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Local analytic reduction of families of diffeomorphisms

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ABSTRACT

We study local analytic simplification of families of analytic maps near a hyperbolic fixed point. A particularly important application of the main result concerns families of hyperbolic saddles, where Siegel's theorem is too fragile, at least in the analytic category. By relaxing on the formal normal form we obtain analytic conjugacies. Since we consider families, it is more convenient to state some results for analytic maps on a Banach space; this gives no extra complications. As an example we treat a family passing through a $1:-1$ resonant saddle.

1. Introduction and motivation

We want to explore the limits of analytic simplification, by means of changes of variables (i.e. a conjugacy), of a dynamical system described by a diffeomorphism, in the neighbourhood of a fixed point p. In bifurcation theory it is assumed that this diffeomorphism moreover depends on external parameters. Hence the dependence of the change of variables on the parameter will also be of importance. Such a local analysis is often needed as a starting point for understanding more difficult global phenomena. For instance, if there is a saddle-type fixed point p, it is important to have a good local model in order to study the orbits in the vicinity of the fixed point.

We shall consider a family of diffeomorphisms $F_\mu : \mathbb{C}^n \mapsto \mathbb{C}^n$ fixing the origin. We want to look for a (parameter dependent) change of variables U_μ such that $G_\mu = U_\mu^{-1} \circ F_\mu \circ U_\mu$ has as few terms as possible in its Taylor expansion. Later on we will be more precise on the meaning of 'as few terms as possible' and the dependence on the parameter μ. It is our aim to look for conjugacies remaining as much as possible in the same smoothness category as the diffeomorphism.

It is well known that the arithmetic relations between the eigenvalues of the linear part at the fixed point determine to a great extent the kind of normal form that can be obtained by a conjugacy. In a generic family these relations may vary greatly if the parameter changes, so this has an influence on the normal form. If we start from an analytic diffeomorphism, we look for a 'simplest possible' analytic normal form and conjugacy. Ideally this would be: the linear part of the diffeomorphism, or at least some polynomial form. Unfortunately, for a general parameter-dependent saddle this is highly non-generic, even on the level of formal Taylor series. For example in two real dimensions, if the eigenvalues are λ_1 and λ_2, with $0 < \lambda_1 < 1 < \lambda_2$, the ratio $\log\lambda_1/\log\lambda_2$ may pass through rational and irrational values, giving an obstruction for a polynomial analytic normal form, even on the formal level.

One can then reduce analytically to a polynomial normal form up to a 'finitely flat remainder', that is: a remainder term of finite order in the space variable. In the context of vector fields this approach was already studied in H. Dulac's memoir [11] for planar systems; a generalization can be found in [19].

Our methods below allow to give an explicit and sharp expression for this flat remainder, that presumably cannot be improved in the general analytic category, especially for families. See [4] for an example in the planar case. If there are

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extra constraints on the system, or if there are no parameters, then a further analytic simplification is sometimes possible [2,6,20,18,17,8]. Another approach is to use finitely smooth (C^k, $k < \infty$) conjugacies in order to eliminate this flat remainder, so one gives up analyticity in general. See e.g. [13,14]. We will not discuss this here.

The method of proof of the principal result closely follows the ideas in [15].

1.1. Setting

We will frequently use multi-index notation, that is: k means $(k_1, \ldots, k_m)$, $|k|$ means $k_1 + \cdots + k_m$ and λ^k means $\lambda_1^{k_1} \ldots \lambda_m^{k_m}$; we denote $\mathbf{1}_j = (0, \ldots, 1, \ldots, 0)$.

We consider a family $F_\mu : \mathbb{C}^m \mapsto \mathbb{C}^m$ of local analytic diffeomorphisms, depending on μ in some set of parameters Λ, with $F_\mu(0) = 0$ for all parameter values $\mu \in \Lambda$. For example for hyperbolic fixed points it is not restrictive to assume that the fixed point is at the origin for all μ near some given parameter value $\mu = \mu_0$.

We consider the Taylor series of F_μ which converges on some polydisk. Let $F_\mu(z) = A_\mu z + f_\mu(z)$ where $A_\mu = D_z F_\mu(0)$ is the linear part of F_μ at zero and $f_\mu(z) := F_\mu(z) - A_\mu z$, so that $D_z f_\mu(0) = 0$. In order to explain the ideas we assume, for simplicity, that A_μ is semi-simple, although this hypothesis will not be necessary in the principal result in Section 1.3. Then there is a μ-dependent basis such that $A_\mu = \mathrm{diag}(\lambda_1(\mu), \ldots, \lambda_m(\mu))$. Let us fix the parameter for this moment. Then the eigenvalues λ of A are called *resonant* if there exist $(k, j) \in \mathbb{N}^m \times \mathbb{N}$ with $|k| > 1$ and $R(\lambda, k, j) = 0$, where the function R is defined as

$$R(\lambda, k, j) = \lambda_j - \lambda^k. \tag{1}$$

Conversely, if for all $(k, j) \in \mathbb{N}^m \times \mathbb{N}$ one has $R(\lambda, k, j) \neq 0$ then the eigenvalues are called *non-resonant*. A term $x^k \mathbf{1}_j$ in the Taylor series of F, is called *resonant* if $R(\lambda, k, j) = 0$ and *non-resonant* if $R(\lambda, k, j) \neq 0$.

Classical results for the conjugacy problem with a fixed parameter are theorems by Poincaré and Siegel, see for example [1]. The theorem by Poincaré, in the real case, assumes that the eigenvalues of A are located either inside the unit circle or outside the unit circle. Then F is locally linearizable, that is $G = A$, by an analytic change of coordinates in the absence of resonance. In the presence of resonance there is an analytic transformation which conjugates F to a polynomial $G = A + p$ containing resonant terms only. In case of parameter-dependency, this was studied in [5] and [12] for vector fields respectively diffeomorphisms.

There are only a few results for the complementary situation. That is, parameter dependent systems with parameters in an open set, where eigenvalues are located on either side of the unit circle and possibly resonant. Here we mention [16] and [7] where analytic normal forms are presented for particular two- and four-dimensional systems. Also see [9] and references therein.

Although comparable results in the analytic category, like the ones in this paper, are already known for *vector fields* [3], we have experienced that the usual passage from the 'vector fields case' to the 'diffeomorphisms case' is not at all as classical as could be expected; particular issues appear such as in Section 2.3.

To our knowledge there are two ways to proceed in general. Either enlarge the transformation group or allow a more general normal form. Enlarging the transformation group will almost inevitably mean losing smoothness. In our approach, we will keep analyticity and allow a 'slight tolerance' to the formal normal form.

