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Abstract—Port-based network modeling of a large class of complex physical systems leads to dynamical systems known as port-Hamiltonian systems. The key ingredient of any port-Hamiltonian system is a power-conserving interconnection structure (mathematically formalized by the geometric notion of a Dirac structure) linking the pairs of conjugate port variables of the various ports corresponding to energy storage (defined by a Hamiltonian function depending on energy variables), resistive effects, external interaction, etc. The interconnection of port-Hamiltonian systems defines a new port-Hamiltonian system with Dirac structure determined by the Dirac structures of the constituent parts. For thermodynamic systems this framework needs modification by extending the space of energy variables, as used for port-Hamiltonian systems, into a space of energy and co-energy variables together with an additional coordinate needed for the formulation of the energy. Geometrically this extended space is formalized as a contact manifold. The thermodynamic properties of the system are given by a Legendre submanifold of the contact manifold. Furthermore a contact Hamiltonian is defined, related to the internal power-conserving interconnection structure, whose resulting dynamics leaves invariant the Legendre submanifold. Finally, interaction contact Hamiltonians are defined together with port-conjugated pairs of input and output variables modeling the interaction of the system with its environment. Interconnection of such thermodynamic systems is shown to lead to a thermodynamic system with the same structure.

1. Introduction

The work reported in this brief paper is part of a continuing line of research on the geometric and coordinate-free formulation of network models of physical systems. The aim is to provide a geometric framework for the systematic description of complex physical systems with interacting components stemming from different physical domains (mechanical, electromagnetic, hydraulic, etc.) The derived type of mathematical models are thought to be of importance for analysis and simulation, since they make explicit the underlying physical characteristics of the system (energy balance, existence of conserved quantities, volume preservation, etc.). Furthermore, they form a natural starting point for *control* where the dynamical properties of the system are sought to be influenced by the interconnection with additional control components.

The present work is devoted to such a geometric network description of open *thermodynamic systems*. In this case the established port-Hamiltonian framework for the description of complex physical systems needs modifications by extending the space of energy and co-energy variables into a contact manifold.

2. Control Contact Systems

In this section we briefly recall the basic concepts of contact geometry (following [5] and [1]) in the context of thermodynamics (see [6] and the references herein). Afterwards, we shall recall the definition of control contact systems. These systems are control systems where the drift vector field and the input vector fields are contact vector fields satisfying the condition that they leave invariant some Legendre submanifold.

They are also an extension of port-Hamiltonian systems defined with respect to Dirac structures [8, 3], systems associated with reversible thermodynamics transformations [6], and systems associated with irreversible thermodynamic transformations [4].

First we recall the canonical state space, called *thermodynamic phase space*, in which the thermodynamic properties of a system are defined. It has a canonical structure, called *contact structure*, which plays an analogous role as the symplectic structure for Lagrangian or Hamiltonian systems.

Let $\mathcal{N}$ be an n-dimensional, connected, differentiable manifold of class C^∞ and define the associated thermodynamic phase space $\mathcal{T}$ as the 1-jet bundle from $\mathcal{N}$ into $\mathbb{R}$. $\mathcal{T}$ may be identified with $\mathbb{R} \times T^*\mathcal{N}$ ([5]) whose elements are denoted (x^0, x, p). It has a canonical contact structure defined by the contact form

$$\theta = \mathrm{d}x^0 - \sum_{k=1}^{n} p_k \mathrm{d}x^k, \qquad (1)$$

where d denotes the exterior derivative. We recall the following useful characterization of a contact form [5].

Proposition 1 *A 1-form θ on a $2n + 1$-dimensional*

manifold is a contact form if and only if $\theta \wedge (d\theta)^n$ is a volume form.

Such structures appear in the differential-geometric representation of thermodynamic systems [6, 4]. Indeed, x^0 is associated with a thermodynamic potential, such as the energy U, the enthalpy H, etc., and (x^i, p_i) denotes the pairs of conjugated extensive and intensives variables. In this case the contact form is closely related to the Gibbs' relation obtained from the vanishing of the contact form : $dU = T dS - P dV + \mu_i dN^i$. Actually the Gibbs' relation corresponds to the definition of a canonical submanifold of a contact structure, called *Legendre submanifold*, and playing an analogous role as Lagrangian submanifolds for symplectic structures.

