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Discontinuous stabilization of nonlinear systems: Quantized and switching controls

Francesca Ceragioli and Claudio De Persis

Abstract— In this paper we consider the classical problem of stabilizing nonlinear systems in the case the control laws take values in a discrete set. First, we present a robust control approach to the problem. Then, we focus on the class of dissipative systems and rephrase classical results available for this class taking into account the constraint on the control values. In this setting, feedback laws are necessarily discontinuous and solutions of the implemented system must be considered in some generalized sense. The relations with the problems of quantized and switching control are discussed.

I. INTRODUCTION

Recently, the literature about switched, quantized and hybrid systems ([21], [11], [20], [29], [28]) has given a new perspective to the classical problem of stabilization. In fact, on one hand, since systems considered are more general, there is a wider choice of control strategies (see, e.g., [25], [26], [30]). On the other hand, the new models often take into account some constraints which are important for applications. In this paper the basic assumption is that control laws take values only in a discrete set U. Since a vector field is naturally associated to each admissible control value, the system can be seen as a family of vector fields with a rule which governs the switching among them. We consider switching rules which depend only on the state variable so that they can also be interpreted as discontinuous feedback laws. Controlling with a discrete set of input values has been deeply explored in the literature on quantized control. As in [11] and [15] for linear systems, our design of the control values follows a logarithmic law, so that the resulting control law is simpler to implement than in other approaches ([20], [22], [10]) and it does not introduce an exceedingly large number of quantization levels (cf. [9] for a different approach to stabilization of nonlinear systems using a minimal number of "quantization levels"). On the other hand, differently from Ishii-Francis in [15] and Cepeda-Astolfi in [6], we do not couple our switched controller with a dwell-time logic, the latter being an approach which turns out useful to avoid chattering-like phenomena.

One of the aims of the paper is in the spirit of the situation in which U needs to be chosen appropriately. We study conditions which guarantee that, given a continuous stabilizing feedback law, the celebrated logarithmic quantization does not cancel the stabilizing effect. A first proposition

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can be viewed as a discontinuous version of the results about stability under vanishing perturbations (e.g. [19]). A second proposition is a nonlinear version of a result in [13] (see also [11], [15]). Previous results for nonlinear systems have appeared in [23]. Our contribution differs from the latter in two ways. First, we put a special emphasis on how the solution of the closed loop system should be intended. Second, a connection between the coarseness of the quantizer and a finite L_2-gain problem is obtained in the robust control setting pointed out by [13]. Further, it is shown how, trading off global asymptotic stability against semi-global practical stability, it is possible to overcome the limitation on the coarseness of the quantizer by appropriate redesign of the control law. A logarithmic quantizer requires an infinite number of quantization levels to guarantee asymptotic stability. Nevertheless, it is possible to cope with a finite number of quantization levels and obtain semi-global practical stability without affecting the coarseness of the quantizer. This is discussed as well.

As clearly pointed out in [13], in problems of stabilization under logarithmic quantization, the uncertainty introduced by the quantizer is a sector bounded uncertainty. An effective way to deal with stability of nonlinear systems in the presence of sector bounded uncertainties is to rely on the theory of dissipative systems. This simple observation motivates the second aim of the paper, namely to show how some classical results on feedback stabilization of nonlinear dissipative systems can be restated in this setting. The idea of extending stabilization results which use dissipativity to "non-classical" systems is not new, but there is still not a wide literature on the subject. To the best of the authors' knowledge, the most complete paper on the subject is [14]. In this paper hybrid systems which generate left continuous dynamical systems are considered: our approach is quite different, since we do not assume uniqueness of solutions of the implemented systems. Relations with the paper [24] are discussed as well. Despite of what the robust control approach allows to do, characterizing the coarsest quantizer in the dissipativity framework is harder. Nevertheless, for a special class of dissipative systems, namely the passive ones, we give conditions under which asymptotic stabilizability can be achieved with a finite and "minimal" number of quantization levels.

We remark that, once the classical control laws have been "quantized", i.e. approximated by new control laws taking values in U, the new feedbacks are necessarily discontinuous, and solutions of the implemented systems must be intended in some generalized sense. A number of different notions of

generalized solutions have been proposed. Here we focus on Krasowskii solutions. However, motivated by [2], [1], [27], [4], [7], it is possible to restate the results in terms of Carathéodory solutions, which may be more appropriate for the problem at hand. We refer the reader to [8].

