4-1974

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Let R be a Bezout ring (a commutative ring in which all finitely generated ideals are principal), and let M be a finitely generated R-module. We will study questions of the following sort: (A) If every localization of M can be generated by n elements, can M itself be generated by n elements? (B) If $M \oplus R^m \cong R^n$ for some m, n, is M necessarily free? (C) If every localization of M has an element with zero annihilator, does M itself have such an element? We will answer these and related questions for various familiar classes of Bezout rings. For example, the answer to (B) is "no" for general Bezout rings but "yes" for Hermite rings (defined below). Also, a Hermite ring is an elementary divisor ring if and only if (A) has an affirmative answer for every module M.

1. Preliminaries. All rings in this discussion are commutative with 1, and modules are unital. If A is an $m \times n$ matrix over R, let $\text{Ker}(A)$ be the set of columns $X \in R^n$ for which $AX = 0$ and let $\text{Im}(A)$ be the submodule of R^m generated by the columns of A. The $k \times k$ identity matrix is denoted by I_k, and A^T is the transpose of A. We say the R-module M is "named" by the matrix A if $M \cong \text{Coker}(A) = R^m / \text{Im}(A)$.

We define the *Fitting invariants* $F_i(A)$ of an $m \times n$ matrix A as follows: $F_i(A)$ is the ideal of R generated by the $(m + 1 - i) \times (m + 1 - i)$ minors of A, for $0 \le i \le m$ and $F_i(A) = R$ for $i > m$. If A names the module M, we put $F_i(M) = F_i(A)$. (For a proof that this is unambiguous, consult [6, p. 146] for example.) The following well-known lemma will be useful later:

LEMMA 1.1. *Let M be any finitely presented module. then $F_1(M) \subseteq (0: M)$. If M is free of rank r, then $F_r(M) = 0$ and $F_{r+1}(M) = R$.*

For M a finitely generated R-module, we denote by $\mu(R, M)$ the minimum number of generators required for M over R.

PROPOSITION 1.2. *A commutative ring R is Bezout if and only if every finitely presented R-module can be named by a square matrix.*

Proof. For the "if" implication, let $a, b \in R$ and let $M = R/(a, b)$. Let A be a square matrix that names M. Then $(a, b) =$

$F_1(M) = (\det A)$, a principal ideal. The "only if" implication is a consequence of the next lemma, communicated to us by Lawrence Levy.

LEMMA 1.3.   *Let R be a ring in which $\mu(R, I) \leqq k$ for every finitely generated ideal I. Let M and N be finitely generated R-modules, with $M \subseteq N$.   Then $\mu(R, M) \leqq k \cdot \mu(R, N)$.*

Proof. It is easy to see that we may assume $N = R^n$. For $M \subseteq R^n$, we show that $\mu(R, M) \leqq kn$, by induction on n, the case $n = 1$ being true by hypothesis.   Suppose $\{m_\alpha\}$ is a finite set of generators for M and set $m_\alpha = (u_\alpha, r_\alpha)$, where $u_\alpha \in R^{n-1}$, $r_\alpha \in R$. Let π be the projection from R^n onto its first $n - 1$ coordinates; then by induction $\pi(M)$ can be generated by $k(n - 1) = t$ elements $v_1, v_2, \cdots, v_t$.   Choose $s_j \in R$ so that $(v_j, s_j) \in M$ and choose $a_{\alpha j}$ so that $u_\alpha = \Sigma_j a_{\alpha j} v_j$.   Then $m_\alpha = \Sigma_j a_{\alpha j} (v_j, s_j) + (0, r_\alpha - \Sigma_j a_{\alpha j} s_j)$.   The ideal generated by the $r_\alpha - \Sigma_j a_{\alpha j} s_j$ may be generated by k elements $b_1, b_2, \cdots, b_k$ of R.   Clearly, the nk elements $(v_1, s_1), \cdots, (v_t, s_t), (0, b_1), \cdots, (0, b_k)$ generate M.

We remark that the proof of (1.3) works perfectly well in the noncommutative case, thus answering a question of Pierce [8, §25, (1)].

