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Renardel de Lavalette, G.R.

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The interpolation theorem in fragments of logics

by G.R. Renardel de Lavalette

Mathematisch Instituut, Universiteit van Amsterdam, Postbus 20239,
1000 HE Amsterdam, the Netherlands

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ABSTRACT

In the first part of this paper, we prove that there are continuously many fragments of intuitionistic propositional calculus (**IpC**) which fail to have the interpolation property, thereby extending a result of J.I. Zucker. Our proof makes use of the Rieger-Nishimura lattice. The second part is devoted to transferring this result to fragments of classical predicate calculus (**CPC**): this is done by giving a translation T of fragments of **IpC** in fragments of **CPC** which preserves the interpolation property.

1. INTRODUCTION

The Interpolation Theorem for **CPC** (classical predicate calculus) has been stated and proved for the first time by Craig [Cr]. Schütte [Sch] gives a proof for **IPC** (intuitionistic predicate calculus). Since then, the Interpolation Theorem (IT for short) has been shown to hold or to fail in quite a lot of logics (modal, higher-order, many-sorted, etc). There is an extensive literature on the subject which we shall not attempt to survey here.

In this paper, we are only interested in the IT in fragments of propositional and predicate logic. Ville proved that the IT holds in any fragment of **CpC** (**p** for propositional); see [K & K], Chapter 1, Exercises. Zucker [Z] gives an example of a fragment of **IpC** for which interpolation fails; this was the starting-point for our investigations which led to the present paper.

In Section 2 we extend Zucker's result in that we give a set of $2^{\aleph_0}$ fragments of **IpC** for which the IT fails. Section 3 is devoted to transferring this to fragments of **CPC**: we define a translation T which maps every fragment of **IpC** on one of **CPC**, preserving the IT in both directions.

I wish to thank Prof. Troelstra and Jeff Zucker for interesting me in this subject and for their comments on earlier versions of this paper.

2. $2^{\aleph_0}$ FRAGMENTS OF INTUITIONISTIC PROPOSITIONAL LOGIC FOR WHICH INTERPOLATION FAILS

2.1. We start introducing some notation for this and the next section. For propositional logic, we use the language $\boldsymbol{L}_0$, containing the connectives $\wedge$, $\vee$, $\rightarrow$, the propositional constants $\bot$ and $\top$, and $\boldsymbol{V}=\{p_1,p_2,\dots\}$, the set of propositional variables; $p,q,r,q_1,\dots$ are metavariables for elements of $\boldsymbol{V}$.

Formulae are defined as usual. ***Form*** is the set of all formulae of $\boldsymbol{L}_0$. $\vdash_c$ ($\vdash_i$) denote classical (intuitionistic) derivability, $\equiv_c$ and $\equiv_i$ are used for derivable equivalence; we drop the subscript if that causes no confusion.

If A is a formula of some logic, then $PV(A)$ is the set of predicate or propositional variables occurring in A; similar for FV (free individual variables) and BV (bound individual variables).

$A[B/C]$ stands for the formula A' which is formed by substituting B for every occurrence of C in A.

2.2. DEFINITION. i) If $A\in\boldsymbol{Form}$, $PV(A)\subset\{q_1,\dots,q_n\}$, $n\geq 0$, $q_1,\dots,q_n$ all different, then $\lambda q_1\cdots q_n\bullet A$ is a *propositional* (*n-ary*) *connective* abstracted from A.

ii) ***Con*** (***Con***^{n}) is the set of (n-ary) connectives.

iii) If $c\in\boldsymbol{Con}^n$, $c=\lambda q_1\cdots q_n\bullet A$ and $B_1,\dots,B_n\in\boldsymbol{Form}$, then

$$c(B_1,\dots,B_n)\underset{\text{def}}{=}A[B_1,\dots,B_n/q_1,\dots,q_n].$$

2.3. DEFINITION. i) Let $\boldsymbol{C}\subset\boldsymbol{Con}$. We define the *propositional fragment* $[\boldsymbol{C}]$ as the smallest subset of ***Form*** satisfying:

a) $\boldsymbol{V}\subset[\boldsymbol{C}]$, and

b) if $c\in\boldsymbol{C}\cap\boldsymbol{Con}^n$ and $A_1,\dots,A_n\in[\boldsymbol{C}]$, then $c(A_1,\dots,A_n)\in[\boldsymbol{C}]$.

We shall often write $[c_1,\dots,c_n]$ for $[\{c_1,\dots,c_n\}]$.

ii) ***Frag*** is the set of propositional fragments.

iii) If $\boldsymbol{f},\boldsymbol{g}\in\boldsymbol{Frag}$ then $\boldsymbol{f}$ and $\boldsymbol{g}$ are called equivalent ($\boldsymbol{f}\equiv\boldsymbol{g}$) iff $\forall A\in\boldsymbol{f}\exists B\in\boldsymbol{g}A\equiv B$ and $\forall B\in\boldsymbol{g}\exists A\in\boldsymbol{f}B\equiv A$.

As a simple consequence of the Interpolation Theorem for **IpC**, we see that the following holds:

2.4. THEOREM. Let $A,B\in[\wedge,\vee,\rightarrow,\bot]$, $PV(A)\cap PV(B)\neq\emptyset$, $\vdash A\rightarrow B$. Then there is an $I\in[\wedge,\vee,\rightarrow,\bot]$ such that:

i) $\vdash A\rightarrow I$, $\vdash I\rightarrow B$;

ii) $PV(I)\subset PV(A)\cap PV(B)$.

Briefly: interpolation holds for $[\wedge,\vee,\rightarrow,\bot]$.

More generally, we say that interpolation holds for some fragment $\boldsymbol{f}\in\boldsymbol{Frag}$ iff Theorem 2.4 holds when $[\wedge,\vee,\rightarrow,\bot]$ is replaced by $\boldsymbol{f}$.

REMARK. Theorem 2.4 remains true if we skip the condition $PV(A) \cap \cap PV(B) \neq \emptyset$. The reason we added it lies in the fact that in fragments without nullary connectives, formulae A with $PV(A) = \emptyset$ do not exist.

2.5. Zucker [Z] shows that interpolation fails for $[\delta, \wedge, \rightarrow, \perp]$, where $\delta = \lambda pqr \cdot (p \vee \neg p) \wedge (p \rightarrow q) \wedge (\neg p \rightarrow r)$. He gives two proofs, the first one being syntactical, the second one (due to A.S. Troelstra) using the theory of (finite) Heyting algebras. We shall generalize the method of this second proof to obtain the result mentioned in the title of this section. For information about Heyting algebras we refer to Dummett [D, 5.2].

REMARK. We use the same names for the operators of a Heyting algebra as for the connectives they correspond with: however, it will always be clear from the context which meaning of $\wedge$, $\vee$, $\rightarrow$ is intended; idem for $\perp$ and $\top$. As to newly defined connectives, we suppose corresponding operators for Heyting algebras to be defined, too.