The paper is organized as follows. In Section II, the robust control approach to the quantized stabilization of nonlinear systems is pursued. Section III deals with the dissipativity approach. A special case (passive systems) is dealt with in Section IV. Conclusions are drawn in Section V.

For lack of space all the proofs have been omitted. They can be found in an unabridged version of the paper [8].

Preliminaries We denote by $|\cdot|$ the norm in $\mathbb{R}^n, n \geq 1$ and, if $x_0 \in \mathbb{R}^n$, we use the notation $B_\epsilon(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < \epsilon\}$. Given a set $S \subset \mathbb{R}^n$, the symbols $\overline{\text{co}}\, S, \overset{\circ}{S}, \overline{S}$ denote, respectively, the convex closure of S, i.e. the smallest closed set containing the convex hull of S, the interior of the set S, and its closure. We refer the reader to e.g. [2], [8] for the definition of Krasowskii solutions and for the associated notions of stability.

The input-affine systems we will consider in the paper is

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x)\,, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in U$ ($U \subset \mathbb{R}$) is the input variable, $y \in \mathbb{R}$ is the output variable. In the following we make the following assumptions:

- $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are vector fields of class C^1, $f(0) = 0$;
- $h : \mathbb{R}^n \to \mathbb{R}$ is of class C^1, with $h(0) = 0$;
- $U \subset \mathbb{R}$, $0 \in U$, U symmetric, i.e. if $u \in U$ then also $-u \in U$.

The set $\mathcal{U}$ of admissible inputs is formed by all measurable functions $u : [0, +\infty) \to U$. For each initial state x_0 and each admissible input $u \in \mathcal{U}$, system (1) has a unique local Krasowskii solution.

II. ROBUST CONTROL APPROACH

A. Logarithmic quantizer and global stabilizability results

An important part of the literature about quantized control focuses on techniques which allow to approximate stabilizing feedback by means of control laws which take values in a properly chosen discrete set. In the context of linear systems, the logarithmic quantizer ([11]) had a great success.
Fix $u_0 > 0$, $0 < \rho < 1$, and let $u_i = \rho^i u_0$, $U = \{0, \pm u_i, i \in \mathbb{Z}\}$. Let $\delta = (1 - \rho)/(1 + \rho)$ and ([11], [13])

$$\Psi(y) = \begin{cases} u_i & \dfrac{u_i}{1+\delta} < y \leq \dfrac{u_i}{1-\delta} \\ 0 & y = 0 \\ -\Psi(-y) & y < 0\,. \end{cases} \tag{2}$$

Following [13], one can consider both state and output feedback and, in the latter case, one can further distinguish between the cases of quantized input or quantized measurement. When the full state is measured, the controller

is $u = \Psi(k(x))$. On the other hand, in the presence of input quantization, the dynamic output feedback takes the form

$$\begin{aligned} \dot{\xi} &= f_c(\xi, h(x)) + g_c(\xi, h(x))\Psi(k_c(\xi, h(x))) \\ u &= \Psi(k_c(\xi, h(x))) \end{aligned} \tag{3}$$

whereas, in the presence of output quantization,

$$\begin{aligned} \dot{\xi} &= f_c(\xi) + g_c(\xi)\Psi(h(x)) \\ u &= k_c(\xi) + \ell_c(\xi)\Psi(h(x))\,, \end{aligned} \tag{4}$$

with $\xi \in \mathbb{R}^{n_c}$. The closed-loop system turns out to be

$$\dot{\mathcal{X}} = \mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})\Psi(\mathcal{K}(\mathcal{X}))\,, \quad \mathcal{X} \in \mathbb{R}^N \tag{5}$$

where the actual expressions of $\mathcal{X}$, N, $\mathcal{F}$, $\mathcal{G}$, $\mathcal{K}$ depend on the feedback employed and are understood from the context. The "nominal", i.e. with no quantization, system writes as

$$\dot{\mathcal{X}} = \mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})\mathcal{K}(\mathcal{X})\,, \quad \mathcal{X} \in \mathbb{R}^N\,. \tag{6}$$

