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**Local and global solutions of a differential equation on a curve**

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INTRODUCTION

The curve  $C$  is a non-singular projective curve defined over a complete non-archimedean valued field  $K$  having characteristic zero. After a finite extension of  $K$  the curve admits a stable reduction (see [6]) and we will assume in the sequel that  $C$  has already a stable reduction over  $K$ .

On  $C$  we consider a differential equation of the form

$$(0.1) \quad \frac{d}{dz} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = A \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

where  $A$  is a  $n \times n$ -matrix with coefficients in the function field  $K(C)$  of  $C$  and where  $z \in K(C)$  is transcendental over  $K$ .

The differential equation above can be translated into a connection

$$(0.2) \quad \nabla : V \rightarrow \Omega_{K(C)/K}^1 \otimes V$$

in which  $V$  is an  $n$ -dimensional vectorspace over  $K(C)$  and where  $\Omega_{K(C)/K}^1$  denotes the module of differentials of  $K(C)$  over  $K$ .

The differential equation is called *locally trivial* (with respect to the rigid analytic Grothendieck topology on  $C$ ) if there exists a finite affinoid

(0.3) covering  $\{C_1, \dots, C_s\}$  of  $C$  and for each  $i \in \{1, \dots, s\}$  a fundamental matrix  $F_i$  solving (0.1). The entries of the matrix  $F_i$  are meromorphic functions on  $C_i$  and  $\det(F_i)$  is an invertible meromorphic function on  $C_i$ .

The aim of this paper is to answer a question, posed by *F. Baldassarri*: Suppose that equation (0.1) is locally trivial. Does (0.1) have  $n$  independent solutions in  $K(C)$ ?

In the sequel we will analyse locally trivial equations. Further the differential Galois group of equation (0.1) is studied.

### § 1. LOCAL SYSTEMS ON $C$

A *local system*  $L$  of dimension  $n$  on  $C$  is a sheaf of  $K$ -vectorspaces on  $C$  (with respect to the rigid analytic Grothendieck topology on  $C$ ) such that  $L$  is locally isomorphic to the constant sheaf with fibre  $K^n$ .

Suppose that equation (0.1) is locally trivial. Then (0.1) gives rise to a local system. Indeed; on the intersections  $C_i \cap C_j$  one writes  $F_j = F_i M_{ij}$ . The matrix  $M_{ij}$  is locally constant on  $C_i \cap C_j$  and  $\det(M_{ij})$  is nowhere zero. The  $\{M_{ij}\}$  form a 1-cocycle on  $C$  and determine a local system. This local system is of course equal to the sheaf  $L$  on  $C$  given by:

$$L(U) = \left\{ v \in M(U)^n \mid \frac{d}{dz}(v) = Av \right\}$$

where  $U$  denotes an affinoid part of  $C$  and  $M(U)$  denotes the set of meromorphic functions on  $U$ . In order to understand local systems on  $C$  we have to consider reductions  $R: C \rightarrow Y$  with respect to some pure affinoid  $U$  of  $C$ . Using [6] and [3] Ch. V, one sees that these are pure coverings  $U$  of  $C$  such that

- (a)  $L$  is constant on every affinoid belonging to  $U$ .
- (b) every component of the reduction  $R: C \rightarrow Y$  w.r.t.  $U$  is non-singular and every singular point of  $Y$  is an ordinary double point.

For two reductions  $R_i: C \rightarrow Y_i$  w.r.t. pure coverings  $U_i$  ( $i=1,2$ ) satisfying (a) and (b) there exists a pure covering  $U_3$  finer than  $U_1$  and  $U_2$  such that  $U_3$  satisfies again (a) and (b).

So we find a commutative diagram

$$\begin{array}{ccc}
 C & \xrightarrow{R_1} & Y_1 \\
 R_2 \downarrow & \searrow R_3 & \uparrow \varphi_1 \\
 Y_2 & \xleftarrow{\varphi_2} & Y_3
 \end{array}$$

in which  $R_3: C \rightarrow Y_3$  is the reduction corresponding to  $U_3$  and where  $\varphi_i$  ( $i=1,2$ ) are morphisms of varieties over the residue field of  $K$ . Let  $G_i = G(Y_i)$  ( $i=1,2,3$ ) denote the intersection graph of the components of  $Y_i$ . One knows that the morphisms  $\varphi_i$  ( $i=1,2$ ) are obtained by successive blowing down of  $\mathbb{P}_1$ 's in  $Y_3$ . In particular the fundamental group  $\pi_1(G_3)$  of the graph  $G_3$  is isomorphic to  $\pi_1(G_i)$  ( $i=1,2$ ). This fundamental group is a finitely generated

free group  $\Gamma$  and it does not depend on the choice of the “prestable” reduction of  $C$ . We will call  $\Gamma$  the fundamental group of  $C$ . The group  $\Gamma$  is also the fundamental group of  $Y$ , a reduction of  $C$  satisfying (a) and (b), with respect to the Zariski-topology on  $Y$ . This leads to the following proposition.