A projective module M is said to have (constant) *rank n* provided $\mu(R_P, M_P) = n$ for every P.   It is known [11] that every projective of constant rank is finitely generated. The following lemma (see, for example, the proof of [11, Proposition 1.4]) essentially reduces the study of finitely generated projectives to the case of constant rank:

LEMMA 1.4.   *For every finitely generated projective module M there exist orthogonal idempotents $e_0, \cdots, e_n$ with sum 1, such that Me_k has constant rank k as an Re_k-module for all k.*

It is easy to see that Re_0 (e_0 as above) is the annihilator of M.   In particular, M is faithful if and only if $e_0 = 0$, that is, if and only if $\mu(R_P, M_P) \neq 0$ for all P.   Recall that an element $m \in M$ is *unimodular* provided Rm is a nonzero, free direct summand of M.   The following result is a typical application of (1.4).   The proof will be omitted.

PROPOSITION 1.5.   *These conditions on a ring are equivalent*: (i) *Every finitely generated, faithful projective module has a unimodular element.* (ii) *Every projective module of constant rank is free.* (iii) *Every finitely generated projective module is a direct sum of cyclics.*

A ring R is *Hermite* if every matrix over R is equivalent to an upper triangular matrix.   If every matrix is equivalent to a diagonal

matrix (a_{ij}) in which a_{ii} divides $a_{i+1, i+1}$, R is an *elementary divisor ring*. It has recently been shown [7] that a ring is an elementary divisor ring if and only if every finitely presented module is a direct sum of cyclics. We will see later, in (3.4), that the conditions of (1.5) also have a matrix-theoretic formulation. We will need the following well-known characterization of projective modules in terms of Fitting invariants:

PROPOSITION 1.6. *Let M be a finitely presented R-module. Then M is a rank r projective module if and only if $F_r(M) = 0$ and $F_{r+1}(M) = (1)$. More generally, M is projective if and only if each $F_k(M)$ is generated by an idempotent.*

2. The main theorem. If M is a finitely generated module over a principal ideal ring or a von Neumann regular ring [8, Corollary 13.8], then $\mu(R, M) = \max \mu(R_P, M_P)$, P ranging over all maximal ideals. It seems reasonable to ask which Bezout rings have this property. For Hermite rings, we have a complete answer:

THEOREM 2.1. *The following conditions on a Hermite ring R are equivalent:* (i) $\mu(R, M) = \max_P \mu(R_P, M_P)$, *for every finitely generated R-module M.* (ii) *Every finitely presented, locally cyclic R-module is cyclic.* (iii) *For each $d \in R$, every projective $R/(d)$-module of rank 1 is free.* (iv) *For each ideal $I \subseteq R$, every projective R/I-module of constant rank is free.* (v) *R is an elementary divisor ring.*

Proof. Trivially, (i) implies (ii), and (iv) implies (iii). Since an n-generator projective of constant rank n is clearly free, we see that (i) implies (iv), and (ii) implies (iii). Therefore it will suffice to show that (iii) implies (v) and (v) implies (i).

Assume (iii) holds. In order to prove that R is an elementary divisor ring it will suffice, by the proof of [7, Theorem 3.8], to show that if $(a, b, c) = (1)$ then the module M named by the matrix $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ is cyclic. Set $d = ac$. By (1.1) and (1.6), M is a rank 1 projective $R/(d)$-module; then (iii) implies that M is cyclic.

Finally we show that (v) implies (i). Suppose, first of all, that M is finitely presented. Then by (1.2), M is named by an $n \times n$ diagonal matrix (d_{ij}) in which $(d_{ii}) \subseteq (d_{i+1, i+1})$ for all i. Thus $M \cong R/(d_{11}) \oplus \cdots \oplus R/(d_{mm})$, where $m \leq n$ and $(d_{mm}) \neq R$. Choosing a maximal ideal P containing d_{mm}, we have $\mu(R, M) = \mu(R_P, M_P) = m$, as desired. For the general case, we can appeal to the following lemma:

LEMMA 2.2.   *Let R be any commutative ring, and let M be a finitely generated R-module such that $\mu(R_P, M_P) \leqq n$ for each maximal ideal P.   Then there exists a finitely presented module N, with $\mu(R_P, N_P) \leqq n$ for all P, and a surjection $N \twoheadrightarrow M$.*

