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A normally elliptic Hamiltonian bifurcation

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Dedicated to Klaus Kirchgässner on his sixtieth birthday

1. Setting of the problem, outline

This paper concerns autonomous Hamiltonian systems around an equilibrium point, with a double eigenvalue zero. The main problems already occur in the case of 2 degrees of freedom, where the other eigenvalues form a purely imaginary pair. Therefore, this research is in the Hamiltonian tradition of e.g. Meyer [29], Sanders [39], Van der Meer [46] and Verhulst [49]. It is our aim to describe versal, viz. generic unfoldings of this equilibrium point. Its codimension depends on whether the linear part is semisimple or not. The non-semisimple case has codimension 1 and the semisimple one codimension 3; this means that they will only be met generically if at least 1 respectively 3 parameters are present. The emphasis will be with the semisimple case, which is most degenerate, but which contains the less degenerate, non-semisimple case in a subordinate way. These unfoldings are frequently met in Hamiltonian studies. To fix thoughts we include the following example.

Example. Consider a system of coupled oscillators

$$\ddot{x} + x = f(x, y, \mu)$$
$$\ddot{y} + \varepsilon y = g(x, y, \mu),$$

where $\mu = (\varepsilon, \ldots) \in \mathbb{R}^p$ is a vector of parameters and where f and g contain nonlinearities. For $|\varepsilon| \ll 1$, such a system goes through many high

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order resonances. The unfolding theory, to be developed below, then applies.

The conjugate pair of imaginary eigenvalues gives rise to a formal rotational symmetry in all unfoldings, i.e. a rotational symmetry in their Taylor series. Here the series is considered in dependence on both the phase space variables and the parameters. This is an application of Normal Form Theory, where the terms of the formal power series are changed by canonical coordinate transformations in an inductive process. The symmetry, thus obtained, enables a formal reduction to 1 degree of freedom, around an equilibrium point with a double eigenvalue 0. For similar approaches see some of the above references, also compare e.g. Arnold [6], Takens [43, 44], Broer [10, 11], Golubitsky and Stewart [23] and Broer and Vegter [16]. This Normal Form Theory will be presented in Section 3.

In Section 2, we begin studying the 1 degree of freedom 'backbone'-problem in its own right. Since in the plane the integral curves of the systems are the level sets of the Hamiltonian functions, here the problem reduces to Singularity Theory, e.g. compare Bröcker and Lander [9], Gibson [22], Martinet [28], Poston and Stewart [37] and Thom [45] for these Hamiltonians. It will turn out that the so-called Elliptic and Hyperbolic Umbilic Catastrophes contain all the information we want.

In Section 4 the connection is made between the planar reduction of the symmetric system and the 'backbone'-system of Section 2. It turns out that the formal integral, obtained by normalization, is a *distinguished* parameter in the sense of Golubitsky and Schaeffer [51] and Schecter [41], also compare Wassermann [52]. Now Singularity Theory yields new normal forms, that are polynomial of degree 3, at least in the phase-space variables. We note that here the normalizing transformations no longer need to be canonical. Technical details from Singularity Theory have been collected in an appendix (Section 7). After this we suspend, or dereduce, to the original 4-dimensional setting, so obtaining an integrable, i.e. rotationally symmetric, approximation of the original unfolding.

Thus we obtain a perturbation problem, similar to e.g. Broer [10, 12] or Braaksma and Broer [8]. The perturbation term is of arbitrarily high order, both in the phase space variables and the parameters. This problem is briefly addressed in Section 5.

Remarks. (i) Although in the original family only *generic* restrictions are imposed on the lower order terms, by Normal Form Theory and Singularity Theory, the corresponding unfolding is reduced to an arbitrarily flat perturbation of a normal form, completely determined by a 1 degree of freedom system, which is polynomial of degree 3. This polynomial character may explain why certain 'integrable' characteristics in the unfoldings are so persistent. For a related comment we refer to Verhulst [49].

(ii) Our approach differs from e.g. Van der Meer [46], also compare Duistermaat [19]. In [46, 19] the dynamics is reduced by the energy-momentum map as well as the Moser–Weinstein method. This gives information related to specific periodic solutions, that is also important for the dynamics as a whole. Instead, we just factor out a formal rotational symmetry, and so seem to get a more direct hold on the global dynamics. This procedure is along the same lines as e.g. Broer [10, 11] or Broer and Vegter [16]. It would be interesting to know how both methods compare in this.

In Section 6, the paper is concluded by considerations concerning the general Hamiltonian problem with one double zero singularity. Also the analogue for symplectic maps is discussed.

2. The planar backbone

We consider C^∞ functions $H : \mathbb{R}^2 \to \mathbb{R}$ with the origin as a critical point. The corresponding Hamiltonian vector field, to be denoted by X_H, then has the origin as an equilibrium or singular point. We here recall that

$$dH = \omega(X_H, \cdot),$$

where $\omega = dx \wedge dy$ denotes the standard area 2-form on $\mathbb{R}^2$. For this use of notation, e.g. see Abraham and Marsden [1] and Arnold [5]. In coordinates, for X_H we get the familiar expression

$$\dot{x} = \frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{\partial H}{\partial x}.$$

We recall that, for a vector field X on $\mathbb{R}^2$ to be Hamiltonian with respect to some function H, it is necessary and sufficient for X to have divergence zero. This, in turn, means that its (solution-) flow preserves the standard area mentioned above.

In this section our concern is with the situation where the linear part of X_H at the origin has double eigenvalue zero, and the question becomes what are generic unfoldings of this singularity both in the semisimple and the non-semisimple case. In the former of these cases the corresponding Hamiltonian H, at the origin, vanishes to third order. We claim that the Singularity Theory for the planar Hamiltonian functions provides a good framework for this problem, again compare [9, 22, 28, 37, 45].

In fact, we start considering a C^∞ transformation $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that for two Hamiltonians H and K one has $H = K \circ \Phi$. Such a map Φ is called a *right equivalence* between the functions H and K. The following lemma compares the Hamiltonian vector field X_K with the transformed vector field $\Phi_*(X_H)$, defined by

$$\Phi_*(X_H)(\Phi(p)) := D\Phi(p)(X_H)(p):$$

Lemma 1. $\Phi_*(X_H) = \det D\Phi X_K$.

Proof. We transport the equation $dH = \omega(X_H, \cdot)$ by Φ. Putting $\Phi_*(H) := H \circ \Phi^{-1} = K$ and $\Phi_*(\omega) := (\Phi^{-1})^*(\omega)$ we get $dK = d\Phi_*(H) = \Phi_*(\omega)(\Phi_* X_H, \cdot)$. Here $\Phi_*(\omega) = (\det D\Phi)^{-1}\omega$, whence

$$dK = (\det D\Phi)^{-1}\omega(\Phi_*(X_H), \cdot) = \\ = \omega((\det D\Phi)^{-1}\Phi_*(X_H), \cdot)$$

and $(\det D\Phi)^{-1}\Phi_*(X_H) = X_K$, which immediately proves the lemma. □

Since the vector fields X_H and $\Phi_*(X_H)$ are *conjugate*, the presence of the scalar factor $\det D\Phi$ implies that the Hamiltonian vector fields X_H and X_K are *equivalent*. For this terminology, e.g. see Palis and de Melo [33]. We here recall that conjugacies take integral curves to integral curves in a time-preserving way. Equivalences, however, do take integral curves to integral curves, but without necessarily preserving this time-parametrization. Observe that Φ is a conjugacy between X_H and X_K precisely if $\det D\Phi \equiv 1$, meaning that Φ is both area- and orientation preserving, which in the present setting is the same thing as *canonical* or *symplectic*.

This set up can be widened somewhat by also allowing *left-right equivalences*, which includes transformations in the image space $\mathbb{R}$ of the Hamilton functions. Moreover, in the case where the systems depend on the parameters, we use parameter-dependent (left-right) equivalences on the Hamiltonians, together with reparametrizations. Such a compound transformation is called a *morphism of unfoldings*. For precise definitions, also see Section 7, below.

Remark. Without making an essential difference, the (left-) transformations on the range $\mathbb{R}$ can be restricted to the class of parameter-dependent *translations*, compare [9]. In Section 7, for technical reasons, we take a slightly different though equivalent point of view.

Singularity Theory aims to classify (germs of) generic unfoldings of functions under the equivalence relation provided by these morphisms. From the above it then may be clear that this classification also is relevant for the corresponding Hamiltonian vector fields, even though the classifying diffeomorphisms, i.e. the right equivalences, are not necessarily canonical: it suffices to give a *qualitative* picture of the phase portraits. Note, however, that also many quantative features are kept track of in this way. For similar approaches e.g. see Arnold [6], Broer [10, 11], Broer and Vegter [16], Duistermaat [19] or Van der Meer [46].

2.1. *The linear classification*

To fix thoughts, let us discuss the linear classification, which is quite familiar, speaking in terms of singularities of Hamiltonian vector fields X_H. So we consider the linear part

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = L \begin{pmatrix} x \\ y \end{pmatrix},$$

where L is a 2 by 2 matrix. It is easily seen that L must have trace zero, i.e. $L \in sl(2, \mathbb{R}) = sp(2, \mathbb{R})$. This follows from the fact that $\operatorname{div} X_H = 0$. Moreover, if the Hamiltonian H at the origin has the Hessian matrix

$$D_0^2 H = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

then we have

$$L = \begin{pmatrix} b & c \\ -a & -b \end{pmatrix}.$$

So we consider the 3-dimensional space of matrices $sl(2, \mathbb{R})$, where our morphisms induce the following equivalence relation:

$$L \sim M \Leftrightarrow \exists S \in Gl(2, \mathbb{R}), \exists \kappa \in \mathbb{R} \backslash \{0\} : S \circ L \circ S^{-1} = \kappa M.$$

Note, that only using right equivalences would imply $\kappa = 1$. Moreover, in the case where these right equivalences would be canonical, even $\det S = 1$, i.e. $S \in Sl(2, \mathbb{R}) = Sp(2, \mathbb{R})$. The characteristic polynomial of a matrix $L \in sl(2, \mathbb{R})$ is given by $\lambda^2 + \det(L)$, which directly leads to the following partition in equivalence classes; for similar classifications e.g. compare Gibson [22] and Poston and Stewart [37].

i. *The hyperbolic case:* $\det(L) < 0$, corresponding normal forms:

$$L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad H(x, y) = xy.$$

ii. *The elliptic case:* $\det(L) > 0$, corresponding normal forms:

$$L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad H(x, y) = \frac{1}{2}(x^2 + y^2).$$

iii. *The parabolic case:* $\det(L) = 0$, $L \neq 0$, corresponding normal forms:

$$L = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad H(x, y) = \frac{1}{2} y^2.$$

iv. *The zero case:* $L = 0$, corresponding normal forms:

$$L = 0, \qquad H(x, y) \equiv 0.$$

The situation is illustrated in Fig. 1. The nilpotent variety given by $\det(L) = 0$, which is a cone, corresponding to the cases iii and iv. The complement consists of two open pieces, corresponding to the cases i and ii. The singularities iii and iv are our object of study, iv being the semisimple and iii the non-semisimple case. Geometrically it is evident that the zero case iv has codimension 3, while the parabolic case iii has codimension 1. This also follows from computations as in Arnold [3, 6] in this symplectic setting, e.g. compare Broer [10], as well as from the codimension computations in Singularity Theory, carried out on the level of 2-jets, again see [9, 22, 28, 37, 45]. Note, that by 'normal form' we just mean a preferred, simple member of the corresponding equivalence class. For instance, observe that $xy \sim \frac{1}{2}(x^2 - y^2)$.

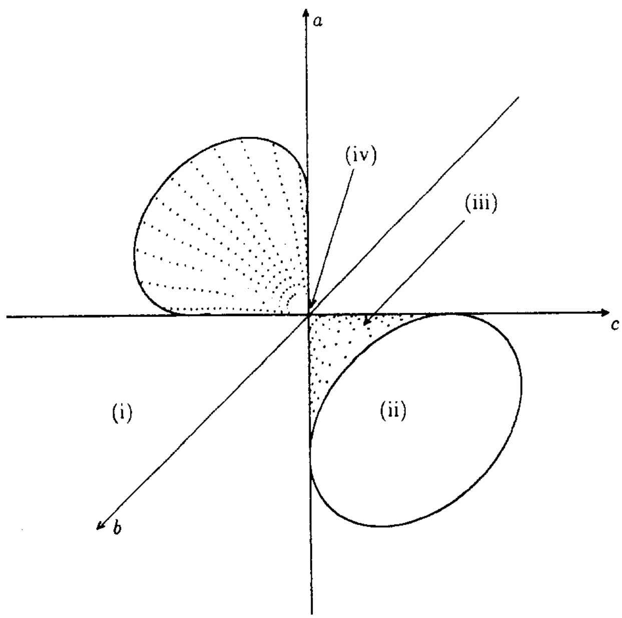


Figure 1
Stratification of $sl(2, \mathbb{R})$ by the equivalence relation $\sim$.

2.2. *The Fold and the Elliptic and Hyperbolic Umbilic Catastrophes*

What happens to the above, linear picture if higher order terms are added? This question is addressed by Singularity Theory, for details we again refer to [9, 22, 28, 37, 45]. The codimension 0 cases i and ii can be handled by the Morse Lemma, which implies that the normal forms given above, also hold with the higher order terms added. The critical points for the Hamiltonian are a saddle-point, respectively a maximum or minimum and the corresponding singularities of the vector field are a saddle-point, respectively a center. These critical points, viz. singularities are called *nondegenerate*. In fact, in these cases *structural stability* holds: the corresponding equivalence classes are open, say, in the C^∞ topology. Notice that here we restrict to the class of functions that in $0 \in \mathbb{R}^2$ have the value 0. The situation is less easy in the cases iii and iv. Let us briefly discuss what Singularity Theory has to say here.

