Unimodal sequences and quantum and mock modular forms

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We show that the rank generating function $U(t;q)$ for strongly unimodal sequences lies at the interface of quantum modular forms and mock modular forms. We use $U(-1;q)$ to obtain a quantum modular form which is "dual" to the quantum form Zagier constructed from Kontsevich's "strange" function $F(q)$. As a result, we obtain a new representation for a certain generating function for L-values. The series $U(i;q)=U(-i;q)$ is a mock modular form, and we use this fact to obtain new congruences for certain enumerative functions.

1. Introduction and Statement of Results

A sequence of integers $\{a_i\}_{i=1}^{s}$ is a strongly unimodal sequence of size n if it satisfies

$$0<a_1<a_2<\ldots<a_k>a_{k+1}>a_{k+2}>\ldots>a_s>0$$

for some k and $a_1+\ldots+a_s=n$. Let $u(n)$ be the number of such sequences. The rank of such a sequence is $s-2k+1$, the number of terms after the maximal term minus the number of terms that precede it.

By letting t (respectively, t^{-1}) keep track of the terms after (resp., before) a maximal term, we find that $u(m,n)$, the number of size n and rank m sequences, satisfies[†]

$$\begin{aligned}U(t;q)&:=\sum_{m,n}u(m,n)t^mq^n=\sum_{n=0}^{\infty}(-tq;q)_n(-t^{-1}q;q)_nq^{n+1}\\&=q+q^2+(t+1+t^{-1})q^3+\ldots,\end{aligned}\quad\textbf{[1.1]}$$

where $(x;q)_n:=(1-x)(1-xq)(1-xq^2)\cdots(1-xq^{n-1})$ for $n\geq 1$ and $(x;q)_0:=1$.

Example: The strongly unimodal sequences of size 5 are: $\{5\}$, $\{1,4\}$, $\{4,1\}$, $\{1,3,1\}$, $\{2,3\}$, $\{3,2\}$, and so $u(5)=6$. Respectively, their ranks are $0,-1,1,0,-1,1$.

The q-series $U(-1;q)$, the generating function for the number of size n sequences with even rank minus the number with odd rank, is intimately related to Kontsevich's strange function[‡]

$$\begin{aligned}F(q)&:=\sum_{n=0}^{\infty}(q;q)_n=1+(1-q)+(1-q)(1-q^2)\\&\quad+(1-q)(1-q^2)(1-q^3)+\ldots.\end{aligned}\quad\textbf{[1.2]}$$

It is strange because it does not converge on any open subset of $\mathbb{C}$, but is well-defined at all roots of unity. Zagier (1) proved that this function satisfies the even "stranger" identity

$$F(q)=-\frac{1}{2}\sum_{n=1}^{\infty}n\chi_{12}(n)q^{\frac{n^2-1}{24}},\quad\textbf{[1.3]}$$

where $\chi_{12}(\bullet)=\left(\frac{12}{\bullet}\right)$. Neither side of this identity makes sense simultaneously. Indeed, the right-hand side[§] converges in the unit disk $|q|<1$, but nowhere on the unit circle. The identity means that $F(q)$ at roots of unity agrees with the radial limit of the right-hand side.

We prove that $U(-1;q)$, which converges in $|q|<1$, also gives $F(q^{-1})$ at roots of unity.

Theorem 1.1. *If q is a root of unity, then $F(q^{-1})=U(-1;q)$.*

Example: Here are two examples: $U(-1;-1)=F(-1)=3$ and $U(-1;i)=F(-i)=8+3i$.

Remark: Theorem 1.1 is analogous to the result of Cohen (2, 3) that $\sigma(q)=-\sigma^*(q^{-1})$ for roots of unity q, for the well-known q-series $\sigma(q)$ and $\sigma^*(q)$ that Andrews et al. (4) defined in their work on partition ranks.

Zagier (1) used Eq. **1.3** to obtain the following identity:

$$e^{-\frac{t}{24}}\sum_{n=0}^{\infty}(1-e^{-t})(1-e^{-2t})\ldots(1-e^{-nt})=\sum_{n=0}^{\infty}\frac{T_n}{n!}\cdot\left(\frac{t}{24}\right)^n,\quad\textbf{[1.4]}$$

where Glaisher's T_n numbers (see Eq. **2.3** and A002439 in ref. 5) are the "algebraic factors" of $L(\chi_{12},2n+2)$. As a companion to Theorem 1.1, we use $U(-1;q)$ to give these same L-values.