Proof.   Choose a surjection $\phi: R^k \twoheadrightarrow M$ with kernel K, and let P be an arbitrary prime ideal of R.   By hypothesis, there are elements $u_1, \cdots, u_n \in R^k$ such that the images of $\phi(u_i)$ generate M_P.   Then there are elements $v_1, \cdots, v_k \in K$ so that the images in R_P^k of $v_1, \cdots, v_k, u_1, \cdots, u_n$ generate R_P^k.   Let $K(P)$ be the sub-module of K generated by the v_i, and set $N(P) = R^k/K(P)$.   Then $N(P)$ is finitely presented, and $\mu(R_P, N(P)_P) \leqq n$.   By [2, Hilfsatz 2] there is a neighborhood $U(P)$ of P (in the Zariski topology on $\mathrm{spec}(R)$) such that $\mu(R_Q, N(P)_Q) \leqq n$ for all $Q \in U(P)$.   Now administer the same treatment to *every* prime P, let $U(P_1), \cdots, U(P_n)$ cover $\mathrm{spec}(R)$, and set $N = R^n/(K(P_1) + \cdots + K(P_n))$.

We remark that there may be primes P for which $\mu(R_P, N_P) > \mu(R_P, M_P)$, even though the same bound n suffices.   In fact, the following proposition tells exactly when N can be chosen so that $\mu(R_P, N_P) = \mu(R_P, M_P)$ for all P.

PROPOSITION 2.3.   *The following conditions on R are equivalent*: (i) *For each finitely generated M there exist a finitely presented N, with $\mu(R_P, N_P) = \mu(R_P, M_P)$ for all P, and a surjection $N \twoheadrightarrow M$.*   (ii) *For each cyclic R-module C there is a finitely presented N, with $\mu(R_P, N_P) = \mu(R_P, C_P)$ for all P.*   (iii) *R has Noetherian prime spectrum.*

Proof.   Certainly (i) implies (ii).   To show that (ii) implies (iii) we have to show that every open set U is quasi-compact.   Now $U = \{P \in \mathrm{spec}(R) | P \not\supseteq I\}$ for some ideal I.   Let $C = R/I$ and choose N as in (ii).   Then $U = \{P \in \mathrm{spec}(R) | N_P = 0\} = \{P \in \mathrm{spec}(R) | P \not\supseteq F_1(N)\}$,   by (1.1), and the latter set is quasi-compact since $F_1(N)$ is finitely generated.   Finally, to prove (iii) implies (i), let M be any finitely generated module, and set $V_n = \{P \in \mathrm{spec}(R) | \mu(R_P, M_P) \leqq n\}$.   Then each V_n is quasi-compact (since $\mathrm{spec}(R)$ is Noetherian) and open in $\mathrm{spec}(R)$, by [2, Hilfsatz 2].   It follows that the sets $U_n = \{P \in \mathrm{spec}(R) | \mu(R_P, M_P) = n\}$ are open and closed in the patch topology on $\mathrm{spec}(R)$, [12].   As in the proof of (2.2), choose a surjection $\phi: R^k \twoheadrightarrow M$, and fix a prime P, say $P \in U_n$.   Choose u_i, v_j as in the proof of (2.2) and define $K(P), N(P)$ and $U(P)$ as before.   The proof is completed by replacing $U(P)$ by $U(P) \cap U_n$, and then using compactness of $\mathrm{spec}(R)$ relative to the patch topology.

It is unknown whether the conditions of Theorem 2.1 are equivalent to the apparently weaker condition: Every projective of constant rank is free. If so, it would follow that every semihereditary Bezout ring is an elementary divisor ring. (Reason: Over any semi-hereditary ring, every projective module, finitely generated or not, is a direct sum of finitely generated ideals, [5].) In particular, this would answer the long-standing question, mentioned in [4] and [1, p. 290]: "Is every Bezout domain an elementary divisor ring?"

We remark that there are Hermite rings that are not elementary divisor rings [3]. Thus by passing to an appropriate homomorphic image, we obtain an example of a Hermite ring with a rank 1 projective that is not free.

We conclude this section with an example of an elementary divisor ring with a projective module (necessarily non-finitely generated), that is not a direct sum of cyclics. Let $\mathbf{R}^+$ be the space of nonnegative real numbers, and let R be the ring of continuous real-valued functions on $\beta \mathbf{R}^+ - \mathbf{R}^+$, where β is the Stone-Cech compactification. This space is connected [3, 2.8], so R has no nontrivial idempotents. By [3, 3.3, 4.1, 4.2] R is an elementary divisor ring. Obviously R is not a domain, so we can apply:

PROPOSITION 2.4. *Let R be a semiprime Bezout ring with no nontrivial idempotents. If every projective ideal of R is a direct sum of principal ideals, then R is a domain.*

Proof. Since the annihilator of a projective principal ideal is generated by an idempotent, every projective ideal of R is free. In particular, projective ideals are finitely generated. By [11], every finitely generated flat module is projective. Let P be any minimal prime of R. Since every localization of R is a valuation domain, we see that R/P is flat. Hence $P = 0$ as desired.

