# Closedness and Adjoints of Products of Operators, and Compressions

Azizov, T.Ya.; Dijksma, A.

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**Integral Equations and Operator Theory**

# Closedness and Adjoints of Products of Operators, and Compressions

T. Ya. Azizov and A. Dijksma

**Abstract.** We reprove and slightly improve theorems of Nudelman and Stenger about compressions of maximal dissipative and self-adjoint operators to subspaces of finite codimension and discuss related results concerning the closedness and the adjoint of a product of two operators on a Hilbert space.

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**Keywords.** Banach space, Hilbert space, Krein space, conjugate, adjoint, closure, product, compression, symmetric, self-adjoint, dissipative, maximal dissipative, polar decomposition, dense set, codimension.

## 1. Introduction

The motivation to begin the research for this note is a recent result of Nudelman [19] of 2011: *The compression of a densely defined maximal dissipative operator in a Hilbert space to a subspace of finite codimension is densely defined maximal dissipative.* It is a generalization of an older theorem of Stenger [21] of 1968 : *The compression of a self-adjoint operator in a Hilbert space to a subspace of finite codimension is self-adjoint.* For another proof of this theorem we refer to [12, Lemma 1]. Shortly after the publication of Stenger's paper, in reaction to this paper, there appeared a number of papers dealing with the closely related questions: *When is the product of two closed operators closed*? and *When is the adjoint of a product of two operators the reverse product of the adjoints of the operators*? We mention here the papers [4,5,12–14,20] from the period 1968–1972. Earlier results are contained in the papers [6,15] of 1963 and in the book [11] of 1966, where more references can be found. Later, in 1976/77, a detailed analysis related to the second question appeared in [8]. In this paper we reprove and slightly improve the theorems of Nudelman and Stenger, see Theorem 3.2 and 3.3 in Sect. 3, and reprove for Hilbert space operators answers to the above mentioned questions, see Theorem 4.1 and 4.3 in Sect. 4. Theorem 4.1 deals with the first question and is a special case of [11, Theorem IV.2.7(i)] due to Goldberg. Theorem 4.3

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concerns the second question and is a special case of the theorem in [20] due to Schechter. Our proofs of these theorems are based on results from Sect. 2, which we think are new. We make use of the polar decomposition of an operator. This notion was previously used in the proofs of [5, Theorem 12 and Theorem 13] and [4, Corollary 2] which state sufficient conditions, different from the ones in Theorem 4.3, under which the equality $(ST)^* = T^*S^*$ holds, where $S$ and $T$ are densely defined operators on a Hilbert space and for example $S^*$ denotes the adjoint of $S$.

In Sect. 2 we prove two theorems, Theorems 2.2 and 2.4, about a spectral connection between an operator and its compression to a space of finite codimension and we prove a theorem, Theorem 2.3, concerning the closedness and the adjoint of a product of two operators. These results are formulated in a Banach space setting and are applied in the last two sections which both deal with operators on spaces with an inner product. In Sect. 3 we prove the theorems of Nudelman and Stenger in the format of if and only if statements and their Krein space analogs. Nudelman's result is a direct consequence of Theorem 2.2. In the proofs of the theorems in Sect. 4 we apply Theorems 2.3 and 2.4.

In another note [3] we plan to discuss the theorems of Nudelman and Stenger using an associated kernel function and to generalize them to linear relations.

In the paper $E\dot{+}F$ stands for the direct sum of two linear spaces $E$ and $F$, $\sigma_p(T)$ and $\rho(T)$ for the point spectrum and the resolvent set of an operator $T$, and $\overline{D}$ and $\overline{T}$ for the closure of a set $D$ and of a closable operator $T$. Further, $\operatorname{dom} T$, $\operatorname{ran} T$ and $\ker T$ stand for domain, range and kernel (null space) of an operator $T$, $T'$ denotes the conjugate of an operator $T$ on a Banach space and a subspace is a closed linear subset.

## 2. Preliminaries

**Lemma 2.1.** *Let $E$ be a Banach space and let $P$ be a projection in $E$ such that* $\operatorname{codim}\operatorname{ran} P =: \varkappa < \infty$. *For a subspace $\widetilde{L}$ of $E$ the following statements are equivalent.*

($i$) *There is a subspace $L \subset E$ with $L \subset \widetilde{L}$,* $\operatorname{codim} L = \varkappa$ *and* $L \cap \ker P = \{0\}$.