Definition 1 *A* Legendre submanifold *of a contact manifold* $(\mathcal{T}, \theta)$ *is an n-dimensional submanifold of $\mathcal{T}$ that is an integral manifold of θ.*

Legendre submanifolds are locally generated by a generating function.

Theorem 1 ([1]) *For a given set of canonical coordinates and any partition $I \cup J$ of the set of indices $\{1, \ldots, n\}$ and for any differentiable function $F(x^I, p_J)$ of n variables, $i \in I, j \in J$, the formulas*

$$x^0 = F - p_J \frac{\partial F}{\partial p_J}, \quad x^J = -\frac{\partial F}{\partial p_J}, \quad p_I = \frac{\partial F}{\partial x^I} \quad (2)$$

define a Legendre submanifold of $\mathbb{R}^{2n+1}$. Conversely, every Legendre submanifold of $\mathbb{R}^{2n+1}$ can be defined in a neighborhood of every point by these formulas, for at least one of the 2^n possible choices of the subset I.

Consider the particular case of a generating function F which is a differentiable function on $\mathcal{N}$. The Legendre submanifold is then the set

$$\mathcal{L}_F := \left\{ x^0 = F(x), x, p = \frac{\partial F}{\partial x}(x) \right\}. \quad (3)$$

For thermodynamic systems, the generating functions are potentials such as U, H, etc., while the associated Legendre submanifold defines the thermodynamic properties of some system, for instance of an ideal mixture of perfect gas.

Finally we recall the definition of the class of vector fields, called contact vector fields, which preserve the contact structure. They may be characterized using the following result.

Proposition 2 ([5]) *A vector field X on $(\mathcal{T}, \theta)$ is a* contact vector field *if and only if there exists a differentiable function ρ such that*

$$\mathcal{L}(X)\,\theta = \rho\,\theta, \quad (4)$$

where $\mathcal{L}(X)$ denotes the Lie derivative with respect to the vector field X.

It is worth noting that the set of contact vector fields forms a Lie subalgebra of the Lie algebra of vector fields.

Analogously to the case of Hamiltonian vector fields, there exists a mapping between contact vector fields and differentiable functions on $\mathcal{T}$. This mapping associates to a contact field X a function called *contact Hamiltonian*. Conversely, to every function f on $\mathcal{T}$ one associates a contact vector field denoted X_f and expressed in canonical coordinates as follows

$$\begin{aligned} X_f = \quad & \left(f - \sum_{k=1}^n p_k \frac{\partial f}{\partial p_k} \right) \frac{\partial}{\partial x^0} + \frac{\partial f}{\partial x^0} \left(\sum_{k=1}^n p_k \frac{\partial}{\partial p_k} \right) \\ & + \sum_{k=1}^n \left(\frac{\partial f}{\partial x^k} \frac{\partial}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial}{\partial x^k} \right). \end{aligned}$$
$$(5)$$

Furthermore this mapping is an isomorphism from the Lie algebra structure of the contact vector fields to the Lie algebra structure on the space of contact Hamiltonians defined by the following bracket:

$$\{f, g\} = i([X_f, X_g])\theta, \quad (6)$$

where i denotes the interior product and $[\cdot, \cdot]$ is the usual Lie bracket of vector fields.

We recall next the definition, given in [2], of a class of systems called *control contact systems* which extend port-Hamiltonian systems defined with respect to a Poisson bracket [8] and reversible thermodynamic transformations [6] [4]. The definition of a control contact system is obtained by augmenting the internal contact vector field with additional input vector fields.

Definition 2 *Let $(\mathcal{T}, \theta)$ be a contact manifold and $\mathcal{L}$ a Legendre submanifold. A conservative control contact system on $\mathcal{T}$ is determined by an input space $\mathcal{U} = \mathbb{R}^m$ and input functions u_j, $j = 1, .., m$, together with $m+1$ contact Hamiltonian functions : K_0 the internal contact Hamiltonian and K_j, $j = 1, \ldots, m$ the interaction contact Hamiltonians, all satisfying the invariance condition : $K_{l|\mathcal{L}} \equiv 0$, $l = 0, \ldots, m$. The dynamics of the conservative control system is given by the differential equation*

$$\frac{d}{dt}(x^0, x, p)^t = X_{K_0} + \sum_{j=1}^m u_j X_{K_j}. \quad (7)$$

In the context of thermodynamic systems, this system may be interpreted as follows. The Legendre submanifold $\mathcal{L}$ represents the thermodynamic properties of the system. The internal contact Hamiltonian K_0 represents the law giving the fluxes in the closed system due to non-equilibrium conditions *in* the system (for instance due to heat conduction or chemical reaction

kinetics). Finally the interaction contact Hamiltonians K_j provide the flows due to the non-equilibrium of the system with its environment.