In the parabolic case iii the Splitting Lemma, compare the above references, tells us that, after carrying out an approximate morphism, we only need to consider 1-parameter families of the form

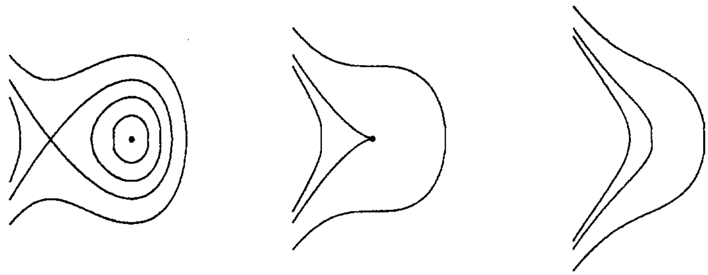
$$H^\mu(x, y) = \frac{1}{2} y^2 + V^\mu(x),$$

where $V^0(x) = O(x^3)$ as $x \to 0$. So, the Splitting Lemma splits the unfolding in a 'Morse part' $\frac{1}{2}y^2$, see above, and a part $V^\mu(x)$ containing a more degenerate singularity. The morphism needed to obtain this split form, does not involve a change in the parameters. A *universal unfolding* is given by

$$H^\mu(x, y) = \frac{1}{2} y^2 + \frac{1}{3} x^3 + \mu x,$$

where the 'potential' function $V^\mu(x) = \frac{1}{3}x^3 + \mu x$ 'is' a *Fold Catastrophe*, again compare the above references. Also here, this universal unfolding is a normal form, i.e. a preferred member of its equivalence class, now considered in the world of all 1-parameter unfoldings of the present parabolic singularity. Moreover, in this set of 1-parameter families the universal unfolding is structurally stable, again meaning that its equivalence class is open. Finally, generic unfoldings of the parabolic singularity with more than 1 parameter can be *reduced* to the case with 1 parameter, carrying out a projection in the parameter-direction. In this way, a suitable morphism leads to the case where all the other parameters are 'mute': the unfolding does not explicitly depend on those.

In Fig. 2 we depicted the corresponding phase portraits: For $\mu > 0$ no singularities exist, while for $\mu < 0$ a (nondegenerate) saddle-point and center are present, their distance to the origin being of the order $\sqrt{-\mu}$. For $\mu = 0$ the origin occurs as the 'new', parabolic singularity. We shall refer to this bifurcation as the *Saddle-Center* or *Hamiltonian Saddle-Node bifurcation.*



$\mu < 0$ $\mu = 0$ $\mu > 0$

Figure 2
The Saddle-Center or Hamiltonian Saddle-Node bifurcation.

In the zero case iv, the situation is even more complicated. Since the linear part L completely vanishes here, we don't have a convenient splitting. However, Singularity Theory now provides the following two universal unfoldings:

$$H^{\mu,\nu,\kappa}(x, y) = x^2 y \pm \frac{1}{3} y^3 + \mu(x^2 \mp y^2) + \nu x + \kappa y,$$

the 'upper signs' corresponding to the *Hyperbolic* and the 'lower signs' to the *Elliptic Umbilic Catastrophe*. In another classification, see Arnold [4], these unfoldings are labeled $D_4^{\pm}$. Again universality means being a preferred member of ones equivalence class of 3-parameter unfoldings, where the number of parameters always safely can be reduced to 3. And also here the families are structurally stable in the above sense, so there are two open classes of 3-parameter unfoldings.

We conclude this section with brief descriptions of the two umbilic catastrophes, mainly referring to [37, 45].

2.3. Description of the Elliptic Umbilic Catastrophe

We begin by giving the catastrophe set $\mathscr{C}$ of the Elliptic Umbilic Catastrophe in the parameter space. This is the set of parameter-points (μ, ν, κ) where the Hamiltonian $H^{\mu,\nu,\kappa}$ has a degenerate critical point. The set $\mathscr{C}$, depicted in Fig. 3, is given parametrically by

$$\mu^2 = x^2 + y^2, \qquad \nu = -2\mu x - 2xy, \qquad \kappa = -2\mu y + y^2 - x^2,$$

with x and y as parameters. Observe that it has a cone-like structure, with a curvilinear triangle as base, the edges of which meet in cusps. The axis of the 'cone' is the μ-axis and the triangle shrinks quadratically in $|\mu|$ as $\mu \to 0$. Moreover, the parameter μ, for $\mu \neq 0$ only effects the characteristic size of

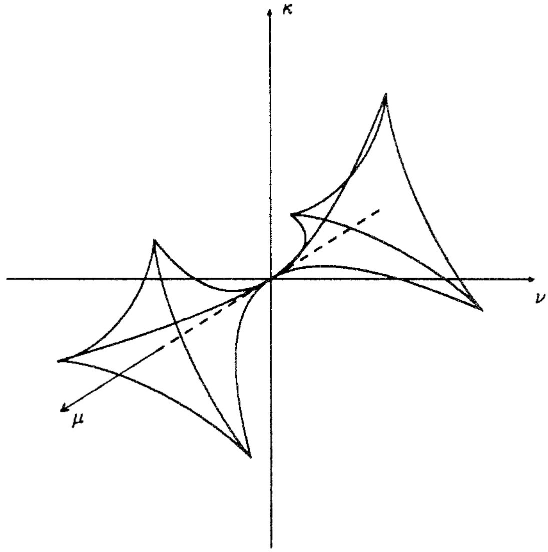


Figure 3
Catastrophe set of the Elliptic Umbilic.

the phase-portraits, while its sign governs the orientation of the flow. For $\mu = 0$ the central, umbilical singularity occurs at the origin.

In Fig. 4 the phase-portraits are given relative to the 2-dimensional section $\mu = 1$. We restricted to one sector of the diagram, the other phase portraits easily follow by symmetry considerations. Also the central, umbilical singularity is depicted, as it occurs at $(\mu, \nu, \kappa) = 0$. The corresponding critical point for the function $H^{0,0,0}(x, y) = x^2y - \frac{1}{3}y^2$ is called *monkey saddle*. Observe that upon transversal crossing of the edges of the triangle the Hamiltonian Saddle-Node, described above, occurs subordinately. In the 3-dimensional bifurcation diagram these edges correspond to 2-dimensional sheets. Moreover, there exist three subordinate local 2-parameter subfamilies, transversal to the cusp-curves corresponding to the vertices of the triangle, admitting a Splitting Lemma approach. In fact, the 'potential' function $V^{\mu,\nu}$ then undergoes a *Dual Cusp Catastrophe*. The corresponding Hamiltonian family, in its own right, has normal form

$$H^{\mu,\nu}(x, y) = \frac{1}{2}y^2 - \frac{1}{4}x^4 + \frac{1}{2}\mu x^2 + \nu x.$$

It is straightforward to give both the bifurcation diagram and the phase portraits for this normal form family. Finally we wish to point at the subordinate *heteroclinic bifurcations*, occurring at the 'bisectors' of the triangle. At the center of the triangle we find coincidence of two, and hence three, of such bifurcations. In this way, the bifurcation set extends the catastrophe set.

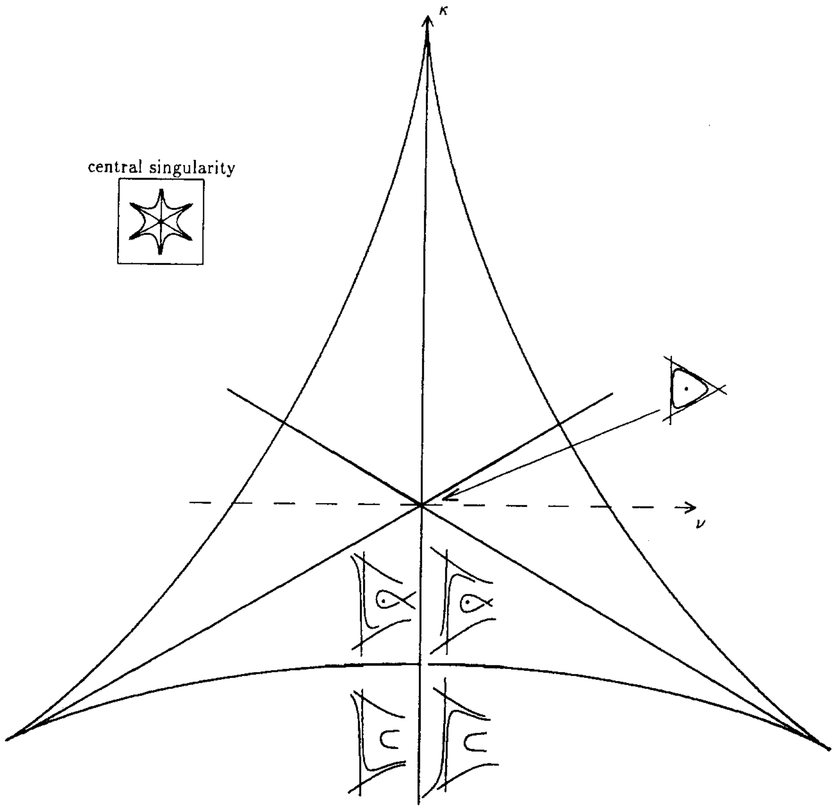


Figure 4
Bifurcation diagram of the Elliptic Umbilic, $\mu \neq 0$ fixed.

2.4. Description of the Hyperbolic Umbilic Catastrophe

Next we turn to the Hyperbolic Umbilic Catastrophe. Also here we denote the catastrophe set by $\mathscr{C}$, see Fig. 5. This time it is given by

$$\mu^2 = y^2 - x^2, \qquad \nu = -2\mu x - 2xy, \qquad \kappa = 2\mu y - y^2 - x^2,$$

again with x and y regarded as parameters. The story is much the same as before. Again the set $\mathscr{C}$ has a cone-like structure, but now with a base consisting of the disjoint union of a smooth curve and a cusp-line. Also the role of the parameter μ is the same as before. In Fig. 6 we show the phase portraits related to the section $\mu = 1$, as well as the central umbilic singularity. Observe that again all kinds of subordinate bifurcations occur. Apart from Dual Cusp Catastrophes, here we also have the 'ordinary' Cusp Catastrophe, with normal form

$$H^{\mu,\nu}(x, y) = \frac{1}{2}y^2 + \frac{1}{4}x^4 + \frac{1}{2}\mu x^2 + \nu x.$$

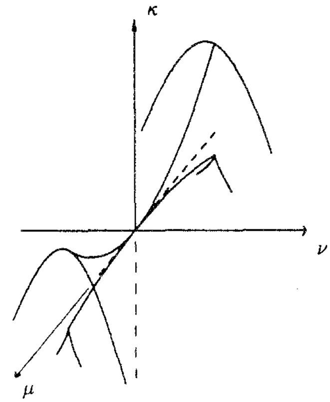


Figure 5
Catastrophe set of the Hyperbolic Umbilic.

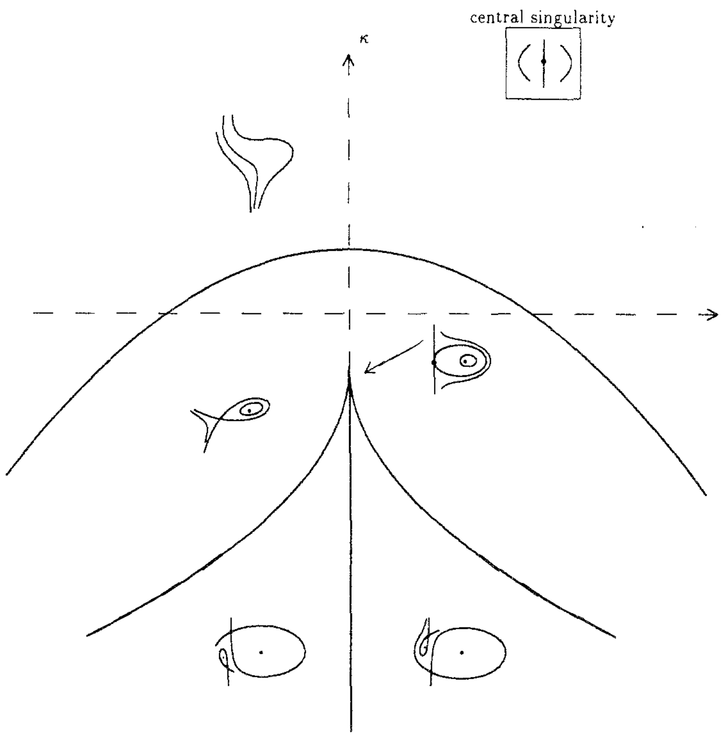


Figure 6
Bifurcation diagram of the Hyperbolic Umbilic, $\mu \neq 0$ fixed.

Again bifurcation set as well as phase portraits are easily given. Also note that, on the 'bisector' of the cusp a subordinate heteroclinic bifurcation occurs.

Many of the above, 2-dimensional phase portraits also can be found in Andronov et al. [2]. For an extensive discussion of both umbilical catastrophes in the context of planar gradient vector fields we refer to Vegter [47].

3. A Hamiltonian normal form: formal reduction to 1 degree of freedom

We now return to the 4-dimensional setting. To begin with, we notice that a direct approach applying Singularity Theory to the Hamilton functions does not give much information on the dynamics, but only on the foliation of their 3-dimensional level sets and its lower dimensional singularities. Therefore, instead, we present a Formal Normal Form permitting the formulation of a perturbation problem related to the 2-dimensional situation of Section 2. The relevant normal form theory started with Poincaré [35] and Birkhoff [7], for more recent references see Gustavson [24], Moser [30, 31], Takens [43, 44], Arnold [5], Sanders [39], Broer [11, 10], Sanders and Verhulst [40], Van der Meer [46] and Broer and Vegter [16].

3.1. The Normal Form

To be more precise, we endow $\mathbb{R}^4$ with coordinates (x_1, y_1, x_2, y_2) and the natural symplectic form $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, considering a C^∞ family of Hamiltonian functions H^μ, where $\mu \in \mathbb{R}^p$ is a vector of parameters. As before, for any Hamiltonian H the corresponding Hamiltonian vector field X_H is given by $dH = \omega(X_H, \cdot)$, which in coordinates means

$$\dot{x}_j = \frac{\partial H}{\partial y_j}, \qquad \dot{y}_j = -\frac{\partial H}{\partial x_j},$$

for $j = 1, 2$.

