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U-duality p-Branes in Diverse Dimensions

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ABSTRACT

U-duality p-branes in toroidally compactified type II superstring theories in space-time dimensions $10 > D \geq 4$ can be constructed explicitly based on the conjectured U-duality symmetries and the corresponding known single-charge super p-brane configurations. As concrete examples, we first construct explicitly the $SL(3,Z)$ superstrings and $SL(3,Z) \times SL(2,Z)$ 0-branes as well as their corresponding magnetic duals in $D = 8$. For the $SL(3,Z)$ superstrings (3-branes), each of them is characterized by a triplet of integers corresponding to the electric-like (magnetic-like) charges associated with the three 2-form gauge potentials present in the theory. For the $SL(3,Z) \times SL(2,Z)$ 0-branes (4-branes), each of them is labelled by a pair of triplets of integers corresponding to the electric-like (magnetic-like) charges associated with the two sets of three 1-form gauge potentials. The string (3-brane) tension and central charge are shown to be given by $SL(3,Z)$ invariant expressions. It is argued that when any two of the three integers in the integral triplet are relatively prime to each other, the corresponding string (3-brane) is stable and does not decay into multiple strings (3-branes) by a 'tension gap' or 'charge gap' equation. Similar results hold also for the 0-branes (4-branes). Alongwith the $SL(2,Z)$ dyonic membranes of Izquierdo et. al., these examples provide a further support for the conjectured $SL(3,Z) \times SL(2,Z)$ U-duality symmetry in this theory. Moreover, the study of these examples along with the previous ones provides us a recipe for constructing the U-duality p-branes of various supergravity theories in diverse dimensions. Constructions for these U-duality p-branes are also given.

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1 Introduction

Supergravity theories in diverse dimensions have long been known to possess certain non-compact global symmetry groups, i.e., the Cremmer-Julia symmetry groups [1,2]. Since these theories are the long wavelength limit of various (dimensionally reduced) string theories, the discrete subgroups of these groups have been conjectured to be promoted to the full quantum string theories and have been named as U-duality groups [3]. From the string theory point of view, each of these groups usually contains a perturbative T-duality group [4] as well as a non-perturbative strong-weak duality group [5,6] as its subgroups. For example, the theory we are going to consider explicitly in detail the $N = 2$, $D = 8$ supergravity theory, which is the low energy effective action of T^2-compactified type IIA/IIB* string theory, has a global $SL(3, R) \times SL(2, R)$ Cremmer-Julia symmetry. The corresponding quantum type II string theory in $D = 8$ has been conjectured to possess the discrete U-duality group $SL(3, Z) \times SL(2, Z)$. Since a U-duality symmetry transforms the string coupling constant in a non-trivial way, it interchanges the strong and weak coupling regimes of the same theory. Thus this symmetry is by nature non-perturbative and generally it is difficult to prove the conjecture in the perturbative framework of string theory. However, there exist certain BPS saturated states as classical solutions [6,7,8] in these theories whose masses and the charges do not receive any quantum corrections due to some non-renormalization theorems of the underlying supersymmetric theories. Thus these states are very useful to identify the non-perturbative symmetry group of the quantum string theory.

Given a U-duality symmetry for a particular system, it is clearly artificial to consider p-brane solutions carrying either electric or magnetic charges associated with only a single $(p + 2)$-form field strength unless this field strength is a singlet under the U-duality symmetry, as pointed out in [9]. In general, we expect that there is an infinite family of such solutions forming U-duality multiplets. In this paper, we first construct explicitly the $SL(3, Z)$ BPS saturated string-like and 3-brane-like (the magnetic dual of string) solutions, and the $SL(3, Z) \times SL(2, Z)$ BPS saturated particle-like and 4-brane-like (the magnetic dual of 0-brane) solutions by using the symmetry of the toroidally compactified type II string theory in $D = 8$. These particular constructions, combined with the previous studies [9,10,11,12], provide us a recipe for constructing the general U-duality p-brane

*In lower dimensions, type IIA and type IIB theories are equivalent by a perturbative T-duality symmetry. But for definiteness, we will consider compactification of type IIA theory to $D = 8$.

solutions in diverse dimensions. We then apply this recipe to construct all U-duality p-brane solutions, preserving half of the spacetime supersymmetry, of various supergravity theories in diverse dimensions. In exactly the same fashion, U-duality p-brane solutions preserving less than half of spacetime supersymmetry [13] can also be constructed. The key to all such constructions of U-duality p-brane solutions is the scalar matrices each of which parametrizes the corresponding Cremmer-Julia coset G/H in various supergravity theories.

The $SL(3,Z)$ superstring and super 3-brane solutions as well as the $SL(3,Z) \times SL(2,Z)$ superparticle and super 4-brane solutions in $D = 8$ are in a sense complementary to the dyonic membrane solutions of Izquierdo et al [12]. The membranes are associated with a doublet representation of the $SL(2,Z)$ T-duality group and are inert under the $SL(3,Z)$ group. On the other hand, the strings as well as the 3-branes are associated with a triplet representation of the strong-weak $SL(3,Z)$ duality group and are inert under the T-duality $SL(2,Z)$ group. The particles or 0-branes as well as the 4-branes are, however, associated with both a triplet representation of the $SL(3,Z)$ and a doublet representation of the $SL(2,Z)$ groups. Combined with their magnetic duals, the $SL(3,Z)$ strings and $SL(3,Z) \times SL(2,Z)$ 0-branes along with the $SL(2,Z)$ dyonic membranes of Izquierdo et. al. complete all the U-duality p-branes in $D = 8$. Therefore, the $SL(3,Z)$ string-like solutions, the $SL(3,Z) \times SL(2,Z)$ 0-brane solutions along with their magnetic duals and the dyonic membrane solutions provide a strong support in favor of the conjectured U-duality symmetry in $D = 8$ quantum type II string theory. The string (3-brane) solutions are characterized by a triplet of integers corresponding to the electric-like (magnetic-like) charges associated with the three two-form gauge fields (one from the NSNS sector and the other two from the RR sector) present in the theory. The 0-brane (4-brane) solutions, on the other hand, are characterized by a pair of triplets of integers corresponding to the electric-like (magnetic-like) charges associated with the two sets of three 1-form gauge potentials (one set of 1-form gauge potentials has Kaluza-Klein origin and the other set has its origin in the dimensional reductions of the antisymmetric tensors in higher dimensions). We will show that both the string (3-brane) tension and central charge associated with a general string (3-brane) solution are given by $SL(3,Z)$ invariant expressions. The mass and the central charge associated with a general 0-brane (4-brane) solution are given by $SL(3,Z) \times SL(2,Z)$ invariant expressions. As stated earlier, these physical quantities remain unrenormalized in the full quantum theory and therefore provide a strong indication that $SL(3,Z) \times SL(2,Z)$ is indeed a symmetry of $D=8$ theory. We will also show that when any of the three pairs of integers are coprime, the corresponding string (3-brane) is stable as it is prevented from decaying into multiple strings (3-branes) by a 'tension gap' or 'charge gap' equation. Similar conclusions hold also for the 0-branes and 4-branes. This is actually true for all U-duality p-branes of supergravity theories in diverse dimensions.

We organize the remaining sections of this paper as follows: In section 2, we give a brief discussion of $D=8$ NSNS strings which will provide a starting point for the construction of $SL(3,Z)$ strings. We demonstrate in section 3, using the $D=8$ maximal supergravity as an example, how to write a dimensionally reduced bosonic action in a manifest Cremmer-Julia symmetry invariant form if this symmetry is realized at the level of action. This process also determines how the various fields transform under the underlying Cremmer-Julia symmetry and the corresponding scalar coset matrix which are all important for the construction of U-duality p-branes. In particular, we provide a way to determine the scalar coset matrix when the underlying Cremmer-Julia symmetry is not realized at the level of action but at the level of equations of motion. Based on the discussion given in section 3, we give a detail construction of the $SL(3,Z)$ strings in section 4. Various properties of these strings are discussed and the construction of the corresponding magnetic dual $SL(3,Z)$ 3-brane solutions are also given. In section 5, we construct the $SL(3,Z) \times SL(2,Z)$ 0-branes and the magnetic dual 4-branes, which completes the construction of all U-duality p-branes in $D=8$ type II theory. Our final section consists of the construction of U-duality p-branes of various supergravity theories in diverse dimensions.

2 NSNS Strings: A Brief Review

Since we will make use of the NSNS string solution of Dabholkar et al [7] in $D=8$, let us give a brief discussion of this solution. The low energy bosonic action common to all string theories in $D=8$ has the form:

$$S_8 = \int d^8x\sqrt{-g}\left[R - \frac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi - \frac{1}{12}e^{-2\Phi/\sqrt{3}}\tilde{F}^{(1)}_{\mu\nu\lambda}\tilde{F}^{(1)\,\mu\nu\lambda}\right] \tag{1}$$

Here $g = \det(g_{\mu\nu})$, $g_{\mu\nu}$ being the canonical metric which is related to the eight dimensional string metric by $G_{\mu\nu} = e^{\Phi/\sqrt{3}}g_{\mu\nu}$. R is the scalar curvature with respect to the canonical metric, Φ is the eight dimensional dilaton and $\tilde{F}^{(1)}_{\mu\nu\lambda}$ is the field strength associated with the Kalb-Ramond antisymmetric tensor field $A^{(1)}_{\mu\nu}$. The equations of motion following from (1) admit a two parameter family of black string solution as given below [8]:

$$\begin{aligned} ds^2 &= -\left(1-\frac{r_+^4}{r^4}\right)\left(1-\frac{r_-^4}{r^4}\right)^{-1/3} dt^2 + \left(1-\frac{r_-^4}{r^4}\right)^{2/3} (dx^1)^2 \\ &\quad + \left(1-\frac{r_+^4}{r^4}\right)^{-1}\left(1-\frac{r_-^4}{r^4}\right)^{-5/6} dr^2 + r^2\left(1-\frac{r_-^4}{r^4}\right)^{1/6} d\Omega_5^2 \\ e^{2\Phi} &= \left(1-\frac{r_-^4}{r^4}\right)^{2/\sqrt{3}}, \qquad \tilde{F}_3^{(1)} = 4\,(r_+r_-)^2 * e^{2\Phi/\sqrt{3}}\epsilon_5 \end{aligned} \tag{2}$$

Here $d\Omega_5^2$ is the metric on the unit 5-dimensional sphere and ϵ_5 is the corresponding volume form. The '$*$' denotes the Hodge dual operation. r_+ and r_- are the two parameters representing the two horizons with $r_+ \geq r_-$ and are related to the charge and the mass of the black string solution. In the extremal limit when $r_+ = r_-$, the solution becomes supersymmetric saturating the BPS condition. By introducing the isotropic coordinate $\rho^4 = r^4 - r_-^4$, the solution in the extremal limit can be written as:

$$\begin{aligned} ds^2 &= \left(1+\frac{Q}{4\rho^4}\right)^{-2/3}\left[-(dt)^2 + (dx^1)^2\right] + \left(1+\frac{Q}{4\rho^4}\right)^{1/3}\left(d\rho^2 + \rho^2 d\Omega_5^2\right) \\ e^{-2\Phi} &= \left(1+\frac{Q}{4\rho^4}\right)^{2/\sqrt{3}} = A(\rho), \qquad \tilde{F}_3^{(1)} = QA^{-1/\sqrt{3}}(\rho) * \epsilon_5, \end{aligned} \tag{3}$$

where $Q = 4r_-^4$. Eq.(3) represents precisely the string solution constructed by Dabholkar et al [7] and we notice that this solution is the extremal limit of the black string solution of ref.[8] in a new coordinate. Q in Eq.(3) is the electric charge associated with the gauge field $A^{(1)}_{\mu\nu}$ and is defined as $Q = \frac{1}{\pi^3}\int_{S^5} *e^{-2\Phi/\sqrt{3}}\tilde{F}_3^{(1)}$. Note that this charge is quantized in some basic units since there also exists magnetically charged 3-brane solution in this theory [6,14]. It should be remarked here that the solution (3) has a singularity at $\rho = 0$ since the volume of the 5-sphere vanishes and the curvature blows up at that point [6]. So, the string solution (3) has been been obtained by coupling the supergravity action (1) to a macroscopic string source. This type of solution are usually called the 'fundamental' solution.