We now give two propositions which state sufficient conditions for a stabilizing feedback law to be "quantizable" by means of the logarithmic quantizer (2). As a first step we consider Krasowskii solutions of the system in which the quantized feedback law is implemented. More precisely, Krasowskii solutions of (5) are absolutely continuous functions which satisfy the following differential inclusion:

$$\dot{\mathcal{X}} \in \mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})K(\Psi(\mathcal{K}(\mathcal{X}))) \subseteq$$
$$\begin{cases} \mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})\{1 + \lambda\delta, \lambda \in [-1,1]\}\mathcal{K}(\mathcal{X}) & \mathcal{X} \neq 0 \\ 0 & \mathcal{X} = 0. \end{cases}$$

In fact, let $\mathcal{X}$ be such that $\mathcal{K}(\mathcal{X}) > 0$ (analogous considerations can be repeated for $\mathcal{K}(\mathcal{X}) < 0$). Since $(1 - \delta)\mathcal{K}(\mathcal{X}) \leq \Psi(\mathcal{K}(\mathcal{X})) \leq (1 + \delta)\mathcal{K}(\mathcal{X})$ then for all $v \in K(\Psi(\mathcal{K}(\mathcal{X})))$, $(1 - \delta)\mathcal{K}(\mathcal{X}) \leq v \leq (1 + \delta)\mathcal{K}(\mathcal{X})$, i.e. $v \in \{\mathcal{K}(\mathcal{X})(1 + \lambda\delta), \lambda \in [-1,1]\}$. Hence, given $\mathcal{V} : \mathbb{R}^N \to \mathbb{R}$, for any $\mathcal{X} \in \mathbb{R}^N$ and any $v \in K(\Psi(\mathcal{K}(\mathcal{X})))$, we will need to study $\nabla\mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})v)$. We can rewrite $\mathcal{K}(\mathcal{X}) - v = \lambda\delta\mathcal{K}(\mathcal{X})$, for some $\lambda \in [-1, 1]$, so that

$$\begin{aligned} \nabla\mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})v) \leq\ &\nabla\mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})\cdot \\ &\mathcal{K}(\mathcal{X}) - \mathcal{G}(\mathcal{X})\lambda\delta\mathcal{K}(\mathcal{X}))\,. \end{aligned}$$

Proposition 1: Assume that there exist $\mathcal{V}, \mathcal{W} : \mathbb{R}^N \to \mathbb{R}$ continuous positive definite, $\mathcal{V}$ is of class C^1 and radially unbounded, $\mathcal{K} : \mathbb{R}^n \to \mathbb{R}$ continuous such that for all $\mathcal{X} \in \mathbb{R}^N$,

$$\nabla\mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})\mathcal{K}(\mathcal{X})) \leq -\mathcal{W}(\mathcal{X}) \tag{7}$$

and assume moreover that there exists $\alpha > 0$ such that for all $\mathcal{X} \in \mathbb{R}^N$

$$\alpha|\nabla\mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})\mathcal{K}(\mathcal{X})| \leq \mathcal{W}(\mathcal{X}). \tag{8}$$

Then for every $\delta < \alpha$ the closed-loop system (5) is globally asymptotically stable at the origin with respect to Krasowskii solutions.

Remark. Proposition 1 recalls results about stability under vanishing perturbations which are collected in [19] (Section 5.1, page 204). In fact, the term $\lambda\delta\mathcal{K}(\mathcal{X})$, $\lambda \in [-1, 1]$, can be seen as a discontinuous vanishing perturbation affecting

system (6). Bearing in mind this, it is not difficult to realize that, if (6) is exponentially stable, then system (5) is exponentially stable as well.

If a stabilizing feedback and a Lyapunov function are known, but condition (8) is not satisfied, one can turn to the following proposition, which is inspired by the linear discrete time scenario studied in [13], and shows a connection between the coarsness of the quantizer and a finite L_2-gain problem.

Proposition 2: Assume that system (6) is asymptotically stable and there exist $\mathcal{V} : \mathbb{R}^N \to \mathbb{R}$ of class C^1, positive definite and radially unbounded and $\gamma > 0$ such that for all $\mathcal{X} \in \mathbb{R}^N$,

$$
\begin{aligned}
&\nabla \mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})\mathcal{K}(\mathcal{X})) \\
&+ \frac{1}{4\gamma^2}|\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})|^2 + \mathcal{K}^2(\mathcal{X}) \leq 0.
\end{aligned} \tag{9}
$$

Then for any $\delta \leq 1/\gamma$ the closed-loop system (5) is globally asymptotically stable at the origin with respect to Krasowskii solutions.

