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An SL(2, Z) Multiplet of Type IIB Super Five-Branes

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ABSTRACT

It is well-known that the low energy string theory admits a non-singular solitonic super five-brane solution which is the magnetic dual to the fundamental string solution. By using the symmetry of the type IIB string theory, we construct an SL(2, Z) multiplet of magnetically charged super five-branes starting from this solitonic solution. These solutions are characterized by two integral three-form charges (q_1, q_2) and are stable when the integers are coprime. We obtain an expression for the tension of these (q_1, q_2) five-branes as envisaged by Witten. The SL(2, Z) multiplets of black strings and black fivebranes and the existence of similar magnetic dual solutions of strings in type II string theory in $D < 10$ have also been discussed.

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String theories in the long wavelength limit are described by various kinds of supergravity theories in $D = 10$. The supergravity equations of motion are well-known [1–3] to admit various black p-brane solutions which are essentially the black hole solutions of the dimensionally reduced low energy string effective action spatially extended to the ten dimensional theory. These solutions are usually characterized by two parameters related to the charge and the mass of the black p-branes. In the extremal limit as usual in the Reissner-Nordstrom black hole, the charge and the ADM mass of the p-branes are related to each other and the solutions become supersymmetric saturating the BPS bound. The extremal solutions of string effective actions are particularly interesting since their masses and the charges can be calculated exactly due to certain non-renormalization theorems of the underlying supersymmetric theories. Thus although these solutions are obtained from the low energy effective theory, they are quite useful to identify certain non-perturbative symmetries of string theory.

Low energy effective action of any string theory admits a fundamental string solution [4] and its magnetic dual the non-singular solitonic five-brane solution [5–9]. These are the extremal limit, as we have mentioned, of the black string and black five-brane solutions of the corresponding supergravity equations of motion. Since it is known that the equations of motion of type IIB supergravity theory is invariant [10] under an SL(2, R) group one can construct a more general string like as well as five-brane solutions using this symmetry group. We should like to mention that the SL(2, R) symmetry of type IIB string theory is a non-perturbative symmetry. It transforms the string coupling constant in a non-trivial way and therefore mixes up the perturbative and non-perturbative effects of Type IIB string theory. A discrete subgroup of this SL(2, R) group has been conjectured to be the exact symmetry group of the quantum type IIB string theory [11]. Using this symmetry, Schwarz [12] has constructed an SL(2, Z) multiplet of string like solutions in type IIB string theory starting from the fundamental string solution. Both the string tension and the charge were shown to be given by the SL(2, Z) covariant expressions. Since the tension and the charge of these extremal solutions remain unrenormalized, it provides a strong support in favor of the conjecture that SL(2, Z) is an exact symmetry group of the quantum theory.

Given the symmetry of the type IIB theory, it is natural to expect that the solitonic five-brane solution should also form an SL(2, Z) multiplet as pointed out by Schwarz in ref.[12]. In this paper, we construct* an infinite family of magnetically charged super five-branes, permuted by SL(2, Z) group, starting from the solitonic five-brane solution of string theory. These solutions are characterized by a pair of integers corresponding to the magnetic charges associated with the two three-form field strengths present in the NSNS and RR sector of the spectrum. When these two integers (q_1, q_2) are relatively prime to each other, the five-brane solutions are shown to be stable and can be regarded as bound state [15] configuration of q_1 solitonic five-branes with q_2 D5-branes [16]. The magnetic charge as well as the five-brane tension are shown to be given by SL(2, Z) covariant expressions. This provides more evidence that SL(2, Z) is indeed an exact symmetry group of the quantum theory. The expression for the tension of these (q_1, q_2) super five-branes has been envisaged by Witten [15] some time ago. We then discuss that similar magnetic dual solutions of strings also exist in lower dimensional type II theory.

The low energy effective action common to any ten dimensional string theory has the form:

$$S = \int d^{10}x\sqrt{-g}\left[R - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{12}e^{-\phi}H^{(1)}_{\mu\nu\lambda}H^{(1)\,\mu\nu\lambda}\right] \tag{1}$$

