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Robust stabilization of nonlinear systems by quantized and ternary control

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ABSTRACT

Results on the problem of stabilizing a nonlinear continuous-time minimum-phase system by a finite number of control or measurement values are presented. The basic tool is a discontinuous version of the so-called semi-global backstepping lemma. We derive robust practical stabilizability results by quantized and ternary controllers and apply them to some control problems. Estimates on the required bandwidth are also provided.

1. Introduction

The problem of controlling systems through a limited bandwidth channel has recently raised a great interest in the community, as thoroughly surveyed in [1]. A possible approach to the problem for continuous-time systems consists in partitioning (a subset of) the state space into a finite number of regions and transmitting information whenever the state crosses one of the boundaries. The resulting system is known as a quantized control system, and the focus of this paper is on this class of systems. Many authors have contributed to the topic, and we refer the interested reader to [1] for an exhaustive bibliography. Among the papers which are important to our derivations we recall [2–5].

In [2], adopting a time-varying quantization, and relying on input-to-state stability of the system, the author shows asymptotic convergence to the origin. In [4,5], the role of static logarithmic quantization [3] to prove practical semi-global stabilizability of nonlinear stabilizable systems has been investigated. The two papers mainly differ in the type of solution adopted. In particular, the paper [5] establishes a few connections between quantized control and discontinuous control systems, investigating Carathéodory and Krasowskii solutions in the context of quantized control systems, while the authors of [4] propose a hysteresis-based implementation of the quantized control. The two papers also present results which rely on notions of robustness different from input-to-state stability. Finally in [4], an adaptive control scheme for nonlinear continuous-time uncertain systems is proposed.

In this paper, we establish a few results on the problem of stabilizing a nonlinear continuous-time system by quantized control robustly with respect to uncertainties. A discontinuous version of the *semi-global backstepping* lemma of [6], in which the measured state is logarithmically quantized, is applied to show that *minimum-phase* nonlinear systems, possibly with *uncertain parameters*, can be robustly semi-globally practically stabilized by a quantized function of *partial-state* measurements. The control techniques introduced in the papers previously discussed cannot be directly applied to the problem considered here, and the resulting quantized control we propose is new to the best of our knowledge. In the scenario in which the feedback information travels through a finite-bandwidth channel, it is important to calculate the bandwidth needed to implement the quantized controller. The solution we propose has the additional advantage of allowing us to estimate an upper bound on the required *bandwidth*. We also show that semi-global practical stabilization is possible even using a simple *switched ternary* controller. The backstepping lemma of [6] has played a fundamental role in the design of many robust nonlinear control schemes [7]. We conclude the paper by presenting a few examples where the quantized backstepping lemma is used to solve some of these robust nonlinear control problems in the presence of quantization.

Preliminary facts are presented in Section 2. In Section 3, the semi-global backstepping tool in the presence of quantization is proven. An upper bound on the bandwidth associated with the quantized control scheme we propose is estimated in Section 4. The ternary controller is introduced in Section 5. Some examples are illustrated in Section 6.

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2. Preliminaries

The system we focus our attention on is of the form

$$\begin{aligned}\dot{x} &= F(x,\mu) + G(x,\mu)\zeta \\ \dot{\zeta} &= q(x,\zeta,\mu) + b(x,\zeta,\mu)u\end{aligned} \tag{1}$$

with $x \in \mathbb{R}^{n-1}$, $\zeta \in \mathbb{R}$, μ an unknown parameter ranging over the compact set $\mathcal{P}$, $u \in \mathbb{R}$, $b(x,\zeta,\mu) \geq b_0 > 0$ for all (x,ζ,μ). The role of these kinds of systems is to solve many important control problems and how several classes of nonlinear control systems can be reduced to the form (1) will be emphasized later on (cf. Section 6 in this paper, and also [6,7], Chapters 11 and 12). We suppose that the upper subsystem satisfies the following property ([6], see also [7]), which claims that the upper subsystem with $\zeta = 0$ is asymptotically stable with a given region of attraction:

Definition. The system $\dot{x} = F(x,\mu)$, $x \in \mathbb{R}^{n-1}$, satisfies a Uniform Lyapunov Property (ULP) if there exists an open set $\mathcal{A} \subset \mathbb{R}^{n-1}$, a real number $c \geq 1$, a continuously differentiable definite positive function $V : \mathcal{A} \to \mathbb{R}_+$ such that $\Gamma_{c+1} := \{x : V(x) \leq c+1\} \subset \mathcal{A}$ and $\frac{\partial V}{\partial x}F(x,\mu) < 0$, for all $x \in \Gamma_{c+1}$, $x \neq 0$, for all $\mu \in \mathcal{P}$. $\triangleleft$

Introduce the Lyapunov function [6]

$$W(x,\zeta) = \frac{cV(x)}{c+1-V(x)} + \frac{d\zeta^2}{d+1-\zeta^2}$$

defined on the set $\{x : V(x) < c+1\} \times \{\zeta : \zeta^2 < d+1\}$, for some $d \geq 1$, and definite positive and proper therein. For an arbitrary $\sigma > 0$, consider the set $S = \{(x,\zeta) : \sigma \leq W(x,\zeta) \leq c^2+d^2+1\}$. The set is well defined, because if $W(x,\zeta) \leq c^2+d^2+1$, then $V(x) < c+1$ and $\zeta^2 < d+1$. In [6] (see also [8]) it is proven that a linear high-gain *partial-state* feedback $u = \bar{k}\zeta$ exists which makes $\dot{W}(x,\zeta)$ negative on S (thus allowing the authors to conclude that any trajectory starting in S is attracted by $\Omega_\sigma := \{(x,\zeta) : W(x,\zeta) \leq \sigma\}$). In this paper, we are interested in carrying out an analogous investigation in the case in which the feedback information ζ is available in a "limited" form, namely it undergoes quantization.

Following [6], consider the derivative $\dot{W}(x,\zeta) = (\partial W/\partial x)\dot{x} + (\partial W/\partial \zeta)\dot{\zeta}$. It is possible to obtain the following inequality to hold for all $(x,\zeta) \in \Omega_{c^2+d^2+1}$:

$$\begin{aligned}\dot{W}(x,\zeta) \leq\ & \frac{c}{c+1}\frac{\partial V}{\partial x}F(x,\mu) + w(x,\zeta,\mu)\zeta \\ & + 2\frac{d(d+1)}{(d+1-\zeta^2)^2}\zeta b(x,\zeta,\mu)u,\end{aligned} \tag{2}$$

where $w(x,\zeta,\mu) = \frac{c(c+1)}{(c+1-V(x))^2}\frac{\partial V}{\partial x}G(x,\mu) + 2\frac{d(d+1)}{(d+1-\zeta^2)^2}q(x,\zeta,\mu)$.