Remark. Roughly speaking, Proposition 2 states that, if we know a stabilizing static or dynamic, state or output feedback controller such that we can solve (9) for some $\mathcal{V}$, then any quantization of the control input or of the measured output by means of a function Ψ in a sector bound whose "amplitude" is smaller than $1/\gamma$ does not cancel the stabilizing effect of the controller. The fulfillment of the inequality (9) implies the existence of a controller which renders the L_2-gain of an appropriate system less than or equal to γ. Observe that, in [13], it is shown how, for linear discrete-time systems, the inverse of the smallest γ for which this L_2-gain attenuation problem is solvable gives the coarsest quantizer for which (quadratic) stabilization via quantized feedback is achievable. Related results are found in [18], Lemma 1 and Remark 4.

B. Overcoming the limitation of the quantization density

We have seen so far that, unless a solution is found to the inequality (9), it may be difficult to asymptotically stabilize a nonlinear system using a coarse quantization. To overcome this limitation, we resort here to a different approach. Given the uncertainty due to the quantization, is it possible to devise a control law that, besides stabilizing, is able to actively counteract the quantization error? The (positive) answer is provided by the following statement.

Proposition 3: Assume there exist $\mathcal{V}, \mathcal{W} : \mathbb{R}^N \to \mathbb{R}$ continuous positive definite, $\mathcal{V}$ of class C^1 and radially unbounded, $\mathcal{K} : \mathbb{R}^n \to \mathbb{R}$ continuous such that (7) holds for all $\mathcal{X} \in \mathbb{R}^N$. For any pair $0 < r < R$, for any $\delta \in (0, 1)$, there exist $u_0 \geq 0$ and a continuous function $\tilde{\mathcal{K}} : \mathbb{R}^N \to \mathbb{R}$ such that for any Krasowskii solution φ of

$$
\dot{\mathcal{X}} = \mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})\Psi(\tilde{\mathcal{K}}(\mathcal{X})) , \tag{10}
$$

if $\varphi(0) \in \overline{B_R(0)}$, then there exists $T > 0$ such that $\varphi(t) \in \overline{B_r(0)}$ for all $t \geq T$.

Remark. Compared with Proposition 2, the result states that, even for those δ which are not smaller than $1/\gamma$, it is possible

to stabilize, although not asymptotically, the system under feedback quantization.

It is seen from the latter proposition that we must trade off global asymptotic stability against semi-global practical stability in order to stabilize the system without posing any constraint on the quantization density. Nevertheless, in a special noticeable case pointed out below, it is possible to recover asymptotic stability.

Corollary 1: Let the hypothesis of Proposition 3 hold, with $\mathcal{K}$ continuously differentiable, and additionally assume that $\mathcal{K}$ renders the closed loop system locally exponentially stable. Then, for any $R > 0$ and any $\delta \in (0, 1)$, there exists a continuously differentiable function $\tilde{\mathcal{K}} : \mathbb{R}^n \to \mathbb{R}$ such that the closed-loop system (10) is locally asymptotically stable at the origin with respect to Krasowskii solutions, and for any Krasowskii solution φ, $\varphi(0) \in \overline{B_R(0)}$ implies $\lim_{t \to \infty} \varphi(t) = 0$.

C. Semi-global practical stabilization by means of finite valued feedback laws

The results of the previous sub-section require an infinite number of quantization levels. Here we investigate the case in which only a finite number of quantization levels can be used. This problem has been deeply investigated in the case of linear discrete-time systems in [11], [12]. When a continuous stabilizing feedback law is known, it is relatively easy for nonlinear continuous-time systems to obtain semi-global practical stabilization by quantizing such feedback law. We introduce the truncated version of (2):

$$
\Psi_f(y) = \begin{cases}
u_0 & u_0/(1+\delta) < y \\
u_i & u_i/(1+\delta) < y \leq u_i/(1-\delta) , \\
& \qquad\qquad\qquad 1 \leq i \leq j \\
0 & 0 \leq y \leq u_j/(1+\delta) \\
-\Psi(-y) & y < 0 ,
\end{cases}
$$

with j to determine.