Here $g = (\det g_{\mu\nu})$, $g_{\mu\nu}$ being the canonical metric which is related to string metric by $G_{\mu\nu} = e^{\phi/2}g_{\mu\nu}$. R is the scalar curvature with respect to the canonical metric, ϕ is the dilaton and $H^{(1)}_{\mu\nu\lambda}$ is the field strength associated with the Kalb-Ramond antisymmetric tensor field $B^{(1)}_{\mu\nu}$. These are the massless modes which couple to any string theory. For type II strings these massless modes belong to the NSNS sector of the spectrum. The equations of motion following from (1) are:

$$\nabla_\mu\left(e^{-\phi}H^{(1)\,\mu\nu\lambda}\right) = 0 \tag{2}$$

$$\nabla^2\phi + \frac{1}{12}e^{-\phi}\left(H^{(1)}\right)^2 = 0 \tag{3}$$

$$R_{\mu\nu} - \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{4}e^{-\phi}H^{(1)}_{\mu\rho\sigma}H^{(1)\;\rho\sigma}_{\nu} + \frac{1}{48}e^{-\phi}\left(H^{(1)}\right)^2 g_{\mu\nu} = 0 \tag{4}$$

These low energy field equations can be solved by using certain ansatz on the metric and the field strength $H^{(1)}_{\mu\nu\lambda}$. By demanding that the metric be static, spherically symmetric which becomes flat asymptotically with a regular horizon, one can obtain both the electrically charged black string solution and the magnetically charged black five-brane solution

*Earlier attempt for the construction of SL(2, Z) multiplet of five-brane solution of type IIB string theory were made in [13,14], but the solutions described there are incomplete.

from (2) – (4) as given below [3]

$$
\begin{aligned}
ds^2 &= -\left(1-\frac{r_+^6}{r^6}\right)\left(1-\frac{r_-^6}{r^6}\right)^{-1/4} dt^2 + \left(1-\frac{r_-^6}{r^6}\right)^{3/4} (dx^1)^2 \\
&\qquad + \left(1-\frac{r_+^6}{r^6}\right)^{-1}\left(1-\frac{r_-^6}{r^6}\right)^{-11/12} dr^2 + r^2\left(1-\frac{r_-^6}{r^6}\right)^{1/12} d\Omega_7^2 \\
e^{2\phi} &= \left(1-\frac{r_-^6}{r^6}\right), \qquad H^{(1)} = 6\,(r_+ r_-)^3 * e^{\phi}\epsilon_7
\end{aligned}
\tag{5}
$$

for the string solution and

$$
\begin{aligned}
ds^2 &= -\left(1-\frac{r_+^2}{r^2}\right)\left(1-\frac{r_-^2}{r^2}\right)^{-3/4} dt^2 + \left(1-\frac{r_-^2}{r^2}\right)^{1/4} \delta_{ij} dx^i dx^j \\
&\qquad + \left(1-\frac{r_+^2}{r^2}\right)^{-1}\left(1-\frac{r_-^2}{r^2}\right)^{-3/4} dr^2 + r^2\left(1-\frac{r_-^2}{r^2}\right)^{1/4} d\Omega_3^2 \\
e^{-2\phi} &= \left(1-\frac{r_-^2}{r^2}\right), \qquad H^{(1)} = 2\,(r_+ r_-)\,\epsilon_3
\end{aligned}
\tag{6}
$$

for the five-brane solution. Here $i,\, j = 1, 2, \ldots, 5$. $d\Omega_7^2$ and $d\Omega_3^2$ above are the metric on the unit seven dimensional and three dimensional spheres respectively. ϵ_7 and ϵ_3 are the corresponding volume forms. The '$*$' denotes the Hodge duality transformation. r_+ and r_- are the two parameters related to the mass and the charge of the solutions. Thus eqs. (5) and (6) represent the two parameter family of black string and black five-brane solutions of string theory with an event horizon at $r = r_+$ and an inner horizon $r = r_-$ (where $r_+ \geq r_-$). It is clear from the form of $H^{(1)}$ in (5) that the string solution is electrically charged whereas from (6) we note that the five-brane solution is magnetically charged. We would like to mention that if the supergravity action contains a general $(d+1)$-form field strength then the magnetically charged solution can be obtained from electrically charged solution by using the duality transformation $\phi \to -\phi$, $e^{-\phi} * H^{(1)} \to H^{(1)}$ and $d \to 8-d$. Thus the five-brane solution can be obtained from the string solution using this duality transformation. Since, e^{ϕ} is the string coupling constant, the magnetically charged five-brane solution is a non-perturbative solution of weakly coupled string theory. Note that the solutions (5) and (6) are written assuming that the dilaton vanishes in the asymptotic limit, but we will restore the asymptotically constant value of the dilaton when we write the more general type IIB solution. Finally, we note that for $r_+ > r_-$, the solutions are non-extremal and therefore non-BPS, but when $r_+ = r_-$ the solutions become supersymmetric saturating the BPS limit.