Because of the ULP property, if the state belongs to $S_0 = \{(x,\zeta) \in S : \zeta = 0\}$, then $\dot{W}(x,\zeta) < 0$. By continuity, there exists a neighborhood U of S_0 where the sum of the first two terms $\frac{c}{c+1}\frac{\partial V}{\partial x}F(x,\mu) + w(x,\zeta,\mu)\zeta$ on the right-hand side of the inequality above remains strictly negative. Without loss of generality, we can suppose that a constant $\eta > 0$ exists such that $U = \{(x,\zeta) \in S : |\zeta| < \eta\}$ (see Fig. 1). Then, to show that $\dot{W}(x,\zeta)$ is negative on S, it is enough to investigate the sign of $\dot{W}(x,\zeta)$ on $\tilde{S} := S \setminus U$ only.

3. Stabilization by quantized control

In what follows, we consider the case in which the control $\bar{k}\zeta$, or the measurement ζ, is quantized by a logarithmic quantizer. Let

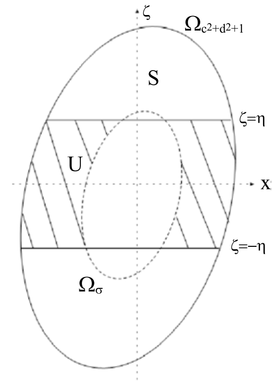

Fig. 1. The figure represents the sets of interest in the paper. The outer contour is the boundary of $\Omega_{c^2+d^2+1}$, while the inner contour is the boundary of Ω_σ. S is the region between the two. The two horizontal segments are the sets of points such that $\zeta = \pm\eta$. The open set U is emphasized by oblique solid lines. The regions at the top, center and bottom, delimited by the boundary of $\Omega_{c^2+d^2+1}$ and the two horizontal solid lines, are respectively Ω_-, Ω_0, Ω_+.

$u_0 \in \mathbb{R}_+$, $j \in \mathbb{N}$ and $0 < \delta < 1$ be constants to be designed. Also let $u_i = \rho^i u_0$, with $\rho = \frac{1-\delta}{1+\delta}$ [3]. The following map is the quantizer

$$\Psi(r) = \begin{cases} u_i & \frac{1}{1+\delta}u_i < r \leq \frac{1}{1-\delta}u_i,\ 0 \leq i \leq j \\ 0 & 0 \leq r \leq \frac{1}{1+\delta}u_j \\ -\Psi(-r) & r < 0, \end{cases} \tag{3}$$

and $u = -\Psi(\bar{k}\zeta)$ is the quantized input. We do not define the quantizer for $\bar{k}\zeta > (1-\delta)^{-1}u_0$, since u_0 will be designed in such a way that this bound is never exceeded. Observe that it is *equivalent* to consider either the quantized control law $u = -\Psi(\bar{k}\zeta)$ or the control law $u = -\bar{k}\bar{\Psi}(\zeta)$, function of the *quantized partial-state* $\bar{\Psi}(\zeta)$, provided that $\bar{\Psi}$ is appropriately defined. As a matter of fact, define $\bar{\Psi}$ as Ψ in (3), but with a new set of quantization levels $\bar{u}_i$ (instead of u_i) defined as $\bar{u}_i = \rho^i\bar{u}_0$, with $\bar{u}_0 = \bar{k}^{-1}u_0$. Then, it is easy to show that $\bar{k}\bar{\Psi}(\zeta) = \Psi(\bar{k}\zeta)$, and all the results drawn with $u = -\Psi(\bar{k}\zeta)$ also hold for $u = -\bar{k}\bar{\Psi}(\zeta)$. In what follows, we only refer to the quantized input $u = -\Psi(\bar{k}\zeta)$.

Observe that the quantizer has $2j+3$ quantization levels, with $u_0, j, \bar{k}$ to be determined. Of course, the size of the *deadzone* of the quantizer, i.e. the region around the zero where $\Psi = 0$, decreases as j increases. The parameter δ can be viewed as a function of the quantization *density* (see [3]), and we do not assume any constraint on its value (cf. the remark below to see why, for open-loop unstable systems, assuming that $\delta \in (0,1)$ does not appear to be restrictive). The closed-loop system is

$$\begin{aligned}\dot{x} &= F(x,\mu) + G(x,\mu)\zeta \\ \dot{\zeta} &= q(x,\zeta,\mu) - b(x,\zeta,\mu)\Psi(\bar{k}\zeta).\end{aligned} \tag{4}$$

The following quantities are useful below:

$$\begin{aligned}\bar{w} &= \max_{(x,\zeta)\in\Omega_{c^2+d^2+1},\mu\in\mathcal{P}} |w(x,\zeta,\mu)|, \\ \bar{b} &= \max_{(x,\zeta)\in\Omega_{c^2+d^2+1},\mu\in\mathcal{P}} |b(x,\zeta,\mu)|, \qquad \bar{\zeta} = \max_{(x,\zeta)\in\Omega_{c^2+d^2+1}} |\zeta|.\end{aligned} \tag{5}$$

Observe that the vector field on the right-hand side of (4) is discontinuous and solutions of the system must be intended in some generalized sense. In this section, we focus on Krasowskii solutions, but other types of solutions are possible (see e.g. [5] and the references therein). The main reason to consider Krasowskii solutions lies in the fact that a rather complete Lyapunov theory for the study of the stability of these solutions is available.