3. Stably free modules. We turn now to a very special class of constant rank projectives, the stably free modules. Using (1.1) we see that a module M satisfies $M \oplus R^m \cong R^n$ if and only if $M \cong \ker A$ for some $m \times n$ matrix A with $F_1(A) = (1)$. We want a matrix-theoretic condition for a stably free module to be free. This criterion is well-known in case $m = 1$. The proof of the general case is rather straightforward, and will be omitted.

PROPOSITION 3.1. *Let A be an $m \times n$ matrix with $F_1(A) = (1)$. Then $\ker A$ is free if and only if A can be filled out to an $n \times n$ invertible matrix.*

COROLLARY 3.2.   *Over a Hermite ring every stably free projective module is free.*

Proof. Let A be an $m \times n$ matrix with $F_1(A) = R$. Choose invertible matrices P, Q such that PAQ is lower triangular. Now $m \leq n$, so $PAQ = [B \mid 0]$ for an invertible B. Then $AQ = [P^{-1}B \mid 0]$ can certainly be filled out to an invertible $n \times n$ matrix C; and CQ^{-1} has A as its top m rows.

EXAMPLE 3.3.   A Bezout ring with a stably free projective that is not free.

For any topological space Y, let $C(Y)$ denote the ring of continuous real-valued functions on Y. Let X be the disjoint union of countably many 2-spheres. To be precise, let $X = \cup_n X_n$, where $X_n = \{(a_1, a_2, a_3) \in \mathbf{R}^3$ such that $(a_1 - 3n)^2 + (a_2 - 3n)^2 + (a_3 - 3n)^2 = 1\}$. For $i = 1, 2, 3$, let $\phi_i \in C(X)$ be the (continuous) function defined by $\phi_i(x) = a_i - 3n$, if $x = (a_1, a_2, a_3,) \in X_n$. Each of these "coordinate" functions has a unique extension (still denoted by ϕ_i) to the Stone-Cech compactification βX, and we have $\phi_1^2 + \phi_2^2 + \phi_3^2 = 1$. Set $R = C(\beta X - X)$. Since X is countably compact and locally compact, [3, Theorem 2.7] implies R is Bezout. Let bars denote passage from $C(\beta X)$ to $C(\beta X - X)$ via restriction. Note that X is open in βX, so the map $f \rightarrow \bar{f}$ maps $C(\beta X)$ *onto* R.

Suppose $[\bar{\phi}_1 \bar{\phi}_2 \bar{\phi}_3]$ is the first row of an invertible matrix over R. Then there is a matrix A over $C(\beta X)$ with first row $[\phi_1 \phi_2 \phi_3]$ such that $\bar{\delta}$ is a unit, where $\delta = \det A$. Now it is well-known that for each n, there is a point $x_n \in X_n$ for which $\delta(x_n) = 0$. (To see this, let $[\alpha_1, \alpha_2, \alpha_3]$ be the second row of A. If $\delta(x) \neq 0$ for all $x \in X_n$, then the vector $(\alpha_1(x), \alpha_2(x), \alpha_3(x))$ would be linearly independent of the vector $(\phi_1(x), \phi_2(x), \phi_3(x))$ for each x. Since the latter vector is normal to the tangent plane of X_n at x, the projection of $(\alpha_1(x), \alpha_2(x), \alpha_3(x))$ would define a nonvanishing tangent vector field on X_n.) By compactness, the closure of the set $\{x_n\}$ in βX must contain a point p in $\beta X - X$. Then $\bar{\delta}(p) = \delta(p) = 0$, contradicting the assumption that $\bar{\delta}$ is a unit.

As another application of (3.1) we give a matrix-theoretic characterization of those rings whose projectives of constant rank are free.