In the extremal limit the BPS saturated string solution given in (5) (with $r_+ = r_-$), was constructed previously by Dabholkar et. al. [4]. By going to the isotropic coordinate $\rho^6 = r^6 - r_-^6$, we can rewrite the metric (5) in the extremal limit as,

$$ds^2 = \left(1 + \frac{r_-^6}{\rho^6}\right)^{-3/4} \left[-(dt)^2 + (dx^1)^2\right] + \left(1 + \frac{r_-^6}{\rho^6}\right)^{1/4} \left(d\rho^2 + \rho^2 d\Omega_7^2\right) \tag{7}$$

with $e^{-2\phi} = \left(1 + \frac{r_-^6}{\rho^6}\right)$. This is precisely the solution discussed in ref.[4] and clarifies the relation with the solution described in [3]. It is clear from (7) that in terms of the string metric $G_{\mu\nu} = \left(1 + \frac{r_-^6}{\rho^6}\right)^{-1/4} g_{\mu\nu}$, the above metric becomes flat transverse to the string direction. Also note that the solution (7) is singular since for $\rho \to 0$, the radius of S^7 vanishes and the curvature blows up as ρ^{-2}. This is the reason that string like solution has been constructed [4] by coupling the supergravity action to a macroscopic string source. This BPS saturated singular string like solution is also known as the fundamental string solution. By using the symmetry of the full type IIB string theory including the RR sector, Schwarz has constructed an infinite family of string like solutions starting from this fundamental string solution. In ref.[17], we have pointed out that a similar infinite family of string solutions also exist in $D < 10$ type II string theory.

Let us next look at the black five-brane solution of string theory given in (6). Note that in general the solution is invariant under the symmetry group R $\times$ E(5) $\times$ SO(4) where E(5) is the five dimensional Euclidean group. At the extremal limit, the solution acquires an extra boost symmetry and thus the symmetry group becomes P(6) $\times$ SO(4), where P(6) is the six dimensional Poincare group. So, only at the extremal limit the solution describes the BPS super five-brane and takes the following form:

$$\begin{aligned} ds^2 &= \left(1 - \frac{r_-^2}{r^2}\right)^{1/4} \left[-(dt)^2 + \delta_{ij} dx^i dx^j\right] + \left(1 - \frac{r_-^2}{r^2}\right)^{-7/4} dr^2 + r^2 \left(1 - \frac{r_-^2}{r^2}\right)^{1/4} d\Omega_3^2 \\ Q &= 2r_-^2, \qquad e^{-2\phi} = \left(1 - \frac{Q}{2r^2}\right), \qquad H^{(1)} = Q\epsilon_3 \end{aligned} \tag{8}$$

In terms of the isotropic coordinate $\rho^2 = r^2 - r_-^2$ the solution can be written as,

$$\begin{aligned} ds^2 &= \left(1 + \frac{r_-^2}{\rho^2}\right)^{-1/4} \left[-(dt)^2 + \delta_{ij} dx^i dx^j\right] + \left(1 + \frac{r_-^2}{\rho^2}\right)^{3/4} \left(d\rho^2 + \rho^2 d\Omega_3^2\right) \\ Q &= 2r_-^2, \qquad e^{2\phi} = \left(1 + \frac{Q}{2\rho^2}\right), \qquad H^{(1)} = Q\epsilon_3 \end{aligned} \tag{9}$$

In the string frame $G_{\mu\nu} = \left(1 + \frac{r_-^2}{\rho^2}\right)^{1/4} g_{\mu\nu}$, the metric in (9) reduces to

$$ds^2 = \left[-(dt)^2 + \delta_{ij}dx^i dx^j\right] + \left(1 + \frac{r_-^2}{\rho^2}\right)\left(d\rho^2 + \rho^2 d\Omega_3^2\right) \tag{10}$$

Note that unlike the string solution, the super five-brane solution is regular as $\rho \to 0$ since in this case the radius of S^3 is finite (r_-) and the curvature also remains finite (r_-^{-2}) as $\rho \to 0$.

