency and active power setpoints, respectively. The vector P^* also models active power loads at the buses ([24, Section 2.4]). The vector u_Q is an additional input. The function f accounts for the voltage dynamics/controller and is decided later.

The model (1) with an appropriate selection of f encompasses various models of network-reduced microgrids in the literature, including conventional droop controllers, quadratic droop controllers, and consensus based reactive power control schemes ([44], [33], [29], [34], [30]). We refer the reader to [31] for a compelling derivation of microgrid models from first principles. Our goal here is to provide a unifying framework

for analysis of the microgrid model (1) for different types of voltage controllers, and study frequency regulation, voltage stability, and active as well as reactive power sharing. A key point of our approach is that it does not rely on simplifying and often restrictive premises such as the decoupling assumption and linear approximations. First, we look at the microgrid model (1) in more detail.

Active and reactive power. The active power P_i is given by

$$P_i = \sum_{j \in \mathcal{N}_i} B_{ij} V_i V_j \sin \theta_{ij}, \quad \theta_{ij} := \theta_i - \theta_j \quad (2)$$

and the reactive power by

$$Q_i = B_{ii} V_i^2 - \sum_{j \in \mathcal{N}_i} B_{ij} V_i V_j \cos \theta_{ij}, \quad \theta_{ij} := \theta_i - \theta_j. \quad (3)$$

Note that here $B_{ii} = \hat{B}_{ii} + \sum_{j \in \mathcal{N}_i} B_{ij}$, where $B_{ij} = B_{ji} > 0$ is the susceptance at edge $\{i, j\}$ and $\hat{B}_{ii} \geq 0$ is the shunt susceptance at node i ¹. Hence, $B_{ii} \geq \sum_{j \in \mathcal{N}_i} B_{ij}$ for all i .

It is useful to have compact representations of both active and reactive power. Setting $\Gamma(V) = \text{diag}(\gamma_1(V), \dots, \gamma_M(V))$, $\gamma_k(V) = V_i V_j B_{ij}$, with $k \in \{1, 2, \dots, M\}$ being the index corresponding to the edge $\{i, j\}$ (in short, $k \sim \{i, j\}$), the vector of the active power at all the nodes writes as

$$P = D\Gamma(V)\sin(D^T\theta).$$

where $D = [d_{ik}]$ is the incidence matrix of the graph describing the interconnection structure of the network, and the vector $\sin(\cdot)$ is defined element-wise. Let us now introduce the vector $A_0 = \text{col}(B_{11}, \dots, B_{NN})$. Since $|d_{ik}| \cos(d_{ik}\theta_i + d_{jk}\theta_j) = \cos(\theta_i - \theta_j)$, the vector of reactive power at the nodes takes the form

$$Q = [V][A_0]V - |D|\Gamma(V)\cos(D^T\theta),$$

where $[v]$ represents the diagonal matrix associated to the vector v , and $|D|$ is obtained by replacing each element d_{ij} of D with $|d_{ij}|$.²

Other compact representation is useful as well. To this end, introduce the symmetric matrix

$$A(\cos(D^T\theta)) = \begin{bmatrix} 0 & B_{12} \cos \theta_{12} & \dots & B_{1N} \cos \theta_{1N} \\ B_{21} \cos \theta_{21} & 0 & \dots & B_{2N} \cos \theta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ B_{N1} \cos \theta_{N1} & B_{N2} \cos \theta_{N2} & \dots & 0 \end{bmatrix},$$

¹This shunt susceptance can model a purely inductive load co-located at the inverter i . See [30, Remark II.3] for a discussion on purely inductive loads.

²In fact, denoted by η the vector $D^T\theta$, the entry ij of the matrix $|D|\Gamma(V)\cos(D^T\theta)$ writes as

$$\begin{aligned} [|D|\Gamma(V)\cos(D^T\theta)]_{ij} &= \sum_{k=1}^M |d_{ik}| \gamma_k(V) \cos(\eta_k) \\ &= \sum_{k \sim \{i, j\}} |d_{ik}| V_i V_j B_{ij} \cos(d_{ik}\theta_i + d_{jk}\theta_j) \\ &= \sum_{j \in \mathcal{N}_i} V_i V_j B_{ij} \cos(\theta_i - \theta_j) \end{aligned}$$

where again we are exploiting the identity $\cos(d_{ik}\theta_i + d_{jk}\theta_j) = \cos(\theta_i - \theta_j)$. The vector Q becomes

$$\begin{aligned} Q &= [V][A_0]V - [V]A(\cos(D^T\theta))V \\ &= [V]([A_0] - A(\cos(D^T\theta)))V \\ &=: [V]A(\cos(D^T\theta))V. \end{aligned} \quad (4)$$

As a consequence of the condition $B_{ii} \geq \sum_{j \in \mathcal{N}_i} B_{ij}$ for all i , provided that at least one of the shunt susceptances B_{ii} is non-zero (which is the standing assumption throughout the paper), the symmetric matrix $A(\cos(D^T\theta))$ has all strictly positive eigenvalues and hence is a positive definite matrix. Note that the matrix A can be interpreted as a weighted adjacency matrix of the graph, whereas \mathcal{A} is associated with a loopy Laplacian matrix.

To pursue our analysis, we demonstrate an incremental dissipativity properties of the various microgrid models, with respect to a “synchronous solution”. The notion of dissipativity adopted in this paper is introduced next, and synchronous solutions will be identified afterwards.

Definition 1. System $\dot{x} = f(x, u), y = h(x)$, $x \in \mathcal{X}$, \mathcal{X} the state space, $y, u \in \mathbb{R}^m$, is incrementally cyclo-dissipative with state-dependent supply rate $s(x, u, y)$ and with respect to a given input-state-output triple $(\bar{u}, \bar{x}, \bar{y})$, if there exist a continuously differentiable function $\mathcal{S} : \mathcal{X} \rightarrow \mathbb{R}$, and state-dependent positive semi-definite³ matrices $W, R : \mathcal{X} \rightarrow \mathbb{R}^{m \times m}$, such that for all $x \in \mathcal{X}$, $u \in \mathbb{R}^m$ and $y = h(x)$, $\bar{y} = h(\bar{x})$

$$\frac{\partial \mathcal{S}}{\partial x} f(x, u) + \frac{\partial \mathcal{S}}{\partial \bar{x}} f(\bar{x}, \bar{u}) \leq s(x, u - \bar{u}, y - \bar{y})$$

with

$$s(x, u, y) = -y^T W(x)y + y^T R(x)u. \quad (5)$$

We remark that at this point the function \mathcal{S} is not required to be non-negative nor bounded from below and that the weight matrices W, R are allowed to be state dependent. The use of the qualifier “cyclo” in the definition above stresses the former feature ([40]).

Remark 1. In case the matrices W and R are state independent, some notable special cases of Definition 1 are obtained as follows:

- i) $W \geq 0$, $R = I$, $\mathcal{S} \geq 0$ (incremental passivity)
- ii) $W > 0$, $R = I$, $\mathcal{S} \geq 0$ (output-strict incremental passivity)
- iii) $W \geq 0$, $R = I$ (cyclo-incremental passivity)
- iv) $W > 0$, $R = I$ (output-strict cyclo-incremental passivity).

The synchronous solution. Given the constant vectors \bar{u}_P and \bar{u}_Q , the synchronous solution is defined as the triple

$$(\theta(t), \omega(t), V(t)) = (\bar{\theta}, \bar{\omega}, \bar{V}),$$

where $\bar{\theta} = \mathbb{1}\omega^0 t + \theta^0$, the vectors $\theta^0, \bar{V} \in \mathbb{R}_{>0}^N$ are constant, the scalar ω^0 is constant, $\bar{\omega} = \mathbb{1}\omega^0$, and where

$$\begin{aligned} \mathbf{0} &= -(\bar{\omega} - \omega^*) - K_P(\bar{P} - P^*) + \bar{u}_P \\ \mathbf{0} &= f(\bar{V}, \bar{Q}, \bar{u}_Q). \end{aligned} \quad (6)$$

³A state-dependent matrix $M : \mathcal{X} \rightarrow \mathbb{R}^{m \times m}$ is positive semi-definite if $y^T M(x)y \geq 0$ for all $x \in \mathcal{X}$ and for all $y \in \mathbb{R}^m$. If M is positive semi-definite and $y^T M(x)y = 0 \Leftrightarrow y = \mathbf{0}$ then M is called positive definite.