2.6. We now sketch Troelstra's proof of Zucker's theorem.

We have

$$\exists p \delta(p, q_1, q_2) \equiv q_1 \vee q_2 \equiv \forall r((q_1 \rightarrow r) \wedge (q_2 \rightarrow r)) \rightarrow r;$$

hence

$$\vdash \delta(p, q_1, q_2) \rightarrow (((q_1 \rightarrow r) \wedge (q_2 \rightarrow r)) \rightarrow r), \tag{1}$$

and if $I = I(q_1, q_2)$ is an interpolant for (1), then $I \equiv q_1 \vee q_2$, so it suffices to demonstrate the undefinability of $\vee$ in $[\delta, \wedge, \rightarrow, \perp]$.

Consider the following Heyting algebra, given as a partially ordered system:

$$\perp < a \wedge b < a, b < a \vee b < \top.$$

The set $\{\perp, a \wedge b, a, b, \top\}$ is closed under δ, $\wedge$ and $\rightarrow$, so $\vee$ is not definable in $[\delta, \wedge, \rightarrow, \perp]$. □

2.7. For the generalization, we shall make use of the *Rieger-Nishimura lattice:* this is the Heyting algebra $\boldsymbol{H_R}$ with $X_R = \{\perp = a_{-1}, a_0, a_1, \ldots, b_0, b_1, \ldots, \top\}$ as set of elements, where

$$a_n = a_{n-1} \vee b_{n-1}, \quad n = 1, 2, \ldots,$$

$$b_n = a_n \rightarrow a_{n-1}, \quad n = 0, 1, 2, \ldots.$$

Rieger [R] was the first one to describe $\boldsymbol{H_R}$; better accessible and more informative is Nishimura [N].

$\boldsymbol{H_R}$ is the free Heyting algebra over one generator: this means that if $A(p)$, $B(p)$ are propositional formulae in one variable, then

$$A(p) \equiv B(p) \Leftrightarrow A(a_0) = B(a_0).$$

For a proof, see [N]; there one can also find a list of all equalities of the form $a \wedge b = c$, $a \vee b = c$ and $a \rightarrow b = c$ which hold in $\boldsymbol{H_R}$.

We introduce the connectives

$$\pi_{-1} \underset{\text{def}}{=} \lambda p \bullet \bot; \quad \pi_0 \underset{\text{def}}{=} \lambda p \bullet p;$$

$$\pi_{n+1} \underset{\text{def}}{=} \lambda p \bullet \pi_n(p) \vee (\pi_n(p) \to \pi_{n-1}(p)), \quad n = 0, 1, 2, \ldots .$$

2.8. LEMMA. Let $m \in \mathbb{N}$. Then:

i) $\pi_m(a_n) = a_n$ *if* $m = 1$ *and* $n \geq 1$
$= a_m$ *if* $n = 0$
$= \top$ *if* $n = -1$ *or* $(m \geq 2$ *and* $n \geq 1)$;

ii) $\pi_m(b_n) = a_2$ *if* $(m = 1$ *or* $m = 2)$ *and* $(n = 0$ *or* $n = 1)$
$= b_n$ *if* $m = 1$ *and* $n \geq 2$
$= \top$ *if* $m \geq 3$ *or* $(m = 2$ *and* $n \geq 3)$;

iii) $\pi_m(\top) = \top$.

PROOF. We only prove (i); (ii) can be done the same way, and (iii) is trivial.
$m = 1$ and $n \geq 1$: $\pi_1(a_n) = a_n \vee (a_n \to a_{-1}) = a_n \vee a_{-1} = a_n \vee \bot = a_n$.
$n = 0$: $\pi_0(a_0) = a_0$, $\pi_1(a_0) = a_0 \vee (a_0 \to a_{-1}) = a_1$;
$\pi_{m+1}(a_0) = \pi_m(a_0) \vee (\pi_m(a_0) \to \pi_{m-1}(a_0))$, so if $\pi_m(a_0) = a_m$, $\pi_{m-1}(a_0) = a_{m-1}$, then $\pi_{m+1}(a_0) = a_m \vee (a_m \to a_{m-1}) = a_{m+1}$; with induction, we now get $\pi_m(a_0) = a_m$.
$n = -1$: $\pi_1(a_{-1}) = \bot \vee (\bot \to \bot) = \top$;
$\pi_{m+1}(a_{-1}) = \pi_m(a_{-1}) \vee (\pi_m(a_{-1}) \to \pi_{m-1}(a_{-1}))$, so if $\pi_m(a_{-1}) = \top$, then $\pi_{m+1}(a_{-1}) = \top$; with induction, we get $\pi_m(a_{-1}) = \top$.
$m \geq 2$ and $n \geq 1$: $\pi_2(a_n) = \pi_1(a_n) \vee (\pi_1(a_n) \to \pi_0(a_n)) = a_n \vee (a_n \to a_n) = \top$;
as before, we have by induction $\pi_m(a_n) = \top$. □

2.9. LEMMA. Suppose $a, b, c \in X_R$, $c \notin \{a, b, \top\}$. Then $a \wedge b = c$, $a \to b = c$ or $\pi_n(a) = c$ only in the following cases:

i) $b_n = a_n \to a_{n-1}$,
$b_n = b_{n-1} \to a_{n-1}$,
$b_n = b_{n-1} \to a_{n-2}$,
$b_n = b_{n+1} \to a_{n-1}$;

ii) $a_n = a_{n+1} \wedge b_{n+1}$,
$a_n = b_{n+1} \wedge b_{n+2}$;

iii) $a_n = \pi_n(a_0)$,
$a_2 = \pi_1(b_0)$,
$a_2 = \pi_1(b_1)$,
$a_2 = \pi_2(b_0)$,
$a_2 = \pi_2(b_1)$.

PROOF. Straightforward. □

From now on, we consider in this section only $\boldsymbol{C} \in \boldsymbol{Con}$ of the form $\boldsymbol{C}_{\mathbb{I}} = \{\wedge, \to, \bot\} \cup \{\pi_n : n \in \mathbb{I}\}$, where $\mathbb{I} \subset \mathbb{N}$. We shall prove that there are many non-equivalent fragments of the form $[\boldsymbol{C}_{\mathbb{I}}]$.

2.10. DEFINITION. Let $\boldsymbol{C} \in \boldsymbol{Con}$, $\boldsymbol{H}$ a Heyting algebra, $S \subset X_H$. Then $S^{\boldsymbol{C}}$, the *$\boldsymbol{C}$-closure of S in $\boldsymbol{H}$*, is the smallest set containing S and closed w.r.t. the connectives of $\boldsymbol{C}$.

For $\{a_0\}^{\boldsymbol{C}}$ we shall write $\boldsymbol{H}_R(\boldsymbol{C})$.

2.11. LEMMA. Define $\mathbb{I}(n) = \mathbb{N} - \{n, n+1\}$, $n = 1, 2, \ldots$. Then

i) $\boldsymbol{H}_R(\boldsymbol{C}_{\mathbb{I}(n)}) = X_R$ *if* $n = 2$,

$\quad = X_R - \{a_n, a_{n+1}, b_{n+2}\}$ *if* $n \neq 2$.

ii) $\boldsymbol{H}_R(\boldsymbol{C}_{\mathbb{N}-\{1,2,3\}}) = X_R - \{a_1, a_2, a_3, b_3, b_4\}$.

