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Analytic System Problems and *J*-Lossless Coprime Factorization for Infinite-Dimensional Linear Systems

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ABSTRACT

This paper extends the coprime factorization approach to the synthesis of internally stabilizing controllers satisfying an $\mathbf{H}_\infty$-norm bound to a class of systems with irrational transfer matrices. Using the coprime factorization description, the $\mathbf{H}_\infty$-control problem can be reduced to two stable analytic system problems. Such problems have solutions if and only if a certain J-lossless factorization exists. The full $\mathbf{H}_\infty$-synthesis problem is shown to be equivalent to the solution of two nested J-lossless factorizations. If the irrational transfer matrix has a state-space realization, then the known state-space formulas for the $\mathbf{H}_\infty$-control problem may be recovered using the relationship between J-lossless factorizations and solutions of Riccati equations. However, the results derived here are valid for a larger class of infinite-dimensional systems. © Elsevier Science Inc., 1997

1. INTRODUCTION

This paper uses a coprime factorization approach to the synthesis of internally stabilizing controllers satisfying an $\mathbf{H}_\infty$-norm bound. This reveals the problem to be a stable analytic problem, which motivates our consideration of a general class of analytic system problems of the following type:

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Given $G \in \mathbf{H}_\infty$, find all $Q_1, Q_2 \in \mathbf{H}_\infty$ such that

$$\begin{bmatrix} R \\ I \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \qquad \|R\|_\infty < \gamma.$$

These problems were first motivated and discussed in Helton et al. [41]. Helton first introduced these problems. They cover a very wide class of problems, for example, the Nehari problem ($G_{11} = I$, $G_{21} = 0$, $G_{22} = I$), the model matching problem ($G_{21} = 0$, $G_{22} = I$), and $\mathbf{H}_\infty$-control problems ($G \in H_\infty$). Such problems have been solved by various techniques, such as the operator-theoretic ones in Ball and Helton [6], Flamm and Mitter [27], and Foias and Tannenbaum [28–32]; interpolation, factorization approaches in Ball, Helton, and Verma [7] and Ball and Cohen [5]; J-spectral factorization methods in Green [39] and Green, Glover, Limebeer, and Doyle [40]; and direct state-space methods in Doyle et al. [25], Glover [36], Glover and Doyle [37], and van Keulen [44].

Each of these approaches has its own merits, and each offers different insights into analytic system problems. The approach we take here is the one taken in Green [39], where he showed that the general analytic system problem is equivalent to a J-spectral factorization problem, and in the special case that G is stable, it is equivalent to the existence of an invertible matrix $W \in \mathbf{H}_\infty$ such that GW^{-1} is J-lossless. In Green et al. [40] the authors showed that the latter is equivalent to the existence of a stabilizing, nonnegative solution to a nonstandard algebraic Riccati equation and so rederived the known state-space solution [25, 37] to the $\mathbf{H}_\infty$-control problem. One advantage of this approach is that, at the transfer-matrix level, it gives a solution to both the discrete-time and the continuous-time case for finite-dimensional systems.

In this paper, we shall show that this approach extends to a large class of infinite-dimensional systems. The stable analytic problem is solved for transfer matrices with an impulse response h satisfying $\int_0^\infty e^{\varepsilon t}|h(t)|\,dt < \infty$. The underlying reason for this choice for the class of infinite-dimensional systems is that there is a well-developed theory of Wiener-Hopf factorization and for the Nehari problem for the Wiener algebra [systems with an impulse response satisfying $\int_{-\infty}^{\infty}|h(t)|\,dt < \infty$]; see for example Gohberg et al. [38] and Ball and Helton [3, 4]. The reason for the further restriction on the impulse response is that we also appeal to coprime-factorization results from BGK [10, 11], and the stable Wiener algebra with $\mathbf{L}_1(0,\infty)$-integrable impulse responses does not have satisfactory coprime-factorization properties.

We remark that our class does not permit feedthrough delay terms like e^{-s}, and so it does not include the work in [27–32]. On the other hand, our class does include systems with a state-space realization (A, B, C, D), where A is the infinitesimal generator of a C_0-semigroup, and B and C are allowed to be unbounded to a certain degree. For example, the Pritchard-Salamon class studied in Curtain and Ran [19] is included, and this class allows for very general delay systems and many systems generated by partial differential equations. For an explicit realization of systems with impulse responses satisfying $\int_0^\infty e^{\varepsilon t}|h(t)|\,dt < \infty$, see BGK [10].

An important application of these results on J-lossless coprime factorization is to $\mathbf{H}_\infty$-optimal control problems, and this was carried out for the rational case in Green [39] and [40], enabling state-space formulas for the stabilizing controllers to be obtained. The key to this step was the relationship between J-lossless factorization and the nonnegative, stabilizing solutions to certain Riccati equations. This has been extended to the infinite-dimensional case for bounded realizations in Curtain and Rodriguez [22] and for the Pritchard-Salamon class in Weiss [48]. This means that one can follow the approach in Green [39, Section 6] to obtain the full state-space solution to the $\mathbf{H}_\infty$-optimal control problem for infinite-dimensional systems with bounded input and output operators. Since the steps are all algebraic and a straightforward generalization of those in Green [39, Section 6], we have chosen not to rederive these formulas, but to refer to the alternative derivations in van Keulen [44].

However, obtaining Riccati-equation solutions is not the motivation of this paper. Rather, our results are of interest because they present a completely different approach which applies to a larger class of systems than the Pritchard-Salamon class. For example, they include the class of systems described by parabolic differential equations with Dirichlet boundary control (see Curtain and Ichikawa [18]). Although some special $\mathbf{H}_\infty$-optimal control problems have been recently solved for such systems (see McMillan and Triggiani [45, 46]), the general measurement case has not been resolved. Indeed, a close observation of the two candidate Riccati equations for this case shows that it is unlikely that both will be well posed for parabolic systems with point observation and Dirichlet boundary control. On the other hand, our results are valid for such systems.

The main justification for the coprime factorization approaches is that there are many transfer matrices for which the J-spectral factorization problem cannot be translated into well-posed Riccati equations. So the challenging problem which remains is to develop techniques for solving J-factorization problems in the frequency domain without resorting to Riccati equations.

Since this paper basically generalizes the proofs in Green [39] to a larger class of systems, we have omitted proofs which are straightforward extensions. Of course, extensions to infinite dimensions can be subtle or even false, so for the more discerning reader we have prepared an internal report [23] which includes all the details.

2. NOTATION AND PRELIMINARIES

$\bar{\sigma}(A)$ denotes the maximum singular value of the matrix A and the matrix norm.

$\mathbb{C}_+ = \{s \in \mathbb{C} \mid s + \bar{s} > 0\} \cup \{\infty\}$.

$j\mathbb{R} = \{s \in \mathbb{C} \mid s + \bar{s} = 0\} \cup \{\infty\}$.

$\mathbf{L}_\infty^{p \times q} = \{F : j\mathbb{R} \to \mathbb{C}^{p \times q} \mid \|F\|_\infty = \operatorname{ess\,sup}_{\omega \in \mathbb{R}} \bar{\sigma}(F(j\omega)) < \infty\}$.

$\mathbf{H}_\infty^{p \times q}$ is the space of bounded holomorphic functions from $\mathbb{C}_+$ to $\mathbb{C}^{p \times q}$ with the norm $\|F\|_\infty = \sup_{s \in \mathbb{C}_+} \bar{\sigma}(F(s)) < \infty$.

$\mathbf{H}_2^{p \times q}$ is the space of holomorphic functions $F : \mathbb{C}_+ \to \mathbb{C}^{p \times q}$ such that

$$\|f\|_2^2 = \sup_{s \in \mathbb{C}_+} \int_{-\infty}^{\infty} \bar{\sigma}^2(F(s + j\omega))\, d\omega < \infty.$$

$\mathbf{H}_\infty^{p \times p}$ is a Banach algebra with identity, and $F \in \mathbf{H}_\infty^{p \times p}$ is invertible over $\mathbf{H}_\infty^{p \times p}$ if and only if

$$\inf\left\{\det F(s) \middle| s \in \overline{\mathbb{C}_+}\right\} > 0. \tag{2.1}$$

$$\mathbf{L}_2^p = \left\{f : j\mathbb{R} \to \mathbb{C}^p \middle| \|f\|_2 = \left(\int_{-\infty}^{\infty} |f(j\omega)|^2\, d\omega\right)^{1/2} < \infty\right\}.$$

$M^\sim(s) := [M(-\bar{s})]^*$.

Finally,

$$J_{pq}(\gamma) := \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} \quad \text{for real } \gamma > 0. \tag{2.2}$$

A constant matrix A is said to be J-unitary if

$$A^* J A = J.$$

J-Lossless Matrix Functions

A partitioned matrix $M \in \mathbf{H}_\infty^{(l+m) \times (q+m)}$ is called J-lossless if

$$\begin{aligned} M(s)^* J_{lm}(\gamma) M(s) &\leqslant J_{qm}(\gamma) \quad &&\text{on } \mathbb{C}_+, \\ M(j\omega)^* J_{lm}(\gamma) M(j\omega) &= J_{qm}(\gamma) \quad &&\text{for } \omega \in \mathbb{R}. \end{aligned} \tag{2.3}$$

M is called conjugate J-lossless if $M(s)^*$ is J-lossless.

Notice that a J-lossless matrix need not be square; it is tall ($l \geqslant q$), and it is assumed to be partitioned so that the 2, 2 corner is square.

Linear Fractional Maps

If P is a meromorphic function on $\overline{\mathbb{C}_+}$ with values in $\mathbb{C}^{(l+m)\times(p+q)}$ and K is a meromorphic function on $\overline{\mathbb{C}_+}$ with values in $\mathbb{C}^{q\times m}$, then the linear fractional map $\mathscr{F}(P, K)$ is defined by

$$\mathscr{F}(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}. \tag{2.4}$$

Algebras of Transfer Functions

We shall be concerned with several distinct classes of irrational transfer functions. First we define two classes of stable transfer functions via their impulse responses:

$$\mathscr{A} = \left\{ f : \mathbb{R} \to \mathbb{C} \,|\, f(t) = \begin{cases} f_a(t) + f_0\delta(t), & t \geqslant 0, \\ 0, & t < 0, \end{cases} \right.$$

$$\left. \text{where } f_0 \in \mathbb{C} \text{ and } \int_0^\infty |f_a(t)|\,dt < \infty \right\}, \tag{2.5}$$

$$\hat{\mathscr{A}} = \{\hat{f} \,|\, f \in \mathscr{A}\}, \tag{2.6}$$

where $\hat{}$ denotes the Laplace transform, which is defined on $\overline{\mathbb{C}_+}$. $\mathscr{A}$ and $\hat{\mathscr{A}}$ are commutative algebras with identity. Let

$$\mathscr{A}_- = \left\{ f : \mathbb{R} \to \mathbb{C} \,|\, f(t) = \begin{cases} f_a(t) + f_0\delta(t), & t \geqslant 0, \\ 0, & t < 0, \end{cases} \right.$$

$$\left. \text{where } f_0 \in \mathbb{C} \text{ and } \int_0^\infty e^{\varepsilon t}|f_a(t)|\,dt < \infty \text{ for some } \varepsilon > 0 \right\}, \tag{2.7}$$

$$\hat{\mathscr{A}}_- = \{\hat{f} \,|\, f \in \mathscr{A}_-\}. \tag{2.8}$$

We remark that the symbols $\hat{\mathscr{A}}$ and $\hat{\mathscr{A}}_-$ are often used to denote a larger algebra with infinitely many delay terms, but that class is not appropriate here.

Transfer functions in $\hat{\mathscr{A}}$ and $\hat{\mathscr{A}}_-$ have the following useful properties (see Callier and Desoer [12–14]):

$$\hat{f} \in \hat{\mathscr{A}} \text{ is holomorphic on } \mathbb{C}_+ \text{ and continuous on } j\mathbb{R}; \tag{2.9}$$

$$\hat{f} \in \hat{\mathscr{A}}_- \text{ is holomorphic on } \overline{\mathbb{C}_+}; \tag{2.10}$$

$$\hat{\mathscr{A}}_- \subset \hat{\mathscr{A}} \subset \mathbf{H}_\infty \quad \text{and} \quad \|\hat{f}\|_\infty \leqslant f_0 + \int_0^\infty |f_a(t)|\, dt; \tag{2.11}$$

$$\hat{f} \in \hat{\mathscr{A}} (\text{or } \hat{\mathscr{A}}_-) \text{ is invertible over } \hat{\mathscr{A}} (\text{or } \hat{\mathscr{A}}_-) \text{ iff (2.1) holds (i.e., iff it is invertible over } \mathbf{H}_\infty); \tag{2.12}$$

$$\hat{f} \in \hat{\mathscr{A}} \text{ has the limit } f_0 \text{ at infinite, i.e., } \{|\hat{f}(s) - f_0| \,|\, |s| \geqslant \rho,\ s \in \overline{\mathbb{C}_+}\} \to 0 \text{ as } \rho \to \infty. \tag{2.13}$$

Our class of unstable irrational systems is the algebra of fractions

$$\hat{\mathscr{B}} := \hat{\mathscr{A}}_- \left[\hat{\mathscr{A}}_\infty\right]^{-1},$$

where $\hat{\mathscr{A}}_\infty$ is the subclass of transfer functions in $\hat{\mathscr{A}}_-$ with the property that they are bounded away from zero at infinity. In view of the property (2.13), $\hat{\mathscr{A}}_\infty$ is the subset of $\hat{\mathscr{A}}_-$ with a nonzero limit at ∞ (the elements of $\hat{\mathscr{A}}_-$ that are nonzero at infinity), and $f \in \hat{\mathscr{B}}$ has only finitely many unstable poles in $\overline{\mathbb{C}_+}$.