Observe that $\bar{\eta} := D^T \bar{\theta} = D^T \theta^0$, and $D^T \bar{\omega} = 0$. Hence, \bar{Q} is constant as \bar{V} and $\bar{\eta}$ are constant. An incremental model of the dynamical system with respect to the synchronous solution can be written as follows

$$\begin{aligned} \frac{d}{dt}(\theta - \bar{\theta}) &= (\omega - \bar{\omega}) \\ T_P \frac{d}{dt}(\omega - \bar{\omega}) &= -(\omega - \bar{\omega}) - K_P(P - \bar{P}) + (u_P - \bar{u}_P) \\ T_Q \frac{d}{dt}(V - \bar{V}) &= f(V, Q, u) - f(\bar{V}, \bar{Q}, \bar{u}_Q). \end{aligned}$$

III. DESIGN OF INCREMENTAL ENERGY FUNCTIONS

A crucial step for the Lyapunov based analysis of the coupled nonlinear model (1) is constructing a storage function. To this end, we exploit the following energy-based function

$$\begin{aligned} U(\theta, \omega, V) &= \frac{1}{2} \omega^T K_P^{-1} T_P \omega - \sum_{\{i,j\} \in E} V_i V_j B_{ij} \cos \theta_{ij} \\ &\quad + \frac{1}{2} \sum_{i=1}^N B_{ii} V_i^2. \end{aligned} \quad (7)$$

Notice that the first term represents the kinetic energy, the second one the reactive power stored in the links and the third one the power associated with the shunt component. Also notice that the last two sums together write as $\frac{1}{2} \sum_{i=1}^N Q_i$. The compact expression of U is therefore

$$\begin{aligned} U(\theta, \omega, V) &= \frac{1}{2} \omega^T K_P^{-1} T_P \omega + \frac{1}{2} \mathbb{1}^T Q \\ &= \frac{1}{2} \omega^T K_P^{-1} T_P \omega + \frac{1}{2} V^T \mathcal{A}(\cos(D^T \theta))V, \end{aligned} \quad (8)$$

where we have exploited (4). Since we are interested in the incremental passivity of the system with respect to the synchronous solution, an incremental storage function is introduced. First, we compute the gradient of the storage function as follows:

$$\begin{aligned} \frac{\partial U}{\partial \omega} &= K_P^{-1} T_P \omega, \quad \frac{\partial U}{\partial \theta_i} = \sum_{j \in \mathcal{N}_i} B_{ij} V_i V_j \sin(\theta_{ij}), \\ \frac{\partial U}{\partial V_i} &= B_{ii} V_i - \sum_{j \in \mathcal{N}_i} B_{ij} V_j \cos \theta_{ij}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial U}{\partial \theta} &= P = D\Gamma(V) \sin(D^T \theta), \\ \frac{\partial U}{\partial V} &= [V]^{-1} Q = [A_0]V - [V]^{-1} D\Gamma(V) \cos(D^T \theta). \end{aligned}$$

In the equality above, we are implicitly assuming that each component of the voltage vector never crosses zero. In fact, we shall assume the following:

Assumption 1. There exists a subset \mathcal{X} of the state space $\mathbb{T}^N \times \mathbb{R}^N \times \mathbb{R}_{>0}^N$ that is forward invariant along the solutions to (1).

Conditions under which this assumption is fulfilled will be provided later in the paper.

Notice that the voltage dynamics identified by f has not yet

been taken into account in the function U . Therefore, to cope with different voltage dynamics (or controllers) we add another component, namely $H(V)$, and define

$$S(\theta, \omega, V) = U(\theta, \omega, V) + H(V). \quad (9)$$

We rest our analysis on the following foundational incremental storage function

$$\begin{aligned} S(\theta, \omega, V) = & S(\theta, \omega, V) - S(\bar{\theta}, \bar{\omega}, \bar{V}) - \left. \frac{\partial S}{\partial \theta} \right|_-^T (\theta - \bar{\theta}) \\ & - \left. \frac{\partial S}{\partial \omega} \right|_-^T (\omega - \bar{\omega}) - \left. \frac{\partial S}{\partial V} \right|_-^T (V - \bar{V}) \end{aligned} \quad (10)$$

where we use the conventional notation

$$\left. \frac{\partial F}{\partial x} \right|_- = \frac{\partial F}{\partial x}(\bar{x}), \quad \left. \frac{\partial F}{\partial x} \right|_-^T = \left(\frac{\partial F}{\partial x}(\bar{x}) \right)^T$$

for a function $F : \mathcal{X} \rightarrow \mathbb{R}$. Note that S can be decomposed as

$$S = \mathcal{U} + \mathcal{H} \quad (11)$$

where

$$\begin{aligned} \mathcal{U}(\theta, \omega, V) = & U(\theta, \omega, V) - U(\bar{\theta}, \bar{\omega}, \bar{V}) - \left. \frac{\partial U}{\partial \theta} \right|_-^T (\theta - \bar{\theta}) \\ & - \left. \frac{\partial U}{\partial \omega} \right|_-^T (\omega - \bar{\omega}) - \left. \frac{\partial U}{\partial V} \right|_-^T (V - \bar{V}) \end{aligned}$$

and

$$\mathcal{H}(V) = H(V) - H(\bar{V}) - \left. \frac{\partial H}{\partial V} \right|_-^T (V - \bar{V}).$$

Observe that as the synchronous solution satisfies $\bar{\omega} \in \mathcal{N}(D^T)$, then

$$\begin{aligned} \left. \frac{\partial U}{\partial \theta} \right|_-^T (\theta - \bar{\theta}) &= (D\Gamma(\bar{V})\mathbf{sin}(D^T\bar{\theta}))^T (\theta - \bar{\theta}) \\ &= (D\Gamma(\bar{V})\mathbf{sin}(D^T\theta^0))^T (\theta - \theta^0) \end{aligned}$$

i.e. the term does not depend explicitly on t . This observation will be useful when differentiating the incremental function by t .

Also note that

$$\left. \frac{\partial \mathcal{U}}{\partial V} \right|_- = \left. \frac{\partial U}{\partial V} - \frac{\partial U}{\partial V} \right|_- = [V]^{-1}Q - [\bar{V}]^{-1}\bar{Q}$$

Moreover, the terms in $\mathcal{U}(\theta, \omega, V)$ which explicitly depends on ω write as

$$\begin{aligned} & \frac{1}{2}\omega^T K_P^{-1}T_P\omega - \frac{1}{2}\bar{\omega}^T K_P^{-1}T_P\bar{\omega} - \bar{\omega}^T K_P^{-1}T_P(\omega - \bar{\omega}) \\ &= \frac{1}{2}(\omega - \bar{\omega})^T K_P^{-1}T_P(\omega - \bar{\omega}). \end{aligned}$$

Hence, the explicit expression of \mathcal{U} is

$$\begin{aligned} \mathcal{U}(\theta, \omega, V) = & \frac{1}{2}(\omega - \bar{\omega})^T K_P^{-1}T_P(\omega - \bar{\omega}) \\ & + \frac{1}{2}V^T \mathcal{A}(\cos(D^T\theta))V - \frac{1}{2}\bar{V}^T \mathcal{A}(\cos(D^T\bar{\theta}))\bar{V} \\ & - (D\Gamma(\bar{V})\mathbf{sin}(D^T\bar{\theta}))^T (\theta - \bar{\theta}) - \bar{Q}[\bar{V}]^{-1}(V - \bar{V}). \end{aligned}$$

Bearing in mind (10) and (11), we notice that

$$\begin{aligned} \frac{\partial S(\theta, \omega, V)}{\partial \theta} &= \left. \frac{\partial U}{\partial \theta} - \frac{\partial U}{\partial \theta} \right|_- = D\Gamma(V)\mathbf{sin}(D^T\theta) - D\Gamma(\bar{V})\mathbf{sin}(D^T\bar{\theta}), \\ \frac{\partial S(\theta, \omega, V)}{\partial \omega} &= \left. \frac{\partial U}{\partial \omega} - \frac{\partial U}{\partial \omega} \right|_- = K_P^{-1}T_P(\omega - \bar{\omega}), \\ \frac{\partial S(\theta, \omega, V)}{\partial V} &= \left. \frac{\partial \mathcal{U}}{\partial V} + \frac{\partial \mathcal{H}}{\partial V} \right|_- \\ &= [V]^{-1}Q - [\bar{V}]^{-1}\bar{Q} + \left. \frac{\partial H}{\partial V} - \frac{\partial H}{\partial V} \right|_- . \end{aligned}$$

The above identities show that the critical points of S occur for $\omega = \bar{\omega}$ and $P = \bar{P}$ which is a desired property. The critical point of S with respect to the V coordinate is determined by the choice of H which depends on the voltage dynamics.

To establish the incremental dissipativity property, we introduce the output variables

$$y = \text{col}(y_P, y_Q) \quad (12)$$

with

$$y_P = T_P^{-1} \frac{\partial S}{\partial \omega} = K_P^{-1}\omega, \quad y_Q = T_Q^{-1} \frac{\partial S}{\partial V},$$

and input variables

$$u = \text{col}(u_P, u_Q). \quad (13)$$

In what follows, we differentiate among different voltage controllers and adjust the analysis accordingly by tuning H .

A. Conventional droop controller

The conventional droop controllers are obtained by setting f in (1) as

$$f(V, Q, u_Q) = -V - K_Q Q + u_Q \quad (14)$$

where $K_Q = [k_Q]$ is a diagonal matrix with positive droop coefficients on its diagonal. Note that u_Q is added for the sake of generality and one can set $u_Q = \bar{u}_Q = K_Q Q^* + V^*$ for nominal constant vectors V^* and Q^* to obtain the well known expression of conventional droop controllers, see e.g. [44]. For this choice of f , we pick the function H in (9) as ([29], [37])

$$H(V) = \mathbf{1}^T K_Q V - c^T \ln(V), \quad (15)$$

with $c \in \mathbb{R}_{>0}^N$. This term has two interesting features. First, it makes the incremental storage function S radially unbounded with respect to V on the positive orthant. Moreover, it shifts the critical points of S as desired. In particular, bearing in mind (11), we have

$$\begin{aligned} \frac{\partial (\mathcal{U}(\theta, \omega, V) + \mathcal{H}(V))}{\partial V} &= [V]^{-1}Q - [\bar{V}]^{-1}\bar{Q} + k_Q \\ &\quad - [V]^{-1}c - k_Q + [\bar{V}]^{-1}\bar{c}, \end{aligned}$$

which, letting $\bar{u}_Q \in \mathbb{R}_{>0}^N$ and setting $c = \bar{Q} + K_Q^{-1}\bar{V}$, yields

$$\begin{aligned} \frac{\partial S}{\partial V} &= [V]^{-1}Q - [V]^{-1}\bar{Q} - [V]^{-1}K_Q^{-1}\bar{V} + K_Q^{-1}\mathbf{1} \\ &= [V]^{-1}K_Q^{-1}(K_Q(Q - \bar{Q}) + V - \bar{V}). \end{aligned}$$

Noting that

$$0 = -\bar{V} - K_Q \bar{Q} + \bar{u}_Q$$

we have

$$\frac{\partial \mathcal{S}}{\partial V} = [V]^{-1} K_Q^{-1} (-T_Q \dot{V} + u_Q - \bar{u}_Q).$$

Hence,

$$T_Q \dot{V} = -K_Q [V] \frac{\partial \mathcal{S}}{\partial V} + u_Q - \bar{u}_Q. \quad (16)$$

In the following subsections we will derive analogous identities and then use those for concluding incremental cyclo-dissipativity of the system.