PROOF. (i) $\supset$: If $i \neq n$, $n+1$, then $a_i = \pi_i(a_0) \in \boldsymbol{H}_R(\boldsymbol{C}_{\mathbb{I}(n)})$; if $i \neq n$, $n+1$, $n+2$, then $b_i = a_i \rightarrow a_{i-1} \in \boldsymbol{H}_R(\boldsymbol{C}_{\mathbb{I}(n)})$; idem for b_n $(= b_{n-1} \rightarrow a_{n-1})$ and b_{n+1} $(= b_n \rightarrow a_{n-1})$. If $n = 2$, then $a_2 = \pi_1(b_0) \in \boldsymbol{H}_R(\boldsymbol{C}_{\mathbb{I}(2)})$; idem for $b_4 = b_3 \rightarrow a_2$ and $a_3 = b_4 \wedge a_4$.

$\subset$: Let, for $x \in X_R$, $d(x)$ be the minimal number of connectives needed to define x (using only a_0 and elements of $\boldsymbol{C}_{\mathbb{I}(n)}$) if that is possible, else ∞. We observe that $a_n = a_{n+1} \wedge b_{n+1} = b_{n+1} \wedge b_{n+2}$, so we only have to prove $d(a_n) = \infty$ if $n \neq 2$.

Suppose $d(a_n) < \infty$, $n \neq 2$. By Lemma 2.9, a_n can only be obtained from b_{n+1} and (a_{n+1} or b_{n+2}), so

(1) $\qquad d(a_{n+1}) < d(a_n)$ or $d(b_{n+2}) < d(a_n)$.

To obtain a_{n+1}, we need (at least) b_{n+2}, hence

(2) $\qquad d(b_{n+2}) < d(a_{n+1})$;

finally, a_{n+1} or a_n is required for b_{n+2}, so

(3) $\qquad d(a_n) < d(b_{n+2})$ or $d(a_{n+1}) < d(b_{n+2})$.

Now (1), (2) and (3) give contradiction, so $d(a_n) = \infty$.
(ii): analogous. $\square$

The following lemma is quite trivial:

2.12. LEMMA. If $\mathbb{I} \subset \mathbb{J}$, then $\boldsymbol{H}_R(\boldsymbol{C}_{\mathbb{I}}) \subset \boldsymbol{H}_R(\boldsymbol{C}_{\mathbb{J}})$. $\square$

The last two lemmata can be combined to get

2.13. LEMMA. Define

$$\mathfrak{J} = \{\mathbb{I} \subset \mathbb{N} \colon 1 \in \mathbb{I} \Leftrightarrow 2 \in \mathbb{I} \text{ and } \forall n \in \mathbb{N}\ (n \notin \mathbb{I} \Rightarrow (n-1 \notin \mathbb{I} \text{ or } n+1 \notin \mathbb{I}))\}.$$

Then $\forall \mathbb{I} \in \mathfrak{J}\, \forall n \in \mathbb{N}\ (n \in \mathbb{I} \Leftrightarrow a_n \in \boldsymbol{H}_R(\boldsymbol{C}_{\mathbb{I}}))$.

PROOF. $\Rightarrow$: trivial.

$\Leftarrow$: Suppose $n \notin \mathbb{I}$. Now $n+1 \notin \mathbb{I}$ or $n-1 \notin \mathbb{I}$, so $\mathbb{I} \subset \mathbb{N} - \{k, k+1\}$ with $k = n-1$ or $k = n$. We distinguish two cases:

i) $k = 2$. Then, because of $1 \in \mathbb{I} \Leftrightarrow 2 \in \mathbb{I}$, even $\mathbb{I} \subset \mathbb{N} - \{1, 2, 3\}$, so (applying Lemma 2.11(ii) and Lemma 2.12) $a_n \notin \boldsymbol{H}_R(\boldsymbol{C}_{\mathbb{I}})$.

ii) $k \neq 2$. Now $a_n \notin \boldsymbol{H}_R(\boldsymbol{C}_{\mathbb{I}})$ be Lemma 2.11(i) and Lemma 2.12. $\square$

COROLLARY. If $\mathbb{I}, \mathbb{J} \in \mathfrak{J}$, $\mathbb{I} \neq \mathbb{J}$, then $[\mathbf{C}_{\mathbb{I}}] \not\equiv [\mathbf{C}_{\mathbb{J}}]$.

We take a subset of $\mathfrak{J}$:

$$\mathfrak{J}' \underset{\text{def}}{=} \{\mathbb{I} \in \mathfrak{J} : \exists n \in \mathbb{N}\, n, n+1, n+2 \in \mathbb{I}\},$$

and show that interpolation fails for all fragments corresponding with elements of $\mathfrak{J}'$; to do this, we need the next lemma.

2.14. LEMMA. Let $X = \{\bot, a_n, a_{n+1}, b_{n+1}, b_{n+2}, b_{n+3}, \top\}$, $n \in \mathbb{N}$. Then

$$X^{C_{\mathbb{I}}} = X \text{ for all } \mathbb{I} \subset \mathbb{N}.$$

PROOF. Simple, using Lemma 2.9. □

2.15. THEOREM. If $\mathbb{I} \in \mathfrak{J}'$, then interpolation fails for $[\mathbf{C}_{\mathbb{I}}]$.

PROOF. Let $\mathbb{I} \in \mathfrak{J}'$, $n \in \mathbb{N}$, n, $n+1$, $n+2 \in \mathbb{I}$. Define

$$\delta_n(p, q_1, q_2) = \pi_{n+2}(p) \wedge (\pi_{n+1}(p) \to q_1) \wedge ((\pi_{n+1}(p) \to \pi_n(p)) \to q_2),$$

$$\varrho(q_1, q_2, r) = ((q_1 \to r) \wedge (q_2 \to r)) \to r.$$

Now

$$\vdash \delta_n(p, q_1, q_2) \to q_1 \vee q_2,$$

because $\pi_{n+2}(p) = \pi_{n+1}(p) \vee (\pi_{n+1}(p) \to \pi_n(p))$, and

$$\vdash q_1 \vee q_2 \to \varrho(q_1, q_2, r),$$

so we have

$$\vdash \delta_n(p, q_1, q_2) \to \varrho(q_1, q_2, r). \tag{1}$$

Suppose $\iota(q_1, q_2)$ is an interpolant for (1). We shall show that $\iota(q_1, q_2)$ cannot be in $[\mathbf{C}_{\mathbb{I}}]$, by considering H_R with the valuation Val_H, (partially) defined by:

$$Val_H(p) = a_0, \qquad Val_H(q_2) = b_{n+1},$$

$$Val_H(q_1) = a_{n+1}, \qquad Val_H(r) = a_{n+2}.$$

This gives

$$\begin{aligned} Val_H(\delta_n(p, q_1, q_2)) &= \delta_n(a_0, a_{n+1}, b_{n+1}) = \pi_{n+2}(a_0) \wedge (\pi_{n+1}(a_0) \to a_{n+1}) \wedge \\ &\wedge ((\pi_{n+1}(a_0) \to \pi_n(a_0)) \to b_{n+1}) = a_{n+2} \wedge (a_{n+1} \to a_{n+1}) \wedge \\ &\wedge (b_{n+1} \to b_{n+1}) = a_{n+2} \wedge \top \wedge \top = a_{n+2}, \\ Val_H(\varrho(q_1, q_2, r)) &= \varrho(a_{n+1}, b_{n+1}, a_{n+2}) = ((a_{n+1} \to a_{n+2}) \wedge \\ &\wedge (b_{n+1} \to a_{n+2})) \to a_{n+2} = (\top \wedge \top) \to a_{n+2} = a_{n+2}; \end{aligned}$$

hence $Val_H(\iota(q_1, q_2)) = \iota(a_{n+1}, b_{n+1}) = a_{n+2}$, which, together with Lemma 2.14, gives $\iota(q_1, q_2) \notin [\mathbf{C}_{\mathbb{I}}]$. □

COROLLARY. There are $2^{\aleph_0}$ non-equivalent fragments of **IpC** for which interpolation fails.