As we will see, the action (1) can be regarded as a special case of the low energy effective action of type II string theory in $D = 8$, when some of the fields are set to zero. So, it is clear that a more general string-like solution than that in (3) exists when we consider the full type II theory. These general solutions can be obtained from (3) by using the symmetry of the type II theory in $D = 8$ as we will show.

3 $SL(3, R)$ Invariant Action

In order to obtain the general p-brane solution the most important object we need is the scalar coset matrix consisting of the scalars of the theory which parametrize the Cremmer-Julia symmetry group modded out by its maximal compact subgroup. One way to obtain this matrix has been outlined in ref.[15]. In this section, we will first show how to write the low energy effective action of $D = 8$ type II theory in $SL(3, R)$ invariant form. This process, in turn, will provide us another way of obtaining the scalar coset matrix in this theory. This method applies in general whenever the Cremmer-Julia symmetry is realized at the level of action. We will show the detail construction of this matrix below.

The type II theory in $D = 8$ can be obtained by a T^2 compactification of $D = 10$ type IIA supergravity theory consisting of a graviton (g_{MN}), a dilaton (ϕ) and a 2-form potential (B_{MN}) in its NSNS sector and a 1-form gauge potential ($\mathcal{A}_M$) and a 3-form gauge potential (A_{MNP}) in its RR sector. As discussed in detail in ref.[16], the toroidally compactified type IIA supergravity theories in $D \leq 9$ can be obtained in general either from the ten dimensional type IIA theory or from the eleven dimensional supergravity by a set of successive 1-step Kaluza-Klein reductions on circles. The same procedure can also be applied to the toroidal compactification of the type IIB supergravity. In each reduction step from $(D + 1)$ to D dimensions, the metric in $(D + 1)$ will give rise to a metric, a Kaluza-Klein vector potential $\mathcal{A}_\mu$, and a "dilatonic" scalar field φ in D dimensions. An n-index gauge potential in $(D+1)$ dimensions will give rise to an n-index gauge potential and an $(n-1)$-index gauge potential in D dimensions. Following the type IIA reduction route to $D = 8$, we have the following bosonic field content: the metric $g_{\mu\nu}$, the ten dimensional type IIA dilaton ϕ together with two additional dilatonic scalars φ_1 and φ_2, one 3-form gauge potential A_3, three 2-form gauge potentials $A_2^{(i)}$ with $A_{\mu\nu}^{(1)} \sim B_{\mu\nu}, A_{\mu\nu}^{(2)} \sim A_{\mu\nu 9}, A_{\mu\nu}^{(3)} \sim A_{\mu\nu 8}$, three 1-form gauge potentials $A_1^{(i)}$ with $A_\mu^{(1)} \sim A_{\mu 89}, A_\mu^{(2)} \sim -B_{\mu 8}, A_\mu^{(3)} \sim B_{\mu 9}$ and another three 1-form gauge potentials $\mathcal{A}_1^{(i)}$ (which can be interpreted as having Kaluza-Klein origin) with $\mathcal{A}_\mu^{(1)} \sim \mathcal{A}_\mu, \mathcal{A}_\mu^{(2)} \sim g_{\mu 9}/g_{99}, \mathcal{A}_\mu^{(3)} \sim g_{\mu 8}/g_{88}$, and four 0-forms (axions) $\chi_1 \sim -g_{89}/g_{88}, \chi_2 \sim -\mathcal{A}_8, \chi_3 \sim -\mathcal{A}_9, \rho \sim B_{89}$. We have used the notation such that the origin of the various fields can be understood from the type IIA theory in $D = 10$. The corresponding Lagrangian using our notation is

$$\begin{aligned}
\mathcal{L} \;=\;& e\Big\{R-\frac{1}{2}\,\Big[(\partial\phi)^2+(\partial\varphi_1)^2+(\partial\varphi_2)^2\Big]-\frac{1}{2}\,e^{-\phi-\frac{3}{\sqrt{7}}\varphi_1-2\sqrt{\frac{3}{7}}\varphi_2}(\partial\rho)^2\\
&-\frac{1}{2}\,\Big[e^{\frac{4}{\sqrt{7}}\varphi_1-2\sqrt{\frac{3}{7}}\varphi_2}(\partial\chi_1)^2+e^{\frac{3}{2}\phi+\frac{1}{2\sqrt{7}}\varphi_1-2\sqrt{\frac{3}{7}}\varphi_2}(\partial\chi_2+\chi_1\partial\chi_3)^2+e^{\frac{3}{2}\phi-\frac{\sqrt{7}}{2}\varphi_1}(\partial\chi_3)^2\Big]\\
&-\frac{1}{12}\,\Big[e^{-\phi+\frac{1}{\sqrt{7}}\varphi_1+\frac{2}{\sqrt{21}}\varphi_2}(F_3^{(1)})^2+e^{\frac{1}{2}\phi-\frac{5}{2\sqrt{7}}\varphi_1+\frac{2}{\sqrt{21}}\varphi_2}(F_3^{(2)})^2+e^{\frac{1}{2}\phi+\frac{3}{2\sqrt{7}}\varphi_1-\frac{4}{\sqrt{21}}\varphi_2}(F_3^{(3)})^2\Big]\\
&-\frac{1}{4}\,\Big[e^{\frac{1}{2}\phi-\frac{5}{2\sqrt{7}}\varphi_1-\frac{5}{\sqrt{21}}\varphi_2}(F_2^{(1)})^2+e^{-\phi+\frac{1}{\sqrt{7}}\varphi_1-\frac{5}{\sqrt{21}}\varphi_2}(F_2^{(2)})^2+e^{-\phi-\frac{3}{\sqrt{7}}\varphi_1+\frac{1}{\sqrt{21}}\varphi_2}(F_2^{(3)})^2\Big]\\
&-\frac{1}{4}\,\Big[e^{\frac{3}{2}\phi+\frac{1}{2\sqrt{7}}\varphi_1+\frac{1}{\sqrt{21}}\varphi_2}(\mathcal{F}_2^{(1)})^2+e^{\frac{4}{\sqrt{7}}\varphi_1+\frac{1}{\sqrt{21}}\varphi_2}(\mathcal{F}_2^{(2)})^2+e^{\sqrt{\frac{7}{3}}\varphi_2}(\mathcal{F}_2^{(3)})^2\Big]\\
&-\frac{1}{48}\,e^{\frac{1}{2}\phi+\frac{3}{2\sqrt{7}}\varphi_1+\sqrt{\frac{3}{7}}\varphi_2}F_4^2\Big\}+\frac{1}{2}\,\rho\,\tilde{F}_4\wedge\tilde{F}_4\\
&-\frac{1}{6}\,\tilde{F}_3^{(i)}\wedge\tilde{F}_3^{(j)}\wedge A_2^{(k)}\epsilon_{ijk}-\tilde{F}_4\wedge\tilde{F}_3^{(i)}\wedge A_1^{(i)},
\end{aligned}\tag{4}$$

where we have defined $e=\sqrt{-g}$, $i,j,k=1,2,3$ and ϵ_{ijk} is totally antisymmetric with $\epsilon_{123}=1$. We follow the notation in ref.[15] that field strengths without tildes include the various Chern-Simons modifications, while field strengths written with tildes do not include the modifications, i.e., $\tilde{F}_n^{(i)}=dA_{n-1}^{(i)}$. The expressions for field strengths without tildes are complicated and given in the appendix. As in ref.[15] the wedge product is defined, for example, as $\tilde{F}_4\wedge\tilde{F}_3^{(i)}\wedge A_1^{(i)}=\frac{1}{4!}\frac{1}{3!}\epsilon^{\mu_1\ldots\mu_8}\tilde{F}_{\mu_1\ldots\mu_4}\tilde{F}^{(i)}_{\mu_5\mu_6\mu_7}A^{(i)}_{\mu_8}$.

Alternatively, the same $D=8$ supergravity can also be obtained from the dimensional reduction of the type IIB supergravity in $D=10$. Actually, it is more convenient to identify the underlying Cremmer-Julia symmetry $SL(3,R)\times SL(2,R)$ if we choose the basis of dilatonic scalars that corresponds to the type IIB reduction route. Moreover, as we will see, one of the advantages in choosing the type IIB basis is that the $SL(2,Z)$ is easily understood as a T duality symmetry since its transformation does not involve the ten dimensional type IIB dilaton at all while the $SL(3,Z)$ is indeed a strong-weak duality symmetry since it contains transformations changing the sign of the dilaton. We therefore choose to work in the type IIB basis from now on.

The type IIA and type IIB reduction routes result in two formulations of the $D=8$ theory that are related to each other by the following orthogonal field redefinitions of the dilatonic scalars ϕ and φ_1 as

$$\begin{pmatrix}\phi\\ \varphi_1\end{pmatrix}_{IIA}=\begin{pmatrix}\frac{3}{4} & -\frac{\sqrt{7}}{4}\\ -\frac{\sqrt{7}}{4} & -\frac{3}{4}\end{pmatrix}\begin{pmatrix}\phi\\ \varphi_1\end{pmatrix}_{IIB},\tag{5}$$

which corresponds to a T-duality transformation. We can therefore obtain the type IIB basis Lagrangian by applying the above relation to the Lagrangian (4). However, before we do so, we need to perform some field redefinitions which will greatly simplify the expressions for the field strengths without tildes given in the appendix and will facilitate the construction for the scalar coset matrix mentioned at the outset of this section.

We first perform the field redefinition $\mathcal{A}_1^{(2)} \to \mathcal{A}_1^{(2)} + \chi_1 \mathcal{A}_1^{(3)}$ and after that we perform $\mathcal{A}_1^{(1)} \to \mathcal{A}_1^{(1)} + \chi_3 \mathcal{A}_1^{(2)} + \chi_2 \mathcal{A}_1^{(3)}$. Now we have,

$$\begin{aligned}
\mathcal{F}_2^{(1)} &= \tilde{\mathcal{F}}_2^{(1)} - \chi_3 \tilde{\mathcal{F}}_2^{(2)} - \chi_2 \tilde{\mathcal{F}}_2^{(3)}, \\
\mathcal{F}_2^{(2)} &= \tilde{\mathcal{F}}_2^{(2)} - \chi_1 \tilde{\mathcal{F}}_2^{(3)}, \\
\mathcal{F}_2^{(3)} &= \tilde{\mathcal{F}}_2^{(3)}. \qquad (6)
\end{aligned}$$

If we introduce a column vector $\mathcal{F}_2$ for the above three 2-form field strengths without tildes and a column vector $\tilde{\mathcal{F}}_2$ for the three field strengths with tildes, we can write the above three equations in a compact form as

$$\mathcal{F}_2 = \lambda_1 \tilde{\mathcal{F}}_2, \qquad (7)$$

where the matrix λ_1 is

$$\lambda_1 = \begin{pmatrix} 1 & -\chi_3 & -\chi_2 \\ 0 & 1 & -\chi_1 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (8)$$