We assume that for $\mu = 0$ the origin of $\mathbb{R}^4$ is a singularity. Then we expand as a Taylor series in (x, y, μ)

$$H^\mu(x, y) = H_2(x, y, \mu) + H_3(x, y, \mu) + \cdots,$$

where H_n is homogeneous of degree n in (x, y, μ). Assuming that the singularity at $(x, y, \mu) = (0, 0, 0)$ has eigenvalues 0 (double) and $\pm i\alpha$, for some real constant $\alpha \neq 0$, it is our aim to normalize or simplify these homogeneous parts, using induction on the degree n. Here 'simple' means 'rotationally symmetric' in a way to be explained below. To this purpose we carry out suitable transformations that preserve the symplectic form ω, i.e. which are *canonical* or *symplectic*.

First, by Williamson's Normal Form, compare Galin [21] and Koçak [27], abbreviating $I := \frac{1}{2}(x_1^2 + y_1^2)$, we normalize the second order part H_2 to:

$$H_2(x, y, \mu) = \alpha I,$$

in the semisimple case and

$$H_2(x, y, \mu) = \alpha I \pm \frac{1}{2} y_2^2$$

in the non-semisimple case. The corresponding (infinitesimally) symplectic matrices, being the linear part of the corresponding Hamiltonian vector fields X_{H^0}, have the respective forms

$$\begin{pmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The codimension of these singularities are the same as the corresponding ones in Section 2, i.e. 3 for the semisimple one and 1 for the non-semisimple one. This directly follows from Section 2 and the fact that pure imaginary eigenvalues in the symplectic setting have open occurrence. For other approaches to this, see the relevant references given at the end of Section 2.1. Recalling that our parameter space is $\mathbb{R}^p$, we fix $p = 3$ in the semisimple and $p = 1$ in the non-semisimple case.

The function I, viz. the Hamiltonian vector field X_I, now will be used to give the rotational symmetry as follows:

Theorem 2. There exists a formal canonical transformation $\hat{\Phi}(x, y, \mu)$ keeping the parameters fixed, and a formal power series $\hat{F}(I, x_2, y_2, \mu)$ such that, formally speaking

$$(\hat{H} \circ \hat{\Phi})(x, y, \mu) = \hat{F}(I, x_2, y_2, \mu).$$

Proof. In fact, let us denote the space of formal power series of Hamiltonian functions by $\prod_{n=2}^{\infty} \mathscr{H}_n\{x, y, \mu\}$, where $\mathscr{H}_n\{x, y, \mu\}$ contains the homogeneous polynomials of degree n. Then, for each n the adjoint action

$$\mathrm{ad}_{H_2} : \mathscr{H}_n\{x, y, \mu\} \to \mathscr{H}_n\{x, y, \mu\}$$

is induced by $\mathrm{ad}_{H_2} : F \mapsto \{H_2, F\}$, where $\{\cdot\,,\cdot\}$ denotes Poisson-brackets, for definitions and further reference see [1, 5]. By the above references [11, 10, 43, 44] it follows that for $n \geq 3$, successively all terms can be normalized into $\ker \mathrm{ad}_I \subset \mathscr{H}_n\{x, y, \mu\}$, using only canonical transformations, that preserve the parameters. Indeed, for each n this transformation can be generated infinitesimally by the Hamiltonian vector field corresponding to an appropriate element of $\mathscr{H}_n\{x, y, \mu\}$. Moreover, the fact that the normalized terms are in $\ker \mathrm{ad}_I$ implies that they Poisson-commute with I, in turn implying that they are equivariant with respect to the circle action generated by the vector field X_I. □

Remark. In the non-semisimple case we can 'improve' the normal form in the following way, compare [46]: If we write $N(x, y) := \pm\frac{1}{2}y_2^2$ so that $H_2 = \alpha I + N$, the homogeneous part of degree n can be 'normalized' into $\ker \mathrm{ad}_I \backslash \mathrm{im}\, \mathrm{ad}_N \subseteq \mathscr{H}_n\{x, y, \mu\}$. Another way to 'improve' the normal form is to incorporate higher order terms of the Hamiltonian in the adjoint action. In the present case, however, since we can apply Singularity Theory in a straightforward manner, only the rotational symmetry is of importance.

Let us see what the formal statement of Theorem 2 means on the level of C^∞ functions:

Corollary 3. There exists a C^∞ canonical transformation Φ, which keeps the parameters fixed, and there exist C^∞ functions $F(I, x_2, y_2, \mu)$ and $P(x, y, \mu)$, such that

1. P is infinitely flat at $(x, y, \mu) = 0$,
2. $(H \circ \Phi)(x, y, \mu) = F(I, x_2, y_2, \mu) + P(x, y, \mu)$.

Proof. The above theorem says, that up to the formal canonical transformation $\hat{\Phi}$, the function H has the symmetric Taylor series $\hat{F}(I, x_2, y_2, \mu)$. If we stop the induction at the order N, we obtain a real analytic transformation Φ_N, such that for the truncation F_N of $\hat{F}$ at the order N, we get $(H \circ \Phi_N)(x, y, \mu) = F_N(I, x_2, y_2, \mu) + O(|(x, y, \mu)|^{N+1})$. Currently however, we work in the C^∞ context, where a theorem of É. Borel is valid. This theorem says the following, e.g. compare [32]: Given any formal power series $\hat{\Phi}$ in the variables (x, y, μ), there exists a C^∞ map Φ with $\hat{\Phi}$ as its Taylor series. A careful look at the level of generating functions ensures us that in this particular case it is also possible to choose Φ canonical. Also we can treat the series $\hat{F}$ in this way. Combining this we get the statement of our Corollary. For details also see [11, 10, 13]. □

Corollary 3 provides the setting for our perturbation problem. In the next section we shall consider the 'unperturbed', symmetric Hamiltonian $F(I, x_2, y_2, \mu)$, later on studying how much can be said when the flat 'perturbation' P is added.

3.2. The integrable case: reduction to 1 degree of freedom

We now consider the vector field associated to the integrable Hamiltonian $F(I, x_2, y_2, \mu)$. As remarked earlier, one consequence of the rotational symmetry is that $\{F, I\} = 0$, i.e. that F and I Poisson-commute, which implies that I is an integral of the system. Our main aim is to use this in order to carry out a reduction to 1 degree of freedom, proceeding as in the Kepler problem, e.g. compare [5]. To this end, let us define a 2π-periodic variable φ by

$$x_1 = \sqrt{2I}\cos\varphi, \qquad y_1 = \sqrt{2I}\sin\varphi.$$

We then have

Corollary 4. In the coordinates (I, φ, x_2, y_2) the Hamiltonian vector field X_F has the form

$$\dot{I} = 0, \qquad \dot{\varphi} = -\frac{\partial F}{\partial I},$$

$$\dot{x}_2 = \frac{\partial F}{\partial y_2}, \qquad \dot{y}_2 = -\frac{\partial F}{\partial x_2}.$$

Proof. A brief computation yields that $\omega = dI \wedge d\varphi + dx_2 \wedge dy_2$, telling us that we remain within the Hamiltonian formalism. This means that in the coordinates (I, φ, x_2, y_2) the vector field X_F has the canonical form as given in the corollary. □

The latter two equations in Corollary 4 constitute the *reduction to* 1 *degree of freedom*: it is a family of planar Hamiltonian vector fields, parametrized by I and μ.

Remarks.

i. The construction of Corollary 4 is quite familiar, for example again see [1, 5]. In fact, one says that φ is canonically conjugate to I. The fact that I is an integral of X_F clearly shows from the Corollary; the variable φ usually is called cyclic. The coordinates (I, φ) often are called Hamiltonian polar coordinates, in this case in the (x_1, y_1)-plane.
ii. Necessarily we have $I \geq 0$, so here the parameter space is a manifold with boundary, being a halfspace of $\mathbb{R}^{1+p}$. From the symmetry it is easy to see, however, that F is smooth at the boundary hyperplane $I = 0$.

4. Generic unfoldings in the integrable case

In this section we consider the integrable Hamiltonian F for its own sake. Writing

$$F^{I,\mu}(x_2, y_2) := F(x_2, y_2, I, \mu),$$

we obtain the family of functions giving the planar reduction of Corollary 4. From now on we abandon the world of canonical transformations and, as in Section 2, work with less rigid equivalences between the systems. In fact, for the planar reductions $F^{I,\mu}$ we use the morphisms based on parameter dependent left-right equivalence and reparametrization. For the duration of Section 4.1, since they do not influence the phase-portraits, we disregard the terms in F that do not depend on (x_2, y_2).

In Section 4.2, when interpreting our results back to the integrable case with 2 degrees of freedom, the parameter I again will be a phase space coordinate. Therefore, in this planar setting we only allow reparametrizations of the form $(I, \mu) \in \mathbb{R}^{1+p} \mapsto (J, \nu) \in \mathbb{R}^{1+p}$, with $J = J(I, \mu)$ and $\nu = \nu(\mu)$. Both these reparametrizations and the corresponding morphisms will be called *restricted.* The parameter I, which has an intrinsic physical meaning, will be called a *distinguished* parameter. We adopt this terminology from Schecter, see [41], who studies unfoldings of vector fields near saddle connections of quasi-hyperbolic singular points. His approach is also based on gearing Singularity Theory to the specific context of the problem. This approach, which is in the related setting of contact-equivalence, mainly follows Golubitsky and Schaeffer [51].

We recall that the restricted parameter I is non-negative. If this property is preserved by the morphisms, we say that they *respect the boundary.* Technically this means that $J(I, \mu) = I\tilde{J}(I, \mu)$, with $\tilde{J}(0, 0) > 0$.

As the main result of this section, we derive normal forms for our planar reductions, viz. the family $F^{I,\mu}$, using restricted morphisms. The heart of our method consists of a standard application of Singularity Theory, using ordinary, unrestricted morphisms. Here the backbone systems of Section 2 play an important role.

4.1. *The reduction to 1 degree of freedom: Singularity Theory revisited*

We begin formulating normal form theorems for both the non-semisimple and the semisimple case.

Theorem 5. In the non-semisimple case, up to restricted morphisms respecting the boundary, generically the family $F^{I,\mu}$ has the form

$$F^{I,\mu}(x_2, y_2) = \frac{1}{2} y_2^2 + \frac{1}{3} x_2^3 + (\mu \pm I)x_2.$$

Observe that this normal form is structurally stable under restricted morphisms: C^3-small changes of the 1-parameter family yield an equivalent family. In fact, we here obtained a *universal unfolding* with respect to restricted morphisms, compare Section 2, also see Section 7, below. In Fig. 7 the bifurcation diagram is depicted. Although it can be extended over the

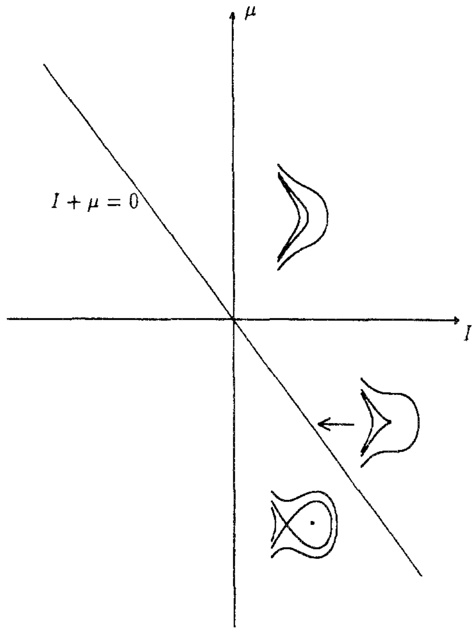


Figure 7
Bifurcation diagram for the reduced integrable Hamiltonian Saddle-Node bifurcation.

boundary $I = 0$, only the part $I \geq 0$ is of importance to us now. Notice that the Hamiltonian Saddle Node bifurcation occurs upon traversing the line $\mu \pm I = 0$.

In the semisimple case the parameter μ is 3-dimensional, writing $\mu = (\mu_1, \mu_2, \mu_3)$ we have:

Theorem 6. In the semisimple case, up to restricted morphisms respecting the boundary, generically the family $F^{I,\mu}$ has one of the forms

$$F^{I,\mu}(x_2, y_2) = x_2^2 y_2 - \frac{1}{3} y_2^3 + (\mu_1 \pm I)(x_2^2 + y_2^2) + \delta_1(I, \mu)x_2 + \delta_2(I, \mu)y_2,$$

$$F^{I,\mu}(x_2, y_2) = x_2^2 y_2 + \frac{1}{3} y_2^3 + (\mu_1 \pm I)(x_2^2 - y_2^2) + \delta_1(I, \mu)x_2 + \delta_2(I, \mu)y_2,$$

for certain coefficient functions $\delta_j : (\mathbb{R} \times \mathbb{R}^3, 0) \to (\mathbb{R}, 0)$, $j = 1, 2$.

The remainder of this subsection will be devoted to a proof of these theorems. As announced before, we apply Singularity Theory to the given familites $F^{I,\mu}$. A standard application of this theory yields normal forms, modulo general, unrestricted morphisms. The first step of our proof will use this result in order to find preliminary normal forms, now modulo restricted morphisms. Secondly, we simplify these forms as far as possible, only using restricted reparametrization.

Remark. A similar program, also in the current setting of left-right equivalences, is followed by Wassermann [52]. In the case of e.g. Theorem 5 his procedure yields the simpler normal form

$$F^{I,\mu}(x_2, y_2) = \frac{1}{2} y_2^2 + \frac{1}{3} x_2^3 + I x_2.$$

This difference has to do with the fact that Wassermann uses a larger class of reparametrizations. First, most importantly, he does not need our constraint that the set $\{I \geq 0\}$ has to be preserved. Second, he has more left-equivalences at his disposal, since these are allowed to depend on the distinguished parameter I. (In our case I originates from state-space, so we cannot use this extra freedom.) A similar remark holds when comparing with the results of [41, 51].