Definition. A curve $\varphi : [0, +\infty) \to \mathbb{R}^n$ is a *Krasowskii solution* of a system of ordinary differential equations $\dot{x} = G(t, x)$, where $G : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$, if it is absolutely continuous and for almost every $t \geq 0$ it satisfies the differential inclusion $\dot{x} \in K(G(t, x))$, where $K(G(t, x)) = \cap_{\delta>0} \overline{\text{co}} G(t, B_\delta(x))$, $\overline{\text{co}} G$ is the convex closure of the set G and $B_\delta(x)$ is the open ball of radius δ centered at x. ◁

In the present case, Krasowskii solutions are absolutely continuous functions which satisfy the differential inclusion (see e.g. [9,5])

$$\begin{pmatrix}\dot{x}\\ \dot{\zeta}\end{pmatrix} \in \begin{pmatrix}F(x,\mu)+G(x,\mu)\zeta\\ q(x,\zeta,\mu)\end{pmatrix} + \left\{\begin{pmatrix}0\\ -b(x,\zeta,\mu)\end{pmatrix} v,\ v \in K(\Psi(\bar{k}\zeta))\right\} \tag{6}$$

where (see [9], Lemma 2, and [5]) $K(\Psi(\bar{k}\zeta))$, with $\bar{k}\zeta \geq 0$ (and symmetrically for $\bar{k}\zeta \leq 0$), coincides with the singleton $\{\Psi(\bar{k}\zeta)\}$ at the points where $\Psi(\bar{k}\zeta)$ is continuous, and with $\overline{\text{co}}\{u_i, u_{i+1}\}$, at the points $\bar{k}\zeta = \frac{u_i}{1+\delta}$, with $0 \leq i \leq j$ and $u_{j+1} := 0$, where $\Psi(r)$ is discontinuous. As a result, $K(\Psi(\bar{k}\zeta))$ satisfies the inclusion:

$$K(\Psi(\bar{k}\zeta)) \subseteq \begin{cases}\{(1+\lambda\delta)\bar{k}\zeta,\ \lambda \in [-1,1]\} & \dfrac{u_j}{1+\delta} < |\bar{k}\zeta| \leq \dfrac{u_0}{1-\delta}\\ \{\lambda(1+\delta)\bar{k}\zeta,\ \lambda \in [0,1]\} & \dfrac{u_j}{1+\delta} \geq |\bar{k}\zeta|.\end{cases} \tag{7}$$

In the analysis below we let v in (6) range over the set on the left-hand side of (7). In this way, the results we establish in the paper hold no matter how the quantization levels are chosen within the sector bound, and in particular this allows us to infer the same results even in the presence of hysteresis (see Section 4).

Remark. Assuming that $\delta \in (0, 1)$ does not appear to be restrictive. In fact, in the differential inclusion above, because of the quantization, the "high-frequency" gain becomes $b(x, \zeta, \mu)(1 + \lambda\delta)$, with $\lambda \in [-1, 1]$ (similar arguments hold for the case when the gain is $b(x, \zeta, \mu)\lambda(1+\delta)$, $\lambda \in [0, 1]$). If we allow δ to be larger than 1, then we should design a control law which stabilizes the system in the presence of a high-frequency gain which takes any value in a set which includes the zero, a task which is considerably difficult, if not impossible, to accomplish for open-loop unstable systems by the class of controllers we consider in the paper. ◁

Then we claim the following version of the so-called "semi-global backstepping lemma" in [6] with quantized feedback:

Proposition 1. *For any $\delta \in (0, 1)$ and any $\sigma \in (0, c^2+d^2+1)$, there exist positive numbers k^*, j^*, and u_0 such that, for any gain $\bar{k} \geq k^*$ and any number of quantization levels $j \geq j^*$, any Krasowskii solution φ of the system (4) is such that, if $\varphi(0) \in \Omega_{c^2+d^2+1}$, then there exists $T > 0$ such that $\varphi(t) \in \Omega_\sigma$ for all $t \geq T$.*

Proof. Let [1]

$$k^* = \frac{d+1}{d}\frac{\bar{w}}{b_0}\frac{1}{\eta(1-\delta)},$$

$$j^* = \left\lceil \log\left(\frac{d^2}{(c^2+d^2+d+1)^2}\frac{\eta}{4}\frac{b_0}{\bar{b}}\right)\log\left(\frac{1-\delta}{1+\delta}\right)^{-1}\right\rceil$$

fix $\bar{k} \geq k^*$ and $j \geq j^*$, and choose $u_0 = (1+\delta)\bar{k}\max_{(x,\zeta)\in\tilde{S}}|\zeta|$. To prove the convergence of the state to Ω_σ, we need to prove [5] that, for any $(x, \zeta) \in S$, for any $v \in K(\Psi(\bar{k}\zeta))$,

$$\begin{aligned}\dot{W}(x,\zeta) &= \frac{c(c+1)}{(c+1-V(x))^2}\frac{\partial V}{\partial x}(F(x,\mu)+G(x,\mu)\zeta)\\ &\quad+\frac{d(d+1)}{(d+1-\zeta^2)^2}2\zeta\cdot(q(x,\zeta,\mu)-b(x,\zeta,\mu)v)\\ &\leq \frac{c}{c+1}\frac{\partial V}{\partial x}F(x,\mu)-2\frac{d(d+1)}{(d+1-\zeta^2)^2}\bar{k}b(x,\zeta,\mu)\zeta^2\\ &\quad+w(x,\zeta,\mu)\zeta-2\zeta\frac{d(d+1)}{(d+1-\zeta^2)^2}b(x,\zeta,\mu)[v-\bar{k}\zeta]<0.\end{aligned} \tag{8}$$

Note first that, if $(x, \zeta) \in \tilde{S}$, where $\tilde{S} = S \setminus U$ is introduced at the end of Section 2, then $|\bar{k}\zeta| \leq u_0$. Hence, depending on the number $j \geq j^*$ of quantization levels, two cases are possible, namely that the set $\hat{S} := \{(x,\zeta) \in S : u_j/(1+\delta) < |\bar{k}\zeta| \leq u_0\}$ is *strictly* contained in $\tilde{S}$ or it is not. Consider the former case. By definition of $K(\Psi(\bar{k}\zeta))$, we have $v - \bar{k}\zeta = \lambda\delta\bar{k}\zeta$, and therefore, for $(x, \zeta) \in \hat{S} \subset \tilde{S}$, the inequality above rewrites as (recall (5))

$$\begin{aligned}\dot{W}(x,\zeta) &\leq \frac{c}{c+1}\frac{\partial V}{\partial x}F(x,\mu)-2\frac{d(d+1)}{(d+1-\zeta^2)^2}\\ &\quad\times(1+\lambda\delta)\cdot\bar{k}b(x,\zeta,\mu)\zeta^2+w(x,\zeta,\mu)\zeta\\ &\leq \frac{c}{c+1}\frac{\partial V}{\partial x}F(x,\mu)-2\frac{d}{d+1}(1-\delta)\bar{k}b_0\zeta^2+\bar{w}|\zeta|.\end{aligned}$$