For theoretical reasons, we also need to consider the larger class of unstable transfer functions, the *Wiener algebra*,

$$\hat{\mathscr{W}} = \left\{\hat{f} \in \mathbf{L}_\infty \middle| \hat{f} = \hat{f}_1 + \hat{f}_2, \text{ where } \hat{f}_1, \hat{f}_2^{\sim} \in \hat{\mathscr{A}}\right\}. \tag{2.14}$$

$\hat{\mathscr{W}}$ is a Banach algebra under pointwise addition, multiplication, and scalar multiplication, and

$$\hat{f}(j\omega) \text{ is bounded and continuous for } \omega \in \mathbb{R}; \tag{2.15}$$

$f \in \hat{\mathscr{W}}$ is invertible over $\hat{\mathscr{W}}$ if and only if

$$\hat{f}(j\omega) \neq 0, \qquad \omega \in \mathbb{R} \cup \{\infty\}; \tag{2.16}$$

$\hat{f}(j\omega)$ is bounded and continuous for $\omega \in \mathbb{R}$; and

$$\hat{f} \text{ has a well-defined limit at infinity.} \tag{2.17}$$

Although $\hat{f}$ may be defined outside $j\mathbb{R}$, this is not important for its properties as a member of $\mathscr{W}$. However, for certain applications the following subalgebra in which the functions are defined and holomorphic on a strip a surrounding $j\mathbb{R}$ is important:

$$\hat{\mathscr{W}}_- = \left\{ \hat{f} \in \mathbf{L}_\infty \middle| \hat{f} = \hat{f}_1 + \hat{f}_2, \text{ where } \hat{f}_1, \hat{f}_2^{\sim} \in \hat{\mathscr{A}}_- \right\}. \tag{2.18}$$

For more properties of $\hat{\mathscr{W}}$ see Gohberg et al. [38, Chapter XII].

Classes of Transfer Matrices

We use the notation $\hat{\mathscr{A}}^{p\times q}$ for the class of $p \times q$ matrices with entries in $\hat{\mathscr{A}}$, and similarly for $\hat{\mathscr{A}}_-^{p\times q}$, $\hat{\mathscr{B}}^{p\times q}$, and $\hat{\mathscr{W}}^{p\times q}$. Where the size of the matrix is unimportant we use $\mathscr{M}\hat{\mathscr{A}}$ to denote a matrix of any size with entries in $\mathscr{M}\hat{\mathscr{A}}$, and similarly for $\mathscr{M}\mathbf{H}_\infty$, $\mathscr{M}\mathbf{L}_\infty$, $\mathscr{M}\hat{\mathscr{B}}$, etc.

We summarize the relevant properties of transfer matrices in $\hat{\mathscr{A}}$, $\mathscr{M}\hat{\mathscr{B}}$, and $\mathscr{M}\hat{\mathscr{W}}$ from Callier and Desoer [12–14] and Gohberg et al. [38, Chapters XII, XXIX, XXX].

$G \in \hat{\mathscr{A}}_-^{p\times p}$ is invertible over $\hat{\mathscr{A}}_-^{p\times p}$ if and only if $\det G$ is invertible over $\hat{\mathscr{A}}_-$ if and only if G is invertible over $\mathbf{H}_\infty^{p\times p}$ [see (2.12)]; (2.19a)

$G \in \mathscr{M}\hat{\mathscr{B}}$ has the representation $G = G_1 + G_2$, where $G_1 \in \mathscr{M}\hat{\mathscr{A}}_-$ and G_2 is a strictly proper rational transfer matrix with all its poles in $\overline{\mathbb{C}_+}$; G has the well-defined limit $G_1(\infty)$ at infinity, the constant part of G_1; (2.19b)

$G \in \hat{\mathscr{B}}^{p\times p}$ is invertible over $\hat{\mathscr{B}}^{p\times p}$ if and only if $G(\infty)$ is a nonsingular matrix; (2.19c)

$M, N \in \mathscr{M}\hat{\mathscr{A}}_-$ are right-coprime over $\mathscr{M}\hat{\mathscr{A}}_-$ if there exist $\hat{X}, \hat{Y} \in \mathscr{M}\hat{\mathscr{A}}_-$ such that $\hat{X}M - \hat{Y}N = I$ in $\overline{\mathbb{C}_+}$, (2.20) or equivalently, $\begin{bmatrix} M(s) \\ N(s) \end{bmatrix}$ has full column rank for every $s \in \overline{\mathbb{C}_+}$ (similarly for left-coprime);

$G \in \mathcal{M}\hat{\mathcal{B}}$ possesses a right-coprime factorization (r.c.f.) $G = NM^{-1}$ over $\mathcal{M}\hat{\mathcal{A}}_-$, where N, M, $\hat{X}, \hat{Y} \in \mathcal{M}\hat{\mathcal{A}}_-$, M is square, and $M(\infty)$ is nonsingular, (2.20) holds, and M can always be chosen to be rational [a similar statement holds for left-coprime factorizations (l.c.f.)]; (2.21)

$G \in \mathcal{M}\hat{\mathcal{B}}$ possesses a doubly coprime factorization $G = \hat{M}^{-1}\hat{N} = NM^{-1}$ over $\mathcal{M}\hat{\mathcal{A}}_-$, where

$$\begin{pmatrix} \hat{X} & -\hat{Y} \\ -\hat{N} & \hat{M} \end{pmatrix} \begin{pmatrix} M & Y \\ N & X \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \text{ in } \overline{\mathbb{C}_+}, \tag{2.22}$$

$\hat{X}, \hat{Y}, X, Y, M, N, \hat{M}, \hat{N} \in \mathcal{M}\hat{\mathcal{A}}_-$, $\hat{M}$ and M are square, $\hat{M}(\infty)$, $M(\infty)$ are nonsingular, and M and $\hat{M}$ can always be chosen to be rational;

A square matrix $G \in \mathcal{M}\hat{\mathcal{W}}$ is invertible over $\mathcal{M}\hat{\mathcal{W}}$ if and only if $\det G(j\omega) \neq 0$ for $\omega \in \mathbb{R} \cup \{\infty\}$; (2.23)

A square matrix $G \in \mathcal{M}\hat{\mathcal{W}}_-$ is invertible over $\mathcal{M}\hat{\mathcal{W}}_-$ if and only if $\det G(j\omega) \neq 0$ for $\omega \in \mathbb{R} \cup \{\infty\}$. (2.24)

We remark that our class of unstable transfer functions will be $\mathcal{M}\hat{\mathcal{B}}$ and the stable class is $\mathcal{M}\hat{\mathcal{A}}_-$. The prime motivation for this choice is the coprime factorization property (2.21) which allows us to carry out an algebraic control synthesis approach just as in the rational case.

Matrix Extension Results

A useful technique used in algebraic control synthesis is to complement the block matrix $[A \;\; B]$ which is left-invertible over some algebra to form a square matrix

$$\begin{bmatrix} C & D \\ A & B \end{bmatrix}$$

which is invertible over the algebra. While this always holds for rational transfer matrices, it does not hold over the classes $\mathcal{M}\hat{\mathcal{A}}$ or $\mathcal{M}\mathbf{H}_\infty$. Another reason for working over $\mathcal{M}\hat{\mathcal{A}}_-$ is that this class does have this property (see Vidyasagar [47, Chapter 8]).

LEMMA 2.1. *Suppose that* $[A\ B]$ *is right-invertible over* $\mathcal{M}\hat{\mathcal{A}}_-$, *where we have partitioned so that* B *is square and* $B(\infty)$ *is nonsingular. Then there exist* $C, D \in \mathcal{M}\hat{\mathcal{A}}_-$ *such that* $\begin{bmatrix} C & D \\ A & B \end{bmatrix}$ *is invertible over* $\mathcal{M}\hat{\mathcal{A}}_-$.

Proof. $G(s) = B^{-1}A \in \mathcal{M}\hat{\mathcal{B}}$, and A, B is a left-coprime factorization. From Callier and Desoer [14, Corollary 2.4] we may conclude that there exist C, D such that $\begin{bmatrix} C & D \\ A & B \end{bmatrix}$ is invertible over $\mathcal{M}\hat{\mathcal{A}}_-$. ■

The following matrix extension result will be needed in Section 4.

LEMMA 2.2. *consider the isometric operator* $B \in \mathbf{L}_\infty^{m\times p}$, *where* $m > p$, *and* $B(j\omega)^*B(j\omega) = I_p$ *for almost all* $\omega \in \mathbb{R}$. *There exists a unitary extension operator* $U = [B\ B_\perp] \in \mathbf{L}_\infty^{m\times m}$ *such that*

$$U(j\omega)^*U(j\omega) = U(j\omega)U(j\omega)^* = I_m.$$

Proof (provided by N. Young of the University of Lancaster). Denote $B = [B_1, \dots, B_p]$, so that $B_k(j\omega) \in \mathbb{C}^m$ for all ω. Then $B_1(j\omega), \dots, B_p(j\omega)$ is an orthonormal set of vectors in $\mathbb{C}^m$ for almost all $\omega \in \mathbb{R}$. The problem is solved if we can find an $F \in \mathbf{L}_\infty^{m\times 1}$ such that $B_1(j\omega), \dots, B_p(j\omega), F(j\omega)$ are linearly independent for almost all $\omega \in \mathbb{R}$. For then we can define the Gram-Schmidt construction

$$B_{p+1}(j\omega) = \frac{H(j\omega)}{\|H(j\omega)\|_{\mathbb{C}^m}},$$

where

$$H(j\omega) = F(j\omega) - \sum_{k=1}^{p} \{B_k(j\omega)^* F(j\omega)\} B_k(j\omega).$$

Clearly $B_{p+1} \in \mathbf{L}_\infty^{m\times 1}$ and

$$[B(j\omega), B_{p+1}(j\omega)]^* [B(j\omega), B_{p+1}(j\omega)] = I_{p+1}.$$

Continuing this process to obtain $B_\perp = [B_{p+1}, \dots, B_m]$ provides the unitary extension $U = [B \ B_\perp]$. It remains to construct an $F \in \mathbf{L}_\infty^{m\times 1}$ such that $B_1(j\omega), \dots, B_p(j\omega)$ and $F(j\omega)$ are linearly independent for almost all $\omega \in \mathbb{R}$.

Let $e_1, \dots, e_m$ be the standard orthonormal basis of $\mathbb{C}^m$, and let S_k be the set of $\omega \in \mathbb{R}$ such that $e_k, B_1(j\omega) \dots, B_p(j\omega)$ are linearly independent.

Expressing the condition $\omega \in S_k$ in terms of certain determinants in the components of the $B_i(j\omega)$ being nonzero shows that S_k is measurable.

Every $\omega \in \mathbb{R}$ is in some S_k, $k = 1, \dots, m$, for otherwise there would exist an ω such that $B_1(j\omega), \dots, B_p(j\omega)$ span a space which containts all $e_1, \dots, e_m$, i.e. they span m, contradicting $p < m$. Hence we may define $F \in \mathbf{L}_\infty^{m\times 1}$ by

$$F(j\omega) = \begin{cases} e_1 & \text{if} \quad \omega \in S_1, \\ e_2 & \text{if} \quad \omega \in S_2 \setminus S_1, \\ \vdots & \\ e_m & \text{if} \quad \omega \in S_m \setminus (S_1 \cup S_2 \cup \cdots \cup S_{m-1}). \end{cases}$$

F has the required linear independence property. ∎

Wiener-Hopf Factorization Results

We need certain factorization results for square, normalized transfer matrices with elements in $\hat{\mathscr{W}}$ or in $\hat{\mathscr{W}}_-$, i.e., $\Phi(\infty) = I$.

A *right canonical Wiener-Hopf factorization* of $\Phi \in \hat{\mathscr{W}}^{n\times n}$ such that $\Phi(\infty) = I$ is

$$\Phi(j\omega) = M(j\omega) N(j\omega), \qquad \omega \in \mathbb{R}, \tag{2.25}$$

where N and $M^\sim$ are invertible over $\hat{\mathscr{A}}^{n\times n}$, and $N(\infty) = M(\infty) = I$.

A *spectral factorization* of $\Phi \in \hat{\mathscr{W}}^{n\times n}$ which satisfies $\Phi(j\omega) = \Phi(j\omega)^*$ and $\Phi(\infty) = I$ is

$$\Phi(j\omega) = N(j\omega)^* N(j\omega), \qquad \omega \in \mathbb{R}, \tag{2.26}$$

where N is invertible over $\hat{\mathscr{A}}^{n\times n}$ and $N(\infty) = I$.

We collect various properties of such factorizations in the following theorem.

THEOREM 2.3.

(i) *A right canonical Wiener-Hopf factorzation* (2.25) *of* $\Phi \in \hat{\mathscr{W}}^{n\times n}$ *with* $\Phi(\infty) = I$ *is unique, if it exists.*

(ii) $\Phi \in \hat{\mathscr{W}}^{n\times n}$ *satisfying* $\Phi(j\omega) = \Phi(j\omega)^*$ *and* $\Phi(\infty) = I$ *posssesses a spectral factorization* (2.26) *if*

$$\det \Phi(j\omega) \neq 0 \qquad \text{for} \quad \omega \in \mathbb{R} \cup \{\infty\}. \tag{2.27}$$

Moreover, it is unique up to multiplication by a constant unitary matrix.

(iii) *If* $\Phi \in \hat{\mathscr{W}}_-^{n\times n}$ *with* $\Phi(\infty) = I$ *has a right-canonical Wiener-Hopf factorization* (2.25), *then* N *and* $M^{\sim}$ *have extensions to invertible elements of* $\hat{\mathscr{A}}_-^{n\times n}$.

(iv) *If* $\Phi \in \hat{\mathscr{W}}_-^{n\times n}$ *satisfies the conditions in (ii), then* N *extends to an invertible element of* $\mathscr{A}_-^{n\times n}$.

Proof. (i): See Gohberg et al. [38, Chapter XXX.9, Theorem 9.2].
(ii), (iv): See Callier and Winkin [15, 16].
(iii): See BGK [10]. ∎

Nehari Theorem
We recall that the *Hankel operator with symbol* $R \in \mathbf{L}_\infty^{p\times q}$ is defined by

$$\Gamma_R f = \pi \Lambda_R f_-$$

for $f \in \mathbf{H}_2^p$, where Λ_R is the multiplication operator on $\mathbf{L}_\infty^{p\times q}$ induced by R, $f_-(s) = f(-s)$, and π is the orthogonal projection from $\mathbf{L}_2^p$ to $\mathbf{H}_2^p$.

