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$D = 11$, $p = 5$

P.S. Howe,

Dept. of Mathematics,

King's College, London, UK.

and

E. Sezgin,

Center for Theoretical Physics,

Texas A & M University,

College Station, Texas, US.

Abstract

The equations of motion of the super five-brane in $D = 11$ dimensions are derived using the formalism of superembeddings. The equations describe highly nonlinear self-interactions of a tensor multiplet in the six dimensional worlsurface, and they have manifest worldsurface local supersymmetry. The geometry of the target space corresponds to $D = 11$ supergravity.

In a recent paper [1] it was shown that all super p-branes preserving half-supersymmetry and with $N=1$ target space supersymmetry, for any dimension D of spacetime, or $N=2$ for $D=10$, can be understood from the perspective of superembeddings of one supermanifold, the worldsurface, into another, the target superspace, and furthermore that the basic superembedding condition which determines the worldsurface multiplets is both very natural and universal. The analysis given in [1] was mainly at the linearised level but is applicable to all branes whether they are type I or type II, where type II branes are those which have physical worldsurface bosons which are not all scalars. Reference [1] follows in the tradition of the doubly supersymmetric approach to supersymmetric extended objects initiated in [2] for which we refer the reader to [3], where the $D=11$ supermembrane was discussed as a supermebedding in a flat target superspace, for a full set of references to the earlier literature. In this paper we briefly report on the full non-linear equations of motion for the 5-brane in $D=11$. Partial results for the bosonic sector have been obtained previously [4, 5, 6, 7], but the superspace approach, as we shall show, gives a systematic method for determining the full system of equations of motion. A detailed discussion of both the 2-brane and the 5-brane in D=11 in arbitrary backgrounds is in preparation.

We consider embeddings $M \hookrightarrow \underline{M}$, where the worldsurface M has (even|odd) dimension (6|16) and the target space, $\underline{M}$, has dimension (11|32). In local coordinates $z^{\underline{M}}$ for $\underline{M}$ and z^M for M the embedded submanifold is given as $z^{\underline{M}}(z)$. We define the embedding matrix $E_A{}^{\underline{A}}$ to be the derivative of the embedding referred to preferred bases on both manifolds:

$$E_A{}^{\underline{A}} = E_A{}^M \partial_M z^{\underline{M}} E_{\underline{M}}{}^{\underline{A}}, \tag{1}$$

where $E_M{}^A$ ($E_A{}^M$) is the supervielbein (inverse supervielbein) which relates the preferred frame basis to the coordinate basis, and the target space supervielbein has underlined indices. The notation is as follows: indices from the beginning (middle) of the alphabet refer to frame (co-ordinate) indices, latin (greek) indices refer to even (odd) components and capital indices to both, non-underlined (underlined) indices refer to M ($\underline{M}$) and primed indices refer to normal directions. We shall also employ a two-step notation for spinor indices; that is, for general formulae a spinor index α (or α') will run from 1 to 16, but to interpret these formulae we shall replace a subscript α by a subscript pair αi and a subscript α' by a pair ${}_i^\alpha$, where $\alpha = 1, \ldots 4$ and $i = 1, \ldots 4$ reflecting the $Spin(1,5) \times USp(4)$ group structure of the $N=2, d=6$ worldsurface superspace. (A lower (upper) α index denotes a left-handed (right-handed) $d=6$ Weyl spinor and the $d=6$ spinors that occur in the theory are all symplectic Majorana-Weyl.)

We shall find it convenient to introduce a basis for the normal bundle, $E_{A'} = (E_{a'}, E_{\alpha'})$, given in terms of a basis of the tangent bundle of the target space $E_{\underline{A}}$ (at any point $p \in M$) by

$$E_{A'} = E_{A'}{}^{\underline{A}} E_{\underline{A}} \ . \tag{2}$$

We can assemble the embedding matrix and the normal matrix into a square matrix which we shall denote by $E_{\overline{A}}{}^{\underline{A}} = (E_A{}^{\underline{A}}, E_{A'}{}^{\underline{A}})$. The inverse of this matrix will be written as $E_{\underline{A}}{}^{\overline{A}} = (E_{\underline{A}}{}^A, E_{\underline{A}}{}^{A'})$