Theorem 1.2. *As a power series in t, we have that*

$$\begin{aligned}e^{\frac{t}{24}}\cdot U(-1;e^{-t})&=\sum_{n=0}^{\infty}\frac{T_n}{n!}\cdot\left(\frac{-t}{24}\right)^n\\&=\frac{6\sqrt{3}}{\pi^2}\cdot\sum_{n=0}^{\infty}\frac{(2n+1)!}{n!}\cdot L(\chi_{12},2n+2)\cdot\left(\frac{-3t}{2\pi^2}\right)^n.\end{aligned}$$

These results are related to the next theorem, which gives a new quantum modular form. Following Zagier[¶] (3), a weight k quantum modular form is a complex-valued function f on $\mathbb{Q}$, or possibly $\mathbb{P}^1(\mathbb{Q})\setminus S$ for some finite set S, such that for all $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$, the function

$$h_\gamma(x):=f(x)-\epsilon(\gamma)(cx+d)^{-k}f\left(\frac{ax+b}{cx+d}\right)$$

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[†]In (7) $u(n)$ is denoted $u^*(n)$ and $U(1;q)$ is denoted $U^*(q)$.

[‡]Zagier credits Kontsevich for relating $F(q)$ to Feynmann integrals in a lecture at Max Planck in 1997.

[§]As Zagier points out in section 6 of ref. 1, the right-hand side of the identity is essentially the "half-derivative" of Dedekind's eta-function, which then suggests that the series may be related to a weight 3/2 modular object.

[¶]Zagier's definition of a quantum modular form is intentionally vague with the idea that sufficient flexibility is required to allow for interesting examples. Here, we modify his defintion to include half-integral weights k and multiplier systems $\epsilon(\gamma)$.

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satisfies a "suitable" property of continuity or analyticity. The $\epsilon(\gamma)$ are roots of unity, such as those in the theory of half-integral weight modular forms when $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. We prove that

$$\phi(x) := e^{-\frac{\pi i x}{12}} \cdot U(-1; e^{2\pi i x}) \qquad \textbf{[1.5]}$$

is a weight $\frac{3}{2}$ quantum modular form. Because $SL_2(\mathbb{Z}) = \langle\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\rangle$ and $\phi(x) - e^{\frac{\pi i}{12}} \cdot \phi(x+1) = 0$, it suffices to consider $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$. The following theorem establishes the desired relationship on the larger domain $\mathbb{Q} \cup \mathbb{H} - \{0\}$, where $\mathbb{H}$ is the upper-half of the complex plane.

Theorem 1.3. *If $x \in \mathbb{Q} \cup \mathbb{H} - \{0\}$, then*

$$\phi(x) + (-ix)^{-\frac{3}{2}}\phi(-1/x) = h(x),$$

where $(ix)^{-\frac{3}{2}}$ is the principal branch and

$$h(x) := \frac{\sqrt{3}}{2\pi i}\int_0^{i\infty} \frac{\eta(\tau)}{(-i(x+\tau))^{\frac{3}{2}}} d\tau - \frac{i}{2} e^{\frac{\pi i x}{6}} (e^{2\pi i x}; e^{2\pi i x})_\infty^2 \cdot \int_0^{i\infty} \frac{\eta(\tau)^3}{(-i(x+\tau))^{\frac{1}{2}}} d\tau.$$

Here, $\eta(\tau) := e^{\frac{\pi i \tau}{12}}(e^{2\pi i \tau}; e^{2\pi i \tau})_\infty$ is Dedekind's eta-function. Moreover, taking $\eta(x) = 0$ for $x \in \mathbb{R}$, $h: \mathbb{R} \to \mathbb{C}$ is a C^∞ function that is real analytic everywhere except at $x = 0$, and $h^{(n)}(0) = (-\pi i/12)^n \cdot T_n$, where T_n is the nth Glaisher number.

Remark: Zagier (1) proved that $e^{\frac{\pi i x}{12}} \cdot F(e^{2\pi i x})$ is a quantum modular form. Theorem 1.3 gives a dual quantum modular form, one whose domain naturally extends beyond $\mathbb{Q}$ to include $\mathbb{H}$. This is somewhat analogous to the situation for $\sigma(q)$ and $\sigma^*(q)$ discussed above. Zagier constructed a quantum modular form from these q-series in example 1 of ref. 3.

