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The Internal Model Principle: Asymptotic Tracking and Regulation in the Behavioral Framework

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***Abstract*— Given a plant, together with an exosystem generating the disturbances and the reference signals, the problem of asymptotic tracking and regulation is to find a controller such that the to-be-controlled plant variable tracks the reference signal regardless of the disturbance acting on the system. If a controller achieves this design objective, we call it a regulator for the plant with respect to the given exosystem. In this paper we formulate the asymptotic tracking and regulation problem in the behavioral framework, with control as interconnection. The problem formulation and its resolution are completely representation free, and specified only in terms of the plant and exosystem dynamics.**

I. Introduction

This paper deals with control in a behavioral context. We consider the problem of finding a free-disturbance stabilizing controller that regulates the tracking error to zero in the presence of a class of exogenous inputs. In other words, we consider the problem of *asymptotic tracking and regulation* in the behavioral framework.

In the behavioral framework, controlling a plant means restricting its behavior to a desired subset of the behavior. This restriction is brought about by interconnecting the plant with a controller that we design. The restricted behavior is then called the controlled behavior, which is required to satisfy the design specifications. In terms of representations, control means that additional laws (e.g., in the form of differential equations representing the controller behavior) are imposed on some of the plant variables. Thus, the plant and controller are interconnected through some of their variables. In our context we do not distinguish between inputs and outputs, so the interconnection does not involve feedback. This idea was introduced by J. C. Willems in [10] in the context of stabilization and pole placement. In this paper we use these ideas to solve the problem of asymptotic tracking and regulation.

Of course, the problem of asymptotic tracking and regulation has been studied before in the literature, in an input-output framework. See for instance Davison and Goldenberg in [3], Francis in [5] and Francis and Wonham in [6]. Many results have been collected by Saberi, Stoorvogel and Sannuti in the book [8]. In these, the concept of internal model principle plays a pivotal role in obtaining a solution to the asymptotic tracking and regulation problem. According to the internal model principle, in order to achieve regulation the controller or the plant must contain the dynamics of the exosystem.

Our work can be seen as the behavioral generalization of [3], [5] and [6]. We use polynomial kernel representations of the plant (see [7]) without any input-output considerations. This problem was initially studied by K. Takaba in [9]. In the work of Takaba only *necessary* conditions were obtained for the existence of a controller which solves the regulation problem. In [4] necessary and sufficient conditions for the existence of a controller which solves the asymptotic tracking and regulation problem were obtained under certain a priori assumptions. It was assumed that the underlying exosystem is anti-stable and that the underlying plant does not annihilate any signal generated by the exosystem. In this paper we generalize these results to the case where the underlying exosystem is just an autonomous system (not necessarily anti-stable) and the underlying plant might annihilate signals generated by the exosystem. Necessary and sufficient conditions for the existence of controllers which solve the asymptotic tracking and regulation problem are expressed in terms of the plant and the exosystem which generates the disturbances and the reference signal. Also a procedure to construct such controllers is given using the polynomial matrices appearing in the kernel representations of the plant and the exosystem.

A. Notation and nomenclature

A few words about the notation and nomenclature used. We use standard symbols for the fields of real and complex numbers $\mathbb{R}$ and $\mathbb{C}$. $\mathbb{C}^-$, and $\bar{\mathbb{C}}_+$ will denote the open left half plane and closed right half plane, respectively. We use $\mathbb{R}^{\mathtt{n}}$, $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}$, etc., for the real linear spaces of vectors and matrices with components in $\mathbb{R}$.

$\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ denotes the set of infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$. $\mathbb{R}[\xi]$ denotes the ring of polynomials in the indeterminate ξ with real coefficients. We use $\mathbb{R}^{\mathtt{n}}[\xi], \mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$, for the spaces of vectors and matrices with components in $\mathbb{R}[\xi]$. Elements of $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$ are called *real polynomial matrices*.

We use the notation $\det(A)$, to denote the determinant of a square matrix A. A square, nonsingular real polynomial matrix R is called *Hurwitz* if all roots of $\det(R)$ lie in the open left half complex plane $\mathbb{C}^-$. It is called *anti-Hurwitz* if all roots of $\det(R)$ lie in the closed right half complex plane $\bar{\mathbb{C}}^+$.

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II. Linear differential systems and polynomial kernel representations

In the behavioral approach to linear systems, a dynamical system is given by a triple $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$, where $\mathbb{R}$ is the time axis, $\mathbb{R}^{\mathtt{w}}$ is the signal space, and the *behavior* $\mathfrak{B}$ is a linear subspace of $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ consisting of all solutions of a set of higher order, linear, constant coefficient differential equations. Such a triple is called a *linear differential system*. More precisely, there exist a positive integer g and a polynomial matrix $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$ such that

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\tfrac{\mathrm{d}}{\mathrm{d}t})w = 0\}.$$

The set of linear differential systems with manifest variable w taking its value in $\mathbb{R}^{\mathtt{w}}$ is denoted by $\mathfrak{L}^{\mathtt{w}}$.

Let $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$ be a polynomial matrix. If the behavior $\mathfrak{B}$ is represented by $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ then we call this a *kernel representation* of $\mathfrak{B}$. Further, a kernel representation is said to be *minimal* if every other kernel representation of $\mathfrak{B}$ has at least g rows. A given kernel representation, $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, is minimal if and only if the polynomial matrix R has full row rank (see [7], Theorem 3.6.4). The number of rows in any minimal polynomial kernel representation of $\mathfrak{B}$ is equal to the *output cardinality* of $\mathfrak{B}$, denoted by $\mathtt{p}(\mathfrak{B})$. This number corresponds to the number of outputs in any input/output representation of $\mathfrak{B}$. We speak of a system as the behavior $\mathfrak{B}$, one of whose representations is given by $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ or just $\mathfrak{B} = \ker(R)$. The '$\frac{\mathrm{d}}{\mathrm{d}t}$' is often suppressed to enhance readability.