PROOF. $\mathfrak{J}' = 2^{\aleph_0}$, for $\{\{1,2,3\} \cup \{n+3 : n \in \mathbb{I}\} : \mathbb{I} \subset \mathbb{N}\} \subset \mathfrak{J}'$. □

REMARK. All fragments considered here have the definability property, which states that implicit definability implies explicit definability. This follows from Kreisel's [Kr] and the fact that $\top (= \bot \rightarrow \bot)$ is definable in our fragments.

3. INTERPOLATION IN FRAGMENTS OF CLASSICAL PREDICATE LOGIC

3.1. First some preliminaries, extending those of 2.1 for predicate logic. $\boldsymbol{L}_1$ is a language for predicate logic, containing $\wedge$, $\vee$, $\rightarrow$, $\forall$, $\exists$, $\bot$, $\top$, the individual variables $x_1, x_2, \ldots$ (metavariables $x, y, z, y_1, y_2, \ldots$) and predicate variables $P_1, P_2, \ldots$ (metavariables $P, Q, R, Q_1, Q_2, \ldots$); ${}^{\#}P$ is the 'arity' of P. If we write $Py_1 \cdots y_n$, we suppose ${}^{\#}P = n$.

Formulae are defined as usual; together they form the set ***FORM***.

For convenience, we shall suppose ***FORM*** to contain only formulae A for which $FV(A) \cap BV(A) = \emptyset$.

When writing $A[B/C]$, we presume $FV(B) \cap BV(A) \subset FV(C)$.

If $x \in BV(A)$ (so $x \notin FV(A)$), $y \notin BV(A) \cup FV(A)$, then we consider $A[y/x]$ to be equal to A. We shall use this convention when, in manipulating with formulae and variables, a formulae A with $x \in BV(A)$ comes into the scope of a quantor $\forall x$, or is subjected to the substitution $[x/y]$; then we tacitly take $A[z/x]$ ($z \notin BV(A) \cup FV(A)$) instead of A.

3.2. DEFINITION. i) If $A \in$ ***FORM***, $FV(A) \subset \{y_1, \ldots, y_n\}$, $n \geq 0$, $y_1, \ldots, y_n$ all different, then $\lambda y_1 \cdots y_n \bullet A$ is an *n-ary predicate* abstracted from A.
ii) $\boldsymbol{PR}^n$ is the set of n-ary predicates. If $B \in \boldsymbol{PR}^n$, then ${}^{\#}B =_{\text{def}} n$.
iii) If $\lambda y_1 \cdots y_n \bullet A \in \boldsymbol{PR}^n$, then

$$\lambda y_1 \cdots y_n \bullet A(z_1, \ldots, z_n) \underset{\text{def}}{=} A[z_i/y_i]_{i=1,\ldots,n}.$$

When P is an n-ary predicate variable we identify P and $\lambda x_1 \cdots x_n \bullet Px_1 \cdots x_n$. We shall use the notation $A[B/C]$ also for substitution of the n-ary predicate B for the n-ary predicate C in A.

3.3. DEFINITION. i) If $A \in$ ***FORM***, $PV(A) \subset \{Q_1, \ldots, Q_k\}$, $FV(A) \subset \{y_1, \ldots, y_n\}$, k, n, ≥ 0 and $Q_1, \ldots, Q_k$, $y_1, \ldots, y_n$ all different, then $\lambda Q_1 \cdots Q_k y_1 \cdots y_n \bullet A$ is a *predicate operator of type* $(k, {}^{\#}Q_1, \ldots, {}^{\#}Q_k, n)$ obtained from A.
ii) ***PRO*** is the set of predicate operators.
iii) If $c = \lambda Q_1 \cdots Q_k y_1 \cdots y_n \bullet A \in$ ***PRO***, $B_i \in \boldsymbol{PR}^{{}^{\#}Q_i}$ $(i = 1, \ldots, k)$, then

$$c(B_1, \ldots, B_k, z_1, \ldots, z_n) \underset{\text{def}}{=} A[B_1, \ldots, B_k/Q_1, \ldots, Q_k][z_i/y_i]_{i=1,\ldots,n}.$$

3.4. DEFINITION. i) To every $A \in$ ***FORM***, we associate the predicate $\bar{A} =_{\text{def}} \lambda y_1 \cdots y_n \bullet A$, where $\{y_1, \ldots, y_n\} = FV(A)$, $y_1, \ldots, y_n$ are all different and ordered according to their leftmost occurrence in A.

ii) To every $A \in \mathbf{FORM}$, we associate the predicate operator $\underline{A} =_{\mathrm{def}}$ $=_{\mathrm{def}} \lambda Q_1 \cdots Q_k y_1 \cdots y_n \bullet A$, where $\{Q_1, \ldots, Q_k\} = PV(A)$, $\{y_1, \ldots, y_n\} = FV(A)$, $Q_1, \ldots, Q_k$ resp. $y_1, \ldots, y_n$ are all different and ordered according to their leftmost occurrence in A.

From now on, we suppose every connective or predicate operator c to be equivalent (disregarding the order of abstracted variables) to $\underline{A}$ for some formulae A (unless c has type $(k, 0, \ldots, 0, n)$ with $n \neq 0$). An example may justify this: if $\underline{A} = \lambda Q_1 \cdots Q_k y_1 \cdots y_n \bullet A$, then $\lambda Q_1 \cdots Q_i Q Q_{i+1} \cdots Q_k y_1 \cdots y_n \bullet A = \underline{A}'$, where A' is obtained from A by replacing the leftmost occurrence of Q_i by $Q_i \wedge \forall x (Qx \cdots x \rightarrow Qx \cdots x)$.

3.5. DEFINITION. Let $\mathbf{C} \subset \mathbf{Con} \cup \mathbf{PRO}$. We define the *fragment* $[\mathbf{C}]$ as the minimal set of $\mathbf{FORM}$ satisfying:

a) if P is a predicate variable, $^{\#}P = n$, then $Py_1 \cdots y_n \in [\mathbf{C}]$;

b) if $c \in \mathbf{C} \cap \mathbf{Con}^n$ and $A_1, \ldots, A_n \in [\mathbf{C}]$, then $c(A_1, \ldots, A_n) \in [\mathbf{C}]$;

c) if $A_1, \ldots, A_k$ are predicates abstracted from formulae of $[\mathbf{C}]$ and if $c \in \mathbf{C} \cap \mathbf{PRO}$ is a predicate operator of type $(k, {}^{\#}A_1, \ldots, {}^{\#}A_k, n)$, then $c(A_1, \ldots, A_k, y_1, \ldots, y_n) \in [\mathbf{C}]$.