Remark: Theorem 1.3 implies that $\Phi(z) := \eta(z)\phi(z)$ behaves analogously to a weight 2 modular form for $SL_2(\mathbb{Z})$ for $z \in \mathbb{H}$ with a suitable error function. Namely, $\Phi(z+1) = \Phi(z)$ and $\Phi(z) - z^{-2}\Phi(-\frac{1}{z}) = \eta(z)h(z)$; see also theorem 1.1 of ref. 6.

It turns out that $U(1;q)$ and $U(\pm i;q)$ also possess deep properties. We have that $U(1;q)$ (7) is a mixed mock modular form, and $U(\pm i;q)$ is a mock theta function (see refs. 8–10). We use these facts to study congruences for certain enumerative functions.

Theorem 1.4. *If $3 < \ell \not\equiv 23$ (mod 24) is prime, $\delta(\ell) := (\ell^2 - 1)/24$ and $\ell \nmid k$, then for all n*

$$u(\ell^2 n + k\ell - \delta(\ell)) \equiv 0 \pmod 2.$$

Example: If $\ell = 7$, then Theorem 1.4 gives $u(49n + a) \equiv 0$ (mod 2) for $a \in \{5, 12, 19, 26, 33, 40\}$.

The nature of Theorem 1.4 suggests the existence of a Hecke-type identity for $U(-1;q)$ analogous to those obtained for $\sigma(q)$ and $\sigma^*(q)$ in ref. 4. Here we obtain such an identity.

Theorem 1.5. *We have that*

$$U(-1;q) = \sum_{n>0} \sum_{6n \geq |6j+1|} (-1)^{j+1} q^{2n^2 - \frac{j(3j+1)}{2}} + 2\sum_{n,m>0} \sum_{6n \geq |6j+1|} (-1)^{j+1} q^{2n^2 + mn - \frac{j(3j+1)}{2}}.$$

These congruences appear to have refinements modulo 4. In analogy with the theory of partition ranks (11–13), we suspect that ranks also "explain" these congruences. Namely, let $u(a,b;n)$ be the number of size n strongly unimodal sequences with rank $\equiv a$ (mod b).

Conjecture 1.6. *If $\ell \equiv 7, 11, 13, 17$ (mod 24) is prime and $\left(\frac{k}{\ell}\right) = -1$, then for all n we have*

$$u(\ell^2 n + k\ell - \delta(\ell)) \equiv 0 \pmod 4. \qquad \textbf{[1.6]}$$

Moreover, for $a \in \{0,1,2,3\}$ we have $u(a,4;\ell^2 n + k\ell - \delta(\ell)) \equiv 0$ (mod 2) and

$$u(0,4;\ell^2 n + k\ell - \delta(\ell)) \equiv u(2,4;\ell^2 n + k\ell - \delta(\ell)) \pmod 4. \qquad \textbf{[1.7]}$$

We have that $u(1,4;n) = u(3,4;n)$, and so the truth of Eq. **1.7** is a proposed explanation of Eq. **1.6**. Therefore, it is natural to study $U(\pm 1;q)$ and the third-order mock theta function (14–16):

$$U(\pm i;q) = \Psi(q) = \sum_{n=1}^\infty \frac{q^{n^2}}{(q;q^2)_n} = \sum_{n=0}^\infty (-q^2;q^2)_n q^{n+1} = \frac{q}{(q^4)_\infty} \cdot \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{6n(n+1)}}{1 - q^{4n+1}}.$$

Using this mock theta function, we are able to obtain the following related congruences.

Theorem 1.7. *If $(Q,6) = 1$, then there are arithmetic progressions $An + B$ such that*

$$u(0,4;An+B) \equiv u(2,4;An+B) \pmod Q.$$

Example: For $Q = 5$, the cusp form in the proof of Theorem 1.7 is annihilated by $T(11^2)$, and so if

$$a(24n-1) := u(0,4;n) - u(2,4;n) \pmod 5$$

[note. $a(n) = 0$ if $n \not\equiv 23$ (mod 24)], then for every $n \equiv 23, 47$ (mod 120) we have that

$$a(121n) - \left(\frac{n}{11}\right)a(n) + a(n/121) \equiv 0 \pmod 5.$$

Because $\left(\frac{n}{11}\right) = 0$ and $a(n/121) = 0$ when $11||n$, this gives congruences such as

$$u(0,2;73205n + 721) \equiv u(2,4;73205n + 721) \pmod 5.$$

2. Quantum Properties of $U(-1;q)$

Here we prove the quantum properties of $U(-1;q)$. We first prove Theorem 1.1 relating the values of Kontsevich's $F(q)$ and $U(-1;q)$ at roots of unity. We then prove Theorem 1.2 giving a new representation of Zagier's L-value generating function, and we conclude with a proof of Theorem 1.3.