($ii$) $P\widetilde{L} = \operatorname{ran} P$.

*Proof.* Assume ($i$). Then $E = L \dot{+} \ker P$ and

$$\operatorname{ran} P = PE = P(L \dot{+} \ker P) = PL \subset P\widetilde{L} \subset \operatorname{ran} P,$$

whence ($ii$).

Assume ($ii$). Denote by $L$ a direct complement of the finite-dimensional subspace $\widetilde{L} \cap \ker P$ in $\widetilde{L}$: $\widetilde{L} = (\widetilde{L} \cap \ker P) \dot{+} L$ (see [11, Theorem II.1.16]). Then

$$L \cap \ker P = L \cap (\widetilde{L} \cap \ker P) = \{0\}.$$

Since $\dim \ker P = \varkappa$, this implies $\operatorname{codim} L \geq \varkappa$. We assume $\operatorname{codim} L > \varkappa$ and derive a contradiction. The assumption implies $\ker P + L \neq E$ and therefore there is a nonzero $y_0 \in E \setminus (\ker P + L)$. From

$$Py_0 \in \operatorname{ran} P \overset{(ii)}{=} P\widetilde{L} = P((\widetilde{L} \cap \ker P) \dot{+} L) = PL$$

we obtain the contradiction

$$0 \neq y_0 \in (\ker P + L) \cap (E \setminus (\ker P + L)) = \{0\}.$$

We conclude that $\operatorname{codim} L = \varkappa$. This proves $(i)$. □

**Theorem 2.2.** *Let $A$ be a linear operator on a Banach space $E$. Let $P$ be a projection in $E$ such that* $\operatorname{codim} \operatorname{ran} P < \infty$ *and let $B$ be the compression of $A$ to* $\operatorname{ran} P$*: $B = PA|_{\operatorname{ran} P \cap \operatorname{dom} A}$. Then*

$$0 \in \rho(A) \text{ and } 0 \notin \sigma_p(B) \Rightarrow 0 \in \rho(B).$$

In this theorem $A$ need not be densely defined.

*Proof of Theorem 2.2.* To show $0 \in \rho(B)$ it suffices to show that (i) $\operatorname{ran} B = \operatorname{ran} P$ and (ii) $B$ is a closed operator on $\operatorname{ran} P$. For these two items and the hypothesis $0 \notin \sigma_p(B)$ imply that $0 \in \rho(B)$. We set $\varkappa = \dim \ker P = \operatorname{codim} \operatorname{ran} P$.

(i) We assume that $\dim(\operatorname{dom} A/\operatorname{dom} B) > \varkappa$ and derive a contradiction. The assumption implies that there is a $(\varkappa + 1)$-dimensional subspace $L_0 \subset \operatorname{dom} A$ such that $L_0 \cap \operatorname{dom} B = \{0\}$. Since $\dim L_0 > \operatorname{codim} \operatorname{ran} P$, we have $L_0 \cap \operatorname{ran} P \neq \{0\}$ which leads to the contradiction

$$\{0\} \neq L_0 \cap \operatorname{ran} P = L_0 \cap \operatorname{dom} B = \{0\}.$$

We conclude that $\varkappa_1 := \dim(\operatorname{dom} A/\operatorname{dom} B) \leq \varkappa$. Therefore there is a subspace $L_1 \subset \operatorname{dom} A$ such that $\dim L_1 = \varkappa_1$ and $\operatorname{dom} A = \operatorname{dom} B \dot{+} L_1$. Define the operator

$$A_1 := A|_{\operatorname{ran} P \cap \operatorname{dom} A} = A|_{\operatorname{dom} B}.$$

Since $0 \in \rho(A)$, we have $E = \operatorname{ran} A_1 \dot{+} AL_1$ and $\dim AL_1 = \varkappa_1$, and hence $\operatorname{ran} A_1$ has codimension $\varkappa_1$. We show that $\varkappa_1 = \varkappa$ by assuming $\varkappa_1 < \varkappa$ and deriving a contradiction. The assumption implies that there is a nonzero $y_0 \in \operatorname{ran} A_1 \cap \ker P$ and therefore there is a nonzero $x_0 \in \operatorname{dom} A_1$ such that $y_0 = A_1 x_0$. Thus $Bx_0 = PA_1x_0 = Py_0 = 0$ which shows that $x_0$ is a nonzero eigenelement of $B$ with eigenvalue 0, contradicting the hypothesis that $0 \notin \sigma_p(B)$. Hence $\operatorname{codim} \operatorname{ran} A_1 = \varkappa$ and $\operatorname{ran} A_1 \cap \ker P = \{0\}$. From Lemma 2.1 with $\widetilde{L} = L = \operatorname{ran} A_1$ it follows that

