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# On the interconnection structures of discretized port Hamiltonian systems

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For numerical simulation and control design purposes, a mixed-finite element method [3] preserving the port Hamiltonian structure of the system has been developed [2]. This method was successfully applied for 1D systems. In this paper, we shall suggest some generalization of this result to higher dimensional spatial domain (3D) using Whitney forms as Galerkin base. The discretization procedure is illustrated on Maxwell's equations.



## 1 Port Hamiltonian framework

Hamiltonian systems with port variables arise from network modelling of distributed parameters systems with boundary energy flows [1]. The port Hamiltonian framework is based on a power-conjugate variables representation (*effort–flux* pairs) linking internal dynamics and boundary exchanges through a power balance. This formulation relies on two objects: a Hamiltonian function (actually the energy function), and a Dirac structure encoding intrinsic properties in a geometric frame.

Let $\Omega$ be an $n$-dimensional smooth manifold, and $\Lambda^k(\Omega)$ be the space of differential $k$-forms on $\Omega$. There exists a natural duality pairing $\langle\cdot\,,\cdot\rangle_\Omega$ (Poincaré duality) between $\alpha \in \Lambda^k(\Omega)$ and $\beta \in \Lambda^{n-k}(\Omega)$ given by

$$\langle \beta, \alpha \rangle_\Omega = \int_\Omega \beta \wedge \alpha\,. \tag{1}$$

Analogous statement still holds if $\Omega$ is replaced by $\partial\Omega$. Therefore, $\alpha$ in $\Lambda^k(\partial\Omega)$ and $\beta$ in $\Lambda^{n-1-k}(\partial\Omega)$ are dual objects in the sense of (1) where integration runs over $\partial\Omega$. Symmetrization of this two pairings leads to a natural non degenerate bilinear symmetric form on the product space $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$ defined as the flux space $\Lambda^p(\Omega) \times \Lambda^q(\Omega) \times \Lambda^{n-p}(\partial\Omega)$ times the effort space $\Lambda^{n-p}(\Omega) \times \Lambda^{n-q}(\Omega) \times \Lambda^{n-q}(\partial\Omega)$, where $p + q = n + 1$. The Stokes-Dirac structure $\mathcal{D}$ follows:

**Theorem 1.1** [1] *The subset $\mathcal{D} \subset \mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$ of elements $(f_p, f_q, f_b, e_p, e_q, e_b)$ satisfying* (2) *is a Dirac structure.*

$$\begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & (-1)^r \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \quad r = pq + 1 \qquad \textit{and} \qquad \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & -(-\mathrm{Id})^{n-q} \end{bmatrix} \begin{bmatrix} e_{p|\partial\Omega} \\ e_{q|\partial\Omega} \end{bmatrix} \tag{2}$$

We assume [1] the total energy $H = \int_\Omega \mathcal{H}$ to be given by the density $\mathcal{H} : \Lambda^p(\Omega) \times \Lambda^q(\Omega) \longrightarrow \Lambda^n(\Omega)$ which satisfies some weak smoothness condition. Time variation of $H$ evaluated along a trajectory $(\alpha_p(t), \alpha_q(t)) \in \Lambda^p(\Omega) \times \Lambda^q(\Omega)$, $t \in \mathbb{R}$, writes for any time $\mathrm{d}H/\mathrm{d}t = \int_\Omega [\delta_p H \wedge \frac{\partial \alpha_p}{\partial t} + \delta_q H \wedge \frac{\partial \alpha_q}{\partial t}]$, where $\delta_x H$ denotes the variational derivative of $H$ w.r.t. $\alpha_x$.

**Definition 1.2** $(-\partial\alpha_p/\partial t,\ -\partial\alpha_q/\partial t,\ f_b,\ \delta_p H,\ \delta_q H,\ e_b) \in \mathcal{D}$ defines a *distributed-parameter port Hamiltonian system.*

A crucial fact is the power balance $\mathrm{d}H/\mathrm{d}t = -\int_{\partial\Omega} e_b \wedge f_b$ which reads: the energy time-variation inside $\Omega$ is equal to the power supplied through its boundary $\partial\Omega$.