COROLLARY 3.4.   *These conditions are equivalent for a ring* R: (i) *Every projective module of constant rank is free.* (ii) *Every* $m \times n$ *matrix* A *with* $F_r(A) = 0$ *and* $F_{r+1}(A) = (1)$ *is equivalent to the* $m \times n$ *matrix with* I_{m-r} *in the top left corner and* 0*'s elsewhere.* (iii) *Every matrix with idempotent Fitting invariants is equivalent to a diagonal matrix.*

Proof. The only nontrivial assertion is (i) implies (ii). Suppose every projective module of constant rank is free, and let A be an $m \times n$ matrix with $F_r(A) = 0$, $F_{r+1}(A) = (1)$. Then $\operatorname{Coker} A \cong R^r$, $\operatorname{Im}(A) \cong R^{m-r}$, and $\operatorname{Ker}(A) \cong R^{n-m+r}$. By filling in the following diagram

$$\begin{array}{ccccccccc} 0 & \to & R^{n-m+r} & \to & R^n & \to & R^{m-r} & \to & 0 \\ & & \cong \downarrow & & \downarrow P & & \cong \downarrow & & \\ 0 & \to & \ker A & \to & R^n & \to & \operatorname{Im} A & \to & 0 \end{array}$$

we obtain an invertible matrix P such that $AP = [B\,|\,0]$ for some $m \times (m-r)$ matrix B. Since $F_1(B^T) = (1)$, B can be filled out to an invertible $m \times m$ matrix $Q = [B\,|\,C]$, by (3.1). Then $Q^{-1}AP = \begin{bmatrix} I_m - 0 \\ 0 \quad 0 \end{bmatrix}$, as desired.

4. Basic elements.

Following Swan [9], we say an element m in a finitely generated module M is basic at a prime P, provided $\mu(R_P, (M/Rm)_P) < \mu(R_P, M_P)$. Equivalently, $m/1$ is part of a minimal generating set for M_P. The element m is said to be *basic* in M provided it is basic at every prime P of R. One can ask the following question, by analogy with the considerations of the preceding two sections: If M is a finitely generated module over a Bezout ring and $M_P \neq 0$ for all P, does M have a basic element? It is not hard to see that the answer is "yes" for Bezout domains. (In fact, in this case M will have a unimodular element.) In general, however, the answer is "no", even for Boolean rings!

For a subset A of the ring R we let $D(A)$ be the (open) subset of $\operatorname{spec}(R)$ consisting of those primes not containing A. If α is an element of R (or of an R-module M) we let $\alpha(P)$ denote the canonical image of α in R_P (or in M_P). Let $Z(\alpha) = \{P \in \operatorname{spec}(R) \,|\, \alpha(P) = 0\} = D(0: \alpha)$. Finally, a point x in a topological space is said to be an *n-point* (n a positive integer) provided x is in the closure of each of n pairwise disjoint open sets.

Our main results on basic elements ·will be for (von Neumann) regular rings. In this context, the condition $M_P \neq 0$ for all P (M finitely generated) simply says that M is faithful. (Indeed, this is true over *any* reduced ring.) Also, since every localization of a regular ring is a field, an element $m \in M$ is basic at P if and only if $m(P) \neq 0$.

LEMMA 4.1. *Let M be a finitely generated module over the regular ring R. Then M has a basic element if (and only if) for every $x \in \operatorname{spec}(R)$ there is an element $m \in M$ such that $x \notin Z(m)^-$.*

Proof. The method of proof is entirely standard, and is implicit throughout [8]: Choose, for each $x \in \text{spec}(R)$, an element m_x such that $m_x(y) \neq 0$ for all y in a neighborhood U_x of x. Choose a finite disjoint refinement $\{V_1, \cdots, V_n\}$ of the open cover $\{U_x\}$, say $V_i \subseteq U_{x_i}$. Finally, choose idempotents e_i such that $V_i = D(e_i)$, and set $b = e_1 x_1 + \cdots + e_n x_n$. Then b is the required basic element.

THEOREM 4.2. *Let R be a regular ring. If $\text{spec}(R)$ has no 3-points, then every two-generator faithful R-module has a basic element. The converse holds if R is a Boolean ring.*

Proof. Suppose $\text{spec}(R)$ has no 3-points, and let $M = R\alpha + R\beta$. Since M is faithful, the open sets $Z(\alpha), Z(\beta), Z(\alpha + \beta)$ are pairwise disjoint. Hence, each point of $\text{spec}(R)$ is outside the closure of at least one of these sets. By (4.1), M has a basic element.