(1.1) PROPOSITION. There are natural bijections between the following three sets:

- ( $\alpha$ ) {Local systems on  $C$  of dimension  $n$ }/isomorphy.
- ( $\beta$ ) { $n$ -dimensional representations of  $\Gamma$ }/equivalence.
- ( $\gamma$ ) {Locally trivial equations (0.1)}/equivalence.

PROOF. For a local system  $L$  on  $C$  and a reduction  $R : C \rightarrow Y$  satisfying (a) and (b) above  $R_*L$  is a local system on  $Y$  with respect to the Zariski topology. Using that  $\Gamma$  is the fundamental group of  $Y$  one easily finds bijections ( $\alpha$ ) $\rightleftharpoons$ ( $\beta$ ).

We have already produced a map ( $\gamma$ ) $\rightarrow$ ( $\alpha$ ). For a local system  $L$  on  $C$  we find a  $n$ -dimensional holomorphic vectorbundle  $E = \mathcal{O}_C \otimes_K L$  with a connection  $\nabla : E \rightarrow \Omega_C^1 \otimes_K E$  such that  $\ker \nabla = L$ . The global meromorphic sections of  $E$  produce then a differential equation in the form (0.2). There is still one point to verify, namely:

Suppose that  $(d/\partial z) - A_i$  ( $i=1,2$ ) induce isomorphic local systems  $L_i$  ( $i=1,2$ ). Then we have to produce a matrix  $S \in Gl(n, K(C))$  such that

$$S^{-1} \left( \frac{d}{dz} - A_1 \right) S = \frac{d}{dz} - A_2.$$

This equation for  $S$  can be written in the form

$$\frac{d}{dz} S = A_1 S - S A_2.$$

This is an equation of type (0.1) and the equation is locally trivial. The corresponding local system is  $\text{Hom}(L_1, L_2)$ . The isomorphism  $\varphi : L_1 \rightarrow L_2$  is a global section of this new local system. Further  $\varphi$  corresponds to the  $S$  with the required properties. This finishes the proof.

The graph  $G = G(Y)$  has a universal covering. This can be used to construct the universal covering  $u : U \rightarrow C$  of  $C$  with respect to the rigid analytic topology. (See [5]). One can transport a local system  $L$  to  $U$ . Then  $u^*L$  is the constant sheaf on  $U$  with fibre  $K^n$ . One finds then:

(1.2) COROLLARY. If equation (0.1) is locally trivial, then the equation has a fundamental matrix of solutions with entries global meromorphic functions on  $U$ .

(1.3) COROLLARY. If the Jacobian variety of  $C$  has good reduction and if (0.1) is locally trivial, then equation (0.1) has  $n$  independent solutions in  $K(C)^n$ .

Statement (1.3) follows from (1.2) since the condition “ $\text{Jac}(C)$  has good reduction” is equivalent to  $\Gamma = \{1\}$  and also to  $U = C$ .

## § 2. THE DIFFERENTIAL GALOIS GROUP

Equation (0.1) always admits a Picard-Vessiot extension. This is a differential field  $L \supset K(C)$  such that

- (a) there is a fundamental matrix of solutions with coefficients in  $L$ .
- (b) no proper subfield of  $L$  has property (a).
- (c)  $K(C)$  and  $L$  have the same field of constants.

The group of differential automorphism of  $L$  over  $K(C)$  is denoted by  $DGal((d/dz) - A)$ . It is an algebraic subgroup of  $Gl(n, K)$  called the differential Galois group. The group does not depend on the choice of the Picard-Vessiot extension  $L$ . (See [4].)

(2.1) PROPOSITION. Let equation  $(d/dz) - A$  of type (0.1) be locally trivial and let  $\varrho: \Gamma \rightarrow Gl(n, K)$  denote the corresponding representation. The  $DGal((d/dz) - A)$  is the smallest algebraic subgroup of  $Gl(n, K)$  containing  $\varrho(\Gamma)$ .

PROOF. There is a fundamental matrix  $F$  with coefficients in the field  $M(U)$  of meromorphic functions on the universal covering  $U$  of  $C$ . Let  $L \subset M(U)$  be the subfield generated over  $K(C)$  by the entries of  $F$ . Then  $L$  is invariant under  $d/dz$  since  $d/dz F = AF$ . One easily sees that  $L$  is a Picard-Vessiot extension of  $K(C)$ . For any  $\gamma \in \Gamma$ , the field  $L$  is invariant under the action of  $\gamma$  on  $M(U)$  since  $\gamma$  commutes with  $d/dz$ . So we have a mapping  $\Gamma \rightarrow DGal(L/K) = DGal((d/dz) - A)$ . Choose a basis  $e_1, \dots, e_n$  of solutions of  $(d/dz) - A$  with coefficients in  $L$ . Then any  $\gamma \in \Gamma$  induces a  $K$ -linear automorphism  $\varrho(\gamma)$  of  $Ke_1 + \dots + Ke_n$ . This representation  $\varrho$  is equivalent to the representation constructed in (1.1). Hence  $\varrho(\Gamma) \subseteq DGal(L/K) \subset Gl(n, K)$ . Let  $G$  denote the smallest algebraic group containing  $\varrho(\Gamma)$ . If  $G \neq DGal(L/K)$  then  $L^G \neq K(C)$ . However  $L^G \subset L^{\varrho(\Gamma)} \subset M(U)^\Gamma = K(C)$ . So  $G = DGal(L/K)$  and the proposition is proved.