2.1 Proof of Theorem 1.1: For ξ a fixed kth root of unity, define the polynomial

$$C(X) = \sum_{n=0}^{k-1} (X - \xi^{-1}) \cdots (X - \xi^{-n}).$$

We have the identity

$$C(\xi^{-1}X) = (X-1)^2 C(X) - X(X^k - 1) + X. \quad \textbf{[2.1]}$$

Define the functions $u_a(X)$ for $a \geq 1$ by

$$(2 - X^k) u_a(\xi^{-a}X) = C(\xi^{-a}X) - (1-X)^2 \cdots (1 - \xi^{-(a-1)}X)^2 C(X).$$

Hence, for $a = k$ we have

$$X^k C(X) = u_k(X). \quad \textbf{[2.2]}$$

Then we have

$$(2 - X^k)(u_{a+1}(X) - u_a(X)) = (1 - \xi X)^2 \cdots (1 - \xi^a X)^2 (C(\xi^a X) - (1 - \xi^{a+1})^2 C(\xi^{a+1} X)).$$

By Eq. **2.1**, we have

$$C(\xi^a X) = (1 - \xi^{a+1} X)^2 C(\xi^{a+1} X) + \xi^{a+1}.$$

Letting $X = 1$ gives $u_{a+1}(1) - u_a(1) = \xi^{a+1}(1-\xi)^2 \cdots (1-\xi^a)^2$. Induction and Eq. **2.2** give

$$C(1) = \sum_{n=0}^{k-1} \xi^{n+1} (1-\xi)^2 \cdots (1-\xi^n)^2.$$

2.2 Proof of Theorem 1.2: By the results of Andrews et al. (6) (see equation 9.2 and propositions 9.2 and 9.3) with $q = e^{-2\pi z}$, we have

$$qv(q) = \sum_{n=0}^{\infty} \frac{q^{n+1}}{(q^{n+1};q)_\infty^2} = \frac{e^{\frac{\pi}{6}(\frac{1}{z} - \frac{3}{2}z)}}{\sqrt{3z}} \int_{-\infty}^{\infty} x e^{-\frac{\pi x^2}{3z}} \cdot \frac{\sinh(\frac{2\pi x}{3})}{\cos(\pi x)} dx \cdot (1 + O(z^N))$$

for any positive N where $v(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^n;q)_\infty^2}$. Because we have $U(-1;q) = (q;q)_\infty^2 qv(q)$ and $(q;q)_\infty^2 = e^{-\frac{\pi}{6}(\frac{1}{z} - z)} z^{-1}(1 + O(z^N))$ for any positive N, we have

$$q^{-\frac{1}{24}} U(-1;q) = \frac{1}{\sqrt{3} z^{\frac{3}{2}}} \int_{\mathbb{R}} x e^{-\frac{\pi x^2}{3z}} \cdot \frac{\sinh(\frac{2\pi x}{3})}{\cos(\pi x)} dx (1 + O(z^N))$$

for any N. The Glaisher's T-numbers are given by

$$\frac{\sinh(\frac{2\pi x}{3})}{\cosh(\pi x)} = \frac{2}{i} \sum_{n=0}^{\infty} \frac{T_n}{(2n+1)!} \left(\frac{i\pi x}{3}\right)^{2n+1}. \quad \textbf{[2.3]}$$

We also have the identity

$$\int_{\mathbb{R}} x^{2j} e^{-\frac{\pi x^2}{3z}} dx = \frac{(2j)!}{2^j j!} \left(\frac{3}{2\pi}\right)^j \sqrt{3z} z^j.$$

Combining these identities and then setting $t = 2\pi z$ completes the proof.