Proposition 4: Assume that there exist $\mathcal{V}, \mathcal{W} : \mathbb{R}^N \to \mathbb{R}$ continuous positive definite, $\mathcal{V}$ of class C^1 and radially unbounded, $\mathcal{K} : \mathbb{R}^n \to \mathbb{R}$ continuous such that (7) holds for all $\mathcal{X} \in \mathbb{R}^N$. For any pair $0 < r < R$, for any $\delta \in (0, 1)$, there exist $u_0 \geq 0$, $j \in \mathbb{N}$, and a continuous function $\tilde{\mathcal{K}} : \mathbb{R}^N \to \mathbb{R}$ such that for any Krasowskii solution φ of

$$
\dot{\mathcal{X}} = \mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})\Psi_f(\tilde{\mathcal{K}}(\mathcal{X})) , \tag{11}
$$

if $\varphi(0) \in \overline{B_R(0)}$ then there exists $T > 0$ such that $\varphi(t) \in \overline{B_r(0)}$ for all $t \geq T$.

Remark. The proof is constructive and expressions for u_0, j and $\tilde{\mathcal{K}}$ are given.

III. STABILIZATION OF DISSIPATIVE SYSTEMS

As the conditions of Proposition 1 and 2 may be quite demanding, whereas Proposition 3 guarantees semi-global practical stabilizability (but see Corollary 1 for a result on semi-global asymptotic stabilizability), we are led to consider special classes of control systems for which classical asymptotic stabilization results are known, and whose

features allow for extensions in case discontinuous control is employed. The remaining part of the paper focuses on a special class of nonlinear single input systems (1), namely those which are *dissipative* with respect to a quadratic supply rate ([17]). The technique used in the proofs does not differ very much from the classical one, but in the novel context further technical assumptions are needed.

The class of dissipative systems is recalled below. In order to deal with possibly discontinuous systems, we need to slightly extend the notion of dissipativity.

Definition 1: System (1) is said to be *dissipative*, respectively *co-dissipative*, with respect to a *quadratic* supply rate

$$q(u, y) = uRu + 2uSy + yQy \qquad (12)$$

with $R, S, Q \in \mathbb{R}$, if there exists a C^1, positive definite and radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ such that, for all $x \in \mathbb{R}^n$, for all $u \in U$, for all $y = h(x)$,

$$\nabla V(x) \cdot (f(x) + g(x)u) \le q(u, y) , \qquad (13)$$

respectively for all $x \in \mathbb{R}^n$, for all $v \in \overline{\mathrm{co}}\, U$, for all $y = h(x)$,

$$\nabla V(x) \cdot (f(x) + g(x)v) \le q(v, y) . \qquad (14)$$

Any function V which verifies either (13) or (14) is said to be a *storage function* for (1).

Remark. Important classes of nonlinear systems are dissipative. If $q(u, y) = uy$ the system is said to be *passive*. In the latter case, since we have assumed $0 \in U$, by taking $u = 0$ in (13), we get that for all $x \in \mathbb{R}^n$, $\nabla V(x) \cdot f(x) \le 0$, which implies that the unforced system

$$\dot{x} = f(x) \qquad (15)$$

is Lyapunov stable. Since system (1) is affine in the input variable u, dissipativity with respect to the supply rate $q(u, y) = uy$ (i.e. passivity) implies co-dissipativity (that we may call *co-passivity*).

Analogously we introduce the notion of dissipativity for a static memoryless system. Since we are interested in the negative interconnection of such systems with (1), we restrict to functions taking values in U.

Definition 2: A system with input $\tilde{u} \in \mathbb{R}$ and output $\tilde{y} \in U$ related by the function $\tilde{y} = \psi(\tilde{u})$, $\psi : \mathbb{R} \to U$, is said to be *dissipative*, respectively *co-dissipative*, with respect to a supply rate $\tilde{q} : \mathbb{R} \times U \to \mathbb{R}$, if for all $\tilde{u} \in \mathbb{R}$, for all $\tilde{y} = \psi(\tilde{u})$, $\tilde{q}(\tilde{u}, \tilde{y}) \ge 0$, respectively for all $\tilde{u} \in \mathbb{R}$, for all $\bar{y} \in K(\psi(\tilde{u}))$, $\tilde{q}(\tilde{u}, \bar{y}) \ge 0$. Equivalently the function ψ is said to be dissipative, respectively co-dissipative.