If necessary, we make distinction between fragments of propositional and predicate logic by writing $[\ ]_p$ and $[\ ]_P$, respectively.

$\mathbf{FRAG}$ is the set of fragments of predicate logic.

REMARK. It is possible to give a more general definition of fragments of predicate logic, namely by dropping the condition $FV(A) \subset \{y_1, \ldots, y_n\}$ in the definition of predicates (Definition 3.2). With such a definition, we get

$$(1) \qquad \mathbf{FORM} \underset{c}{\equiv} [\neg, \wedge, \lambda P \bullet \forall x P x];$$

in the present situation, however, the right-hand side of (1) only contains those closed formulae in which no nested quantification occurs; but we do have

$$\mathbf{FORM} \underset{c}{\equiv} [\{\neg, \wedge\} \cup \{\lambda P_n x_1 \cdots x_{n-1} \bullet \forall x_n P_n x_1 \cdots x_n : n \in \mathbb{N}\}].$$

See also the remark after Definition 3.13.

For some proofs which proceed by formula induction, we need a measure for the complexity of a formula in a fragment.

3.6. DEFINITION. Let $\mathbf{f}$ be some fragment, $\mathbf{f} = [\mathbf{C}]$.

We define $\delta_f : \mathbf{f} \rightarrow \mathbb{N} \cup \{0\}$ as follows:

a) $\delta_f(A) = 0$ if A atomic;

b) if $c \in \mathbf{C}$, then $\delta_f(c(A_1, \ldots, A_k, y_1, \ldots, y_n)) = \max\{\delta_f(A_1), \ldots, \delta_f(A_k)\} + 1$.

CONVENTION. We write δ for δ_f if this can give no confusion; ditto for ϱ_f and σ_f, to be defined later.

3.7. DEFINITION. Let $\boldsymbol{F}\in\boldsymbol{FRAG}$. We say that *interpolation holds in* $\boldsymbol{F}$ iff:

$$\left.\begin{array}{l} A,B\in \boldsymbol{F}\\ \vdash_c A\to B\\ PV(A)\cap PV(B)\neq\emptyset\\ FV(A)\cap FV(B)\neq\emptyset\end{array}\right|\Rightarrow \exists I\in\boldsymbol{F}\text{ with }\left\{\begin{array}{l}\vdash_c(A\to I)\wedge(I\to B)\\ PV(I)\subset PV(A)\cap PV(B)\\ FV(I)\subset FV(A)\cap FV(B)\end{array}\right. \tag{2}$$

REMARK. For the reason why we added the condition $FV(A)\cap FV(B)\neq\emptyset$ in the premiss, see the remark at the end of 2.4.

3.8. Now we set out to define the translation T.

T works roughly as follows: if $\boldsymbol{f}\in\boldsymbol{Frag}$, then $T\boldsymbol{f}\in\boldsymbol{FRAG}$, and there are functions $\varrho:\boldsymbol{f}\to T\boldsymbol{f}$ and $\sigma: T\boldsymbol{f}\to\boldsymbol{f}$ satisfying

$$A,B\in\boldsymbol{f},\ \vdash_i A\to B\Rightarrow\vdash_c\varrho A\to\varrho B, \tag{3}$$

$$A,B\in T\boldsymbol{f},\ \vdash_c A\to B\Rightarrow\vdash_i\sigma A\to\sigma B. \tag{4}$$

To accomplish this, we need a function $\kappa:\boldsymbol{Form}\to\boldsymbol{FORM}$. A well-known candidate which appears to be fit for our purpose is defined by

$$\kappa(p_i)=\forall x_2(Rx_1x_2\to P_ix_2),\ i=1,2,\dots;\ \kappa\bot=\bot;\ \kappa\top=\top;$$

$$\kappa(A\wedge B)=\kappa A\wedge\kappa B;\ \kappa(A\vee B)=\kappa A\vee\kappa B;$$

$$\kappa(A\to B)=\forall x_n(Rx_1x_n\to(\kappa A\to\kappa B)[x_n/x_1]),\text{ where } n \text{ is the smallest index (of } x)>1\text{ not occurring in }\kappa A\to\kappa B.$$

κ translates every formula A of **IpC** into a formula κA, containing unary predicate variables P_i for every p_i in A, one binary predicate variable R and one free variable x_1; κA expresses that, if R is reflexive and transitive, then A is forced in node x_1 of the Kripke model $\boldsymbol{K}=\langle\boldsymbol{U},R,\Vdash\rangle$, where $\boldsymbol{U}$ is the universe over which our individual variables range and the forcing relation $\Vdash$ is determined by

$$x\Vdash p_i\underset{\text{def}}{=}P_ix,\ i=1,2,\dots.$$

See Kripke [K] for more information. In the same article, **IpC** is proved to be complete w.r.t. reflexive and transitive Kripke models, which implies

$$\vdash_i A\to B\Leftrightarrow\vdash_c\forall xRxx\wedge\forall xyz(Rxy\wedge Ryz\to Rxz)\to(\kappa A\to\kappa B), \tag{5}$$

so we are well on our way to (3) and (4).

Two obstacles are still before us:

1) the condition upon R in the right-hand side of (5); this will be eliminated by putting it in the translation ϱ;

2) κ creates two new parameters for which there is no equivalent in the argument, viz. x_1 and R. R in particular causes a lot of trouble in the fragment $T\boldsymbol{f}$; to cope with it, we have to extend the condition $Adm(R)$ on R in ϱ, and to

build up a machinery of definitions and lemmata before we can prove the desired result.

First we need a sentence $Adm(R)$, R a binary predicate, satisfying:

(6) **IpC** is complete w.r.t. the class of Kripke frames $\boldsymbol{K}=\langle \boldsymbol{U}, R\rangle$ which satisfy $Adm(R)$;

(7) $Adm(\lambda xy\bullet Px)\equiv Adm(\lambda xy\bullet Py)\equiv \perp$;

(8) $Adm(R)\wedge Adm(\check{R})\equiv \perp$ $(\check{R} \underset{\text{def}}{=} \lambda xy\bullet Ryx)$.

REMARK. The reasons we want $Adm(R)$ to satisfy (6), (7), (8) are the following. We use Adm in the predicate operators of the fragments $\boldsymbol{Tf}$ to enforce reflexiveness and transitivity of some binary predicate S. Now, if S is not essentially binary (i.e. S is obtained by vacuous abstraction), then (7) will cause formulae in which S is substituted in Adm to collapse. It will appear that R and $\check{R}$ are the only essentially binary predicates in $\boldsymbol{Tf}$; (8) reduces formulae in which R and $\check{R}$ both are substituted in (different instances of) Adm to simpler ones.