B. Quadratic droop controller

Another voltage dynamics proposed in the literature is associated with the quadratic droop controllers of [34], which can be expressed as (1) with

$$f(V, Q, u_Q) = -K_Q Q - [V](V - u_Q), \quad (17)$$

where again $K_Q = [k_Q]$ collects the droop coefficients. The quadratic droop controllers in [34] is obtained by setting $u_Q = V^*$ for some constant vector V^* . Notice however the difference: while [34] focuses on a network preserved micro-grid model in which the equation above models the inverter dynamics and are decoupled from the frequency dynamics, here a fully coupled network reduced model is considered.

Moreover, note that the scaling matrix $[V]$ distinguishes this case from the conventional droop controller. For this case, we adapt the storage function \mathcal{S} by setting

$$H(V) = \frac{1}{2} V^T K_Q^{-1} V. \quad (18)$$

Recall that $\mathcal{S} = \mathcal{U} + \mathcal{H}$. Note that \mathcal{S} is defined on the whole $\mathbb{T}^N \times \mathbb{R}^N \times \mathbb{R}^N$ and not on $\mathbb{T}^N \times \mathbb{R}^N \times \mathbb{R}_{>0}^n$. The resulting function \mathcal{S} can be interpreted as a performance criterion in a similar vein as the cost function in [34]. Clearly we have

$$\frac{\partial \mathcal{S}}{\partial V} = [V]^{-1} Q - [\bar{V}]^{-1} \bar{Q} + K_Q^{-1} (V - \bar{V}). \quad (19)$$

Noting that

$$0 = -K_Q \bar{Q} - [\bar{V}](\bar{V} - \bar{u}_Q)$$

we have

$$[\bar{V}]^{-1} \bar{Q} = -K_Q^{-1} (\bar{V} - \bar{u}_Q).$$

Substituting the latter into (19) yields

$$\begin{aligned} \frac{\partial \mathcal{S}}{\partial V} &= [V]^{-1} Q + K_Q^{-1} V - K_Q^{-1} \bar{u}_Q \\ &= [V]^{-1} K_Q^{-1} (K_Q Q + [V] V - [V] \bar{u}_Q) \\ &= [V]^{-1} K_Q^{-1} (-T_Q \dot{V} + [V] u_Q - [V] \bar{u}_Q) \end{aligned}$$

Hence,

$$T_Q \dot{V} = -K_Q [V] \frac{\partial \mathcal{S}}{\partial V} + [V](u_Q - \bar{u}_Q). \quad (20)$$

C. Reactive current controller

The frequency dynamics of the inverters in microgrids typically mimics that of the synchronous generators known as the *swing equation*. This facilitates the interface of inverters and generators in the grid. To enhance such interface, an idea is to mimic the voltage dynamics of the synchronous generators as well. Motivated by this, we consider the voltage controller identified by

$$f(V, Q, u_Q) = -[V]^{-1} Q + u_Q. \quad (21)$$

This controller aims at regulating the ratio of reactive power over voltage amplitudes, which can be interpreted as “reactive current” ([21]). For this controller, we set

$$H = 0 \quad (22)$$

meaning that $S = U$ and no adaption of the storage function is needed. Clearly, we have

$$\frac{\partial \mathcal{S}}{\partial V} = \frac{\partial \mathcal{U}}{\partial V} = [V]^{-1} Q - [\bar{V}]^{-1} \bar{Q}$$

and it is easy to observe that

$$T_Q \dot{V} = -\frac{\partial \mathcal{S}}{\partial V} + u_Q - \bar{u}_Q \quad (23)$$

where $\bar{u}_Q = [\bar{V}]^{-1} \bar{Q}$ is again the feedforward input guaranteeing the preservation of the steady state.

D. Consensus based controller

In this subsection, we consider another controller which aims at achieving proportional power sharing.

$$f(V, Q, u_Q) = -[V] K_Q L_Q K_Q Q + [V] u_Q \quad (24)$$

where $K_Q = [k_Q]$ is a diagonal matrix and L_Q is the Laplacian matrix of a communication graph which is assumed to be undirected and connected. This controller is a variation of that of [30] in which voltage dynamics are scaled by the voltages at the inverters, namely $[V]$, the reactive power Q is *not* assumed to be independent of the phase variables θ , and an additional input u_Q is introduced. Assuming that $T_Q = I$ for simplicity, we choose H as

$$H(V) = -c^T \ln V \quad (25)$$

where c is a constant vector. Then, we have

$$\frac{\partial \mathcal{S}}{\partial V} = [V]^{-1} Q - [\bar{V}]^{-1} \bar{Q} - [V]^{-1} c + [\bar{V}]^{-1} c.$$

By setting $c = \bar{Q}$, this reduces to

$$\frac{\partial \mathcal{S}}{\partial V} = [V]^{-1} (Q - \bar{Q}). \quad (26)$$

Moreover, defining

$$\bar{u}_Q = K_Q L_Q K_Q \bar{Q}, \quad (27)$$

the dynamics

$$\dot{V} = -[V] K_Q L_Q K_Q Q + [V] u_Q \quad (28)$$

can be rewritten as

$$\dot{V} = -[V] K_Q L_Q K_Q [V] \frac{\partial \mathcal{S}}{\partial V} + [V](u_Q - \bar{u}_Q). \quad (29)$$

E. Incremental dissipativity of microgrid models

In this subsection, we show how the candidate storage functions introduced before allow us to infer incremental dissipativity of the microgrids under the various controllers.

Theorem 1. Assume that the feasibility condition (6) admits a solution and let Assumption 1 hold. Then system (1) with output (12), input (13), and, respectively,

- 1) $f(V, Q, u_Q)$ given by (14);
- 2) $f(V, Q, u_Q)$ given by (17);
- 3) $f(V, Q, u_Q)$ given by (21);
- 4) $f(V, Q, u_Q)$ given by (24);

is incrementally cyclo-dissipative with respect to the synchronous solution $(\bar{\theta}, \bar{\omega}, \bar{V})$, with

- 1) incremental storage function \mathcal{S} defined by (7),(9),(10),(15) and supply rate (5) with weight matrices

$$W(V) = \begin{pmatrix} K_P & \mathbf{0} \\ \mathbf{0} & T_Q K_Q[V] \end{pmatrix}, \quad R = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix};$$

- 2) incremental storage function \mathcal{S} defined by (7),(9),(10),(18) and supply rate (5) with weight matrices

$$W(V) = \begin{pmatrix} K_P & \mathbf{0} \\ \mathbf{0} & T_Q K_Q[V] \end{pmatrix},$$

$$R(V) = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & [V] \end{pmatrix};$$

- 3) incremental storage function \mathcal{S} defined by (7),(9),(10),(22) and supply rate (5) with weight matrices

$$W = \begin{pmatrix} K_P & \mathbf{0} \\ \mathbf{0} & T_Q \end{pmatrix}, \quad R = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix};$$

- 4) incremental storage function \mathcal{S} defined by (7),(9),(10),(25) and supply rate (5) with weight matrices

$$W(V) = \begin{pmatrix} K_P & \mathbf{0} \\ \mathbf{0} & [V] K_Q L_Q K_Q[V] \end{pmatrix},$$

$$R(V) = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & [V] \end{pmatrix}.$$

Proof: 1. Recall that

$$\frac{\partial \mathcal{S}}{\partial \omega} = K_P^{-1} T_P (\omega - \bar{\omega}),$$

$$\frac{\partial \mathcal{S}}{\partial \theta} = D\Gamma(V) \sin(D^T \theta) - D\Gamma(\bar{V}) \sin(D^T \theta^0) = (P - \bar{P}). \quad (30)$$