The basic equation describing the embedding is

$$E_\alpha{}^{\underline{a}} = 0 \ . \tag{3}$$

Its geometrical meaning is that the odd tangent space of the worldsurface is a subspace of the odd tangent space of the target space at each point $p \in M$. As we shall see later, an immediate consequence of (3) is

$$E_\alpha{}^{\underline{\alpha}} E_\beta{}^{\underline{\beta}} T_{\underline{\alpha}\underline{\beta}}{}^{\underline{c}} = T_{\alpha\beta}{}^c E_c{}^{\underline{c}} \ . \tag{4}$$

We can also choose the odd normal tangent bundle (which is not fixed by the embedding) such that

$$E_{\alpha'} = E_{\alpha'}{}^{\underline{\alpha}} E_{\underline{\alpha}} \ , \tag{5}$$

in other words $E_{\alpha'}{}^{\underline{a}} = 0$. This in turn implies that

$$E_{\underline{\alpha}}{}^{a} = E_{\underline{\alpha}}{}^{a'} = 0 \ . \tag{6}$$

As a consequence of (3), (5) and (6), one finds that the inverse E in the even-even sector is the inverse of the even-even part of E and similarly for the odd-odd sector.

Equations (3) and (4) are the fundamental equations. We observe that they do not require a fixed choice of either the worldsurface or target space even tangent bundle, and that no connections are involved. Nevertheless, in order to work out the consequences of these equations it is useful to make appropriate choices of these objects. We shall assume that the target space supergeometry corresponds to on-shell $D = 11$ supergravity. The structure group is $Spin(1, 10)$. All of the components of the torsion vanish except for [9, 10]

$$T_{\underline{\alpha}\underline{\beta}}{}^{\underline{c}} = -i(\Gamma^{\underline{c}})_{\underline{\alpha}\underline{\beta}} \ , \tag{7}$$

$$T_{\underline{a}\underline{\beta}}{}^{\underline{\gamma}} = -\frac{1}{36}(\Gamma^{\underline{bcd}})_{\underline{\beta}}{}^{\underline{\gamma}} H_{\underline{abcd}} - \frac{1}{288}(\Gamma_{\underline{abcde}})_{\underline{\beta}}{}^{\underline{\gamma}} H^{\underline{bcde}} \ , \tag{8}$$

where $H_{\underline{abcd}}$ is totally antisymmetric, and the dimension 3/2 component $T_{\underline{ab}}{}^{\underline{\gamma}}$. $H_{\underline{abcd}}$ is the dimension one component of the closed superspace 4-form H_4 whose only other non-vanishing component is

$$H_{\underline{ab\gamma\delta}} = -i(\Gamma_{\underline{ab}})_{\underline{\gamma\delta}} \ . \tag{9}$$

With these assumptions (4) becomes

$$E_{\alpha}{}^{\underline{\alpha}} E_{\beta}{}^{\underline{\beta}} (\Gamma^{\underline{c}})_{\underline{\alpha}\underline{\beta}} = i T_{\alpha\beta}{}^{c} E_{c}{}^{\underline{c}} \ . \tag{10}$$

The solution to this equation is given by

$$E_{\alpha}{}^{\underline{\alpha}} = u_{\alpha}{}^{\underline{\alpha}} + h_{\alpha}{}^{\beta'} u_{\beta'}{}^{\underline{\alpha}} \ , \tag{11}$$

and

$$E_{a}{}^{\underline{a}} = u_{a}{}^{\underline{a}} - 72 k_{a}{}^{b} u_{b}{}^{\underline{a}} \ , \tag{12}$$

together with

$$T_{\alpha\beta}{}^{c} = -i(\Gamma^{c})_{\alpha\beta} \to -i\eta_{ij}(\gamma^{c})_{\alpha\beta} \ . \tag{13}$$