3.9. DEFINITION.

i) $Adm(R)=_{\text{def}} \forall xRxx\wedge \forall xyz(Rxy\wedge Ryz\rightarrow Rxz)\wedge \exists x\forall yRxy\wedge \forall x\exists y\neg Ryx$.

ii) $A\equiv_S B=_{\text{def}} \vdash_c Adm(S)\rightarrow(A\leftrightarrow B)$, S a binary predicate.

3.10. LEMMA. (6), (7), (8) hold for $Adm(R)$.

PROOF. (7), (8) are easily seen to hold.

Ad (6): If $Adm(R)$, then R is reflexive and transitive, hence $\langle \boldsymbol{U}, R\rangle$ is a Kripke frame for **IpC**; on the other hand, **IpC** is complete w.r.t. the class of finite Kripke trees (for a proof see [K]), and these are quickly transformed in structures with an order relation R satisfying $Adm(R)$: just add branches, if necessary. Conclusion: $Adm(R)$ satisfies (6). ☐

COROLLARY. If $A,B\in \boldsymbol{Form}$, then

$$\vdash_i A\rightarrow B\Leftrightarrow \vdash_c Adm(R)\rightarrow(\kappa A\rightarrow\kappa B).$$

3.11. LEMMA. If $A\in \boldsymbol{Form}$, then $\kappa A\equiv_R \forall yRx_1y\rightarrow\kappa A[y/x_1]$.

PROOF. An immediate consequence of the following property of Kripke models for **IpC**:

$$k\Vdash A \text{ and } Rkk'\Rightarrow k'\Vdash A,$$

which is proved by induction over the complexity of A. ☐

From the substitution property of **CPC** follows:

3.12. LEMMA. $A\equiv_S B\Rightarrow C\equiv_S C[A/B]$. ☐

Now we know enough about κ to define the translation T and the functions ϱ and σ between $\boldsymbol{f}$ and $\boldsymbol{Tf}$, and to prove their characteristics.

IMPORTANT CONVENTION. From now on, our language $\boldsymbol{L}_1$ is supposed to contain as predicate variables only $P_1, P_2, \ldots$ (all unary) and R (binary).

3.13. DEFINITION. i) The function $\tau : \boldsymbol{Con} \rightarrow \boldsymbol{PRO}$ is defined by:

$$\tau\underline{A} = \underline{\kappa A \wedge Adm(R)}.$$

ii) We define $T : \boldsymbol{Frag} \rightarrow \boldsymbol{FRAG}$ as follows:

$$T([\boldsymbol{C}]_p) = [\{\tau c : c \in \boldsymbol{C}\} \cup \{\Box\}]_P, \text{ where}$$

$$\Box \underset{\text{def}}{=} \tau(\lambda p \bullet p) \; (= \lambda PRx \bullet \forall y(Rxy \rightarrow Py) \wedge Adm(R)).$$

REMARK. It is obvious that all predicate operators of a fragment $\boldsymbol{Tf}$, $\boldsymbol{f} \in \boldsymbol{Frag}$, are of type $(n+1, 1, \ldots, 1, 2, 1)$, i.e. look like $\lambda P_1 \cdots P_n R x_1 \bullet A$, A some formula; a direct consequence of this, and of the condition $FV(A) \subset \{y_1, \ldots, y_n\}$ in the definition of predicates, is: all non-atomic formulae in $\boldsymbol{Tf}$ have at most one free variable, so R and $\check{R}$ are the only essentially binary predicates in $\boldsymbol{Tf}$ (as was announced in the remark at the end of 3.8). Without the condition on $FV(A)$, this would not be the case; also the next lemma, which is crucial for the rest of our argument, would vanish.

CONVENTION. $A = A(S)$ and $A = A(S, x_i)$, where $S \in \boldsymbol{PR}^2$ and $A \in \boldsymbol{Tf}$ for some $\boldsymbol{f} \in \boldsymbol{Frag}$, mean: A is not atomic and of the form $c(A_1, \ldots, A_n, S, x_i)$.

It is clear that we have:

(9) $\qquad A \in \boldsymbol{Tf},\ A = A(S) \Rightarrow\ \vdash_c A \rightarrow Adm(S).$

3.14. LEMMA. If $A, B \in \boldsymbol{Tf}$, $A = A(S_1)$, $B = B(S_2)$, then

$$\vdash_c A \rightarrow B \Rightarrow A \equiv \bot \text{ or } S_1 = S_2 = R \text{ or } S_1 = S_2 = \check{R}.$$

PROOF. R and $\check{R}$ are the only binary predicates in $\boldsymbol{Tf}$ without vacuous abstraction, so with Lemma 3.10 we have $R \neq S_1 \neq \check{R} \Rightarrow A(S_1) \equiv \bot$.

Now let $A \not\equiv \bot$, then $S_1 = R$ or $S_1 = \check{R}$. By (9), we have $\vdash_c A \rightarrow Adm(S_1) \wedge \wedge Adm(S_2)$; together with Lemma 3.1 and $A \not\equiv \bot$ this gives $S_1 = S_2$. $\Box$

3.15. LEMMA. If $A \in \boldsymbol{Tf}$, then

$$A = A(S, x_i) \Rightarrow A \equiv_c \Box(\bar{A}, S, x_i).$$

PROOF. By Lemma 3.14, we only have to consider $S = R$ or $S = \check{R}$. We treat $S = R$; $S = \check{R}$ goes analogous. First we observe

(10) $\qquad \vdash A' \Rightarrow\ \vdash A'[\bar{B}/P]$, P a predicate variable, ${}^{\#}\bar{B} = {}^{\#}P$.

Now

$$\begin{aligned} A &= \kappa B[A_1, \dots, A_n, R/P_1, \dots, P_n, R][x_i/x_1] \wedge Adm(R) \\ &\equiv_R \kappa B[A_1, \dots, A_n/P_1, \dots, P_n][x_i/x_1] \\ &\equiv_R \forall y R x_i y \rightarrow (\kappa B[A_1, \dots, A_n/P_1, \dots, P_n][y/x_1]) \\ &\qquad \text{(by Lemma 3.11 and (10))} \\ &\equiv_R \square(\bar{A}, R, x_i); \end{aligned}$$

since $\vdash A \rightarrow Adm(R)$, $\vdash \square(\bar{A}, R, x_i) \rightarrow Adm(R)$, we have $A \equiv \square(\bar{A}, R, x_i)$. ☐

3.16. DEFINITION. Let $\boldsymbol{f} = [\boldsymbol{C}] \in \boldsymbol{Frag}$. We define the function $\varrho_f : \boldsymbol{f} \rightarrow \boldsymbol{Tf}$ as follows:

i) $\varrho_f(p_i) = \square(P_i, R, x_1)$;

ii) if $c \in \boldsymbol{C} \cap \boldsymbol{Con}^n$, $B_1, \dots, B_n \in \boldsymbol{f}$, then

$$\varrho_f(c(B_1, \dots, B_n)) = (\tau c)(\overline{\varrho_f B}_1, \dots, \overline{\varrho_f B}_n, R, x_1).$$

The following lemma characterizes ϱ:

3.17. LEMMA. $A \in [\boldsymbol{C}]_p \Rightarrow \kappa A \equiv_R \varrho A$.

PROOF. Induction over δA:

i) $\delta A = 0$. Then $A = p_i$ for some i: $\kappa A \equiv_R \kappa p_i \wedge Adm(R) = \varrho A$.

ii) $\delta A > 0$. Suppose $A = \underline{B}(B_1, \dots, B_n)$, $\underline{B} = \lambda p_1 \cdots p_n \bullet B \in \boldsymbol{C}$, $B_1, \dots, B_n \in [\boldsymbol{C}]_p$, $\delta B_1, \dots, \delta B_n < \delta A$. Induction hypothesis: $\kappa B_i \equiv_R \varrho B_i$, $i = 1, \dots, n$. For simplicity we suppose $n = 1$. Now

$$\begin{aligned} \varrho A &= (\tau \underline{B})(\overline{\varrho B}_1, R, x_1) && \text{(def. of } \varrho\text{)} \\ &= \underline{\kappa B \wedge Adm(R)}(\overline{\varrho B}_1, R, x_1) && \text{(def. of } \tau\text{)} \\ &\equiv_R \underline{\kappa B}(\overline{\varrho B}_1, R, x_1) && \text{(def. of } \equiv_R\text{)} \\ &= \kappa B[\overline{\varrho B}_1/P_1] && \text{(def. of connective)} \\ &\equiv_R \kappa B[\overline{\kappa B}_1/P_1] && \text{(ind. hyp.)} \end{aligned}$$

$$= \kappa B[\forall x_2 R x_1 x_2 \rightarrow \kappa B_1[x_2/x_1] / \forall x_2 R x_1 x_2 \rightarrow P_1 x_2]$$ (by definition of κ, P_1 occurs only in subformulae of the form $\forall x_2 R x_1 x_2 \rightarrow \rightarrow P_1 x_2$)

$$\begin{aligned} &\equiv_R \kappa B[\kappa B_1 / \forall x_2 R x_1 x_2 \rightarrow P_1 x_2] && \text{(Lemma 3.11, 3.12)} \\ &= \kappa(B[B_1/p_1]) && \text{(def. of } \kappa\text{)} \\ &= \kappa(\underline{B}(B_1)) = \kappa A. \quad \square \end{aligned}$$

Now we simply prove the property of ϱ which was mentioned in (3):

3.18. LEMMA. If $A, B \in \boldsymbol{f}$, then

$$\vdash_i A \rightarrow B \Rightarrow \vdash_c \varrho A \rightarrow \varrho B.$$

PROOF. $\vdash_i A \to B \Rightarrow \vdash_c Adm(R) \to (\kappa A \to \kappa B)$ (Lemma 3.12)

$\Rightarrow \vdash_c Adm(R) \to (\varrho A \to \varrho B)$ (Lemma 3.17)

$\Rightarrow \vdash_c \varrho A \to \varrho B$ (because of (9)). □

3.19. DEFINITION. Let $\boldsymbol{f} = [\boldsymbol{C}]_p \in \boldsymbol{Frag}$.

We define $\sigma_{\boldsymbol{f}}$: $\boldsymbol{Tf} - \{Rx_ix_j : i \neq j\} \to [\boldsymbol{C} \cup \{\bot, \top\}]_p$ as follows:

i) $\sigma P_i x_j = p_i$;

ii) $\sigma R x_i x_i = \top$;

iii) $\sigma(\Box(B, S, x_j)) = \sigma B$ *if* B *atomic and* $(S = R$ *or* $S = \check{R})$
or if $B = B(R)$ *and* $S = R$
or if $B = B(\check{R})$ *and* $S = \check{R}$,
$= \bot$ *otherwise;*

iv) $\sigma((\tau c)(B_1, \ldots, B_n, S, x_j)) = c(A_1, \ldots, A_n)$ *if* $S = R$ *or* $S = \check{R}$
$= \bot$ *otherwise,*
where $A_i = \sigma B_i$ *if* B_i *atomic or* $B_i = B_i(S)$
$= \bot$ *otherwise,* $i = 1, \ldots, n$.

A direct consequence of Definition 3.16 and 3.19 is

3.20. LEMMA. $A \in \boldsymbol{f} \Rightarrow \sigma\varrho A = A$.

PROOF. Simple induction over δA:

i) A atomic: $\sigma\varrho A = \sigma\varrho p_i = \sigma(\Box(P_i, R, x_1)) = \sigma(P_i x_1) = p_i = A$.

ii) $A = c(A_1, \ldots, A_n)$: $\sigma\varrho A = \sigma((\tau c)(\varrho A_1, \ldots, \varrho A_n, R, x_1) = c(\sigma\varrho A_1, \ldots, \sigma\varrho A_n) = c(A_1, \ldots, A_n) = A$ (the third equality is a consequence of the induction hypothesis). □

So σ is a left inverse of ϱ. At the same time, it nearly is a right inverse:

3.21. LEMMA. Let $A = A(S, x_1) \in \boldsymbol{Tf}$. Then

i) $S = R \Rightarrow \varrho\sigma A \equiv A$;

ii) $S = \check{R} \Rightarrow (\varrho\sigma A)[\check{R}/R] \equiv A$;

iii) $R \neq S \neq \check{R} \Rightarrow \varrho\sigma A \equiv A \equiv \bot$.

PROOF. Induction over δA; $A = A(S)$, so $\delta A > 0$. We only prove (i); (ii) goes analogous, and (iii) is trivial, for then $Adm(S) \equiv \bot$.

We distinguish two cases:

a) $A = \Box(\bar{B}, R, x_1)$.

a.1) $B = P_i x$: then $\varrho\sigma A = \varrho p_i = \Box(P_i, R, x_1) = A$.

a.2) $B = Rxx$: then $\varrho\sigma A = \varrho\top = \forall y(Rxy \to \top) \wedge Adm(R) \equiv Adm(R)$, and $A = \forall y(Rxy \to Ryy) \wedge Adm(R) \equiv \top \wedge Adm(R)$, for $\vdash Adm(R) \to \forall y Ryy$, so $\varrho\sigma A = A$.

a.3) $B = B(R)$: then, by induction hypothesis, $\varrho\sigma B \equiv B$; also (from the definition of σ) $\sigma A = \sigma B$, so $B \equiv \varrho\sigma A$; because of Lemma 3.15, $B \equiv \Box(\bar{B}, R, x_1) = A$, so $\varrho\sigma A = A$.

a.4) $B = B(S', y)$, $S' \neq R$, so $B = B' \wedge Adm(S')$; now

$$\begin{aligned} A &= \forall x_2(Rx_1x_2 \to (B' \wedge Adm(S'))[x_2/y]) \wedge Adm(R) \\ &\equiv \forall x_2(Rx_1x_2 \to (B'[x_2/y] \wedge Adm(S') \wedge Adm(R))) \wedge Adm(R) \\ &\equiv \forall x_2(Rx_1x_2 \to \bot) \wedge Adm(R) \quad \text{(by Lemma 3.10)} \\ &\equiv \neg \exists x_2 Rx_1x_2 \wedge Adm(R) \equiv \bot, \text{ for } \vdash Adm(R) \to \forall x_1 \exists x_2 Rx_1x_2; \end{aligned}$$

also $\varrho\sigma A = \varrho\bot \equiv \bot$, so $\varrho\sigma A \equiv A$.