Then

$$\begin{aligned} \frac{d}{dt} \mathcal{S} &= (\omega - \bar{\omega})^T T_P K_P^{-1} \dot{\omega} \\ &+ (D\Gamma(V) \sin(D^T \theta) - D\Gamma(\bar{V}) \sin(D^T \theta^0))^T \dot{\theta} + \left(\frac{\partial \mathcal{S}}{\partial V} \right)^T \dot{V} \\ &= (\omega - \bar{\omega})^T K_P^{-1} (-(\omega - \bar{\omega}) - K_P(P - \bar{P}) + (u_P - \bar{u}_P)) \\ &+ (D\Gamma(V) \sin(D^T \theta) - D\Gamma(\bar{V}) \sin(D^T \theta^0))^T (\omega - \bar{\omega}) \\ &+ \left(\frac{\partial \mathcal{S}}{\partial V} \right)^T T_Q^{-1} (-K_Q[V] \frac{\partial \mathcal{S}}{\partial V} + u_Q - \bar{u}_Q) \\ &= (\omega - \bar{\omega})^T K_P^{-1} (-(\omega - \bar{\omega}) - K_P(P - \bar{P}) + (u_P - \bar{u}_P)) \\ &+ (P - \bar{P})^T (\omega - \bar{\omega}) - \left(\frac{\partial \mathcal{S}}{\partial V} \right)^T T_Q^{-1} K_Q[V] \frac{\partial \mathcal{S}}{\partial V} \\ &+ \left(\frac{\partial \mathcal{S}}{\partial V} \right)^T T_Q^{-1} (u_Q - \bar{u}_Q) \end{aligned}$$

where the chain of equalities hold because of the feasibility condition and (16). Hence

$$\begin{aligned} \frac{d}{dt} \mathcal{S} &= -(\omega - \bar{\omega})^T K_P^{-1} (\omega - \bar{\omega}) + (\omega - \bar{\omega})^T K_P^{-1} (u_P - \bar{u}_P) \\ &- \left(\frac{\partial \mathcal{S}}{\partial V} \right)^T T_Q^{-1} K_Q[V] \frac{\partial \mathcal{S}}{\partial V} + \left(\frac{\partial \mathcal{S}}{\partial V} \right)^T T_Q^{-1} (u_Q - \bar{u}_Q). \end{aligned} \quad (31)$$

Observe now that by definition

$$\frac{\partial \mathcal{S}}{\partial V} = \frac{\partial \mathcal{S}}{\partial V} - \frac{\partial \mathcal{S}}{\partial V} \Big|^-$$

and that $\frac{\partial \mathcal{S}}{\partial V} \Big|^-$ represents the output component $\frac{\partial \mathcal{S}}{\partial V}$ at the synchronous solution. Hence equality (32) at the top of the next page can be established.

We conclude incremental cyclo-dissipativity of system (1), (12), (13), (14) as claimed.

2. If in the chain of equalities defining $\frac{d}{dt} \mathcal{S}$ above, we use (20) instead of (16), we obtain that

$$\begin{aligned} \frac{d}{dt} \mathcal{S} &= -(\omega - \bar{\omega})^T K_P^{-1} (\omega - \bar{\omega}) + (\omega - \bar{\omega})^T K_P^{-1} (u_P - \bar{u}_P) \\ &- \left(\frac{\partial \mathcal{S}}{\partial V} \right)^T T_Q^{-1} K_Q[V] \frac{\partial \mathcal{S}}{\partial V} + \left(\frac{\partial \mathcal{S}}{\partial V} \right)^T T_Q^{-1} [V] (u_Q - \bar{u}_Q) \end{aligned} \quad (33)$$

which shows incremental cyclo-dissipativity of system (1), (12), (13), (17).

3. For this case, adopting the equality (23) results in the equality

$$\begin{aligned} \frac{d}{dt} \mathcal{S} &= -(\omega - \bar{\omega})^T K_P^{-1} (\omega - \bar{\omega}) + (\omega - \bar{\omega})^T K_P^{-1} (u_P - \bar{u}_P) \\ &- \left(\frac{\partial \mathcal{S}}{\partial V} \right)^T T_Q^{-1} \frac{\partial \mathcal{S}}{\partial V} + \frac{\partial \mathcal{S}^T}{\partial V} T_Q^{-1} (u_Q - \bar{u}_Q), \end{aligned} \quad (34)$$

from which incremental cyclo-dissipativity of (1), (12), (13), (21) holds.

4. Finally, in view of (29),

$$\begin{aligned} \frac{d}{dt} \mathcal{S} &= -(\omega - \bar{\omega})^T K_P^{-1} (\omega - \bar{\omega}) + (\omega - \bar{\omega})^T K_P^{-1} (u_P - \bar{u}_P) \\ &- \left(\frac{\partial \mathcal{S}}{\partial V} \right)^T [V] K_Q L_Q K_Q[V] \frac{\partial \mathcal{S}}{\partial V} + \left(\frac{\partial \mathcal{S}}{\partial V} \right)^T [V] (u_Q - \bar{u}_Q) \end{aligned} \quad (35)$$

which implies incremental cyclo-dissipativity of (1), (12), (13), (24). ■

$$\begin{aligned} \frac{d}{dt}\mathcal{S} = & -\left((\omega - \bar{\omega})^T K_P^{-1} \left(\frac{\partial \mathcal{S}}{\partial V} - \frac{\partial \mathcal{S}}{\partial V}\Big|_{-}\right)^T T_Q^{-1}\right) \begin{pmatrix} K_P & \mathbf{0} \\ \mathbf{0} & T_Q K_Q[V] \end{pmatrix} \begin{pmatrix} K_P^{-1}(\omega - \bar{\omega}) \\ T_Q^{-1}\left(\frac{\partial \mathcal{S}}{\partial V} - \frac{\partial \mathcal{S}}{\partial V}\Big|_{-}\right) \end{pmatrix} \\ & + \left((\omega - \bar{\omega})^T K_P^{-1} \left(\frac{\partial \mathcal{S}}{\partial V} - \frac{\partial \mathcal{S}}{\partial V}\Big|_{-}\right)^T T_Q^{-1}\right) \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix} \begin{pmatrix} u_P - \bar{u}_P \\ u_Q - \bar{u}_Q \end{pmatrix}. \end{aligned} \quad (32)$$

IV. FROM CYCLO-DISSIPATIVITY TO DISSIPATIVITY

The dissipation inequalities proven before can be exploited to study the stability of the synchronous solution. Theorem 1 has been established in terms of cyclo-dissipativity rather than dissipativity, i.e. without imposing lower boundedness of the storage function \mathcal{S} . However, in order to conclude the attractivity of the synchronous solution we ask for incremental dissipativity of the system, and require the storage function to posses a strict minimum at the point of interest. To this end, we investigate conditions under which the Hessian of the storage function \mathcal{S} is positive definite at the point of interest, which in this case is identified by the synchronous solution.

It is not difficult to observe that due to the rotational invariance of θ variables, the existence of a *strict* minimum for \mathcal{S} cannot be anticipated. To clear this obstacle, we notice that the phase angles θ appear as relative terms, i.e. $\theta_i - \theta_j$, in (7) and thus in S as well as \mathcal{S} . To make this observation more explicit, we write

$$U(\theta, \omega, V) = U^\eta(D^T \theta, \omega, V)$$

where

$$U^\eta(\eta, \omega, V) = \frac{1}{2}\omega^T K_P^{-1} T_P \omega + \frac{1}{2}V^T \mathcal{A}(\cos(\eta))V.$$

In a similar vein, we define

$$\begin{aligned} \mathcal{U}^\eta(\eta, \omega, V) &= U^\eta(\eta, \omega, V) - U^\eta(\bar{\eta}, \bar{\omega}, \bar{V}) \\ &- \frac{\partial U^\eta}{\partial \eta} \Big|_{-}^T (\eta - \bar{\eta}) - \frac{\partial U^\eta}{\partial \omega} \Big|_{-}^T (\omega - \bar{\omega}) - \frac{\partial U^\eta}{\partial V} \Big|_{-}^T (V - \bar{V}) \end{aligned} \quad (36)$$

to have

$$\mathcal{U}(\theta, \omega, V) = \mathcal{U}^\eta(D^T \theta, \omega, V).$$

Note that with a little abuse of the notation, we replace $\bar{\eta}$ by $D^T \bar{\theta}$ when computing $\mathcal{U}^\eta(D^T \theta, \omega, V)$ from (36). Now, as the function \mathcal{H} in (11) does not depend on θ variables, we define

$$\mathcal{S}^\eta(\eta, \omega, V) = \mathcal{U}^\eta(\eta, \omega, V) + \mathcal{H}(V)$$

to have

$$\mathcal{S}(\theta, \omega, V) = \mathcal{S}^\eta(D^T \theta, \omega, V). \quad (37)$$

Notice that while the solution of interest of $\mathcal{S}(\theta, \omega, V)$ is given by the synchronous solution $(\bar{\theta}, \bar{\omega}, \bar{V})$, that of $\mathcal{S}^\eta(\eta, \omega, V)$ is identified by $(D^T \bar{\theta}, \bar{\omega}, \bar{V})$. It turns out that working with the latter is more convenient, mainly due to the absence of the rotational symmetry in the variables $D^T \theta$.

For a later purpose, also notice that, by (37), the time derivative of \mathcal{S} along the solutions (θ, ω, V) is equal to that of \mathcal{S}^η along $(D^T \theta, \omega, V)$, namely

$$\frac{d}{dt}\mathcal{S}(\theta, \omega, V) = \frac{d}{dt}\mathcal{S}^\eta(D^T \theta, \omega, V).$$

Hence, as a consequence, from the proof of Theorem 1, we infer that

$$\begin{aligned} \frac{d}{dt}\mathcal{S}^\eta(D^T \theta, \omega, V) &= -(\omega - \bar{\omega})^T K_P^{-1}(\omega - \bar{\omega}) + (\omega - \bar{\omega})^T K_P^{-1}(u_P - \bar{u}_P) \\ &- \left(\frac{\partial \mathcal{S}^\eta}{\partial V}\right)^T X(V) \frac{\partial \mathcal{S}^\eta}{\partial V} + \left(\frac{\partial \mathcal{S}^\eta}{\partial V}\right)^T Y(V)(u_Q - \bar{u}_Q), \end{aligned} \quad (38)$$

where $X(V) = T_Q^{-1} K_Q[V]$, T_Q^{-1} or $[V] K_Q^{-1} L_Q K_Q^{-1}[V]$ and $Y(V) = T_Q^{-1}, T_Q^{-1}[V], [V]$ depending on the voltage controller adopted.