$$\operatorname{ran} B = P \operatorname{ran} A_1 = \operatorname{ran} P.$$

(ii) We first show that the operator $A_1$ is closed. Let $x_n \in \operatorname{dom} A_1$ and assume $x_n \to x_0$, $A_1 x_n = Ax_n \to y_0$ as $n \to \infty$. Since $A$ is a closed operator, $x_0 \in \operatorname{dom} A$ and $y_0 = Ax_0$. From $x_n \in \operatorname{ran} P$ and the fact that $\operatorname{ran} P$ is closed, we obtain $x_0 \in \operatorname{ran} P$, that is, $x_0 \in \operatorname{dom} A_1$ and $y_0 = A_1 x_0$. Hence $A_1$ is closed. Since, as shown in (i), $\ker P|_{\operatorname{ran} A_1} = \{0\}$ and $P \operatorname{ran} A_1 = \operatorname{ran} P$, the operator $P|_{\operatorname{ran} A_1} : \operatorname{ran} A_1 \to \operatorname{ran} P$ is bounded and boundedly invertible. Hence $B = PA_1$ is closed. □

The first statement in the next theorem is applied in Sect. 4, see the proof of Theorem 4.1.

**Theorem 2.3.** *Under the assumptions of Theorem 2.2, including the hypotheses* $0 \in \rho(A)$ *and* $0 \notin \sigma_p(B)$*, the operator* $PA$ *is closed. If moreover* $A$ *is densely defined and* $E$ *is a reflexive Banach space, then* $(A'P')' = PA$.

*Proof.* First we show that $PA$ is closed. As shown in the proof of Theorem 2.2, there is a $\varkappa$-dimensional subspace $L_1 \subset \operatorname{dom} A$ such that $\operatorname{dom} A = \operatorname{dom} B \dot{+} L_1$. Since $\operatorname{ran} P \cap L_1 \subset \operatorname{dom} B \cap L_1 = \{0\}$, we have $E = \operatorname{ran} P \dot{+} L_1$. Let $Q$ be the projection onto $\operatorname{ran} P$ parallel to $L_1$. Then the operator $PA(I-Q)$ is bounded, and the operator $PAQ$ is closed because $PAQ = BQ$ and the operator $B$ is a closed. Hence $PA = PAQ + PA(I-Q)$ is closed. This proves the first statement. The assumption that $A$ is densely defined implies that the conjugate $A'$ is well defined. Since $P$ is bounded, $A'P' = (PA)'$. The closedness of $PA$ and the assumption that $E$ is reflexive imply that $PA = (PA)''$. Hence $PA = (A'P')'$. □

Denote by $r(A)$ the set of points of regular type of a closed operator $A$, that is, $\lambda \in r(A)$ if $\ker(A - \lambda) = \{0\}$ and $\operatorname{ran}(A-\lambda) = \overline{\operatorname{ran}(A-\lambda)}$. For $\lambda \in r(A)$ the number $\operatorname{def}_\lambda A := \operatorname{codim} \operatorname{ran}(A - \lambda)$ is called the defect of $A$ in $\lambda$. In particular, $\rho(A) \subset r(A)$ and $\lambda \in r(A)$ is a regular point for $A$ if and only if $\operatorname{def}_\lambda A = 0$.

**Theorem 2.4.** *Let* $A$ *be a closed densely defined linear operator on a Banach space* $E$*. Let* $P$ *be a projection in* $E$ *such that* $\operatorname{codim} \operatorname{ran} P < \infty$ *and let* $B$ *be the compression of* $A$ *to* $\operatorname{ran} P$*. Then*