The following Proposition from [7] relates two minimal kernel representations of a given behavior.

Proposition 2.1: Let $\mathfrak{B}_1 = \ker(R_1)$ and $\mathfrak{B}_2 = \ker(R_2)$ be minimal kernel representations. Then $\mathfrak{B}_1 = \mathfrak{B}_2$ if and only if there exists a unimodular matrix U such that $R_1 = UR_2$.

Definition 2.2: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable w partitioned as $w = (w_1, w_2)$. We will call w_2 *free in* $\mathfrak{B}$ if, for any $w_2 \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2})$, there exists w_1 such that $(w_1, w_2) \in \mathfrak{B}$.

The following result was shown in [7]:

Proposition 2.3: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1, w_2). Let a minimal kernel representation of $\mathfrak{B}$ be given by $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + R_2(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$. Then w_2 is free in $\mathfrak{B}$ if and only if the polynomial matrix R_1 has full row rank.

Definition 2.4: A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is called *autonomous* if it has no free variables, equivalently, $\mathtt{p}(\mathfrak{B}) = \mathtt{w}$. It is called *stable* if for all $w \in \mathfrak{B}$ we have $\lim_{t\to+\infty} w(t) = 0$.

The following Proposition was shown in [7].

Proposition 2.5: If $\mathfrak{B} = \ker(R)$, then $\mathfrak{B}$ is autonomous if and only if R has full column rank and is stable if and only if $R(\lambda)$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^{+}$. Note that a stable behavior is necessarily autonomous.

We denote the set of all linear autonomous differential systems with w variables by $\mathfrak{L}^{\mathtt{w}}_{\mathrm{aut}}$.

Definition 2.6: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{aut}}$. Then $\mathfrak{B}$ is called *anti-stable* if for all non-zero $w \in \mathfrak{B}$ we have either $\lim_{t\to+\infty} w(t) \neq 0$ or $\lim_{t\to+\infty} w(t)$ does not exist.

Proposition 2.7: If $\mathfrak{B} = \ker(R)$, then $\mathfrak{B}$ is anti-stable if and only if $R(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}^{-}$.

Definition 2.8: A function of the form $H(t) = \sum_{i=1}^{N}\sum_{j=1}^{n_i} A_{ij}t^{j-1}e^{\lambda_i t}$ is called a *Bohl function*, i.e., a Bohl function is a finite sum of products of polynomials and exponentials. In the real case, a Bohl function is a finite sum of products of polynomials, real exponentials, sines, and cosines. A function $H(t)$ is called *stable Bohl* if it is Bohl and $\lim_{t\to+\infty} H(t) = 0$. A function $H(t)$ is called *anti-stable Bohl* if it is Bohl and for non-zero $H(t)$ we have either $\lim_{t\to+\infty} H(t) \neq 0$ or $\lim_{t\to+\infty} H(t)$ does not exist. Then we have the following Proposition.

Proposition 2.9: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{aut}}$. Then

1) every $w \in \mathfrak{B}$ is a Bohl function,
2) if $\mathfrak{B}$ is stable then every $w \in \mathfrak{B}$ is a stable Bohl function, and
3) if $\mathfrak{B}$ is anti-stable then every $w \in \mathfrak{B}$ is a anti-stable Bohl function.

Now we have the following Proposition.

Proposition 2.10: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{aut}}$. Then there exists a stable $\mathfrak{B}_{\mathrm{s}} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{aut}}$, and an anti-stable $\mathfrak{B}_{\mathrm{a}} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{aut}}$ such that $\mathfrak{B} = \mathfrak{B}_{\mathrm{s}} \oplus \mathfrak{B}_{\mathrm{a}}$.

Proof: We skip the proof due to space limitations. □

We now recall from [7] the definitions of stabilizability and detectability.

Definition 2.11: A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be *stabilizable*, if for every $w \in \mathfrak{B}$, there exists $w' \in \mathfrak{B}$ such that $w'(t) = w(t)$ for $t \leq 0$, and $\lim_{t\to+\infty} w'(t) = 0$.

The following result was shown in [7]:

Proposition 2.12: If $\mathfrak{B} = \ker(R)$ is a minimal kernel representation of $\mathfrak{B}$, then $\mathfrak{B}$ is stabilizable if and only if $R(\lambda)$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^{+}$.

Definition 2.13: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with plant variable $w = (w_1, w_2)$. We say that w_2 is *observable* from w_1 in $\mathfrak{B}$ if, whenever (w_1, w_2), $(w_1, w_2') \in \mathfrak{B}$, then $w_2 = w_2'$. We say that w_2 is *detectable* from w_1 in $\mathfrak{B}$ if, whenever (w_1, w_2), $(w_1, w_2') \in \mathfrak{B}$, then $\lim_{t\to+\infty}(w_2 - w_2')(t) = 0$.

The following result was shown in [7]:

Proposition 2.14: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1, w_2). Let a minimal kernel representation of $\mathfrak{B}$ be given by $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + R_2(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$. In $\mathfrak{B}$, w_2 is detectable from w_1 if and only if $R_2(\lambda)$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^{+}$.

Definition 2.15: Let $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\bullet}$, $C \in \mathbb{R}^{\bullet\times\mathtt{n}}$. We call the pair (A, B) *stabilizable* if the behavior defined by $\ker\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}I - A & -B \end{pmatrix}$ is stabilizable and we call the pair (C, A) *detectable* if the behavior defined by $\ker\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}I - A \\ C \end{pmatrix}$ is stable.