We would like to point out that the low energy string effective action we have considered in (1) can be regarded as a special case of more general type IIB action when the RR fields are included. Let us recall that the massless states of type IIB string theory in the bosonic sector consist of a graviton ($G_{\mu\nu}$), a dilaton (ϕ) and an antisymmetric tensor field ($B^{(1)}_{\mu\nu}$) as NSNS gauge fields whereas in the RR sector it has another scalar (χ), another antisymmetric tensor field ($B^{(2)}_{\mu\nu}$) and a four-form gauge field ($A^+_{\mu\nu\rho\sigma}$) whose field strength is self-dual. It is well-known that the equations of motion of type IIB supergravity theory is invariant under an SL(2, R) group known as the supergravity duality group [10]. The four-form gauge field is a singlet under this duality group and it couples to a self-dual three-brane whose form has been derived in ref.[3,18]. Since we are interested in the five-brane solution we set the corresponding five-form field strength to zero in what follows. In this case the type IIB supergravity equations of motion can be derived from the following covariant action [19],

$$\begin{aligned}\tilde{S}_{\rm IIB} \;&=\; \int d^{10}x\sqrt{-G}\left[e^{-2\phi}\left(R + 4\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{12}H^{(1)}_{\mu\nu\lambda}H^{(1)\,\mu\nu\lambda}\right)\right.\\ &\left.\quad -\frac{1}{2}\nabla_\mu\chi\nabla^\mu\chi - \frac{1}{12}\left(H^{(2)}_{\mu\nu\lambda} + \chi H^{(1)}_{\mu\nu\lambda}\right)\left(H^{(2)\,\mu\nu\lambda} + \chi H^{(1)\,\mu\nu\lambda}\right)\right]\end{aligned} \tag{11}$$

We can rewrite the action (11) in the Einstein frame as follows:

$$\begin{aligned}S_{\rm IIB} \;&=\; \int d^{10}x\sqrt{-g}\left[R - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{2}e^{2\phi}\nabla_\mu\chi\nabla^\mu\chi\right.\\ &\left.\quad -\frac{1}{12}\left(e^{-\phi}H^{(1)}_{\mu\nu\lambda}H^{(1)\,\mu\nu\lambda} + e^{\phi}\left(H^{(2)}_{\mu\nu\lambda} + \chi H^{(1)}_{\mu\nu\lambda}\right)\left(H^{(2)\,\mu\nu\lambda} + \chi H^{(1)\,\mu\nu\lambda}\right)\right)\right]\end{aligned} \tag{12}$$

It is to be noted that (12) reduces precisely to the effective action (1) when the RR fields are set to zero. Since (1) is a special case of (12), the five-brane solution obtained from (1) (given in (6), (8) or (9)) can also be generalized for the type IIB theory. We are going to construct in the following these generalized solution of five-branes of type IIB theory.

The construction will be facilitated if we write the action (12) in the manifestly SL(2, R) invariant form as given below [12,19]:

$$S_{\rm IIB} = \int d^{10}x\sqrt{-g}\left[R + \frac{1}{4}{\rm tr}\,\nabla_\mu\mathcal{M}\nabla^\mu\mathcal{M}^{-1} - \frac{1}{12}\mathcal{H}^T_{\mu\nu\lambda}\mathcal{M}\mathcal{H}^{\mu\nu\lambda}\right] \tag{13}$$

where $\mathcal{M} \equiv \begin{pmatrix} \chi^2 + e^{-2\phi} & \chi \\ \chi & 1 \end{pmatrix} e^\phi$ represents an SL(2, R) matrix and $\mathcal{H}_{\mu\nu\lambda} \equiv \begin{pmatrix} H^{(1)}_{\mu\nu\lambda} \\ H^{(2)}_{\mu\nu\lambda} \end{pmatrix}$. Also the superscript '$T$' represents the transpose of a matrix. The action (13) can be easily seen to be invariant under the following global SL(2, R) transformations:

$$\mathcal{M} \to \Lambda\mathcal{M}\Lambda^T, \qquad \mathcal{H}_{\mu\nu\lambda} \to \left(\Lambda^{-1}\right)^T\mathcal{H}_{\mu\nu\lambda}, \qquad g_{\mu\nu} \to g_{\mu\nu} \tag{14}$$

where $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $ad - bc = 1$, represents a global SL(2, R) transformation matrix. It is easy to check that under the transformation (14), the complex scalar $\lambda = \chi + ie^{-\phi}$ and the two field strengths $H^{(1)}_{\mu\nu\lambda}$ and $H^{(2)}_{\mu\nu\lambda}$ transform as,

$$\begin{aligned} \lambda &\to \frac{a\lambda + b}{c\lambda + d} \\ H^{(1)}_{\mu\nu\lambda} &\to dH^{(1)}_{\mu\nu\lambda} - cH^{(2)}_{\mu\nu\lambda} \\ H^{(2)}_{\mu\nu\lambda} &\to -bH^{(1)}_{\mu\nu\lambda} + aH^{(2)}_{\mu\nu\lambda} \end{aligned} \tag{15}$$