with $\eta_{ij} = -\eta_{ji}$ being the $USp(4)$ invariant tensor. In addition, $u_{\alpha}{}^{\underline{\alpha}}$ and $u_{\alpha'}{}^{\underline{\alpha}}$ together make up a 32×32 matrix $u_{\overline{\alpha}}{}^{\underline{\alpha}}$ which is an element of $Spin(1, 10)$ and $u_{a}{}^{\underline{a}}$, $u_{a'}{}^{\underline{a}}$ together make up the corresponding element of the Lorentz group which we shall denote by $u_{\overline{a}}{}^{\underline{a}}$, so that

$$u_{\overline{\alpha}}{}^{\underline{\alpha}} u_{\overline{\beta}}{}^{\underline{\beta}} (\Gamma^{\underline{a}})_{\underline{\alpha}\underline{\beta}} = i(\Gamma^{\overline{a}})_{\overline{\alpha}\overline{\beta}} u_{\overline{a}}{}^{\underline{a}} \ . \tag{14}$$

The tensor $h_{\alpha}{}^{\beta'}$ is given by

$$h_{\alpha}{}^{\beta'} \to h_{\alpha i \beta}{}^{j} = \delta_i{}^j (\gamma^{abc})_{\alpha\beta} h_{abc} \ , \tag{15}$$

where h_{abc} is self-dual. Finally, k_{ab}, defined by

$$k_{ab} = h_{a}{}^{cd} h_{bcd} \ , \tag{16}$$

is symmetric and traceless. Given the above solution one can find another by acting with the group $Spin(1,5)\times USp(4)$ on the $d=6$ odd indices α and with the Lorentz group on the vector indices. There is also the freedom to make Weyl rescalings; since the theory is invariant under these transformations we can, in particular, take the conformal factor to be equal to one.

The odd-odd and even-even components of the normal matrix $E_{A'}{}^{\underline{A}}$ can be chosen to be

$$E_{\alpha'}{}^{\underline{\alpha}} = u_{\alpha'}{}^{\underline{\alpha}} \ , \tag{17}$$

and

$$E_{a'}{}^{\underline{a}} = u_{a'}{}^{\underline{a}} \ , \tag{18}$$

and the inverses in the odd-odd and even-even sectors are

$$E_{\underline{\alpha}}{}^{\alpha} = u_{\underline{\alpha}}{}^{\alpha} \ , \qquad E_{\underline{\alpha}}{}^{\alpha'} = u_{\underline{\alpha}}{}^{\alpha'} - u_{\underline{\alpha}}{}^{\beta} h_{\beta}{}^{\alpha'} \ , \tag{19}$$

and

$$E_{\underline{a}}{}^{a} = u_{\underline{a}}{}^{b} (m^{-1})_b{}^a \ , \qquad E_{\underline{a}}{}^{a'} = u_{\underline{a}}{}^{a'} \ , \tag{20}$$

where we have introduced the inverses of the group matrices u with similar index conventions, and where

$$m_a{}^b = \delta_a{}^b - 72 k_a{}^b \ . \tag{21}$$

We will assume that the matrix m is invertible. The special configurations of h for which $\det m$ vanishes require special care. Such singular points in the field space presumably correspond to a new kind of phase transition. This point deserves further study, and we hope to address it in the future. It may be related to the singular points in field space which arise in the context of the Dirac-Born-Infeld action.

The above results show how the odd tangent spaces of the worldsurface are related to the odd tangent spaces of the target space. However, the even tangent space of the world surface is not fixed. We can choose it, and the worldsurface connection (which takes its values in the Lie algebra $\mathfrak{so}(1,5)\oplus\mathfrak{usp}(4)$), so that the torsion constraints on the worldsurface take a convenient form. In this instance they turn out to be the constraints of $N=2, d=6$ conformal supergravity. In what follows we shall not need all the details of this, but we note that

$$T_{\alpha b}{}^{c} = T_{\alpha\beta}{}^{\gamma} = T_{ab}{}^{c} = 0 \ . \tag{22}$$