The following version of the Nehari theorem is a special case of the more general results in Ball and Helton [3, 4]; it is the continuous-time analogue of the result in Section 4 of [2]. We restrict our attention to two cases: firstly, when the transfer functions are in the Wiener algebra $\mathscr{M}\hat{\mathscr{W}}$, and secondly, when they are in the subalgebra $\mathscr{M}\hat{\mathscr{W}}_-$.

THEOREM 2.4. *Suppose that* $R(-s) \in \hat{\mathscr{A}}^{p\times q}$, *and denote its associated Hankel operator by* Γ_R. *Then the following statements are equivalent*:

(i) *one has*

$$\|\Gamma_R\| < \gamma; \tag{2.28}$$

(ii) *there exists* $Q \in \hat{\mathscr{A}}^{p\times q}$ *such that*

$$\|R + Q\|_\infty < \gamma; \tag{2.29}$$

(iii) *there exists a matrix function* $\Theta \in \hat{\mathscr{W}}^{(p+q)\times(p+q)}$

$$\Theta(s) = \begin{pmatrix} \theta_{11}(s) & \Theta_{12}(s) \\ \theta_{21}(s) & \theta_{22}(s) \end{pmatrix} \tag{2.30}$$

such that all solutions of $Q \in \hat{\mathscr{A}}^{p\times q}$ *of*

$$\|R + Q\|_\infty \leqslant \gamma \tag{2.31}$$

are parametrized by

$$Q(s) + R(s) = (\theta_{11}(s)H(s) + \theta_{12}(s))(\theta_{21}(s)H(s) + \theta_{22}(s))^{-1}, \tag{2.32}$$

where $H \in \hat{\mathscr{A}}^{p\times q}$ *and* $\|H\|_\infty \leqslant 1$.

The matrix function Θ *is any which satisfies*

$$\Theta^{\sim}(s)\begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix}\Theta(s) = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} \quad \text{for} \quad s \in j\mathbb{R}, \tag{2.33}$$

$\Theta_{22}^{-1} \in \hat{\mathscr{A}}^{p\times p}$ *and*

$$\Theta(s)\mathbf{H}_2^{p+q} = \begin{bmatrix} I_p & R(s) \\ 0 & I_q \end{bmatrix}\mathbf{H}_2^{p+q}. \tag{2.34}$$

Moreover, Θ *and* $\Theta^{-1} \in \hat{\mathscr{W}}^{(p+q)\times(p+q)}$, *and for any* $H \in \hat{\mathscr{A}}^{p\times q}$ *satisfying* $\|H\|_\infty \leqslant 1$, *there hold* $Q + R \in \hat{\mathscr{W}}^{p\times q}$ *and* $(\theta_{21}H + \theta_{22})^{-1} \in \hat{\mathscr{A}}^{q\times q}$.

In the sequel it proves more convenient to introduce another matrix function W which replaces Θ.

LEMMA 2.5. *Under the assumptions of Theorem 2.4, there exists a matrix function* $\Theta \in \hat{\mathscr{W}}^{(p+q)\times(p+q)}$ *satisfying* (2.33) *and* (2.34) *if and only if there exists a* $W \in \hat{\mathscr{A}}^{(p+q)\times(p+q)}$ *such that* W^{-1} *and* $W_{11}^{-1} \in \mathscr{M}\hat{\mathscr{A}}$ *and the following holds for* $s = j\omega$, $\omega \in \mathbb{R}$:

$$\begin{bmatrix} I_p & 0 \\ R^\sim & I_q \end{bmatrix}\begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix}\begin{bmatrix} I_p & R \\ 0 & I_q \end{bmatrix} = W^\sim J_{pq}(\gamma)W. \tag{2.35}$$

In this case

$$\Theta(s) = \begin{bmatrix} I_p & R(s) \\ 0 & I_q \end{bmatrix} W(s)^{-1}\begin{bmatrix} I_p & 0 \\ 0 & \dfrac{1}{\gamma}I_q \end{bmatrix}. \tag{2.36}$$

Proof.

(1) From Theorem 2.4, $\Theta^{-1} \in \mathscr{M}\hat{\mathscr{W}}$ and with Θ defined by (2.36), it is straightforward to verify the equivalence of the factorizations (2.35) and (2.33).

(2) Let us now interpret (2.34) in terms of properties of W. With Θ given by (2.36), (2.34) becomes

$$W^{-1}(s)\mathbf{H}_2^{p+q} = \mathbf{H}_2^{p+q},$$

and this holds if and only if W and $W^{-1} \in \mathbf{H}_\infty^{(p+q)\times(p+q)}$. But from (2.36), it follows that $W^{-1} \in \mathscr{M}\hat{\mathscr{W}}$. So W, W^{-1} are stable elements of the Wiener algebra $\mathscr{M}\mathscr{W}$ and hence they are in $\mathscr{M}\hat{\mathscr{A}}$. Hence the property (2.34) for Θ is equivalent to the property that W and $W^{-1} \in \hat{\mathscr{A}}^{(p+q)\times(p+q)}$.

(3) We now show that $\Theta_{22}^{-1} \in \hat{\mathscr{A}}^{p\times p}$ if and only if $W_{11}^{-1} \in \hat{\mathscr{A}}^{p\times p}$. Note that

$$W(s)^{-1} = \begin{pmatrix} \Theta_{11} - R\Theta_{21} & -\dfrac{1}{\gamma}(\Theta_{12} - R\Theta_{22}) \\ \Theta_{21} & -\dfrac{1}{\gamma}\Theta_{22} \end{pmatrix},$$

which shows that $(W^{-1})_{22} = -(1/\gamma)\Theta_{22}$. Now a well-known result from matrix algebra (Kailath [43, p. 656]) is that $W_{11}^{-1} = (W^{-1})_{11} - (W^{-1})_{12}(W^{-1})_{22}^{-1}(W^{-1})_{21}$. Thus $W_{11}^{-1} \in \hat{\mathscr{A}}^{p\times p}$ if $\Theta_{22}^{-1} \in \hat{\mathscr{A}}^{p\times p}$. The converse follows on exchanging the roles of Θ and W (see [23] for details). ∎

A direct consequence of this lemma is an alternative formulation of solutions to the Nehari problem (c.f. Curtain and Zwart [24, Theorem 3.5]).

THEOREM 2.6. *Suppose that $R(-s) \in \hat{\mathscr{A}}^{p\times q}$ and there exists a $W \in \hat{\mathscr{A}}^{(p+q)\times(p+q)}$ such that $W^{-1} \in \hat{\mathscr{A}}^{(p+q)\times(p+q)}$ and $W_{11}^{-1} \in \hat{\mathscr{A}}^{p\times q}$ satisfy*

$$\begin{bmatrix} I_p & 0 \\ R^{\sim} & I_q \end{bmatrix} J_{pq}(\gamma) \begin{bmatrix} I_p & R \\ 0 & I_q \end{bmatrix} = W^{\sim} J_{pq}(\gamma) W \qquad \text{for} \quad s = j\omega, \quad \omega \in \mathbb{R}. \tag{2.37}$$

Then the set of matrices $Q \in \hat{\mathscr{A}}^{p\times q}$ such that $\|Q + R\|_\infty \leqslant \gamma$ for $\gamma > \|\Gamma_R\|$ is parametrized by

$$Q = Q_1 Q_2^{-1}, \qquad \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = W^{-1} \begin{bmatrix} U \\ I_q \end{bmatrix}, \tag{2.38}$$

where $U \in \hat{\mathscr{A}}^{p\times q}$ satisfies $\|U\|_\infty \leqslant \gamma$.
Moreover, $Q_2^{-1} \in \hat{\mathscr{A}}^{p\times p}$.

Proof. See [23]. ∎

Since J-spectral factorizations are not unique, it is important to show that the properties that W is invertible over $\mathscr{M}\hat{\mathscr{A}}$ and $W_{11}^{-1} \in \mathscr{M}\hat{\mathscr{A}}$ are shared by all J-spectral factorizations.

LEMMA 2.7. *Suppose that W is invertible over $\mathscr{M}\hat{\mathscr{A}}$. Then*

(i) *$Y, Y^{-1} \in \mathscr{M}\hat{\mathscr{A}}$ satisfy $Y^{\sim} J Y = W^{\sim} J W$ on $s = j\omega$, $\omega \in \mathbb{R}$, if and only if $Y = AW$, where A is a constant, J-unitary matrix. (J denotes $J_{pq}(\gamma)$.)*

(ii) *If*

$$\begin{bmatrix} I & 0 \\ R^{\sim} & I \end{bmatrix} J_{pq}(\gamma) \begin{bmatrix} I & R \\ 0 & I \end{bmatrix} = W^{\sim} J_{pq}(\gamma) W = Y^{\sim} J_{pq}(\gamma) Y \quad (2.39)$$

for $s = j\omega$, $\omega \in \mathbb{R}$, *and* $R^{\sim}, W, W^{-1}, Y, Y^{-1} \in \mathscr{M}\hat{\mathscr{A}}$, *then* $Y_{11}^{-1} \in \mathscr{M}\hat{\mathscr{A}}$ *if and only if* $W_{11}^{-1} \in \mathscr{M}\hat{\mathscr{A}}$.

Proof. (i): If Y satisfies $Y^{\sim} JY = W^{\sim} JW$ on $s = j\omega$, then

$$JYW^{-1} = (Y^{\sim})^{-1} W^{\sim} J \qquad \text{on} \quad s = j\omega.$$

The right-hand side is holomorphic on $\operatorname{Re} s < 0$, and the left-hand side is holomorphic on $\operatorname{Re} s > 0$. As elements of the Wiener algebra $\mathscr{M}\hat{\mathscr{W}}$ they are equal (the two-sided Laplace transform is unique), and so they must be both equal to a constant matrix. $A = YW^{-1}$ is constant, and it satisfies $A^* JA = J$.

(ii): (2.35) implies that $W_{11}^{\sim} W_{11} - \gamma^2 W_{21}^{\sim} W_{21} = I$ on $s = j\omega$, and $W, W_{11}^{-1} \in \mathscr{M}\hat{\mathscr{A}} \subset \mathscr{M}\mathbf{H}_\infty$ shows that $\|W_{21} W_{11}^{-1}\|_\infty < 1/\gamma$. Let $A = YW^{-1}$, and recall from (i) that $A^* JA = J$ and so $A^{-1} = J^{-1} A^* J$ and $AJ^{-1} A^* = J^{-1}$. The $(1, 1)$ block of this yields

$$A_{11} A_{11}^* - \gamma^{-2} A_{12} A_{12}^* = I,$$

and so A_{11} is nonsingular and $\| A_{11}^{-1} A_{12} \| < \gamma$. Now $A_{12}, W_{21}, W_{11}^{-1} \in \mathscr{M}\hat{\mathscr{A}} \subset \mathscr{M}\mathbf{H}_\infty$, and there holds $\| A_{11}^{-1} A_{12} W_{21} W_{11}^{-1} \|_\infty < 1$. This shows that $I + A_{11}^{-1} A_{12} W_{21} W_{11}^{-1}$ is invertible over $\mathscr{M}\mathbf{H}_\infty$ and hence over $\mathscr{M}\hat{\mathscr{A}}$ [see (2.19)]. Its inverse is $W_{11}(A_{11} W_{11} + A_{12} W_{21})^{-1} A_{11}$, and so $Y_{11} = A_{11} W_{11} + A_{12} W_{21}$ is invertible over $\mathscr{M}\hat{\mathscr{A}}$. ∎

We now examine the consequences of assuming that $R(-s) \in \mathscr{M}\hat{\mathscr{A}}_-$, a smoother class of transfer matrices.

LEMMA 2.8. *Suppose that* $R(-s) \in \mathscr{M}\hat{\mathscr{A}}_-$.

(i) *In Lemma* 2.5, $W, W^{-1}, W_{11}^{-1} \in \mathscr{M}\hat{\mathscr{A}}_-$.

(ii) *In Theorem* 2.4 $\Theta_{11}, \Theta_{12} \in \mathscr{M}\hat{\mathscr{W}}_-$, $\theta_{21}, \theta_{22} \in \mathscr{M}\hat{\mathscr{A}}_-$, *and for any* $H \in \hat{\mathscr{A}}_-^{p \times p}$ *satisfying* $\|H\|_\infty \leqslant 1$, *there holds* $(\theta_{21} H + \theta_{22})^{-1} \in \hat{\mathscr{A}}_-^{q \times q}$.

(iii) *In Theorem* 2.6, $Q_2^{-1} \in \hat{\mathscr{A}}_-^{q \times q}$, *provided that* $U \in \hat{\mathscr{A}}_-^{p \times q}$.

(iv) *Lemma* 2.7 *holds with* $\mathscr{M}\hat{\mathscr{A}}$ *replaced by* $\mathscr{M}\hat{\mathscr{A}}_-$.

(v) *Moreover*, (2.34) *and* (2.35) *hold on a strip surrounding the imaginary axis.*

Proof. (i): In Lemma 2.5 we established the existence of a J-spectral factorization

$$G(j\omega)^* JG(j\omega) = W(j\omega)^* JW(j\omega),$$

where

$$G = \begin{bmatrix} I_p & R \\ 0 & I_q \end{bmatrix}.$$

Since $W, W^{-1} \in \mathscr{M}\hat{\mathscr{A}}$, we may conclude that $W(\infty)$ exists and is invertible. Define the matrix $\Phi \in \mathscr{M}\hat{\mathscr{W}}_-$ by

$$\Theta = W(\infty)^{-*} G^{\sim} JGW(\infty)^{-1} J^{-1}.$$

Then

$$\Phi(j\omega) = W(\infty)^{-*} W^{\sim}(j\omega) JW(j\omega) W(\infty)^{-1} J^{-1},$$

and this is a right canonical Wiener-Hopf factorization of Φ, identifying $N = JWW(\infty)^{-1}J^{-1}$ and $M^{\sim} = WW(\infty)^{-1}$. By Theorem 2.3(i) and (iii) it follows that W has an extension to $\mathscr{M}\hat{\mathscr{A}}_-$, and by (2.19) $W^{-1}, W_{11}^{-1} \in \mathscr{M}\hat{\mathscr{A}}_-$.