Remark 2. An alternative way to get rid of the rotational invariance is to set the voltage angle of one node of the network as the reference, and rest the analysis on the reduced order system, see [29] for more details. Another way, is to express the dynamics of the system using directly (η, ω, V) variables, similarly as in [38]. However, we do not adopt this approach here in order to better contrast our results with others in the literature on oscillator synchronization that are mostly working with (θ, ω, V) coordinates [14], [32], [27], [20]. ■

To proceed with the analysis, first note that

$$\frac{\partial \mathcal{S}^\eta(\eta, \omega, V)}{\partial \eta} = \Gamma(V) \sin(\eta) - \Gamma(\bar{V}) \sin(\bar{\eta}), \quad (39)$$

$$\frac{\partial \mathcal{S}^\eta(\eta, \omega, V)}{\partial \omega} = \frac{\partial \mathcal{S}(\theta, \omega, V)}{\partial \omega}, \quad \frac{\partial \mathcal{S}^\eta(\eta, \omega, V)}{\partial V} = \frac{\partial \mathcal{S}(\theta, \omega, V)}{\partial V}, \quad (40)$$

where $\eta = D^T \theta$. Hence, $(\bar{\eta}, \bar{\omega}, \bar{V})$ is a critical point of \mathcal{S}^η . Next, we compute the Hessian as

$$\frac{\partial^2 \mathcal{S}^\eta}{\partial(\eta, \omega, V)^2} = \begin{bmatrix} \Gamma(V)[\cos(\eta)] & \mathbf{0} & * \\ \mathbf{0} & K_P^{-1} T_P & \mathbf{0} \\ [V]^{-1} |D| \Gamma(V)[\sin(\eta)] & \mathbf{0} & \mathcal{A}(\cos(\eta)) + \frac{\partial^2 \mathcal{H}}{\partial V^2} \end{bmatrix}. \quad (41)$$

Clearly, the matrix above is positive definite if and only if

$$\begin{bmatrix} \Gamma(V)[\cos(\eta)] & [\sin(\eta)] \Gamma(V) |D|^T [V]^{-1} \\ [V]^{-1} |D| \Gamma(V)[\sin(\eta)] & \mathcal{A}(\cos(\eta)) + \frac{\partial^2 \mathcal{H}}{\partial V^2} \end{bmatrix} > 0. \quad (42)$$

Notice that in all the previously studied cases, the matrix $\frac{\partial^2 \mathcal{H}}{\partial V^2}$ is diagonal. In particular,

$$\begin{aligned} \frac{\partial^2 \mathcal{H}}{\partial V^2} &= K_Q + [V]^{-2} [c], & \frac{\partial^2 \mathcal{H}}{\partial V^2} &= K_Q^{-1}, \\ \frac{\partial^2 \mathcal{H}}{\partial V^2} &= 0, & \frac{\partial^2 \mathcal{H}}{\partial V^2} &= [V]^{-2} [c], \end{aligned} \quad (43)$$

for conventional droop, quadratic droop, reactive current controller, and consensus based protocol, respectively. Now, let

$$\frac{\partial^2 H}{\partial V^2} := [h(V)]. \quad (44)$$

and $h(V) = \text{col}(h_i(V_i))$. Then, the following result, which establishes decentralized conditions for checking the positive definiteness of the Hessian, can be proven:

Proposition 1. *Let $\bar{V} \in \mathbb{R}_{>0}^N$ and $\bar{\eta} \in (-\frac{\pi}{2}, \frac{\pi}{2})^m$. If for all $i = 1, 2, \dots, N$,*

$$m_{ii} := \hat{B}_{ii} + \sum_{k \sim \{i, \ell\} \in E} B_{i\ell} \left(1 - \frac{\bar{V}_\ell \sin^2(\bar{\eta}_k)}{\bar{V}_i \cos(\bar{\eta}_k)} \right) + h_i(\bar{V}_i) > 0, \quad (45)$$

and

$$m_{ii} > \sum_{k \sim \{i, \ell\} \in E} B_{i\ell} |\cos(\bar{\eta}_k)| \left(1 + \frac{\bar{V}_i}{\bar{V}_j} \tan^2(\bar{\eta}_k) \right), \quad (46)$$

then

$$\frac{\partial^2 S^\eta}{\partial(\eta, \omega, V)^2} \Big|_- > 0. \quad (47)$$

Proof: The proof is given in the appendix. ■

Remark 3. The result shows that the two conditions (45) and (46) for positive definiteness are met provided that at the point $(\bar{\eta}, \bar{\omega}, \bar{V})$ the relative voltage phase angles are small enough and the voltages magnitudes are approximately the same. This is a remarkable property, stating that if the equilibria of interest are characterized by small relative voltage phases and similar voltage magnitudes, then they are minima of the incremental storage function $\mathcal{S}(\theta, \omega, V)$, and equivalently *isolated* minima of $\mathcal{S}^\eta(\eta, \omega, V)$. ■

Remark 4. The Hessian of energy functions has always played an important role in stability studies of power networks (see e.g. [39], and [29] for a microgrid stability investigation). Conditions for assessing the positive definiteness of the Hessian of an energy function associated to power networks have been reported in the literature since [39], and used even recently to study e.g. the convexity of the energy function ([17]). Our conditions however are different and hold for more general energy functions. ■

Remark 5. It is interesting to establish a connection with existing studies on oscillator synchronization arising in different contexts. Once again, this connection leverages the use of the energy function. If the coupling between any pair of nodes i, j is represented by a single variable v_{ij} , modeling e.g. a dynamic coupling, instead of the product of the voltage variables $V_i V_j$, then a different model arises. To obtain this, we focus for the sake of simplicity on oscillators without inertia, and replace the previous energy function (7) with

$$U(\theta, v) = -\frac{1}{2} \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} v_{ij} B_{ij} \cos(\theta_j - \theta_i) + \frac{1}{2} \sum_{\{i, j\} \in E} v_{ij}^2.$$

Then

$$\frac{\partial U}{\partial v_{ij}} = -B_{ij} \cos(\theta_j - \theta_i) + v_{ij},$$

and the resulting (gradient) system becomes

$$\begin{aligned} \dot{\theta}_i &= \sum_{j \in \mathcal{N}_i} v_{ij} B_{ij} \sin(\theta_j - \theta_i), \quad i = 1, 2, \dots, N \\ \dot{v}_{ij} &= -(B_{ij} \cos(\theta_j - \theta_i) - v_{ij}), \quad \{i, j\} \in E, \end{aligned}$$

which arises in oscillator networks with so-called plastic coupling strength ([27], [20], [22]) and in the context of flocking with state dependent sensing ([27], [16], [32]). Although stability analysis of equilibria have been carried out for these systems, the investigation of the methods proposed in this paper in those contexts is still unexplored and deserves attention.

V. ATTRACTIVITY OF THE SYNCHRONOUS SOLUTION

In this section, we establish the attractivity of the synchronous solution, which amounts to the frequency regulation ($\bar{\omega} = \omega^*$) with optimal properties. Moreover, we investigate voltage stability and reactive power sharing in the aforementioned voltage controllers.

Recall from (6) that for the synchronous solution we have

$$0 = -K_P(D\Gamma(\bar{V})\sin(D^T\theta^0) - P^*) + \bar{u}_P. \quad (48)$$

Among all possible vectors \bar{u}_P satisfying the above, we look for the one that minimizes the quadratic cost function

$$C(\bar{u}_P) = \frac{1}{2} \bar{u}_P^T K_P^{-1} \bar{u}_P$$

This choice is explicitly computed as

$$\bar{u}_P = -\mathbb{1} \frac{\mathbb{1}^T P^*}{\mathbb{1}^T K_P^{-1} \mathbb{1}}. \quad (49)$$

Then, substituting (49) into (48),

$$\bar{P} = P^* - K_P^{-1} \mathbb{1} \frac{\mathbb{1}^T P^*}{\mathbb{1}^T K_P^{-1} \mathbb{1}}, \quad (50)$$

or, component-wise,

$$\bar{P}_i = P_i^* - (k_P)_i^{-1} \frac{\mathbb{1}^T P^*}{\mathbb{1}^T K_P^{-1} \mathbb{1}},$$

where $K_P = [k_P]$. In the case of droop coefficients selected proportionally ([33], [15], [1], [6], [38]), i.e.

$$(k_P)_i P_i^* = (k_P)_j P_j^*,$$

for all i, j , we conclude that

$$(k_P)_i \bar{P}_i = (k_P)_j \bar{P}_j \quad (51)$$

which accounts for the desired active power sharing based on the diagonal elements of K_P as expected. Bearing in mind (50), the feasibility condition (6) reduces to the following assumption:

Assumption 2. *There exists constant vectors $\bar{V} \in \mathbb{R}^N$ and $\theta^0 \in \mathbb{T}^N$ such that*

$$D\Gamma(\bar{V})\sin(D^T\theta^0) = (I - K_P^{-1} \frac{\mathbb{1}\mathbb{1}^T}{\mathbb{1}^T K_P^{-1} \mathbb{1}})P^* \quad (52)$$

and

$$0 = f(\bar{V}, [\bar{V}]\mathcal{A}(\cos(D^T\theta^0))\bar{V}, \bar{u}_Q). \quad (53)$$

Remark 6. Similar to [38, Remark 5] it can be shown that if the assumption above is satisfied then necessarily $\bar{V} \in \mathbb{R}_{>0}^N$. Furthermore, in case the network is a tree, it is easy to observe that (52) is satisfied if and only if there exists $\bar{V} \in \mathbb{R}_{>0}^N$ such that

$$\|\Gamma(\bar{V})^{-1}D^\dagger(I - K_P^{-1}\frac{\mathbb{1}\mathbb{1}^T}{\mathbb{1}^TK_P^{-1}\mathbb{1}})P^*\|_\infty < 1,$$

with D^\dagger denoting the left inverse of D . In the case of the quadratic voltage droop and consensus based reactive power controllers, explicit expressions of the voltage vector \bar{V} can be given (see Subsection V-A), in which case the condition above becomes dependent on the voltage phase vector at the equilibrium θ^0 only.