We would like to point out here that unlike the electrically charged string solution, the magnetic charges associated with $H^{(1)}_{\mu\nu\lambda}$ and $H^{(2)}_{\mu\nu\lambda}$ of the five-brane should transform in the same way as the field strengths themselves. This follows from the fact that Noether charge (or the electrical charge) of the string solution is conserved due to the equation of motion following from (13) whereas the topological charge (or the magnetic charge) of the five-brane is conserved due to Bianchi identity. Therefore the magnetic charges of the five-branes transform as $\mathcal{Q} \to (\Lambda^{-1})^T\mathcal{Q}$ or in components,

$$\begin{aligned} Q^{(1)} &\to dQ^{(1)} - cQ^{(2)} \\ Q^{(2)} &\to -bQ^{(1)} + aQ^{(2)} \end{aligned} \tag{16}$$

Note that the original solution (8) or (9) had one charge Q associated with $H^{(1)} = Q\epsilon_3$ and this charge was quantized in some basic units. Now after the transformation (16) Q will no longer remain quantized. So, in order to recover the charge quantization [20] we modify the original charge by $\Delta^{1/2}_{(q_1,q_2)}Q$, where $\Delta_{(q_1,q_2)}$ is an arbitrary constant which will be determined later. The construction of the SL(2, R) matrix Λ can be motivated such that it properly reproduces the asymptotic value of the complex scalar $\lambda_0 = \chi_0 + ie^{-\phi_0}$, after the transformation, where ϕ_0 and χ_0 are the asymptotic value of the dilaton and the RR scalar. The relevant SL(2, R) transformation matrix then takes the form [12,17]

$$\Lambda = \begin{pmatrix} e^{-\phi_0}\cos\alpha + \chi_0 \sin\alpha & -e^{-\phi_0}\sin\alpha + \chi_0\cos\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} e^{\phi_0/2} \tag{17}$$

Here α is an arbitrary parameter which will be fixed from the charge quantization condition. Then from (16), we find the charges associated with $H^{(1)}_{\mu\nu\lambda}$ and $H^{(2)}_{\mu\nu\lambda}$ to be given as,

$$\begin{aligned} Q^{(1)} &= e^{\phi_0/2}\cos\alpha \Delta^{1/2}_{(q_1,q_2)} Q \\ Q^{(2)} &= \left(e^{-\phi_0/2}\sin\alpha - \chi_0 e^{\phi_0/2}\cos\alpha\right)\Delta^{1/2}_{(q_1,q_2)} Q \end{aligned} \tag{18}$$

By demanding that the charges be quantized we find,

$$\begin{aligned} \cos\alpha &= e^{-\phi_0/2}\Delta^{-1/2}_{(q_1,q_2)} q_1 \\ \sin\alpha &= e^{\phi_0/2}\left(q_2 + q_1\chi_0\right)\Delta^{-1/2}_{(q_1,q_2)} \end{aligned} \tag{19}$$

where q_1 and q_2 are integers. Using $\cos^2\alpha + \sin^2\alpha = 1$, (19) determines the value of $\Delta_{(q_1,q_2)}$ as,

$$\begin{aligned} \Delta_{(q_1,q_2)} &= e^{-\phi_0} q_1^2 + \left(q_2 + q_1\chi_0\right)^2 e^{\phi_0} \\ &= (q_1,\ q_2)\, \mathcal{M}_0 \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \end{aligned} \tag{20}$$

where $\mathcal{M}_0 = \begin{pmatrix} \chi_0^2 + e^{-2\phi_0} & \chi_0 \\ \chi_0 & 1 \end{pmatrix} e^{\phi_0}$. It is clear from (20) that $\Delta_{(q_1,q_2)}$ is SL(2, Z) covariant. Therefore, the charge of a (q_1, q_2) five-brane is given by an SL(2, Z) covariant expression

$$\begin{aligned} Q_{(q_1,q_2)} &= \Delta^{1/2}_{(q_1,q_2)} Q \\ &= \sqrt{e^{-\phi_0} q_1^2 + \left(q_2 + q_1\chi_0\right)^2 e^{\phi_0}}\; Q \end{aligned} \tag{21}$$