(ii): $\Theta_{11}, \Theta_{12} \in \mathscr{M}\hat{\mathscr{W}}_-$ and $\Theta_{21}, \Theta_{21} \in \mathscr{M}\hat{\mathscr{A}}_-$ follow from (i) and (2.36) in Lemma 2.5. Next, $\theta_{21}H + \theta_{22} \in \mathscr{M}\hat{\mathscr{A}}_-$, and it is invertible over $\mathscr{M}\mathbf{H}_\infty$ and thus also over $\in \mathscr{M}\hat{\mathscr{A}}_-$ [see (2.19a)].

(iii): This follows from (ii), since $Q_2 = \gamma(\theta_{21}H + \theta_{22})$.

(iv): This follows because $\mathscr{M}\hat{\mathscr{A}}_-$ is a subalgebra of $\mathscr{M}\hat{\mathscr{A}}$ [see (2.7)].

(v): This follows by analytic continuation, since $W, W^{\sim}, R, R^{\sim}, Y, Y^{\sim}$ are holomorphic on $[-\varepsilon + j\omega, \varepsilon + j\omega]$ for some $\varepsilon > 0$. ∎

In other words, Theorems 2.4 and 2.6 can be modified by replacing $\hat{\mathscr{A}}$ by $\hat{\mathscr{A}}_-$ and $\hat{\mathscr{W}}$ by $\hat{\mathscr{W}}_-$. Finally, it is easy to verify that Theorems 2.4 and 2.6 remain true for $R(-s) \in \hat{\mathscr{W}}^{p\times q}$. We state this as a corollary.

COROLLARY 2.9. *Suppose that $R(-s) \in \hat{\mathscr{W}}^{p\times q}$ and there exists a $W \in \hat{\mathscr{A}}^{(p+q)\times(p+q)}$ such that W^{-1} and $W_{11}^{-1} \in \hat{\mathscr{A}}^{(p+q)\times(p+q)}$ and $\hat{\mathscr{A}}^{q\times q}$ respectively satisfy* (2.37). *Then the set of all matrices $Q \in \hat{\mathscr{A}}^{p\times q}$ such that $\|Q + R\|_\infty \leqslant \gamma$ for $\gamma > \|\Gamma_R\|$ is parametrized by* (2.38).

Proof.

(1) First we show that Theorem 2.4 remains the same for $R \in \hat{\mathscr{W}}^{p \times p}$. Decompose $R = R_u + R_s$, where $R_u^{\sim}, R_s \in \hat{\mathscr{A}}^{p \times q}$. Then $R_u + R_s + Q = R_u + Q_s$, where $Q_s = Q + R_s \in \hat{\mathscr{A}}^{p \times q}$ and $R_u^{\sim} \in \hat{\mathscr{A}}^{p \times q}$. So applying Theorem 2.4 to R_u, we obtain all solutions Q_s to $\|R_u + Q_s\|_\infty = \|R + Q\|_\infty < \gamma$ as

$$Q_s = -R_u + (\Theta_{11}H + \Theta_{12})(\Theta_{21}H + \Theta_{22})^{-1}.$$

So all $Q \in \hat{\mathscr{A}}^{p \times q}$ satisfying $\|R + Q\|_\infty < \gamma$ are given by

$$Q = Q_s - R_s = -R + (\Theta_{11}H + \Theta_{12})(\Theta_{21}H + \Theta_{22})^{-1}.$$

(2) Next we note that the proofs of Lemma 2.5 and Theorem 2.6 are essentially algebraic and are valid for a general $R \in \hat{\mathscr{W}}^{p \times q}$. ■

Finally, we remark that using Lemma 2.8 it follows that if we take $R \in \hat{\mathscr{W}}^{p \times q}$, then Corollary 2.9 holds with $W, W^{-1} \in \hat{\mathscr{A}}_-^{(p+q) \times (p+q)}$, $W_{11}^{-1} \in \hat{\mathscr{A}}_-^{q \times q}$, and $U, Q \in \hat{\mathscr{A}}_-^{p \times q}$.

3. STABILIZATION THEORY

In this section, we develop an implicit characterization of stabilizing controllers $K \in \mathscr{M}\hat{\mathscr{B}}$ for the plant $P \in \mathscr{M}\hat{\mathscr{B}}$, just as in Green [39] for the rational case. Most of the arguments are algebraic and apply equally well to any algebra of transfer matrices with a coprime factorization property. Complete proofs are given in Curtain and Green [23].

First we recall the definition of internal stability of the configuration in Figure 1.

DEFINITION 3.1. Consider $P \in \hat{\mathscr{B}}^{(l+m) \times (p+q)}$ and $K \in \hat{\mathscr{B}}^{q \times m}$ as in Figure 1, i.e.

$$\begin{bmatrix} z \\ y \end{bmatrix} = P \begin{bmatrix} w \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ v_1 \end{bmatrix}, \qquad u = Ky + v_2. \tag{3.1}$$

We say (P, K) is well posed if the nine transfer matrices from w, v_1, v_2 to z, y, u are in $\mathscr{M}\hat{\mathscr{B}}$. (P, K) is internally stable if it is well posed and if these nine transfer matrices are in $\mathscr{M}\hat{\mathscr{A}}_-$. In this case we say that K stabilizes P. We say P is stabilizable if such a K exists.

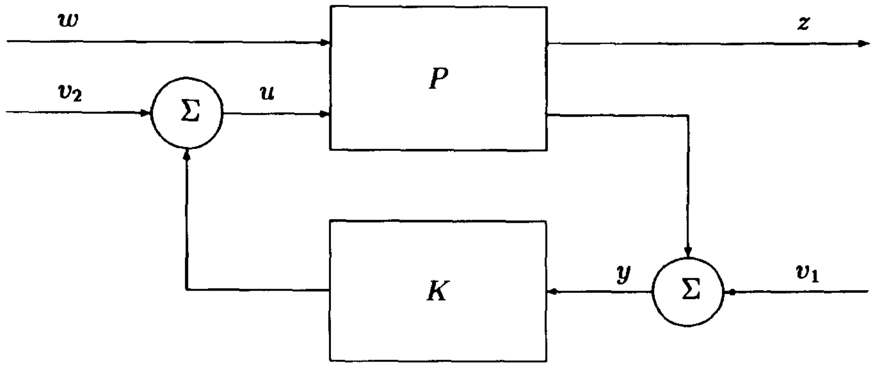

FIG. 1. Internal stability for (P, K).

We shall use the following implicit parametrization of all stabilizing controllers in the sequel.

LEMMA 3.2. *Suppose that* $P \in \hat{\mathscr{B}}^{(l+m)\times(p+q)}$ *is stabilizable and has a right-coprime factorization*

$$P = NM^{-1} \qquad \textit{with} \quad M = \begin{bmatrix} I & 0 \\ M_{21} & M_{22} \end{bmatrix}.$$

Then $K \in \mathscr{M}\hat{\mathscr{B}}$ *stabilizes* P *if and only if* $K = \hat{K}_2^{-1}\hat{K}_1$, *where*

$$\begin{bmatrix} \hat{K}_1 & \hat{K}_2 \end{bmatrix} \begin{bmatrix} N_{21} & -N_{22} \\ -M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} Q & I_q \end{bmatrix}, \tag{3.2}$$

Q, $\hat{K}_1$, $\hat{K}_2 \in \mathscr{M}\hat{\mathscr{A}}_-$, *and* $\hat{K}_2$ *is square with* $\hat{K}_2(\infty)$ *nonsingular.*

In this case,

$$\begin{pmatrix} \mathscr{F}(P, K) \\ I_p \end{pmatrix} = \begin{pmatrix} N_{11} & N_{12} \\ I_p & 0 \end{pmatrix} \begin{pmatrix} I_p \\ Q \end{pmatrix}. \tag{3.3}$$

The obvious dual result follows.

COROLLARY 3.3. *Suppose that $P \in \hat{\mathscr{B}}^{(l+m)\times(p+q)}$ is stabilizable and that P has a left-coprime factorization*

$$P = \hat{M}^{-1}\hat{N}, \qquad where \quad \hat{M} = \begin{bmatrix} I_l & \hat{M}_{12} \\ 0 & \hat{M}_{22} \end{bmatrix}.$$

Then $K \in \mathscr{M}\hat{\mathscr{B}}$ stabilizes P if and only if $K = K_1 K_2^{-1}$, where

$$\begin{bmatrix} \hat{Q} \\ I_m \end{bmatrix} = \begin{bmatrix} \hat{N}_{12} & -\hat{M}_{12} \\ -\hat{N}_{22} & \hat{M}_{22} \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}, \tag{3.4}$$

Q, K_1, $K_2 \in \mathscr{M}\hat{\mathscr{A}}_-$, and K_2 is square with $K_2(\infty)$ nonsingular.
In this case,

$$\begin{bmatrix} \mathscr{F}(P, K) & I_l \end{bmatrix} = \begin{bmatrix} I_l & \hat{Q} \end{bmatrix} \begin{bmatrix} \hat{N}_{11} & I_l \\ \hat{N}_{21} & 0 \end{bmatrix}. \tag{3.5}$$

The above parametrizations of stabilizing controllers are implicit, in contrast to the more usual Youla parametrization (see Vidyasagar [47]). The advantage of this implicit parametrization is that it can be used to decompose the $\mathbf{H}_\infty$ controller synthesis as a two-stage procedure, where each stage involves the solution of a stable analytic system problem of the type considered in Section 4.

This synthesis is carried out in Section 5, and the solution of stable analytic system problems is considered in the next section.

4. STABLE ANALYTIC SYSTEMS

For the $\mathbf{H}_\infty$ problem we need to obtain solutions to a class of analytic system problems of the following type. For a given real number $\gamma > 0$ and given $G_{11}, G_{12}, G_{21}, G_{22} \in \mathscr{M}\hat{\mathscr{A}}_-$ consider

$$\begin{bmatrix} R \\ I \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}. \tag{4.1}$$

The problem is to find the set of all $R \in \mathscr{M}\hat{\mathscr{W}}_-$ such that $\|R\|_\infty < \gamma$ and there exist $Q_1, Q_2 \in \mathscr{M}\hat{\mathscr{A}}_-$ satisfying (4.1).

If G_{22} is square and $\det G_{22}(\infty) \neq 0$, we show that (4.1) can be reduced to an equivalent model matching problem of the type

$$\begin{bmatrix} R \\ I \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}. \tag{4.2}$$

Then using the factorization results from Theorem 2.3, we reduce the solution of (4.2) to that of the Nehari problem of the type

$$\begin{bmatrix} R \\ I \end{bmatrix} = \begin{bmatrix} I & G_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}. \tag{4.3}$$

Solutions for such problems in terms of J-spectral factorizations are summarized in Section 2 (see Theorems 2.4, 2.6, Lemmas 2.5, 2.7, and Corollary 2.9). Finally, we reformulate the solutions to (4.1) in terms of a J-lossless condition which are suitable for the $\mathbf{H}_\infty$ control synthesis in Section 5.

First we give conditions under which (4.1) can be reduced to the model matching form (4.2).

LEMMA 4.1. *Consider the analytic system problem* (4.1) *for a* $G \in \mathcal{M}\hat{\mathcal{A}}_-$, *where* G_{22} *is square and* $\det G_{22}(\infty) \neq 0$. *The problem of finding* $R \in \mathcal{M}\mathcal{W}_-$, Q_1 *and* $Q_2 \in \mathcal{M}\hat{\mathcal{A}}_-$ *is equivalent to the model matching problem*

$$\begin{bmatrix} R \\ I \end{bmatrix} = \begin{bmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ 0 & I \end{bmatrix} M \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \tag{4.4}$$

for certain $\tilde{G}_{11}, \tilde{G}_{12} \in \mathcal{M}\hat{\mathcal{A}}_-$ *and* $M, M^{-1} \in \mathcal{M}\hat{\mathcal{A}}_-$.

Proof. From (4.1) we see that the analytic system problem has a solution if and only if $[G_{21} \;\; G_{22}]$ is right-invertible over $\mathcal{M}\hat{\mathcal{A}}_-$. Since G_{22} is square and nonsingular at infinity, then by Lemma 2.1 there exist $A, B \in \mathcal{M}\hat{\mathcal{A}}_-$ such that

$$M = \begin{bmatrix} A & B \\ G_{21} & G_{22} \end{bmatrix} \quad \text{and} \quad M^{-1} \in \mathcal{M}\hat{\mathcal{A}}_-.$$

Writing now

$$G = (GM^{-1})M = \begin{bmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ 0 & I \end{bmatrix} M$$

proves the lemma. ■

We remark that this lemma does not necessarily hold if we only seek $Q_1, Q_2 \in \mathscr{M}\hat{\mathscr{A}}$ (or $\mathscr{M}\mathbf{H}_\infty$), since Lemma 2.1 on the existence of a complement to $[G_{21} \;\; G_{22}]$ relies on the existence of left and right factorizations. These are not guaranteed in $\mathscr{M}\hat{\mathscr{A}}$ or $\mathscr{M}\mathbf{H}_\infty$.

We now solve the model matching problem (4.2).

THEOREM 4.2. *Suppose that*

$$G = \begin{bmatrix} B & A \\ 0 & I_q \end{bmatrix} \in \hat{\mathscr{A}}^{(l+q)\times(p+q)}$$

is left-invertible over $\mathscr{W}^{(l+q)\times(p+q)}$. *The following are equivalent statements*:

(i) *There exists a* $Q \in \hat{\mathscr{A}}^{p\times q}$ *such that*

$$\|A + BQ\|_\infty < \gamma. \tag{4.5}$$

(ii) *There exists a* $W \in \hat{\mathscr{A}}^{(p+q)\times(p+q)}$ *such that* $W^{-1} \in \hat{\mathscr{A}}^{(p+q)\times(p+q)}$, $W_{11}^{-1} \in \hat{\mathscr{A}}^{p\times p}$, *and*

$$G^{\sim} J_{lq}(\gamma) G = W^{\sim} J_{pq}(\gamma) W \qquad on \quad s = j\omega, \quad \omega \in \mathbb{R}. \tag{4.6}$$

Furthermore, if such a W *exists, the set of all* $Q \in \hat{\mathscr{A}}^{p\times q}$ *satisfying* $\|A + BQ\|_\infty \leqslant \gamma$ *is given by*

$$Q = Q_1 Q_2^{-1}, \qquad \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = W^{-1} \begin{bmatrix} U \\ I \end{bmatrix} \tag{4.7}$$

whenever $U \in \hat{\mathscr{A}}^{p\times q}$ *satisfies* $\|U\|_\infty \leqslant \gamma$. *The latter implies that* $Q_2^{-1} \in \hat{\mathscr{A}}^{q\times q}$.