1.2. An example

In order to fix the ideas of the reader we consider an example of an analytic family of saddles in $\mathbb{R}^2$ passing through a $1 : -1$ resonance. By this we mean a family

$$F(x, \mu) = \left(x_1 \left(\lambda_1(\mu) + \sum_{|k| \geqslant 1} f_k^1(\mu) x^k \right), x_2 \left(\lambda_2(\mu) + \sum_{|k| \geqslant 1} f_k^2(\mu) x^k \right) \right)$$

where for each value of the parameter the numbers $\lambda_1(\mu), \lambda_2(\mu)$ are, for simplicity, real and positive. Suppose that for $\mu = \mu_0$ the condition

$$\frac{\log(\lambda_1(\mu_0))}{\log(\lambda_2(\mu_0))} = -1$$

holds and that the family depends analytically on the parameter μ. When the parameter is fixed at $\mu = \mu_0$, we see that the resonant terms $f_k^1(\mu_0) x^k$, $f_k^2(\mu_0) x^k$ of $F(x, \mu_0)$ correspond to those $k = (k_1, k_2)$ for which $k_1 = k_2$. Let us write $u = x_1 x_2$. A consequence of our main result will be the following: given any $N \in \mathbb{N}$ there exists an analytic change of variables, depending moreover analytically on the parameter μ near μ_0, conjugating $F(., \mu)$ to

$$G(x, \mu) = \begin{cases} x_1(\lambda_1(\mu) + b_0^1(\mu, u) + \sum_{s \geqslant 1} u^{Ns} (x_2^s b_s^1(\mu, u) + x_1^s b_s^2(\mu, u))), \\ x_2(\lambda_2(\mu) + b_0^2(\mu, u) + \sum_{s \geqslant 1} u^{Ns} (x_2^s c_s^1(\mu, u) + x_1^s c_s^2(\mu, u))), \end{cases} \tag{2}$$

where all the occurring functions are analytic. Note that, if we put $\mu = \mu_0$ and if we truncate the foregoing expression to $\hat{G}(x, \mu_0) = (x_1(\lambda_1(\mu_0) + b_0^1(\mu_0, u)), x_2(\lambda_2(\mu_0) + b_0^2(\mu_0, u)))$, then we have the usual normal form at $\mu = \mu_0$. Moreover the 'remainder' $R(x, \mu) = G(x, \mu) - \hat{G}(x, \mu)$ is N-flat in u and we obtain an explicit form for this remainder R.

1.3. Results

Due to the presence of parameters it will be more convenient to state the principal result in the context of Banach spaces E. This will not complicate the exposition at all. We have of course $E = \mathbb{C}^n$ or $\mathbb{R}^n$ in mind as main cases. Suppose that an analytic function fixing the origin $F : E \to E$ is given, where $E = E^1 \oplus \cdots \oplus E^n$ is a direct sum of Banach spaces, and let A be the linear part of F. Suppose also that each E_i is an invariant subspace for F (we will comment on this assumption later).

We shall now use the usual formalism of symmetric multilinear maps on direct sums of vector spaces, including multi-index notation: see Section 2.2 for details.

We suppose that F is analytic near 0, that is: for a certain $\delta > 0$, the Taylor series of F converges to F for all $\|x\| \leqslant 2\delta$ and, due to the invariance of the splitting, we can write:

$$F(x) = \sum_{j=1}^{n} \sum_{k \in \mathbb{N}^n} F^j_{k+1_j}\big(x^{k+1_j}\big).$$

Furthermore, it follows that A is block-diagonal with respect to the direct sum splitting of E, i.e. $A = A^1 \oplus \cdots \oplus A^n$, with each A^i a continuous linear map $A^i : E^i \to E^i$. Put $\lambda_i = \|A^i\|$, $\tilde{\lambda}_i = \|(A^i)^{-1}\|$, $\rho = \max_{i=1}^n\{\lambda_i.\tilde{\lambda}_i\}$ and let $D, C \in \mathbb{R}$ be fixed, such that $0 < D\rho < 1$ and $0 < C\rho < 1$. We introduce the good set as

$$G_{D,C} = \big\{k \in \mathbb{N}^n \,\big|\, \lambda^k \leqslant D^{|k|} \text{ or } \tilde{\lambda}^k \leqslant C^{|k|}\big\},$$

and the bad set as its complement

$$B_{D,C} = \mathbb{N}^n \setminus G_{D,C}.$$

We give some brief comments on the value of ρ. If $E = \mathbb{C}^n$ and $A^i = [a_i]$, $i = 1, \ldots, n$ (i.e. A is a diagonal matrix) then $\rho = 1$; this is also true if each A^i is a multiple of the identity map. In the case that A^i is a Jordan block we can assume, up to a linear change of variables, that ρ is arbitrarily close to 1. On the other hand, if the variation of the spectrum of A^i is large, then ρ can be large compared to 1. In the case of matrices the factor ρ is known as the condition number.

Our main result is the following:

Theorem 1. *Suppose E is a Banach space that admits a direct sum decomposition $E = E^1 \oplus \cdots \oplus E^n$. Suppose that $F : E \to E$ is an analytic function for which its Taylor series converges to F for all $\|x\| \leqslant 2\delta$. Suppose that each E_i is an invariant subspace for F, and let A be the linear part of F. Then there exists an analytic near identity transformation U (i.e. its linear part $DU(0)$ is the identity), convergent for each $\|x\| \leqslant \delta$, such that:*

(i) *U contains only terms in the good set, i.e.*

$$U(x) = x + \sum_{j=1}^{n} \sum_{\substack{k \in G_{D,C} \\ |k| \geqslant 1}} u^j_{k+1_j}\big(x^{k+1_j}\big).$$

(ii) *The conjugation $G = U^{-1} \circ F \circ U$ contains only terms in the bad set, i.e.*

$$G(x) = Ax + \sum_{j=1}^{n} \sum_{\substack{k \in B_{D,C} \\ |k| \geqslant 1}} g^j_{k+1_j}\big(x^{k+1_j}\big).$$

Moreover, $G(x)$ converges for each $\|x\| \leqslant \delta$.

As a consequence we obtain the following theorem:

Theorem 2. *Suppose that $F : \mathbb{C}^n(\Lambda) \to \mathbb{C}^n(\Lambda)$ is a parameter dependent analytic function leaving invariant each coordinate axis, so F is of the form*

$$F(x) = \sum_{j=1}^{n} \sum_{k \in \mathbb{N}^n} F^j_{k+1_j}(\mu) x^{k+1_j}.$$

Suppose that F depends continuously (resp. C^k, C^∞, C^ω) on the parameter and that its Taylor series converges to F for all $\|x\| \leqslant 2\delta$. Hence the linear part A of f is semi-simple i.e. $A = \operatorname{diag}(\lambda_1(\mu), \ldots, \lambda_n(\mu))$. Define the good set as

$$G_{D,C} = \left\{ k \in \mathbb{N}^n \;\middle|\; \left|\lambda(\mu_0)\right|^k \leqslant D^{|k|} \text{ or } \frac{1}{|\lambda(\mu_0)|^k} \leqslant C^{|k|} \right\},$$

and the bad set $B_{D,C}$ as its complement.