The consequences of the embedding equations can now be analysed systematically by going through a set of identities which arises from pulling back the defining equation of the target space torsion 2-form to the worldsurface. These are

$$\nabla_A E_B{}^{\underline{C}} - (-1)^{AB} \nabla_B E_A{}^{\underline{C}} + T_{AB}{}^{C} E_C{}^{\underline{C}} = (-1)^{A(B+\underline{B})} E_B{}^{\underline{B}} E_A{}^{\underline{A}} T_{\underline{AB}}{}^{\underline{C}} \ , \tag{23}$$

where ∇ is covariant with respect to both the worldsurface and target space structure groups. The dimension zero component of (23) is simply (4). We shall not give all the details of the analysis of the rest of these equations here but restrict our attention to indicating how the equations of motion arise. At dimension one-half one finds

$$\nabla_\alpha E_\beta{}^{\underline{\gamma}} + \nabla_\beta E_\alpha{}^{\underline{\gamma}} = i(\Gamma^c)_{\alpha\beta} E_c{}^{\underline{\gamma}} \ . \tag{24}$$

Defining

$$\chi_a{}^{\alpha'} = E_a{}^{\underline{\alpha}} E_{\underline{\alpha}}{}^{\alpha'} \ , \tag{25}$$

one finds, by right-multiplying equation (24) by $E_{\underline{\alpha}}{}^{\alpha'}$, the Dirac equation for the spin one-half fermions

$$(\gamma^a)^{\alpha\beta}\chi_{a\beta}{}^{j}=0\ . \qquad (26)$$

At dimension one one has

$$\nabla_a E_\beta{}^{\underline{\gamma}}-\nabla_\beta E_a{}^{\underline{\gamma}}+T_{a\beta}{}^{\delta}E_\delta{}^{\underline{\gamma}}=E_\beta{}^{\underline{\beta}}E_a{}^{\underline{a}}T_{\underline{a}\underline{\beta}}{}^{\underline{c}}\ . \qquad (27)$$

Right-multiplying by $E_{\underline{\gamma}}{}^{\gamma'}$, one observes that the term involving the worldsurface torsion drops out, and that by using the Dirac equation one can derive

$$\eta^{ab}K_{ab}{}^{c'}=\frac{1}{8}(\gamma^{c'})^{jk}(\gamma^a)^{\beta\gamma}Z_{a\beta j\gamma k}\ , \qquad (28)$$

and

$$\hat{\nabla}^c h_{abc}=-\frac{\eta^{jk}}{96}((\gamma_{[a})^{\beta\gamma}Z_{b],\beta\gamma,jk}+\frac{1}{2}(\gamma_{ab}{}^{c})^{\beta\gamma}Z_{c\beta j\gamma k})\ , \qquad (29)$$

where

$$Z_{a\beta}{}^{\gamma'}=E_\beta{}^{\underline{\beta}}E_a{}^{\underline{a}}T_{\underline{a}\underline{\beta}}{}^{\underline{\gamma}}E_{\underline{\gamma}}{}^{\gamma'}-E_a{}^{\underline{\gamma}}\nabla_\beta E_{\underline{\gamma}}{}^{\gamma'}\ . \qquad (30)$$

The first term in Z is therefore simply a projection of the dimension one torsion; the second term involves the product of a dimension 1/2 E with the odd derivative of a dimension zero E. Both of these quantities are determined from the dimension 1/2 equations, and all dimension 1/2 quantities are expressible in terms of χ. This term is therefore bilinear in χ but also has a somewhat complicated dependence on h_{abc}. The hatted covariant derivative in (29) is defined as follows:

$$\hat{\nabla}_a h_{bcd}=\nabla_a h_{bcd}-3X_{a,[b}{}^{e}h_{cd]e}\ , \qquad (31)$$

with

$$X_{a,b}{}^{c}=(\nabla_a u_b{}^{\underline{c}})u_{\underline{c}}{}^{c}\ . \qquad (32)$$

The left-hand side of (28) is part of the second fundamental form of the surface which we define to be

$$K_{AB}{}^{C'}=(\nabla_A E_B{}^{\underline{C}})E_{\underline{C}}{}^{C'}\ . \qquad (33)$$

The spinor, scalar and tensor equations of motion are the leading components in the worldsurface θ-expansions of equations (26),(28) and (29), respectively. To see that this identification is correct we can appeal to the linearised case where

$$\chi_a{}^{\gamma'}\ \rightarrow\ \partial_a\Theta^{\gamma'}\ , \qquad (34)$$

$$K_{ab}{}^{c'}\ \rightarrow\ \partial_a\partial_b X^{c'}\ , \qquad (35)$$

in a physical gauge, $X^{a'}$ and $\Theta^{\alpha'}$ being the transverse coordinate superfields which describe the excitations of the brane. (These are not independent superfields however, because of (4), see [1].)