Proof. G is left-invertible if and only if B is. This holds if and only if $\det B^{\sim} B(j\omega) \neq 0$ for $\omega \in \mathbb{R} \cup \{\infty\}$. So applying Theorem 2.3(ii) to $S = B^{\sim} B$, we obtain the spectral factorization of $B^{\sim} B = S = B_0^{\sim} B_0$, where

B_0, $B_0^{-1} \in \mathcal{M}\hat{\mathcal{A}}$. Let $B_i = BB_0^{-1}$, and note that $B_i^{\sim} B_i = I_p$. Now by Lemma 2.2 there exists a unitary extension $[B_i \;\; B_{\perp}]$. So we have

$$\|A + BQ\|_\infty < \gamma \quad \Leftrightarrow \quad \left\| A + \begin{bmatrix} B_i & B_{\perp} \end{bmatrix} \begin{bmatrix} B_0 Q \\ 0 \end{bmatrix} \right\|_\infty < \gamma$$

$$\Leftrightarrow \quad \left\| \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} + \begin{bmatrix} B_0 Q \\ 0 \end{bmatrix} \right\|_\infty < \gamma,$$

where $R = [B_i \;\; B_{\perp}]^{\sim} A$ and we have used the fact that $[B_i \;\; B_{\perp}]$ is unitary. Thus

$$\|A + BQ\|_\infty < \gamma \quad \Leftrightarrow \quad \|R_2\|_\infty < \gamma$$

and

$$(R_1 + B_0 Q)^{\sim} (R_1 + B_0 Q) + R_2^{\sim} R_2 < \gamma^2 I_q \qquad \text{on} \quad s = j\omega.$$

Let

$$\Phi = \gamma^2 I_q - R_2^{\sim} R_2 = \gamma^2 I_q - A^{\sim}\left[I_l - B(B^{\sim} B)^{-1} B^{\sim} \right] A. \quad (4.8)$$

Then $\Phi \in \mathcal{M}\mathcal{W}$ satisfies $\Phi^* = \Phi$ and

$$\Phi(j\omega) > 0 \quad \text{for } \omega \in \mathbb{R} \cup \{\infty\} \quad \Leftrightarrow \quad \|R_2\|_\infty < \gamma.$$

So by Theorem 2.3(ii), there exists $T \in \mathcal{M}\hat{\mathcal{A}}$ such that $T^{-1} \in M\hat{\mathcal{A}}$ and

$$\gamma^2 T^* T = \Phi \qquad \text{on} \quad s = j\omega$$

if and only if $\|R_2\|_\infty < \gamma$.

Summarizing, we have shown the following: there exists a $Q \in \mathcal{M}\hat{\mathcal{A}}$: $\|A + BQ\|_\infty < \gamma$ if and only if there exist $T, \hat{Q} \in \mathcal{M}\hat{\mathcal{A}}$ such that $T^{-1} \in \mathcal{M}\hat{\mathcal{A}}$ and $\|R_1 T^{-1} + \hat{Q}\|_\infty < \gamma$, where

$$\gamma^2 T^{\sim} T = \gamma^2 I_q - A^{\sim}\left[I - B(B^{\sim} B)^{-1} B^{\sim} \right] A, \qquad \hat{Q} = B_0 Q T^{-1}.$$

Observe that $R_1T^{-1} \in \mathcal{M}\hat{\mathcal{W}}$, and so by Corollary 2.9 we obtain: there exists a $Q \in \mathcal{M}\hat{\mathcal{A}}$ such that $\|A + BQ\|_\infty < \gamma$ if and only if there exist $T, X \in \mathcal{M}\hat{\mathcal{A}}$ such that $X^{-1}, T^{-1}, X_{11}^{-1} \in \mathcal{M}\hat{\mathcal{A}}$ and

$$\begin{bmatrix} I_p & 0 \\ (R_1T^{-1})^{\sim} & I_q \end{bmatrix} J_{pq}(\gamma) \begin{bmatrix} I_p & R_1T^{-1} \\ 0 & I_q \end{bmatrix} = X^{\sim} J_{pq}(\gamma) X$$

$$\text{on} \quad s = j\omega, \quad \omega \in \mathbb{R}. \tag{4.9}$$

Now $R_1 = (B_0^{\sim})^{-1}B^{\sim}A$ and

$$G^{\sim}J_{lq}(\gamma)G = \begin{bmatrix} B_0^{\sim} & 0 \\ A^{\sim}BB_0^{-1} & I_q \end{bmatrix}\begin{bmatrix} I_p & 0 \\ 0 & -\Phi \end{bmatrix}\begin{bmatrix} B_0 & (B_0^{\sim})^{-1}B^{\sim}A \\ 0 & I_q \end{bmatrix}$$

$$= \begin{bmatrix} B_0^{\sim} & 0 \\ 0 & T \end{bmatrix}\begin{bmatrix} I_p & R_1T^{-1} \\ 0 & I_q \end{bmatrix}^{\sim} J_{pq}(\gamma)\begin{bmatrix} I_p & R_1T^{-1} \\ 0 & I_q \end{bmatrix}\begin{bmatrix} B_0 & 0 \\ 0 & T \end{bmatrix}, \tag{4.10}$$

and so we see that with

$$W = X\begin{bmatrix} B_0 & 0 \\ 0 & T \end{bmatrix}$$

(4.10) is equivalent to (4.6). B_0 and T are both invertible over $\mathcal{M}\hat{\mathcal{A}}$, and so W has exactly the same properties as X. Thus it remains to show that the existence of a spectral factorization of Φ follows from (4.6). Equation (4.8) shows that $\Phi \in \mathcal{M}\hat{\mathcal{W}}$, and to see that $\Phi(j\omega) > 0$ for $\omega \in \mathbb{R} \cup \{\infty\}$ we examine the following from (4.10) and (4.6):

$$\begin{bmatrix} I_p & 0 \\ 0 & -\Phi \end{bmatrix} = \begin{bmatrix} (B_0^{\sim})^{-1} & 0 \\ -A^{\sim}B(B_0^{\sim}B_0)^{-1} & I_q \end{bmatrix}$$

$$\times W^{\sim}J_{pq}(\gamma)W\begin{bmatrix} B_0^{-1} & -(B_0^{\sim}B_0)^{-1}B^{\sim}A \\ 0 & I_q \end{bmatrix}$$

$$= Y^{\sim}J_{pq}(\gamma)Y,$$

where $Y, Y^{-1} \in \mathcal{M}\hat{\mathcal{W}}$ and $Y_{11}^{-1} \in \mathcal{M}\hat{\mathcal{A}}$. This shows that for $x \in \mathbb{C}^p$, $y \in \mathbb{C}^q$, $\omega \in \mathbb{R}$ we have $\|x\|^2 - \langle y, \Phi(j\omega)y\rangle = \|Y_{11}(j\omega)x + Y_{12}(j\omega)y\|^2 - \gamma^2\|Y_{21}(j\omega)x + Y_{22}(j\omega)y\|^2$, and since $Y_{11}^{-1} \in \mathcal{M}\hat{\mathcal{A}} \subset \mathcal{M}\mathbf{H}_\infty$, we may choose $x = -Y_{11}^{-1}(j\omega)Y_{12}(j\omega)y$ for any $\omega \in \mathbb{R}$. So for all $y \in \mathbb{C}^q$ and all $\omega \in \mathbb{R}$ there holds

$$\langle y, \Phi(j\omega)y\rangle = \|Y_{11}^{-1}(j\omega)Y_{12}(j\omega)y\|^2$$

$$+ \gamma^2\|Y_{22}(j\omega)y - Y_{21}(j\omega)Y_{11}^{-1}(j\omega)Y_{12}(j\omega)y\|^2.$$

This shows that $\Phi(j\omega) = 0 \Leftrightarrow Y_{12}(j\omega) = 0 = Y_{22}(j\omega)$. But the latter would contradict the invertibility of Y in $\mathcal{M}\hat{\mathcal{W}}$. So $\Phi(j\omega) > 0$ for all $\omega \in \mathbb{R}$, and $\Phi(j\infty) > 0$ can be argued similarly.

The parametrization (4.7) follows from Corollary 2.9 and (2.38), since

$$B_0QT^{-1} = \hat{Q} = \hat{Q}_1\hat{Q}_2^{-1}, \qquad \begin{bmatrix} \hat{Q}_1 \\ \hat{Q}_2 \end{bmatrix} = X^{-1}\begin{bmatrix} U \\ I \end{bmatrix},$$

and so $Q = B_0^{-1}\hat{Q}_1\hat{Q}_2^{-1}T = Q_1Q_2^{-1}$ with

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} B_0^{-1}\hat{Q}_1 \\ T^{-1}\hat{Q}_2 \end{bmatrix} = W^{-1}\begin{bmatrix} U \\ I \end{bmatrix}.$$

From Corollary 2.9 we know that $\hat{Q}_2^{-1} \in \hat{\mathcal{A}}^{q\times q}$ and so $Q_2^{-1} \in \hat{\mathcal{A}}^{q\times q}$. ∎

Following the remarks after Corollary 2.9, we can deduce the following alternative version of Theorem 4.2.

COROLLARY 4.3. *Suppose that in Theorem* 4.2 $G \in \hat{\mathcal{A}}_-^{(l+q)\times(p+q)}$ *is left invertible over* $\hat{\mathcal{W}}_-^{(l+q)\times(p+q)}$. *Then* (i) *and* (ii) *remain equivalent on replacing* $\hat{\mathcal{A}}$ *by* $\hat{\mathcal{A}}_-$, *and* (4.6) *holds on the strip* $|\mathrm{Re}\, s| < \varepsilon$ *for some* $\varepsilon > 0$. *Moreover,* $\hat{Q}_2^{-1} \in \hat{\mathcal{A}}_-^{q\times q}$ *if* $U \in \hat{\mathcal{A}}_-^{p\times q}$.

We now give a solution of the general analytic system problem (4.1) in terms of J-spectral factorizations.

THEOREM 4.4. *Consider the analytic system problem* (4.1) *for* $G \in \hat{\mathscr{A}}_-^{(l+m)\times(q+m)}$, *where* G_{22} *is square and* $\det G_{22}(\infty) \neq 0$. *If* G *is left-invertible over* $\mathscr{M}\hat{\mathscr{W}}_-$, *then there exist* $Q_1, Q_2 \in \mathscr{M}\hat{\mathscr{A}}_-$ *and an* $R \in \mathscr{M}\hat{\mathscr{W}}_-$ *with* $\|R\|_\infty < \gamma$ *satisfying* (4.1) *if and only if there exists a* $W \in \mathscr{M}\hat{\mathscr{A}}_-$ *such that*

$$G^{\sim} J_{lm}(\gamma) G = W^{\sim} J_{qm}(\gamma) W \qquad \text{for} \quad |\mathrm{Re}\, s| < \varepsilon, \tag{4.11}$$

where $W, W^{-1} \in \hat{\mathscr{A}}_-^{(q+m)\times(q+m)}$ *and* $(GW^{-1})_{22}$ *is invertible over* $\hat{\mathscr{A}}_-^{m\times m}$. *In this case, all solutions* R *to* (4.1) *are generated by*

$$R = R_1 R_2^{-1}, \qquad \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = GW^{-1}\begin{bmatrix} U \\ I_m \end{bmatrix} \tag{4.12}$$

for $U \in \hat{\mathscr{A}}_-^{q\times m}$ *satisfying* $\|U\|_\infty \leqslant \gamma$, *where* W *is any solution to* (4.11), *and* $R_2^{-1} \in \hat{A}_-^{m\times m}$.

Proof. This follows from Lemma 4.1 as in Green [39, Theorem 3.1]. For a detailed proof see Curtain and Green [23, Theorem 4.5]. ■

In order to reinterpret the J-spectral factorization condition (4.6) as a J-lossless condition, we need the following lemma.

LEMMA 4.5. *Suppose that* $X \in \hat{\mathscr{A}}_-^{(l+m)\times(q+m)}$. *Then* $X_{22}^{-1} \in \hat{\mathscr{A}}_-^{m\times m}$ *and* $X^{\sim} J_{lm}(\gamma) X = J_{mq}(\gamma)$ *for* $|\mathrm{Re}\, s| < \varepsilon$ *for some* $\varepsilon > 0$ *if and only if* X *if* J*-lossless.*

Proof. See Curtain and Rodriguez [22, Lemma 3.2] or Curtain and Green [23, Lemma 4.6]. ■

COROLLARY 4.6. *Consider the analytic system problem* (4.1) *for* $G \in \hat{\mathscr{A}}_-^{(l+m)\times(q+m)}$, *where* G_{22} *is square and* $\det G_{22}(\infty) \neq 0$. *If* G *is left-invertible over* $\mathscr{M}\hat{\mathscr{W}}_-$, *then there exist* $Q_1, Q_2 \in \mathscr{M}\hat{\mathscr{A}}_-$ *and an* $R \in \mathscr{M}\hat{\mathscr{W}}_-$ *with* $\|R\|_\infty < \gamma$ *satisfying* (4.1) *if and only if there exists a* W *such that* $W, W^{-1} \in \mathscr{M}\hat{\mathscr{A}}_-$ *and* GW^{-1} *is* J*-lossless. In this case, the solutions to* (4.1) *with* $\|R\|_\infty \leqslant \gamma$ *are generated by*

$$R = R_1 R_2^{-1}, \qquad \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = GW^{-1}\begin{bmatrix} U \\ I_m \end{bmatrix}, \tag{4.13}$$

where $U \in \mathcal{M}\hat{\mathcal{A}}_-$, $\|U\|_\infty \leqslant \gamma$; *and* W *is such that* $W, W^{-1} \in \mathcal{M}\hat{\mathcal{A}}_-$ *and* GW^{-1} *is J-lossless,* R_2 *and* R_2^{-1} *are in* $\mathcal{M}\hat{\mathcal{A}}_-$ *and all* $Q_1, Q_2 \in \mathcal{M}\hat{\mathcal{A}}_-$ *are generated by*

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} R_2 = W^{-1} \begin{bmatrix} U \\ I_m \end{bmatrix} \tag{4.14}$$

where $U \in \mathcal{M}\hat{\mathcal{A}}_-$ *satisfies* $\|U\|_\infty \leqslant \gamma$.