Note from (14) that the canonical metric does not change under the SL(2, R) transformation. However, since the charge Q is now replaced by $Q_{(q_1,q_2)}$ the metric given in (9) takes the following form:

$$ds^2 = \left(1 + \frac{Q_{(q_1,q_2)}}{2\rho^2}\right)^{-1/4}\left[-(dt)^2 + \delta_{ij}dx^i dx^j\right] + \left(1 + \frac{Q_{(q_1,q_2)}}{2\rho^2}\right)^{3/4}\left(d\rho^2 + \rho^2 d\Omega_3^2\right) \tag{22}$$

The complex scalar field λ changes as

$$\begin{aligned} \lambda &= \frac{a\left(ie^{-\phi}\right)+b}{c\left(ie^{-\phi}\right)+d} \\ &= \frac{\chi_0\Delta_{(q_1,q_2)}A_{(q_1,q_2)}+q_1q_2e^{-\phi_0}\left(A_{(q_1,q_2)}-1\right)+i\Delta_{(q_1,q_2)}A^{1/2}_{(q_1,q_2)}e^{-\phi_0}}{q_1^2e^{-\phi_0}+A_{(q_1,q_2)}e^{\phi_0}\left(\chi_0q_1+q_2\right)^2} \end{aligned} \tag{23}$$

where $A_{(q_1,q_2)} = \left(1+\frac{Q_{(q_1,q_2)}}{2\rho^2}\right)^{-1}$. Note that asymptotically as $\rho\to\infty$, $A_{(q_1,q_2)}\to 1$ and therefore, $\lambda\to\lambda_0$ as expected. The real and imaginary part of (23) give the transformed value of the RR scalar and the dilaton of the theory. Finally, the transformed value of the field strengths $H^{(1)}$ and $H^{(2)}$ can be obtained from (15) as,

$$\begin{aligned} H^{(1)} &= q_1Q\epsilon_3 \\ H^{(2)} &= q_2Q\epsilon_3 \end{aligned} \tag{24}$$

which can be written compactly as follows,

$$\mathcal{H} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} Q\epsilon_3 \tag{25}$$

We can also calculate the tension of a (q_1, q_2) five-brane by calculating the ADM mass per unit five-volume [21]. We note that in general the ADM mass is given by $M = (3r_+^2 - r_-^2)$ and therefore, for the non-extremal case the mass and the charge are independent parameters. But in the extremal case they are related since in that case, $M = 2r_-^2 = Q$. We have seen in (21) that the charge of a (q_1, q_2) five-brane has been modified by $\Delta^{1/2}_{(q_1,q_2)}Q$ and so in order to equate the mass with the charge, mass per unit five-volume i.e. the tension must also satisfy the similar relation:

$$\begin{aligned} T_{(q_1,q_2)} &= \Delta^{1/2}_{(q_1,q_2)}T \\ &= \sqrt{e^{-\phi_0}q_1^2+\left(q_2+q_1\chi_0\right)^2e^{\phi_0}}\;T \\ &= \sqrt{g_s^{-1}q_1^2+\left(q_2+q_1\chi_0\right)^2g_s}\;T \end{aligned} \tag{26}$$

where $g_s = e^{\phi_0}$ in the last expression of (26) denotes the string coupling constant. Thus when $\chi = 0$, the solitonic five-brane or (1, 0) brane tension is proportional to $1/\sqrt{g_s}$ whereas D5-brane or (0, 1) brane tension is proportional to $\sqrt{g_s}$ in the canonical metric. In the string metric, on the other hand, the tension of a general (q_1, q_2) five-brane is given by,

$$T_{(q_1,q_2)} = g_s^{-3/2}\sqrt{g_s^{-1}q_1^2+g_sq_2^2}\;T \tag{27}$$

Here $T_{(q_1,q_2)}$ gets scaled by $g_s^{-3/2}$ because, it has the dimensionality of (length)$^{-6}$. Thus in the string metric, the tension of a solitonic five-brane is proportional to $1/g_s^2$ and the tension of a D5-brane is proportional to $1/g_s$ as expected. This tension formula of a (q_1, q_2) five-brane has been envisaged by Witten in ref.[15].

Thus starting from the solitonic five-brane solution, we have obtained an infinite family of five-brane solutions permuted by SL(2, Z) group in type IIB theory given by the metric and other field configurations in (22) – (25).