With the above redefinitions for $\mathcal{A}_1^{(2)}$ and $\mathcal{A}_1^{(1)}$, we can further perform the field redefinitions for the other three 1-form gauge potentials as $A_1^{(i)} \to A_1^{(i)} - \rho\, \mathcal{A}_1^{(i)}$, the three 2-form gauge potentials as $A_2^{(i)} \to A_2^{(i)} - \frac{1}{2}\rho\, \mathcal{A}_1^{(j)} \wedge \mathcal{A}_1^{(k)} \epsilon_{ijk}$ and the 3-form potential as $A_3 \to A_3 - \rho\, \mathcal{A}_1^{(1)} \wedge \mathcal{A}_1^{(2)} \wedge \mathcal{A}_1^{(3)}$. The 2-form field strengths without tildes can then be written as,

$$\begin{aligned}
F_2^{(1)} &= \tilde{F}_2^{(1)} - \chi_3 \tilde{F}_2^{(2)} - \chi_2 \tilde{F}_2^{(3)} + \rho\, (\tilde{\mathcal{F}}_2^{(1)} - \chi_3 \tilde{\mathcal{F}}_2^{(2)} - \chi_2 \tilde{\mathcal{F}}_2^{(3)}), \\
F_2^{(2)} &= \tilde{\mathcal{F}}_2^{(2)} - \chi_1 \tilde{\mathcal{F}}_2^{(3)} + \rho\, (\tilde{\mathcal{F}}_2^{(2)} - \chi_1 \tilde{\mathcal{F}}_2^{(3)}), \\
F_2^{(3)} &= \tilde{F}_2^{(3)} + \rho\, \tilde{\mathcal{F}}_2^{(3)}, \qquad (9)
\end{aligned}$$

Similarly, the three 3-form field strengths without tildes are,

$$\begin{aligned}
F_3^{(1)} &= \tilde{F}_3^{(1)} - \left(\tilde{F}_2^{(2)} \wedge \mathcal{A}_1^{(3)} - \tilde{F}_2^{(3)} \wedge \mathcal{A}_1^{(2)}\right), \\
F_3^{(2)} &= \chi_3 \tilde{F}_3^{(1)} + \tilde{F}_3^{(2)} - (\tilde{F}_2^{(3)} \wedge \mathcal{A}_1^{(1)} - \tilde{F}_2^{(1)} \wedge \mathcal{A}_1^{(3)}) - \chi_3 (\tilde{F}_2^{(2)} \wedge \mathcal{A}_1^{(3)} - \tilde{F}_2^{(3)} \wedge \mathcal{A}_1^{(2)}), \\
F_3^{(3)} &= (\chi_2 + \chi_1 \chi_3) \tilde{F}_2^{(1)} + \chi_1 \tilde{F}_2^{(2)} + \tilde{F}_3^{(3)} - (\chi_2 + \chi_1 \chi_3)(\tilde{F}_2^{(2)} \wedge \mathcal{A}_1^{(3)} - \tilde{F}_2^{(3)} \wedge \mathcal{A}_1^{(2)}) \\
&\quad - \chi_1 (\tilde{F}_2^{(3)} \wedge \mathcal{A}_1^{(1)} - \tilde{F}_2^{(1)} \wedge \mathcal{A}_1^{(3)}) - (\tilde{F}_2^{(1)} \wedge \mathcal{A}_1^{(2)} - \tilde{F}_2^{(2)} \wedge \mathcal{A}_1^{(2)}), \qquad (10)
\end{aligned}$$

and the 4-form field strength without tilde is given as,

$$F_4 = \tilde{F}_4 - \tilde{F}_3^{(i)} \wedge \mathcal{A}_1^{(i)} + \frac{1}{2}\epsilon_{ijk}\tilde{F}_2^{(i)} \wedge \mathcal{A}_1^{(j)} \wedge \mathcal{A}_1^{(k)}. \tag{11}$$

We can also write (9) and (10) in compact forms if the corresponding column vectors are introduced, respectively, i.e.,

$$F_2 = \lambda_1(\tilde{F}_2 + \rho\,\tilde{\mathcal{F}}_2), \tag{12}$$

$$F_3 = \lambda_2(\tilde{F}_3 - G_3), \tag{13}$$

and

$$F_4 = \tilde{F}_4 - \tilde{F}_3^T \wedge \mathcal{A}_1 + \frac{1}{2}\epsilon_{ijk}\tilde{F}_2^{(i)} \wedge \mathcal{A}_1^{(j)} \wedge \mathcal{A}_1^{(k)}. \tag{14}$$

In the above, the matrix λ_1 is given by Eq.(8) and the matrix λ_2 is

$$\lambda_2 = \begin{pmatrix} 1 & 0 & 0 \\ \chi_3 & 1 & 0 \\ (\chi_2 + \chi_1\chi_3) & \chi_1 & 1 \end{pmatrix}. \tag{15}$$

Also, the components of the column vector G_3 are $G_3^{(i)} \equiv \epsilon_{ijk}\tilde{F}_2^{(j)} \wedge \mathcal{A}_1^{(k)}$ and 'T' denotes the transposition.

Using the above field redefinitions the Lagrangian (4) can be written in the type IIB basis as follows,

$$\begin{aligned}
\mathcal{L} \;=\; & e\,R - \frac{1}{2}\,e\,\left[(\partial\sigma)^2 + e^{2\sigma}(\partial\rho)^2\right] - \frac{1}{48}\,e\,e^{-\sigma}F_4^2 + \frac{1}{2}\rho\,F_4 \wedge F_4 \\
& -\frac{1}{2}\,e\,\left[(\partial\phi)^2 + (\partial\varphi)^2 + e^{-\phi-\sqrt{3}\varphi}(\partial\chi_1)^2 + e^{\phi-\sqrt{3}\varphi}(\partial\chi_2 + \chi_1\partial\chi_3)^2 + e^{2\phi}(\partial\chi_3)^2\right] \\
& -\frac{1}{12}\,e\,\left[e^{-\phi+\frac{1}{\sqrt{3}}\varphi}(F_3^{(1)})^2 + e^{\phi+\frac{1}{\sqrt{3}}\varphi}(F_3^{(2)})^2 + e^{-\frac{2}{\sqrt{3}}\varphi}(F_3^{(3)})^2\right] - \frac{1}{6}\,\tilde{F}_3^{(i)} \wedge \tilde{F}_3^{(j)} \wedge A_2^{(k)}\epsilon_{ijk} \\
& -\frac{1}{4}\,e\,e^{\sigma}\,\left[e^{\phi-\frac{1}{\sqrt{3}}\varphi}(F_2^{(1)})^2 + e^{-\phi-\frac{1}{\sqrt{3}}\varphi}(F_2^{(2)})^2 + e^{\frac{2}{\sqrt{3}}\varphi}(F_2^{(3)})^2\right] \\
& -\frac{1}{4}\,e\,e^{-\sigma}\,\left[e^{\phi-\frac{1}{\sqrt{3}}\varphi}(\mathcal{F}_2^{(1)})^2 + e^{-\phi-\frac{1}{\sqrt{3}}\varphi}(\mathcal{F}_2^{(2)})^2 + e^{\frac{2}{\sqrt{3}}\varphi}(\mathcal{F}_2^{(3)})^2\right] \\
& -\tilde{F}_4 \wedge \tilde{F}_3^{(i)} \wedge A_1^{(i)}.
\end{aligned} \tag{16}$$

In obtaining the above Lagrangian, we have further made the following field redefinitions

$$\begin{aligned}
\varphi_1 &= \sqrt{\frac{3}{7}}\varphi + \frac{2}{\sqrt{7}}\sigma, \\
\varphi_2 &= \frac{2}{\sqrt{7}}\varphi - \sqrt{\frac{3}{7}}\sigma,
\end{aligned} \tag{17}$$

and have dropped surface terms.

We now re-express the above action in a manifestly $SL(3,R)$ invariant form using the compact forms for various field strengths given in Eqs.(7) and (12)–(14). This process also determines the scalar matrix parametrizing the coset $SL(3,R)/SO(3)$. Let us demonstrate how to achieve this by the following example. The kinetic terms for the 3-form field strengths in the above Lagrangian can be re-expressed as

$$-\frac{1}{12}e\,F_3^T\begin{pmatrix} e^{-\phi+\frac{1}{\sqrt{3}}\varphi} & 0 & 0 \\ 0 & e^{\phi+\frac{1}{\sqrt{3}}\varphi} & 0 \\ 0 & 0 & e^{-\frac{2}{\sqrt{3}}\varphi}\end{pmatrix}F_3 = -\frac{1}{12}e\,(\tilde{F}_3-G_3)^T\mathcal{M}_3(\tilde{F}_3-G_3), \qquad (18)$$

where Eq.(13) has been used and the scalar matrix $\mathcal{M}_3$ parametrizing the coset $SL(3,R)/SO(3)$ is

$$\begin{aligned}\mathcal{M}_3 &= \lambda_2^T\begin{pmatrix} e^{-\phi+\frac{1}{\sqrt{3}}\varphi} & 0 & 0 \\ 0 & e^{\phi+\frac{1}{\sqrt{3}}\varphi} & 0 \\ 0 & 0 & e^{-\frac{2}{\sqrt{3}}\varphi}\end{pmatrix}\lambda_2, \\ &= e^{\frac{\varphi}{\sqrt{3}}}\begin{pmatrix} \begin{array}{c} e^{-\phi}+\chi_3^2e^{\phi} \\ +(\chi_2+\chi_1\chi_3)^2e^{-\sqrt{3}\varphi}\end{array} & \begin{array}{c}\chi_3e^{\phi} \\ +\chi_1(\chi_2+\chi_1\chi_3)e^{-\sqrt{3}\varphi}\end{array} & (\chi_2+\chi_1\chi_3)e^{-\sqrt{3}\varphi} \\ \begin{array}{c}\chi_3e^{\phi} \\ +\chi_1(\chi_2+\chi_1\chi_3)e^{-\sqrt{3}\varphi}\end{array} & e^{\phi}+\chi_1^2e^{-\sqrt{3}\varphi} & \chi_1e^{-\sqrt{3}\varphi} \\ (\chi_2+\chi_1\chi_3)e^{-\sqrt{3}\varphi} & \chi_1e^{-\sqrt{3}\varphi} & e^{-\sqrt{3}\varphi}\end{pmatrix} \end{aligned} \qquad (19)$$

A different but equivalent form of this scalar coset matrix has been given in ref.[17]. Now the same procedure can be applied to the kinetic terms for the 2-form field strengths $F_2^{(i)}$ and $\mathcal{F}_2^{(i)}$, respectively. Using this technique we end up with the following compact form for the Lagrangian of $D=8$ type II theory,

$$\begin{aligned}\mathcal{L} &= e\,R+\frac{1}{4}e\,\mathrm{tr}\nabla_\mu\mathcal{M}_2\nabla^\mu\mathcal{M}_2^{-1}-\frac{1}{48}e\,e^{-\sigma}F_4^2+\frac{1}{2}\rho\,F_4\wedge F_4 \\ &+\frac{1}{4}e\,\mathrm{tr}\nabla_\mu\mathcal{M}_3\nabla^\mu\mathcal{M}_3^{-1}-\frac{1}{12}e\,(\tilde{F}_3-G_3)^T\mathcal{M}_3(\tilde{F}_3-G_3)-\frac{1}{6}\tilde{F}_3^{(i)}\wedge\tilde{F}_3^{(j)}\wedge A_2^{(k)}\epsilon_{ijk} \\ &-\frac{1}{4}e\,e^{\sigma}(\tilde{F}_2+\rho\tilde{\mathcal{F}}_2)^T\mathcal{M}_3^{-1}(\tilde{F}_2+\rho\tilde{\mathcal{F}}_2)-\frac{1}{4}e\,e^{-\sigma}\tilde{\mathcal{F}}_2^T\mathcal{M}_3^{-1}\tilde{\mathcal{F}}_2-\tilde{F}_4\wedge\tilde{F}_3^T\wedge A_1, \end{aligned} \qquad (20)$$

where the scalar matrix $\mathcal{M}_2$ parametrizes the coset $SL(2,R)/SO(2)$ and is given as,

$$\mathcal{M}_2 = e^{\sigma}\begin{pmatrix} e^{-2\sigma}+\rho^2 & \rho \\ \rho & 1\end{pmatrix}. \qquad (21)$$

It is not difficult to check that the above action is invariant under the following $SL(3,R)$ transformations:

$$
\begin{aligned}
& g_{\mu\nu} \to g_{\mu\nu}, \qquad F_4 \to F_4, \qquad \mathcal{M}_2 \to \mathcal{M}_2, \\
& \mathcal{M}_3 \to \Lambda_3 \mathcal{M}_3 \Lambda_3^T, \qquad \mathcal{A}_1 \to \Lambda_3 \mathcal{A}_1, \qquad A_1 \to \Lambda_3 A_1, \\
& A_2 \to (\Lambda_3^{-1})^T A_2, \qquad G_3 \to (\Lambda_3^{-1})^T G_3,
\end{aligned} \tag{22}
$$

where Λ_3 is a global $SL(3,R)$ matrix.