(2.2) EXAMPLE. According to (1.1) there are only interesting examples is the reduction of  $C$  has a non-trivial fundamental group. For the Tate-curve  $C = K^*/\langle q \rangle$  ( $0 < |q| < 1$ ) a locally trivial differential equation corresponds to a homomorphism  $\varrho: \Gamma = \langle q \rangle \rightarrow Gl(n, K)$ . So the equation is determined by a single matrix  $\varrho$  (the generator of  $\Gamma$ ). If we want the differential equation to be irreducible than the matrix has the form

$$\begin{pmatrix} c & 1 & & \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ & & & 1 \\ & & & & c \end{pmatrix} \text{ with } c \in K^*.$$

The corresponding connection  $\nabla: K(C)^n \rightarrow \Omega^1 \otimes K(C)^n$  can be found expli-

citely. Namely: for a suitable basis  $e_1, \dots, e_n$  of  $K(C)^n$  the  $\nabla$  is given by

$$\begin{aligned}\nabla e_1 &= \omega \otimes e_1 \\ \nabla e_2 &= \omega \otimes (e_2 + e_1) \\ &\vdots \\ \nabla e_n &= \omega \otimes (e_n + e_{n-1})\end{aligned}$$

The differential form  $\omega$  has poles of order 1 at the two points  $u(1), u(c)$  of  $C$  with residues 1 and  $-1$ . Here  $u: K^* \rightarrow C$  is again the universal covering of the curve  $C$ . As yet  $\omega$  is only determined upto holomorphic differentials on  $C$ . The precise choice of  $\omega$  can be explained with theta functions.

Put

$$\theta(z) = \prod_{n \geq 0} (1 - q^n z^{-1}) \prod_{n > 0} (1 - q^n z) \text{ and put } \theta_c(z) = \theta(c^{-1}z).$$

Then  $\omega = d\eta/\eta$  with  $\eta = \theta_1/\theta_c$ . The Picard-Vessiot extensions  $L$  of  $K(C)$  can also be calculated. For  $n=1$ ,  $L = K(C)(\eta)$ , note that  $\eta(qz) = c\eta(z)$ . For  $n=2$ ,  $L = K(C)(\eta, f)$  where

$$f = \sum_{n \geq 0} \frac{1}{1 - q^{-n}z} + \sum_{n > 0} \frac{1}{1 - q^{-n}z^{-1}}.$$

We note that  $f(qz) - f(z) = 1$ . For  $n > 2$  similar, but more complicated, expressions can be found.

In the above we have classified all differential equations on  $K^*/\langle q \rangle$  which are locally trivial.

(2.3) EXAMPLE.  $C$  a Mumford curve of genus  $g \geq 2$ .

The fundamental group  $\Gamma$  is a free group on  $g$  generators. This group has many representations in  $Gl(n, K)$  if  $n > 1$ . For 1-dimensional representation  $\varrho: \Gamma \rightarrow K^*$  however one can again write down explicitly the corresponding differential equation  $\nabla: K(C) \rightarrow \Omega^1 \otimes K(C) = \Omega^1$ .

This is given by  $\nabla(g) = dg + g\omega$  where  $\omega$  is a differential on  $C$  of the form  $d\eta/\eta$  where  $\eta \in M(U)$  is a thetafunction satisfying  $\gamma(\eta) = \varrho(\gamma)\eta$  for all  $\gamma \in \Gamma$ .

(2.4) EXAMPLE.  $C$  a Mumford curve of genus  $g > 2$ . Let  $E$  denote a vectorbundle on  $C$  of rank  $n$  and degree 0. Suppose that  $E$  is indecomposable. According to A. Weil and Atiyah there exists a connection  $\nabla: E \rightarrow \Omega_{C/K}^1 \otimes E$ . (See [1.8].) This  $\nabla$  is however not unique. If there would be a choice of  $\nabla$  such that  $\nabla$  is locally trivial then one finds a representation (using (1.1))  $\varrho: \Gamma \rightarrow Gl(n, K)$  which induces the vectorbundle  $E$ . And so the  $p$ -adic version of A. Weil's theorem on vectorbundles would have been proved.

There seems however not much hope for the construction above since the condition "locally trivial" is difficult to obtain. For stable vectorbundles  $E$  of degree zero on  $C$  it has been proved that  $E$  is obtained from a representation  $\Gamma \rightarrow Gl(n, K)$ . (See [2] and [7].)

In particular  $E$  admits a connection  $\nabla$  which is locally trivial.

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