Since h_{abc} is self-dual, (29) also implies a modified bosonic Bianchi identity for the tensor fields. However, in order to introduce a corresponding 2-form potential it is necessary to find a superspace 3-form H_3 which obeys a 4-form Bianchi identity. From earlier results [4, 5, 6], and by comparison with Dirichlet-branes in $D=10$ [8], we know that the identity we should expect to hold should have the form

$$dH_3=-\tfrac{1}{24}H_4\ , \qquad (36)$$

where H_4 is the target space 4-form pulled back onto the worldsurface. It follows that H_3 can be written locally as

$$H_3 = dB_2 - \tfrac{1}{24}C_3 \ , \tag{37}$$

where B_2 is a super two-form potential on the worldsurface, and C_3 is the pullback of the target space super three-form such that $H_4 = dC_3$.

The Bianchi identity (36) was verified at the linearised level in [1]. The claim is that (36) is indeed satisfied provided that the only non-vanishing component of H_3 is the one with purely even indices, H_{abc}. To prove this, one has to systematically check the various components of the Bianchi identity (36). Since most of the components of H_3 and H_4 vanish many of these equations are trivially satisfied. The first non-trivial equation arises at dimension zero:

$$-24(\Gamma^c)_{\alpha\beta}H_{abc} = E_\alpha{}^{\underline{\alpha}}E_\beta{}^{\underline{\beta}}E_a{}^{\underline{a}}E_b{}^{\underline{b}}(\Gamma_{\underline{ab}})_{\underline{\alpha\beta}} \ . \tag{38}$$

It is not obvious that the right-hand side of (38) has the same structure as the left-hand side, but it is nevertheless true as one can show directly using (10) and the 'membrane' identity

$$(\Gamma^{\underline{a}})_{(\underline{\alpha\beta}}(\Gamma_{\underline{ab}})_{\underline{\gamma\delta})} = 0 \ . \tag{39}$$

Using (38) one finds the relation between h and H; it is

$$H_{abc} = h_{abc} - 2\cdot 72\, k_a{}^d h_{bcd} + (72)^2 k_a{}^d k_b{}^e h_{cde} \ . \tag{40}$$

This can be rewritten as

$$H_{abc} = m_a{}^d m_b{}^e h_{cde}. \tag{41}$$

It is easy to check that the second and third terms in (40) are indeed antisymmetric and that the second term is anti-self-dual while the third term is self-dual. Thus the actual field strength tensor H_{abc} is not itself self-dual but is determined by its self-dual part. The dimension 1/2 Bianchi identity determines the variation of H_{abc}, $\nabla_\alpha H_{abc}$, in terms of known quantities, i.e. in terms of χ and h, while the dimension one identity is the x-space Bianchi identity in covariantised form. In view of the modified self-duality satisfied by H_{abc} this can be viewed as the equation of motion for the tensor field written in terms of H_{abc} rather than h_{abc}. We emphasise the fact that the Bianchi identity (36) is satisfied automatically provided that we define the components of H_3 as above; it does not contain any new information but enables us to deduce more easily the existence of a 2-form

To summarise we have shown that the embedding condition (3) determines a tensor multiplet on the $N = 2, d = 6$ worldsurface and that the components of this multiplet satisfy their equations of motion. As we have seen, these are somewhat complicated which is related to the fact that the 5-brane resembles a Dirichlet brane in some respects. As we pointed out in [1], the difference between type I and type II embeddings from a geometrical point of view is that for the former there is an adapted basis of the odd tangent bundle of the target space which splits into components tangent and normal to the worldsurface whereas this is not so for type II. In other words, E_α and $E_{\alpha'}$ are not related to $E_{\underline{\alpha}}$ by a $Spin(1,10)$ matrix. The failure of adaptivity is due to the presence of the tensor field h and is a signal of Dirac-Born-Infeld type behaviour; precisely this type of geometrical structure also occurs for Dirichlet branes considered as superembeddings.