2.3 Proof of Theorem 1.3: Define $G(z) := (e^{2\pi i z}; e^{2\pi i z})_\infty U(-1; e^{2\pi i z})$. Theorem 1.1 of ref. 6 gives

$$G(z) - \frac{i}{2}\eta(z)^3 \int_{-\bar{z}}^{i\infty} \frac{\eta(\tau)^3}{(-i(z+\tau))^{\frac{1}{2}}} d\tau + \frac{\sqrt{3}}{2\pi i}\eta(z) \int_{-\bar{z}}^{i\infty} \frac{\eta(\tau)}{(-i(\tau+z))^{\frac{3}{2}}} d\tau$$
$$= z^{-2}\left(G\left(-\frac{1}{z}\right) - \frac{i}{2}\eta\left(-\frac{1}{z}\right)^3 \int_{\frac{1}{z}}^{i\infty} \frac{\eta(\tau)^3}{(-i(-\frac{1}{z}+\tau))^{\frac{1}{2}}} d\tau + \frac{\sqrt{3}}{2\pi i}\eta\left(-\frac{1}{z}\right) \int_{\frac{1}{z}}^{i\infty} \frac{\eta(\tau)}{(-i(\tau - \frac{1}{z}))^{\frac{3}{2}}} d\tau\right). \quad \textbf{[2.4]}$$

Note that using $\eta(-\frac{1}{z}) = \sqrt{-iz}\,\eta(z)$, we have

$$\eta\left(-\frac{1}{z}\right)^3 \int_{\frac{1}{z}}^{i\infty} \frac{\eta(\tau)^3}{(-i(\tau - \frac{1}{z}))^{\frac{1}{2}}} d\tau$$
$$= (\sqrt{-iz})^3 \eta(z)^3 \int_{-\bar{z}}^{0} \frac{\eta(-\frac{1}{\tau})^3}{(-i(-\frac{1}{z} - \frac{1}{\tau}))^{\frac{1}{2}}} \tau^{-2} d\tau$$
$$= (\sqrt{-iz})^3 \eta(z)^3 \int_{-\bar{z}}^{0} \frac{(\sqrt{-i\tau}\,\eta(\tau))^3 (-z\tau)^{\frac{1}{2}}}{(-i(z+\tau))^{\frac{1}{2}}} \tau^{-2} d\tau$$
$$= -z^2 \eta(z)^3 \int_{0}^{-\bar{z}} \frac{\eta(\tau)^3}{(-i(z+\tau))^{\frac{1}{2}}} d\tau. \quad \textbf{[2.5]}$$

Similarly, we have

$$\eta\left(-\frac{1}{z}\right) \int_{\frac{1}{z}}^{i\infty} \frac{\eta(\tau)}{(-i(\tau - \frac{1}{z}))^{\frac{3}{2}}} d\tau = -z^2 \eta(z) \int_{0}^{-\bar{z}} \frac{\eta(\tau)}{(-i(z+\tau))^{\frac{3}{2}}} d\tau. \quad \textbf{[2.6]}$$

Combining Eqs. **2.4–2.6** gives

$$G(z) - z^{-2} G\left(-\frac{1}{z}\right) = \frac{\sqrt{3}}{2\pi i}\eta(z) \int_{0}^{i\infty} \frac{\eta(\tau)}{(-i(z+\tau))^{\frac{3}{2}}} d\tau - \frac{i}{2}\eta(z)^3 \int_{0}^{i\infty} \frac{\eta(\tau)^3}{(-i(z+\tau))^{\frac{1}{2}}} d\tau.$$

Dividing by $\eta(z)$ and using its modular transformation property give the result for $x \in \mathbb{H}$.

For $x \in \mathbb{Q}$, note that $(e^{2\pi i x}; e^{2\pi i x})_\infty = 0$. Moreover, Zagier, in the discussion after the theorem of section 6 of ref. 1, explains how the integral $\int_0^\infty \eta(z)(z+x)^{-\frac{3}{2}} dz$ is real analytic for real x.

3. Congruence Properties and the Hecke-Type Identity

We first prove Theorem 1.4 on the parity of $u(n)$, and we then prove Theorem 1.5 giving the Hecke-type identity for $U(-1;q)$. We then conclude this section with the proof of Theorem 1.7.