Then there exist a neighbourhood $\widetilde{\Lambda}$ of μ_0 and an analytic near identity transformation U such that:

(i) *U contains only terms in the good set, i.e.*

$$U(x) = x + \sum_{j=1}^{n} \sum_{\substack{k \in G_{D,C} \\ |k| \geqslant 1}} u^j_{k+1_j}(\mu) x^{k+1_j}.$$

(ii) *The conjugation $G = U^{-1} \circ F \circ U$ does not contain any term in the good set, i.e.*

$$G(x) = Ax + \sum_{j=1}^{n} \sum_{\substack{k \in B_{D,C} \\ |k| \geqslant 1}} g^j_{k+1_j}(\mu) x^{k+1_j}.$$

1.4. *Method of proof*

Write $F = A + f$ where A is the linear part A and f is the higher order part. We assume that A is already in some standard form so we do not perform linear transformations, that is, we let $U = \mathrm{id} + u$ be a near identity transformation. Thus A is also the linear part of G and we write $G = A + g$.

Inspired by [15] we use the following approach. We write the conjugacy problem as

$$0 = F \circ U - U \circ G = A \circ u - u \circ (A + g) + f \circ (\mathrm{id} + u) - g. \tag{3}$$

With appropriate open parts of Banach spaces V, W, X and Z, to be defined in Section 3, we introduce the functional

$$\mathcal{F} : V \times W \times X \to Z : (f, g, u) \mapsto A \circ u - u \circ (A + g) + f \circ (\mathrm{id} + u) - g \tag{4}$$

and we try to solve $\mathcal{F}(f, g, u) = 0$ for (g, u) given the map f. We will do this by an application of the implicit function theorem. The main difficulty in applying this theorem is to prove that $\mathcal{F}$ is well defined between appropriate function spaces, and is C^1 in (f, g, u). In order to achieve this result we need some machinery which will be reviewed in Section 2. In Section 5.1 we will come back to the functioning of parameters in our setting.

2. Analytic functions on Banach spaces

We make extensively use of the theory of analytic maps between Banach spaces, see [10] for background. Our approach is based on that of [15].

2.1. *Local analytic functions and power series*

We define $\mathcal{A}_\delta(E, F)$ as the set of C^∞ functions f between the Banach spaces E and F for which the Taylor series converges absolutely to f for all $\|x\| \leqslant \delta$. The following definitions explain this in more detail.

Definition 1. We define $\mathcal{L}^k(E, F)$ to be the space of k-multilinear symmetric mappings $f_k : E^k \to F : (x_1, x_2, \ldots, x_k) \to f_k(x_1, x_2, \ldots, x_k)$, i.e.

$$f_k(x_1, \ldots, x_i, \ldots, x_k) = f_k(x_{\varphi(1)}, \ldots, x_{\varphi(i)}, \ldots, x_{\varphi(k)})$$

for all $x_i \in E$ and all permutations $\varphi \in S_k$.

Using the norm $\|f_k\| := \sup_{x \in E} \frac{\|f_k(x, \ldots, x)\|}{\|x\|^k}$, it is a standard result that $\mathcal{L}^k(E, F)$ becomes a Banach space.

We now introduce the analogue of formal power series for Banach spaces, and define analytic functions as those power series that converge absolutely on a certain neighbourhood of the origin.

Definition 2. We define formal power series and analytic functions $E \to F$ as follows.

(i) We denote by $\mathcal{P}(E,F)$ the set of formal power series $f = \sum_{k\geqslant 0} f_k$, where $f_k \in \mathcal{L}^k(E,F)$.
(ii) $\mathcal{A}(E,F)$ is the set of formal power series $f = \sum_{k\geqslant 0} f_k$, where $f_k \in \mathcal{L}^k(E,F)$ are such that there exists a $\delta > 0$ for which $\sum_{k\geqslant 0} \|f_k\|\delta^k < \infty$. (Note that this condition is equivalent with $\overline{\lim}_{k\to\infty} \sqrt[k]{\|f_k\|} < \infty$.) We will refer to $\mathcal{A}(E,F)$ as the set of analytic functions from E to F.
(iii) $\mathcal{A}_\delta(E,F)$ is the subset of $\mathcal{A}(E,F)$ for which $\|f\|_\delta := \sum_{k\geqslant 0} \|f_k\|\delta^k < \infty$, for some $\delta > 0$. We will refer to $\mathcal{A}_\delta(E,F)$ as the set of analytic functions with radius of convergence at least δ.

Note that for each $x \in E$, with $\|x\| \leqslant \delta$, the power series $\sum_{k\geqslant 0} f_k(x,\dots,x)$ converges absolutely since

$$\sum_{k\geqslant 0} \big\| f_k(x,\dots,x)\big\| \leqslant \sum_{k\geqslant 0} \|f_k\|\|x\|^k \leqslant \|f\|_\delta.$$

Hence we can define the analytic function

$$f : B_E(0;\delta) \to F : x \mapsto \sum_{k\geqslant 0} f_k(x,\dots,x).$$

It is clear that the Taylor series at 0 of this function corresponds to our formal series $\sum_{k\geqslant 0} f_k$. This allows us to switch from the function view to the power series view and back. During the remainder of this article, we will switch between these two views without further notice.

Using the definitions above, we can now state the following proposition which we need later on. A proof can be found e.g. in [15].

Proposition 3. *If $B_\delta := \{g \in \mathcal{A}_\eta(D,E) \mid g(0) = 0$ and $\|g\|_\eta < \delta\}$, for some fixed $\eta > 0$, then the composition operator*

$$O : \mathcal{A}_\delta(E,F) \times B_\delta \to \mathcal{A}_\eta(D,F) : (f,g) \mapsto f \circ g$$

is C^1.

2.2. Direct sum splitting of an analytic function

Let X be a Banach space and let E be a Banach space which is a direct sum of the Banach spaces $E_1, E_2, \dots, E_n$. Then an element x of $E = E_1 \oplus \cdots \oplus E_n$ can be written in a unique way as $x = \pi_1(x) + \cdots + \pi_n(x) = x_1 + \cdots + x_n$, with $x_i \in E_i$, and $\pi_i : E \to E_i$ is the projection on the i-th component. Let now $f_k \in \mathcal{L}^k(E,X)$. Analogous to $\mathbb{C}^n$ we can try to expand f_k in homogeneous polynomials of degree k. Indeed, with the use of the multinomium of Newton, it is readily verified that

$$f_k\big((x_1 + \cdots + x_n)^k\big) = \sum_{\substack{l\in\mathbb{N}^n \\ |l|=k}} \binom{k}{l} f_k\big(x_1^{l_1} x_2^{l_2} \dots x_n^{l_n}\big),$$

where $\binom{k}{l} = \frac{k!}{l_1!\dots l_n!}$ are the multinomial coefficients and $|l| = l_1 + \cdots + l_n$. Note that in the formula above we deliberately used the power notations $x^k = (\underbrace{x,\dots,x}_{k})$ for $k \in \mathbb{N}$ and

$$x^l = x_1^{l_1} \dots x_n^{l_n} = (\underbrace{x_1,\dots,x_1}_{l_1}, \dots, \underbrace{x_n,\dots,x_n}_{l_n})$$

for $l = (l_1,\dots,l_n) \in \mathbb{N}^n$. Define now for each $l = (l_1,\dots,l_n) \in \mathbb{N}^n$,

$$f_l := \binom{|l|}{l} f_{|l|} \circ (\underbrace{\pi_1,\dots,\pi_1}_{l_1}, \dots, \underbrace{\pi_n,\dots,\pi_n}_{l_n}),$$

then clearly $f_l \in \mathcal{L}^{|l|}(E,X)$. Furthermore $f_k = \sum_{l\in\mathbb{N}^n, |l|=k} f_l$ and a general $f \in \mathcal{P}(E,X)$, can be decomposed as $f = \sum_{l\in\mathbb{N}^n} f_l$. If X also admits a direct sum splitting $X_1 \oplus \cdots \oplus X_m$, then we can further split this function into its components, and this formula becomes

$$f = \sum_{i=1}^m \sum_{l\in\mathbb{N}^n} f_l^i,$$

where $f_l^i = \pi_i \circ f_l$. As an analogy to the situation in $\mathbb{C}^n$, we will refer to f_l^i as a term (monomial) in x^l or as a term (monomial) with degree l.