The choice of k^* above gives $\dot{W}(x, \zeta) \leq -\bar{w}\eta$. Consider now the subset of points in $\tilde{S}$ such that $|\bar{k}\zeta| \leq u_j/(1+\delta)$. Such a set is non-void because $\hat{S} \subset \tilde{S}$ by hypothesis. For these points, we have

$$|v-\bar{k}\zeta| \leq |\bar{k}\zeta| \leq u_j/(1+\delta) \leq \frac{u_0}{1+\delta}\left(\frac{1-\delta}{1+\delta}\right)^j,$$

and the bound on $\dot{W}(x, \zeta)$ becomes

$$\begin{aligned}\dot{W}(x,\zeta) &\leq \frac{c}{c+1}\frac{\partial V}{\partial x}F(x,\mu)-\frac{d}{d+1}\bar{k}b_0\zeta^2\\ &\quad+2\frac{u_0}{1+\delta}\left(\frac{1-\delta}{1+\delta}\right)^j\frac{(c^2+d^2+d+1)^2}{d(d+1)}\bar{b}|\zeta|.\end{aligned}$$

The choice of u_0 and j^* guarantees that for $j \geq j^*$ the last term on the right-hand side of the inequality is not larger than the second term, and this gives $\dot{W}(x,\zeta) \leq -\frac{1}{2}\frac{d}{d+1}\bar{k}b_0\eta^2 \leq -\frac{\bar{w}\eta}{2(1-\delta)}$.

Consider now the case when $\hat{S} \not\subset \tilde{S}$, and let first $(x, \zeta) \in \hat{S} \cap \tilde{S}$. This case is the same as $(x, \zeta) \in \hat{S}$ when $\hat{S} \subset \tilde{S}$. Then as before

$$\begin{aligned}\dot{W}(x,\zeta) &\leq \frac{c}{c+1}\frac{\partial V}{\partial x}F(x,\mu)+w(x,\zeta,\mu)\zeta\\ &\quad-2\frac{d}{d+1}(1-\delta)\bar{k}b_0\zeta^2<0.\end{aligned}$$

On the other hand, if $(x, \zeta) \notin \hat{S} \cap \tilde{S}$, then necessarily $(x, \zeta) \in U$. Then we have

$$\dot{W}(x,\zeta) \leq \frac{c}{c+1}\frac{\partial V}{\partial x}F(x,\mu)+w(x,\zeta,\mu)\zeta - \begin{cases}2\dfrac{d}{d+1}(1-\delta)\bar{k}b(x,\zeta,\mu)\zeta^2, & \dfrac{u_j}{1+\delta} < |\bar{k}\zeta| \leq u_0\\ 0, & \dfrac{u_j}{1+\delta} \geq |\bar{k}\zeta|.\end{cases}$$

Since the sum of the first two terms on the right-hand side is negative because $(x, \zeta) \in U$, and the third term is always non-positive, we see that $\dot{W}(x, \zeta) < 0$. This ends the proof. □

Remark. The constant k^* differs from the one in [6,7] by the presence of the factor $(1 - \delta)^{-1}$. That is, as expected, the error due to quantization is counteracted by raising the controller gain.

[1] The symbol $\lceil r \rceil$ denotes the smallest integer not smaller than r. The definition of $j \geq j^*$ yields that $(\frac{1-\delta}{1+\delta})^j \leq \frac{d^2}{(c^2+d^2+d+1)^2}\frac{\eta}{4}\frac{b_0}{\bar{b}}$.

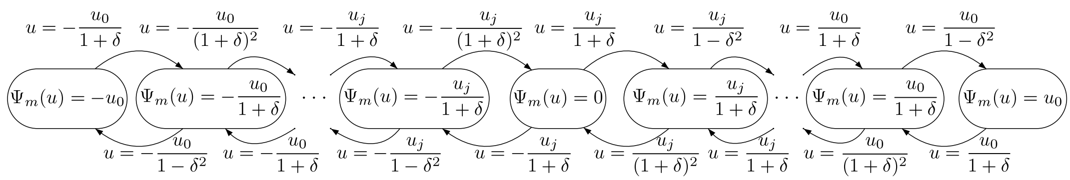

Fig. 2. The graph illustrates how the function $\Psi_m(u)$ takes values depending on u, $u=\bar{k}\zeta$. Each edge connects two nodes, and is labeled with the condition (*guard*) which triggers the transition from the starting node to the destination node.

Furthermore, it is interesting to observe that the constant j^*, that is the number of quantization levels, only depends on the size of the domain of attraction and of the target set. ◁

4. An estimate on the bandwidth

The stabilization technique examined in the previous section has two main ingredients: the selection of the set of control values, and the switching law which schedules them. A possible performance measure of the control law can then be taken the number of times the controller switches to a new value within an interval of time divided by the length of the interval itself. Such a measure is given the name of *bandwidth*, because of the obvious implication in the scenario in which the quantized control is fed back to the process through a finite-bandwidth communication channel. The Krasowskii solutions considered above do not exclude the possibility to have accumulation of switching points in finite time. To circumvent the possible occurrence of chattering or Zeno phenomenon, we introduce a modified quantizer following [4]. The modified quantizer is obtained from (3) to which, for each quantization level, a new one is added, to obtain the following multi-valued map (see [4], Figure 3.4, for a pictorial representation of the map):

$$\Psi_m(u)=\begin{cases} u_i & \frac{1}{1+\delta}u_i<u\le\frac{1}{1-\delta}u_i,\ 0\le i\le j\\ \frac{1}{1+\delta}u_i & \frac{1}{(1+\delta)^2}u_i<u\le\frac{1}{(1+\delta)(1-\delta)}u_i,\ 0\le i\le j\\ 0 & 0\le u\le\frac{1}{1+\delta}u_j\\ -\Psi_m(-u) & u<0.\end{cases} \tag{9}$$

Since the map above is multi-valued, we need to specify the law according to which $\Psi_m(u(t))$ changes its value as $u(t)$ evolves. This law is illustrated by the graph in Fig. 2. At time $t=0$, we set $\Psi_m(u(0))=\Psi(u(0))$. This value of $\Psi_m(u(0))$ identifies a node of the graph. If the value of $u(0)$ fulfills one of the conditions of the edges leaving the node, then a transition is triggered and the quantizer takes the new value – which is denoted by $\Psi_m(u(0^+))$ – given by the destination node. For $t>0$, $\Psi_m(u(t))$ remains constant until $u(t)$ triggers a transition of $\Psi_m(u(t))$ to the new value, denoted by $\Psi_m(u(t^+))$, again chosen according to the graph of Fig. 2. We refer to [4], Section 3, for further details on the switching mechanism.