Remark. An important class of static memoryless systems are sector bounded systems. More precisely the system is said to be *sector bounded* if it is dissipative with respect to

$$\tilde{q}(\tilde{u}, \tilde{y}) = (\tilde{y} - \alpha\tilde{u})(\beta\tilde{u} - \tilde{y}) \qquad (16)$$

for some pair of real numbers $\beta > \alpha > 0$. Note that if $\psi : \mathbb{R} \to U$ is dissipative with respect to the previous $\tilde{q}$, i.e. it is sector bounded, then it is locally bounded, continuous at

zero, $\psi(\tilde{u}) = 0$ if and only if $\tilde{u} = 0$, and moreover it is also co-dissipative, i.e. we may say it is also *co-sector bounded*.

To infer *asymptotic* stabilizability for dissipative systems, the following property is relevant.

Definition 3: Let $Z_h = \{x \in \mathbb{R}^n : h(x) = 0\}$. System (1) is said to be *zero-state detectable* if for any Carathéodory solution $\varphi : [0, +\infty) \to \mathbb{R}$ of the system (15) such that $\varphi(t) \in Z_h$ for all $t \ge 0$, it holds $\lim_{t \to +\infty} \varphi(t) = 0$.

The previous notion of zero-state detectability may be strengthened when dealing with discontinuous systems and *strong zero-state detectability* may be considered. We refer the reader to [24] for a definition of the notion.

It is not hard to characterize conditions under which dissipative systems can be asymptotically stabilized. This descends from a result about asymptotic stability of interconnected dissipative systems (see e.g. [17]), where one of the systems is a *memoryless* dissipative function. The usual assumption on the function ψ is to be sufficiently regular (for instance, locally Lipschitz). Our first result shows that such assumption can be removed. On the other hand, the assumption of co-dissipativity must be adopted.

Proposition 5: Let $U \subset \mathbb{R}$ be given. Assume that

(i) system (1) is co-dissipative with respect to the quadratic supply rate (12) and zero-state detectable;

(ii) $\psi : \mathbb{R} \to U$ is measurable, locally bounded, continuous at 0 with $\psi(\tilde{u}) = 0$ if and only if $\tilde{u} = 0$, and ψ is co-dissipative with respect to the quadratic supply rate

$$\tilde{q}(\tilde{u}, \tilde{y}) = \tilde{u}\tilde{R}\tilde{u} + 2\tilde{y}\tilde{S}\tilde{u} + \tilde{y}\tilde{Q}\tilde{y} \qquad (17)$$

(iii) there exists $a > 0$ such that the matrix

$$M = \begin{bmatrix} Q + a\tilde{R} & -S + a\tilde{S} \\ -S + a\tilde{S} & R + a\tilde{Q} \end{bmatrix}$$

is definite negative.

Then the closed loop system

$$\dot{x} = f(x) - g(x)\psi(h(x)) , \qquad (18)$$

is globally asymptotically stable at the origin with respect to Krasowskii solutions.

Remark. One of the interests of this result lies in the possibility of stabilizing nonlinear systems using quantized feedback. In fact, consider a *discontinuous sector-bounded* static nonlinearity ψ satisfying (16). A possible function of this kind is the one defined in (2). It is straightforward to determine conditions on Q, R, S under which any nonlinear system (1) co-dissipative with respect to a supply rate (12) in negative feedback interconnection with a sector bounded memoryless nonlinearity is such that $M < 0$, and, as such, is asymptotically stabilizable under the hypothesis of zero-state detectability. Note that dissipative systems which satisfy conditions as discussed above can be well unstable when $u = 0$. In some cases, when it is not possible to find $a > 0$ such that $M < 0$, similar arguments can be applied: a result analogous to the previous one can be given the form of

the *nonlinear discontinuous* version of the celebrated *circle criterion* [17]. The result is omitted and can be found in [8].

IV. STABILIZATION OF PASSIVE SYSTEMS VIA SMALL INPUTS

The results of the previous section require continuity of ψ at the origin. Investigating the case in which this requirement is not met is particularly interesting if the sector bound is such that $\alpha = 0$ and $\beta = +\infty$. In the case of passive systems, we derive the following propositions in which stabilization is obtained by means of feedback laws taking values in a finite set U. A previous result on stabilization of passive systems which makes use of Krasowskii solutions was proved in [24].