We will in this situation use the supnorm $\|f\| = \max_{i\in\{1,\dots,n\}} \|f^i\|$ instead of the sumnorm.

2.3. A class of formal (semi-)groups in $\mathcal{P}(E,E)$

Suppose that E is a Banach space with direct sum splitting $E = E_1 \oplus \cdots \oplus E_n$ and suppose that $f \in \mathcal{P}(E,E)$. As explained in Section 2.2, we can decompose f as

$$f = \sum_{i=1}^{n} \sum_{k \in \mathbb{N}^n} f_k^i.$$

Let now $K \subset \mathbb{N}^n$, and define $\mathcal{P}_K(E,E)$, the set of formal series adapted to K, as

$$\mathcal{P}_K(E,E) := \left\{ f \in \mathcal{P}(E,E) \;\middle|\; f = \sum_{i=1}^{n} \sum_{k \in K} f^i_{1_i+k} \right\},$$

where $1_i = (0,\dots,1,\dots,0)$ is the i-th unit vector. Intuitively, the term $f^i_{1_i+k}$, where $k \in K$, correspond to a term $x_i x^k = x_i x_1^{k_1} \dots x_n^{k_n}$ in the classical Taylor series. Note that if $l = (\dots,0,\dots)$, with a zero at the i-th entry, then $f^i_l = 0$. As a consequence each E_i is an invariant subspace.

With respect to the composition of maps it is natural to require that the subset K of $\mathbb{N}^n$ is a semigroup, i.e. for every $k_1, k_2 \in K$ also $k_1 + k_2 \in K$. We shall call a semigroup in $\mathbb{N}^n$ a *cone*.

Lemma 4. *Let $K \subset \mathbb{N}^n$ be a cone. Then $\mathcal{P}_K(E,E)$ forms a semigroup under composition.*

Proof. Let K be a cone and let g and h be elements of $\mathcal{P}_K(E,E)$. We show that their composition $g \circ h$ remains in $\mathcal{P}_K(E,E)$. Since on the formal level the composition is defined as

$$\sum_{i=1}^{n} \sum_{k \in \mathbb{N}^n} g_k^i \circ \sum_{i=1}^{n} \sum_{k \in \mathbb{N}^n} h_k^i = \sum_{i=1}^{n} \sum g_k^i \big(h^1_{l_1^1}, \dots, h^1_{l^1_{k_1}}, \dots, h^n_{l^n_{k_1}}, \dots, h^n_{l^n_{k_1}}\big),$$

where the sum ranges over all indices k and l_i^j for which $l_1^i + \cdots + l_{k_n}^i = k_i$, for each $1 \leqslant i \leqslant n$. We take now a general term in the substitution. Hence every term appearing in the formal composition looks like:

$$g_k^i \big(h^1_{l_1^1}, \dots, h^1_{l^1_{k_1}}, h^2_{l_1^2}, \dots, h^2_{l^2_{k_2}}, \dots, h^n_{l_1^n}, \dots, h^n_{l^n_{k_n}}\big). \tag{5}$$

Since $g \in \mathcal{P}_K(E,E)$, it follows that $k = 1_i + \tilde{k}$; where $\tilde{k} \in K$. Furthermore, since also $h \in \mathcal{P}_K(E,E)$; it follows that for each $h^\alpha_{l^\alpha_\beta}$ we have that $l^\alpha_\beta = 1_\alpha + m^\alpha_\beta$ where $m^\alpha_\beta \in K$. The term given by formula (5) is clearly a term of degree

$$\begin{aligned} l_1^1 + \cdots + l_{k_n}^n &= 1_1 + m_1^1 + \cdots + 1_1 + m^1_{k_1} + \cdots + m^n_{k_n} &(6)\\ &= \underbrace{1_1 + \cdots + 1_1}_{k_1} + \cdots + \underbrace{1_n + \cdots + 1_n}_{k_n} + m_1^1 + \cdots + m^1_{k_1} + \cdots + m^n_{k_n} &(7)\\ &= k + \gamma, &(8) \end{aligned}$$

where $\gamma = m_1^1 + \cdots + m^n_{k_n} \in K$ since K is a semigroup and $k = 1_i + \tilde{k}$. Hence $k + \gamma = 1_i + \tilde{k} + \gamma = 1_i + \hat{k}$, where $\tilde{k} + \gamma = \hat{k} \in K$. Since this is an arbitrary term, it follows that the composition $g \circ h \in \mathcal{P}_K(E,E)$. $\square$

Let $\mathcal{D}_K(E,E)$ be the subset of $\mathcal{P}_K(E,E)$ for which the linear part is invertible. In a completely analogous way as in Lemma 4 one proves concerning inversion:

Lemma 5. *Let $K \subset \mathbb{N}^n$ be a cone. Then $\mathcal{D}_K(E,E)$ forms a group under composition.*

Now we define for each cone K the subspaces $\mathcal{D}_{K,\delta}(E,E) = \mathcal{A}_\delta(E,E) \cap \mathcal{D}_K(E,E)$. If the cone is $K = \mathbb{N}^n$, we will use the notation $\mathcal{D}_\delta(E,E)$. Note that if $F_1, F_2 \in \mathcal{D}_{K,\delta}(E,E)$ have a linear part which can be split as in Section 2.2, i.e. the linear parts have the form

$$A^1 \oplus \cdots \oplus A^n, \tag{9}$$

then the same is true for the composition $F_1 \circ F_2$.

3. Proof of Theorem 1

Let us first rephrase the main Theorem 1 to our current setting.