To achieve the optimal input (49), we consider the following active power controller ([33], [15], [6])

$$\begin{aligned} \dot{\xi} &= -L_P\xi + K_P^{-1}(\omega^* - \omega) \\ u_P &= \xi \end{aligned} \quad (54)$$

where the matrix L_P is the Laplacian matrix of an undirected and connected communication graph. For the choice of the voltage/reactive power control u_Q , we set $u_Q = \bar{u}_Q$ where \bar{u}_Q is a constant vector enforcing the setpoint for the voltage dynamics. The role of this setpoint will be made clear in Subsection V-A. Then, the main result of this section is as follows:

Theorem 2. Suppose that Assumption 2 and condition (47), with $\bar{\omega} = \omega^*$, hold. Let u_P be given by (54) and $u_Q = \bar{u}_Q \in \mathbb{R}^N$. Then the solutions of (1) locally converge to the set of points where $\omega = \omega^*$ and $u_P = \bar{u}_P$ with \bar{u}_P being the optimal input (49). Moreover, the following statements hold:

- (i) For conventional droop controller (14), the vectors V and Q locally converge to the constant vectors \bar{V} and \bar{Q} satisfying

$$K_Q\bar{Q} + \bar{V} = \bar{u}_Q$$

- (ii) For quadratic droop controller (17), the vectors V and Q locally converge to the constant vectors \bar{V} and \bar{Q} satisfying

$$K_Q[\bar{V}]^{-1}\bar{Q} + \bar{V} = \bar{u}_Q$$

- (iii) For reactive current controller (21), the vectors V and Q locally converge to the constant vectors \bar{V} and \bar{Q} satisfying

$$[\bar{V}]^{-1}\bar{Q} = \bar{u}_Q$$

- (iv) For consensus based reactive power controller (24), the vector V locally converges to a constant vector \bar{V} , and Q converges to a constant vector $\bar{Q} = [\bar{V}]\mathcal{A}(\cos(D^T\theta^0))\bar{V}$ satisfying

$$L_QK_Q\bar{Q} = K_Q\bar{u}_Q.$$

Moreover, for all $t \geq 0$,

$$\mathbb{1}^TK_Q^{-1}\ln(V(t)) = \mathbb{1}^TK_Q^{-1}\ln(\bar{V}) = \mathbb{1}^TK_Q^{-1}\ln(V(0)).$$

In case $\bar{u}_Q = 0$, then

$$\bar{Q} = K_Q^{-1}\mathbb{1}\frac{\bar{V}^T\mathcal{A}(\cos(D^T\theta^0))\bar{V}}{\mathbb{1}^TK_Q^{-1}\mathbb{1}}.$$

Proof: The desired synchronous solution in this case is characterized by $\bar{\omega} = \omega^*$, \bar{u}_P given by (49), $\bar{\xi} = K_P^{-1}\bar{u}_P$, and the corresponding \bar{V} and \bar{Q} satisfying (6). Define the incremental storage function $\mathcal{C}_P(\xi) = \frac{1}{2}(\xi - \bar{\xi})^T(\xi - \bar{\xi})$. Notice that $\bar{\xi} \in \text{im } \mathbb{1}$. Then

$$\begin{aligned} \frac{d}{dt}\mathcal{C}_P &= -(\xi - \bar{\xi})^TL_P(\xi - \bar{\xi}) - (\xi - \bar{\xi})^T(\omega - \bar{\omega}) \\ &= -(\xi - \bar{\xi})^TL_P(\xi - \bar{\xi}) - (u_P - \bar{u}_P)^TK_P^{-1}(\omega - \bar{\omega}). \end{aligned}$$

Observe that, by setting $u_Q = \bar{u}_Q$ and bearing in mind (38), the equalities (31), (33), (35) and (34) can be written in a unified manner as

$$\begin{aligned} \frac{d}{dt}\mathcal{S}^\eta(D^T\theta, \omega, V) &= -(\omega - \bar{\omega})^TK_P^{-1}(\omega - \bar{\omega}) \\ &\quad - \left(\frac{\partial\mathcal{S}^\eta}{\partial V}\right)^TX(V)\frac{\partial\mathcal{S}^\eta}{\partial V} + (\omega - \bar{\omega})^TK_P^{-1}(u_P - \bar{u}_P) \end{aligned}$$

where X is a positive (semi)-definite matrix suitably chosen according to the underlying voltage dynamics. Now taking $\mathcal{S}^\eta + \mathcal{C}_P$ as the Lyapunov function, we have

$$\begin{aligned} \frac{d}{dt}\mathcal{S}^\eta + \frac{d}{dt}\mathcal{C}_P &= -(\omega - \bar{\omega})^TK_P^{-1}(\omega - \bar{\omega}) \\ &\quad - \left(\frac{\partial\mathcal{S}^\eta}{\partial V}\right)^TX(V)\frac{\partial\mathcal{S}^\eta}{\partial V} - (\xi - \bar{\xi})^TL_P(\xi - \bar{\xi}). \end{aligned} \quad (55)$$

By local strict convexity of $\mathcal{S}^\eta + \mathcal{C}_P$ (thanks to (47)), we can construct a forward invariant compact level set around the desired synchronous equilibrium $(D^T\bar{\theta}, \bar{\omega}, \bar{V})$ and apply LaSalle's invariance principle. Notice in particular that on this forward invariant set $V(t) \in \mathbb{R}_{>0}^N$ for all $t \geq 0$. Then the solutions are guaranteed to converge to the largest invariant set where

$$\begin{aligned} \omega &= \bar{\omega} \\ 0 &= L_P(\xi - \bar{\xi}) \\ 0 &= \left(\frac{\partial\mathcal{S}^\eta}{\partial V}\right)^TX(V)\frac{\partial\mathcal{S}^\eta}{\partial V} \end{aligned} \quad (56)$$

Recall that $\bar{\xi} \in \text{im } \mathbb{1}$. Hence, on the invariant set, $L_P\xi = 0$ and thus $\xi = \gamma\mathbb{1}$ for some $\gamma \in \mathbb{R}$. Note that, by (54), γ has to be constant given the fact that $\omega = \omega^*$ and $L_P\xi = 0$. Also note that

$$u_P = K_P(D\Gamma(V)\sin(D^T\theta^0) - P^*)$$

on the invariant set. Multiplying both sides of the above equality by $\mathbb{1}^TK_P^{-1}$ yields $\gamma N = -\mathbb{1}^TP^*$. Therefore, $\xi = \frac{1}{N}\mathbb{1}\mathbb{1}^TP^*$, and u_P converges to the optimal input \bar{u}_P given by (49).

By (31) and (33), the matrix $X(V)$ is equal to $T_Q^{-1}K_Q[V]$ for both the droop controller and quadratic droop controller. Hence, $K_Q[V]\frac{\partial\mathcal{S}^\eta}{\partial V} = 0$ from the third equality in (56). Then, by (16) and (20), and (40), we obtain that $\dot{V} = 0$ on the invariant set for these controllers. Similarly by (34), the matrix $X = T_Q^{-1}$ for the reactive current controller, which by (23) results again in $\dot{V} = 0$. Consequently on the invariant set, we have

$$0 = f(\bar{V}, \bar{Q}, \bar{u}_Q).$$

This together with the isolation of the minima (47) proves the statements (i), (ii), and (iii) in Theorem 2.

For the consensus based reactive power controller, we have $X(V) = [V]K_Q L_Q K_Q [V]$ as evident from (35). Hence, by (26) and the third equality in (56), on the invariant set we obtain that

$$L_Q K_Q Q = L_Q K_Q \bar{Q}. \quad (57)$$

Substituting the above into the corresponding voltage dynamics (28) yields

$$\dot{V} = -[V]K_Q L_Q K_Q \bar{Q} + [V]\bar{u}_Q$$

Hence, by (27), we have $\dot{V} = 0$ on the invariant set, and thus V converges to a constant vector \tilde{V} . Then obviously Q is equal to a constant vector, namely $\tilde{Q} = [\tilde{V}]\mathcal{A}(\cos(D^T \theta^0))\tilde{V}$. By (27) and (57), the vector \tilde{Q} satisfies

$$L_Q K_Q \tilde{Q} = K_Q^{-1} \bar{u}_Q. \quad (58)$$

Recalling that $\bar{u}_Q = K_Q L_Q K_Q \bar{Q}$ (see (27)), the voltage dynamics can be written as

$$\dot{V} = -[V]K_Q L_Q K_Q (Q - \bar{Q}).$$

Hence, we have

$$\frac{d}{dt}(\mathbb{1}^T K_Q^{-1} \ln V) = \mathbb{1}^T K_Q^{-1} [V]^{-1} [V]K_Q L_Q K_Q (Q - \bar{Q}) = 0,$$

as $\mathbb{1}^T L_Q = 0$, which proves that $\mathbb{1}^T K_Q^{-1} \ln(V)$ is a conserved quantity.