Proof. Apply Theorem 4.4 to obtain a solution of (4.1) with $Q_1, Q_2 \in \mathcal{M}\hat{\mathcal{A}}_-$, $R \in \mathcal{M}\hat{\mathcal{W}}_-$, $R_2^{-1} \in \mathcal{M}\hat{\mathcal{A}}_-$, where $\|R\|_\infty < \gamma$ if and only if there exists a $W \in \mathcal{M}\hat{\mathcal{A}}_-$ such that $W^{-1}, (GW^{-1})_{22}^{-1} \in \mathcal{M}\hat{\mathcal{A}}_-$ and

$$G^{\sim} J_{lm}(\gamma) G = W^{\sim} J_{qm}(\gamma) W \qquad \text{for} \quad |\mathrm{Re}\, s| < \varepsilon.$$

Thus

$$(GW^{-1})^{\sim} J_{lm}(\gamma) GW^{-1} = J_{qm}(\gamma) \qquad \text{for} \quad |\mathrm{Re}\, s| < \varepsilon. \tag{4.15}$$

Lemma 4.5 shows that (4.15) and $(GW^{-1})_{22}^{-1} \in \mathcal{M}\hat{\mathcal{A}}_-$ hold if and only if GW^{-1} is J-lossless.

Equation (4.13) and $R_2^{-1} \in \mathcal{M}\hat{\mathcal{A}}_-$ follow from Theorem 4.4.

From (4.1) we obtain

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = G \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} R_2,$$

and with (4.13), we obtain (4.14), since G is left-invertible. ■

We remark that another version of this theorem holds assuming that G_{21} is square and $\det G_{21}(\infty) \neq 0$. It can be deduced from Corollary 4.6 by swapping columns. Of course, then the off-diagonal blocks of W will be square.

Notice that although we have proved that $R_2^{-1} \in \mathcal{M}\hat{\mathcal{A}}_-$, this will not be true for the inverse of $Q_2 = [(W^{-1})_{21}U + (W^{-1})_{22}]R_2^{-1}$ in general. In the applications to control, the candidate for the controller will be $Q_1 Q_2^{-1}$, and so it is important to derive conditions under which $Q_2(\infty)$ is nonsingular. In this direction, we have the following technical lemmas, whose proof should be the same as in the finite-dimensional case (see Lemmas 3.6, 3.7 in Green [39]). Unfortunately, the proof of Lemma 3.6 in [39] is incorrect, so we include a correct proof here.

Lemma 4.7. *Suppose that $Z, Z^{-1} \in \mathcal{M}\hat{\mathcal{A}}_{-}$. Then there exists a constant matrix U such that $U^*U < \gamma^2 I$ and*

$$[0 \quad I]Z\begin{bmatrix} U \\ I \end{bmatrix}(\infty)$$

is nonsingular if and only if there exist $Y, Y^{-1} \in \mathcal{M}\hat{\mathcal{A}}_{-}$ such that $YJ^{-1}Y^* = ZJ^{-1}Z^*$ and $\det Y_{22}(\infty) \neq 0$.

Proof. If Y exists, then by Lemma 2.7, $Y = ZA$, where $AJ^{-1}A^* = J^{-1}$ or equivalently $A^*JA = J$ and $A^{-1} = J^{-1}A^*J$. The (1, 1) block of $A^*JA = J$ yields

$$A_{11}^* A_{11} - \gamma^2 A_{21}^* A_{22} = I,$$

and A_{11} is invertible with

$$\gamma^2 A_{11}^{-*} A_{21}^* A_{21} A_{11}^{-1} = I - A_{11}^{-*} A_{11}^{-1}. \tag{4.16}$$

Let $U = \gamma^2 A_{11}^{-*} A_{21}^*$ to obtain $UU^* < \gamma^2 I$. Now

$$\begin{aligned} [0 \quad I]Z\begin{bmatrix} U \\ I \end{bmatrix}(\infty) &= [0 \quad I]YA^{-1}\begin{bmatrix} U \\ I \end{bmatrix}(\infty) \\ &= \begin{bmatrix} Y_{21}(\infty) & Y_{22}(\infty) \end{bmatrix} J_{-1} A^* J \begin{bmatrix} U \\ I \end{bmatrix} \\ &= \begin{bmatrix} Y_{21}(\infty) & Y_{22}(\infty) \end{bmatrix} \begin{bmatrix} A_{11}^* U - \gamma^2 A_{21}^* \\ -\dfrac{1}{\gamma^2} A_{12}^* U + A_{22}^* \end{bmatrix} \\ &= Y_{22}(\infty)\left(-A_{12}^* A_{11}^{-*} A_{21}^* + A_{22}^* \right), \end{aligned}$$

which will be nonsingular if $S = A_{22} - A_{21} A_{11}^{-1} A_{12}$ is.

Suppose on the contrary that there exists an $x \neq 0$ such that $Sx = 0$. Then the (1, 2) block of $A^*JA = J$ yields

$$A_{11}^* A_{12} x = \gamma^2 A_{21}^* A_{21} A_{11}^{-1} A_{12} x$$

and

$$A_{12}x = \gamma^2 (A_{11}^{-1})^* A_{21}^* A_{21} A_{11}^{-1} A_{12} x$$
$$= [I - (A_{11}^{-1})^* A_{11}^{-1}] A_{12} x \qquad \text{from (4.16).}$$

Thus $A_{12}x = 0$ and $A_{22}x = 0$. But examining the (2, 2) block of $A^*JA = J$ shows that A_{22} is invertible and so $x = 0$. Hence S is nonsingular.

Conversely, if U exists, set $Y = ZA$, where

$$A = \begin{bmatrix} (\gamma^2 I - UU^*)^{-1/2} & 0 \\ 0 & (\gamma^2 I - U^*U)^{-1/2} \end{bmatrix} \begin{bmatrix} \gamma I & \gamma U \\ \gamma^{-1} U^* & \gamma I \end{bmatrix},$$

and verify that $AJ^{-1}A^* = J^{-1}$ and that

$$Y_{22} = [0 \quad I] Y \begin{bmatrix} 0 \\ I \end{bmatrix}$$
$$= [0 \quad I] Z \begin{bmatrix} U \\ I \end{bmatrix} \gamma (\gamma^2 I - U^*U)^{-1/2}.$$

Thus $Y_{22}(\infty)$ is nonsingular. ■

Identifying Z with W^{-1}, we obtain the following result.

COROLLARY 4.8. *Suppose that* $G \in \hat{\mathcal{M}\mathcal{A}}_-$ *under the assumptions of Corollary* 4.6. *Suppose that in the parametrization of* (4.13)–(4.14) *there exists a* $U \in \hat{\mathcal{M}\mathcal{A}}_-$ *with* $\|U\|_\infty < \gamma$ *such that* $Q_2(\infty)$ *is nonsingular. Then there exists a* $\overline{W}$ *satisfying the conditions of Corollary* 4.6 *with the additional property that* $\det(\overline{W}^{-1})_{22}(\infty) \neq 0$. *Consequently, with this choice of* $\overline{W}$, $\det Q_2(\infty) \neq 0$ *for every strictly proper* $U \in \hat{\mathcal{M}\mathcal{A}}_-$ $(U(\infty) = 0)$.

Proof. Apply Lemma 4.7 to $Z = W^{-1}$, $Y = \overline{W}^{-1}$, noting that $Q_2 R_2 = Z_{21}U + Z_{22}$ is nonsingular at ∞. ■

Under certain conditions on G we can guarantee that W_{21} (or W_{22}) is strictly proper.

LEMMA 4.9. *Consider* $G \in \hat{\mathscr{A}}_{-}^{(l+m)\times(q+m)}$, *where* G_{21} *(respectively,* G_{22}*) is strictly proper and there exists a* $W \in \hat{\mathscr{A}}_{-}^{(q+m)\times(q+m)}$ *satisfying* $W^{-1} \in \hat{\mathscr{A}}_{-}^{(q+m)\times(q+m)}$ *and* GW^{-1} *is J-lossless. Then there exists another* $\overline{W}$ *with the same properties and such that in addition,* $\overline{W}_{21}$ *(respectively,* $\overline{W}_{22}$*) is strictly proper.*

Proof. Consider the $G_{12}(\infty) = 0$ case. There holds

$$(W^*JW)(\infty) = G^*JG(\infty),$$

and since $W(\infty)$ is nonsingular, $[G^*JG(\infty)]_{11} > 0$. Taking the Schur complement, we obtain $\overline{W}(\infty)$ such that $\overline{W}_{21}(\infty) = 0$. Define $\overline{W} = \overline{W}(\infty)W(\infty)^{-1}W$. ∎

We remark that in the case that G_{21} is strictly proper, we can achieve $W_{21}(\infty) = 0$. Since W is invertible, we must have $W_{22}(\infty)$ and $(W^{-1})_{22}(\infty)$ nonsingular. So by Corollary 4.8 we can obtain well-posed controllers by choosing a strictly proper U. This applies to the model matching problem (4.2).

Finally, we quote a last technical lemma we need in Section 5.

LEMMA 4.10. *Suppose that* $X \in \hat{\mathscr{A}}_{-}^{(l+m)\times(q+m)}$ *is J-lossless, and define G and W*

$$W = \begin{bmatrix} I & 0 \\ X_{21} & X_{22} \end{bmatrix}^{-1}, \qquad G = XW = \begin{bmatrix} G_{11} & G_{12} \\ 0 & I \end{bmatrix}.$$

(i) *If* $U \in \mathscr{M}\hat{\mathscr{A}}_{-}$ *with* $\|U\|_\infty \leqslant \gamma$, *then* $X_{21}U + X_{22}$ *is invertible over* $\mathscr{M}\hat{\mathscr{A}}_{-}$ *and* $(X_{11}U + X_{12})(X_{21}U + X_{22})^{-1} \in \mathscr{M}\hat{\mathscr{A}}_{-}$ *with* $\mathbf{H}_\infty$*-norm* $\leqslant \gamma$.

(ii) *If* $Q \in \mathscr{M}\hat{\mathscr{A}}_{-}$ *is such that* $\|G_{11}Q + G_{12}\|_\infty < \gamma$, *then* $I - X_{21}Q$ *is invertible over* $\in \mathscr{M}\hat{\mathscr{A}}_{-}$ *and* $Q(I - X_{21}Q)^{-1}X_{22} = (I - QX_{21})^{-1}QX_{22} \in \mathscr{M}\hat{\mathscr{A}}_{-}$ *with* $\mathbf{H}_\infty$*-norm* $\leqslant \gamma$.

Proof. This is similar to Lemma 3.4 in Green [39]. It is proved in detail in Lemma 4.7 in Curtain and Green [23]. ∎

5. $\mathbf{H}_\infty$-CONTROL PROBLEMS

In this section we synthesize the results in Section 3 on stabilizing controllers with the results in Section 4 on stable analytic system problems. The type of control problems we consider are those of finding a stabilizing controller K for a plant P such that the closed-loop system satisfies a norm constraint $\|\mathscr{F}(P, K)\|_\infty < \gamma$; these are termed $\mathbf{H}_\infty$-control problems in the literature.

Our approach is to reduce the general $\mathbf{H}_\infty$-control problem to two simpler ones of the output estimation type. So we first obtain solutions for the latter. As it happens, it is more convenient to obtain solutions to the dual disturbance feedforward problem and then appeal to duality to solve the output estimation problem.

THEOREM 5.1 (Disturbance feedforward problems). *Consider $P \in \hat{\mathscr{B}}^{(l+m)\times(m+q)}$ which has a left-coprime factorization*

$$P = \begin{bmatrix} I_l & \hat{M}_{12} \\ 0 & \hat{M}_{22} \end{bmatrix}^{-1} \begin{bmatrix} 0 & \hat{N}_{12} \\ I_m & \hat{N}_{22} \end{bmatrix}$$

where $G = \begin{bmatrix} \hat{N}_{12} & -\hat{M}_{12} \\ -\hat{N}_{22} & \hat{M}_{22} \end{bmatrix}$ *is left-invertible over $\mathscr{M}\hat{\mathscr{W}}_-$.*

There exists a stabilizing controller $K \in \mathscr{M}\hat{\mathscr{B}}$ for P such that $\|\mathscr{F}(P, K)\|_\infty < \gamma$ if and only if P has a right-coprime factorization

$$P = NM^{-1}, \qquad \text{with} \quad \begin{bmatrix} N_{11} & N_{12} \\ M_{11} & M_{12} \end{bmatrix} \ J\text{-lossless and} \ \det N_{22}(\infty) \neq 0. \tag{5.1}$$

In this case, all stabilizing controllers $K \in \mathscr{M}\hat{\mathscr{B}}$ such that $\|\mathscr{F}(P, K)\|_\infty \leqslant \gamma$ are given by $K = K_1 K_2^{-1}$, where

$$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} M_{21} & M_{22} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} U \\ I_m \end{bmatrix} \tag{5.2}$$

for $U \in \mathscr{M}\hat{\mathscr{A}}_-$ with $\|U\|_\infty \leqslant \gamma$ and $\det K_2(\infty) \neq 0$.

Proof. This follows the proof of Theorem 3.8 in Green [39], where we use Corollary 3.6 on the parametrization of stabilizing controllers, and Theorem 4.8 on solutions to stable analytic system problems. For a complete proof see Theorem 5.1 in Curtain and Green [23]. ■

While disturbance feedforward problems reduce to stable analytic system problems of the type (4.1), output estimation problems reduce to a transposed problem. The solution is most efficiently obtained by taking the transpose of Theorem 5.1.