At the end we shall give a geometric, though not completely decisive argument, saying that the umbilical normal forms of Theorem 6 are not structurally stable, but that arbitrarily small changes of the coefficient-functions $(\delta_1(I, \mu), \delta_2(I, \mu))$ may yield families that are not equivalent under restricted morphisms.

In the Appendix, cf. Section 7, we consider this problem more in general, studying how (uni-) versality under restricted morphisms relates to (uni-) versality under unrestricted morphisms. We shall show that a family depending on a distinguished parameter I has a universal unfolding with respect to restricted morphisms precisely if, considered as a family parametrized by I it is a versal unfolding with respect to unrestricted morphisms. In the case of Theorem 6, the 1-parameter family $F^{I,0}$ is not versal, since here any versal family must have at least 3 parameters. This then confirms that the normal forms of Theorem 6 are not structurally stable.

4.1.1. Proofs of Theorems 5 and 6

We begin deriving preliminary normal forms, under restricted morphisms.

1. In the non-semisimple case of Theorem 5, we can apply the Splitting Lemma, compare Section 2, yielding a parameter-preserving morphism, that gives the form $\bar{F}^{I,\mu}(x_2, y_2) = \frac{1}{2}y_2^2 + V^{I,\mu}(x_2)$, for some family $V^{I,\mu}$ of 'potentials'. From Section 2 we also know that a normal form for these potential functions is the 1-parameter family $N^{\bar{\varepsilon}}(x_2) := \frac{1}{3}x_2^3 + \bar{\varepsilon} x_2$, the so-called *Fold*. So, under the usual (generic) transversality conditions, we are given an unrestricted morphism

$$\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R} : (x_2, I, \mu) \mapsto (H(x_2, I, \mu), \bar{\varepsilon}(I, \mu)),$$

with $\bar{\varepsilon}(0, 0) = 0$ such that

$$N^{\bar{\varepsilon}(I,\mu)}(H(x_2, I, \mu)) = V^{I,\mu}(x_2).$$

In other words, if we consider the slightly adapted form

$$\bar{N}^{I,\mu}(x_2) := \frac{1}{3} x_2^3 + \bar{\varepsilon}(I, \mu) x_2$$

we conclude by inspection that the *restricted* morphism

$$\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}^2 : (x_2, I, \mu) \mapsto (H(x_2, I, \mu), I, \mu),$$

satisfies

$$\bar{N}^{I,\mu}(H(x_2, I, \mu)) = V^{I,\mu}(x_2).$$

This procedure leaves us with a preliminary normal form

$$\bar{F}^{I,\mu}(x_2, y_2) = \frac{1}{2} y_2^2 + \frac{1}{3} x_2^3 + \bar{\varepsilon}(I, \mu) x_2.$$

2. In the semisimple case of Theorem 6 a completely similar procedure yields the preliminary normal forms

$$\bar{F}^{I,\mu}(x_2, y_2) = x_2^2 y_2 \pm \frac{1}{3} y_2^3 + \bar{\varepsilon}(I, \mu)(x_2^2 \mp y_2^2) + \bar{\delta}_1(I, \mu) x_2 + \bar{\delta}_2(I, \mu) y_2,$$

related to the *Elliptic* and the *Hyperbolic Umbilic*, again see Section 2.

So our preliminary normal forms have rather general coefficients, depending on the parameters. The question now is, how far these coefficients can be further simplified, using restricted reparametrizations.

To this purpose, more generally, we consider a map

$$h : \mathbb{R}^k \times \mathbb{R}^c \to \mathbb{R}^c,$$

representing the unfolding coefficients as a vector. On $\mathbb{R}^k \times \mathbb{R}^c$ we use the coordinates (I, μ), where $I = (I_1, I_2, \ldots, I_k)$ are distinguished and $\mu = (\mu_1, \mu_2, \ldots, \mu_c)$ external (unfolding) parameters. In the above examples we have

1. $k = c = 1$ and $h = \bar{\varepsilon}$;
2. $k = 1$, $c = 3$ and $h = (\bar{\varepsilon}, \bar{\delta}_1, \bar{\delta}_2)$.

The following lemma will prove useful for our simplification purposes, since it provides us with a normal form of the unfolding coefficients, and hence for the unfolding itself.

Lemma 7. Let $k \leq c$, and let $h_* : \mathbb{R}^k \times \mathbb{R}^c \to \mathbb{R}^c$ be a map such that the derivatives $D_\mu h_*(0, 0)$ and $D_I h_*(0, 0)$ have the maximal ranks c and k, respectively. Let $\pi : \mathbb{R}^c \to \mathbb{R}^k$ be a linear projection onto some k-dimensional subspace of $\mathbb{R}^c$ such that $D_I(\pi \circ h_*)(0, 0)$ has rank k. Then for any map $h : \mathbb{R}^k \times \mathbb{R}^c \to \mathbb{R}^c$ there is a local, restricted reparametrization $\phi : \mathbb{R}^k \times \mathbb{R}^c \to \mathbb{R}^k \times \mathbb{R}^c$, defined near $(0, 0) \in \mathbb{R}^k \times \mathbb{R}^c$, that maps the set $\{(I, \mu) \in \mathbb{R}^k \times \mathbb{R}^c \mid I_1 = 0, \ldots, I_k = 0\}$ onto itself, such that

$$\pi(h(I, \mu)) = \pi(h_*(\phi(I, \mu))).$$

Moreover, if also the derivatives $D_\mu h(0, 0)$ and $D_I(\pi \circ h)(0, 0)$ have the maximal ranks c and k, respectively, there is even an invertible reparametrization ϕ with these properties.

In applications of this lemma the map h_* plays the role of a normal form. Under generic assumptions on h_* and h we may take for the map π the canonical projection onto the first k coordinates and hence the identity-map in the case of $k = c$. Then the lemma says that the first k components of any map h can be brought into normal form h_*.

Before presenting its proof, let us indicate how Lemma 7 allows us to find the normal forms for the Hamiltonian families of Theorems 5 and 6. The normal form for the non-semisimple case is obtained by applying the lemma to the map $h_*(I, \mu) = \mu \pm I$. The coefficient of I is taken to be positive if the partial derivative with respect to I of $\bar{\varepsilon}$ at $(I, \mu) = (0, 0)$ is positive. In this way we achieve that ϕ even preserves the half-plane $\{(I, \mu) \mid I \geq 0\}$.

In the semisimple case we have $h = (\bar{\varepsilon}, \bar{\delta}_1, \bar{\delta}_2)$, and we apply the lemma to the map $h_*(I, \mu) = (\mu_1 + I, \mu_2, \mu_3)$. In fact only the first component of h_* matters here. Generically we may assume that $(\partial\bar{\varepsilon}/\partial I)(0, 0) \neq 0$, so π may be taken equal to the projection onto the first coordinate. Applying Lemma 7 yields an *invertible* restricted reparametrization $(I, \mu) \mapsto \phi(I, \mu)$, such that

$$\bar{\varepsilon}(\phi(I, \mu)) = \mu_1 + I.$$

Taking $\delta_i = \bar{\delta}_i \circ \phi^{-1}$, for $i = 1, 2$, we see that

$$F^{\phi(I,\mu)} = \bar{F}^{I,\mu},$$

with $F^{I,\mu}(x_2, y_2) = x_2^2 y_2 + \frac{1}{3}y_2^3 + (\mu_1 + I)(x_2^2 - y_2^2) + \delta_1(I, \mu)x_2 + \phi_2(I, \mu)y_2$. The cases with the minus-signs are treated similarly. This finishes the proof of Theorems 5 and 6, leaving us with the task of proving Lemma 7.

Proof of Lemma 7. Since $D_\mu h_*(0, 0)$ has maximal rank, the Implicit Function Theorem guarantees that the system of c equations

$$h_*(0, \bar{\mu}) = h(0, \mu)$$

has a unique solution $\bar{\mu} = \phi_0(\mu) \in \mathbb{R}^c$ for μ near $0 \in \mathbb{R}^c$. Since $D_I(\pi \circ h_*)(0, 0)$ has also maximal rank we see that

$$\pi \circ h_*(I, \phi_0(\mu)) = \pi \circ h(0, \mu) \Leftrightarrow I = 0.$$

Again applying the Implicit Function Theorem we get a unique solution $\bar{I} = \phi_1(I, \mu) \in \mathbb{R}^k$ for the system of k equations

$$\pi \circ h_*(\bar{I}, \phi_0(\mu)) = \pi \circ h(I, \mu)$$

for (I, μ) near $(0, 0) \in \mathbb{R}^k \times \mathbb{R}^c$. In particular we now have that $\pi \circ h_*(\phi_1(0, \mu), \phi_0(\mu)) = \pi \circ h(0, \mu)$, so we conclude that $\phi_1(0, \mu) = 0$. In other words, the *restricted* reparametrization $\phi : \mathbb{R}^k \times \mathbb{R}^c \to \mathbb{R}^k \times \mathbb{R}^c$, defined by $\phi(I, \mu) := (\phi_1(I, \mu), \phi_0(\mu))$, satisfies $\pi \circ h = \pi \circ h_* \circ \phi$, and maps the set $\{0\} \times \mathbb{R}^c$ onto itself.

Finally consider the case in which also the derivatives $D_\mu h(0, 0)$ and $D_I(\pi \circ h)(0, 0)$ have the maximal ranks c and k, respectively. Since $h_*(0, \phi_0(\mu)) = h(0, \mu)$, it follows that ϕ_0 is locally invertible near $0 \in \mathbb{R}^c$. Similarly the map $I \mapsto \phi_1(I, 0)$ is locally invertible, since $\pi h_*(\phi_1(I, 0)) = \pi h(I, 0)$. Therefore the reparametrization ϕ is invertible. □

4.1.2. Miscellaneous remarks

This subsection is concluded, discussing various aspects of the Theorems 5 and 6.

A geometric picture, structural stability?

We start presenting a geometric picture of the situation at hand. This picture came about in a discussion with Duistermaat. Indeed, in the setting of Theorems 5 and 6 we have $k = 1$. Hence, the map h defines a family of half-curves

$$\{I \geq 0 \mapsto h(I, \mu) \in \mathbb{R}^p\}_\mu,$$

parametrized by $\mu \in \mathbb{R}^p$. Here $p = 1$ in the non-semisimple and $p = 3$ in the semisimple case. The range $\mathbb{R}^p$ exactly is the parameter-space of the corresponding backbone-system of Section 2. This family of half-curves, by construction is invariant under restricted reparametrization. The fact that $D_\mu h(0, 0)$ has maximal rank implies that we can simplify the starting points $I = 0$ of our curves: in Lemma 7 we e.g. can take $h_*(0, \mu) = \mu$. The fact that also $D_I h(0, 0)$ has maximal rank, implies that we can further simplify the I-dependence of exactly one of the components of h: we chose $\varepsilon(I, \mu) = \mu_1 \pm I$. We conclude that, in this way, no further simplification of the normal form is possible.

A question is whether techniques as used by Wassermann [51] might reduce the coefficients δ_j further to a polynomial form. In that case the normal form only has a finite number of moduli. However, the current discussion casts serious doubt on this possibility. Also see an earlier remark concerning the role of the set $\{I = 0\}$.

Notice, however that we are constructing our restricted morphisms in a specific way, first reducing to a polynomial form in (x_2, y_2) and then carrying out suitable reparametrizations. At this point it is not completely clear why further simplification would not be possible in a more general approach. As said before, in the Appendix (Section 7), diving a little into Singularity Theory, we shall take away this doubt.

An analogue in higher degree of freedom

As we shall see in Section 6.1 below, an analogue exists of the present situation in higher degree of freedom. There, at the central equilibrium point again we have a double eigenvalue 0, but now we have k pairs $\pm i\alpha_1, \pm i\alpha_2, \ldots, \pm i\alpha_k$, of normally elliptic eigenvalues, where strong resonances are excluded. By similar techniques as used here, we then find a planar reduction as above, now with k distinguished parameters $I = (I_1, I_2, \ldots, I_k)$. In this case, for $k \geq 3$, by the first part of Lemma 7, we obtain a normal form $h_*(I, \mu) := (\mu_1 + I_1, \mu_2 + I_2, \mu_3 + I_3)$, modulo restricted reparametrization. In fact, the corresponding unfoldings

$$F^{I,\mu}(x_2, y_2) = x_2^2 y_2 \pm \frac{1}{3} y_2^3 + (\mu_1 + I_1)(x_2^2 \mp y_2^2) + (\mu_2 + I_2)x_2 + (\mu_3 + I_3)y_2$$

are universal with respect to restricted morphisms.

Remarks.

i. In the Appendix (Section 7) we shall provide a general, but quite simple device for obtaining a universal unfolding with respect to restricted morphisms, given an 'ordinary' universal unfolding, i.e. with respect to unrestricted morphisms. Let us roughly describe how this goes. Indeed, let a family of maps $(x, I) \in (\mathbb{R}^n \times \mathbb{R}^k) \mapsto f(x, I) \in \mathbb{R}$ be given, where we consider $f(x, I)$ as an unfolding of $f_0(x) := f(x, 0)$. Now, if $f = f(x, I)$ is a universal unfolding of f_0 with respect to unrestricted morphisms respecting the set $\{I = 0\}$, then a universal unfolding of $f(x, I)$ in the restricted sense is given by $F: \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$, defined by

$$F(x, I, \mu) = f(x, I) + \sum_{j=1}^{k} \mu_j \frac{\partial f}{\partial I_j}(x, 0),$$

where, as before, I is the distinguished and μ the external parameter. Both the above result for $k \geq 3$ and Theorem 5 are direct consequences of this, also see Section 7;

ii. In the higher dimensional analogue the distinguished parameters $I_1, I_2, \ldots, I_k$ all are non-negative. Therefore, as in the Theorems 5 and 6, the restricted morphisms can be also required to 'respect the boundary' $\{I_1 = 0\} \cup \{I_2 = 0\} \cup \cdots \cup \{I_k = 0\}$. However, in that case there is no hope of finding simple universal unfoldings. An example like $h(I_1, I_2, \mu_1, \mu_2) = (I_1 + \mu_1, I_2 + I_1 + \mu_2)$, as it may occur for $k = 2$, easily convinces the reader of this fact. In our set-up we only require the intersection $\{I = 0\} = \{I_1 = 0\} \cap \{I_2 = 0\} \cap \cdots \cap \{I_k = 0\}$ to be preserved.