The first observation is that the result proven in the previous section continues to hold even in the presence of the modified quantizer. As a matter of fact, Proposition 1 was proven by showing that the derivative

$$\begin{pmatrix}\frac{\partial W}{\partial x} & \frac{\partial W}{\partial \xi}\end{pmatrix}\begin{pmatrix}F(x,\mu)+G(x,\mu)\zeta\\ q(x,\zeta,\mu)-b(x,\zeta,\mu)v\end{pmatrix} \tag{10}$$

was strictly negative for all $(x,\zeta)\in S$, $\mu\in\mathcal{P}$ and $v\in K(\Psi(\bar{k}\zeta))$. Now, if we adopt the modified quantizer defined above, the closed-loop system becomes the switched system

$$\begin{aligned}\dot{x}&=F(x,\mu)+G(x,\mu)\zeta\\ \dot{\zeta}&=q(x,\zeta,\mu)-b(x,\zeta,\mu)\Psi_m(\bar{k}\zeta),\end{aligned} \tag{11}$$

and proving the stability of the (unique) solution amounts simply to show that (10) is still negative when v is replaced by $\Psi_m(\bar{k}\zeta)$. This is an immediate consequence of the result in the previous section:

Corollary 1. *For any $\delta\in(0,1)$ and any $\sigma\in(0,c^2+d^2+1)$, there exist positive numbers k^*, j^*, and u_0 such that, for any gain $\bar{k}\ge k^*$ and any number of quantization levels $j\ge j^*$, the unique solution φ of the system (11), is such that, if $\varphi(0)\in\Omega_{c^2+d^2+1}$, then there exists $T>0$ such that $\varphi(t)\in\Omega_\sigma$ for all $t\ge T$.*

Proof. By definition of Ψ_m and $K(\Psi(\bar{k}\zeta))$, for each $|\bar{k}\zeta|\le u_0(1-\delta)^{-1}$, $\Psi_m(\bar{k}\zeta)\in K(\Psi(\bar{k}\zeta))$. This ends the proof. □

Now we make the notion of bandwidth more precise. Let first $0=t_0,t_1,t_2,\ldots$ be the sequence of switching times, that is the times at which the control law $u=-\Psi_m(\bar{k}\zeta)$ changes its value, and $B(t_\kappa)$ the number of quantization levels used to encode the control at time t_κ (but we could equivalently use the number of bits used to encode the value transmitted at time t_k). For each t, let κ be the largest integer for which $t\ge t_\kappa$, and define the average data rate over the interval $[t_0,t]$ as the quantity $R_{av}[t_0,t]=(\sum_{j=0}^{\kappa}B(t_j))/(t-t_0)$, that is the total number of values taken by the quantized control on the interval $[t_0,t]$, divided by the duration of the interval. Moreover, we denote as the average data rate the limit $R_{av}=\limsup_{t\to\infty}R_{av}[t_0,t]=\limsup_{t\to\infty}(\sum_{j=0}^{\kappa}B(t_j))/(t-t_0)$. Under the conditions of Corollary 1, the following result provides a bound on the average data rate needed to guarantee semi-global practical stabilization. In the statement, we refer to the following quantities

$$\bar{q}=\max_{(x,\zeta)\in\Omega_{c^2+d^2+1},\mu\in\mathcal{P}}|q(x,\zeta,\mu)|,\qquad k_0=\frac{d+1}{d}\frac{\bar{w}}{b_0}\frac{1}{\eta}. \tag{12}$$

Proposition 2. *Let $\bar{k}=k^*$. The unique solution φ of the system (11) is such that, if $\varphi(0)\in\Omega_{c^2+d^2+1}$, then there exists $T>0$ such that $\varphi(t)\in\Omega_\sigma$ for all $t\ge T$, and the associated average data rate is not greater than $\frac{4j+1}{DT_m}$, where $DT_m=\frac{1}{(\zeta\rho^{j-1})^{-1}\bar{q}+k_0\bar{b}}\frac{\delta}{1+\delta}$.*

Proof. The proof boils down to estimating a lower bound on the time which elapses between two consecutive switching times (*inter-switching* time). The estimate is found by focusing on the function of time $\Psi_m(k\zeta(t))$, and in particular on the value $|\Psi_m(k\zeta(t^+))|$ and where t^+ denotes $|\Psi_m(k\zeta(t))|$ soon after the switching. Then, the smallest distance to be covered by $k\zeta(t)$ before a new switching takes place, and the largest velocity at which the function $k\zeta(t)$ evolves are computed. It is enough to carry out these calculations in three cases. If $|\Psi_m(k\zeta(t^+))|=u_i$, with $0\le i\le j$, then $|u|$ remains equal to u_i for not less than

$$\frac{\frac{u_i}{1+\delta}\frac{\delta}{1-\delta}}{\bar{k}(\bar{q}+\bar{b}u_i)}=\frac{\bar{\zeta}}{\bar{q}+\bar{k}(1-\delta)\bar{b}\bar{\zeta}\rho^{i-1}}\rho^i\frac{\delta}{1-\delta},$$

where we have exploited the definition of u_i, u_0, $\rho=(1-\delta)(1+\delta)^{-1}$, and (12). Observe that by the definition of k^* in Proposition 1,

$\bar{k}(1-\delta) = k^*(1-\delta) = k_0$. Hence, the bound above becomes equal to

$$\frac{\bar{\zeta}}{\bar{q} + k_0\bar{b}\bar{\zeta}\rho^{i-1}}\rho^i\frac{\delta}{1-\delta}. \tag{13}$$