$$0 \notin \sigma_p(A) \text{ and } 0 \in \rho(B) \Rightarrow 0 \in \rho(A).$$

*Proof.* We use the same notation as in the proof of Theorem 2.2. Recall $A_1 = A|_{\operatorname{dom} B}$ and $\varkappa = \operatorname{codim} \operatorname{ran} P$. Since $0 \in \rho(B)$, the range $\operatorname{ran} A_1$ is closed in $E$ and the operator $P|_{\operatorname{ran} A_1}$ is a bijection from $\operatorname{ran} A_1$ onto $\operatorname{ran} P$. It follows that $E = \ker P \dot{+} \operatorname{ran} A_1$ and $\operatorname{codim} \operatorname{ran} A_1 = \varkappa$. The inclusion $\operatorname{ran} A_1 \subset \operatorname{ran} A$ implies that $\operatorname{ran} A$ is closed. Hence, by the assumption that $\ker A = \{0\}$, we have $0 \in r(A)$ and

$$\operatorname{def}_0 A = \operatorname{codim} \operatorname{ran} A \leq \operatorname{codim} \operatorname{ran} A_1 = \varkappa.$$

To show that $0 \in \rho(A)$ it suffices to show that $\operatorname{def}_0 A = 0$. We prove this equality by showing that the assumption $\operatorname{def}_0 A > 0$ yields a contradiction. The assumption implies that there a subspace $D$ with $\dim D = \varkappa - \operatorname{def}_0 A < \varkappa$ such that $\operatorname{ran} A = D \dot{+} \operatorname{ran} A_1$ and hence such that

$$\operatorname{dom} A = A^{-1} D \dot{+} \operatorname{dom} A_1.$$

From $\operatorname{codim} \operatorname{ran} P = \varkappa$ it follows that there is a $\varkappa$-dimensional subspace $E_0 \subset E'$ orthogonal to $\operatorname{ran} P$, which means that for all functionals $e' \in E_0$ we have $e'(Px) = 0,\ x \in E$. The inclusion $\operatorname{dom} A_1 \subset \operatorname{ran} P$ implies $E_0$ is also orthogonal to $\operatorname{dom} A_1$. Since $\dim A^{-1} D = \dim D < \varkappa$, there is a nonzero element $e' \in E_0$ orthogonal to $A^{-1}D$, hence $e'(\operatorname{dom} A) = \{0\}$. Since $A$ is densely defined, $e'(E) = \{0\}$. Thus we have obtained the contradiction that the nonzero element $e'$ is zero. □

*Remark* 2.5. If we do not suppose that the operator $A$ is densely defined, then the implication in Theorem 2.4 does not hold. This is clear from the proof of the theorem, if we take $A = A|_{\operatorname{ran} P \cap \operatorname{dom} A}$ with $P \neq I$.

In the sequel we use the following lemmas.

**Lemma 2.6.** *If $D$ is a dense linear subset of a Banach space $E$ and $G$ is a closed linear subset of $E$ of finite codimension, then $D \cap G$ is dense in $G$.*

**Lemma 2.7.** *If $T$ is a closed densely defined operator on a Banach space, then* $\operatorname{ran} T$ *is closed if and only if* $\operatorname{ran} T'$ *is closed.*

For proofs of the first lemma see [9, Lemma 2.1] or [11, Lemma IV.2.8] and for proofs of the second lemma see [16, Lemma 324], [18, Theorem 5.1] or [11, Theorem IV.1.2].

## 3. Compressions

An operator $T$ on a Hilbert space with inner product $(\cdot, \cdot)$ is called *dissipative* if $\operatorname{Im}(Tf, f) \geq 0$, $f \in \operatorname{dom} T$, and it is *maximal dissipative* if it is not properly contained in another dissipative operator. For the following lemma we refer to [2, Corollary 2.2.5 and Lemma 2.2.8], see also [17, Subsection V.3.10].

**Lemma 3.1.** *For a densely defined operator $T$ in a Hilbert space the following statements are equivalent.*

(1) *$T$ is maximal dissipative.*
(2) *$T$ is dissipative and $\rho(T) \cap C_- \neq \emptyset$.*
(3) *$T$ is dissipative and $\mathbb{C}_- \subset \rho(T)$.*

The implication $(i) \Rightarrow (ii)$ in the theorem below is due to Nudelman [19]. It is a direct consequence of Theorem 2.2. The implication $(ii) \Rightarrow (i)$ seems to be new and follows from Theorem 2.4.