Algorithms

A drawback of many applications of Singularity Theory is its lack of constructiveness regarding the normalizing transformations. This is unlike the situation in the Normal Form Theory of Section 3. The lack of constructiveness is felt the strongest, when dealing with concrete examples or applications, e.g. see Section 1. In such cases, one would for instance like to compute the coefficients δ_j, $j = 1, 2$, of Theorem 6, which needs the keeping track of all normalizing transformations. Similarly, in the above case with k pairs of elliptic eigenvalues, one likes to know where the normalizing morphism takes the boundary set $\{I_1 = 0\} \cup \{I_2 = 0\} \cup \cdots \cup \{I_k = 0\}$.

Therefore, generally speaking, one wishes to have a good (algorithmic) knowledge of these normal form transformations. It will turn out that this knowledge exists, even to the level of formula manipulation. In [17] we shall come back to this.

4.2. Interpretation of the planar results to the integrable case

Now we interpret the results found in the previous section, for the dynamics in the 4-dimensional phase space. In the present symmetric case, we have the integral I, facilitating our considerations. In the perturbation analysis to follow, generically I no longer is an integral. In all Hamiltonian cases, however, the Hamiltonian itself is an integral: the 'energy'. In the integrable case this is F^μ and in the 'perturbed' case $H^\mu = F^\mu + P^\mu$. In order to be able to give a convenient perturbation analysis we therefore describe the integrable dynamics both regarding the level sets of I and those of F^μ.

4.2.1. Restricting to level sets of I

For each value of I we find ourselves in the corresponding level set of I. For $I \neq 0$ this level set is diffeomorphic to the 3-dimensional space $\mathbb{S}^1 \times \mathbb{R}^2$, coordinatized by (φ, x_2, y_2). Note that these level sets foliate $\mathbb{R}^4$, except for the (x_2, y_2)-plane which is the level set $I = 0$. For $I \neq 0$ the dynamics is given by

$$\dot{x}_2 = \frac{\partial F^\mu}{\partial y_2}, \qquad \dot{y}_2 = -\frac{\partial F^\mu}{\partial x_2},$$

$$\dot{\varphi} = -\frac{\partial F^\mu}{\partial I},$$

where we recall that the first two equations are the planar reduction and that $(\partial F^\mu/\partial I)|_0 = \alpha$, recalling that $\alpha \neq 0$. Observe that here we took the original meaning of F, so abandoning the form in Section 4.1, where the 'constant terms' were deleted.

It follows that the non-critical, i.e. non-zero, level sets of I admit an invariant foliation $L_{I,\mu}$, the leaves of which are the intersections with the level sets of the integrable Hamiltonian F^μ. The integral curves of the planar reductions from Section 4.1 are the intersections of these leaves with the section $\varphi = 0$. Note, moreover, that this foliation is invariant under all rotations $(\varphi, x_2, y_2) \mapsto (\varphi + \beta, x_2, y_2)$. From this we see that the geometry of $L_{I,\mu}$, as a 1-parameter family of 2-dimensional leaves (cylinders, tori) with 1-dimensional singularities (circles), by the Theorems 5 and 6, is completely determined by Section 2.2.

Next let us discuss the dynamics in the non-zero level sets of I. First observe that the corresponding restriction of the integrable vector field X_{F^μ} has *divergence zero*: it preserves the volume form $d\varphi \wedge dx_2 \wedge dy_2$. Then, regarding the dynamics in the various leaves, we summarize

Proposition 8. The 'integrable dynamics' in the leaves of $L_{I,\mu}$ is determined as follows by the planar reduction:

1. The singularities in the planar reductions give rise to 1-dimensional, singular leaves, being circles with periodic dynamics. Their normal linear behavior is given by the linear behavior of the reduced singularities;
2. The regular curves of the reduction yield 2-dimensional Lagrangian leaves, which are either cylinders or tori. In the cylinders the dynamics 'spirals', while in the tori, up to smooth equivalence, the motion is parallel. The normal linear part of the tori identically vanishes.

Proof. Most of the statements are obvious. The 'spiralling' on the invariant cylinders just means that there exists a Lyapunov function, the level sets of which are transverse to those of φ.

Concerning the dynamics in the tori we have to show that up to a smooth equivalence the restricted vector field is constant. In fact, multiplication of the vector field with a suitable C^∞ function yields that $\dot{\varphi} \equiv -\alpha$, while inside the (x_2, y_2)-plane we can parametrize the closed orbit by a 2π-periodic coordinate, proportional to time. If this coordinate is called ψ, the pair (φ, ψ) provides an equivalence as desired. □

Remark. Another way to express what is going on is saying that, restricted to a non-zero level of I, the (area preserving) Poincaré map of the section $\varphi = 0$ is the flow over time 1 of a planar Hamiltonian vector field, equivalent to the reduction of Section 4.1.

To fix thoughts, let us discuss what happens qualitatively in the non-semisimple case for $I \neq 0$. In fact, this is exactly the integrable Hamiltonian Saddle-Node bifurcation of closed orbits, compare Fig. 8. For values of (I, μ) with $I + \mu > 0$ we get an invariant foliation of cylinders with spiralling flow. On the line $I + \mu = 0$ one of the cylinders exhibits a closed integral curve of parabolic type. For $I + \mu < 0$ this closed orbit falls apart into two of these, one of elliptic type and the other hyperbolic. The stable and unstable manifold of the hyperbolic closed orbit coincide, enclosing a solid 2-torus. This solid torus is foliated by parallel invariant 2-tori, shrinking down to the elliptic closed orbit.

The interpretation for $I \neq 0$ in the semisimple case is similar: Just take any of the planar vector fields from the Figs. 4, 6 and add the rotational component $\dot{\varphi} = -\alpha + h.o.t$, yielding the integrable dynamics on $\mathbb{S}^1 \times \mathbb{R}^2$.

Finally, what happens in the zero-level $I = 0$ is easy to describe: here the dynamics exactly is the same as the reduced dynamics of Section 4.1.

4.2.2. *Restricting to energy levels*

The aim of this subsection is to show that, if we restrict to the level sets of F^{μ}, the same qualitative analysis as before applies. We begin defining the

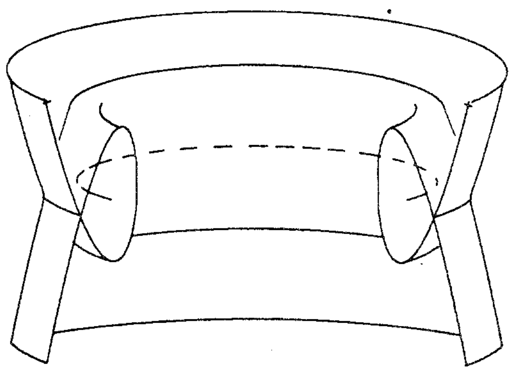


Figure 8
The integrable Hamiltonian Saddle-Node bifurcation of closed orbits for $\mu \pm I < 0$.

2-parameter foliation L_μ as follows: it is just $L_{I,\mu}$, where $I \neq 0$ also is allowed to vary.

Next consider the map M from the product of phase space and parameter space to itself, given by

$$M : ((I, \varphi, x_2, y_2), \mu) \mapsto ((F^\mu(I, x_2, y_2), \varphi, x_2, y_2), \mu).$$

Since $(\partial F^\mu / \partial I)|_0 = \alpha$, it follows that M is a local diffeomorphism. Let $M = ((E, \varphi, x_2, y_2), \mu)$ be its decomposition in component functions. Note that both I and E are considered as (polar-) coordinate functions.

Observing that $F = E \circ M$, we see that the leaves of L_μ are given by

$$I = c_I, \qquad E = c_E,$$

where c_I and c_E are constants. From this symmetric formulation it follows that, restricted to a fixed level set of E, the foliation L_μ is given by the levels of the function I. Let us denote this restricted foliation by $L_{E,\mu}$. An equation for this restriction then is given by

$$I = G^\mu(E, x_2, y_2),$$

where $G := I \circ M^{-1}$. Also let us define $G^{E,\mu}(x_2, y_2) := G^\mu(E, x_2, y_2)$.

We end this section by comparing the families $F^{I,\mu}$ and $G^{E,\mu}$ of planar functions for parameter-values (I, μ) and (E, μ) near $(0, 0)$. Here observe that both I and E are distinguished parameters. From the above we see that for x_2, y_2 and μ fixed, their levels correspond by the map $M|_{\varphi=0}$. However, this does not give us a restricted morphism in the formal sense of Sections 4.1 and 7.

Nevertheless such a morphism *can* be obtained along the following lines. First, it is not hard to see that the central singularities $G^{0,0}$ and $F^{0,0}$ are equivalent in the common sense. Subsequently, the unfolding $G^{E,0}$ can be treated as in Theorems 5 and 6, yielding similar normal forms for $G^{E,\mu}$. In the non-semisimple case this unfolding will be again universal, and hence—after identifying properly—equivalent to $F^{I,\mu}$.

Remark. At this point we come back to the remark following Proposition 8. Again, one can restrict to any of the levels of E, at hand, considering the Poincaré map of the section $\varphi = 0$, which also in this case is area preserving. We here recall, that the restriction of any Hamiltonian vector field to an energy level preserves an appropriate volume, cf. [1]. Moreover, in the present integrable case this map is the flow over time 1 of a planar Hamiltonian vector field, equivalent to normal forms as in Theorems 5 and 6. Following the convention of Broer and Takens [13], such maps, that correspond to flows, are called *integrable*.

5. Generic unfoldings in the near integrable case

In this section we consider the *original* system, which may be viewed as 'perturbed' from the integrable normal form truncation studied before. We have to say here that the fine-structure is very intricate, in particular for parameter values near the bifurcations sets. Many theoretical questions have not (yet) been solved in this respect, therefore, our treatment necessarily will be somewhat more sketchy.

So we consider the 'perturbed' family of Hamilton functions

$$H^{\mu}(x, y) = F^{\mu}(I, x_2, y_2) + P(x, y, \mu),$$

compare Corollary 3, where the perturbation function P is infinitely flat at $((x, y), \mu) = 0$. This implies that P is small in the C^{∞}-topology, where its size is controlled by the diameter of the neighborhood of $((x, y), \mu) = 0$ in $\mathbb{R}^4 \times \mathbb{R}^p$ under consideration. The question is, what can be said of this perturbed system, in view of the results of Theorems 5, 6.

One way to formulate the perturbation problem, is to compare in the respective energy levels, the Poincaré maps with respect to the section $\varphi = 0$. In both cases this leads to an area preserving map in the (x_2, y_2)-plane, where the unperturbed one is integrable as described above. The question then is which dynamical features are persistent under small perturbation and which are not.

Generically the function I is not an integral of H^{μ}, and H^{μ} is not even emooth in I at $I = 0$. In order to avoid difficulties, from $\mathbb{R}^4 \times \mathbb{R}^p$ we take away a wedge, given by

$$|I| \leq c|\mu|,$$

where $c > 0$ is an arbitrary small constant. Let us denote the complement of this wedge by C_c, from now on restricting to this.

5.1. Persistent features

We start with some dynamical phenomena that are persistent under *any* flat perturbation P. Here this means that they can be found for both (x_2, y_2) and μ sufficiently near 0.

5.1.1. Closed orbits

By the Implicit Function Theorem all hyperbolic and elliptic closed orbits do survive the perturbation. Also the type then is persistent, as well as the local behaviour of the stable and unstable manifolds.

5.1.2. KAM-tori

The integrable approximations in a number of cases exhibit families of invariant 2-tori with parallel dynamics, cf. Proposition 8. In the reduction to 1 degree of freedom such a family corresponds to a 'cylinder' of closed orbits. For the integrable Poincaré map, cf. the end of the previous section, these closed orbits are invariant circles. In all cases the cylinder on one side is limited by an elliptic equilibrium and on the other by a graph of saddle-connections.

For the persistence of the 2-tori we need to study the frequency-ratio of the tori. Compare Arnold [5], Moser [30, 31], or Pöschel [36]. For the integrable Poincaré map this ratio is equal to the rotation number. The Twist Condition requires that this rotation number varies with the position of the invariant circle.

KAM-theory, for details again see [5, 30, 31, 36], says that under the Twist Condition certain tori, with diophantine frequency-ratio persist. Moreover, their union has positive measure, even in each energy-level.

Near the elliptic fixed points the Poincaré map can be expanded in Birkhoff Normal Form, depending on the number of resonances in the eigenvalues, e.g. see [7, 30]. If the first nonlinear term does not vanish, the elliptic point is density point (in the sense of Lebesgue) of quasi-periodic orbits, compare [36].

The condition on this coefficient can be seen as a kind of local Twist Condition. These Twist Conditions have to be verified on the Normal Forms of Theorems 5 and 6, but this would be outside the scope of the present paper. This investigation involves study of the *period integrals*. In the non-semisimple case of the Hamiltonian Saddle Node this integral is known to be monotonous, see Chow and Sanders [18]. For other special cases e.g. see Broer [10, 12].

Remarks.

i. In Broer [11, 10, 12] 1-parameter families of volume preseving vector fields are studied. In dimension 3 generically two cases can be distinguished, which in a certain sense are 'contained' in the present umbilical unfoldings.
ii. In the real analytic case the frequency-ratio or rotation number also is an analytic function. Because of the limits of cylinders mentioned earlier, its image is some halfline. Then it follows that in this case the Twist Condition holds almost everywhere. For a similar argument see [42].

5.2. Non-persistent features

Next we come to some features that will change under a *generic* flat perturbation P. We have to point out, however, that P is very close to the zero-function in the C^∞-topology. Therefore, the phenomena under consideration can be destroyed by arbitrarily small perturbations in sufficiently small neighborhoods of $((x_2, y_2), \mu) = 0$. These phenomena are called *flat*, compare Broer and Vegter [15] or Broer and Takens [13].