Equality (58) and $\bar{u}_Q = 0$ yields

$$\tilde{Q} = \alpha K_Q^{-1} \mathbb{1}$$

for some $\alpha \in \mathbb{R}$. In fact, $\alpha \in \mathbb{R}_{>0}$, since for the synchronous solution the above writes as

$$[\tilde{V}]\mathcal{A}(\cos(D^T \theta^0))\tilde{V} = \alpha K_Q^{-1} \mathbb{1}.$$

Multiplying both sides of the above equality by $\mathbb{1}^T$ yields

$$\tilde{V}^T \mathcal{A}(\cos(D^T \theta^0))\tilde{V} = \alpha \mathbb{1}^T K_Q^{-1} \mathbb{1},$$

thus completing the proof. ■

A. Power sharing

Theorem 2 portrays the asymptotic behavior of the microgrid models discussed in this paper. An immediate interesting consequence is the achievement of *frequency regulation*, *voltage stability*, and *optimal active power sharing* for the coupled nonlinear microgrid model (1). Note that active power sharing is guaranteed by the convergence of P to \bar{P} that satisfies (51). Next, we take a closer look at other consequences and implications of Theorem 2 for different voltage dynamics.

1) *Conventional droop controller*: From the first statement of Theorem 2, it readily follows that

$$\frac{(k_Q)_i \bar{Q}_i + \bar{V}_i}{(k_Q)_j \bar{Q}_j + \bar{V}_j} = \frac{(\bar{u}_Q)_i}{(\bar{u}_Q)_j}.$$

Therefore, the ratio on the left hand side of the above can be arbitrarily assigned by an appropriate choice of \bar{u}_Q , for each $i, j \in \{1, 2, \dots, N\}$. This results in a partial reactive power

sharing for the droop controlled inverters.

2) *Quadratic droop controller*: From the second statement of Theorem 2, we obtain that

$$\frac{(k_Q)_i \bar{Q}_i + \bar{V}_i^2}{(k_Q)_j \bar{Q}_j + \bar{V}_j^2} = \frac{(\bar{u}_Q)_i}{(\bar{u}_Q)_j},$$

which again results in a partial reactive power sharing by an appropriate choice of \bar{u}_Q . Moreover, in this case, the voltage variables at steady-state are explicitly given by

$$\bar{V} = (I + K_Q \mathcal{A}(\cos(D^T \theta^0)))^{-1} \bar{u}_Q.$$

3) *Reactive current controller*: In this case, the third statement of Theorem 2 yields

$$\frac{\bar{Q}_i}{\bar{V}_i} = \frac{(\bar{u}_Q)_i}{(\bar{u}_Q)_j} = \left(\frac{\bar{V}_j}{\bar{V}_i}\right) \left(\frac{\bar{Q}_i}{\bar{Q}_j}\right).$$

The first equality provides the exact reactive current sharing, whereas the second equality can be interpreted as a mixed voltage and reactive power sharing condition. Moreover, the voltage variables at steady-state are given by

$$\bar{V} = \mathcal{A}^{-1}(\cos(D^T \theta^0)) \bar{u}_Q.$$

4) *Consensus based reactive power controller*: In this case, the exact reactive power sharing can be achieved as evident from the fourth statement of Theorem 2, with $\bar{u}_Q = 0$. In particular, we have

$$(k_Q)_i \tilde{Q}_i = (k_Q)_j \tilde{Q}_j, \quad (59)$$

which guarantees proportional reactive power sharing according to the elements of k_Q as desired. Notice that the quantity $\mathbb{1}^T K_Q^{-1} \ln V$ is a conserved quantity in this case. Hence, the point of convergence for the voltage variables is primarily determined by the initialization $V(0)$.

B. Power sharing and lossy lines

Under appropriate conditions, power sharing properties of the consensus based controller are preserved in the presence of lossy transmission lines that are homogeneous, namely whose impedances Z_{ij} equal $|Z_{ij}|e^{\sqrt{-1}\phi}$, with $\phi \in [0, \frac{\pi}{2}]$. Consistently, let us consider the case of shunt components at the buses that are a series interconnection of a resistor and an inductor whose impedance is $r_{ii} + \sqrt{-1}x_{ii}$. The active and reactive power associated to this shunt element are given by

$$s_{ii}^\ell = p_{ii}^\ell + \sqrt{-1}q_{ii}^\ell = \frac{V_i^2}{r_{ii}^2 + x_{ii}^2} r_{ii} + \sqrt{-1} \frac{V_i^2}{r_{ii}^2 + x_{ii}^2} x_{ii}$$

Assuming homogeneity of the shunt elements, i.e. $r_{ii} + \sqrt{-1}x_{ii} = \sqrt{r_{ii}^2 + x_{ii}^2} e^{\sqrt{-1} \arctan \frac{x_{ii}}{r_{ii}}} = |Z_{ii}| e^{\sqrt{-1} \arctan \phi}$, where $\phi = \arctan \frac{x_{ii}}{r_{ii}}$ for all i , then

$$s_{ii}^\ell = p_{ii}^\ell + \sqrt{-1}q_{ii}^\ell = \frac{V_i^2}{|Z_{ii}|} \cos \phi + \sqrt{-1} \frac{V_i^2}{|Z_{ii}|} \sin \phi.$$

Active and reactive power exchanged between buses i and j are (see e.g. [44], [24])

$$p_{ij}^\ell = \frac{V_i^2}{|Z_{ij}|} \cos \phi - \frac{V_i V_j}{|Z_{ij}|} \cos \phi \cos \theta_{ij} + \frac{V_i V_j}{|Z_{ij}|} \sin \phi \sin \theta_{ij}$$

and

$$q_{ij}^\ell = \frac{V_i^2}{|Z_{ij}|} \sin \phi - \frac{V_i V_j}{|Z_{ij}|} \sin \phi \cos \theta_{ij} - \frac{V_i V_j}{|Z_{ij}|} \cos \phi \sin \theta_{ij}$$

Then the total active and reactive power “supplied” by the inverter to the network is equal to

$$\begin{aligned} P_i^\ell &= \sum_{j \in \mathcal{N}_i} p_{ij}^\ell + p_{ii}^\ell = \\ & \begin{bmatrix} \sin \phi & \cos \phi \end{bmatrix} \left[\begin{array}{c} \sum_{j \in \mathcal{N}_i} \frac{V_i V_j}{|Z_{ij}|} \sin \theta_{ij} \\ V_i^2 \left(\frac{1}{|Z_{ii}|} + \sum_{j \in \mathcal{N}_i} \frac{1}{|Z_{ij}|} \right) - \sum_{j \in \mathcal{N}_i} \frac{V_i V_j}{|Z_{ij}|} \cos \theta_{ij} \end{array} \right] \\ Q_i^\ell &= \sum_{j \in \mathcal{N}_i} q_{ij}^\ell + q_{ii}^\ell = \\ & \begin{bmatrix} -\cos \phi & \sin \phi \end{bmatrix} \left[\begin{array}{c} \sum_{j \in \mathcal{N}_i} \frac{V_i V_j}{|Z_{ij}|} \sin \theta_{ij} \\ V_i^2 \left(\frac{1}{|Z_{ii}|} + \sum_{j \in \mathcal{N}_i} \frac{1}{|Z_{ij}|} \right) - \sum_{j \in \mathcal{N}_i} \frac{V_i V_j}{|Z_{ij}|} \cos \theta_{ij} \end{array} \right] \end{aligned}$$

Bearing in mind (2), (3), we observe that

$$\begin{bmatrix} P_i^\ell \\ Q_i^\ell \end{bmatrix} = R(\phi) \begin{bmatrix} P_i \\ Q_i \end{bmatrix} \quad (60)$$

where

$$R(\phi) = \begin{bmatrix} \sin \phi & \cos \phi \\ -\cos \phi & \sin \phi \end{bmatrix}.$$

Hence, under the conditions of Theorem 2,⁴ the closed-loop system in which the voltage is controlled via a reactive power consensus scheme, namely

$$\begin{aligned} \dot{\theta} &= \omega \\ T_P \dot{\omega} &= -(\omega - \omega^*) - K_P (P^\ell \sin \phi - Q^\ell \cos \phi - P^*) + u_P \\ &= -(\omega - \omega^*) - K_P (P - P^*) + u_P \\ T_Q \dot{V} &= [V] K_Q L_Q K_Q (P^\ell \cos \phi + Q^\ell \sin \phi) \\ &= [V] K_Q L_Q K_Q Q \end{aligned} \quad (61)$$

defined by means of the measured active and reactive power P^ℓ, Q^ℓ in the presence of lossy lines and shunt elements, and where P, Q, P^ℓ, Q^ℓ are related via (60) and u_P is defined as in (49), guarantees convergence of P and Q to respectively \bar{P} and \bar{Q} satisfying (51) and (59). The implementation of the dynamics (61), (60) requires the knowledge of the parameter ϕ , which is assumed to be available.