Proposition 6: Let $U = \{0, \epsilon, -\epsilon\}, \epsilon \in \mathbb{R}, \epsilon > 0$. Assume that system (1) is passive. Moreover assume that one of the following condition holds:

(i) for each $x \in Z_h, x \neq 0$,

$$\nabla h(x) \cdot (f(x) + \lambda g(x)) \neq 0 , \qquad (19)$$

for any $\lambda \in [-\epsilon, \epsilon]$;

(ii) system (1) is zero-state detectable and for each $x \in Z_h, x \neq 0$, for which there exists $\lambda \in [-\epsilon, \epsilon]$ such that

$$\nabla h(x) \cdot (f(x) + \lambda g(x)) = 0 , \qquad (20)$$

either $\lambda = 0$ or $g(x) = 0$;

(iii) system (1) is strongly zero-state detectable.

Then, the closed-loop system

$$\dot{x} = f(x) - \epsilon g(x)\mathrm{sgn}h(x) \qquad (21)$$

is globally asymptotically stable at the origin with respect to Krasowskii solutions.

Remark. A few observations are in order:

- To the purpose of establishing a deeper connection with the literature in switched control, Proposition 6 (i) can be given an alternative form. Define the vector fields

$$f_1(x) = f(x) - \epsilon g(x) , \quad f_2(x) = f(x) + \epsilon g(x) ,$$

and correspondingly the system

$$\dot{x} = uf_1(x) + (1 - u)f_2(x) ,$$

with u taking values in the set $\{0, 1\}$. Suppose there exists a C^1 positive definite and radially unbounded function V and a real number $\alpha \in (0, 1)$ such that $\nabla V(x)(\alpha f_1(x) + (1 - \alpha)f_2(x)) \leq 0$ for all $x \in \mathbb{R}^n$. Assume additionally that, for any $x \neq 0$ such that $\nabla V(x)f_1(x) = \nabla V(x)f_2(x)$, and for any $\beta \in [0, 1]$, $\nabla(\nabla V(x)(f_1(x) - f_2(x)))(\beta f_1(x) + (1 - \beta)f_2(x))) \neq 0$. Then, there exists a static state (discontinuos) feedback $u = k(x)$ with values in the set $\{0, 1\}$ such that the closed-loop system is globally asymptotically stable at the origin with respect to Krasowskii solutions. When both f_1 and f_2 are linear, the result is Theorem 1 in [2]. The reader is referred to the latter reference for a thorough discussion of the result within the framework of switched control.

- The main difference of the latter proof (ii) with respect to the one of Proposition 5 lies in guaranteeing that the solution φ_Γ asymptotically tends to zero as $t \to \infty$ notwithstanding the fact that it is evolving on the *discontinuity* manifold $h(x) = 0$. To this purpose, condition (ii) plays a fundamental role. In fact, the second one of the following examples shows that zero-state detectability alone is not enough in order to get asymptotic stability of system (21).

Example. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_1 u \qquad (22) \\ y &= x_1 x_2. \end{aligned}$$

Let $U = \{0, \epsilon, -\epsilon\}$, with $\epsilon < 1$. Condition (i) of Proposition 6 holds, then the feedback law $u = k(x) = -\epsilon\,\mathrm{sgn}(x_1 x_2)$ stabilizes the system asymptotically with respect to Krasowskii solutions.

Zero-state detectability does not suffice to prove stabilizability of passive systems with respect to Krasowskii solutions.

Example. Consider again system (22) with $U = \{0, -1, 1\}$. Condition (i) of Proposition 6 does not hold on the x_1-axis. It is immediate to verify that the system is passive with storage function $V(x) = x^T x/2$. The system is also zero-state detectable. As a matter of fact, let $u(t) = 0$ and $y(t) = 0$ for all $t \geq 0$. Then $x_1(t)x_2(t) = 0$ for all $t \geq 0$. Then $\dot{x}_1(t)x_2(t) + x_1(t)\dot{x}_2(t) = x_2^2(t) - x_1^2(t) = 0$ which implies $x_2(t) = \pm x_1(t)$ then $x_1(t) = 0, x_2(t) = 0$ for all $t \geq 0$. In this case the feedback controller is given by $u = -\mathrm{sgn}(x_1 x_2)$. Note that $K(f(\cdot) - g(\cdot)\mathrm{sgn}h(\cdot))(x_1, 0) = \{(0, -(1-\lambda)x_1), \lambda \in [-1, 1]\}$, then, by taking $\lambda = 1$, we get that the points on the x_1-axis are equilibrium positions for the closed-loop system, which is not asymptotically stable.