THEOREM 5.2 (Output estimation problems). *Consider $P \in \mathcal{M}\hat{\mathcal{B}}$ which has a right-coprime factorization*

$$P = \begin{bmatrix} 0 & I_q \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} I_p & 0 \\ M_{21} & M_{22} \end{bmatrix}^{-1},$$

where

$$G = \begin{bmatrix} N_{21} & -N_{22} \\ -M_{21} & M_{22} \end{bmatrix}$$

is right-invertible over $\mathcal{M}\hat{\mathcal{W}}_-$. There exists a stabilizing controller $K \in \mathcal{M}\hat{\mathcal{B}}$ for P such that $\|\mathcal{F}(P, K)\|_\infty < \gamma$ if and only if P has a left-coprime factorization $P = \hat{M}^{-1}\hat{N}$, with

$$\begin{bmatrix} \hat{N}_{11} & \hat{M}_{11} \\ \hat{N}_{21} & \hat{M}_{21} \end{bmatrix}$$

conjugate J-lossless and $\det \hat{N}_{22}(\infty) \neq 0$.

In this case, all stabilizing controllers $K \in \mathcal{M}\hat{\mathcal{B}}$ such that $\|\mathcal{F}(P, K)\|_\infty \leqslant \gamma$ are given by $K = \hat{K}_2^{-1}\hat{K}_1$, where $\hat{K}_1$, $\hat{K}_2$ satisfy

$$\begin{bmatrix} \hat{K}_1, \hat{K}_2 \end{bmatrix} = \begin{bmatrix} U & I_q \end{bmatrix} \begin{bmatrix} \hat{M}_{12} & \hat{N}_{12} \\ \hat{M}_{22} & \hat{N}_{22} \end{bmatrix}$$

for $U \in \mathcal{M}\hat{\mathcal{A}}_-$ with $\|U\|_\infty \leqslant \gamma$ and $\det \hat{K}_{22}(\infty) \neq 0$. ■

We now proceed to consider more general $\mathbf{H}_\infty$-control problems. First we show that the existence of a stabilizing controller K such that $\|\mathscr{F}(P, K)\|_\infty < \gamma$ implies that a certain analytic system of the model matching type (4.2) must have a solution, and consequently a certain J-lossless factorization must exist.

LEMMA 5.3. *Suppose that $P \in \hat{\mathscr{B}}^{(l+m)\times(p+q)}$ has a right-coprime factorization $P = YX^{-1}$ such that*

$$X = \begin{bmatrix} I_p & 0 \\ X_{21} & X_{22} \end{bmatrix},$$

Y_{22}, X_{22} are right-coprime, and Y_{12} is left-invertible over $\mathscr{M}\hat{\mathscr{W}}_-$. If there exists a stabilizing controller $K \in \mathscr{M}\hat{\mathscr{B}}$ for P such that $\|\mathscr{F}(P, K)\|_\infty < \gamma$, then P has a right-coprime factorization $P = NM^{-1}$ with

$$\begin{bmatrix} N_{11} & N_{12} \\ M_{11} & M_{12} \end{bmatrix}$$

J-lossless and det $M_{21}(\infty) \neq 0$.

Proof. This generalizes Lemma 4.1 in Green [39]. For a detailed proof see Lemma 5.3 in Curtain and Green [23]. ∎

Using left-coprime factorizations of P, one obtains the following dual result.

LEMMA 5.4. *Suppose that $P \in \hat{\mathscr{B}}^{(l+m)\times(p+q)}$ has a left-coprime factorization $P = \hat{X}^{-1}\hat{Y}$ such that*

$$\hat{X} = \begin{bmatrix} I_l & \hat{X}_{12} \\ 0 & \hat{X}_{22} \end{bmatrix},$$

$\hat{Y}_{22}, \hat{X}_{22}$ are left-coprime, and $\hat{Y}_{21}$ is right-invertible over $\mathscr{M}\hat{\mathscr{W}}_-$. If there exists a stabilizing controller $K \in \mathscr{M}\hat{\mathscr{B}}$ such that $\|\mathscr{F}(P, K)\|_\infty < \gamma$, then P has

a left-coprime factorization $P = \hat{M}^{-1}\hat{N}$, *with*

$$\begin{bmatrix} \hat{N}_{11} & \hat{M}_{11} \\ \hat{N}_{21} & \hat{M}_{21} \end{bmatrix}$$

conjugate J-lossless and det $\hat{M}_{12}(\infty) \neq 0$. ■

The key step for considering a more general $\mathbf{H}_\infty$-control problem is to show that K is a stabilizing controller for P with $\|\mathscr{F}(P, K)\|_\infty < \gamma$ if and only if K is a stabilizing controller for P_+ with $\|\mathscr{F}(P_+, K)\|_\infty < \gamma$, where P_+ corresponds to an output estimation problem which we have already solved in Theorem 5.2. Since this step was not proved completely in Green [39, Theorem 4.3], we give a complete proof here.

THEOREM 5.5. *Suppose that* $P \in \hat{\mathscr{B}}^{(l+m)\times(p+q)}$ *has a right-coprime factorization* $P = NM^{-1}$ *such that*

$$\begin{bmatrix} N_{11} & N_{12} \\ M_{11} & M_{12} \end{bmatrix}$$

is J-lossless and det $M_{21}(\infty) \neq 0$. *The following are equivalent statements*:

(i) $K \in \mathscr{M}\hat{\mathscr{B}}$ *is a stabilizing controller for* P *such that* $\|\mathscr{F}(P, K)\|_\infty \leqslant \gamma$.

(ii) $K \in \mathscr{M}\hat{\mathscr{B}}$ *is a stabilizing controller for* $P_+ \in \hat{\mathscr{B}}^{(q+m)\times(p+q)}$ *such that* $\|\mathscr{F}(P_+, K)\|_\infty \leqslant \gamma$, *where* P_+ *is given by*

$$P_+ = \begin{bmatrix} I_q & 0 \\ N_{21} & N_{22} \end{bmatrix}\begin{bmatrix} 0 & I_p \\ M_{21} & M_{22} \end{bmatrix}^{-1} = \begin{bmatrix} 0 & I_q \\ N_{22} & N_{21} \end{bmatrix}\begin{bmatrix} I_p & 0 \\ M_{22} & M_{21} \end{bmatrix}^{-1}. \tag{5.3}$$

(iii) *Moreover*, $\|\mathscr{F}(P, K)\|_\infty < \gamma$ *if and only if* $\|\mathscr{F}(P_+, K)\|_\infty < \gamma$.

Proof.

(1) Since

$$\begin{bmatrix} N_{11} & N_{12} \\ M_{11} & M_{12} \end{bmatrix}$$

is J-lossless, by Lemma 4.5 M_{12} is invertible over $\mathcal{M}\hat{\mathcal{A}}_-$. Consider a new factorization of $P = \overline{N}\overline{M}^{-1}$, given by

$$\begin{bmatrix} \overline{N} \\ \overline{M} \end{bmatrix} := \begin{bmatrix} N \\ M \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ I & 0 \end{bmatrix}^{-1}. \tag{5.4}$$

Notice that

$$\overline{M} = \begin{bmatrix} I & 0 \\ \overline{M}_{21} & \overline{M}_{22} \end{bmatrix}.$$

Applying Lemma 3.2, we see that K stabilizes P if and only if $K = \overline{K}_2^{-1}\overline{K}_1$, where

$$\begin{bmatrix} \overline{K}_1 & \overline{K}_2 \end{bmatrix} \begin{bmatrix} \overline{N}_{21} & -\overline{N}_{22} \\ -\overline{M}_{21} & \overline{M}_{22} \end{bmatrix} = \begin{bmatrix} Q & I \end{bmatrix}, \tag{5.5}$$

with $\overline{K}_1, \overline{K}_2 \in \mathcal{M}\hat{\mathcal{A}}_-$ and $\det \overline{K}_{22}(\infty) \neq 0$, and

$$\begin{bmatrix} \mathcal{F}(P, K) \\ I \end{bmatrix} = \begin{bmatrix} \overline{N}_{11} & \overline{N}_{12} \\ I & 0 \end{bmatrix} \begin{bmatrix} I \\ Q \end{bmatrix}. \tag{5.6}$$

(2) Suppose (i) holds, and let $K = \overline{K}_2^{-1}\overline{K}_1$ be such that $\overline{K}_1, \overline{K}_2$ satisfy (5.5) with $\|\mathcal{F}(P, K)\|_\infty \leqslant \gamma$. Using Lemma 4.10(ii) with

$$X = \begin{bmatrix} N_{11} & N_{12} \\ M_{11} & M_{12} \end{bmatrix}, \qquad W = \begin{bmatrix} I & 0 \\ M_{11} & M_{12} \end{bmatrix}^{-1}, \qquad G = XW = \begin{bmatrix} \overline{N}_{12} & \overline{N}_{11} \\ 0 & I \end{bmatrix},$$

from (5.4) it follows that since $\|G_{12} + G_{11}Q\|_\infty = \|\mathcal{F}(P, K)\|_\infty \leqslant \gamma$ [from (5.6)], $I - M_{11}Q$ is invertible over $\mathcal{M}\hat{\mathcal{A}}_-$ and $R = (I - QM_{11})^{-1}QM_{12} \in \mathcal{M}\hat{\mathcal{A}}_-$ with norm $< \gamma$. Now premultiplying (5.5) by $(I - QM_{11})^{-1}$ and postmultiplying it by

$$\begin{bmatrix} M_{12} & -M_{11} \\ 0 & I \end{bmatrix}$$

yields

$$\begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} N_{22} & -N_{21} \\ -M_{22} & M_{21} \end{bmatrix} = \begin{bmatrix} R & I \end{bmatrix}, \tag{5.7}$$

where $[K_1, K_2] = (I - QM_{11})^{-1}[\bar{K}_1, \bar{K}_2]$ and $R = (I - QM_{11})^{-1}QM_{12}$. From the above we see that $R \in \mathcal{M}\hat{\mathcal{A}}_-$ with $\|R\|_\infty \leqslant \gamma$, $K_1, K_2 \in \mathcal{M}\hat{\mathcal{A}}_-$, and $\det \bar{K}_2(\infty) \neq 0$. So applying Lemma 3.5 to P^+, we see that $K_2^{-1}K_1 = \bar{K}_2^{-1}\bar{K}_1 = K$ stabilizes P_+ and that $\|\mathcal{F}(P_+, K)\|_\infty = \|R\|_\infty \leqslant \gamma$.

(3) Suppose now that (ii) holds, and let $K = K_2^{-1}K_1$ stabilize P_+. By Lemma 3.2, (5.7) holds and $\|\mathcal{F}(P_+, K)\|_\infty = \|R\|_\infty \leqslant \gamma$. We apply Lemma 4.10(i) to

$$X = \begin{bmatrix} N_{11} & N_{12} \\ M_{11} & M_{12} \end{bmatrix}$$

as before and conclude that $M_{11}R + M_{12}$ is invertible over $\mathcal{M}\hat{\mathcal{A}}_-$ and $(N_{11}R + N_{12})(M_{11}R + M_{12})^{-1} \in \mathcal{M}\hat{\mathcal{A}}_-$ with norm $< \gamma$. Since $M_{12}^{-1} \in \mathcal{M}\hat{\mathcal{A}}_-$, it follows that $(I + M_{12}^{-1}M_{11}R)^{-1} \in \mathcal{M}\hat{\mathcal{A}}_-$ and equivalently, $(I + RM_{12}^{-1}M_{11})^{-1} \in \mathcal{M}\hat{\mathcal{A}}_-$. Define

$$\left[\bar{K}_1, \bar{K}_2\right] = \left(I + RM_{12}^{-1}M_{11}\right)^{-1}[K_1, K_2]$$

and $Q = (I + RM_{12}^{-1}M_{11})^{-1}RM_{12}^{-1} = R(M_{11}R + M_{12})^{-1}$, and verify that (5.5) and (5.6) hold with $\mathcal{F}(P, K) = (N_{11}R + N_{12})(M_{11}R + M_{12})^{-1}$ having norm $< \gamma$. So by Lemma 3.2, K stabilizes P.

(4) We now prove (iii). Consider

$$\begin{bmatrix} z \\ y \end{bmatrix} = P\begin{bmatrix} w \\ u \end{bmatrix} = N\begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \qquad \text{where} \quad \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = M^{-1}\begin{bmatrix} w \\ u \end{bmatrix}.$$

There holds

$$\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ M_{11} & M_{12} \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

and the J-lossless property of $\begin{bmatrix} N_{11} & N_{12} \\ M_{11} & M_{12} \end{bmatrix}$ gives

$$\|z\|^2 - \gamma^2\|w\|^2 = \|\alpha\|^2 - \gamma^2\|\beta\|^2. \tag{5.8}$$

Note that

$$\begin{bmatrix} \alpha \\ y \end{bmatrix} = \begin{bmatrix} I_q & 0 \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

$$\begin{bmatrix} \beta \\ u \end{bmatrix} = \begin{bmatrix} 0 & I \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

and so

$$\begin{bmatrix} \alpha \\ y \end{bmatrix} = P_+ \begin{bmatrix} \beta \\ u \end{bmatrix}.$$

With $u = Ky$, it is readily verified that

$$\alpha = \mathscr{F}(P_+, K)\beta \quad \text{and} \quad z = \mathscr{F}(P, K)w.$$

Now

$$\begin{aligned} \|\mathscr{F}(P, K)\omega\|^2 - \|\mathscr{F}(P_+, K)\beta\|^2 &= \|z\|^2 - \|\alpha\|^2 \\ &= \gamma^2\left(\|\omega\|^2 - \|\beta\|^2\right) \qquad \text{from (5.8)}. \end{aligned}$$

If $\|\mathscr{F}(P, K)\omega\|^2 < \gamma^2\|\omega\|^2$, then

$$\|\mathscr{F}(P_+, K)\beta\|^2 + \gamma^2\left(\|\omega\|^2 - \|\beta\|^2\right) < \gamma^2\|\omega\|^2$$

and so

$$\|\mathscr{F}_+(P, K)\beta\|^2 \leqslant \gamma^2\|\beta\|^2.$$

Conversely, if $\|\mathscr{F}(P, K)\beta\|^2 \leqslant \gamma^2\|\beta\|^2$, then $\|\mathscr{F}(P, K)\omega\|^2 \leqslant \gamma\|\omega\|^2$. This proves that $\|\mathscr{F}(P_+, K)\|_\infty \leqslant \gamma$ is equivalent to $\|\mathscr{F}(P, K)\|_\infty \leqslant \gamma$. ■

So under certain factorization conditions on P, we have reduced the $\mathbf{H}_\infty$-control problem for P to an output estimation problem solved in Theorem 5.2. This results in the solution to the general $\mathbf{H}_\infty$-control problem for our class of infinite-dimensional systems.