Presently our considerations are based on Robinson [38], who gives Kupka Smale Theorems for conservative systems in the C^∞-context. For real analytic analogues we refer to Broer and Tangerman [14].

5.2.1. Coinciding separatrices

For the integrable Poincaré maps there are lots of cases where the stable and unstable manifolds of a saddle point coincide. According to [38] this is not a generic property. In fact, for generic area preserving maps these 'separatrices split'.

If this happens automatically transversal hetero- and homoclinic points occur, giving rise to various types of Horseshoes and the corresponding chaos. Compare, for example, Moser [30, 31].

5.2.2. Resonant tori

Let us consider the cylinders of invariant circles as they may occur for the integrable Poincaré map. If the Twist Condition holds, see above, there is a dense union of these circles with a rational rotation number. Each of these circles is a continuum of closed orbits, all with the same period. Again according to [38], this is not a generic property. In fact, for generic area preserving maps the closed orbits of bounded period are isolated.

Generically these closed orbits again are either elliptic or hyperbolic and also only transversal hetero- and homoclinic points do occur.

Another, related, matter is the Poincaré–Birkhoff Fixed Point Theorem, see [7, 30], implying that in between any two KAM-circles with rotation numbers $\delta_1 < \delta_2$, and any rational number $\delta_1 < \varrho < \delta_2$ there exist periodic points with rotation number ϱ.

Remarks.

i. According to Broer and Takens [13], due to the flatness, in the C^∞-situation the dynamics can be rather complicated. For instance, sufficiently near a diophantine circle, any number of closed orbits with a given rotation number ϱ may generically occur. It is not clear whether this also holds true in the real analytic case.
ii. In real analytic cases sometimes also more explicit, viz. exponential, estimates can be given, on the splitting of the separatrices, for instance compare Holmes, Scheurle and Marsden [26] or Fontich and Simó [20]. A question is how to apply these methods in the present setting.

6. Generalizations

We conclude this paper with some general remarks. First we consider the case of a double zero eigenvalue with more than one pair of normally elliptic eigenvalues and second the case with a general normal eigenvalue configuration. In both cases, under certain conditions, reduction to the planar case of Section 2 is possible. Finally we give some remarks on a similar situation concerning symplectic maps.

6.1. More normal ellipticity

A first question is what changes if in the central singularity, next to the double eigenvalue 0, one has pure imaginary eigenvalues $\pm i\alpha_1, \pm i\alpha_2, \ldots, \pm i\alpha_k$, for $k \geq 2$. Here *resonances* between the α_j come into play. To be precise, if for $\langle m, \alpha \rangle := \sum_{j=1}^{k} m_j \alpha_j$, for a given $N \in \mathbb{N}$ one has

$$1 \leq \sum_{j=1}^{k} |m_j| \leq N \Rightarrow \langle m, \alpha \rangle \neq 0,$$

then a Normal Form result holds completely similar to the conclusion of Corollary 3, for general reference also compare, for instance [7, 11, 10, 43, 44]. In fact one then finds a canonical transformation Φ, keeping the parameters fixed, such that

$$(H \circ \Phi)(x, y, \mu) = F(I_1, I_2, \ldots, I_k, x_{k+1}, y_{k+1}) + P(x, y, \mu),$$

where for the perturbation term P one has the finite flatness $P(x, y, \mu) = O(|(x, y, \mu)|^N)$. So, up to an N-flat perturbation, one finds k integrals $I_1, I_2, \ldots, I_k$, providing a k-torus symmetry. Factoring out this symmetry, for $N \geq 4$, gives a reduction as before, with the same planar backbone and a similar perturbation analysis, see Sections 2 and 5 and, in particular, Section 4. At the end of Section 4.1, for $k \geq 3$, polynomial normal forms are obtained, with coefficients that are linear in the I_j and in the unfolding parameters.

In particular all this holds in the case where $N = \infty$, where, by the Borel Theorem, the term P again becomes infinitely flat, compare Section 3.1. Notice that in that case the frequencies α_j have to be independent over the rationals, i.e. $\langle m, \alpha \rangle = 0 \Leftrightarrow m = 0$.

Remarks.

i. In the case where strong resonances are present, the analysis becomes more complicated. In that case a straightforward application of Singularity Theory as before, does not apply. As an example consider the 3 degrees of freedom case with eigenvalues 0 and ± 1, each having multiplicity 2. Here the difficulties of the present paper are combined with those of Van der Meer [46].

ii. The approach of this subsection formally is of importance for certain problems with infinitely many degrees of freedom where $k = \infty$. It is not yet clear to us however, how the asymptotics for $k \to \infty$ can be given a sensible meaning.

6.2. *Normal hyperbolicity*

A second question is how to deal with non-imaginary eigenvalues. Then the usual reduction to the *center manifold* applies, e.g. see Hirsch, Pugh and Shub [25]. This means that there exists a normally hyperbolic center manifold W^c, which is tangent to the eigenspace corresponding to the imaginary eigenvalues and invariant under the flow. 'Reduction' then means, restriction of the whole bifurcational analysis to this center manifold W^c: all the interesting dynamics takes place in here. E.g. compare Palis and Takens [34].

To be precise, let us assume that the phase space is $\mathbb{R}^{2n}$, with the natural symplectic form $\omega = \sum_{k=1}^{n} dx_k \wedge dy_k$. Let X_H be a Hamiltonian vector field on $\mathbb{R}^{2n}$, with the origin as a singularity. We also assume that there are $2m$ purely imaginary eigenvalues, counting multiplicities, so dim $W^c = 2m$.

Our first aim is to point out the quite familiar fact that W^c inherits a symplectic structure, by restricting ω to it. This restriction is the pull-back $i^*\omega$, where $i : W^c \to \mathbb{R}^{2n}$ denotes the Inclusion Map. For this use of notation, again see [1, 5]. We have

Lemma 9. The pull-back $i^*\omega$ defines a symplectic form on W^c, which is preserved by the restriction of X_H to W^c. Moreover, there exist local coordinates $x_1, \ldots, x_m, y_1, \ldots, y_m$ on W^c, such that

$$i^*\omega = \sum_{k=1}^{m} dx_k \wedge dy_k.$$

Proof. Both $X_H \mid W^c$ and $i^*\omega$ are restrictions to W^c, while both W^c and ω are invariant under the flow of X_H. The point then is to show that $i^*\omega$ is a symplectic form.

In order to do so, we only have to prove nondegeneracy, which can be checked in the origin. In fact, this directly follows from Williamson's Normal Form, again see [21, 27]: Just split off the non-imaginary eigenvalues in a separate 'Jordan'-block. The remaining $2m$-dimensional space then corresponds to the elliptic eigenvalues, and is the tangent space of W^c.

Finally, the existence of coordinates as claimed in the lemma, follows by inspecting the proof of Darboux' Theorem, compare [1, 5]. □

Reduction to the center manifold W^c means restriction of H and X_H to W^c. By the local coordinates of Lemma 9, we can pass to $\mathbb{R}^{2m}$ with the natural symplectic structure. One problem here is the differentiability of the center manifold. On one hand, for any $k \in \mathbb{N}$ there is a neighborhood of the origin, where W^c is at least of class C^k. On the other hand examples show that W^c does not have to be C^∞. Nevertheless, with some care, a bifurcational analysis as before can be given, e.g. compare Vegter [48], also see [17].

6.3. *Symplectic maps*

The analogue for symplectic maps is the fixpoint with a double 1 eigenvalue. Now there is a Normal Form Theorem [11, 10, 13, 30, 43], saying that the map formally is the flow of a vector field over time 1. To be precise, assume that $T^\mu: \mathbb{R}^2 \to \mathbb{R}^2$ is a C^∞ family of symplectic diffeomorphism, again with $\mu \in \mathbb{R}^p$. Also assume that $T^0(0) = 0$ and that $D_0 T^0$ has double eigenvalue 1.

Theorem 10. There exists a C^∞, parameter preserving, symplectic transformation Φ^μ and a C^∞ Hamiltonian vector field X^μ, such that

$$(\Phi^{-1} \circ T \circ \Phi)^\mu = X_1^\mu + P^\mu,$$

where X_1 denotes the flow over time 1 of X and where P is infinitely flat at 0, both in phase and parameter space.

As a consequence, the vector field X^0 has the origin as a singularity with a double zero eigenvalue, so it is subject to the Unfolding Theory of Section 2. In the non-semisimple case this leads to the Saddle Node bifurcation for symplectic maps, a perturbation analysis similar to Section 5 has to be carried out. For a different approach see Meyer [29]. In the semisimple case the Elliptic and Hyperbolic Umbilic Catastrophes again play a role. The remarks of Sections 6.1 and 6.2, *mutatis mutandis*, also apply in this case.

Remark. Theorem 10 is a special formulation of a more general result, see Takens [43]. Let us sketch how this generalization runs. We consider the fixed point 0 of T^0, where the derivative $D_0 T^0$ has only eigenvalues on the complex unit circle. Note, that this can always be achieved by restricting to a center manifold. Let S denote the semisimple part of this derivative, then a symplectic (or canonical) transformation Φ^μ and a Hamiltonian vector field X^μ exist, satisfying both

1. $S_* X^\mu = X^\mu$, i.e., X^μ is equivariant with respect to the group generated by S;
2. $(\Phi^{-1} \circ T \circ \Phi)^\mu = S \circ X_1^\mu + P^\mu$, where P is flat as before.

This result especially is of interest in the 'resonant' case, where S has eigenvalues that are roots of unity. For an application with $S = -\mathrm{Id}$, involving period-doubling, see Broer and Vegter [16]. Also notice, that Theorem 10 just covers the case where $S = \mathrm{Id}$.

The validity of this result is not at all restricted to dimension 2, but holds for arbitrary symplectic fixed points. In fact, as in above the vector field case, the approach generalizes to any setting where some appropriate structure has to be preserved, compare [11, 10, 44]. This preservation is suitably expressed in terms of Lie algebra's of vector fields and the corresponding Lie groups of diffeomorphisms. The structures we have in mind are given by a volume or a symplectic form, or by a symmetry group.

7. Appendix

We pick up the line of thought, left at the end of Section 4.1. To this purpose $\mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^c$ is endowed with coordinates $x = (x_1, \ldots, x_n)$, $I = (I_1, \ldots, I_k)$ and $\mu = (\mu_1, \ldots, \mu_c)$. Here I and μ are parameters, where I is distinguished. For precise definitions, see below. We shall consider families of functions $f: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$, depending on x and I, as well as unfoldings $F: \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^c \to \mathbb{R}$ of these, so with $F(x, I, 0) = f(x, I)$.

In particular, we are interested in the case, where the family $f(x, I)$, as an unfolding of $f_0(x) := f(x, 0)$, is (uni-)versal in the 'ordinary' sense, i.e. with respect to general, unrestricted morphisms. Indeed, the main result of this section is

Theorem 11. Let $f: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ be a family depending on a distinguished parameter $I \in \mathbb{R}^k$. Let $f_0: \mathbb{R}^n \to \mathbb{R}$, defined by $f_0(x) = f(x, 0)$, have codimension c. Then

1. f has a universal unfolding with respect to restricted morphisms if and only if f, considered as an unfolding of f_0, is versal with respect to non-restricted morphisms.
2. Any universal unfolding of f (if it exists) has c external parameters.
3. If $f = f(x, I)$ is a universal unfolding of f_0 with respect to unrestricted morphisms, then $k = c$ and $F: \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^c \to \mathbb{R}$, defined by

$$F(x, I, \mu) = f(x, I) + \sum_{j=1}^{c} \mu_j \frac{\partial f}{\partial I_j}(x, 0)$$

is a universal unfolding of f with respect to restricted morphisms respecting the set $\{I = 0\}$.

Below we shall give a proof of Theorem 11. Preceding this proof, we present a brief overview of some terminology and basic results form 'classical'

Singularity Theory. This will provide us with the general context, in which the present study becomes self-contained. As before, our main sources here are Bröcker and Lander [9], Gibson [22], Martinet [28], Poston and Stewart [37] and Thom [45].

Before undertaking this, however, let us examine some examples, so obtaining universal unfoldings for the families of Hamiltonian functions in this paper.

Examples

1. For $k \geq 3$, consider the family

   $$f(x, I_1, \ldots, I_{k-2}) = x^k + I_1 x + \cdots + I_{k-2} x^{k-2},$$

   for a special case of this, compare Theorem 5. This family, parametrized by $I = (I_1, \ldots, I_{k-2}) \in \mathbb{R}^{k-2}$, is a universal unfolding in the unrestricted sense of the function f_0, given by $f_0(x) = x^k$. Therefore Theorem 11 applies, so the family

   $$F(x, \mu_1, \ldots, \mu_{k-2}, I_1, \ldots, I_{k-2}) = x^k + (\mu_1 + I_1)x + \cdots (\mu_{k-2} + I_{k-2})x^{k-2},$$

   is a universal unfolding of f in the context of restricted morphisms.
2. Next we consider the family

   $$f(x, y, I_1, I_2, I_3) = x^2 y \pm \frac{1}{3} y^3 + I_1(x^2 \mp y^2) + I_2 x + I_3 y,$$

   cf. Theorem 6, which is a universal unfolding of the Elliptic viz. Hyperbolic Umbilic $f_0(x, y) = x^2 y \pm \frac{1}{3} y^3$. Using Theorem 11 we see that

   $$F(x, y, I_1, I_2, I_3, \mu_1, \mu_2, \mu_3) = x^2 y \pm \frac{1}{3} y^3 + (\mu_1 + I_1)(x^2 \mp y^2) + (\mu_2 + I_2)x + (\mu_3 + I_3)y$$

   is a universal unfolding of the family f, with respect to restricted morphisms. Compare the end of Section 4.1.

In particular it now rigourously follows that the umbilic families don't have versal unfoldings with respect to restricted morphisms, if the number of distinguished parameters is less than 3. Again compare the end of Section 4.1.