The stability of these (q_1, q_2) five-brane solutions can be understood along the same line as in the case of string solutions [17,21]. Since the tension of a (q_1, q_2) five-brane is given in (26), it can be easily checked that when $\chi = 0$, the five-brane tension satisfy the following triangle inequality

$$T_{(p_1,p_2)} + T_{(q_1,q_2)} \geq T_{(p_1+q_1,p_2+q_2)} \tag{28}$$

Such relation is quite typical of a BPS state. The equality holds when $p_1q_2 = p_2q_1$ or in other words when $p_1 = nq_1$ and $p_2 = nq_2$, where n is any integer. Thus when q_1 and q_2 are relatively prime, the inequality prevents the five-brane state to decay into five-branes of lower masses. Since the charge of a (q_1, q_2) five-brane also satisfies similar relation (21) it can be readily checked again that when q_1, q_2 are coprime the charge conservation can not be satisfied if the five-brane decay into multiple five-branes. Thus (q_1, q_2) five-brane configuration with q_1, q_2 relatively prime, describes a bound state configuration of q_1 solitonic five-branes with q_2 D5-branes.

Note here, as discussed by Witten, that unlike the string solution, the D5-branes themselves do not form bound states as the six dimensional super Yang-Mills theory does not contain vacua with a mass gap. On the other hand, D5-branes when combined with solitonic five-branes do form bound states and has been discussed qualitatively by Witten in ref.[15].

The SL(2, Z) multiplet of black fivebranes can be obtained similarly. Eqs. (14)–(21) and Eqs. (23)–(25) remain the same but (22) should be replaced by (6) with r_+ and r_- now given by $2(r_+r_-) = \Delta^{1/2}_{(q_1,q_2)}Q$. The same procedure applies to the construction of the SL(2, Z) multiplet of black strings. These multiplets may be useful in studying the physics of black strings and fivebranes.

Finally, we would like to mention that similar infinite family of magnetic dual solutions of strings also exist in $D > 4$. The low energy effective action common to any string theory in D dimensions has the form:

$$S_D = \int d^D x \sqrt{-g} \left[R - \frac{4}{D-2} \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{12} e^{-\frac{8}{D-2}\phi} H^{(1)}_{\mu\nu\lambda} H^{(1)\,\mu\nu\lambda} \right] \tag{29}$$

The equations of motion following from (29) are given as:

$$\nabla_\mu \left(e^{-\frac{8}{D-2}\phi} H^{(1)\,\mu\nu\lambda} \right) = 0 \tag{30}$$

$$\nabla^2 \phi + \frac{1}{12} e^{-\frac{8}{D-2}\phi} \left(H^{(1)} \right)^2 = 0 \tag{31}$$

$$R_{\mu\nu} - \frac{4}{D-2} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{4} e^{-\frac{8}{D-2}\phi} H^{(1)}_{\mu\rho\sigma} H^{(1)\ \rho\sigma}_{\nu} + \frac{1}{6(D-2)} e^{-\frac{8}{D-2}\phi} \left(H^{(1)} \right)^2 g_{\mu\nu} = 0 \tag{32}$$

By using the same ansatz on the metric as before one can obtain the magnetic dual solution of the string in $D > 4$ dimensions of the form:

$$ds^2_D = A^{-\frac{2}{D-2}} \left[-(dt)^2 + \delta_{ij} dx^i dx^j \right] + A^{\frac{D-4}{D-2}} \left(d\rho^2 + \rho^2 d\Omega_3^2 \right) \tag{33}$$

where A is a function of radial coordinate ρ only whose explicit form is given in [9] as:

$$e^{2\phi} = (A(\rho))^{\sqrt{\frac{D-2}{8}}} = \left(1 + \frac{r_-^2}{\rho^2} \right)^{\sqrt{\frac{D-2}{8}}} \tag{34}$$

where r_- is the charge of the dual object. Also, the field strength $H^{(1)} = Q\epsilon_3$. The magnetic dual object is a 4-brane in $D = 9$, a 3-brane in $D = 8$, a 2-brane in $D = 7$, a string in $D = 6$ and a 0-brane (a particle) in $D = 5$.

Since it is known that the toroidally compactified type IIB string theory also possesses the SL(2, R) invariance [23,24] (This symmetry can also be obtained in $D \leq 9$ from toroidally compactified M-theory, see, e.g., [25].) with the same transformation properties of $\mathcal{H}_{\mu\nu\lambda}$ and the complex scalar λ and since the reduced action can be shown to be given by (29) when RR fields are set to zero, we can straightforwardly use the SL(2, Z) rotation to find the SL(2, Z) family of the dual solutions of strings starting from (33) and (34). The solutions in this case are very similar as in the ten dimensional case. (For the detailed construction of string solution in $D < 10$ see [17].) The corresponding black SL(2, Z) multiplet for each of the dual objects can be constructed in a similar fashion described above.