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1, w_2). Often we are interested only in the behavior of one of the components, say the variable w_1, obtained by projecting $\mathfrak{B}$ onto the first component w_1. This behavior $\mathfrak{B}_{w_1}$ is defined by $\mathfrak{B}_{w_1} := \{w_1 \mid \exists w_2 \text{ such that } (w_1, w_2) \in \mathfrak{B}\}$. If $\mathfrak{B} = \ker(R_1\ R_2)$ is a kernel representation, then a kernel representation for $\mathfrak{B}_{w_1}$ is obtained as follows: choose a unimodular matrix U such that $UR_2 = \begin{pmatrix} R_{12} \\ 0 \end{pmatrix}$, with R_{12} full row rank, and conformably partition $UR_1 = \begin{pmatrix} R_{11} \\ R_{21} \end{pmatrix}$. Then $\mathfrak{B}_{w_1} = \ker(R_{21})$ (see [7], section 6.2.2).

III. Review of stabilization by interconnection

In this section we will briefly recall the notion of stabilization by interconnection. We will first look at the full interconnection case, i.e. the case when all the plant variables are available for interconnection.

Definition 3.1: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ be a plant behavior. A controller for $\mathcal{P}$ is a system behavior $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$. The full interconnection of $\mathcal{P}$ and $\mathcal{C}$ is defined as the system with behavior $\mathcal{P} \cap \mathcal{C}$. This behavior is called the *controlled behavior*, and is also an element of $\mathfrak{L}^{\mathtt{w}}$. The full interconnection is called *regular* if $\mathtt{p}(\mathcal{P} \cap \mathcal{C}) = \mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C})$. In that case we call $\mathcal{C}$ a *regular controller*.

In full interconnection, the regularity condition is equivalent to: $\mathcal{C}$ does not re-impose restrictions on the plant variable w that are already present in the laws of $\mathcal{P}$ (see [10]).

A given plant is stabilizable if and only if we can stabilize it by interconnecting it with a suitable controller, called a stabilizing controller, which is defined as follows [11].

Definition 3.2: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$. A controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ is said to be a *stabilizing controller* if the behavior $\mathcal{P} \cap \mathcal{C}$ is stable and the interconnection is regular.

The following result was shown in [10].

Proposition 3.3: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$. Then there exists a stabilizing controller for $\mathcal{P}$ if and only if $\mathcal{P}$ is stabilizable.

Next we will look at the so called partial interconnection case, in which only a pre-specified subset of the plant variables is available for interconnection. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ be a linear differential system, with system variable (w, c), where w takes its values in $\mathbb{R}^{\mathtt{w}}$ and c in $\mathbb{R}^{\mathtt{c}}$. The variable w should be interpreted as the variable to-be-controlled, the variable c as the one through which we can interconnect the plant with a controller, called the control variable. Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ (to be interpreted as a controller behavior) with variable c.

Definition 3.4: The interconnection of $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ through c is defined as the system behavior $\mathcal{P} \wedge_c \mathcal{C} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, given by $\mathcal{P} \wedge_c \mathcal{C} = \{(w, c) \mid (w, c) \in \mathcal{P} \text{ and } c \in \mathcal{C}\}$. The behavior $\mathcal{P} \wedge_c \mathcal{C}$ is called *the full controlled behavior*. The behavior $(\mathcal{P} \wedge_c \mathcal{C})_w \in \mathfrak{L}^{\mathtt{w}}$ that is obtained by eliminating c from $\mathcal{P} \wedge_c \mathcal{C}$ is called *the manifest controlled behavior*. The interconnection of $\mathcal{P}$ and $\mathcal{C}$ through c is called *regular* if $\mathtt{p}(\mathcal{P} \wedge_c \mathcal{C}) = \mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C})$. $\mathcal{C}$ is then called a *regular controller*.

In partial interconnection, the regularity condition is equivalent to: $\mathcal{C}$ does not re-impose restrictions on the control variable c that are already present in the laws of $\mathcal{P}$ (see [1] and [2]).

Given $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1, w_2), in this paper we use the notation $\mathcal{N}_{w_1}(\mathcal{P})$ to indicate the behavior obtained by putting $w_2 = 0$ and projecting onto the variable w_1 i.e., $\mathcal{N}_{w_1}(\mathcal{P}) = \{w_1 \mid (w_1, 0) \in \mathcal{P}\}$.

The following Proposition on polynomial matrices will be useful in the paper.

Proposition 3.5: Let $A \in \mathbb{R}[\xi]^{\mathtt{p}\times\mathtt{p}}$ be Hurwitz and $B \in \mathbb{R}[\xi]^{\mathtt{q}\times\mathtt{q}}$ be anti-Hurwitz. Then for any $C \in \mathbb{R}[\xi]^{\mathtt{p}\times\mathtt{q}}$ there exists a solution (X, Y) of the equation $AX + YB = C$.

Proof: We skip the proof due to space limitations. □

In the next section we will formulate the asymptotic tracking and regulation problem studied in this paper.

IV. Asymptotic tracking and regulation

In this section we will introduce the problem of asymptotic tracking and regulation in a behavioral context, with control by general, regular, interconnection.

We start with a plant behavior $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$, with plant variable (w, c, v). The system variable has been partitioned into w, c and v. These variables represent the to-be-controlled variable (like tracking error), the interconnection variable (like sensor measurements and actuator inputs), and external disturbances and reference signals, respectively. The interconnection variable c is the system variable through which we are allowed to interconnect $\mathcal{P}$ with a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$. As the variable v represents a reference signal and external disturbances we assume it to be free in $\mathcal{P}$. In addition to the plant $\mathcal{P}$, let an exosystem $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}$ which generates the disturbance and the reference signal be given.

Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$. Then the interconnection of the plant $\mathcal{P}$ with $\mathcal{C}$ is given by

$$\mathcal{P} \wedge_c \mathcal{C} = \{(w, c, v) \mid (w, c, v) \in \mathcal{P} \text{ and } c \in \mathcal{C}\}. \quad (1)$$

Then we have the following definition for a free-disturbance stabilizing controller.