b) $A = (\tau c)(\bar{B}_1, \ldots, \bar{B}_n, R, x_1)$. Without loss of generality, we suppose that $n = 3$, B_1 is atomic, $B_2 = B_2(R)$ and $B_3 = B_3(S')$ with $S' \neq R$. Now

$$\begin{aligned} A &\equiv (\tau c)(\bar{B}_1, \bar{B}_2, \overline{B_3 \wedge Adm(S')}, R, x_1) \wedge Adm(R) \\ &\equiv (\tau c)(\bar{B}_1, \bar{B}_2, \bar{\bot}, R, x_1) \wedge Adm(R) \quad (\text{here } \bar{\bot} = \lambda x \bullet \bot) \end{aligned}$$

and

$$\begin{aligned} \varrho\sigma A &= \varrho(c(\sigma B_1, \sigma B_2, \bot)) \\ &= \varrho(c(p, \sigma B_2, \bot)) \quad (p = p_i \text{ (if } B_1 = P_i x) \text{ or } \top \text{(if } B_1 = Rxx)) \\ &= (\tau c)(\overline{\varrho p}, \overline{\varrho\sigma B}_2, \overline{\varrho\bot}, R, x_1) \\ &\equiv (\tau c)(\bar{B}_1, \bar{B}_2, \bar{\bot}, R, x_1) \quad (\text{ind. hyp. and def. of } \varrho), \end{aligned}$$

so $\varrho\sigma A \equiv A$. ☐

Now it is simple to prove the desired property of σ:

3.22. LEMMA. If $f \in \boldsymbol{Frag}$, $A, B \in Tf$, $A = A(R, x_1)$, $B = B(R, x_1)$, then:

$$\vdash_c A \to B \Rightarrow \vdash_i \sigma A \to \sigma B.$$

PROOF.

$$\begin{aligned} \vdash_c A \to B &\Rightarrow \vdash_c \varrho\sigma A \to \varrho\sigma B && \text{(Lemma 3.21)} \\ &\Rightarrow \vdash_c Adm(R) \to (\kappa\sigma A \to \kappa\sigma B) && \text{(Lemma 3.17)} \\ &\Rightarrow \vdash_i \sigma A \to \sigma B && \text{(corollary of Lemma 3.10).} \end{aligned}$$ ☐

Finally, we can prove the theorem we did all the work for:

3.23. THEOREM. Let $f = [\boldsymbol{C}]_p \in \boldsymbol{Frag}$. Then
i) interpolation holds in $f \Rightarrow$ interpolation holds in Tf;
ii) interpolation holds in $Tf \Rightarrow$ interpolation holds in $[\boldsymbol{C} \cup \{\top, \bot\}]_p$.

PROOF. i) Suppose $A, B \in Tf$, $\vdash_c A \to B$, $PV(A) \cap PV(B) \neq \emptyset \neq FV(A) \cap FV(B)$.

Case 1. A is atomic. Then B is atomic, for B not atomic implies $B \equiv B \wedge \wedge Adm(S)$ for some $S \in \boldsymbol{PR}^2$, and then $\vdash A \to Adm(S)$ which, together with A atomic, yields $A = \bot$, so $PV(A) = \emptyset$. From A, B atomic and $\vdash A \to B$ follows $A = B$, so A is interpolant.

Case 2. B is atomic, A not. We have the following subcases:
a) $B = P_i x_j$ or $B = Rx_i x_i$. Then B is interpolant.
b) $B = Rx_i x_j,\ i \neq j$.
b.1) $A = A(R, x_j)$. Then $\vdash A \to \exists y \forall x Rxy$. But $A \equiv A \wedge Adm(R)$, so $\vdash A \to \forall y \exists x \neg Rxy$; hence $A \equiv \bot$. A is not atomic, so $A = c(A_1, \dots, A_n, R, x_j)$ for some $c \in \boldsymbol{C}$. Now

$$A_0 \underset{\text{def}}{=} c(\overline{Rxx}, \dots, \overline{Rxx}, \lambda xy \bullet Rxx, x_j)$$

is an interpolant, for $A_0 \equiv Adm(\lambda xy \bullet Rxx) \equiv \bot$.
b.2) $A = A(R, x_i)$. Then $\vdash A \to \forall y Rx_i y$. Together with $\vdash A \to (\forall y Rx_i y \to A[y/x_i])$ (a consequence of Lemma 3.15), this gives $\vdash A \to \forall x_i A$, which yields even $\vdash A \to \forall x_i y Rx_i y$; since $A \equiv A \wedge Adm(R)$ and $\vdash Adm(R) \to \forall x \exists y \neg Ryx$, we now get $A \equiv \bot$, and A_0 is interpolant.
b.3) $A = A(\check{R}, x_j)$. Analogous to (b.2).
b.4) $A = A(\check{R}, x_i)$. Analogous to (b.1).
b.5) $A = A(S),\ R \neq S \neq \check{R}$. Then $A \equiv \bot$ and A_0 is interpolant.

Case 3. $A = A(S_1, x_i),\ B = B(S_2, x_i)$. According to Lemma 3.14, there are three possibilities:
a) $S_1 = S_2 = R$. We take for simplicity $i = 1$ ($i \neq 1$ requires some substitution). $\vdash_c A \to B$, hence, with Lemma 3.22, $\vdash_i \sigma A \to \sigma B$. Let $I \in \mathbf{f}$ be an interpolant of this last implication. Then (Lemma 3.18) $\vdash_c \varrho\sigma A \to \varrho I$ and $\vdash_c \varrho I \to \varrho\sigma B$. Now Lemma 3.21 says $\varrho\sigma A \equiv A,\ \varrho\sigma B \equiv B$, so ϱI is interpolant for $\vdash_c A \to B$.
b) $S_1 = S_2 = \check{R}$. Analogous to (a).
c) $A \equiv \bot$. Then A_0 is interpolant.

ii) Suppose $A, B \in [\boldsymbol{C} \cup \{\top, \bot\}]_p,\ \vdash_i A \to B,\ PV(A) \cap PV(B) \neq \emptyset$.

Case 1. A (or B) is atomic. Then A (or B) is interpolant.

Case 2. A nor B is atomic. By Lemma 3.18, $\vdash_c \varrho A \to \varrho B$. Let $I \in \mathbf{Tf}$ be an interpolant of this last implication. Then $I \equiv \bot$ (and $\bot$ is interpolant for $A \to B$), or I is not atomic (for $\vdash_c I \to \varrho B$, so $\vdash_c I \to Adm(R)$). In this last case, $\vdash_i \sigma\varrho A \to \sigma I$ and $\vdash_i \sigma I \to \sigma\varrho B$, because of Lemma 3.22. But (Lemma 3.20) $\sigma\varrho A = A,\ \sigma\varrho B = B$, hence σI is interpolant for $A \to B$. □

COROLLARY. There are $2^{\aleph_0}$ fragments of **CPC** for which interpolation fails.

PROOF. Follows immediately from Theorem 2.18 and 3.23(ii). □

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