Proposition 6. *Let K be a cone and let $F \in \mathcal{D}_{K,2\delta}(E,E)$ be an analytic diffeomorphism on the Banach space E. Suppose that E admits a direct sum composition $E = E_1 \oplus \cdots \oplus E_n$ and suppose that each E_i is an invariant subspace for F. Then an analytic near-identity coordinate transform $U \in \mathcal{D}_{K\cap G_{D,C},\delta}(E,E)$ exists, such that $G = U^{-1} \circ F \circ U \in \mathcal{D}_{K\cap B_{D,C},\delta}(E,E)$.*

We will proceed in two stages. First we will remove terms in a somewhat smaller good set

$$G_D = \{k \in \mathbb{N}^m \mid \|A\|^k \leqslant D^{|k|}\}, \tag{10}$$

where $0 < \rho D < 1$. The corresponding bad set is

$$B_D = \{k \in \mathbb{N}^m \mid \|A\|^k > D^{|k|}\}, \tag{11}$$

and we will first prove:

Proposition 7. *Let F be as in Proposition 6. Then an analytic near-identity coordinate transform $U \in \mathcal{D}_{K\cap G_D,\delta}(E,E)$ exists, such that $G = U^{-1} \circ F \circ U \in \mathcal{D}_{K\cap B_D,\delta}(E,E)$.*

As already explained in Section 1.4, we have to solve $\mathcal{F}(f,g,u) = 0$ for g and u for a given map f, where the functional $\mathcal{F}$ was defined in (4). To solve this functional equation we use an appropriate version of the implicit function theorem, which we now state.

Theorem 8 *(Implicit Function Theorem). Let V, W, X be open in the Banach spaces $\overline{V}$, $\overline{W}$, $\overline{X}$ and let $\overline{Z}$ be a Banach space. Suppose that $\mathcal{F} : V \times W \times X \to Z$ is C^1, $\mathcal{F}(0,0,0) = 0$ and that $D_{(g,u)}\mathcal{F} : \overline{W} \times \overline{X} \to \overline{Z} : (g,u) \mapsto D\mathcal{F}(0,0,0).(0,g,u)$ is an isomorphism of Banach spaces. Then there exists open neighbourhoods $V_1 \subset V$, $W_1 \subset W$, $X_1 \subset X$ of zero, such that for each $f \in V_1$ there exists a unique $(g,u) \in W_1 \times X_1$ with $\mathcal{F}(f,g,u) = 0$.*

We now introduce the appropriate Banach spaces and well chosen open subsets of them.

Definition 3. The Banach spaces $\overline{V}$, $\overline{W}$, $\overline{X}$ and $\overline{Z}$ and their corresponding open parts V, W, X and Z are defined as follows

$$\overline{V} = V = \{f \in \mathcal{A}_{K,2\delta}(E,E) \mid f_0 = 0,\ f_1 = 0\},$$
$$\overline{W} = \left\{g \in \mathcal{A}_{K,\delta}(E,E) \;\middle|\; g = \sum_{j=1}^{n} \sum_{k\in B_D, |k|\geqslant 1} g^j_{k+1_j}\right\},$$
$$W = \left\{g \in \overline{W} \;\middle|\; \left(\frac{\|g^j\|_\delta}{\|A^j\|}\right) < (1-D)\delta,\ \text{for each } j = 0,1,\ldots,n\right\},$$
$$\overline{X} = \left\{u \in \mathcal{A}_{K,\delta}(E,E) \;\middle|\; u = \sum_{j=1}^{n} \sum_{k\in G_D, |k|\geqslant 1} u^j_{k+1_j}\right\},$$
$$X = \{u \in \overline{X} \mid \|u\|_\delta < \delta\},$$
$$\overline{Z} = Z = \{h \in \mathcal{A}_{K,\delta}(E,E) \mid h_0 = 0,\ h_1 = 0\}.$$

Three crucial points in the proof are: (1) the fact that $\mathcal{F}$ is well defined, (2) the continuous differentiability of the functional $\mathcal{F}$ and (3) the fact that its derivative is an isomorphism. We will now state these points as lemmas and prove them.

Lemma 9. *The functional $\mathcal{F}$ is C^1, that is, its Gateaux derivatives are continuous.*

Proof. Since $\|A \circ u\| \leqslant \|A\|\|u\|$, it follows that the part $(f,g,u) \mapsto A \circ u$ is C^1, it is also clear that the part $(f,g,u) \mapsto -g$ is C^1. Because $\|\mathrm{id} + u\|_\delta \leqslant \|\mathrm{id}\|_\delta + \|u\|_\delta < 2\delta$, it follows directly from Proposition 3 that $(f,g,u) \mapsto f \circ (\mathrm{id} + u)$ is C^1. The part $(f,g,u) \mapsto u \circ (A + g)$ is more difficult. First let's make a short calculation:

$$\begin{aligned}
u^j \circ (A+g) &= \sum_{k\in G_D} u^j_{k+1_j}(A+g)^{k+1_j} \\
&= \sum_{(k,j)\in G_D} u^j_{k+1_j}\big(\big(A^j+g^j\big), \big(A^1+g^1\big)^{k_1}, \dots, \big(A^n+g^n\big)^{k_n}\big) \\
&= \sum_{(k,j)\in G_D} \frac{\|A^j\|\|A^1\|^{k_1}\dots\|A^n\|^{k_n}}{D^{|k|+1}} u^j_{k+1_j}\left(\left(\frac{DA^j}{\|A^j\|}+\frac{D}{\|A^j\|}g^j\right), \left(\frac{DA^1}{\|A^1\|}+\frac{D}{\|A^1\|}g^1\right)^{k_1}, \dots, \right. \\
&\quad \left. \left(\frac{DA^n}{\|A^n\|}+\frac{D}{\|A^n\|}g^n\right)^{k_n}\right).
\end{aligned}$$

The map

$$u = \sum_{j=1}^n \sum_{k\in G_D} u^j_{k+1_j} \mapsto u' := \sum_{j=1}^n \sum_{k\in G_D} \frac{\|A^j\|\|A^1\|^{k_1}\dots\|A^n\|^{k_n}}{D^{|k|+1}} u^j_{k+1_j}$$

is clearly linear. It is also continuous since

$$\sum_{k\in G_D} \frac{\|A^j\|\|A^1\|^{k_1}\dots\|A^n\|^{k_n}}{D^{|k|+1}} \big\|u^j_{k+1_j}\big\|\delta^k \leqslant \sum_{k\in G_D} \|A\|_{\sup}\big\|u^j_{k+1_j}\big\|\delta^k \leqslant \|A\|_{\sup}\big\|u^j\big\|_\delta.$$

Here $\|A\|_{\sup} := \max_{j\in\{1,\dots,n\}}\|A^j\|$. We now use Proposition 3 a second time to ensure that

$$\left(u'^j, \left(\sum_{i=1}^n \frac{DA^i}{\|A^i\|}+\frac{D}{\|A^i\|}g^i\right)\right) \mapsto u'^j \circ \left(\sum_{i=1}^n \frac{DA^i}{\|A^i\|}+\frac{D}{\|A^i\|}g^i\right)$$

is C^1. This is justified since

$$\left\|\frac{DA^i}{\|A^i\|}+\frac{D}{\|A^i\|}g^i\right\|_\delta < D\delta + (1-D)\delta = \delta.$$

Hence this mapping is C^1. Adding the individual C^1 pieces finishes the proof. □