**Theorem 3.2.** *Let $T$ be a closed densely defined dissipative operator in a Hilbert space $E$. Let $P$ be an orthogonal projection in $E$ with* $\operatorname{codim} \operatorname{ran} P < \infty$ *and let $S$ be the compression of $T$ to* $\operatorname{ran} P$: $S = PT|_{\operatorname{ran} P \cap \operatorname{dom} T}$. *Then*

$(i)$ *$T$ is maximal dissipative in $E$* $\Leftrightarrow$ $(ii)$ *$S$ is maximal dissipative in* $\operatorname{ran} P$.

*Proof.* We fix a complex number $\lambda$ with $\operatorname{Im} \lambda < 0$ and set $A := T - \lambda$ and $B := PA|_{\operatorname{ran} P \cap \operatorname{dom} A} = S - \lambda$. Since $T$ is closed, $A$ is closed; since $T$ is dissipative, $0 \notin \sigma_p(A)$. On account of Lemma 2.6, $S$ is densely defined; since $S$ is dissipative, $0 \notin \sigma_p(B)$.

Assume $(i)$. Then, by Lemma 3.1, $\lambda \in \rho(T)$, that is, $0 \in \rho(A)$. Theorem 2.2 implies $0 \in \rho(B)$. Hence $\lambda \in \rho(S)$ and Lemma 3.1 implies $(ii)$.

Assume $(ii)$. Then, by Lemma 3.1, $\lambda \in \rho(S)$, hence $0 \in \rho(B)$. Theorem 2.4 then implies, $0 \in \rho(A)$, that is, $\lambda \in \rho(T)$. Lemma 3.1 implies $(i)$. □

A densely defined operator $T$ on a Hilbert space is called *symmetric* if $T \subset T^*$, it is called *self-adjoint* if equality prevails. The only if statement in the next theorem is due to Stenger [21].

**Theorem 3.3.** *Let $T$ be a closed densely defined symmetric operator in a Hilbert space $E$. Let $P$ be an orthogonal projection in $E$ with* $\operatorname{codim}\operatorname{ran} P < \infty$ *and let $S$ be the compression of $T$ to* $\operatorname{ran} P$. *Then $T$ is self-adjoint in $E$ if and only if $S$ is self-adjoint in* $\operatorname{ran} P$.

*Proof.* The theorem immediately follows from Theorem 3.2 because an operator $T$ is self-adjoint if and only if both $T$ and $-T$ are maximal dissipative. □

Theorems 3.2 and 3.3 also hold in a Krein space setting. We assume the reader is familiar with operator theory in spaces with an indefinite metric as in [2], see also [1,7].

**Theorem 3.4.** *Let $T$ be a closed densely defined dissipative (symmetric) operator in a Krein space $E$. Let $P$ be an orthogonal projection in $E$ with* $\operatorname{codim}\operatorname{ran} P < \infty$ *and let $S$ be the compression of $T$ to* $\operatorname{ran} P$. *Then $T$ is maximal dissipative (self-adjoint) in $E$ if and only if $S$ is maximal dissipative (self-adjoint) in* $\operatorname{ran} P$.

*Proof.* Denote by $[\,\cdot\,,\,\cdot\,]$ the indefinite inner product on $E$. Let $J$ be a fundamental symmetry on $E$ such that $J|_{\operatorname{ran} P}$ is a fundamental symmetry on $\operatorname{ran} P$ or, equivalently, such that $PJ = JP$. Then in the inner product $(x, y) := [Jx, y]$ $E$ and $\operatorname{ran} P$ are Hilbert spaces and $P$ is the Hilbert space orthogonal projection in $E$ onto $\operatorname{ran} P$. Since $T$ is dissipative (maximal dissipative, self-adjoint) in the Krein space $E$ if and only if $JT$ is dissipative, (maximal dissipative, self-adjoint) in the Hilbert space $E$ and $S$ is dissipative (maximal dissipative, self-adjoint) in the Krein space $\operatorname{ran} P$ if and only if $JS$ is dissipative, (maximal dissipative, self-adjoint) in the Hilbert space $\operatorname{ran} P$, the theorem follows directly from Theorem 3.2 and 3.3 and the equalities

$$JS = JPT|_{\operatorname{ran} P\cap \operatorname{dom} T} = PJT|_{\operatorname{ran} P\cap \operatorname{dom} JT}.$$

□

The above theorems can be generalized to linear relations (multi-valued operators). This will be proved in another note [3].