The invariance condition only implies that the control contact system obeys the first principle. The above formulation of the invariance condition comes from the following theorem.

Theorem 2 ([6]) *Let $(\mathcal{T}, \theta)$ be a contact manifold and $\mathcal{L}$ a Legendre submanifold. Then X_f is tangent to $\mathcal{L}$ if and only if f is identically zero on $\mathcal{L}$.*

3. Port Contact Systems

In this section we recall the losslessness properties of conservative control contact systems and the definition of port outputs conjugated to the inputs. In the same way as for input-output and port-Hamiltonian systems, the differential geometric structure of the system induces the energy balance equation from which the definition of the port-conjugated outputs follows. However, contrary to input-output and port-Hamiltonian systems, for conservative control contact systems the energy balance will only be considered on its restriction to the Legendre submanifold (where the thermodynamic properties are satisfied).

Let us first compute the time derivative of a differentiable real function V on $\mathcal{T}$ with respect to a conservative control contact system. A straightforward calculation leads to the following balance equation [2].

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \sum_{j=1}^{m} u_j \, y_V^j + \sigma_V, \tag{8}$$

where y_V^j is the *V-conjugated output variable* :

$$y_V^j = \{K_j, V\} + V \frac{\partial K_j}{\partial x^0}, \tag{9}$$

and σ_V is a *source term* defined by :

$$\sigma_V = \{K_0, V\} + V \frac{\partial K_0}{\partial x^0}. \tag{10}$$

We now define a conserved quantity for which the source term is zero. However, there is no reason to require the source term to be zero on the entire state space but rather on the Legendre submanifold (see [2] for more details and examples).

Definition 3 *A conserved quantity of a control contact system is a real-valued function V on $\mathcal{T}$ such that $\sigma_{V|\mathcal{L}} = 0$.*

Using the definition of a port-conjugated output (9), we are now able to define a *port contact system* in the same manner as in the input-output Hamiltonian case.

Definition 4 *A port contact system is a control contact system (Definition 3), with the additional condition that there exists a generating function U of a Legendre submanifold that is a conserved quantity (i.e. $\sigma_{U|\mathcal{L}_U} = 0$), completed with the U-conjugated outputs defined in (9). The port contact system is denoted by $(\mathcal{N}, U, K_j)$.*

4. Interconnection of port contact systems

In this section we consider the interconnection or composition of port contact systems. We only consider, as a first step towards more general cases, the composition by Dirac structures. Thereby we generalize the composition of port-Hamiltonian systems by Dirac structures [9] [7]. *Dirac structures* on a vector space $\mathcal{V}$ are vector subspaces $\mathcal{D}$ of $\mathcal{V} \times \mathcal{V}^*$ satisfying the isotropy and co-isotropy condition $\mathcal{D}^\perp = \mathcal{D}$ where

$$\mathcal{D}^\perp = \{ (v, v^*) \in \mathcal{V} \times \mathcal{V}^* \, | < v^* \mid \hat{v} > + < \hat{v}^* \mid v >= 0 \\ \forall \ (\hat{v}, \hat{v}^*) \in \mathcal{D}\} \tag{11}$$

with $<|>$ denoting the natural pairing between $\mathcal{V}$ and its dual space $\mathcal{V}^*$. They generalize the notions of Poisson as well as of pre-symplectic vector spaces. Physically they correspond to a power-continuous interconnection structure appearing in the modelling of reversible physical systems.