Conversely, assume R is Boolean, and z is in the closure of each of the pairwise disjoint open sets $D(A_i)$, where A_i is an ideal of R, $i = 1, 2, 3$. We consider the module constructed in the proof of [8, Proposition 20.1]. Let $T = \{(a_1 + a_3, a_2 + a_3) \in R \oplus R \mid a_i \in A_i\}$ and set $M = (R \oplus R)/T$. Let α_1 and α_2 be the images in M of $(1, 0)$ and $(0, 1)$, respectively. It is easily seen that $A_i = (0 : \alpha_i)$, $i = 1, 2$, and $A_3 = (0 : \alpha_1 + \alpha_2)$; that is, $D(A_i) = Z(\alpha_i)$ and $D(A_3) = Z(\alpha_1 + \alpha_2)$.

Suppose $\alpha = e_1\alpha_1 + e_2\alpha_2$ is a basic element of M. If, say, $e_1(z) = 0$ and $e_2(z) = 1$, then $N = Z(e_1) \cap Z(1 - e_2)$ is a neighborhood of z, and hence meets $Z(\alpha_2)$. But $\alpha(x) = 0$ for all $x \in N \cap Z(\alpha_2)$, a contradiction. The other two cases $(e_1(z) = 1, e_2(z) = 0; e_1(z) = e_2(z) = 1)$ are handled similarly.

One can easily produce Boolean rings whose spectra contain 3-points. In fact, Pierce [8, Lemma 21.5] has shown that every infinite Boolean ring has a homomorphic image whose spectrum has a 3-point. It follows that every infinite Boolean ring has a 2-generator module M such that the annihilator of each element of M is strictly larger than the annihilator of M.

It is unknown to the authors whether the converse of (4.2) holds in any regular ring. There seems to be no way of extending the proof given above for Boolean rings. For example, if none of the residue fields of R has characteristic 2, then $\alpha_1 - \alpha_2$ is a basic element of M.

Can "two-generator" be replaced by "finitely-generated" in (4.2)? This seems plausible, but, again, no straight-forward extension of the method of proof of (4.2) seems to work.

A module M over a regular ring is a generator of the category of R-modules if and only if M has a unimodular element, [13, 4.2]. Carl Faith has asked whether being a generator is a local property for finitely generated modules. The next theorem gives a negative answer.

THEOREM 4.3.   *A regular ring R is self-injective if and only if finitely generated faithful R-module has a unimodular element.*

Proof.   Suppose R is self-injective, and let $m_1, \cdots, m_k$ generate M. Then the map $R \to M^k$ taking r to $(rm_1, \cdots, rm_k)$ is a split monomorphism. By the remark above, M has a unimodular element.

Now suppose R is not self-injective. There are two cases to consider. *Case* 1: $\mathrm{spec}(R)$ contains a 2-point. Say $P \in D(I)^- \cap D(J)^-$ where I and J are ideals of R with $I \cap J = 0$. Then, by [10, Proposition 2.6], $(0: I) + (0: J) \subseteq P$. Hence, by [10, Theorem 1.4], the faithful module $R/I \oplus R/J$ has no unimodular element. *Case* 2: $\mathrm{spec}(R)$ contains no 2-points. Let M be the two-generator module $\Gamma(X, \mathscr{A})$, where $X = \mathrm{spec}(R)$ and $\mathscr{A}$ is the sheaf of modules constructed in [8, p. 107]. This module is nonsingular ("torsion-free" in the terminology of [8]), but not projective. Also, M is faithful, by formula (1) of [8, p. 107]. Suppose, by way of contradiction, that m is a unimodular element of M, that is, $M = Rm \oplus M'$. Then M' is locally cyclic, hence cyclic by (2.1), say $M' = R/I$. Now, M' is nonsingular, so $D(I)$ is the interior of its own closure, by [8, Corollary 22.2]. But since $\mathrm{spec}(R)$ has no 2-points it follows that $D(I)$ is both open and closed, that is, I is generated by an idempotent. Therefore M is projective, a contradiction.

It is well-known that a Boolean ring R is self-injective if and only if $\mathrm{spec}(R)$ has no 2-points. (See, for example, [8, Corollary 24.2].) thus for Boolean rings, we have the following parallel with (4.2):

COROLLARY 4.4.   *Let R be a Boolean ring. Then $\mathrm{spec}(R)$ has no 2-points if and only if every two-generator faithful R-module has a unimodular element.*

In view of (4.2) and (4.4), it is worth remarking that there exist Boolean rings whose spectra have 2-points but no 3-points [8, pp. 92, 93].

Acknowledgment.   The authors are grateful to Thomas S. Shores for several helpful conversations related to this research.

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