## 4. Closedness and Adjoints of Operator Products

In the proofs of the theorems in this section the polar decomposition of an operator plays a key role. Recall (see for example [17,22]) that the polar decomposition of a closed densely defined operator $T$ in a Hilbert space is the factorization $T = U|T|$, where $|T| = \sqrt{T^*T}$ and $U$ is the partial isometry with initial space $(\ker U)^\perp = \overline{\operatorname{ran}|T|}$ and final space $\operatorname{ran} U = \overline{\operatorname{ran} T}$.

The following theorem is essentially a Hilbert space version of [11, Theorem IV.2.7(i)] or [10, Proposition XVII.3.2]. We give a different proof.

**Theorem 4.1.** *Let $S$ and $T$ be closed densely defined operators on a Hilbert space. If* $\operatorname{ran} S$ *is closed and* $\dim\ker S < \infty$, *then $ST$ is a closed operator.*

*Proof.* Let $S = U|S|$ and $T^* = V|T^*|$ be the polar decompositions of $S$ and $T^*$. Let $P$ be the orthogonal projection onto $\operatorname{ran} S^*$ which, by Lemma 2.7, is closed. Then $S = SP = U|S|P$ and hence $ST = (U|S|)(P|T^*|)V^*$. We claim that $P|T^*|$ is closed. If the claim is true, then, since $V^*$ is bounded and $(U|S|)|_{\operatorname{ran} P}$ is boundedly invertible, the above equality implies that the operator $ST$ is closed. It remains to prove the claim. For that we apply Theorem 2.3 with $A = |T^*| + 1$ and $P$ as defined above. We verify the assumptions in the theorem: Since $(\operatorname{ran} P)^\perp = \ker S$, $\operatorname{codim} \operatorname{ran} P < \infty$. $A$ is a closed operator defined with dense domain $\operatorname{dom} A = \operatorname{dom} T^*$ and, since $|T^*|$ is nonnegative, we have $0 \in \rho(A)$. We assume that $0 \in \sigma_p(B)$ and derive a contradiction. The assumption implies that there is a nonzero $x \in \operatorname{dom} B$ such that $Bx = PAx = 0$ or, equivalently, $P|T^*|x = -Px$. Denote by $(\,\cdot\,,\,\cdot\,)$ the inner product in the Hilbert space. Then, since $\operatorname{dom} B \subset \operatorname{ran} P$ and $x \neq 0$, we obtain the contradiction:

$$0 \le (|T^*|x, x) = (P|T^*|x, x) = -(Px, x) = -(x, x) < 0.$$

This implies $0 \notin \sigma_p(B)$. Thus the conditions of Theorem 2.3 are satisfied and hence, by the first statement in this theorem, the operator $PA = P(|T^*| + 1)$ is closed. This readily implies that $P|T^*|$ is closed. □

**Lemma 4.2.** *Let $A$ and $B$ be densely defined operators on a Hilbert space such that the product $AB$ is also densely defined. Then*

$$(AB)^* = B^* A^* \tag{4.1}$$

*if $B$ satisfies one of the following conditions*:

(*a*) $0 \in \rho(B)$.
(*b*) *$B$ is a partial isometry with* $\dim \mathit{ker} B^* < \infty$ *and* $\ker B^* \subset \ker A$.
(*c*) *$B$ is an orthogonal projection with dim* $\ker B < \infty$.

*Proof.* Since $A$, $B$ and $AB$ are densely defined, their adjoints $A^*$, $B^*$ and $(AB)^*$ are well defined operators and we have $B^*A^* \subset (AB)^*$. Let $E$ be the Hilbert space in which $A$ and $B$ act and denote by $(\,\cdot\,,\,\cdot\,)$ and $\|\cdot\|$ the inner product and corresponding norm of $E$.

(*a*) To prove (4.1) it suffices to show $\operatorname{dom}(AB)^* \subset \operatorname{dom} B^*A^*$. Let $y \in \operatorname{dom}(AB)^*$. This means, by definition, that the linear functional $x \mapsto (ABx, y)$ is continuous on $\operatorname{dom} AB$, that is, there exists a finite $c > 0$ such that:

$$|(ABx, y)| \le c\|x\|, \quad x \in \operatorname{dom} AB.$$

Since $0 \in \rho(B)$, we have $\operatorname{dom} AB = B^{-1}\operatorname{dom} A$, hence $B\operatorname{dom} AB = \operatorname{dom} A$ and $\|x\| = \|B^{-1}Bx\| \le \|B^{-1}\|\|Bx\|$. It follows that $y \in \operatorname{dom} A^*$, $(ABx, y) = (Bx, A^*y)$ and

$$|(Bx, A^*y)| \le c\|x\|, \quad x \in B^{-1}\operatorname{dom} A.$$

Since $\overline{\operatorname{dom} A} = E$ and $B^{-1}$ is continuous, this inequality can be extended by continuity to all $x \in \operatorname{dom} B$. Hence $A^*y \in \operatorname{dom} B^*$. So $y \in \operatorname{dom} B^*A^*$ and $\operatorname{dom}(AB)^* \subset \operatorname{dom} B^*A^*$. Thus (4.1) holds in this case.