As we have demonstrated above, the $SL(3,R)$ group is indeed a global symmetry of the theory which is realized at the level of Lagrangian. The corresponding discrete subgroup $SL(3,Z)$ must contain all the non-perturbative U-duality symmetries since it transforms the ten dimensional type IIB dilaton ϕ and in particular, it contains transformations which reverse the sign of the dilaton, while the discrete subgroup $SL(2,Z)$ of the $SL(2,R)$ does not transform the dilaton at all (see the following discussion).

The $SL(2,R)$ can be fully realized only at the level of equations of motion since it rotates the equation of motion and the Bianchi identity for the 4-form field strength F_4. The above Lagrangian also implies that the $SL(2,R)$ transforms only the scalars (σ,ρ) while leaves the rest of the scalars inert (This will not be true if the type IIA basis is used instead). Therefore, the discrete $SL(2,Z)$ acts like the usual S-duality $SL(2,Z)$ as an electric/magnetic duality symmetry but unlike the S-duality $SL(2,Z)$ it is merely a T-duality symmetry. Even though the $SL(2,R)$ has long been conjectured to be a symmetry of the $D=8$ supergravity by Cremmer and Julia and its discrete subgroup $SL(2,Z)$ is believed to be one of $SL(2,Z)$ factors in the T-duality symmetry $O(2,2;Z) \equiv [SL(2,Z) \times SL(2,Z)]/Z_2$ for the T^2-compactified type II string theory, to our knowledge this $SL(2,R)$ invariance has not been demonstrated explicitly at the level of equations of motion of the supergravity. We postpone to give a demonstration of this symmetry explicitly elsewhere [18]. When the 1-form potentials A_1 and $\mathcal{A}_1$ are both set to zero, showing the $SL(2,R)$ symmetry is not different from that of the classical S-duality $SL(2,R)$ in $D=4$ [5,19]. Actually, Izquierdo et al [12] just employed both the $SL(2,R)$ and the $SL(2,Z)$ symmetries to construct the dyonic membranes in the case of vanishing A_1 and $\mathcal{A}_1$.

Given that the $SL(2,R)$ is indeed a symmetry of the $D=8$ supergravity theory, can we construct the scalar coset matrix $\mathcal{M}_2$ in a similar fashion as we did for the matrix $\mathcal{M}_3$? The answer turns out to be true in general whenever we have field strengths which transform among themselves (without the need of introducing their duals) in a representation of the underlying group. In other words, one should be able to see that certain terms containing these field strengths in the Lagrangian are invariant under the underlying global symmetry transformation. In the present case, we know that the 1-from potential A_1 and $\mathcal{A}_1$ each transform as a triplet of the global $SL(3,R)$ while some combination of the two transforms as a doublet of the $SL(2,R)$. Examining the kinetic terms for $\tilde{F}_2$ and $\tilde{\mathcal{F}}_2$ in the Lagrangian already suggests to us the similarity with what we know about the strong-weak $SL(2,R)$ case in $D=10$ type IIB theory [9]. If we write $\mathcal{M}_3=\nu\nu^T$ with

$$\nu = e^{\frac{\varphi}{2\sqrt{3}}}\left(\begin{array}{ccc} e^{-\phi/2} & \chi_3 e^{\phi/2} & (\chi_2+\chi_1\chi_3)e^{-\frac{\sqrt{3}}{2}\varphi} \\ 0 & e^{\phi/2} & \chi_1 e^{-\frac{\sqrt{3}}{2}\varphi} \\ 0 & 0 & e^{-\frac{\sqrt{3}}{2}\varphi} \end{array}\right), \tag{23}$$

and introduce a 2-form doublet

$$f_2 = \left(\begin{array}{c} \nu^{-1}\mathcal{F}_2 \\ \nu^{-1}F_2 \end{array}\right), \tag{24}$$

then the kinetic terms for both $\tilde{\mathcal{F}}_2$ and $\tilde{F}_2$ in the Lagrangian can be written in the following simple compact form as,

$$-\frac{1}{4}e\, f_2^T\mathcal{M}_2 f_2, \tag{25}$$

where the scalar matrix $\mathcal{M}_2$ parametrizes the coset $SL(2,R)/SO(2)$ and is given precisely by Eq.(21). The above compact form is invariant under the following $SL(2,R)$ transformations:

$$g_{\mu\nu}\to g_{\mu\nu}, \qquad \mathcal{M}_2\to\Lambda_2\mathcal{M}_2\Lambda_2^T, \qquad f_2\to(\Lambda_2^T)^{-1}f_2, \tag{26}$$

where Λ_2 is a global $SL(2,R)$ element.

In summary, the bosonic action of the D-dimensional supergravity in $10>D\geq 4$, obtained from either $D=11$ supergravity or $D=10$ type IIB supergravity by the dimensional reduction on torus, can be written in a manifest Cremmer-Julia symmetry invariant form if this symmetry is realized at the level of action, through redefining various fields algebraically and performing certain necessary dualizations for field strengths. This process also determines the corresponding scalar coset matrix as demonstrated in the above for $\mathcal{M}_3$ in $D=8$. If the underlying Cremmer-Julia symmetry cannot be fully realized at the level of action, we should seek certain field strengths which transform among themselves (without need to introduce their duals) in a certain representation of the symmetry. Then the kinectic terms for these field strengths can be written in an invariant form of the symmetry. This can also be used to determine the scalar coset matrix as we did above for the $SL(2,R)$ case. These scalar coset matrices are important for us to construct U-duality multiplets in $10 > D \geq 4$ in section 6. In the following two sections, we will present two explicit examples of the construction of U-duality p-brane multiplets in $D = 8$. First, we will show how to construct $SL(3,Z)$ strings (also the magnetic dual 3-branes) and then we show how to obtain $SL(3,Z) \times SL(2,Z)$ 0-branes (and the dual 4-branes) in this theory. These examples demonstrate the general features of U-duality p-branes of various supergravity theories in diverse dimensions.

4 $SL(3,Z)$ Strings and 3-Branes

In this case, we need to keep the metric $g_{\mu\nu}$, the scalars parametrizing the scalar coset matrix $\mathcal{M}_3$ and the three 2-form gauge potentials $A_2^{(i)}$ in the Lagrangian (16) or (20). The rest of fields in the Lagrangian can be consistently set to zero. The corresponding action can then be written as follows:

$$\begin{aligned} S_8 &= \int d^8x \Big[\sqrt{-g}\Big(R + \frac{1}{4}\mathrm{tr}\,\nabla_\mu\mathcal{M}_3\nabla^\mu\mathcal{M}_3^{-1} - \frac{1}{12}\tilde{F}_3^T\mathcal{M}_3\tilde{F}_3\Big) \\ &\qquad - \frac{1}{2.6^3}\epsilon^{\mu_1\ldots\mu_8}\tilde{F}^{(i)}_{\mu_1\mu_2\mu_3}\tilde{F}^{(j)}_{\mu_4\mu_5\mu_6}A^{(k)}_{\mu_7\mu_8}\epsilon_{ijk}\Big] \end{aligned} \tag{27}$$

where all the notations are explained in the previous section. As mentioned in the previous section, the $SL(2,R)$ factor in the Cremmer-Julia $SL(3,R) \times SL(2,R)$ symmetry is merely a classical T-duality symmetry while the $SL(3,R)$ contains all the classical non-perturbative U-duality symmetry. Actually, the $SL(3,R)$ contains a strong-weak $SL(2,R)$ and a T-duality $SL(2,R)$ as its subgroups. This $SL(2,R)$ along with the T-duality $SL(2,R)$ symmetry just mentioned forms the complete classical T-duality group $SL(2,R) \times SL(2,R) \simeq SO(2,2)$ of the eight dimensional theory. The strong-weak $SL(2,R)$ is actually being inherited from the ten dimensional type IIB theory. One way to understand the nature of the two $SL(2,R)$ subgroups is to examine the scalar coset matrix $\mathcal{M}_3$ in (19). If we set $\varphi = \chi_1 = \chi_2 = 0$, then $\mathcal{M}_3$ is equivalent to the ten dimensional type IIB scalar matrix parametrizing the strong-weak coset $SL(2,R)/SO(2)$ since χ_3 is the RR scalar χ in the type IIB theory. In order to see the T-duality $SL(2,R)$ subgroup, we cannot simply set the dilaton to zero. What we should do instead is to set the shifted dilaton to zero since it is well known that the dilaton is shifted under T-duality.

In the present context, the shifted dilaton is $\tilde{\phi} = \phi - \varphi/\sqrt{3}$ which is proportional to the eight dimensional dilaton as we will see. If we set $\tilde{\phi} = \chi_2 = \chi_3 = 0$, then the $\mathcal{M}_3$ is equivalent to a scalar matrix which does not involve the new dilaton $\tilde{\phi}$ and at the same time parametrizes a $SL(2,R)/SO(2)$ coset. Therefore this $SL(2,R)$ must correspond to a T-duality $SL(2,R)$. If we set fields $A_2^{(2)} = A_2^{(3)} = 0$ and $\chi_1 = \chi_2 = \chi_3 = 0$, then the action (27) is reduced to

$$S_8 = \int d^8x\sqrt{-g}\left[R - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{12}e^{-\frac{2}{\sqrt{3}}\Phi}(\tilde{F}_3^{(1)})^2 - \frac{1}{2}(\partial\Psi)^2\right], \tag{28}$$

where we have made the field redefinitions:

$$\begin{aligned}\Phi &= \frac{\sqrt{3}}{2}\phi - \frac{1}{2}\varphi,\\ \Psi &= \frac{1}{2}\phi + \frac{\sqrt{3}}{2}\varphi,\end{aligned} \tag{29}$$

with Φ the eight dimensional dilaton. It is easy to see that the NSNS string solution considered in section 2 continues to be a NSNS string solution of the above action with $\Psi = 0$.

To construct the $SL(3,Z)$ strings (or U-duality p-branes in general), we always start with zero asymptotic values for the scalars, i.e., here $\mathcal{M}_{30} = I$ with I the unit matrix, and a pure NSNS string (or a pure NSNS p-brane). Here $\mathcal{M}_3$ is denoted as $\mathcal{M}_{30}$ when the scalars take their asymptotic values, i.e., the subscript '0' denotes the asymptotic value. Depending on the charge carried by the NSNS string to be a quantized unit charge or just an arbitrary classical one, there exist two methods which can be used to construct the $SL(3,Z)$ strings. In the former case, a compensating factor needs to be introduced to the initial unit charge by hand such that the transformed charge triplet obtained by a partially given classical $SL(3,R)$ transformation acting on the initial charge triplet, can remain to be quantized. In the latter case, an initial charge triplet with the arbitrary classical NSNS charge as its only non-vanishing component is transformed by the same $SL(3,R)$ transformation to a general charge triplet. Then we impose the charge quantization on the transformed charge triplet due to the existence of 3-branes, the magnetic duals of strings. The two methods produce the same general $SL(3,Z)$ string solution but they have different implications. For the former method, we sandwich a classical $SL(3,R)$ transformation between quantum mechanically allowable intial and final string configurations. As a consequence, the mass of the final string configuration is different from that of the the initial configuration by the compensating factor introduced by hand while the $SL(3,R)$ transformation preserves the mass. This bizarre phenomenon is entirely due to the unnatural use of the method which requires to introduce the compensating factor by hand. We do not have this problem with the second method. Therefore, we will employ it to construct the $SL(3,Z)$ strings in the following.