With similar arguments it can be shown that a lower bound on the time $|u|$ remains equal to $u_i(1+\delta)^{-1}$ given by: $\frac{\bar{\zeta}}{\bar{q}+k_0\bar{b}\bar{\zeta}\rho^{i-1}\frac{1}{1+\delta}}\rho^i\frac{\delta}{1-\delta}$. Finally a lower bound on the dwell time when $|\Psi_m(k\zeta(t^+))| = 0$ is $\frac{\bar{\zeta}}{\bar{q}}\rho^j\frac{2}{1+\delta}$. Hence, comparing the three estimates above it is seen that the inter-switching time cannot be less than (13) with $i = j$. Indeed, the bound (13) is a monotonically decreasing function of i and hence the minimum is reached at $i = j$ and is equal to

$$DT_m = \frac{\bar{\zeta}}{\bar{q} + k_0\bar{b}\bar{\zeta}\rho^{j-1}}\rho^j\frac{\delta}{1-\delta} = \frac{1}{\frac{\bar{q}}{\bar{\zeta}\rho^{j-1}} + k_0\bar{b}}\frac{\delta}{1+\delta}.$$

Now let $t \in [t_\kappa, t_{\kappa+1}), \kappa \geq 1$. Then $\kappa+1$, the number of switchings in the interval $[t_0, t]$, satisfies $\kappa \leq \frac{t-t_0}{DT_m}$. Then

$$\begin{aligned} R_{av} &= \limsup_{t\to\infty} R_{av}[t_0, t] = \limsup_{t\to\infty} \frac{\sum_{\ell=0}^{\kappa}(4j+1)}{t-t_0} \\ &= \limsup_{t\to\infty} \frac{(\kappa+1)(4j+1)}{t-t_0} \\ &= \limsup_{t\to\infty} \left(\frac{\kappa}{t-t_0} + \frac{1}{t-t_0}\right)(4j+1) \\ &\leq \limsup_{t\to\infty} \left(\frac{1}{DT_m} + \frac{1}{t-t_0}\right)(4j+1) = \frac{4j+1}{DT_m}. \quad \square \end{aligned}$$

Remark. A special case is when $q(x, \zeta, \mu)$ is identically zero. In this case it is seen that $|\bar{k}\zeta(t)|$ is a monotonically decreasing function as long as $u_j(1+\delta)^{-2} < |\bar{k}\zeta(t)| \leq u_0(1-\delta)^{-1}$. Hence, at some finite time $\bar{t}$, the control becomes equal to zero, and $\zeta(t)$ remains equal to the value $\zeta(\bar{t})$ for all $t \geq \bar{t}$. As a result, switching stops, and for $t \geq \bar{t}$, the number κ of switchings in any interval of time $[t, t_0]$ remains constant and finite. Hence, $R_{av} = 0$. We conclude that in the special case $q(x, \zeta, \mu) = 0$, the control law $u = -\Psi_m(\bar{k}\zeta)$ guarantees semi-global practical stability with an *average data rate* equal to *zero*. ◁

5. Ternary controller

In this section we remark that Proposition 1 can be also obtained by using a ternary controller. Let η be the positive constant introduced at the end of Section 2 in the definition of the neighborhood U, and introduce the following sets:

$$\begin{aligned} \Omega_- &= \{(x, \zeta) \in \Omega_{c^2+d^2+1} : \zeta \geq \eta\}, \\ \Omega_0 &= \{(x, \zeta) \in \Omega_{c^2+d^2+1} : |\zeta| < \eta\}, \\ \Omega_+ &= \{(x, \zeta) \in \Omega_{c^2+d^2+1} : \zeta \leq -\eta\}. \end{aligned}$$

These sets are depicted in Fig. 1. Assume without loss of generality that η is small enough such that Ω_-, Ω_+ are not void. We propose the following controller. (Similar elementary controllers have been studied in [10] for a different class of nonlinear systems.) At the initial time $t = 0$, assume that $(x, \zeta) \in \Omega_{c^2+d^2+1}$, and set the control value as

$$u(0) = \begin{cases} -\bar{k} & \text{if } \zeta(0) \geq \eta \\ 0 & \text{if } |\zeta(0)| < \eta \\ \bar{k} & \text{if } \zeta(0) \leq -\eta. \end{cases} \tag{14}$$

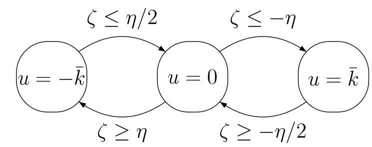

Fig. 3. The ternary switched controller.

As in the previous section, for $t \geq 0$, the controller is chosen according to the law

$$u(t^+) = \begin{cases} -\bar{k} & \text{if } ([u(t) = 0] \wedge [\zeta(t) \geq \eta]) \\ & \quad \vee([u(t) = -\bar{k}] \wedge [\zeta(t) > \eta/2]) \\ & \quad ([u(t) = -\bar{k}] \wedge [\zeta(t) \leq \eta/2]) \\ 0 & \text{if } \quad \vee([u(t) = \bar{k}] \wedge [\zeta(t) \geq -\eta/2]) \\ & \quad \vee([u(t) = 0] \wedge [|\zeta(t)| < \eta]) \\ \bar{k} & \text{if } ([u(t) = 0] \wedge [\zeta(t) \leq -\eta]) \\ & \quad \vee([u(t) = \bar{k}] \wedge [\zeta(t) < -\eta/2]), \end{cases} \tag{15}$$

with $\bar{k} > 0$ a parameter to be designed. This law could also be described by an automaton analogous to the one in Fig. 2 but with three states only (see Fig. 3).

The stability result with the ternary controller reads as follows:

Proposition 3. *There exists a choice of $\bar{k}$ such that the Lyapunov function $W(x, \zeta)$, computed along any trajectory of the closed-loop system (1), (14), (15) which starts in S, satisfies $\dot{W}(x(t), \zeta(t)) < 0$ for all $(x(t), \zeta(t)) \in S$.*

Proof. It has already been proven that, for $(x, \zeta) \in U$, $\dot{W}(x(t), \zeta(t)) < 0$ with $u = 0$. On the other hand, for $u = -\bar{k}\,\mathrm{sgn}\,\zeta$,

$$\begin{aligned} \dot{W}(x, \zeta) &= \frac{c(c+1)}{(c+1-V(z))^2}\frac{\partial V}{\partial x}F(x, \mu) + w(x, \zeta, \mu)\zeta \\ &\quad - 2\frac{d(d+1)}{(d+1-\zeta^2)^2}b(x, \zeta, \mu)\bar{k}|\zeta| \\ &\leq \frac{c}{c+1}\frac{\partial V}{\partial x}F(x, \mu) + |w(x, \zeta, \mu)||\zeta| - 2\bar{k}b_0\frac{d}{d+1}|\zeta|, \end{aligned}$$

and choosing $\bar{k} \geq \frac{1}{b_0}\frac{d+1}{d}\bar{w}$, with $\bar{w}$ as in (5), we have, for all $(x, \zeta) \in \Omega_{c^2+d^2+1}$ such that $|\zeta| \geq \eta$, $\dot{W}(x, \zeta, u) \leq -\bar{w}|\zeta| \leq -\bar{w}\eta$. This concludes the proof. □