Definition 4.1: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$. Assume v is free in $\mathcal{P}$. Then a $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is called a *free-disturbance stabilizing* controller for $\mathcal{P}$ if

1) the interconnection $\mathcal{P} \wedge_c \mathcal{C}$ is regular,
2) v is free in $\mathcal{P} \wedge_c \mathcal{C}$,
3) for all $(w, c, 0) \in \mathcal{P} \wedge_c \mathcal{C}$ we have $\lim_{t\to+\infty}(w(t), c(t)) = (0, 0)$, i.e., $\mathcal{N}_{(w,c)}(\mathcal{P}) \wedge_c \mathcal{C}$ is stable.

Condition (2.) in the above definition asks the controller not to put any restrictions on the variable v which represents the reference signal and external disturbances acting on the system. Condition (1.) about the regularity of the interconnection $\mathcal{P} \wedge_c \mathcal{C}$ will make sure that $\mathcal{C}$ does not re-impose restrictions on the control variable c that are already present in the laws of $\mathcal{P}$. Condition (3.) asks the controller to drive the plant variables w and c to zero if $v = 0$, i.e., if the disturbance is absent. We have the following Theorem.

Theorem 4.2: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$. Assume v is free in $\mathcal{P}$. Then there exists a disturbance free stabilizing controller for $\mathcal{P}$ if and only if

1) $\mathcal{N}_{(w,c)}(\mathcal{P})$ is stabilizable, and
2) w is detectable from (c, v) in $\mathcal{P}$.

Proof: We skip the proof due to space limitations. □

The interconnection of the plant $\mathcal{P}$ with the exosystem $\mathcal{E}$ and controller $\mathcal{C}$ is given by

$$\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} = \{(w, c, v) \mid (w, c, v) \in \mathcal{P}, v \in \mathcal{E} \text{ and } c \in \mathcal{C}\}.$$

We have the following definition of a regulator.

Definition 4.3: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w+c+v}}$. Assume v is free in $\mathcal{P}$. Then $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is called a *regulator* for $\mathcal{P}$ with respect to $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}$, if

1) $\mathcal{C}$ is a free-disturbance stabilizing controller for $\mathcal{P}$
2) for all $(w, c, v) \in \mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C}$ we have $\lim_{t\to+\infty} w(t) = 0$, i.e., $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w$ is stable.

Condition (2.) in the above definition asks the controller to achieve regulation of the system variable w.

We now formulate the main problem of this paper:

Problem: Given a plant $\mathcal{P} \in \mathfrak{L}^{\mathtt{w+c+v}}$ with system variable (w, c, v), with v free in $\mathcal{P}$, and an autonomous system $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$ with system variable v, find necessary and sufficient conditions for the existence of a regulator $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ for $\mathcal{P}$ with respect to $\mathcal{E}$.

As a first step in resolving Problem, we will show that without loss of generality we can assume that in $\mathcal{P} \wedge_v \mathcal{E}$, the interconnection of plant and exosystem, v is observable from (w, c), equivalently, $\mathcal{E} \cap \mathcal{N}_v(\mathcal{P}) = 0$. Indeed, the following lemma shows that signals $v \in \mathcal{E} \cap \mathcal{N}_v(\mathcal{P})$ are completely decoupled from the (w, c)-behavior of $\mathcal{P} \wedge_v \mathcal{E}$:

Lemma 4.4: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w+c+v}}$, $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$. Let $\mathcal{E}' \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$ be such that $\mathcal{E} = \mathcal{E}' \oplus (\mathcal{E} \cap \mathcal{N}_v(\mathcal{P}))$. Then $(\mathcal{P} \wedge_v \mathcal{E})_{(w,c)} = (\mathcal{P} \wedge_v \mathcal{E}')_{(w,c)}$.

Proof: As $\mathcal{E}' \subseteq \mathcal{E}$, it is straightforward that $(\mathcal{P} \wedge_v \mathcal{E}')_{(w,c)} \subseteq (\mathcal{P} \wedge_v \mathcal{E})_{(w,c)}$. We now prove the converse inclusion. Let $(w, c) \in (\mathcal{P} \wedge_v \mathcal{E})_{(w,c)}$. Then there exist a v such that $(w, c, v) \in \mathcal{P}$ and $v \in \mathcal{E}$. As $\mathcal{E} = \mathcal{E}' \oplus (\mathcal{E} \cap \mathcal{N}_v(\mathcal{P}))$ there exist $v_1 \in \mathcal{E}'$ and $v_2 \in \mathcal{E} \cap \mathcal{N}_v(\mathcal{P})$ such that $v = v_1 + v_2$. Hence $(w, c, v_1 + v_2) \in \mathcal{P}$, and $v_2 \in \mathcal{N}_v(\mathcal{P})$, equivalently, $(0, 0, v_2) \in \mathcal{P}$. From linearity we have $(w, c, v_1 + v_2) - (0, 0, v_2) \in \mathcal{P}$. Therefore $(w, c, v_1) \in \mathcal{P}$. Hence we have $(w, c, v_1) \in \mathcal{P} \wedge_v \mathcal{E}'$, which implies that $(w, c) \in (\mathcal{P} \wedge_v \mathcal{E}')_{(w,c)}$. We conclude that $(\mathcal{P} \wedge_v \mathcal{E})_{(w,c)} \subseteq (\mathcal{P} \wedge_v \mathcal{E}')_{(w,c)}$. □

Since, obviously, for any direct summand $\mathcal{E}'$ as above we have $\mathcal{E}' \cap \mathcal{N}_v(\mathcal{P}) = 0$, the following theorem shows that for the solvability of Problem the assumption $\mathcal{E} \cap \mathcal{N}_v(\mathcal{P}) = 0$ can indeed be made without loss of generality:

Theorem 4.5: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w+c+v}}, \mathcal{E} \in \mathfrak{L}^{\mathtt{v}}$. Let $\mathcal{E}' \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$ be such that $\mathcal{E} = \mathcal{E}' \oplus (\mathcal{E} \cap \mathcal{N}_v(\mathcal{P}))$. Then $\mathcal{C}$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$ if and only if $\mathcal{C}$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}'$.