7.1. Elements from 'classical' Singularity Theory

Classical Singularity Theory deals with families of functions depending on a number of external parameters. Usually a specific value of the parameter corresponds to a function that has a degenerate singularity. One of the main issues is to determine *universal* unfoldings of such degenerate functions. These may be considered as a *model* for the set of all functions that can be obtained by perturbing the function with the degenerate singularity, in the sense that any function obtained by such small perturbations is equivalent to a function contained in the universal family.

To make these ideas more precise consider a function $f: \mathbb{R}^n \to \mathbb{R}$. An unfolding of f is a function $F: \mathbb{R}^n \times \mathbb{R}^c \to \mathbb{R}$ such that $F(x, 0) = f(x)$. Since we apply this theory to the study of Hamiltonian systems the value of functions at the origin of $\mathbb{R}^n$ is irrelevant. Therefore we assume throughout this section that $f(0) = 0$, and that any unfolding F satisfies $F(0, \mu) = 0$ for all $\mu \in \mathbb{R}^c$. Occasionally this will be expressed by the notation $F: (\mathbb{R}^n \times \mathbb{R}^c, 0) \to (\mathbb{R}, 0)$. This is no serious restriction, since one obtains similar results in case the value at the origin does matter. One usually needs one additional unfolding parameter to account for the variation of the 0-level of unfoldings. We refer to [17] for a more complete discussion.

Remark. Another way to vary 0-levels, is admitting transformations of the range $\mathbb{R}$, compare Section 2. The corresponding morphisms were called *left-right equivalences*. As has been said there, this context does not change if only translations in $\mathbb{R}$ are allowed. Universal unfoldings with respect to this wider class of morphisms again have one parameter less than in the present setting.

It should also be noted that our method yields *local* unfoldings. Therefore we shall assume that an unfolding $F: \mathbb{R}^n \times \mathbb{R}^c \to \mathbb{R}$ is only defined for (x, μ) near $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^c$.

We continue giving a formal definition of *morphism*, tailored for our set-up. To this purpose let $F: \mathbb{R}^n \times \mathbb{R}^c \to \mathbb{R}$ and $G: \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ be unfoldings of a fixed function f. Then a morphism from G to F is a pair of smooth functions (H, h), where

1. $H: \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n \times \mathbb{R}^c$, with $H(x, 0) = x$,
2. $h: \mathbb{R}^d \to \mathbb{R}^c$, with $h(0) = 0$,

such that

$$G(x, v) = F(H(x, v), h(v)).$$

The unfolding G is said to be *induced* from F by the morphism (H, h). The unfolding F is called a *versal unfolding* of the function f if any other unfolding of f is induced from F by a suitable morphism. A *universal* unfolding is a versal unfolding with a minimal number of parameters.

Let $\mathscr{E}_n$ denote the ring of functions $f: \mathbb{R}^n \to \mathbb{R}$, or, more precisely, the ring of *germs* at $0 \in \mathbb{R}^n$ of such functions. For $f \in \mathscr{E}_n$ let $J(f)$ denote the *Jacobian ideal*, viz. the set consisting of all combinations

$\alpha_1(\partial f/\partial x_1) + \cdots + \alpha_n(\partial f/\partial x_n)$, where $\alpha_1, \ldots, \alpha_n$ range over $\mathscr{E}_n$. The *ideal* $\mathscr{M}_n$ consists of functions vanishing at the origin. If f has a singularity (critical point) at $0 \in \mathbb{R}^n$ then $J(f) \subset \mathscr{M}_n$. In this case the *codimension* of f is the dimension of the real vector space $\mathscr{M}_n/J(f)$. Note that here we deviate from the usual definition of codimension, being the dimension of $\mathscr{E}_n/J(f)$. Our definition reflects the fact that we only consider functions whose value at $0 \in \mathbb{R}^n$ is equal to 0.

One of the main results from Singularity Theory is the following so-called 'universal unfolding theorem', e.g. compare [9, 22, 28, 37, 45].

Theorem 12. Let $f: (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a function with a singularity at $0 \in \mathbb{R}^n$. The c-parameter family $F: (\mathbb{R}^n \times \mathbb{R}^c, 0) \to (\mathbb{R}, 0)$, is a universal unfolding of f if and only if c is equal to the codimension of f and

$$\left\{ \frac{\partial F}{\partial \mu_1}\bigg|_{\mu=0} \bmod J(f), \ldots, \frac{\partial F}{\partial \mu_c}\bigg|_{\mu=0} \bmod J(f) \right\} \tag{1}$$

is a basis for the real vector space $\mathscr{M}_n/J(f)$. The unfolding F is versal if $\mathscr{M}_n/J(f)$ is generated by (1).

Here $\partial F/\partial \mu_j|_{\mu=0}$ is defined by $\partial F/\partial \mu_j|_{\mu=0}(x) = (\partial F/\partial \mu_j)(x, 0)$. To avoid clumsy notation we shall drop the 'mod $J(f)$', so $\partial F/\partial \mu_1|_{\mu=0}$ is also used to denote $\partial F/\partial \mu_1|_{\mu=0} \bmod J(f)$. It will be clear from the context which interpretation is meant.

According to Theorem 12 the family F is a versal unfolding if and only if

$$\mathscr{M}_n = J(f) + \mathbb{R}\left\{ \frac{\partial F}{\partial \mu_1}\bigg|_{\mu=0}, \ldots, \frac{\partial F}{\partial \mu_c}\bigg|_{\mu=0} \right\}. \tag{2}$$

It is easy to show that this condition is *necessary*. Indeed, in order to see that, let $\theta: \mathbb{R}^n \to \mathbb{R}$ be a function with $\theta(0) = 0$ and consider the 1-parameter unfolding $G: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ of f defined by

$$G(x, v) = f(x) + v\theta(x).$$

Since F is versal there is a morphism (H, h) from G to F, i.e. such that

$$G(x, v) = F(H(x, v), h(v)).$$

Taking partial derivatives with respect to v on both sides of the latter equality, after putting $v = 0$ and using the facts that $H(x, 0) = x$ and $h(0) = 0$, we obtain:

$$\begin{aligned} \theta(x) &= D_x F(H(x, 0), h(0)) \cdot \frac{\partial H}{\partial v}(x, 0) + D_\mu F(H(x, 0), h(0)) \cdot \frac{\partial h}{\partial v}(0) \\ &= \sum_{j=1}^{n} \alpha_j(x) \frac{\partial f}{\partial x_j}(x) + \sum_{j=1}^{c} \beta_j \frac{\partial F}{\partial \mu_j}(x, 0), \end{aligned} \tag{3}$$

where $\alpha_j(x)$ is the j-th component of $(\partial H/\partial v)(x, 0)$ and β_j is the j-th component of $(\partial b/\partial v)(0)$. This proves (2) and hence the necessity of this condition. The proof of its sufficiency—omitted here—is much harder, since this is based on the Malgrange–Mather Preparation Theorem, again see the above references, in particular [9, 28].

As a direct consequence of Theorem 12 we see that any two universal unfoldings of f have the same number of parameters. The following lemma shows that there is an *invertible* morphism between any two universal unfoldings of f.

Lemma 13. let $F_1, F_2 : \mathbb{R}^n \times \mathbb{R}^c \to \mathbb{R}$ be universal unfoldings of $f : \mathbb{R}^n \to \mathbb{R}$ and let $(x, \mu) \mapsto (H(x, \mu), h(\mu))$ be any morphism from F_2 to F_1. Then $h : \mathbb{R}^c \to \mathbb{R}^c$, near $0 \in \mathbb{R}^c$, is a local diffeomorphism.

Proof. Take partial derivatives with respect to μ_i, $1 \leq i \leq d$, on both sides of the equation $F_2(x, \mu) = F_1(H(x, \mu), h(\mu))$, and set $\mu = 0$. We then get

$$\left.\frac{\partial F_2}{\partial \mu_i}\right|_{\mu=0} = \sum_{j=1}^{n} \left.\frac{\partial H_j}{\partial \mu_i}\right|_{\mu=0} \frac{\partial f}{\partial x_j} + \sum_{j=1}^{c} \frac{\partial h_j}{\partial \mu_i}(0) \left.\frac{\partial F_1}{\partial \mu_i}\right|_{\mu=0}.$$

Since both $\{\partial F_1/\partial \mu_1|_{\mu=0}, \ldots, \partial F_1/\partial \mu_c|_{\mu=0}\}$ and $\{\partial F_2/\partial \mu_1|_{\mu=0}, \ldots, \partial F_2/\partial \mu_c|_{\mu=0}\}$ are bases of $\mathcal{M}_n/J(f)$, we see that the matrix $((\partial h_j/\partial \mu_i)(0))_{i,j=1,\ldots,d}$ is invertible. This proves that h is invertible near $0 \in \mathbb{R}^c$. □

If a versal unfolding coincides with a universal unfolding on a linear subspace of the space of parameters, then the morphism between them can be chosen equal to the identity on that subspace. More precisely:

Lemma 14. Let $F : \mathbb{R}^n \times \mathbb{R}^c \to \mathbb{R}$ be a universal unfolding of $f : \mathbb{R}^n \to \mathbb{R}$. Let $G : \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^l \to \mathbb{R}$ be an unfolding of f such that $G(x, \mu, 0) = F(x, \mu)$. Then there is a morphism $(x, \mu, v) \mapsto (H(x, \mu, v), h(\mu, v))$ from G to F such that

$$H(x, \mu, 0) = x, \; h(\mu, 0) = \mu.$$

Proof. Since F is universal there is a morphism (K, k) from G to F. Let $K_{\mu v}$ and k_v be defined by $K_{\mu,v}(x) = K(x, \mu, v)$ and $k_v(\mu) = k(\mu, v)$, respectively. Since $K_{0,0}(x) = x$, we see that $K_{\mu,v}$ is invertible. Furthermore $F(K_{\mu,0}(x), k_0(\mu)) = G(x, \mu, 0) = F(x, \mu)$, so k_0 is invertible according to Lemma 13. The map (H, h), defined by $H(x, \mu, v) = K_{\mu,0}^{-1}(K_{\mu,v}(x))$ and $h(\mu, v) = k_0^{-1}(k_v(\mu))$, is a morphism from G to F satisfying the conditions stated in the lemma. □

The following result is a parametrized version of a well-known property of submersions, the so-called 'lifting property'.

Lemma 15. If the morphism $(\Psi, \psi) : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n \times \mathbb{R}^c$ is a submersion, then for any morphism $(\Phi, \phi) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^c$ there is a morphism $(\Lambda, \lambda) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^l$ such that

$$(\Psi, \psi) \circ (\Lambda, \lambda) = (\Phi, \phi).$$

There is one more technical result to be needed in the sequel.

Lemma 16. (Wall, [50]) If $\phi_1, \ldots, \phi_k \in \mathscr{E}_n$ generate an ideal $\mathscr{I}$ of finite codimension and for $\alpha_1, \ldots, \alpha_k \in \mathscr{E}_n$ we have $\sum_{j=1}^{k} \alpha_j \phi_k = 0$ index $j = 0$, then $\alpha_j \in \mathscr{I}$.

7.2. Unfoldings with distinguished parameters

Let us now consider a family of functions $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$, depending on k distinguished parameters $I = (I_1, \ldots, I_k)$. An unfolding of f then is a function $F : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^c \to \mathbb{R}$ such that $F(x, I, 0) = f(x, I)$ for (x, I) near $0 \in \mathbb{R}^n \times \mathbb{R}^k$. The additional parameters $\mu \in \mathbb{R}^c$ will be called *external.*

As we have seen in Section 4.1, we only allow transformations in which unfolding parameters don't depend on distinguished parameters, although distinguished parameters are allowed to depend on unfolding parameters. More formally let $F : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^c \to \mathbb{R}$ and $G : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^d \to \mathbb{R}$ be unfoldings of f. Then a *restricted morphism* from G to F is a triple of smooth functions (H, K, h), where

1. $H : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^d \to \mathbb{R}^n$, with $H(x, I, 0) = x$,
2. $K : \mathbb{R}^k \times \mathbb{R}^d \to \mathbb{R}^k$, with $K(I, 0) = I$ and $K(0, \mu) = 0$,
3. $h : \mathbb{R}^d \to \mathbb{R}^c$, with $h(0) = 0$.

such that the following equality holds:

$$G(x, I, \nu) = F(H(x, I, \nu), K(I, \nu), h(\nu)).$$

The condition $K(0, \mu) = 0$ reflects the fact that I usually is a special parameter that might only have a physical interpretation if its value is non-negative: in our case the I_js are non-negative action variables. Therefore our context is more restricted than e.g. Wassermann's, see [52]. Therefore we at least want to preserve the set $I = 0$ under restricted morphisms. The unfolding G is said to be *induced* from F by the restricted morphism (H, K, h).

Remark. Here and elsewhere, the condition that the (restricted) morphisms are the identity on the singular object, and therefore near-identity maps, is very common. Again compare the above references. This seems a further restriction, but it does not affect the codimension of the singularity.

In fact, from the proof of the necessity of the condition in Theorem 12, we see that this is not the case, since there is not any restriction on functions like the α_j.

The unfolding F is called a *versal unfolding with respect to restricted morphisms* of the function f, if any other unfolding of f is induced from F by a suitable restricted morphism. A *universal* unfolding in this context is a versal unfolding with a minimal number of external parameters. Compare the definitions in the previous subsection.

At this moment all terms in the formulation of Theorem 11 have been properly defined. The remainder of this section is devoted to its proof.

7.3. Proof of Theorem 11

We begin this subsection with some tools. These results show—among other things—that the number of external parameters in a (uni)versal unfolding of a family $f(x, I)$ with respect to restricted morphisms is not smaller than the codimension of the function $f_0 = f|_{I=0}$. Therefore any versal unfolding of f (with respect to restricted morphisms) with exactly this number of external parameters is *universal.*

Lemma 17. Let $F: \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^d \to \mathbb{R}$ be a versal unfolding with respect to restricted morphisms of the family $f: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$. Then the family $F_0: \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$, defined by $F_0(x, \mu) = F(x, 0, \mu)$, is a versal unfolding unfolding with respect to unrestricted morphisms of the function $f_0: \mathbb{R}^n \to \mathbb{R}$ given by $f_0(x) = f(x, 0)$.