Remark. The ternary controller can be viewed as a controller of the form $u = -\bar{k}\,\mathrm{sgn}\,\zeta$ plus hysteresis, with $\mathrm{sgn}\,\zeta = 1$ for $\zeta > 0$ and $\mathrm{sgn}\zeta = -1$ for $\zeta \leq 0$. Since the function sgn cannot be described in terms of the quantizer (3) (the sgn function is discontinuous at zero while Ψ is continuous at zero), the ternary controller (15) cannot be obtained from the quantized controller $u = -\Psi(\bar{k}\zeta)$ by simply adding hysteresis. This implies that Proposition 3 cannot be derived directly from Corollary 1, and its proof requires a (minor) modification of the arguments in the proof of Proposition 1. ◁

Remark. As for quantized control, it is possible to give an estimate on the bandwidth needed to implement the ternary controller. Using the same arguments of Section 4, one can show that an upper bound on the average data rate is $6(\bar{q}\eta^{-1} + \bar{b}k_0)$ where k_0 is the quantity defined in (12). The two estimates on the data rate using quantized and ternary control are similar but it is difficult to compare the two control laws. Arguably, in some cases, the ternary control above is easier to implement than the quantized control (cf. [11] to see how binary control is more robust to changes for linear scalar systems). On the other hand, while the state is approaching the target set, the amplitude of the changes in the values taken by the quantized controller are less pronounced than the ternary control and in some cases, to keep the state in the vicinity of the origin, the quantized controller may require a minor effort than the ternary controller. ◁

6. Applications

In this section we emphasize a number of cases to which the previous results apply.

6.1. Systems with uniform relative degree ≥ 1

It is well known that a nonlinear input-affine system is said to have a *uniform* relative degree r if it has a relative degree r at x_0 for each $x_0 \in \mathbb{R}^n$. It is also well known that there exists a globally defined diffeomorphism which changes the system into one of the following form (see e.g. Proposition 9.1.1. in [12]):

$$\begin{aligned} \dot z &= f(z,\xi_1) \\ \dot\xi_i &= \xi_{i+1}, \quad 1 \le i \le r-1 \\ \dot\xi_r &= \bar q(z,\xi) + \bar b(z,\xi)u \\ y &= \xi_1 \end{aligned} \tag{16}$$

with $z \in \mathbb{R}^{n-r}$, $y \in \mathbb{R}$ the output of the system, and $\bar b(z,\xi) \ge b_0 > 0$ for all (z,ξ). Systems like the one above restricted to the components $z, \xi_1, \ldots, \xi_{r-1}$, with ξ_r viewed as an input, can be always stabilized by means of a linear high-gain partial-state feedback ([12], Theorem 9.3.1), provided that the origin $z = 0$ is a globally asymptotically stable equilibrium point for $\dot z = f(z, 0)$, i.e. system (16) is minimum phase. As a matter of fact, for any $R > 0$, there exists a linear "control law" $\xi_r = -a\xi$, with a a row vector depending on R and $\xi = (\xi_1 \ldots \xi_{r-1})^{\mathrm T}$, such that every solution of

$$\begin{aligned} \dot z &= f(z,\xi_1) \\ \dot\xi_i &= \xi_{i+1}, \quad 1 \le i \le r-2 \\ \dot\xi_{r-1} &= -a\xi \end{aligned} \tag{17}$$

starting from the cube in $\mathbb{R}^{n-1}$ whose edges are $2R$ long, asymptotically converges to the origin. Perform the change of coordinates $\xi_r = -a\xi + \zeta$, let $x = (z^{\mathrm T}\,\xi^{\mathrm T})^{\mathrm T}$, and rewrite (16) as

$$\begin{aligned} \dot x &= F(x) + G\zeta \\ \dot\zeta &= q(x,\zeta) + b(x,\zeta)u, \end{aligned} \tag{18}$$

where $F(x)$ is the vector field on the right-hand side of (17), and G, q, b are understood from the context. The system $\dot x = F(x)$ satisfies the ULP property. In the case $r = 1$, the system (16) is already in the form (18) with $\zeta = \xi_1$. We conclude that both Proposition 1, Corollary 1 and Proposition 3 can be applied to system (16) to obtain:

Proposition 4. *Consider a minimum-phase nonlinear system of the form (16). For any $R > 0$ and any $\varepsilon > 0$, there exist quantized feedback laws $u = -\Psi(\bar k\zeta)$, $u = -\Psi_m(\bar k\zeta)$, or a ternary feedback law (14), (15), with $\zeta = a\xi + \xi_r$, and a time $T > 0$, such that any trajectory φ of the closed-loop system which starts in the cube centered at the origin of side $2R$ lies in the cube centered at the origin of side 2ε for all $t \ge T$.*

Remark. A trivial consequence of the Proposition is that, similarly to the non-quantized case, for minimum-phase relative-degree-one nonlinear systems, a static quantized output feedback suffices to stabilize the system. In fact, for these systems, ζ coincides with the output y of the system. ◁

In the remaining subsections, we shall refer to systems for which similar results apply as *semi-globally practically stabilizable* systems.