To conclude, we have constructed in this paper an SL(2, Z) family of super five-brane solutions in type IIB string theory starting from the known non-singular solitonic five-brane solution. These solutions are characterized by a pair of integers corresponding to the magnetic charges associated with the two three-form field strengths in the NSNS and RR sector of the theory. We have shown that both the charge and the tension of a general (q_1, q_2) five-brane are given by SL(2, Z) covariant expressions. This provides more evidence in support of the conjecture that SL(2, Z) is an exact symmetry group of the quantum type IIB string theory. When the integers q_1 and q_2 are relatively prime, we have shown that the five-brane is stable since it is prevented from decaying into multiple five-branes by a triangle inequality relation of both the tension as well as the charge associated with the five-brane. We have obtained the tension formula for a general (q_1, q_2) five-brane as envisaged by Witten in ref.[15]. We have discussed that a similar family of the magnetic-dual solutions of the string also exists in each of $D > 4$ dimensions in type II string theory. We also discussed how to obtain the corresponding SL(2, Z) multiplets for black strings and its dual objects for $10 \geq D > 4$.

References:

1. G. W. Gibbons, Nucl. Phys. B207 (1982) 337; G. W. Gibbons and K. Maeda, Nucl. Phys. B298 (1988) 741.

2. D. Garfinkle, G. Horowitz and A. Strominger, Phys. Rev. D43 (1991) 3140, Erratum: Phys. Rev. D45 (1992) 3888.

3. G. Horowitz and A. Strominger, Nucl. Phys. B360 (1991) 197.

4. A. Dabholkar and J. A. Harvey, Phys. Rev. Lett. 63 (1989) 478; A. Dabholkar, G. Gibbons, J. A. Harvey and F. Ruiz Ruiz, Nucl. Phys. B340 (1990) 33.

5. A. Strominger, Nucl. Phys. B343 (1990) 167.

6. S. J. Rey, Phys. Rev. D43 (1991) 526.

7. C. G. Callan, J. A. Harvey and A. Strominger, Nucl. Phys. B359 (1991) 611.

8. M. J. Duff and J. X. Lu, Nucl. Phys. B354 (1991) 141.

9. M. J. Duff, R. R. Khuri and J. X. Lu, Phys. Rep. 259 (1995) 213.

10. B. Julia, in *Supergravity and Superspace*, Eds. S. W. Hawking and M. Rocek, Cambridge University Press, 1981.

11. C. M. Hull and P. K. Townsend, Nucl. Phys. B438 (1995) 109.

12. J. H. Schwarz, Phys. Lett. B360 (1995) 13 (see revised version, hep-th/9508143).

13. E. Bergshoeff, H. J. Boonstra and T. Ortin, Phys. Rev. D53 (1996) 7206.

14. R. Poppe and S. Schwager, Phys. Lett. B393 (1997) 51.

15. E. Witten, Nucl. Phys. B460 (1996) 335.

16. J. Polchinski, Phys. Rev. Lett. 75 (1995) 4724; J. Polchinski, S. Chaudhuri and C. V. Johnson, *Notes on D-Branes*, hep-th/9602052; J. Polchinski, *TASI Lectures on D-Branes*, hep-th/9611050.

17. S. Roy, *SL(2, Z) Multiplets of Type II Superstrings in* $D < 10$, hep-th/9706165 (to appear in Phys. Lett. B).

18. M. J. Duff and J. X. Lu, Phys. Lett. B273 (1991) 409.

19. C. Hull, Phys. Lett. B357 (1995) 545.

20. R. I. Nepomechie, Phys. Rev. D31 (1985) 1921; C. Teitelboim, Phys. Lett. B167 (1986) 69.

21. J. X. Lu, Phys. Lett. B313 (1993) 29.

22. J. H. Schwarz, *Lectures on Superstring and M-Theory Dualities*, hep-th/9607201.

23. J. Maharana, Phys. Lett. B402 (1997) 64.

24. S. Roy, *On S-Duality of Toroidally Compactified Type IIB String Effective Action*, hep-th/9705016.

25. E. Cremmer, B. Julia, H. Lu and C. Pope, *Dualisation of Dualities. I.*, hep-th/9710119.