($b$) Consider the matrix representations of $A$ and $B$ relative to the two orthogonal decompositions $E = \ker B \oplus \operatorname{ran} B^*$ and $E = \ker B^* \oplus \operatorname{ran} B$:

$$B = \begin{bmatrix} 0 & 0 \\ 0 & B_1 \end{bmatrix} : \begin{bmatrix} \ker B \\ \operatorname{ran} B^* \end{bmatrix} \to \begin{bmatrix} \ker B^* \\ \operatorname{ran} B \end{bmatrix},$$

where $B_1$ is bounded and boundedly invertible, and

$$A = \begin{bmatrix} 0 & A_{01} \\ 0 & A_1 \end{bmatrix} : \begin{bmatrix} \ker B^* \\ \operatorname{ran} B \end{bmatrix} \to \begin{bmatrix} \ker B \\ \operatorname{ran} B^* \end{bmatrix}.$$

Since $\operatorname{dom} A = \ker B^* \oplus (\operatorname{ran} B \cap \operatorname{dom} A)$, we have that $\operatorname{dom} A_{01} = \operatorname{dom} A_1 = (\operatorname{ran} B \cap \operatorname{dom} A)$. By Lemma 2.6, these domains are dense in $\operatorname{ran} B$. Thus $A_{01}B_1$ and $A_1B_1$ are densely defined and

$$(AB)^* = \begin{bmatrix} 0 & 0 \\ (A_{01}B_1)^* & (A_1B_1)^* \end{bmatrix} : \begin{bmatrix} \ker B \\ \operatorname{ran} B^* \end{bmatrix} \to \begin{bmatrix} \ker B \\ \operatorname{ran} B^* \end{bmatrix}.$$

Arguments similar to the ones in part ($a$) can be used to show that $(A_{01}B_1)^* = B_1^*A_{01}^*$ and $(A_1B_1)^* = B_1^*A_1^*$. These equalities imply (4.1).

($c$) As in ($a$) to prove (4.1) it suffices to show that $\operatorname{dom}(AB)^* \subset \operatorname{dom} BA^* = \operatorname{dom} A^*$. Let $g \in \operatorname{dom}(AB)^*$. This means, by definition, that the linear functional $f \mapsto (ABf, g)$ is continuous in $\operatorname{dom} AB$. Our aim is to check that the functional $f \mapsto (Af, g)$ is continuous in $\operatorname{dom} A$. Since $A$ is densely defined, Lemma 2.6 implies that $\operatorname{dom} A|_{\operatorname{ran} B}$ is dense in $\operatorname{ran} B$ and therefore there is a subspace $L \subset \operatorname{dom} A$ such that $\operatorname{dom} A = \operatorname{dom} A|_{\operatorname{ran} B} \dot{+} L$, $\dim L = \dim \ker B < \infty$ and $L \cap \operatorname{ran} B = \{0\}$, hence also $E = \operatorname{ran} B \dot{+} L$. Denote by $Q$ the projection onto $L$ parallel to $\operatorname{ran} B$. Let $f \in \operatorname{dom} A$. Then $Qf \in \operatorname{dom} A$, $B(I - Q)f = (I - Q)f \in \operatorname{dom} A$ and

$$(Af, g) = (AB(I - Q)f, g) + (AQf, g).$$

The first summand on the right is continuous in $(I - Q)f$ and hence in $f$ and the second summand is continuous in $f$, because $L$ is finite-dimensional. Thus $f \mapsto (Af, g)$ is continuous on $\operatorname{dom} A$, that is, $g \in \operatorname{dom} A^*$. This proves (4.1). □

The next theorem is essentially the theorem in [20] (see also [5, Theorem 6] and [8, p. 306]) in a Hilbert space setting. It is the same as [11, Proposition 2], but our proof of it is different.