Proof. Let $G: \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ be an unfolding of f_0. We have to show that there is an unrestricted morphism from G to F_0.

To this end introduce the unfolding $\bar{G}: \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}$, of the family f, defined by $\bar{G}(x, I, v) = G(x, v) + f(x, I) - f_0(x)$. We consider $I \in \mathbb{R}^k$ as a distinguished parameter of $\bar{G}$. Since F is universal there is a restricted morphism $(x, I, v) \mapsto (H(x, I, v), K(I, v), h(v))$ such that $\bar{G}(x, I, v) = F(H(x, I, v), K(I, v), h(v))$. Taking $I = 0$ and using the fact that $K(0, v) = 0$, we get: $G(x, v) = F(H(x, 0, v), 0, h(v))$, so $(x, v) \mapsto (H(x, 0, v), 0, h(v))$ is a morphism from G to F_0. □

Theorem 12 has a counterpart in the present context of unfoldings of families with distinguished parameters. However, we don't need such a strong result in our approach: a *necessary* condition for versality of foldings will do. As before, the Malgrange–Mather Preparation Theorem is needed for the sufficiency. The relevant condition is expressed in the following result.

Lemma 18. If $F: \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^d \to \mathbb{R}$ is a versal unfolding with respect to restricted morphisms of the family $f: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$, then for every function $\theta: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ such that $\theta(0, I) = 0$, the equation

$$\theta(x, I) = \sum_{j=1}^{n} \alpha_j(x, I) \frac{\partial f}{\partial x_j}(x, I) + \sum_{j=1}^{d} \beta_j \frac{\partial F}{\partial \mu_j}(x, I, 0) + \sum_{j=1}^{k} \gamma_j(I) \frac{\partial f}{\partial I_j}(x, I) \tag{4}$$

has a solution $\alpha_j: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$, for $1 \le j \le n$, $\beta_j \in \mathbb{R}$, for $1 \le j \le d$ and $\gamma_j: \mathbb{R}^k \to \mathbb{R}$, with $\gamma_j(0) = 0$.

The proof is quite similar to the partial proof of Theorem 12, see above, and therefore it is omitted here. As a technical issue we mention that the condition $\gamma_j(0) = 0$ corresponds to the fact that the component K of a restricted morphism (H, K, h) vanishes if the distinguished parameter is equal to 0.

We now have all the tools needed for the proof of Theorem 11 at our disposal. Note that Part 2 is an immediate consequence of Lemma 17. We shall first prove Part 3, since this will be used for Part 1.

7.3.1. Proof of Theorem 11, Part 3

Let f be a universal unfolding of f_0 with respect to unrestricted morphisms. In particular we then have $k = c$, where c is the codimension of f_0. We have to show that F is a universal unfolding of f with respect to restricted morphisms. Our approach bears some resemblance with the proofs of the Theorems 5 and 6, cf. Section 4.1.

Indeed, let $G: \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^d \to \mathbb{R}$ be an arbitrary unfolding of the family f, with distinguished parameter $I \in \mathbb{R}^c$. Since f is a universal unfolding of f_0 there is a unrestricted morphism $(x, I, \nu) \in \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^d \mapsto (H(x, I, \nu), h(I, \nu)) \in \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^c$ such that $G(x, I, \nu) = f(H(x, I, \nu), h(I, \nu))$.

Similarly there is an unrestricted morphism $(x, I, \mu) \in \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^c \mapsto (M(x, I, \mu), N(I, \mu)) \in \mathbb{R}^n \times \mathbb{R}^c$ such that $F(x, I, \mu) = f(M(x, I, \mu), N(I, \mu))$. Since $F(x, I, 0) = f(x, I)$ we may assume that $M(x, I, 0) = x$ and $N(I, 0) = I$, see Lemma 14. In particular $D_1 N(0, 0)$ has maximal rank.

In order to see that also $D_\mu N(0, 0)$ has maximal rank, we argue as follows. Since f is a universal unfolding of f_0, the necessity part of Theorem 12 tells us that

$$\mathscr{M}_n = J(f_0) + \mathbb{R}\left\{ \left.\frac{\partial f}{\partial I_1}\right|_{I=0}, \ldots, \left.\frac{\partial f}{\partial I_c}\right|_{I=0} \right\}.$$

Since $(\partial f / \partial I_j)(x, 0) = (\partial F / \partial \mu_j)(x, 0, 0)$ for $1 \le j \le c$, we invoke the sufficiency part of Theorem 12 in order to conclude that $(x, \mu) \mapsto F(x, 0, \mu)$ is a universal unfolding of f_0. Since $(x, \mu) \mapsto (M(x, 0, \mu), N(0, \mu))$ is a morphism from $F|_{I=0}$, the map $\mu \mapsto N(0, \mu)$ is locally invertible near $\mu = 0$, cf. Lemma 13. Therefore $D_\mu N(0, 0)$ indeed has maximal rank.

Next we apply Lemma 7 to obtain a *restricted* reparametrization $\phi : \mathbb{R}^c \times \mathbb{R}^c \to \mathbb{R}^c \times \mathbb{R}^c$, such that $h = N \circ \phi$.

Since $M(x, 0, 0) = x$, the map $x \mapsto M(x, I, \mu)$ is locally invertible near $x = 0 \in \mathbb{R}^n$ and for (I, μ) near $(0, 0) \in \mathbb{R}^c \times \mathbb{R}^c$. So there is a map $(x, I, \mu) \mapsto M^{\text{inv}}(x, I, \mu)$ such that $M(M^{\text{inv}}(x, I, \mu), I, \mu) = x$. Therefore $f(x, N(I, \mu)) = F(M^{\text{inv}}(x, I, \mu), I, \mu)$.

Combining these results we get

$$\begin{aligned} G(x, I, v) &= f(H(x, I, v), N(\phi(I, v))) \\ &= F(M^{\text{inv}}(H(x, I, v), \phi(I, v)), \phi(I, v)), \end{aligned}$$

and we have obtained a *restricted* morphism from G to F, which ends the proof that F is universal in the restricted sense.

7.3.2. *Proof of Theorem 11, Part 1 (sufficiency)*

Now, assuming that f is a versal unfolding of f_0, we have to show that f has a universal unfolding with respect to restricted morphisms.

According to Theorem 12, the real vector space $\mathcal{M}/J(f_0)$ is generated by $\partial f/\partial I_1|_{I=0}, \ldots, \partial f/\partial I_k|_{I=0}$. Without loss of generality we may assume that $\{\partial f/\partial I_1|_{I=0}, \ldots, \partial f/\partial I_c|_{I=0}\}$ is a basis for this vector space. Therefore the c-parameter family $\bar{f}: \mathbb{R}^n \times \mathbb{R}^c \to \mathbb{R}$, defined by $\bar{f}(x, \bar{I}) = f(x, \bar{I}, 0)$, is a *universal* unfolding of f_0. Here $(\bar{I}, 0)$ stands for $(I_1, \ldots, I_c, 0, \ldots, 0) \in \mathbb{R}^c \times \mathbb{R}^{k-c}$. In view of the first part of the proof there is a universal unfolding $\bar{F}: \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^c \to \mathbb{R}$ of $\bar{f}$ with respect to restricted morphisms. Using $\bar{F}$ we shall construct a universal unfolding for f.

According to the lifting property Lemma 15, there exists a morphism $(\Psi, \psi) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^c$ such that

1. $f(x, I) = \bar{f}(\Psi(x, I), \psi(x, I))$;
2. $\Psi(x, \bar{I}, 0) = x$ and $\psi(\bar{I}, 0) = \bar{I}$.

In particular, note that (Ψ, ψ) is a submersion. We now show that the family $F: \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^c \to \mathbb{R}$, defined by $F(x, I, \mu) := (\bar{F}(\Psi(x, I), \psi(I), \mu)$, is a versal unfolding of f with respect to restricted morphisms. This immediately implies that F is universal, since the number c of external parameters is equal to the codimension of f_0, and therefore minimal, see Lemma 17. Occasionally it will be convenient to omit function arguments. We then write e.g. $F = \bar{F} \circ (\Psi, \psi, 1_c)$.

Let us first check that F is indeed an unfolding of f, indeed, $F(x, I, 0) = \bar{F}(\Psi(x, I), \psi(I), 0) = \bar{f}(\Psi(x, I), \psi(I)) = f(x, I)$.

Next consider an arbitrary unfolding $G : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}$ of f. All we have to do, is finding a restricted morphism from G to F. Indeed, since $(\Psi, \psi, 1_p) : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^p$ is a submersion, there exists a family $\bar{G} : \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^p \to \mathbb{R}$ such that $G(x, I, v) = \bar{G}(\Psi(x, I), \psi(I), v)$. It is again easy to check that $\bar{G}$ is an unfolding of $\bar{f}$. In fact, $\bar{G}(x, \bar{I}, 0) = \bar{G}(\Psi(x, \bar{I}, 0), \psi(\bar{I}, 0), 0) = G(x, (\bar{I}, 0), 0) = f(x, \bar{I}, 0) = \bar{f}(x, \bar{I})$.

Therefore there exists a restricted morphism $(\bar{H}, \bar{K}, \bar{h}) : \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^p \to \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^c$ from $\bar{G}$ to $\bar{F}$.

Using the fact that $(\Psi, \psi, 1_p)$ is a submersion, we may apply a parametrized version of Lemma 15 to lift morphisms via $(\Psi, \psi, 1_p)$. Lifting the restricted morphism $(\bar{H}, \bar{K}, \bar{h}) \circ (\Psi, \psi, 1_p)$ we obtain a restricted morphism $(H, K, h) : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^p$ such that $(\bar{H}, \bar{K}, \bar{h}) \circ (\Psi, \psi, 1_p) = (\Psi, \psi, 1_p) \circ (H, K, h)$. Since now

$$\begin{aligned} G &= \bar{G} \circ (\Psi, \psi, 1_p) \\ &= \bar{F} \circ (\bar{H}, \bar{K}, \bar{h}) \circ (\Psi, \psi, 1_p) \\ &= \bar{F} \circ (\Psi, \psi, 1_p) \circ (H, K, h) \\ &= F \circ (H, K, h), \end{aligned}$$

we have obtained a restricted morphism from G to F, so F is versal.

7.3.3. Proof of Theorem 11, Part 1 (necessity)

Let $F : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^d \to \mathbb{R}$ be a versal unfolding of f with respect to restricted morphisms. We will show that f is a versal unfolding of f_0. In view of Theorem 12, to this end we have to prove that

$$\mathcal{M}_n = J(f_0) + \mathbb{R}\left\{ \left.\frac{\partial f}{\partial I_1}\right|_{I=0}, \ldots, \left.\frac{\partial f}{\partial I_k}\right|_{I=0} \right\}. \tag{5}$$

So let $\eta \in \mathcal{M}_n$, i.e. η is a function $\mathbb{R}^n \to \mathbb{R}$ defined near $0 \in \mathbb{R}^n$.

Consider the function $\theta : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ defined by $\theta(x, I) = I_1 \eta(x)$. According to Lemma 18 there are $\alpha_j : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$, $1 \le j \le n$, $\beta_j \in \mathbb{R}$, $1 \le j \le d$ and $\gamma_j : \mathbb{R}^k \to \mathbb{R}$, $1 \le j \le k$ with $\gamma_j(0) = 0$, such that

$$\theta(x, I) = \sum_{j=1}^n \alpha_j(x, I) \frac{\partial f}{\partial x_j}(x, I) + \sum_{j=1}^d \beta_j \frac{\partial F}{\partial \mu_j}(x, I, 0) + \sum_{j=1}^k \gamma_j(I) \frac{\partial f}{\partial I_j}(x, I). \tag{6}$$

Putting $I = 0$ we get

$$0 = \sum_{j=1}^n \alpha_j(x, 0) \frac{\partial f_0}{\partial x_j}(x) + \sum_{j=1}^d \beta_j \frac{\partial F}{\partial \mu_j}(x, 0, 0). \tag{7}$$

Since $(x, \mu) \mapsto F(x, 0, \mu)$ is a universal unfolding of f_0, cf. Lemma 17, the set $\{\partial F/\partial \mu_1|_{(I,\mu)=(0,0)}, \ldots, \partial F/\partial \mu_d|_{(I,\mu)=(0,0)}\}$ is a basis for the real vector space $\mathcal{M}_n/J(f_0)$, cf. Theorem 12. Hence $d = c$, and $\beta_1 = \cdots = \beta_d = 0$. Therefore (7) reduces to $\sum_{j=1}^n \alpha_j(x, 0)(\partial f_0/\partial x_j)(x) = 0$. Now Lemma 16 allows us to conclude that $\alpha_j|_{I=0} \in J(f_0)$, for $1 \le j \le n$. Taking partial derivatives with respect to I_1 at $I = 0$ on both sides of equation (6) we get:

$$\eta(x) = \sum_{j=1}^{n} \alpha_j(x, 0) \frac{\partial^2 f}{\partial x_j \partial I_1}(x, 0) + \sum_{j=1}^{n} \frac{\partial \alpha_j}{\partial I_1}(x, 0) \frac{\partial f_0}{\partial x_j}(x) + \sum_{j=1}^{k} \frac{\partial \gamma_j}{\partial I_1}(0) \frac{\partial f}{\partial I_j}(x, 0).$$

Since $\alpha_j|_{I=0} \in J(f_0)$, for $1 \le j \le n$, it follows that $\eta \in J(f_0) + \mathbb{R}\{\partial f/\partial I_1|_{I=0}, \ldots, \partial f/\partial I_k|_{I=0}\}$. In other words, we have proved (5). □

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Abstract

A universal local bifurcation analysis is presented of an autonomous Hamiltonian system around a certain equilibrium point. This central equilibrium has a double zero eigenvalue, the other eigenvalues being in general position. Main emphasis is given to the 2 degrees of freedom case where these other eigenvalues are purely imaginary. By normal form techniques and Singularity Theory unfoldings are obtained, having 'integrable' approximations related to the Elliptic and Hyperbolic Umbilic Catastrophes.

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