We now calculate the Gateaux derivatives and find:

$$\begin{aligned}
&D_u\mathcal{F}(0,0,0).u = \lim_{t\to 0}\frac{A\circ tu - tu\circ A}{t} = A\circ u - u\circ A, \\
&D_f\mathcal{F}(0,0,0).f = \lim_{t\to 0}\frac{tf\circ \mathrm{id}}{t} = f, \qquad D_g\mathcal{F}(0,0,0).g = \lim_{t\to 0}\frac{-tg}{t} = -g.
\end{aligned} \tag{12}$$

Using these derivatives, we are now able to prove:

Lemma 10. *$D_{(g,u)}\mathcal{F}(0,0,0) : \overline{W}\times\overline{X} \to \overline{Z} : (g,u) \mapsto D\mathcal{F}(0,0,0).(0,g,u)$ is an isomorphism of Banach spaces.*

Proof. We split $D_{(g,u)}\mathcal{F}(0,0,0)$ in its 'good' and its 'bad' part. Since $A^j\circ u = \sum_{k\in G_D} A^j \circ u^j_{k+1_j}$ and $u^j\circ A = \sum_{k\in G_D} u^j_{k+1_j}\circ (A,\dots,A)$, it follows that the projection on the good and bad cone yields $\pi_{G_D}(A\circ u - u\circ A - g) = A\circ u - u\circ A$ and $\pi_{B_D}(A\circ u - u\circ A - g) = -g$. Hence, in order to show that $D_{(g,u)}\mathcal{F}(0,0,0)$ is an isomorphism, it is sufficient to show that $\mathcal{G}_1 : W\to W : g\mapsto -g$ and $\mathcal{G}_2 : X\to X : u\mapsto (A\circ u - u\circ A)$ are isomorphisms. It is clear that $\mathcal{G}_1$ is an isomorphism. It remains to show that $\mathcal{G}_2$ is an isomorphism. Now

$$A\circ u - u\circ A = \sum_{i=1}^n A^i\circ u^i - u^i\circ A = \sum_{i=1}^n A^i\big(u^i - \big(A^i\big)^{-1}\circ u^i\circ A\big) = A^i\sum_{i=1}^n (\mathrm{id}-R_i)\big(u^i\big),$$

where $R_i : \overline{X}\to\overline{X} : u\mapsto (A^i)^{-1}\circ u^i\circ A$. If we can show that $\|R_i\| < 1$, then it follows that $\mathrm{id}-R_i$ and hence also $A^i(\mathrm{id}-R_i)$ is an isomorphism, which completes the proof. It remains to show that $\|R_i\| < 1$. This is true since

$$\begin{aligned}
\big\|\big(A^i\big)^{-1}\circ u^i\circ A\big\|_\delta &= \sum_{k\in G_D}\big\|\big(A^i\big)^{-1}\circ u^i_{k+1_i}\big(A^i, \underbrace{A^1,\dots,A^1}_{k_1},\dots,\underbrace{A^n,\dots,A^n}_{k_n}\big)\big\|\delta^{|k|+1} \\
&\leqslant \sum_{k\in G_D}\big\|\big(A^i\big)^{-1}\big\|\big\|u^i_{k+1_i}\big\|\big\|A^i\big\|\big\|A^1\big\|^{k_1}\dots\big\|A^n\big\|^{k_n}\delta^{|k|+1}
\end{aligned}$$

$$\leqslant \sum_{k\in G_D} \rho D^k \|u^i_{k+1_i}\| \delta^{|k|+1} \leqslant \rho D \sum_{k\in G_D} \|u^i_{k+1_i}\| \delta^{|k|+1} \leqslant \rho D \|u^i\|_\delta$$

and since, by assumption, $\rho D < 1$. □

We are now in a position to prove Proposition 7.

Proof of Proposition 7. According to Theorem 8, with the help of Lemma 10, we can show that there exists a small r such that this theorem is true for all $F = A + f$ with $\|f\|_{2\delta} < r$. Suppose now that $\|f\|_{2\delta} \geqslant r$. We apply now classical rescaling. Choose $0 < \gamma < 1$ such that $\tilde f = \gamma^{-1} f \circ (\gamma \mathrm{id}) = \gamma^{-1} \sum_{(k,j)\in\mathbb{N}^2\times\{1,\dots,n\}} \gamma^{|k|} f^j_k$ has a norm $\|\tilde f\| < r$. Let now $\tilde u$, $\tilde g$ be the solution of the equation $\mathcal{F}(\tilde f, \tilde g, \tilde u) = 0$ and define $u := \gamma \tilde u \circ (\gamma^{-1}\mathrm{id})$ and $g := \gamma \tilde g \circ (\gamma^{-1}\mathrm{id})$. Then it is clear that

$$0 = \gamma \mathcal{F}(\tilde f, \tilde g, \tilde u) \circ (\gamma^{-1}\mathrm{id}) = \mathcal{F}(f, g, u).$$

This concludes the proposition. □

As a corollary we can now complete the proof of our main result.

Proof of Proposition 6. *Step* 1. We first invert F and apply then Proposition 7 to F^{-1}, with $K = \mathbb{N}^n$. Note that F^{-1} corresponds to the same factor ρ as F, since reversing the roles of A and A^{-1} does not alter the value of ρ. Hence we know that the reduction G does not contain any term outside the cone

$$B_C = \{k \in \mathbb{N}^m \mid \|A^{-1}\|^k > C^{|k|}\}.$$

Using Lemma 5, we see that the same is true for G^{-1}, since B_C is a cone.

Step 2. We rename G^{-1}, our previous reduction from Step 1, again as F. Then F contains only terms in the cone $K = B_C$, it follows that there exists a reduction to a certain G containing only terms in the cone $K \cap B_D = B_C \cap B_D = B_{D,C}$. □

4. An invariant manifold theorem

The methods from the preceding section allow us to obtain the well-known stable and unstable manifold theorems for analytic diffeomorphisms, as well as the smooth dependence on possible parameters.

Let us first describe the situation in $\mathbb{C}^2$ in order to fix the ideas of the reader. Suppose that $F(x, y) = (\lambda_1 x, \lambda_2 y) + O(|(x, y)|^2)$ is given, where $|\lambda_1| < 1$, $|\lambda_2| > 1$, and we want to find a stable manifold for F. We try to find a coordinate transform $U = \mathrm{id} + O((x, y)^2)$ such that in new coordinates $G(x, y) = U^{-1} \circ F \circ U(x, y)$ leaves the $y = 0$ plane invariant. This is equivalent to

$$G(x, 0) = \big(\lambda_1 x + O\big((x, y)^2\big), 0\big).$$

The inverse image of the plane $y = 0$ is then an invariant (stable) manifold of F. This is precisely what we will do in a slightly more general context. Let $E = E_1 \oplus E_2$ be a direct sum of Banach spaces and $F = A + \sum_{k\geqslant 2} F_k \in \mathcal{A}_\delta(E, E)$ with diagonal linear part A. Hence, using the notations of Section 2.2, $A = F_1 = F^1_{(1,0)} + F^2_{(0,1)} = A^1 + A^2$. Suppose now that $\|A^1\| < 1$ and $\|(A^2)^{-1}\| < 1$. Choose $\|A^1\| < D < 1$ and define the bad set

$$BS := \{(k, j) \in \mathbb{N}^2 \times \{1, 2\},\ |k| = k_1 + k_2 \geqslant 2 \mid (k, j) \neq ((k_1, 0), 2)\},$$

i.e. if $(k, 2) \in BS$, then $k_2 \geqslant 1$; and the good set, the set of terms that we are trying to remove (see below for more details),

$$GS := \{(k, j) \in \mathbb{N}^2 \times \{1, 2\},\ |k| = k_1 + k_2 \geqslant 2 \mid (k, j) = ((k_1, 0), 2)\}.$$