We first seek a most general $SL(3,R)$ transformation Λ_{30} such that it maps the zero asymptotic values of the scalars to arbitrary given ones, i.e., mapping $\mathcal{M}_{30} = I$ to $\mathcal{M}_{30} = \Lambda_{30} I \Lambda_{30}^T = \Lambda_{30}\Lambda_{30}^T$. Note that we can write in general $\Lambda_{30} = \nu_{30} R$ with ν_{30} a 3×3 matrix in the coset $SL(3,R)/SO(3)$ and R a 3×3 $SO(3)$ matrix. Using the facts that in general $\mathcal{M}_3 = \nu\nu^T$ and $RR^T = R^T R = I$, we must have, from the above and from Eq.(23) with scalars taking their asymptotic values,

$$\nu_{30} = e^{\frac{\varphi_0}{2\sqrt{3}}} \begin{pmatrix} e^{-\phi_0/2} & \chi_{30} e^{\phi_0/2} & (\chi_{20} + \chi_{10}\chi_{30}) e^{-\sqrt{3}\varphi_0/2} \\ 0 & e^{\phi_0/2} & \chi_{10} e^{-\sqrt{3}\varphi_0/2} \\ 0 & 0 & e^{-\sqrt{3}\varphi_0/2} \end{pmatrix}. \tag{30}$$

The explicit form of the $SO(3)$ matrix R is not needed in what follows but we here write it in any case in terms of the three Euler angles (α, β, γ):

$$R = \begin{pmatrix} \cos\alpha\cos\beta & -\sin\alpha\cos\gamma - \cos\alpha\sin\beta\sin\gamma & \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma \\ \sin\alpha\cos\beta & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\cos\alpha\sin\gamma - \sin\alpha\sin\beta\cos\gamma \\ \sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{pmatrix}. \tag{31}$$

As discussed in section 2, the NSNS string configuration, carrying an arbitrary classical charge $Q_{(q_1,q_2,q_3)} = \Delta^{1/2}_{(q_1,q_2,q_3)} Q_0$ with $\Delta_{(q_1,q_2,q_3)}$ an as yet undetermined dimensionless factor and Q_0 the charge unit which may be taken as the quantized unit charge, is associated with the non-vanishing NSNS gauge potential $A_2^{(1)}$. The general string configuration which we are going to construct requires all three 2-form gauge potentials $A_2^{(1)}, A_2^{(2)}, A_2^{(3)}$ to be non-zero. Associated with this configuration is a Noether (or electric-like) charge triplet

$$\mathcal{Q} \equiv \begin{pmatrix} Q^{(1)} \\ Q^{(2)} \\ Q^{(3)} \end{pmatrix}, \tag{32}$$

where

$$Q^{(i)} = \int_{S^5} \left((\mathcal{M}_3)_{ij} * \tilde{F}_3^{(j)} + \frac{1}{2}\epsilon_{ijk} A_2^{(j)} \wedge \tilde{F}_3^{(k)} \right), \tag{33}$$

with S^5 the asymptotic 5-sphere. It follows that the charge triplet should transform as $\mathcal{Q} \to \Lambda_3 \mathcal{Q}$. Therefore, with the $SL(3,R)$ transformation $\Lambda_{30} = \nu_{30} R$ acting on the initial

NSNS charge, we have the following transformed charges:

$$\begin{aligned}
Q^{(1)} &= (\Lambda_{30})_{11}\Delta^{1/2}_{(q_1,q_2,q_3)}Q_0 \\
&= \Big[e^{-\phi_0/2+\varphi_0/2\sqrt{3}}\,R_{11} + \chi_{30}e^{\phi_0/2+\varphi_0/2\sqrt{3}}\,R_{21} \\
&\qquad + (\chi_{20}+\chi_{10}\chi_{30})\,e^{-\sqrt{3}\varphi_0/2}\,R_{31}\Big]\Delta^{1/2}_{(q_1,q_2,q_3)}\,Q_0, \\
Q^{(2)} &= (\Lambda_{30})_{21}\Delta^{1/2}_{(q_1,q_2,q_3)}Q_0 \\
&= \Big[e^{\phi_0/2+\varphi_0/2\sqrt{3}}\,R_{21} + \chi_{10}e^{-\varphi_0/\sqrt{3}}\,R_{31}\Big]\,\Delta^{1/2}_{(q_1,q_2,q_3)}\,Q_0 \\
Q^{(3)} &= (\Lambda_{30})_{31}\Delta^{1/2}_{(q_1,q_2,q_3)}Q_0 \\
&= e^{-\varphi_0/\sqrt{3}}\,R_{31}\Delta^{1/2}_{(q_1,q_2,q_3)}\,Q_0
\end{aligned} \tag{34}$$

Given the vacuum moduli and the three charges $Q^{(i)}$ $(i = 1, 2, 3)$, we have three $SO(3)$ group parameters and the additional $\Delta_{(q_1,q_2,q_3)}$ to be fixed. However, we have only three equations in (34). This is in contrary to the case for the $SL(2, Z)$ strings [9] or fivebranes [11] of $D = 10$ type IIB theory where under the similar conditions the $SL(2, R)$ parameters are completely fixed. Surprisingly, we find that the most important factor $\Delta_{(q_1,q_2,q_3)}$ can nevertheless be completely determined as will be demonstrated below. Then it can be seen from (34) that we can only determine two of the three $SO(3)$ group parameters. This seems to imply that our general string solution will contain an arbitrary parameter. But again to our surprise we find that the general solution has nothing to do with this arbitrary group parameter and all the relevant physical quantities can be uniquely determined as we will show below.

Solving the $SO(3)$ matrix elements R_{11}, R_{21}, R_{31} from (34), we have

$$\begin{aligned}
R_{11} &= e^{\phi_0/2-\varphi_0/2\sqrt{3}}\,\Delta^{-1/2}_{(q_1,q_2,q_3)}\,\frac{Q^{(1)} - \chi_{30}Q^{(2)} - \chi_{20}Q^{(3)}}{Q_0}, \\
R_{21} &= e^{-\phi_0/2-\varphi_0/2\sqrt{3}}\,\Delta^{-1/2}_{(q_1,q_2,q_3)}\,\frac{Q^{(2)} - \chi_{10}Q^{(3)}}{Q_0}, \\
R_{31} &= e^{\varphi_0/\sqrt{3}}\,\Delta^{-1/2}_{(q_1,q_2,q_3)}\,\frac{Q^{(3)}}{Q_0}.
\end{aligned} \tag{35}$$

Using the orthogonal relation $R_{ki}R_{kj} = \delta_{ij}$ for $i = j = 1$, i.e., $R_{11}^2 + R_{21}^2 + R_{31}^2 = 1$, we can fix the $\Delta_{(q_1,q_2,q_3)}$ as

$$\Delta_{(q_1,q_2,q_3)} = e^{2\varphi_0/\sqrt{3}}\left(\frac{Q^{(3)}}{Q_0}\right)^2 + e^{-\phi_0-\varphi_0/\sqrt{3}}\left(\frac{Q^{(2)} - \chi_{10}Q^{(3)}}{Q_0}\right)^2$$

$$+e^{\phi_0-\varphi_0/\sqrt{3}}\left(\frac{Q^{(1)}-\chi_{30}Q^{(2)}-\chi_{20}Q^{(3)}}{Q_0}\right)^2,$$

$$= \left(Q^{(1)}/Q_0, Q^{(2)}/Q_0, Q^{(3)}/Q_0\right)\mathcal{M}_{30}^{-1}\begin{pmatrix} Q^{(1)}/Q_0 \\ Q^{(2)}/Q_0 \\ Q^{(3)}/Q_0 \end{pmatrix}, \qquad (36)$$

where

$$\mathcal{M}_{30}^{-1} = e^{-\frac{\varphi_0}{\sqrt{3}}}\begin{pmatrix} e^{\phi_0} & -\chi_{30}e^{\phi_0} & -\chi_{20}e^{\phi_0} \\ -\chi_{30}e^{\phi_0} & \chi_{30}^2 e^{\phi_0}+e^{-\phi_0} & -\chi_{10}e^{-\phi_0}+\chi_{20}\chi_{30}e^{\phi_0} \\ -\chi_{20}e^{\phi_0} & \chi_{20}\chi_{30}e^{\phi_0}-\chi_{10}e^{-\phi_0} & \chi_{20}^2 e^{\phi_0}+\chi_{10}^2 e^{-\phi_0}+e^{\sqrt{3}\varphi_0} \end{pmatrix}. \qquad (37)$$

From (36), it is clear that $\Delta_{(q_1,q_2,q_3)}$ is $SL(3,R)$ invariant.

By now we have constructed a most general $D=8$ string configuration carrying classical charges given by the charge triplet. The central charge (therefore the ADM mass per unit length as well as the tension measured in Einstein metric) associated with this string is $Q_{(q_1,q_2,q_3)} = \Delta^{1/2}_{(q_1,q_2,q_3)}Q_0$ with $\Delta_{(q_1,q_2,q_3)}$ as given in Eq.(36). The metric continues to be given by the one in Eq.(3) but now with $Q = Q_{(q_1,q_2,q_3)}$. The three 3-form field strengths are now given by the triplet

$$\begin{pmatrix} \tilde{F}_3^{(1)} \\ \tilde{F}_3^{(2)} \\ \tilde{F}_3^{(3)} \end{pmatrix} = (\Lambda_{30}^T)^{-1}\begin{pmatrix} \Delta^{1/2}_{(q_1,q_2,q_3)}\, Q_0\, A^{-1/\sqrt{3}}(\rho)\, *\epsilon_5 \\ 0 \\ 0 \end{pmatrix},$$

$$= \mathcal{M}_{30}^{-1}\begin{pmatrix} Q^{(1)} \\ Q^{(2)} \\ Q^{(3)} \end{pmatrix} A^{-\frac{1}{\sqrt{3}}}(\rho) * \epsilon_5, \qquad (38)$$

where we have used $\mathcal{M}_{30}^{-1} = (\Lambda^T{}_{30})^{-1}\Lambda_{30}^{-1}$,

$$\mathcal{Q} = \Lambda_{30}\begin{pmatrix} \Delta^{1/2}_{(q_1,q_2,q_3)}Q_0 \\ 0 \\ 0 \end{pmatrix}, \qquad (39)$$

and $A(\rho)$ is given in Eq.(3). So far all the above quantities are independent of the undetermined arbitrary $SO(3)$ group parameter. Our last step to complete the construction of the general classical string solution is to determine all the scalars appearing in $\mathcal{M}_3$ as given by Eq.(19). This can be achieved by the following matrix equation:

$$\mathcal{M}_3 = A^{-\frac{1}{2\sqrt{3}}}(\rho)\Lambda_{30}\begin{pmatrix} A^{\frac{\sqrt{3}}{2}}(\rho) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\Lambda_{30}^T, \qquad (40)$$

$$= A^{-\frac{1}{2\sqrt{3}}}(\rho)\nu_{30}\begin{pmatrix} R_{11}^2B(\rho)+1 & R_{11}R_{21}B(\rho) & R_{11}R_{31}B(\rho) \\ R_{21}R_{11}B(\rho) & R_{21}^2B(\rho)+1 & R_{21}R_{31}B(\rho) \\ R_{31}R_{11}B(\rho) & R_{31}R_{21}B(\rho) & R_{31}^2B(\rho)+1 \end{pmatrix}\nu_{30}^T, \qquad (41)$$

where $B(\rho) = A^{\sqrt{3}/2}(\rho) - 1$. Let us examine a few things here. As $\rho \to \infty$, $A(\rho) \to 1$ and $B(\rho) \to 0$. So, from (41), we first have $\mathcal{M}_3 \to \nu_{30}\nu_{30}^T = \mathcal{M}_{30}$ as expected. Second, the right side of (41) is completely fixed, independent of the undetermined $SO(3)$ group parameter, since R_{11}, R_{21}, R_{31} are completely fixed by Eq.(35). Furthermore, the scalars appearing in $\mathcal{M}_3$ can be determined without even a sign ambiguity. Look at the structure of $\mathcal{M}_3$ matrix in (19). From the above equation, we can simply read out φ first. We can then fix χ_1. Then ϕ and $\chi_2 + \chi_1\chi_3$. With all these known, we can then determine χ_3. Applying this χ_3 and the known χ_1 to the known $\chi_2+\chi_1\chi_3$, we finally fix χ_2. It follows that all scalars for this solution are independent of the undetermined $SO(3)$ parameter. We therefore confirm our claim that the $SL(3,Z)$ multiplet of string solutions can be obtained without any arbitrariness eventhough one of the $SO(3)$ parameter remains undetermined. We will not present the explicit expressions for each of the scalars here.