In the sequel we only consider a particular Dirac structure defined as the graph of a skew-symmetric map $J : \mathcal{V}^* \longrightarrow \mathcal{V}$. Consider now two differential manifolds $\mathcal{N}_1$ and $\mathcal{N}_2$ of respective dimensions n_1 and n_2, with coordinates $x_i = (x_i^1, \ldots, x_i^{n_i})$ for $i = 1, 2$. Each 1-jet space $\mathcal{T}^i$ over $\mathcal{N}_i$ is endowed with a canonical contact structure whose contact form is denoted by θ_i (recall that $\mathcal{T}^i = \mathbb{R} \times T^*\mathcal{N}_i$). We now construct the composed state space in the same way. Denote by $\mathcal{N}$ the whole product space, i.e. $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$. Then, the 1-jet bundle over $\mathcal{N}$, called $\mathcal{T}$, is also endowed with a canonical contact form θ whose local expression is

$$\theta = \mathrm{d}x^0 - \sum_{j=1}^{n_1+n_2} p_j \, \mathrm{d}x^j, \tag{12}$$

where $x^j = x_1^j$ and $p_j = p_j^1$ if $1 \le j \le n_1$, else $x^j = x_2^{j-n_1}$ and $p_j = p_{j-n_1}^2$ if $n_1 + 1 \le j \le n_1 + n_2$.

According to Definition 4, consider two port contact systems $(\mathcal{N}_i, U_i, K_j^i)$ on $\mathcal{T}^i$ with contact Hamiltonian K_i defined as

$$K_i = K_0^i + \sum_{j=1}^{n_i} u_j^i \, K_j^i, \tag{13}$$

satisfying the invariance condition with respect to the conserved quantity U_i, $i = 1, 2$. We define the new

(conserved) generating function U of the Legendre submanifold of the composed state space as $U_1 + U_2$.

Denote by m the number of input variables involved in the interconnection ($m \leq min(m_1, m_2)$). Without loss of generality we may suppose that the first m variables are involved in the interconnection.

Proposition 3 *The composition of two port contact systems $(\mathcal{N}_i, U_i, K_j^i)_{i=1,2}$ with respect to a Dirac structure $\mathcal{D}$ is the port contact system determined by the contact Hamiltonian $K = K_1 + K_2$ and by the following relations (determined by $\mathcal{D}$) on the port-conjugated variables*

$$\begin{pmatrix} u_j^1 & u_j^2 \end{pmatrix}^t = \begin{pmatrix} 0 & J \\ -J^t & 0 \end{pmatrix} \begin{pmatrix} y_{U_1}^j & y_{U_2}^j \end{pmatrix}^t, \qquad (14)$$

where J is an antisymmetric matrix of full rank m.

It is obvious to see that the invariance condition is satisfied by K on $\mathcal{L}_U$. We now show that U is a conserved quantity of the interconnected system thus obtained, when restricted to the Legendre submanifold $\mathcal{L}_U$. Indeed, let us compute its time-derivative

$$\frac{dU}{dt}_{|\mathcal{L}_U} = \sum_{j=1}^m \left[u_1^j \{K_j^1, U_1\} + u_2^j \{K_j^2, U_2\} \right]_{|\mathcal{L}_U}, \quad (15)$$

which is zero by (14) and (9). It may be noted that in the case when both port contact systems are lifted port-Hamiltonian systems (in the sense of [3]) then the composed system is the lifted port-Hamiltonian obtained by the interconnection of the port-Hamiltonian systems through the Dirac structure defined in (14).

Furthermore, the dynamics of the base variables x_i *restricted* to $\mathcal{L}_U$ is

$$\dot{x}_i = -\frac{\partial K_{0,i}}{\partial p^i} - (-1)^i J\, y_{U_i} \frac{\partial K^i}{\partial p^i}. \qquad (16)$$

This corresponds to the feedback interconnection of the two port contact systems, restricted to $\mathcal{L}_{U_1}$ and $\mathcal{L}_{U_2}$ respectively, by the modulated feedback (14).

It is important to note that this is no more true for trajectories lying outside the Legendre submanifold $\mathcal{L}_U$ of the composed system. Indeed the projection onto the x axis provides extra terms in eqn. (16).

5. Conclusions

A key property of port-Hamiltonian systems is the fact that the interconnection of port-Hamiltonian systems defines a new port-Hamiltonian system with Dirac structure determined by the Dirac structures of the constituent parts and the interconnection Dirac structure. In this paper we have shown that the interconnection of port contact systems via a skew-symmetric relation between the involved port-conjugated input and output variables leads again to a port contact system with the same structure. Obviously, this is only a first step towards a general theory of interconnection of port contact systems. Furthermore, it is to be expected that the general interconnection of port contact systems will lead to more insight into a full extension of the port-Hamiltonian framework to the thermodynamic case.

Acknowledgments

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