**Theorem 4.3.** *Let $S$ and $T$ be densely defined operators on a Hilbert space. If $T$ is closed and* $\operatorname{ran} T$ *is closed and has finite codimension, then $ST$ is a densely defined operator and*

$$(ST)^* = T^*S^*. \tag{4.2}$$

*Proof.* Let $E$ be the Hilbert space on which $S$ and $T$ act and denote by $(\,\cdot\,,\,\cdot\,)$ and $\|\cdot\|$ the inner product and corresponding norm of $E$. First we show that $ST$ is densely defined. Consider the operator $T_1 = T|_{\operatorname{dom} T \cap \operatorname{ran} T^*}$. It is a bijection onto $\operatorname{ran} T$ and from $T\operatorname{dom} ST = \operatorname{dom} S \cap \operatorname{ran} T$ it follows that

$$\operatorname{dom} ST = T_1^{-1}(D) \oplus \ker T, \quad D := \operatorname{dom} S \cap \operatorname{ran} T.$$

We claim that $T_1^{-1}(D)$ is dense in $\operatorname{ran} T^*$, which is closed by Lemma 2.7. Assuming the claim is correct, we obtain from the above equality that

$$\overline{\operatorname{dom} ST} = \overline{T_1^{-1}(D)} \oplus \ker T = \operatorname{ran} T^* \oplus \ker T = E,$$

that is, $ST$ is densely defined. It remains to prove the claim. We prove it by showing that if $x \in \operatorname{ran} T^*$ is orthogonal to $T_1^{-1}(D)$, then $x = 0$. Let $x = T^* y$, $y \in E$, and assume $x \perp T_1^{-1}(D)$. Then

$$\{0\} = (x, T_1^{-1}(D)) = (T^* y, T_1^{-1}(D)) = (y, T T_1^{-1}(D)) = (y, D),$$

which shows that $y \in D^\perp$. By Lemma 2.6, $D$ is dense in $\operatorname{ran} T$, hence $D^\perp = (\operatorname{ran} T)^\perp = \ker T^*$. It follows that $x = T^* y = 0$. This completes the proof of the claim.

We now prove the equality (4.2). Let $T^* = V|T^*|$ be the polar decomposition of $T^*$. Then $\operatorname{dom} |T^*| = \operatorname{dom} T^*$, $\operatorname{ran} |T^*| = \operatorname{ran} T$, and $\ker V = \ker |T^*| = \ker T^*$ and $T = |T^*| V^*$, see [17, Section VI.2.7]. Thus $\ker V$ is finite dimensional and contained in $\ker S|T^*|$; this is needed below when we apply Lemma 4.2 $(b)$. Denote by $R$ the closed densely defined linear operator on $E$ such that

$$R = \begin{cases} |T^*| & \text{on } \operatorname{ran} T \cap \operatorname{dom} T^*, \\ I & \text{on } \ker T^*, \end{cases}$$

and let $P$ be the orthogonal projection onto $\operatorname{ran} T$. Then $R$ is self-adjoint and boundedly invertible, $\dim \ker P < \infty$ and $|T^*| = PR = RP$. By the first part of this proof, $ST$, $S|T^*|$ and $SP$ are densely defined operators and therefore the equality (4.2) follows from Lemma 4.2:

$$\begin{aligned} (ST)^* &= (S|T^*|V^*)^* \overset{(b)}{=} V^{**}(S|T^*|)^* = V(SPR)^* \\ &\overset{(a)}{=} VR(SP)^* \overset{(c)}{=} VRPS^* = V|T^*|S^* = T^* S^*. \end{aligned}$$

□

*Remark* 4.4. In Theorem 4.3 we do not claim that the closed operator $T^* S^*$ is densely defined, or, what is equivalent in this case, that $ST$ is closable. If $T^* S^*$ is densely defined, then $(T^* S^*)^* = \overline{ST}$. So, if, in addition, $ST$ is closed, then $(T^* S^*)^* = ST$.

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T. Ya. Azizov
Department of Mathematics
Voronezh State University
Universitetskaya pl. 1
Voronezh 394006, Russia
e-mail: azizov@math.vsu.ru

A. Dijksma (✉)
Johann Bernoulli Institute of Mathematics and Computer Science
University of Groningen
P.O. Box 407
9700 AK Groningen, The Netherlands
e-mail: a.dijksma@rug.nl

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