Our general classical string solution also preserves half of the spacetime supersymmetry as the original pure NSNS string since the global $SL(3,R)$ transformation commutes with the supersymmetry transformation. Therefore, our general string solution continues to be BPS which implies that the ADM mass per unit length, the central charge $Q_{(q_1,q_2,q_3)}$ and the string tension measured in Einstein metric are all the same in proper units.

So far we have only constructed the most general classical string solution in the sense that the three charges $Q^{(1)}, Q^{(2)}, Q^{(3)}$ can be arbitrary. Due to the presence of the magnetic duals of strings, i.e., the 3-branes, each of the three charges must be quantized [14] separately in terms of the unit charge Q_0. For example, the magnetic-like charges $P^{(1)} \neq 0, P^{(2)} = 0, P^{(3)} = 0$ carried by a 3-brane must imply that $Q^{(1)}$ is quantized in terms of the unit charge Q_0. So the charge triplet for a general quantum-mechanically allowable string solution is

$$\mathcal{Q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} Q_0, \tag{42}$$

where q_1, q_2, q_3 are three integers. In terms of the unit charge Q_0, the charge triplet should remain to be an integral triplet under quantum-mechanically allowable transformation. This necessarily breaks the continuous $SL(3,R)$ symmetry to a discrete $SL(3,Z)$ whose elements take only integral values.

The most general quantum-mechanically allowable string configuration can be obtained simply by imposing $Q^{(1)} = q_1 Q_0, Q^{(2)} = q_2 Q_0, Q^{(3)} = q_3 Q_0$ in the above classical string configuration. For example,

$$\Delta_{(q_1,q_2,q_3)} = (q_1,q_2,q_3)\,\mathcal{M}_{30}^{-1}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}, \tag{43}$$

which is now $SL(3,Z)$ invariant. Therefore, the ADM mass per unit length $M_{(q_1,q_2,q_3)}$, the central charge $Q_{(q_1,q_2,q_3)}$ and the string tension $T_{(q_1,q_2,q_3)}$ measured in Einstein metric are all $SL(3,Z)$ invariant. In proper units, we can set all three equal in which case we can take Q_0 as the fundamental string tension T. Then for a (q_1,q_2,q_3)-string, we have

$$\begin{aligned} M_{(q_1,q_2,q_3)} &= Q_{(q_1,q_2,q_3)} = T_{(q_1,q_2,q_3)} = \Delta^{1/2}_{(q_1,q_2,q_3)} T \\ &= \sqrt{e^{\frac{2\varphi_0}{\sqrt{3}}} q_3^2 + e^{-\phi_0-\frac{\varphi_0}{\sqrt{3}}}(q_2-\chi_{10}q_3)^2 + e^{\phi_0-\frac{\varphi_0}{\sqrt{3}}}(q_1-\chi_{30}q_2-\chi_{20}q_3)^2}\, T \end{aligned} \tag{44}$$

The (q_1,q_2,q_3)-string tension measured in string metric is

$$\begin{aligned} T_{(q_1,q_2,q_3)} &= e^{-\Phi_0/\sqrt{3}} \Delta^{1/2}_{(q_1,q_2,q_3)} T \\ &= \sqrt{e^{-\phi_0+\sqrt{3}\varphi_0} q_3^2 + e^{-2\phi_0}(q_2-\chi_{10}q_3)^2 + (q_1-\chi_{30}q_2-\chi_{20}q_3)^2}\, T \end{aligned} \tag{45}$$

where $\Phi_0 = \sqrt{3}\phi_0/2 - \varphi_0/2$.

Let us make a few comments about the above tension formula. For simplicity, we set $\chi_{10} = \chi_{20} = \chi_{30} = 0$. We note that the tension for $(1,0,0)$-string is proportional to 1 which is expected since this is a NSNS string. The tension for $(0,1,0)$-string is proportional to $e^{-\phi_0} = 1/g_s$, i.e., inversely to the $D = 10$ type IIB string coupling constant. This is also expected since this string is a D-string [20] in $D = 10$ and this tension relation can be easily verified through simple dimensional reduction of the $D = 10$ D-string σ-model action to $D = 8$. The tension for the $(0,0,1)$-string is, however, proportional to $e^{-\phi_0/2+\sqrt{3}\varphi_0/2} = e^{-(\Phi_0-\varphi_0)/\sqrt{3}}$, a strange behavior. We would naively expect that this is also a D-string, i.e., the tension should be inversely proportional to either the $D = 10$ string coupling constant or $D = 8$ string coupling constant which is $e^{\sqrt{3}\Phi_0}$ [6]. It turns out that this string is a $D = 10$ type IIB D-threebrane with two of its spatial dimensions wrapped on the two compactified dimensions when we go from 10 to 8.

Let us see how this tension can be obtained from the $D = 10$ D-threebrane σ-model action. In $D = 10$ the action for the D-threebrane to the lowest order (ignoring the world volume vector fields) can be represented in string metric as

$$S_3 \sim \int d^4\xi\, e^{-\phi}\sqrt{-\gamma}\gamma^{ij}\partial_i X^M \partial_j X^N g_{MN}, \tag{46}$$

where the factor $e^{-\phi}$ indicates that the threebrane tension is inversely proportional to the string coupling constant, i.e., $\sim 1/g_s$, and the so-called worldvolume induced metric $\gamma_{ij} = \partial_i X^M \partial_j X^N g_{MN}$ with $i = 0, 1, 2, 3$. In order to obtain the $(0, 0, 1)$-string in $D = 8$ from the threebrane in $D = 10$, we have to adopt the so-called double dimensional reduction procedure [21], i.e., identifying two worldvolume spatial dimensions with the two compactified spatial dimensions of spacetime, i.e., $\xi^3 = z^9, \xi^2 = z^8$ with z^8, z^9 the two compactified space-like dimensions of spacetime. When we compactify the $D = 10$ type IIB supergravity theory to $D = 8$, the Einstein metric is

$$ds_{10}^2 = e^{-\varphi/2\sqrt{3}} ds_8^2 + e^{\sqrt{3}\varphi/2} \left[(dz^8)^2 + (dz^9)^2\right], \tag{47}$$

where ds_8^2 is the eight dimensional Einstein metric and the Kaluza-Klein vectors are ignored here. The ten dimensional string metric (also the eight dimensional string metric) is

$$\begin{aligned} ds_{10}^2(\text{string metric}) &= e^{\phi/2}\, ds_{10}^2, \\ &= e^{\Phi/\sqrt{3}} ds_8^2 + e^{\phi/2+\sqrt{3}\varphi/2} \left[(dz^8)^2 + (dz^9)^2\right], \end{aligned} \tag{48}$$

where Φ is the eight dimensional dilaton. With these metric relations and assuming that all the fields are independent of z^8 and z^9, we have $\gamma_{22} = \gamma_{33} = e^{\phi/2+\sqrt{3}\varphi/2}$. Now the D-threebrane action in $D = 10$ goes to the $(0, 0, 1)$-string action in $D = 8$ as

$$S_3 \to S_1 \sim \int d^2\xi e^{-\phi} \gamma_{22} \sqrt{-\gamma} \gamma^{ij} \partial_i X^\mu \partial_j X^\nu g_{\mu\nu}, \tag{49}$$

where $i = 0, 1$. From the above, we have the $(0, 0, 1)$-string tension proportional to $e^{-\phi_0}(\gamma_{22})_0 = e^{-\phi_0/2+\sqrt{3}\varphi_0/2}$, which is exactly the same as that given by our tension formula (45).

As discussed earlier, the global $SL(3, R)$ contains a strong-weak $SL(2, R)$ subgroup (corresponding to $\varphi = \chi_1 = \chi_2 = 0$) and a T-dual $SL(2, R)$ subgroup (corresponding to $\tilde{\phi} = \chi_2 = \chi_3 = 0$). Similarly, we expect that the quantum $SL(3, Z)$ contains a strong-weak $SL(2, Z)$ subgroup and a T-dual $SL(2, Z)$ subgroup. Evidence for this can be provided by the tension formula (45). When $\varphi = \chi_1 = \chi_2 = 0$, we recover the tension formula for the type IIB $SL(2, Z)$ (q_1, q_2)-string as discussed in ref.[9]. Also for $\tilde{\phi} = \chi_2 = \chi_3 = 0$, we have the formula for the T-dual $SL(2, Z)$ (q_2, q_3)-strings. For the (q_2, q_3)-string, the tension for $(q_2, 0)$-string is inversely proportional to the tension for $(0, q_3)$-string. This inverse relation is actually $1/R \to R$, a typical T-duality relation, with $R = e^{\varphi_0/\sqrt{3}}$ the compactification radius measured in string metric. In other words, the $(q_2, 0)$-string carries momentum modes while $(0, q_3)$-string carries winding modes with respect to the compactifications.

The 3-form field strength triplet is now given as,

$$\tilde{F}_3 = \mathcal{M}_{30}^{-1} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} Q_0 A^{-\frac{1}{\sqrt{3}}}(\rho) * \epsilon_5, \tag{50}$$

As mentioned earlier, the metric in Eq.(3) retains the same form but now with $Q = Q_{(q_1,q_2,q_3)}$:

$$ds^2 = \left(1 + \frac{Q_{(q_1,q_2,q_3)}}{4\rho^4}\right)^{-2/3} \left[-dt^2 + (dx^1)^2\right] + \left(1 + \frac{Q_{(q_1,q_2,q_3)}}{4\rho^4}\right)^{1/3} \left[d\rho^2 + \rho^2 d\Omega_5^2\right] \tag{51}$$

The above (q_1, q_2, q_3)-string configuration encodes all the information about the $SL(3, Z)$ multiplets of the $D = 8$ strings. Note that for given asymptotic values of the scalars, i.e., for a given vacuum, each of the infinitely many integral triplets (q_1, q_2, q_3) gives a different value for the $\Delta_{(q_1,q_2,q_3)}$ which cannot be related to each other by a $SL(3, Z)$ transformation since it is invariant by such a transformation. Further, this $\Delta_{(q_1,q_2,q_3)}$ measures the mass per unit length, the central charge and the tension. Therefore, we can use this factor to label different $SL(3, Z)$ multiplets. Within each such multiplet, we have a collection of infinitely many discrete vacua and a collection of infinitely many integral charge triplets. Each of such vacua and its corresponding integral charge triplet are obtained from the given initial vacuum $\mathcal{M}_{30}$ and the given initial charge triplet (q_1, q_2, q_3) by a particular $SL(3, Z)$ transformation. Picking a special vaccum in such a multiplet will break the $SL(3, Z)$ spontaneously. In other words, all the string configurations in such a multiplet are physically equivalent. The physically inequivalent string configurations are those with different $\Delta_{(q_1,q_2,q_3)}$ values which correspond to different integral triplet (q_1, q_2, q_3) for a fixed $\mathcal{M}_{30}$, i.e., a fixed vaccum.