THEOREM 5.6. *Suppose that $P \in \hat{\mathscr{B}}^{(l+m)\times(p+q)}$ has a right-coprime factorization $P = YX^{-1}$ and a left-coprime factorization $P = \hat{X}^{-1}\hat{Y}$ such that*

$$X = \begin{bmatrix} I_p & 0 \\ X_{21} & X_{22} \end{bmatrix},$$

$$Y_{22},\ X_{22} \text{ are right-coprime, and } Y_{12} \text{ is left-invertible over } \mathscr{M}\hat{\mathscr{W}}_-, \tag{5.9}$$

$$\hat{X} = \begin{bmatrix} I_l & \hat{X}_{12} \\ 0 & \hat{X}_{22} \end{bmatrix},$$

$$\hat{X}_{22},\ \hat{Y}_{22} \text{ are left-coprime, and } \hat{Y}_{12} \text{ is right-invertible over } \mathscr{M}\hat{\mathscr{W}}_-. \tag{5.10}$$

There exists a stabilizing controller $K \in \mathscr{M}\hat{\mathscr{B}}$ for P such that $\|\mathscr{F}(P, K)\|_\infty < \gamma$ if and only if the following two conditions hold: P has a right-coprime factorization $P = NM^{-1}$ with

$$\begin{bmatrix} N_{11} & N_{12} \\ M_{11} & M_{12} \end{bmatrix} \text{ J-lossless and } \det M_{21}(\infty) \neq 0, \tag{5.11}$$

and P_+ has a left-coprime factorization $P_+ = \hat{M}^{-1}\hat{N}$ with

$$\begin{bmatrix} \hat{N}_{11} & \hat{M}_{11} \\ \hat{N}_{21} & \hat{M}_{21} \end{bmatrix} \text{ conjugate J-lossless and } \det \hat{N}_{22}(\infty) \neq 0, \tag{5.12}$$

where

$$P_+ = \begin{bmatrix} I_q & 0 \\ N_{21} & N_{22} \end{bmatrix}\begin{bmatrix} 0 & I_p \\ M_{21} & M_{22} \end{bmatrix}^{-1} \in \hat{\mathscr{B}}^{(l+m)\times(p+q)}. \tag{5.13}$$

In this case, K is a stabilizing controller for P such that $\|\mathscr{F}(P, K)\|_\infty \leqslant \gamma$ if and only if $K = \hat{K}_2^{-1}\hat{K}_1$, where

$$\left[\hat{K}_1, \hat{K}_2\right] = \begin{bmatrix} U & I_q \end{bmatrix}\begin{bmatrix} \hat{M}_{12} & \hat{N}_{12} \\ \hat{M}_{22} & \hat{N}_{22} \end{bmatrix}, \qquad U \in \hat{\mathscr{A}}^{q\times m} \tag{5.14}$$

with $\|U\|_\infty \leqslant \gamma$ and $\det \hat{K}_2(\infty) \neq 0$.

Equivalently, $K = \mathscr{F}(S, U)$, *where*

$$S = \begin{bmatrix} \hat{N}_{12} & I_m \\ \hat{N}_{22} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \hat{M}_{12} & 0 \\ \hat{M}_{22} & I_q \end{bmatrix}. \tag{5.15}$$

Proof. This follows the proof of Theorem 4.4 in Green [39]. For details see Curtain and Green [23, Theorem 5.6]. ∎

We remark that in Green [39] it is shown that the assumptions (5.9) and (5.10) reduce to the usual rank conditions imposed for solving the standard $\mathbf{H}_\infty$ optimal-control problem using a state-space approach. For the case that G has a realization as a Pritchard-Salamon system, (5.9) and (5.10) will reduce to the so-called "invariant zeros conditions" on p. 134 in Van Keulen [44]. Furthermore, using the results on J-spectral factorizations in terms of Riccati equations from Weiss [48], one can rederive the state-space Riccati-equation solution given in [44]. However, this will not be possible for more general classes of systems, as the Riccati equations need not be well posed (cf. the parabolic case in [19]).

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REFERENCES

1 V. M. Adamjan, D. Z. Arov, and M. G. Krein, Infinite block Hankel matrices and related extension problems, *AMS Transl.* III:133–156 (1987).

2 J. A. Ball, *Invariant Subspace Representations, Unitary Interpolants and Factorization Indices*, Oper. Theory Adv. Appl. 12, Birkhäuser, Basel, 1984, pp. 11–38.

3 J. A. Ball and J. W. Helton, A Beurling-Lax theorem for the Lie group $U(m, n)$ which contains most classical interpolation theory, *J. Operator Theory* 8:107–142 (1983).

4 J. A. Ball and J. W. Helton, Beurling-Lax representation using classical Lie groups with many applications III: Groups preserving two bilinear forms, *Amer. J. Math.* 108:95–174 (1986).

5 J. A. Ball and N. Cohen, Sensitivity minimization in an H^∞ norm: Parametrization of all suboptimal solutions, *Internat. J. Control* 46:785–816 (1987).

6 J. A. Ball and J. W. Helton, *Shift Invariant Subspace, Passivity and Reproducing Kernels and* $\mathbf{H}^\infty$*-Optimization*, Oper. Theory Adv. Appl. 55, Birkhaüser, Basel, 1988.

7 J. A. Ball, J. W. Helton, and M. Verma, A factorization principle for stabilization of linear control systems, *Internat. J. Robust and Nonlinear Control* 1(4):229–294 (1991).

8 J. A. Ball and A. C. M. Ran, Optimal Hankel norm model reductions and Wiener-Hopf factorization, I: The canonical case, *SIAM J. Control Optim.* 25:362–383 (1987).

9 H. Bart, I. Gohberg, and M. A. Kaashoek, *Minimal Factorization of Matrix and Operator Functions*, Birkhäuser, Basel, 1979.

10 H. Bart, I. Gohberg, and M. A. Kaashoek, Wiener-Hopf equations with symbols analytic in a strip, in *Constructive Methods of Wiener-Hopf Factorization*, (I. Gohberg and M. A. Kaashoek, Eds.), Oper. Theory Adv. Appl. 21, Birkhäuser, 1986, pp. 39–74.

11 H. Bart, I. Gohberg, and M. A. Kaashoek, Wiener-Hopf factorization, Inverse Fourier transform and exponentially dichotomous operators, *J. Funct. Anal.* 86:1–42 (1986).

12 F. M. Callier and C. A. Desoer, An algebra of transfer functions for distributed linear time-invariant systems, *IEEE Trans. Circuits and Systems* 25:651–663 (1978); Corrections, 26:320 (1979).

13 F. M. Callier and C. A. Desoer, Simplifications and new connections on an algebra of transfer functions of distributed linear time-invariant systems, *IEEE Trans. Circuits and Systems* CAS-27:320–323 (1980).

14 F. M. Callier and C. A. Desoer, Stabilization, tracking and disturbance rejection in multivariable convolution systems, *Ann. Soc. Sci. Bruxelles* 94(I):7–51 (1980).

15 F. M. Callier and J. Winkin, The spectral factorization problem for SISO distributed systems, modelling, robustness and sensitivity reduction, in *Control Systems* (R. F. Curtain, Ed.), NATO Adv. Sci. Inst. Ser. F Comput. Systems Sci., Springer-Verlag, Berlin, 1986, pp. 463–489.

16 F. M. Callier and J. Winkin, On spectral factorization and LQ-optimal regulation for multivariable distributed systems, *Internat. J. Control* 52:55–75 (1990).

17 R. F. Curtain, Robust stabilizability of normalized coprime factors; the infinite-dimensional case, *Internat. J. Control* 51:1173–1190 (1990).

18 R. F. Curtain and A. Ichikawa, The Nehari problem for parabolic systems with Dirichlet boundary control, *Integral Equations Operator Thoery*, submitted for publication.

19 R. F. Curtain and A. Ran, Explicit formulas for Hankel norm approximations of infinite-dimensional systems, *Integral Equations Operator Theory* 13:455–469 (1989).

20 R. F. Curtain and K. Glover, Robust stabilization of infinite dimensional systems by finite dimensional controllers, *Systems Control Lett.* 7:41–47 (1986).

21 R. F. Curtain, H. Logemann, S. Townley, and H. J. Zwart, Well-posedness, Stabilizability and Admissibility for Pritchard-Salamon Systems, Rep. 260, Inst. für Dynamische Systeme, Univ. of Bremen, Germany, 1992.

22 R. F. Curtain and A. Rodriguez, Necessary and sufficient conditions for J-spectral factorizations with a J-lossless property for infinite-dimensional systems in continuous and discrete time, *Linear Algebra Appl.* 203:327–358 (1994).

23 R. F. Curtain and M. Green, Analytic System Problems and *J*-Lossless Coprime Factorization of Infinite-Dimensional Linear Systems, Rep. W-9218, Mathematics Inst., Univ. of Groningen, The Netherlands.
24 R. F. Curtain and H. J. Zwart, The Nehari problem for Pritchard-Salamon systems: A direct approach, *Integral Equations Operator Theory* 18:130–153 (1994).
25 J. C. Doyle, K. Glover, P. Khargonekar and B. Francis, State-space solutions to standard H_2 and H_∞ control problems, *IEEE Trans. Automat. Control* AC-34(8):831–847 (1989).
26 H. Dym, *J Contractive Matrix Functions, Reproducing Kernel Hilbert Spaces and Interpolation*, CBMS Regional Conf. Ser. in Math. 71, Amer. Math. Soc., Providence, 1989.
27 D. S. Flamm, and S. K. Mitter, $\mathbf{H}^\infty$-sensitivity for delay systems: Part I, *Systems Control Lett.* 9:17–24 (1987).
28 C. Foias and A. Tannenbaum, On the Nehari problem for a certain class of $\mathbf{L}_\infty$-functions appearing in control theory, *J. Funct. Anal.* 74:146–159 (1987).
29 C. Foias and A. Tannenbaum, On the four block problem I, in *Oper. Theory Adv. Appl.* 32, Birkhäuser, Basel, 1988, pp. 93–112.
30 C. Foias and A. Tannenbaum, On the four block problem II, *Integral Equations Operator Theory* 11:726–767 (1988).
31 C. Foias, A. Tannenbaum, and G. Zames, Weighted sensitivity minimization for delay systems, *IEEE Trans. Automat. Control* AC-31:763–766 (1986).
32 C. Foias, A. Tannenbaum, and G. Zames, On the $\mathbf{H}^\infty$-optimal sensitivity problem for systems with delays, *SIAM J. Control Optim.* 25:686–705 (1987).
33 B. A. Francis, *A Course in H_∞-Control Theory*, Lecture Notes in Control and Inform. Sci. 88, Springer-Verlag, Berlin, 1987.
34 B. A. Francis and J. C. Doyle, Linear control theory with an H_∞-optimality criterion, *SIAM J. Control Optim.* 25:815–844 (1987).
35 Y. Genin, P. Van Dooren, and T. Kailath, On Σ-lossless transfer functions and related questions, *Linear Algebra Appl.* 50:251–274 (1984).
36 K. Glover, All optimal Hankel norm approximations of linear multivariable systems and their L_∞-error bounds, *Internat. J. Control* 39:1115–1193 (1984).
37 K. Glover and J. Doyle, State-space formulae for all stabilizing controllers that satisfy a H^∞-norm bound and relations to risk sensitivity, *Systems Control Lett.* 11:167–172 (1988).
38 I. Gohberg, S. Goldberg, and M. Kaashoek, *Classes of Linear Operators*, Vols. 1, 2, Birkhaüser, 1990, 1993.
39 M. Green, $\mathbf{H}^\infty$ controller synthesis by *J*-lossless coprime factorization, *SIAM J. Control Optim.* 30:522–547 (1992).
40 M. Green, K. Glover, D. J. N. Limebeer, and J. Doyle, A *J*-spectral factorization approach to $\mathbf{H}_\infty$ control, *SIAM J. Control Optim.* 28:1350–1371 (1990).
41 J. W. Helton et al., *Operator Theory, Analytic Functions, Matrices and Electrical Engineering*, Conference Board of the Mathematical Sciences, Regional Conf. Ser. in Math. 68, Amer. Math. Soc., 1987.
42 E. Hille, *Analytic Function Theory*, Vol. 1, Chelsea, New York, 1984.

43 T. Kailath, *Linear Systems*, Prentice-Hall, Englewood Cliffs, N.J., 1980.
44 B. A. M. Van Keulen, *H_∞-Control for Distributed Parameter Systems: A State-Space Approach*, Birkhäuser, Basel, 1993.
45 C. McMillan and R. Triggiani, Min-max game theory and algebraic Riccati equations for boundary control problems with continuous input-output solution map. Part I: The stable case, in: *Marcel Dekker Notes in Pure and Applied Mathematics* (Clement and Lumer, Eds.), 1992, Chapter 32, pp. 377–403.
46 C. McMillan and R. Triggiani, Min-max game theory and algebraic Riccati equations for boundary control problems with analytic semigroups. Part I: The stable case, in *Marcel Dekker Notes in Pure and Applied Mathematics*, (Elworthy et al., Eds.), 1993, Chapter 47, pp. 757–780.
47 M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, MIT Press, Cambridge, Mass., 1995.
48 M. Weiss, Riccati Equations in Hilbert Spaces: A Popov Function Approach, Ph.D. Thesis, Univ. of Groningen, The Netherlands, 1994.
49 G. Zames and B. A. Francis, Feedback, minimax sensitivity and optimal robustness, *IEEE Trans. Automat. Control* AC-28:585–601 (1983).

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