We will look for a coordinate transform $U = \mathrm{id} + \sum_{(k,j)\in GS} u^j_k$ containing only good terms, that conjugates F to $G = U^{-1} \circ F \circ U$, such that $G = A + \sum_{(k,j)\in BS} g^j_k$ contains only bad terms. Here the set of bad terms is chosen exactly as the set of terms that are still left (i.e. unremoved) in the Taylor expansion of G, thus note that when G contains only bad terms, then, for $x_1 \in E_1$

$$\pi_2 \circ G(x_1) = 0,$$

because $k_2 \geqslant 1$ if $j = 2$. Hence G leaves E_1 invariant. As explained in the introduction, this problem is equivalent to finding a zero of the functional equation

$$\mathcal{F} : V \times W \times X \to Z : (f, g, u) \mapsto A \circ u - u \circ (A + g) + f \circ (\mathrm{id} + u) - g,$$

a problem that we can try to solve in a similar way as in Section 3. We set

$$\overline{V} = V = \{f \in \mathcal{A}_{2\delta}(E,E) \mid f_0 = 0,\ f_1 = 0\},$$
$$\overline{W} = \left\{g \in \mathcal{A}_\delta(E,E) \;\middle|\; g = \sum_{(k,j)\in BS} u_k^j\right\}, \qquad W = \{g \in \overline{W} \mid \|g\|_\delta < (1-D)\delta\},$$
$$\overline{X} = \left\{u \in \mathcal{A}_\delta(E,E) \;\middle|\; u = \sum_{(k,j)\in GS} u_k^j\right\}, \qquad X = \{u \in \overline{X} \mid \|u\|_\delta < \delta\},$$
$$\overline{Z} = Z = \{h \in \mathcal{A}_\delta(E,E) \mid h_0 = 0,\ h_1 = 0\}.$$

Lemma 11. *$\mathcal{F}$ is C^1.*

Proof. We use the same technique as in Lemma 9.

Since $\|A \circ u\| \leqslant \|A\|\|u\|$, it follows that the part $(f,g,u) \mapsto A \circ u$ is C^1, it is also clear that the part $(f,g,u) \mapsto -g$ is C^1. Because $\|\mathrm{id} + u\|_\delta \leqslant \|\mathrm{id}\|_\delta + \|u\|_\delta < 2\delta$, it follows directly from Proposition 3 that $(f,g,u) \mapsto f \circ (\mathrm{id} + u)$ is C^1. We take now a closer look to the composition

$$\begin{aligned} u \circ (A+g) &= \sum_{(k,1)\in G} u_k^1(\underbrace{A+g,\dots,A+g}_{|k|}) + \sum_{(k,2)\in G} u_k^2(\underbrace{A+g,\dots,A+g}_{|k|}) \\ &= \sum_{(k,2)\in G} u_k^2(\underbrace{A+g,\dots,A+g}_{|k|}) = \sum_{k_1 \geqslant 2} u^2_{(k_1,0)}\big(\underbrace{A^1+g^1,\dots,A^1+g^1}_{k_1}\big). \end{aligned}$$

Since $\|A^1 + g^1\|_\delta \leqslant \|A^1\|\delta + \|g^1\|_\delta < (D + (1-D))\delta = \delta$, we use Proposition 3 to conclude that $u^2 \circ (A^1 + g^1)$ is C^1. Since the projections $u \mapsto u^2$ and $g \mapsto g^1$ are C^1, it follows that the composition $(u,g) \mapsto (u^2, g^1) \mapsto u^2 \circ (A^1 + g^1) = u \circ (A+g)$ is also C^1. Adding the individual C^1 pieces finishes the proof. □

Lemma 12. *$D_{(g,u)}\mathcal{F}(0,0,0) : \overline{W} \times \overline{X} \to Z : (g,u) \mapsto D\mathcal{F}(0,0,0).(0,g,u)$ is an isomorphism of Banach spaces.*

Proof. The differential $D_{(g,u)}\mathcal{F}(0,0,0)$ is given by the same formulas as in (12). We split $D_{(g,u)}\mathcal{F}(0,0,0)$ in its good and its bad part. Since $A \circ u = \sum_{(k,j)\in GS} A^j \circ u_k^j$ and $u \circ A = \sum_{(k,j)\in GS} u_k^j \circ (A,\dots,A)$, it follows that k_2 remains 0 in the second components of these parts. Hence

$$\pi_{GS}(A \circ u - u \circ A - g) = A \circ u - u \circ A, \qquad \pi_{BS}(A \circ u - u \circ A - g) = -g.$$

Hence, in order to show that $D_{(g,u)}\mathcal{F}(0,0,0)$ is an isomorphism, it is sufficient to show that $\mathcal{G}_1 : \overline{W} \to \overline{W} : g \mapsto -g$ and $\mathcal{G}_2 : \overline{X} \to \overline{X} : u \mapsto (A \circ u - u \circ A)$ are isomorphisms. It is clear that $\mathcal{G}_1$ is an isomorphism. It remains to show that $\mathcal{G}_2$ is an isomorphism. Because $u \in X$, it follows that $u = \sum_{(k,1)\in G} u_k^1 + \sum_{(k,2)\in G} u_k^2 = \sum_{k_1 \geqslant 2} u^2_{(k_1,0)} = u^2$. Hence

$$A \circ u - u \circ A = A^2 \circ u^2 - u^2 \circ A^1 = \big(A^2\big)\big(u^2 - \big(A^2\big)^{-1} \circ u^2 \circ A^1\big) = \big(A^2\big) \circ (\mathrm{id} - M)\big(u^2\big),$$

where $M : \overline{X} \to \overline{X} : u^2 \mapsto (A^2)^{-1} \circ u^2 \circ A^1$. Because $\|u^2 \circ A^1\| \leqslant \|u^2\|$ for any $u^2 \in \overline{X}$, it follows that

$$\frac{\|M(u^2)\|}{\|u^2\|} = \frac{\|(A^2)^{-1} \circ u^2 \circ A^1\|}{\|u^2\|} \leqslant \frac{\|(A^2)^{-1}\|\|u^2 \circ A^1\|}{\|u^2\|} \leqslant \big\|\big(A^2\big)^{-1}\big\|.$$

Hence

$$\|M\| = \sup_{u^2} \frac{\|M(u^2)\|}{\|u^2\|} \leqslant \big\|\big(A^2\big)^{-1}\big\|,$$

where $\|(A^2)^{-1}\| < 1$ and it follows that $\mathrm{id} + M$ is an isomorphism. Hence $\mathcal{G}_2$ is also an isomorphism. □