3.1 Proof of Theorem 1.4: By theorem 1 of ref. 14 (see Eq. **1.2**), we have that

$$U(-1;q) = \frac{1}{(q;q)_\infty} \cdot \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(1+q^n) q^{\frac{3n^2+n}{2}}}{(1-q^n)^2} - \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} + 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n q^{\frac{n^2+n}{2}}}{1-q^n}\right).$$

If $spt(n)$ is the smallest parts partition function of Andrews, then by theorem 4 of ref. 17 we have:

$$S(q) := \sum_{n=0}^{\infty} spt(n)q^n$$
$$= \frac{1}{(q;q)_\infty}\left(\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2} + \sum_{n=1}^{\infty}\frac{(-1)^n q^{\frac{3n^2+n}{2}}(1+q^n)}{(1-q^n)^2}\right).$$

We have used the elementary fact that

$$\sum_{n=1}^{\infty}\sum_{d|n} dq^n = \sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2} = \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}. \qquad \textbf{[3.1]}$$

We have $U(-1;q) \equiv S(q) \pmod 2$, and so the theorem follows from theorem 1.2 in ref. 18[||].

3.2 Proof of Theorem 1.5: We prove Theorem 1.5 using the method of Bailey pairs. As usual, we let $(a)_n := (a;q)_n$. Two sequences (α_n, β_n) form a Bailey pair for a if

$$\beta_n = \sum_{r=0}^{n}\frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}$$
$$\alpha_n = \frac{(1-aq^{2n})(a)_n(-1)^n q^{\frac{n(n-1)}{2}}}{(1-a)(q)_n}\sum_{j=0}^{n}(q^{-n};q)_j(aq^n;q)_j q^j\beta_j.$$

The following Bailey pair is central to the proof of Theorem 1.5.

Lemma 3.1. *If $\beta_n = 1$ and $\alpha_0 = 1$ and for $n > 0$*

$$\alpha_n = (1-q^{2n})q^{2n^2-n}\left(\frac{q-2}{1-q} + \sum_{j=2}^{n}(-1)^j\frac{(1-q^{2j-1})}{(1-q^j)(1-q^{j-1})}q^{-\frac{3j(j-1)}{2}}\right),$$

then (α_n, β_n) is a Bailey pair with respect to 1.

Proof: We apply theorem 8 of ref. 19 with $\beta_n = 1$ for all n. By letting $b, c, d \to 0$, and then letting $a = 1$, one obtains the lemma. Some care is required for the $j = 0$ and $j = 1$ terms.

The following is Bailey's Lemma (for example, see ref. 19).

Lemma 3.2. *(Bailey's Lemma). If α_n and β_n form a Bailey pair relative to a, then*

$$\sum_{n\geq 0}\frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n}{(aq/\rho_1)_n(aq/\rho_2)_n}\alpha_n$$
$$= \frac{(aq)_\infty(aq/\rho_1\rho_2)_\infty}{(aq/\rho_1)_\infty(aq/\rho_2)_\infty}\sum_{n\geq 0}(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n\beta_n.$$

Proof of Theorem 1.5: By Lemma 3.2 with $\rho_1 = x$, $\rho_2 = x^{-1}$ and $a = 1$, Lemma 3.1 gives

$$\sum_{n\geq 0}(x)_n(x^{-1})_n q^n = \frac{(xq)_\infty(x^{-1}q)_\infty}{(q)_\infty^2} + \frac{(x)_\infty(x^{-1})_\infty}{(q)_\infty^2}$$
$$\times \sum_{n\geq 1}\frac{q^n}{(1-xq^n)(1-x^{-1}q^n)}\cdot\alpha_n.$$

Dividing by $(1-x)(1-x^{-1})$ and collecting the $n = 0$ terms give

$$\sum_{n>0}(xq)_{n-1}(x^{-1}q)_{n-1}q^n = \frac{1}{(1-x)(1-x^{-1})}$$
$$\cdot\left(\frac{(xq)_\infty(x^{-1}q)_\infty}{(q)_\infty^2} - 1\right) + \frac{(xq)_\infty(x^{-1}q)_\infty}{(q)_\infty^2}$$
$$\times\sum_{n\geq 1}\frac{q^n}{(1-xq^n)(1-x^{-1}q^n)}\cdot\alpha_n. \qquad \textbf{[3.2]}$$