6.2. Robust quantized stabilization of nonlinear systems

In this section we propose a quantized controller to stabilize nonlinear systems of the form

$$\begin{aligned} \dot z &= F(\mu)z + G(\xi_1,\mu)\xi_1 \\ \dot\xi_i &= q_{i0}(\xi_1,\ldots,\xi_i,\mu)z + \sum_{j=1}^{i} q_{i,j}(\xi_1,\ldots,\xi_i,\mu)\xi_j \\ &\quad + b_i(\xi_1,\ldots,\xi_i,\mu)\xi_{i+1}, \quad 1 \le i \le r-1 \\ \dot\xi_r &= q_{r0}(\xi_1,\ldots,\xi_r,\mu)z + \sum_{i=1}^{r} q_{ri}(\xi_1,\ldots,\xi_r,\mu)\xi_i \\ &\quad + b_r(\xi_1,\ldots,\xi_r,\mu)u, \end{aligned} \tag{19}$$

where $z \in \mathbb{R}^{n-r}$, and $b_i(z,\xi_1,\ldots,\xi_i,\mu) \ge b_{i0} > 0$ for all $(z,\xi_1,\ldots,\xi_i) \in \mathbb{R}^{n-r+i}$ and $\mu \in \mathcal{P}$. We also assume that, for all $\mu \in \mathcal{P}$, there exists $P(\mu) = P^{\mathrm T}(\mu) > 0$ such that $F^{\mathrm T}(\mu)P(\mu) + P(\mu)F(\mu) \le -I$. The first fact we recall is the following [13,7]:

Lemma 1. *Set $\xi = (\xi_1,\ldots,\xi_{r-1})$. There exists an $(r-1)\times(r-1)$ matrix $M(\xi)$ and a $1\times(r-1)$ vector $\delta(\xi)$ of smooth functions such that $\xi^{\mathrm T}M(\xi)\xi$ is a positive definite and proper function, and the function $V(z,\xi) = z^{\mathrm T}P(\mu)z + \xi^{\mathrm T}M(\xi)\xi$ satisfies*

$$\begin{pmatrix} \dfrac{\partial V}{\partial x} & \dfrac{\partial V}{\partial \xi} \end{pmatrix} \cdot \begin{pmatrix} f(z,\xi_1,\mu) \\ q_{10}(\xi_1,\mu)z + q_{11}(\xi_1,\mu)\xi_1 + b_1(\xi_1,\mu)\xi_2 \\ \cdots \\ q_{r-1,0}(\xi,\mu)z + \sum_{i=1}^{r-1} q_{r-1,i}(\xi,\mu)\xi_i + b_{r-1}(\xi,\mu)\delta(\xi)\xi \end{pmatrix}$$
$$\le -\varepsilon V(z,\xi).$$

By the change of coordinates $\zeta = \xi_r - \delta(\xi)\xi$, letting as before $x^{\mathrm T} = (z^{\mathrm T},\xi^{\mathrm T})^{\mathrm T}$, it is immediate to see that we are in the setting of Proposition 1 or Proposition 3, and systems of the form (19) can be semi-globally practically stabilized by a quantized or ternary controller.

6.3. A simple output-feedback switched stabilization scheme

Consider the nonlinear system

$$\begin{aligned} \dot x &= F(\mu)x + G(y,\mu)y + \bar g(\mu)\gamma(y)u \\ \dot y &= H(\mu)x + K(y,\mu)y, \end{aligned} \tag{20}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}$ the measured output, and $\gamma(y)$ a smooth function bounded away from zero. Under appropriate conditions, namely ([14,15], and also [7], Section 11.3) (i) the system has a well-defined uniform relative degree $r \ge 2$ and (ii) its zero dynamics is globally asymptotically stable, one can prove that, for the system above, to which it is appended the additional dynamics

$$\begin{aligned} \dot\xi_i &= -\lambda_{i-1}\xi_i + \xi_{i+1}, \quad 2 \le i \le r-1 \\ \dot\xi_r &= -\lambda_{r-1}\xi_r + \gamma(y)u, \end{aligned} \tag{21}$$

there exists a change of coordinates $z = T(x, y, \xi, \mu)$, linear in (x, y, ξ, μ), which transforms the extended system into

$$\begin{aligned} \dot z &= \tilde F(\mu)z + \tilde G(y,\mu)y \\ \dot y &= \tilde H(\mu)z + \tilde K(y,\mu)y + b(\mu)\xi_2 \\ \dot\xi_i &= -\lambda_{i-1}\xi_i + \xi_{i+1}, \quad 2 \le i \le r-1 \\ \dot\xi_r &= -\lambda_{r-1}\xi_r + \gamma(y)u, \end{aligned}$$

with $b(\mu)$ bounded away from zero. This system is in the form (19), and therefore there exists a quantized or a ternary controller depending on $y, \xi_2, \ldots, \xi_r$ for it. The appended dynamics (21) with u given by (3), (9) or (14), (15), and $\zeta = \xi_r - \delta(\xi)\xi$, $\xi = (y, \xi_2, \ldots, \xi_{r-1})$, is a dynamic output feedback controller which semi-globally practically stabilizes the system (20). The

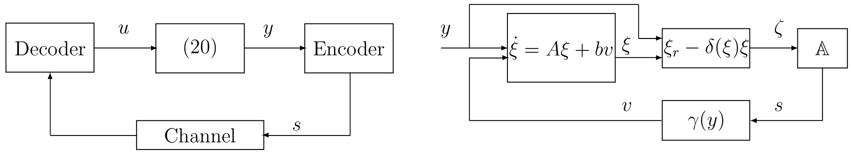

Fig. 4. In the picture on the left, the switched output feedback controller for system (20) is implemented through a network. The encoder is depicted in the picture on the right. The block labeled with $\mathbb{A}$ is the automaton depicted in Fig. 2 (quantized control) or the automaton described in Fig. 3 (ternary control). The device which converts the values generated by $\mathbb{A}$ into packets of bits which can be transmitted through the network is not depicted for the sake of simplicity.

implementation of the closed-loop system through a network in the case of quantized or ternary controller is illustrated in Fig. 4. The decoder, on the other hand, is a device which carries out the inverse operation with respect to the encoder, and is not depicted for the sake of simplicity.

Compared with [16], the solution proposed here does not require a copy of the system to control, and in fact applies to a class of systems which, although less general than the class in [16], present model uncertainty. Moreover, the dynamics of the "encoder" on the sensor side is *linear*, and therefore it requires less computational effort than in [16]. A similar class as (20) was considered in [17], where the output is quantized with no pre-processing. However, in that paper, the control law u must be designed so as to guarantee input-to-state stability with respect to state measurement errors, a task which may be considerably harder than designing the control law as in Lemma 1. Observe that we do *not* employ a dense quantization, that is we do not require a small quantization error (the quantization density can be any number in (0, 1)) to compensate for the lack of input-to-state stability.

7. Conclusion

We have discussed results on the problem of stabilizing nonlinear systems using a finite number of control values and in the presence of parametric uncertainty. These results are instrumental to solve important control problems by quantized feedback, a few of which have been presented in the paper. The tools presented in the paper are suitable to tackle other important control problems by quantized feedback such as the output regulation problem, a topic on which future research could focus.

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