## 2 Discretization of Maxwell's equations

In this setting, we address 3D Maxwell's equations. The electric and magnetic fields induction $(d, b) = (\alpha_p, \alpha_q)$ are modelled as two-forms, and the electric and magnetic fields intensity are given by the one-forms $e, h$. The Hamiltonian is $H = \frac{1}{2}\int_\Omega e \wedge d + h \wedge b$, hence $\delta_p H = e$ and $\delta_q H = h$. Maxwell's equations, in $\Omega \subset \mathbb{R}^3$ without current and where we do take boundary fluctuations into account, write as the distributed-parameters port Hamiltonian systems

$$\begin{bmatrix} -\dot{d} \\ -\dot{b} \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e \\ h \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} e_{|\partial\Omega} \\ h_{|\partial\Omega} \end{bmatrix}. \tag{3}$$

The energy-balance equation on the domain $\Omega$ writes $\mathrm{d}H/\mathrm{d}t = -\int_{\partial\Omega} e \wedge h$, where $e \wedge h$ is known as the *Poynting vector*. We shall use this set of equations to perform a discretization scheme preserving the geometric structure.

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Whitney forms are suitable bases for electromagnetic fields approximation [4]. Consider an oriented simplicial complex $K$ associated with a meshing of $\Omega \subset \mathbb{R}^3$ by a finite number of tetrahedra. Incidence matrices provide intrinsic discrete differential operators $G, R, D$ related to weak – grad, rot and div – operators [3]. Let $T$ be a tetrahedron delimited by nodes $\{a_1a_2a_3a_4\}$. The Whitney edge-element associated with $e = \{a_ia_j\}$ is $w_e = \lambda_i \mathrm{d}\lambda_j - \lambda_j \mathrm{d}\lambda_i$, and the Whitney face-element associated with $f = \{a_ia_ja_k\}$ is $w_f = 2(\lambda_i \mathrm{d}\lambda_j \wedge \mathrm{d}\lambda_k + \lambda_j \mathrm{d}\lambda_k \wedge \mathrm{d}\lambda_i + \lambda_k \mathrm{d}\lambda_i \wedge \mathrm{d}\lambda_j)$, where $\lambda_l$ denotes the barycentric coordinate associated with node $a_l$. Electromagnetic fields are then interpolated (according to their degree) as follows:

$$e = \sum_{j=1\ldots6} E_j\, w_{e_j} \qquad h = \sum_{j=1\ldots6} H_j\, w_{e_j} \qquad ; \qquad -\dot{d} = \sum_{i=1\ldots4} \dot{D}_i\, w_{f_i} \qquad -\dot{b} = \sum_{i=1\ldots4} \dot{B}_i\, w_{f_i}\ . \tag{4}$$

The discrete analog of the pairing $\langle \cdot, \cdot \rangle_\Omega$ given equation (1) between discrete 1-form $\alpha$ and 2-form $\beta$ writes

$$\langle \alpha, \beta \rangle \quad = \quad \int_T \alpha \wedge \beta \quad = \quad \int_T (\sum_{j=1}^{4} \alpha_j\, w_{e_j}) \wedge (\sum_{i=1}^{6} \beta_i\, w_{f_i}) \quad = \quad \{\alpha\}^t\, Y\, \{\beta\}\,, \tag{5}$$

where elements $Y_{ji}$'s are $\int_T w_{e_j} \wedge w_{f_i}$, and $\{x\}$ denotes the column vector $(X_1, \ldots)^t$ of degrees of freedom. This matrix $Y$ is neither square nor full rank (even in the general case), thus providing a **degenerate product which is an obstacle** to define a Dirac structure. We suggest to reduce the state, using the discrete conservation laws, to achieved a non degenerate pairing.