Proof: Using Lemma 4.4, we have $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w = ((\mathcal{P} \wedge_v \mathcal{E})_{(w,c)} \wedge_c \mathcal{C})_w = ((\mathcal{P} \wedge_v \mathcal{E}')_{(w,c)} \wedge_c \mathcal{C})_w = (\mathcal{P} \wedge_v \mathcal{E}' \wedge_v \mathcal{C})_w$. Therefore $\mathcal{C}$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$ if and only if $\mathcal{C}$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}'$. □

It is easy to show that the condition $\mathcal{E} \cap \mathcal{N}_v(\mathcal{P}) = 0$ is equivalent to the condition that v is observable from (w, c) in $\mathcal{P} \wedge_v \mathcal{E}$.

Also the following theorem will be instrumental in solving Problem.

Theorem 4.6: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w+v}}$ with system variable (w, v). Assume v is free in $\mathcal{P}$. Let $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$ be an anti-stable system with system variable v. Then $(\mathcal{P} \wedge_v \mathcal{E})_w$ is stable if and only if the following conditions hold.

1) $\lim_{t\to+\infty} w(t) = 0$ for all $(w, 0) \in \mathcal{P}$, i.e., $\mathcal{N}_w(\mathcal{P})$ is stable, and
2) $(0, v) \in \mathcal{P}$ holds for all $v \in \mathcal{E}$, i.e., $\mathcal{E} \subseteq \mathcal{N}_v(\mathcal{P})$.

Proof: (if) $(w, v) \in \mathcal{P} \wedge_v \mathcal{E}$ implies $(w, v) \in \mathcal{P}$ and $v \in \mathcal{E}$. As $(0, v) \in \mathcal{P}$ for all $v \in \mathcal{E}$, from linearity, we have $(w, v) - (0, v) \in \mathcal{P}$. Therefore $(w, 0) \in \mathcal{P}$. Since we have $\lim_{t\to+\infty} w(t) = 0$ for all $(w, 0) \in \mathcal{P}$, we conclude that $\lim_{t\to+\infty} w(t) = 0$ holds for all $(w, v) \in \mathcal{P} \wedge_v \mathcal{E}$.

(only if) We have $\{(w, 0) \mid (w, 0) \in \mathcal{P}\} \subseteq \mathcal{P} \wedge_v \mathcal{E}$. Since $\lim_{t\to+\infty} w(t) = 0$ for all $(w, v) \in \mathcal{P} \wedge_v \mathcal{E}$, we obtain $\lim_{t\to+\infty} w(t) = 0$ for all $(w, 0) \in \mathcal{P}$.

Let $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w + R_2(\frac{\mathrm{d}}{\mathrm{d}t})v = 0$ be a minimal representation of $\mathcal{P}$. Let $v \in \mathcal{E}$. As v is free in $\mathcal{P}$ there exists a w such that

$$R_1(\tfrac{\mathrm{d}}{\mathrm{d}t})w = -R_2(\tfrac{\mathrm{d}}{\mathrm{d}t})v. \tag{2}$$

As $(\mathcal{P} \wedge_v \mathcal{E})_w$ is stable, w is a stable Bohl function. Hence, the LHS of Equation (2) is a stable Bohl function. Also, since $\mathcal{E}$ is anti-stable, v is either identically equal to 0 or anti-stable Bohl. This implies that the RHS of Equation (2) is either identically equal to 0, or an anti-stable Bohl function. Equation (2) thus implies that $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w = -R_2(\frac{\mathrm{d}}{\mathrm{d}t})v = 0$. Consequently, $(w, 0) \in \mathcal{P}$. From linearity we have $(w, v) - (w, 0) \in \mathcal{P}$, which implies that $(0, v) \in \mathcal{P}$. Therefore $v \in \mathcal{N}_v(\mathcal{P})$. □

Remark 4.7: Condition 2) of Theorem 4.6 provides a version of the so called internal model principle in the behavioral setting. That is, in order to achieve regulation of the variable w subject to all exogenous signals $v \in \mathcal{E}$, the plant $\mathcal{P}$ must contain the dynamics of $\mathcal{E}$, expressed by $\mathcal{E} \subseteq \mathcal{N}_v(\mathcal{P})$.

As regulation is an asymptotic condition, intuitively the stable part of the exosystem does not affect regulation. From the following Theorem, we in fact show that we can reduce the general problem to the case when the exosystem is anti-stable.

Theorem 4.8: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w+c+v}}$ and $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$. Let $\mathcal{E} = \mathcal{E}_{\mathrm{s}} \oplus \mathcal{E}_{\mathrm{a}}$ where $\mathcal{E}_{\mathrm{s}} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$ is stable and $\mathcal{E}_{\mathrm{a}} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$ is anti-stable. Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$. Then the following statements are equivalent.

1) $\mathcal{C}$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$.
2) $\mathcal{C}$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}_{\mathrm{a}}$.

Proof: We skip the proof due to space limitations. □

Using Theorems 4.5 and 4.8, in the remaining paper with out loss of generality we make the following assumptions on exosystem $\mathcal{E}$.

Assumptions :

A1. $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$ is an anti-stable system, and

A2. v is observable from (w, c) in $\mathcal{P} \wedge_v \mathcal{E}$, i.e., $\mathcal{E} \cap \mathcal{N}_v(\mathcal{P}) = 0$.