We note also that the geometry of the worldsurface is induced in the sense that the components of the worldsurface supervielbein can be expressed in terms of target space fields and the embedding matrix $\partial_M Z^{\underline{M}}$, up to gauge transformations.

The embedding condition (3) leads directly to the equations of motion. This is not surprising in view of the fact that it is not known how to construct an off-shell version of the tensor multiplet. However, modulo difficulties with self-duality, one might hope to find a 'component' action, in other words a Green-Schwarz type action. This is a very interesting question and is currently under study.

In the case of Dirichlet branes, which form an interesting class of type II branes, Green-Schwarz type κ invariant actions have been recently found in [12, 13]. It would be interesting to understand how the superembedding formalism is related to the Green-Schwarz formalism for Dirichlet and other type II branes. For type I branes, κ-symmetry is related to odd worldsurface diffeomorphisms as was first pointed out in [2]. In fact, for all branes, under an infinitesimal worldsurface diffeomorphism $\delta Z^M = -V^M$ the variation of the embedding expressed in a preferred frame basis is

$$\delta Z^{\underline{A}} \equiv \delta Z^{\underline{M}} E_{\underline{M}}{}^{\underline{A}} = v^A E_A{}^{\underline{A}} \; . \tag{42}$$

For an odd transformation ($v^a = 0$) one has

$$\delta z^{\underline{a}} = 0 \qquad \delta z^{\underline{\alpha}} = v^{\alpha} E_{\alpha}{}^{\underline{\alpha}} \; . \tag{43}$$

The vanishing of the even variation $\delta Z^{\underline{a}}$ is typical of κ-symmetry and follows from the basic embedding condition (3).

Finally, we mention the fact that we have assumed that the target space geometry corresponds to on-shell $D = 11$ supergravity. It may be possible to derive this rather than take it as an input and there are indications that this should be so. In particular we know that the requirement of κ symmetry for the 2-brane in the Green Schwarz formalism forces the equations of motion [11]. However, the situation is more complicated here in that one is trying to determine the form of the embedding matrix and the dimension zero torsion on both the worldsurface and on the target space from (3).

References

[1] P.S. Howe and E. Sezgin, *Superbranes*, hep-th/9607227.

[2] D. Sorokin, V. Tkach and D.V. Volkov , *Superparticles, twistors and Siegel symmetry*, Mod. Phys. Lett. **A4** (1989) 901.

[3] I.A. Bandos, D. Sorokin, M. Tonin, P. Pasti and D. Volkov, *Superstrings and supermembranes in the doubly supersymmetric geometrical approach*, Nucl. Phys. **B446** (1995) 79, hep-th/9501113.

[4] P.K. Townsend, *D-branes from M-branes*, hep-th/9512062.

[5] O. Aharony, *String theory dualities from M theory*, hep-th/9604103.

[6] E. Bergshoeff, M. de Roo and T. Ortin, *The eleven dimensional fivebrane*, hep-th/9606118.

[7] E. Witten, *Five-brane effective action in M-theory*, hep-th/9610234.

[8] J. Polchinski, *Dirichlet-branes and Ramond-Ramond charges*, hep-th/9510017.

[9] E. Cremmer and S. Ferrara, *Formulation of 11-dimensional supergravity in superspace*, Phys. Lett. **91B** (1980) 61.

[10] L. Brink and P.S. Howe, *Eleven dimensional supergravity on the mass shell in superspace*, Phys. Lett. **91B** (1980) 384.

[11] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Supermembranes and eleven-dimensional supergravity*, Phys. Lett. **189B** (1987) 75.

[12] M. Cederwall, A. von Gussich, B.E.W. Nilsson and A. Westerberg, *The Dirichlet super-three-brane in ten-dimensional type IIB supergravity*, hep-th/9610148.

[13] M. Aganagic, C. Popescu and J.H. Schwarz, *D-brane actions with local κ symmetry*, hep-th/9610249.