Since $R$ is a $(4\times6)$-matrix with rank 3, we shall reduce the state by a quotient with its kernel. According to Linear Algebra, it exists a full column rank $(4 \times 3)$-matrix $\overline{R}$ such that $\mathrm{im}\,\overline{R} = \mathrm{im}\,R$. Thus, there exists a 3-vector $\dot{D}$ (resp. $\dot{B}$) such that the conservation law $\{-\dot{d}\} = -R\{h\}$ writes $\{-\dot{d}\} = \overline{R}\dot{D}$ (resp. $\{-\dot{b}\} = R\{e\}$ writes $\{-\dot{b}\} = \overline{R}\dot{B}$). As a consequence, the relation reduces to $\dot{D} = -[\overline{R}^t\overline{R}]^{-1}\overline{R}^t R\{h\} := -\widetilde{R}\{h\}$ (resp. $\dot{B} = \widetilde{R}\{e\}$), since $\overline{R}^t\overline{R}$ is invertible. In the same way, internal efforts variables express $E = [Y\overline{R}]^t\{e\} := M\{e\}$ and $H = M\{h\}$. The power $P_{\text{int}}$ inside an element $T$ is now computed as

$$P_{\text{int}} = \langle \dot{D}, E\rangle + \langle \dot{B}, H\rangle = \{h\}^t[-\widetilde{R}^t M + M^t\widetilde{R}]\{e\}\,, \quad \text{where } -\widetilde{R}^t M + M^t\widetilde{R} \quad \textbf{is full rank}\,. \tag{6}$$

Notice this power in the reduced variables equals the power computed with original variables: $\langle -\dot{d}, e\rangle = \{-\dot{d}\}^t(Y^t\{e\}) = [\overline{R}\dot{D}]^t(Y^t\{e\}) = \dot{D}^t([Y\overline{R}]^t\{e\}) = \langle \dot{D}, E\rangle$, *idem* $\langle -\dot{b}, h\rangle = \langle \dot{B}, H\rangle$.
On the other hand, the power *through* the boundary of an element $T$ is naturally given by the boundary wedge product. It is represented by the skewsymmetric full rank $(6 \times 6)$-matrix $Y_{\text{b}}$ with coefficients $\int_{\partial T} w_e \wedge w_{e'}$. Hence the boundary power $P_{\text{b}}$ is given by $\{e\}^t\, Y_{\text{b}}\, \{h\}$, and corresponds to the discrete Poynting vector computed at the boundary.

**Assertion.** The linear transformation $T_L : \mathbb{R}^6 \times \mathbb{R}^6 \longrightarrow \mathbb{R}^{12} \times \mathbb{R}^{12}$ given by

$$\left[\begin{array}{ccccccc} \dot{D} & E & F_b & \vdots & \dot{B} & H & E_b \end{array}\right]^t = \left[\begin{array}{ccccccc} -\widetilde{R} & 0 & 0 & \vdots & 0 & M & \mathrm{Id} \\ 0 & M & \mathrm{Id} & \vdots & \widetilde{R} & 0 & 0 \end{array}\right]^t \left[\begin{array}{c} \{e\} \\ \cdots \\ \{h\} \end{array}\right] := \left[\begin{array}{c} M_{\mathcal{E}} \\ M_{\mathcal{F}} \end{array}\right]^t \left[\begin{array}{c} \{e\} \\ \{h\} \end{array}\right] \tag{7}$$

defines an image representation of the discretized Stokes-Dirac structure associated with system (3).

The proof follows from the construction above. Since $\mathcal{D} = \mathrm{im}(T_L)$, one has to (and can easily) check the necessary and sufficient conditions: *(i)* $\mathrm{rank}\left(\left[\begin{array}{c} M_{\mathcal{E}} \\ M_{\mathcal{F}} \end{array}\right]\right)$ =12, and *(ii)* $M_{\mathcal{E}}(M_{\mathcal{F}})^t + M_{\mathcal{F}}(M_{\mathcal{E}})^t = [0]$.

## 3 Conclusion

A mixed-finite element method based on Whitney forms has been applied on Maxwell's equations (3) expressed in port Hamiltonian setting. The main obstacle is the definition of a discrete non degenerate (power) product between internal *effort-flux* variables. This is due to the non maximality rank condition of (5). The state space vectored out by the kernel of the weak grad operator allows the construction of the suitable internal power product (6). Adding the (naturally non degenerate) boundary product leads to the image representation (7) of a discretized Stokes-Dirac structure which approximates (3). The discretization procedure thus preserves the geometric structure of the equations.

The discrete structure obtained can be completed by the discretization of the Hamiltonian, using the same approximation bases. Furthermore, the procedure presented here is generalizable to higher dimension.

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