Finally, we would like to discuss the stability of a general $(q_1,\ q_2,\ q_3)$ string (Discussion of the stability for Type IIB $SL(2, Z)$ strings is given in [9,22]). We have noted that the tension of such a string is given by $T_{(q_1,q_2,q_3)} = \Delta_{(q_1,q_2,q_3)}^{1/2} T$ and so, it can be easily checked that the tensions satisfy the following triangle inequality relation, irrespective of the vacuum moduli,

$$T_{(q_1,q_2,q_3)} + T_{(p_1,p_2,p_3)} \geq T_{(q_1+p_1,q_2+p_2,q_3+p_3)} \tag{52}$$

where the equality holds if and only if $p_1 q_2 = p_2 q_1$, $p_2 q_3 = p_3 q_2$ and $p_1 q_3 = p_3 q_1$, i.e., when, $p_1 = n q_1$, $p_2 = n q_2$, $p_3 = n q_3$, with n being an integer. So, when any two of the three integers q_1, q_2, q_3 are relatively prime to each other the string would be stable as the (q_1, q_2, q_3)-string in that case will be prevented from decaying by the inequality relation (52) called as the 'tension gap' equation. The same conclusion can be drawn when the central charge triangle inequality relation (now called 'charge gap' equation) and the charge conservation are employed. We therefore conclude that all physically inequivalent stable (q_1, q_2, q_3)-string configurations are those corresponding to all possible integral triplets (q_1, q_2, q_3) with any two of the three integers in each triplet relatively prime, and with $\mathcal{M}_{30}$ belonging to the fundamental region of $SL(3, Z)$, i.e., $SL(3, Z)\backslash SL(3, R)/SO(3)$.

The magnetic dual of a string in $D = 8$ is a 3-brane. The $SL(3, Z)$ family of 3-branes can be constructed following the same steps as we did in ref.[11] for the $SL(2, Z)$ five-branes, the magnetic duals of strings in $D = 10$ type IIB theory. We will not repeat these steps here but merely present the $SL(3, Z)$ (p_1, p_2, p_3)-threebrane configuration associated with a magnetic-like integral charge triplet denoted as p. The Δ-factor in this case is,

$$\Delta_{(p_1,p_2,p_3)} = (p_1, p_2, p_3)\, \mathcal{M}_{30} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}. \tag{53}$$

The mass per unit 3-brane volume $M_{(p_1,p_2,p_3)}$, the central charge $Q_{(p_1,p_2,p_3)}$ and the tension $T_{(p_1,p_2,p_3)}$ measured in Einstein metric are

$$\begin{aligned} M_{(p_1,p_2,p_3)} &= Q_{(p_1,p_2,p_3)} = T_{(p_1,p_2,p_3)} = \Delta^{1/2}_{(p_1,p_2,p_3)} Q_0, \\ &= \Big\{ e^{-\phi_0+\varphi_0/\sqrt{3}} p_1^2 + e^{\phi_0+\varphi_0/\sqrt{3}} (p_2 + \chi_{30} p_1)^2 \\ &\qquad + e^{-2\varphi_0/\sqrt{3}} \left[p_3 + \chi_{10} p_2 + (\chi_{20} + \chi_{10}\chi_{30}) p_1\right]^2 \Big\}^{1/2} Q_0, \end{aligned} \tag{54}$$

where Q_0 is the unit magnetic charge which can be taken as the fundamental 3-brane tension T.

The 3-form field strength triplet is

$$\tilde{F}_3 = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} Q_0 \epsilon_3. \tag{55}$$

Similarly, the scalars are determined uniquely by the following matrix equation

$$\mathcal{M}_3 = A^{\frac{1}{2\sqrt{3}}}(\rho) \Lambda_{30} \begin{pmatrix} A^{-\frac{\sqrt{3}}{2}}(\rho) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Lambda_{30}^T, \tag{56}$$

$$= A^{\frac{1}{2\sqrt{3}}}(\rho)\nu_{30} \begin{pmatrix} R_{11}^2 B(\rho)+1 & R_{11}R_{21}B(\rho) & R_{11}R_{31}B(\rho) \\ R_{21}R_{11}B(\rho) & R_{21}^2 B(\rho)+1 & R_{21}R_{31}B(\rho) \\ R_{31}R_{11}B(\rho) & R_{31}R_{21}B(\rho) & R_{31}^2 B(\rho)+1 \end{pmatrix} \nu_{30}^T, \tag{57}$$

where $B(\rho) = A^{-\sqrt{3}/2}(\rho) - 1$ with $A(\rho) = \Big(1 + Q_{(p_1,p_2,p_3)}/2\rho^2\Big)^{2/\sqrt{3}}$ and

$$\begin{aligned} R_{11} &= \Delta^{-1/2}_{(p_1,p_2,p_3)}\, e^{-\phi_0/2+\varphi_0/2\sqrt{3}}\, p_1, \\ R_{21} &= \Delta^{-1/2}_{(p_1,p_2,p_3)}\, e^{\phi_0/2+\varphi_0/2\sqrt{3}}\, (p_2 + \chi_{30} p_1), \\ R_{31} &= \Delta^{-1/2}_{(p_1,p_2,p_3)}\, e^{-\varphi_0/\sqrt{3}}\, [p_3 + \chi_{10} p_2 + (\chi_{20} + \chi_{10}\chi_{30})\, p_1]. \end{aligned} \tag{58}$$

The metric is

$$ds^2 = \left(1 + \frac{Q_{(p_1,p_2,p_3)}}{2\rho^2}\right)^{-1/3} \Big[-(dt)^2 + (dx^i)^2\Big] + \left(1 + \frac{Q_{(p_1,p_2,p_3)}}{2\rho^2}\right)^{2/3} \Big[(d\rho)^2 + \rho^2\, d\Omega_3^2\Big], \tag{59}$$

with $i = 1, 2, 3$. In string metric, the tension for a (p_1, p_2, p_3)-threebrane is

$$\begin{aligned} T_{(p_1,p_2,p_3)} &= \Big\{ e^{-3\phi_0+\sqrt{3}\varphi_0} p_1^2 + e^{-\phi_0+\sqrt{3}\varphi_0} (p_2 + \chi_{30} p_1)^2 \\ &\qquad + e^{-2\phi_0}\, [p_3 + \chi_{10} p_2 + (\chi_{20} + \chi_{10}\chi_{30}) p_1]^2 \Big\}^{1/2} T. \end{aligned} \tag{60}$$

Using the brane σ-model action approach discussed before, we can understand this tension formula easily from the facts that the $(1, 0, 0)$-threebrane is a $D = 10$ type IIB NSNS 5-brane [23] wrapped on the two compactified dimensions and $(0, 1, 0)$-threebrane a wrapped $D = 10$ type IIB RR 5-brane [24] while $(0, 0, 1)$-threebrane is obtained by simple dimensional reduction of the $D = 10$ type IIB threebrane [8,25].

As for the $SL(3, Z)$ strings, similar results can also be obtained on multiplets and stability for the 3-branes. The corresponding $SL(3, Z)$ black strings and black 3-branes can also be constructed similarly.

5 $SL(3, Z) \times SL(2, Z)$ 0-Branes and 4-Branes

Given that $SL(3, R) \times SL(2, R)$ is the Cremmer-Julia symmetry of the $D = 8$ theory, we cannot resist to give a complete construction of all U-duality p-branes in this theory. In this section, we will present the last two U-duality p-branes, namely, U-duality 0-branes and 4-branes. Let us discuss the 0-branes first.

We will construct the $SL(3,Z) \times SL(2,Z)$ multiplets of 0-branes from the following known 0-brane configuration preserving half of the spacetime supersymmetry [6],

$$
\begin{aligned}
ds^2 &= -\left(1+\frac{Q}{5\rho^5}\right)^{-5/6}(dt)^2 + \left(1+\frac{Q}{5\rho^5}\right)^{1/6}\left[d\rho^2+\rho^2 d\Omega_6^2\right], \\
e^{-2\tilde{\Phi}} &= \left(1+\frac{Q}{5\rho^5}\right)^{\sqrt{7/3}} = A(\rho), \;\; \tilde{\mathcal{F}}_2^{(1)} = QA^{-\sqrt{7/12}}(\rho) * \epsilon_6, \qquad (61)
\end{aligned}
$$

which is the solution of the following action

$$
S = \int d^8x\sqrt{-g}\left[R - \frac{1}{2}(\partial\tilde{\Phi})^2 - \frac{1}{4}\, e^{-\sqrt{7/3}\tilde{\Phi}}(\tilde{\mathcal{F}}_2^{(1)})^2\right]. \qquad (62)
$$

In order to obtain the above action from our action (16), we are forced to take

$$
\sigma = -\phi = \sqrt{\frac{3}{7}}\,\tilde{\Phi}, \;\; \varphi = \sqrt{\frac{1}{7}}\,\tilde{\Phi}, \qquad (63)
$$

and we also need to set the rest of the fields not relevant for us to zero. In other words, to have a 0-brane solution which preserves half of the spacetime supersymmetry, we are forced to have non-vanishing σ field. This turns out to be the key to have a complete construction for the $SL(3,Z)\times SL(2,Z)$ 0-brane solutions. Otherwise, the $SL(2,Z)$ factor would have been trivial.

We could construct the $SL(3,Z) \times SL(2,Z)$ multiplets of 0-branes by following the same route as we did for the $SL(3,Z)$ strings. But from the study of the $SL(3,Z)$ strings along with the examples studied previously in [9,11,12], we learn that a U-duality p-brane configuration can be determined completely by the underlying symmetry properties without the need to follow the detail steps as, for example, we did for the $SL(3,Z)$ strings, once a particular p-brane configuration is known. In other words, we can simply write down a U-duality p-brane configuration based on the underlying symmetry properties. We will use here the latter method to write down the 0-brane solution. This specific example will also serve the purpose of demonstrating the method for constructing the general U-duality p-branes of various supergravity theories in diverse dimensions which will be presented in the following section.

Starting from the above particular 0-brane solution, we write down first the $SL(3,Z)$ 0-branes involving the 2-from field strengths $\tilde{\mathcal{F}}_2^{(i)}$ with $i = 1,2,3$. The $\Delta_{(q_1,q_2,q_3)}$-factor for this $SL(3,Z)$ 0-brane is

$$
\Delta_{(q_1,q_2,q_3)} = (q_1, q_2, q_3)\,\mathcal{M}_{30}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}, \qquad (64)
$$

which is $SL(3,Z)$ invariant as follows from (22). Here $\mathcal{M}_{30}$ is the scalar coset matrix given by Eq.(19) with the scalars taking their asymptotic values. The ADM mass and the central charge are

$$M_{(q_1,q_2,q_3)} = Q_{(q_1,q_2,q_3)} = \Delta^{1/2}_{(q_1,q_2,q_3)} Q_0, \tag{65}$$

where Q_0 is the unit electric charge. The 2-form field strength triplet is now given as,

$$\tilde{\mathcal{F}}_2 = \mathcal{M}_{30} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} Q_0 A^{-\sqrt{7/12}}(\rho) * \epsilon_6. \tag{66}$$

As we did for the $SL(3,Z)$ strings, the scalars can be uniquely determined by the following matrix equation

$$\begin{aligned} \mathcal{M}_3 &= A^{1/\sqrt{21}}(\rho)\,\Lambda_{30} \begin{pmatrix} A^{-\sqrt{3/7}}(\rho) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Lambda_{30}^T, \\ &= A^{1/\sqrt{21}}(\rho)\,\nu_{30} \begin{pmatrix} R_{11}^2 B(\rho)+1 & R_{11}R_{21}B(\rho) & R_{11}R_{31}B(\rho) \\ R_{21}R_{11}B(\rho) & R_{21}^2 B(\rho)+1 & R_{21}R_{31}B(\rho) \\ R_{31}R_{11}B(\rho) & R_{31}R_{21}B(\rho) & R_{31}^2 B(\rho)+1 \end{pmatrix} \nu_{30}^T, \end{aligned} \tag{67}$$

where $B(\rho) = A^{-\sqrt{3/7}}(\rho) - 1$. The same discussion as for the $SL(3,Z)$ strings applies here. The corresponding metric for the (q_1,q_2,q_3)-particle continues to be given by the metric in (61) but now with $Q = Q_{(q_1,q_2,q_3)}$.