To simplify the α_n, we have that

$$\frac{1-q^{2j-1}}{(1-q^j)(1-q^{j-1})} = \frac{1}{2}\cdot\left(\frac{1+q^j}{1-q^j} + \frac{1+q^{j-1}}{1-q^{j-1}}\right),$$

which in turn implies that

$$\sum_{j=2}^{n}(-1)^j\frac{(1-q^{2j-1})}{(1-q^j)(1-q^{j-1})}q^{-\frac{3j(j-1)}{2}} = \frac{1}{2}\cdot\frac{1+q}{1-q}\cdot q^{-3} + \frac{(-1)^n}{2}$$
$$\cdot\frac{1+q^n}{1-q^n}\cdot q^{-\frac{3n(n-1)}{2}} + \frac{1}{2}\sum_{j=2}^{n-1}(-1)^{j+1}(1+q^j)q^{-\frac{3j(j-1)}{2}}\frac{1-q^{3j}}{1-q^j}.$$

Thus, $\alpha_0 = 1$, and for $n \geq 1$ we have

$$\alpha_n = (1-q^{2n})q^{2n^2-n}\left(\frac{q-2}{1-q} + \frac{1}{2}\left(\frac{1+q}{1-q}\cdot q^{-3} + \frac{(-1)^n(1+q^n)q^{-\frac{3n(n-1)}{2}}}{1-q^n}\right.\right.$$
$$\left.\left. + \sum_{j=2}^{n-1}(-1)^{j+1}(1+q^j)(1+q^j+q^{2j})q^{-\frac{3j(j+1)}{2}}\right)\right)$$
$$= (1-q^{2n})q^{2n^2-n}\left(-1 + \sum_{j=1}^{n-1}(-1)^{j+1}(1+q^j)q^{-\frac{j(3j+1)}{2}}\right.$$
$$\left. + \frac{(-1)^n q^{-\frac{3n(n-1)}{2}+n}}{1-q^n}\right)$$
$$= (1-q^{2n})q^{2n^2-n}\left(\sum_{j=-n+1}^{n-1}(-1)^{j+1}q^{-\frac{j(3j+1)}{2}}\right) + (-1)^n(1+q^n)q^{\frac{n(n+3)}{2}}$$
$$= (1-q^{2n})q^{2n^2-n}\left(\sum_{j=-n}^{n-1}(-1)^{j+1}q^{-\frac{j(3j+1)}{2}}\right) + (-1)^n(1+q^n)q^{\frac{n(n+1)}{2}}.$$

We note that

$$\lim_{x\to 1}\frac{1}{(1-x)(1-x^{-1})}\left(\frac{(xq)_\infty(x^{-1}q)_\infty}{(q)_\infty^2} - 1\right) = \sum_{n>0}\frac{q^n}{(1-q^n)^2}$$
$$= \sum_{n>0}\frac{(-1)^{n+1}q^{\frac{n(n+1)}{2}}(1+q^n)}{(1-q^n)^2}.$$

Now, insert these facts in Eq. **3.2**, let $x \to 1$, and use the identity $\frac{1+q^n}{1-q^n} = 1 + 2\sum_{m\geq 1}q^{mn}$.

3.3 Proof of Theorem 1.7: We give a sketch because it is analogous to theorem 1.5 of ref. 12 and theorem 1 of ref. 20. We have

$$U(\pm i;q) = \Psi(q) = \sum_{n=0}^{\infty}(u(0,4;n) - u(2,4;n))q^n,$$

where $\Psi(q)$ is one of Ramanujan's third-order mock theta functions. We have that $q^{-1}\Psi(q^{24})$ is the holomorphic part of a weight $1/2$ harmonic Maass form whose shadow is a unary theta function. Using quadratic and trivial twists modulo Q, one obtains a weight $1/2$ weakly holomorphic modular form. By work of

[||]Theorem 1.2 in ref. 18 is not stated correctly in ref. 18. One must replace pm^2 by $p^{4a+1}m^2$ where $\gcd(p,m) = 1$. Recent work by Andrews et al. (23) gives a new proof of this result.

Treneer (21), one obtains weakly holomorphic forms of half-integer weight that are congruent to cusp forms modulo Q. By the Shimura correspondence, we obtain even integer weight cusp forms, which by lemma 3.30 of ref. 22 are annihilated modulo Q by infinitely many Hecke operators $T(p)$. Because the Shimura correspondence is Hecke equivariant, it follows that infinitely many half-integral weight Hecke operators $T(p^2)$ annihilate these cusp forms modulo Q. The proof follows from the formula for the action of these operators.

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