The following Theorem is the main result of this paper. It now provides a solution to Problem.

Theorem 4.9: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$ with system variable (w, c, v). Assume v is free in $\mathcal{P}$. Let $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\text{aut}}$ with system variable v satisfies the assumptions A1 and A2. Then there exists a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$ if and only if the following conditions hold

1) (w, v) is detectable from c in $\mathcal{P} \wedge_v \mathcal{E}$,
2) $\mathcal{N}_{(w,c)}(\mathcal{P})$ is stabilizable, and
3) there exists a polynomial matrix $X(\xi) \in \mathbb{R}[\xi]^{\mathtt{c}\times\mathtt{v}}$ such that for all $v \in \mathcal{E}$ we have $(0, X(\frac{\mathrm{d}}{\mathrm{d}t})v, v) \in \mathcal{P}$.

Proof:

Let $\mathcal{P}$ and $\mathcal{E}$ be given by minimal kernel representations

$$\mathcal{P} = \{(w, c, v) \mid R_1(\tfrac{\mathrm{d}}{\mathrm{d}t})w + R_2(\tfrac{\mathrm{d}}{\mathrm{d}t})c + R_3(\tfrac{\mathrm{d}}{\mathrm{d}t})v = 0\} \quad (3)$$

and

$$\mathcal{E} = \{v \mid V(\tfrac{\mathrm{d}}{\mathrm{d}t})v = 0\} \quad (4)$$

respectively.

(necessity)

Let $\mathcal{C} = \ker(C)$ be a minimal representation of a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$. Then from Definition 4.3 and using Theorem 4.2, $\mathcal{N}_{(w,c)}(\mathcal{P})$ is stabilizable. We have $\{(w, 0, v) \mid (w, 0, v) \in \mathcal{P} \wedge_v \mathcal{E}\} \subseteq \mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C}$. Therefore using Definition 4.3, for all $(w, 0, v) \in \mathcal{P} \wedge_v \mathcal{E}$ we have $\lim_{t\to+\infty}(w(t), 0) = 0$. Hence for all $(w, 0, v) \in \mathcal{P} \wedge_v \mathcal{E}$, w is a stable Bohl. As v is observable from (w, c) in $\mathcal{P} \wedge_v \mathcal{E}$, for all $(w, 0, v) \in \mathcal{P} \wedge_v \mathcal{E}$ and w stable Bohl we have v stable Bohl. Therefore for all $(w, 0, v) \in \mathcal{P} \wedge_v \mathcal{E}$ we have $\lim_{t\to+\infty}(w(t), v(t)) = 0$, in other words (w, v) is detectable from c in $\mathcal{P} \wedge_v \mathcal{E}$.

We have

$$\mathcal{P} \wedge_c \mathcal{C} = \ker \begin{pmatrix} R_1 & R_2 & R_3 \\ 0 & C & 0 \end{pmatrix}. \quad (5)$$

v is free in $\mathcal{P} \wedge_c \mathcal{C}$ and $\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C})$ stable implies that $\begin{pmatrix} R_1 & R_2 \\ 0 & C \end{pmatrix}$ is Hurwitz. There exists a unimodular matrix U such that $U\begin{pmatrix} R_1 & R_2 & R_3 \\ 0 & C & 0 \end{pmatrix} = \begin{pmatrix} \tilde{R}_{11} & 0 & \tilde{R}_{13} \\ \tilde{R}_{21} & \tilde{R}_{22} & \tilde{R}_{23} \end{pmatrix}$ where $\tilde{R}_{11}$ and $\tilde{R}_{22}$ are Hurwitz. Therefore from Proposition 2.1 we have

$$\mathcal{P} \wedge_c \mathcal{C} = \ker \begin{pmatrix} \tilde{R}_{11} & 0 & \tilde{R}_{13} \\ \tilde{R}_{21} & \tilde{R}_{22} & \tilde{R}_{23} \end{pmatrix}, \quad (6)$$

$$(\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} = \ker \begin{pmatrix} \tilde{R}_{11} & \tilde{R}_{13} \end{pmatrix}, \text{ and} \quad (7)$$

$$\mathcal{N}_v((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)}) = \ker(\tilde{R}_{13}). \quad (8)$$

From Proposition 3.5, as $\tilde{R}_{22}$ is Hurwitz and V is anti-Hurwitz there exists a solution $(X, \tilde{Y}_2)$ of the equation

$$\tilde{R}_{22}X + \tilde{R}_{23} = \tilde{Y}_2 V. \quad (9)$$

We have $\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} = \ker \begin{pmatrix} \tilde{R}_{11} & 0 & \tilde{R}_{13} \\ \tilde{R}_{21} & \tilde{R}_{22} & \tilde{R}_{23} \\ 0 & 0 & V \end{pmatrix}$. It is easy to see that $\begin{pmatrix} \tilde{R}_{11} & 0 & \tilde{R}_{13} \\ \tilde{R}_{21} & \tilde{R}_{22} & \tilde{R}_{23} \\ 0 & 0 & V \end{pmatrix}$ has full row rank. Then we have $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_{(w,v)} = (\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E} = \ker \begin{pmatrix} \tilde{R}_{11} & \tilde{R}_{13} \\ 0 & V \end{pmatrix}$. From Theorem 4.6 $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w = ((\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_{(w,v)})_w = ((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E})_w$ stable implies that $\mathcal{E} \subseteq \mathcal{N}_v((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)})$. Hence from Equations (8) and (4) there exists a polynomial matrix $\tilde{Y}_1$ such that

$$\tilde{R}_{13} = \tilde{Y}_1 V. \quad (10)$$

Using Equations (9) and (10) we have

$$\begin{pmatrix} 0 \\ \tilde{R}_{22} \end{pmatrix} X + \begin{pmatrix} \tilde{R}_{13} \\ \tilde{R}_{23} \end{pmatrix} = \begin{pmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \end{pmatrix} V. \quad (11)$$