In the example below condition (ii) of Proposition 6 applies.
Example. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_1^k x_2^\ell u \qquad (23) \\ y &= x_1^k x_2^{\ell+1} \end{aligned}$$

where k, ℓ are integers not smaller than 1. Choosing as storage function $V(x) = x^T x/2$, it is immediately shown that the system is passive. It is also zero-state detectable. We have $h(x) = x_1^k x_2^{\ell+1}$ and

$$\begin{aligned} \nabla h(x)(f(x) + \lambda g(x)) = \\ kx_1^{k-1}x_2^{\ell+2} + (\ell+1)x_1^k x_2^\ell(-x_1 + x_1^k x_2^\ell \lambda) . \end{aligned}$$

Any $x \neq 0$ such that $h(x) = 0$ can be equal either to $(x_1, 0)$ or $(0, x_2)$. In the former case, $\nabla h(x)(f(x) + \lambda g(x)) = 0$ for any λ, and $g(x) = 0$. In the latter case, if $k > 1$, then again $\nabla h(x)(f(x) + \lambda g(x)) = 0$ for any λ, and $g(x) = 0$, whereas, if $k = 1$, $\nabla h(x)(f(x) + \lambda g(x)) = kx_2^{\ell+2}$, and therefore
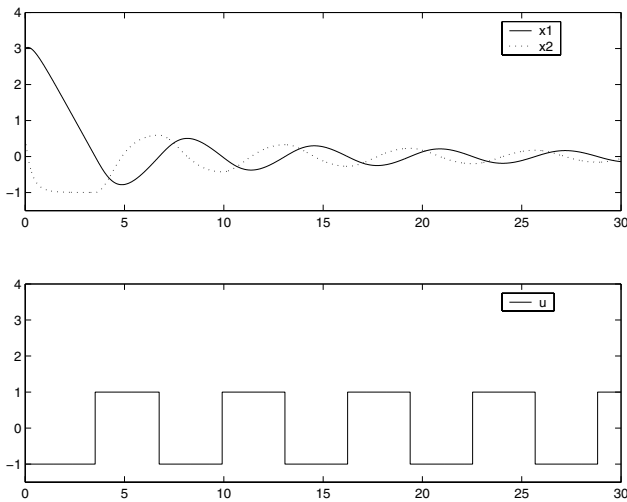
Fig. 1. Switched control of the system $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 + x_1 x_2 u$, $y = x_1 x_2^2$. Note that in this case for every initial condition there exists a unique Krasowskii solution.

never equal to zero. We conclude that the feedback $u = -\text{sgn} x_1^k x_2^{\ell+1}$ globally asymptotically stabilizes the system with respect to Krasowskii solutions. Moreover, along the two switching manifolds $x_1 = 0$ and $x_2 = 0$, the vector field is continuous, and no sliding mode will arise. Finally, we observe that in the polar coordinates (ρ, θ), the system satisfies the equation

$$\dot{\theta} = -1 - \frac{x_1^{k+1} x_2^{\ell}}{\rho^2} \text{sgn} x_1^k x_2^{\ell+1} \, ,$$

which, bearing in mind that $V(\rho, \theta) = \rho^2/2$ and that V is monotone non increasing along any trajectory of the system, yields

$$|\dot{\theta}| \leq 1 + \rho^{k+\ell-1}(0) \, .$$

This allows to conclude that, after a switching has occurred, a certain amount of time (*dwell time*) must elapse before a new switching can take place (see also [7]). Simulation results are reported in Figure 1 in the case $k = \ell = 1$.

V. CONCLUSIONS

For switched and quantized systems, it is interesting to study how the powerful stabilization techniques developed within the robust control framework and for dissipative systems can be reinterpreted. In this paper we give a contribution in this direction in the case of single input affine systems: the multi input case may be analogously treated. For quantized nonlinear control systems, state quantization deserves further attention. Investigating conditions which guarantee that switching occurs not too fast is also of importance.

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