Let us assume that

$$\frac{(k_P)_i}{(k_P)_j} = \frac{(k_Q)_i}{(k_Q)_j}, \quad \forall i, j. \quad (62)$$

Then, by relation (60) at steady state,

$$\begin{aligned} \bar{P}_i^\ell &= \bar{P}_i \sin \phi + \bar{Q}_i \cos \phi \\ &= \frac{(k_P)_j}{(k_P)_i} \bar{P}_j \sin \phi + \frac{(k_Q)_j}{(k_Q)_i} \bar{Q}_j \cos \phi \\ &= \frac{(k_P)_j}{(k_P)_i} \bar{P}_j^\ell. \end{aligned}$$

⁴In these conditions, whenever relevant, the susceptances \hat{B}_{ii}, B_{ij} should be replaced by $|Z_{ii}|^{-1}, |Z_{ij}|^{-1}$.

Similarly, for the reactive power

$$\begin{aligned} \tilde{Q}_i^\ell &= -\bar{P}_i \cos \phi + \bar{Q}_i \sin \phi \\ &= -\frac{(k_P)_j}{(k_P)_i} \bar{P}_j \cos \phi + \frac{(k_Q)_j}{(k_Q)_i} \bar{Q}_j \sin \phi \\ &= \frac{(k_Q)_j}{(k_Q)_i} (-\bar{P}_j \cos \phi + \bar{Q}_j \sin \phi) \\ &= \frac{(k_Q)_j}{(k_Q)_i} \tilde{Q}_j^\ell. \end{aligned}$$

The previous arguments can be formalized as follows:

Proposition 2. Suppose that Assumption 2 with $f(V, Q, u_Q) = -[V] K_Q L_Q K_Q Q$ and condition (47), with $\bar{\omega} = \omega^*$ and \hat{B}_{ii}, B_{ij} replaced by $|Z_{ii}|^{-1}, |Z_{ij}|^{-1}$, respectively, hold. Let u_P be given by (54). Then the solutions of (61) locally converge to the set of points where $\omega = \omega^*$ and $u_P = \bar{u}_P$ with \bar{u}_P being the optimal input (49). Moreover, for all $t \geq 0$,

$$\mathbf{1}^T K_Q^{-1} \ln(V(t)) = \mathbf{1}^T K_Q^{-1} \ln(\tilde{V}) = \mathbf{1}^T K_Q^{-1} \ln(V(0)).$$

Finally, P^ℓ, Q^ℓ converge to constant vectors $\bar{P}^\ell, \tilde{Q}^\ell$ that satisfy

$$\begin{aligned} (k_P)_i \bar{P}_i^\ell &= (k_P)_j \bar{P}_j^\ell \\ (k_Q)_i \tilde{Q}_i^\ell &= (k_Q)_j \tilde{Q}_j^\ell, \end{aligned} \quad (63)$$

provided that (62) holds.

C. Dynamic extension

Another interesting feature is that thanks to the incremental passivity property the static controller $u_Q = \bar{u}_Q$ can be extended to a dynamic controller. To see this note that the incremental input-output pair, associated with u_Q , appears in the time derivative of the storage function \mathcal{S}^η as

$$(u_Q - \bar{u}_Q)^T R \frac{\partial \mathcal{S}^\eta}{\partial V}.$$

Clearly this term is vanished by applying the feedforward input $u_Q = \bar{u}_Q$. But an alternative way to compensate for this term is to introduce the dynamic controller

$$\begin{aligned} \dot{\lambda} &= -R \frac{\partial \mathcal{S}^\eta}{\partial V} \\ u_Q &= \lambda \end{aligned} \quad (64)$$

Then, denoting the steady state value of λ by $\bar{\lambda}$, the incremental storage function $\mathcal{C}_Q = \frac{1}{2}(\lambda - \bar{\lambda})^T (\lambda - \bar{\lambda})$ satisfies

$$\frac{d}{dt} \mathcal{C}_Q = -(\lambda - \bar{\lambda})^T R \frac{\partial \mathcal{S}}{\partial V} = -(u_Q - \bar{u}_Q)^T R \frac{\partial \mathcal{S}^\eta}{\partial V}$$

Therefore, the same convergence analysis can be constructed based on the storage function $\mathcal{S}^\eta + \mathcal{C}_P + \mathcal{C}_Q$, and thus the result of Theorem 2 extends to the case of dynamic voltage/reactive power controller (64).

VI. CONCLUSIONS

We have presented a systematic design of incremental Lyapunov functions for the analysis and the design of network-reduced models of microgrids. Our results encompass existing ones and lift restrictive conditions, thus providing a powerful framework where microgrid control problems can be naturally cast. The method deals with the fully nonlinear model of microgrids and no linearization is carried out.

Two major extensions can be envisioned. The first one is the investigation of similar techniques for network-preserved models of microgrids. Early results show that this is feasible and will be further expanded in a follow-up publication. The second one is how to use the obtained incremental passivity property to interconnect the microgrid with dynamic controllers and obtain a better understanding of voltage control. Examples of these controllers are discussed in [35] but many others can be proposed and investigated.

A more general question is how the set-up we have proposed can be extended to deal with other control problems that are formulated in the microgrid literature. Furthermore, the proposed controllers exchange information over a communication network and would be interesting to assess the impact of the communication layer on the results. In that regard, the use of Lyapunov functions is instrumental in advancing such research, since powerful Lyapunov-based techniques for the design of complex networked cyber-physical systems are already available (see e.g. [11]).

APPENDIX

Proof of Proposition 1. For the sake of notational simplicity, in this proof we omit the bar from all V, θ . Note that by assumption $\Gamma(V)[\cos(\eta)]$ is nonsingular. Then the Hessian is positive definite, or equivalently (42) holds, if and only if $\Gamma(V)[\cos(\eta)]$ and

$$\Psi(\eta, V) := \mathcal{A}(\cos(\eta)) + [h(V)] - [V]^{-1} |D[\Gamma(V)[\sin(\eta)]^2 \cdot [\cos(\eta)]^{-1} |D^T[V]^{-1} > 0.$$

Introduce the diagonal weight matrix

$$W(V, \eta) := \Gamma(V)[\sin(\eta)]^2 [\cos(\eta)]^{-1}.$$

For each $k \sim \{i, j\} \in E$, its k th diagonal element is

$$W_k(V_i, V_j, \eta_k) := B_{ij} V_i V_j \frac{\sin^2(\eta_k)}{\cos(\eta_k)}.$$

Furthermore, it can be verified that

$$\begin{aligned} & [|D[\Gamma(V)[\sin(\eta)]^2 [\cos(\eta)]^{-1} |D^T[V]^{-1}]_{ij} \\ &= \begin{cases} \sum_{k \sim \{i, \ell\} \in E} B_{i\ell} V_i V_\ell \frac{\sin^2(\eta_k)}{\cos(\eta_k)} & \text{if } i = j \\ B_{ij} V_i V_j \frac{\sin^2(\eta_k)}{\cos(\eta_k)} & \text{if } i \neq j, \end{cases} \end{aligned}$$

from which

$$\begin{aligned} & [[V]^{-1} |D[\Gamma(V)[\sin(\eta)]^2 [\cos(\eta)]^{-1} |D^T[V]^{-1}]_{ij} \\ &= \begin{cases} \sum_{k \sim \{i, \ell\} \in E} B_{i\ell} \frac{V_\ell \sin^2(\eta_k)}{V_i \cos(\eta_k)} & \text{if } i = j \\ B_{ij} \frac{V_i \sin^2(\eta_k)}{V_j \cos(\eta_k)} & \text{if } i \neq j. \end{cases} \end{aligned}$$

On the other hand, for $m \sim \{i, j\} \in E$,

$$\begin{aligned} & [\mathcal{A}(\cos(\eta)) + [h(V)]]_{ij} \\ &= \begin{cases} \hat{B}_{ii} + \sum_{k \sim \{i, \ell\} \in E} B_{i\ell} + h_i(V_i), & \text{if } i = j \\ -B_{ij} \cos(\eta_m), & \text{if } i \neq j. \end{cases} \end{aligned}$$

Suppose that each diagonal entry of matrix $\Psi(\eta, V)$ is positive, that is for each $i = 1, 2, \dots, N$,

$$\begin{aligned} m_{ii} &:= \hat{B}_{ii} + \sum_{\ell=1, \ell \neq i}^N B_{i\ell} + h_i(V_i) - \sum_{k \sim \{i, \ell\} \in E} B_{i\ell} \frac{V_\ell \sin^2(\eta_k)}{V_i \cos(\eta_k)} \\ &= \hat{B}_{ii} + \sum_{k \sim \{i, \ell\} \in E} B_{i\ell} \left(1 - \frac{V_\ell \sin^2(\eta_k)}{V_i \cos(\eta_k)} \right) + h_i(V_i) > 0. \end{aligned}$$

Notice that this holds true because of condition (45). Assume also that, for each $i = 1, 2, \dots, N$,

$$\begin{aligned} m_{ii} &> \sum_{k \sim \{i, \ell\} \in E} B_{i\ell} \left| \cos(\eta_k) + \frac{V_i \sin^2(\theta_k)}{V_j \cos(\theta_k)} \right| \\ &= \sum_{k \sim \{i, \ell\} \in E} B_{i\ell} |\cos(\eta_k)| \left(1 + \frac{V_i}{V_j} \tan^2(\eta_k) \right), \end{aligned}$$

which is condition (46). Then by Gershgorin theorem all the eigenvalues of the matrix $\Psi(\eta, V)$ have strictly positive real parts and the Hessian is positive definite.

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