Multiplying both sides with U^{-1} in the above equation we obtain

$$\begin{pmatrix} R_2 \\ C \end{pmatrix} X + \begin{pmatrix} R_3 \\ 0 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} V \quad (12)$$

where $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} := U^{-1} \begin{pmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \end{pmatrix}$. Then we have

$$R_2 X + R_3 = Y_1 V. \quad (13)$$

Since $\mathcal{E} = \ker(V)$, for all $v \in \mathcal{E}$ we then have $\begin{pmatrix} R_2 & R_3 \end{pmatrix} \begin{pmatrix} X(\frac{\mathrm{d}}{\mathrm{d}t})v \\ v \end{pmatrix} = 0$, i.e., $(0, X(\frac{\mathrm{d}}{\mathrm{d}t})v, v) \in \mathcal{P}$.

(sufficiency)

Let $\mathcal{P}$ be given by the Equation (3). There exists a unimodular matrix U such that $U\begin{pmatrix} R_1 & R_2 & R_3 \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \end{pmatrix}$, where R_{11} has full row rank. Therefore from Proposition 2.1 we have

$$\mathcal{P} = \ker \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \end{pmatrix}, \quad (14)$$

$$\mathcal{N}_{(w,c)}(\mathcal{P}) = \ker \begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{pmatrix}, \quad (15)$$

$$(\mathcal{N}_{(w,c)}(\mathcal{P}))_c = \ker(R_{22}), \quad (16)$$

and

$$\mathcal{P} \wedge_v \mathcal{E} = \ker \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \\ 0 & 0 & V \end{pmatrix}. \quad (17)$$

There exists a polynomial matrix $X(\xi) \in \mathbb{R}[\xi]^{\mathtt{c}\times\mathtt{v}}$ such that for all $v \in \mathcal{E}$ $(0, X(\frac{\mathrm{d}}{\mathrm{d}t})v, v) \in \mathcal{P}$. Hence $V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0$ implies $\begin{pmatrix} R_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{pmatrix} X(\frac{\mathrm{d}}{\mathrm{d}t})v + \begin{pmatrix} R_{13}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ R_{23}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{pmatrix} v = 0$. Therefore there exists a polynomial matrix $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ such that

$$\begin{pmatrix} R_{12} \\ R_{22} \end{pmatrix} X + \begin{pmatrix} R_{13} \\ R_{23} \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} V. \quad (18)$$

This implies

$$R_{22}X + R_{23} = Y_2 V. \quad (19)$$

From Equation (15), $\mathcal{N}_{(w,c)}(\mathcal{P})$ stabilizable implies that $\begin{pmatrix} R_{11}(\lambda) & R_{12}(\lambda) \\ 0 & R_{22}(\lambda) \end{pmatrix}$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+$, which in turn implies that $R_{22}(\lambda)$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+$. From Equation (16) we conclude that $(\mathcal{N}_{(w,c)}(\mathcal{P}))_c$ stabilizable. From Proposition 3.3 there exists a $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ such that $(\mathcal{N}_{(w,c)}(\mathcal{P}))_c \cap \mathcal{C}$ is stable and regular. Factor R_{22} as $R_{22} = DK$ where D is Hurwitz and $K(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Let S be such that $\begin{pmatrix} K \\ S \end{pmatrix}$ is unimodular. Then for an arbitrary polynomial matrix F and an arbitrary Hurwitz polynomial matrix H of suitable dimensions, it is easy to verify that

$$C = FR_{22} + HS \quad (20)$$

serves as a stabilizing controller for $(\mathcal{N}_{(w,c)}(\mathcal{P}))_c$. Note that $\begin{pmatrix} R_{22} \\ C \end{pmatrix}$ is Hurwitz for all C given by the Equation (20). From Equation (17), (w, v) is detectable from c in $\mathcal{P} \wedge_v \mathcal{E}$ implies that $\begin{pmatrix} R_{11}(\lambda) & R_{13}(\lambda) \\ 0 & R_{23}(\lambda) \\ 0 & V(\lambda) \end{pmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$. This implies that R_{11} is square nonsingular and Hurwitz and $\begin{pmatrix} R_{23}(\lambda) \\ V(\lambda) \end{pmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$. As $V(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}^-$ (use the fact that V is anti-Hurwitz) we conclude that $\begin{pmatrix} R_{23}(\lambda) \\ V(\lambda) \end{pmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$. Hence there always exists a solution (F, M) for the equation

$$FR_{23} + MV = HSX. \tag{21}$$

We now prove that any controller given by $\mathcal{C} = \ker(C)$ where $C = FR_{22} + HS$ with F satisfying the Equation (21) acts as a regulator. The following identities hold true.

$$\begin{aligned} CX &= FR_{22}X + HSX \\ &= FR_{22}X + FR_{23} + MV \text{ (from Equation (21))} \\ &= FY_2V + MV\text{(from Equation (19))}. \end{aligned}$$

We have

$$CX = WV, \tag{22}$$

where $W := FY_2 + M$. We have

$$\mathcal{P} \wedge_c \mathcal{C} = \ker \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \\ 0 & C & 0 \end{pmatrix}, \tag{23}$$

$$\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C}) = \ker \begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \\ 0 & C \end{pmatrix}. \tag{24}$$