In a similar way as in the proof of Proposition 7 we can show:

Corollary 13. *Let $F : E \to E$ be an analytic diffeomorphism, $F(0) = 0$, with diagonal linear part $F^1 = A^1 + A^2$. Suppose that $\|A^1\| < 1$ and $\|(A^2)^{-1}\| < 1$, then there exists a coordinate transform $U : E \to E$, $U = \mathrm{id} + u$ with $u = O(2)$, such that $G = U^{-1} \circ F \circ U$ has the E_1 plane as an invariant manifold or equivalently $G = \sum_{(k,j)\in BS} g_k^j$.*

5. Examples

5.1. The situation in $\mathbb{C}^n$

We will explain how Theorem 2 follows from Theorem 1. We work with parameter dependent analytic power series F_μ, where the parameter varies in an open set Λ centered around μ_0. More precisely:

Definition 4. We define $\mathbb{C}^n(\Lambda)$ to be the space of power series $\sum_{n\geqslant 0} f_n(\mu)x^n$, where $f_n(\mu)$ is a bounded analytic function on the open domain Λ, such that

$$\sum_{n\geqslant 0} \|f_n\|_\infty \delta^n < \infty \tag{13}$$

for a certain $\delta > 0$. We define $\mathbb{C}^n(\Lambda)_\delta$ as the subset of $\mathbb{C}^n(\Lambda)$ for which (13) holds.

It is standard to show that $\mathbb{C}^n(\Lambda)_\delta$ is a Banach space for the norm defined by the left-hand side of (13).

Remark 1. There are other possible choices for the definition of $\mathbb{C}^n(\Lambda)$: for example if the coefficients f_n depend C^k ($0 \leqslant k < \infty$) on the parameter μ; one then uses a C^k norm for f_n.

We choose for a fixed $0 < D < 1 < C$ a neighbourhood $\widetilde{\Lambda}$ of μ_0 such that $\rho < \frac{1}{D}$ and $\rho > \frac{1}{C}$. This is possible since $\rho = \max_{i=1}^n \{\sup_{\mu\in\widetilde{\Lambda}} |\lambda_i(\mu)|.\sup_{\mu\in\widetilde{\Lambda}} \frac{1}{|\lambda_i(\mu)|}\}$, is close to 1 if $\widetilde{\Lambda}$ is chosen small enough. Hence we can apply Theorem 1 with $E = \mathbb{C}^n(\widetilde{\Lambda})$ and obtain Theorem 2.

5.2. A $1:-1$ resonant saddle

We reconsider the example from Section 1.2 and explain how expression (2) can be obtained from the main result. We consider a family F_μ passing through a $1:-1$ resonance in modulus; by this we mean a family

$$F_\mu(x) = \begin{cases} x_1(\lambda_1(\mu) + \sum_{|k|\geqslant 2} f_k^1(\mu)x^k), \\ x_2(\lambda_2(\mu) + \sum_{|k|\geqslant 2} f_k^2(\mu)x^k), \end{cases}$$

where

$$\frac{\log(|\lambda_1(\mu_0)|)}{\log(|\lambda_2(\mu_0)|)} = -1, \tag{14}$$

and the series are convergent on a sufficient small neighbourhood around the origin. Note that condition (14) concerns the moduli of the eigenvalues: this is necessary in order to apply our main result; omitting the modulus in (14) would lead us to questions of a completely different nature, like for example in the case of elliptic fixed points.

Using Theorem 2, we see that for any $0 < D < 1$ we can conjugate F in an analytic way to a form

$$G(x) = \begin{cases} x_1(\lambda_1(\mu) + \sum_{k\in K} g_k^1(\mu)x^k), \\ x_2(\lambda_2(\mu) + \sum_{k\in K} g_k^2(\mu)x^k), \end{cases}$$

where $K = B_{D,D}$ is a cone containing the resonant line. The closer D is chosen to 1, the smaller the cone. Note also that if $f_k^i(\lambda)$ is continuous (resp. analytic on a neighbourhood with fixed radius) and the supremum norm is considered, then also the coefficients $g_k^i(\mu)$ are continuous (resp. analytic on a neighbourhood with fixed radius). Since we supposed a $1:-1$ resonance in modulus, the main resonant equation at μ_0, is given by

$$\left(|\lambda_1(\mu_0)|, |\lambda_2(\mu_0)|\right)^k = 1 \quad \Leftrightarrow \quad k_1 - k_2 = 0.$$

The only thing we still need to do is describing the terms inside the cone determined by $(N, N+1)$ and $(N+1, N)$. Note that the terms in the upper part of this cone determined by $(N, N+1)$ and $(1, 1)$ correspond to linear combinations of these two vectors

$$r(N, N+1) + s(1, 1) = (A, B),$$

such that r, s are positive real numbers and A, B are natural numbers. Since $B - A = (rN + r + s) - (rN + s) = r$, it follows that r is a natural number. Hence it follows that also $s = A - rN$ is a natural number. It follows that any couple $(A, B) \in \mathbb{N}^2$ in the upper part of this cone can be expressed as $r(N, N+1) + s(1, 1)$, where r, s are natural numbers. In the same way in can be shown that any $(A, B) \in \mathbb{N}^2$ in the lower cone determined by $(N+1, N)$ and $(1, 1)$ can be expressed as $r(N+1, N) + s(1, 1)$, where r, s are natural numbers. Hence

$$G(x) = \begin{cases} x_1(\lambda_1(\mu) + b_0^1(\mu, x_1x_2) + \sum_{s\geqslant 1, r\geqslant 0}(g^1_{(r,s)}(\mu)(x_1x_2)^r(x_1^N x_2^{N+1})^s + h^1_{(r,s)}(\mu)(x_1x_2)^r(x_1^{N+1}x_2^N)^s)), \\ x_2(\lambda_2(\mu) + b_0^2(\mu, x_1x_2) + \sum_{s\geqslant 1, r\geqslant 0}(g^2_{(r,s)}(\mu)(x_1x_2)^r(x_1^N x_2^{N+1})^s + h^1_{(r,s)}(\mu)(x_1x_2)^r(x_1^{N+1}x_2^N)^s)), \end{cases}$$

or, when putting $u = x_1x_2$ and $b_s^i(\mu, u) = \sum_{r\geqslant 0} g^i_{(r,s)}(\mu)u^r$ and $c_s^i(\mu, u) = \sum_{r\geqslant 0} h^i_{(r,s)}(\mu)u^r$, we obtain

$$G(x) = \begin{cases} x_1(\lambda_1(\mu) + b_0^1(\mu, u) + \sum_{s\geqslant 1} u^{Ns}(x_2^s b_s^1(\mu, u) + x_1^s b_s^2(\mu, u))), \\ x_2(\lambda_2(\mu) + b_0^2(\mu, u) + \sum_{s\geqslant 1} u^{Ns}(x_2^s c_s^1(\mu, u) + x_1^s c_s^2(\mu, u))). \end{cases}$$

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