As C is chosen such that $\begin{pmatrix} R_{22} \\ C \end{pmatrix}$ is Hurwitz, we have $\begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \\ 0 & C \end{pmatrix}$ square, nonsingular and Hurwitz. Therefore from Equation (23), the interconnection $\mathcal{P} \wedge_c \mathcal{C}$ is regular, from Equation (24), $\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C})$ is stable, and also from Proposition 2.3, v is free in $\mathcal{P} \wedge_c \mathcal{C}$. We have

$$\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} = \left\{ (w,c,v) \;\middle|\; \begin{array}{l} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t})w + R_{12}(\frac{\mathrm{d}}{\mathrm{d}t})c + R_{13}(\frac{\mathrm{d}}{\mathrm{d}t})v = 0, \\ R_{22}(\frac{\mathrm{d}}{\mathrm{d}t})c + R_{23}(\frac{\mathrm{d}}{\mathrm{d}t})v = 0, \\ C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0, \qquad V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0 \end{array} \right\}.$$

Substituting Equation (18) into the above equation yields

$$\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} = \left\{ (w,c,v) \;\middle|\; \begin{array}{l} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t})w + R_{12}(\frac{\mathrm{d}}{\mathrm{d}t})(c - X(\frac{\mathrm{d}}{\mathrm{d}t})v) \\ \qquad + Y_1V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0, \\ R_{22}(\frac{\mathrm{d}}{\mathrm{d}t})(c - X(\frac{\mathrm{d}}{\mathrm{d}t})v) + Y_2V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0, \\ C(\frac{\mathrm{d}}{\mathrm{d}t})(c - X(\frac{\mathrm{d}}{\mathrm{d}t})v) + CX(\frac{\mathrm{d}}{\mathrm{d}t})v = 0, \\ V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0 \end{array} \right\}.$$

It further follows from Equation (22) that

$$\begin{aligned} &\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} \\ &= \left\{ (w,c,v) \;\middle|\; \begin{array}{l} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t})w + R_{12}(\frac{\mathrm{d}}{\mathrm{d}t})(c - X(\frac{\mathrm{d}}{\mathrm{d}t})v) \\ \qquad + Y_1V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0, \\ R_{22}(\frac{\mathrm{d}}{\mathrm{d}t})(c - X(\frac{\mathrm{d}}{\mathrm{d}t})v) + Y_2V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0, \\ C(\frac{\mathrm{d}}{\mathrm{d}t})(c - X(\frac{\mathrm{d}}{\mathrm{d}t})v) + WV(\frac{\mathrm{d}}{\mathrm{d}t})v = 0, \\ V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0 \end{array} \right\}. \\ &= \left\{ (w,c,v) \;\middle|\; \begin{array}{l} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t})w + R_{12}(\frac{\mathrm{d}}{\mathrm{d}t})(c - X(\frac{\mathrm{d}}{\mathrm{d}t})v) = 0, \\ R_{22}(\frac{\mathrm{d}}{\mathrm{d}t})(c - X(\frac{\mathrm{d}}{\mathrm{d}t})v) = 0, \\ C(\frac{\mathrm{d}}{\mathrm{d}t})(c - X(\frac{\mathrm{d}}{\mathrm{d}t})v) = 0, \; V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0 \end{array} \right\}. \end{aligned}$$

From the above, we see that, for all $(w, c, v) \in \mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C}$, $(w, c - X(\frac{\mathrm{d}}{\mathrm{d}t})v)$ belongs to $\ker \begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \\ 0 & C \end{pmatrix}$.

Since $\begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \\ 0 & C \end{pmatrix}$ is Hurwitz, $\lim_{t \to +\infty}(w(t), c(t) - X(\frac{\mathrm{d}}{\mathrm{d}t})v(t)) = 0$ holds for all $(w, c, v) \in \mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C}$. This clearly implies that $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w$ is stable. □

V. Conclusion

In this paper we have formulated and resolved the problem of asymptotic tracking and regulation in a completely representation free manner. We used the theory of behavioral control for this purpose. In the behavioral context, controllers act on the plant using general interconnection, without a priori input-output considerations. Given a plant and exosystem, we have established necessary and sufficient conditions for the existence of a regulator only in terms of the plant and exosystem dynamics.

References

[1] M.N. Belur and H.L. Trentelman, "Stabilization, pole placement and regular implementability," *IEEE Transactions on Automatic Control*, Vol. 47, nr. 5, pp. 735 - 744, 2002.

[2] M.N. Belur, *Control in a Behavioral Context*, Doctoral Dissertation, University of Groningen, The Netherlands, 2003.

[3] E.J. Davison and A. Goldenberg, " The robust control of a general servomechanism problem: the servo compensator," *Automatica* , 11, pp. 461-471, 1975.

[4] S. Fiaz, K. Takaba and H.L. Trentelman, "Tracking and regulation in the behavioral framework", *Proceedings of Mathematical Theory of Networks and System*, Budapest, Hungary, 2010.

[5] B.A. Francis, "The linear multivariable regulator problem, *SIAM Journal on Control and Optimization*, Vol. 15, No.3, pp. 486-505, 1977.

[6] B.A. Francis and W.M. Wonham, "The internal model principle for linear multivariable regulators," *Applied mathematics and optimization*, Vol. 2, No.2, pp. 170-194, 1975.

[7] J.W. Polderman and J.C. Willems, *Introduction to Mathematical Systems Theory: a Behavioral Approach*, Springer-Verlag, Berlin, 1997.

[8] A. Saberi, A.A. Stoorvogel, and P. Sannuti, *Control of linear systems with regulation and input constraints*, Communication and Control Engineering Series, Springer Verlag, 2000.

[9] K. Takaba, "A note on the regulation problem in the behavioral Framework", *Proceedings of ICROS-SICE International Joint Conference 2009*, Fukuoka, Japan, August 18-21, 2009.

[10] J.C. Willems, "On interconnection, control, and feedback," *IEEE Transactions on Automatic Control*, Vol. 42, pp. 326-339, 1997.

[11] J.C. Willems and H.L. Trentelman, "Synthesis of dissipative systems using quadratic differential forms - part I," *IEEE Transactions on Automatic Control*, Vol. 47, nr. 1, pp. 53 - 69, 2002.