We now start to construct the $SL(3,Z)\times SL(2,Z)$ 0-brane directly from the initial 0-brane configuration. If we denote q as the integral electric charge triplet associated with the 2-from field strength triplet $\tilde{\mathcal{F}}_2$ and q' as the integral electric charge triplet associated with the 2-form field strength triplet $\tilde{F}_2$, we then have the $\Delta_{(q,q')}$-factor as

$$\Delta_{(q,q')} = \left(q^T, q'^T\right) \mathcal{M}_{20}^{-1} \begin{pmatrix} \mathcal{M}_{30} q \\ \mathcal{M}_{30} q' \end{pmatrix}, \tag{68}$$

which is $SL(3,Z)\times SL(2,Z)$ invariant. Here $\mathcal{M}_{20}$ is the scalar coset matrix given by Eq.(21) with the scalars taking their asymptotic values.

The ADM mass $M_{(q,q')}$ and the central charge $Q_{(q,q')}$ are given by

$$\begin{aligned} M_{(q,q')} &= Q_{(q,q')} = \Delta^{1/2}_{(q,q')} Q_0, \\ &= \left[e^{\sigma_0}\left(q-\rho_0 q'\right)^T \mathcal{M}_{30}\left(q-\rho_0 q'\right) + e^{-\sigma_0} q'^T \mathcal{M}_{30} q'\right]^{1/2} Q_0, \\ &= \Big\{ e^{\sigma_0-\phi_0+\varphi_0/\sqrt{3}}\left(q_1-\rho_0 q'_1\right)^2 + e^{\sigma_0+\phi_0+\varphi_0/\sqrt{3}}\left[\left(q_2-\rho q'_2\right)+\chi_{30}\left(q_1-\rho_0 q'_1\right)\right]^2 \end{aligned}$$

$$+e^{\sigma_0-2\varphi_0/\sqrt{3}}\left[(q_3-\rho_0q_3')+\chi_{10}\left(q_2-\rho_0q_2'\right)+\left(\chi_{20}+\chi_{10}\chi_{30}\right)\left(q_1-\rho_0q_1'\right)\right]^2$$
$$+e^{-\sigma_0-\phi_0+\varphi_0/\sqrt{3}}q'^2_1+e^{-\sigma_0+\phi_0+\varphi_0/\sqrt{3}}\left(q_2'+\chi_{30}q_1'\right)^2$$
$$+e^{-\sigma_0-2\varphi_0/\sqrt{3}}\left[q_3'+\chi_{10}q_2'+\left(\chi_{20}+\chi_{10}\chi_{30}\right)q_1'\right]^2\Bigg\}^{1/2}Q_0. \tag{69}$$

The six 2-form field strengths $\tilde{\mathcal{F}}_2^{(i)}$ and $\tilde{F}_2^{(i)}$ with $i=1,2,3$ are given by

$$\begin{pmatrix}\tilde{\mathcal{F}}_2\\ \tilde{F}_2\end{pmatrix}=\mathcal{M}_{20}^{-1}\begin{pmatrix}\mathcal{M}_{30}q\\ \mathcal{M}_{30}q'\end{pmatrix}Q_0A^{-\sqrt{7/12}}(\rho)*\epsilon_6. \tag{70}$$

The scalars parametrizing the coset $SL(3,R)/SO(3)$ continue to be given by Eq.(67) but now the $SO(3)$ elements appearing in the equation take the form,

$$\begin{aligned}
R_{11} &= \pm\Delta_{(q,q')}^{-1/2}\,e^{-\phi_0/2+\varphi_0/2\sqrt{3}}\left[e^{\sigma_0}\left(q_1-\rho_0q_1'\right)^2+e^{-\sigma_0}q'^2_1\right]^{1/2},\\
R_{21} &= \pm\Delta_{(q,q')}^{-1/2}e^{\phi_0/2+\varphi_0/2\sqrt{3}}\left\{e^{\sigma_0}\left[(q_2-\rho_0q_2')+\chi_{30}\left(q_1-\rho_0q_1'\right)\right]^2+e^{-\sigma_0}\left(q_2'+\chi_{30}q_1'\right)^2\right\}^{1/2},\\
R_{31} &= \pm\Delta_{(q,q')}^{-1/2}e^{-\varphi_0/\sqrt{3}}\Big\{e^{\sigma_0}\left[(q_3-\rho_0q_3')+\chi_{10}\left(q_2-\rho_0q_2'\right)+\left(\chi_{20}+\chi_{10}\chi_{30}\right)\left(q_1-\rho_0q_1'\right)\right]^2\\
&\qquad +e^{-\sigma_0}\left[q_3'+\chi_{10}q_2'+\left(\chi_{20}+\chi_{10}\chi_{30}\right)q_1'\right]^2\Big\}^{1/2}.
\end{aligned} \tag{71}$$

The scalars σ and ρ parametrizing the coset $SL(2,R)/SO(2)$ are now given by the following matrix equation

$$\begin{aligned}
\mathcal{M}_2 &= A^{-\sqrt{3/28}}(\rho)\,\Lambda_{20}\begin{pmatrix}A^{\sqrt{3/7}}(\rho) & 0\\ 0 & 1\end{pmatrix}\Lambda_{20}^T,\\
&= A^{-\sqrt{3/28}}(\rho)\,\nu_{20}\begin{pmatrix}C(\rho)\cos^2\alpha+1 & C(\rho)\cos\alpha\sin\alpha\\ C(\rho)\cos\alpha\sin\alpha & C(\rho)\sin^2\alpha+1\end{pmatrix}\nu_{20}^T,
\end{aligned} \tag{72}$$

where $C(\rho)=A^{\sqrt{3/7}}(\rho)-1$, ν_{20} is

$$\nu_{20}=e^{\sigma_0/2}\begin{pmatrix}e^{-\sigma_0} & \rho_0\\ 0 & 1\end{pmatrix}, \tag{73}$$

and $\cos\alpha$ and $\sin\alpha$ are given by

$$\begin{aligned}
\cos\alpha &= \pm\frac{e^{\sigma_0/2}\left(q_1-\rho_0q_1'\right)}{\left[e^{\sigma_0}\left(q_1-\rho_0q_1'\right)^2+e^{-\sigma_0}q'^2_1\right]^{1/2}},\\
\sin\alpha &= \pm\frac{e^{-\sigma_0/2}q_1'}{\left[e^{\sigma_0}\left(q_1-\rho_0q_1'\right)^2+e^{-\sigma_0}q'^2_1\right]^{1/2}}.
\end{aligned} \tag{74}$$

As usual, we will not present the explicit expressions for σ and ρ which can be obtained in a straightforward manner in this simple case. The metric for the $SL(3,Z)\times SL(2,Z)$ 0-brane continues to be given by the one in Eq.(61) but now with $Q=Q_{(q,q')}$. Also we expect that when any two of the three integers in each integral triplet are relatively prime to each other, the 0-brane is stable.

We now present the configuration for a general $SL(3,Z)\times SL(2,Z)$ 4-brane carrying two magnetic-like integral charge triplets p and p'. The Δ-factor is

$$\Delta_{(p,p')}=(p^T,p'^T)\mathcal{M}_{20}\begin{pmatrix}\mathcal{M}_{30}^{-1}p\\ \mathcal{M}_{30}^{-1}p'\end{pmatrix}. \tag{75}$$

The mass per unit 4-brane volume $M_{(p,p')}$, the central charge $Q_{(p,p')}$ and tension $T_{(p,p')}$ measured in Einstein frame are

$$\begin{aligned} M_{(p,p')} &= Q_{(p,p')}=T_{(p,p')}=\Delta^{1/2}_{(p,p')}Q_0,\\ &= \left[e^{-\sigma_0}p^T\mathcal{M}_{30}^{-1}p+e_0^{\sigma}(p'+\rho_0p)^T\mathcal{M}_{30}^{-1}(p'+\rho_0p)\right]^{1/2}Q_0,\\ &= \Big(e^{-\sigma_0}\left\{e^{\phi_0-\varphi_0/\sqrt{3}}(p_1-\chi_{30}p_2-\chi_{20}p_3)^2+e^{-\phi_0-\varphi_0/\sqrt{3}}(p_2-\chi_{10}p_3)^2+e^{2\varphi_0/\sqrt{3}}p_3^2\right\}\\ &\quad +e_0^{\sigma}\Big\{e^{\phi_0-\varphi_0/\sqrt{3}}\left[(p_1'+\rho_0p_1)-\chi_{30}(p_2'+\rho_0p_2)-\chi_{20}(p_3'+\rho_0p_3)\right]^2\\ &\quad +e^{-\phi_0-\varphi_0/\sqrt{3}}\left[(p_2'+\rho_0p_2)-\chi_{10}(p_3'+\rho_0p_3)\right]^2+e^{2\varphi_0/\sqrt{3}}(p_3'+\rho_0p_3)^2\Big\}\Big)^{1/2}Q_0. \end{aligned} \tag{76}$$

The two 2-form field strength triplets are

$$\begin{pmatrix}\tilde{\mathcal{F}}_2\\ \tilde{F}_2\end{pmatrix}=\begin{pmatrix}p\\ p'\end{pmatrix}Q_0\epsilon_2. \tag{77}$$

The metric is

$$ds^2=\left(1+\frac{Q_{(p,p')}}{\rho}\right)^{-1/6}\left[-(dt)^2+(dx^i)^2\right]+\left(1+\frac{Q_{(p,p')}}{\rho}\right)^{5/6}\left[(d\rho)^2+\rho^2d\Omega_2^2\right], \tag{78}$$

with $i=1,2,3,4$.

The scalars parametrizing the coset $SL(3,R)/SO(3)$ are given uniquely by the matrix equation

$$\begin{aligned} \mathcal{M}_3 &= A^{-1/\sqrt{21}}(\rho)\,\Lambda_{30}\begin{pmatrix}A^{\sqrt{3/7}}(\rho)&0&0\\0&1&0\\0&0&1\end{pmatrix}\Lambda_{30}^T,\\ &= A^{-1/\sqrt{21}}(\rho)\,\nu_{30}\begin{pmatrix}R_{11}^2B(\rho)+1&R_{11}R_{21}B(\rho)&R_{11}R_{31}B(\rho)\\ R_{21}R_{11}B(\rho)&R_{21}^2B(\rho)+1&R_{21}R_{31}B(\rho)\\ R_{31}R_{11}B(\rho)&R_{31}R_{21}B(\rho)&R_{31}^2B(\rho)+1\end{pmatrix}\nu_{30}^T, \end{aligned} \tag{79}$$

where $B(\rho) = A^{\sqrt{3/7}}(\rho) - 1$, and the $SO(3)$ elements are

$$
\begin{aligned}
R_{11} &= \pm\Delta_{(p,p')}^{-1/2}\, e^{\phi_0/2-\varphi_0/2\sqrt{3}}\Big\{ e^{-\sigma_0}\,(p_1 - \chi_{30}p_2 - \chi_{20}p_3)^2 \\
&\quad + e_0^{\sigma}\,[(p_1' + \rho_0 p_1) - \chi_{30}(p_2' + \rho_0 p_2) - \chi_{20}(p_3' + \rho_0 p_3)]^2 \Big\}^{1/2}, \\
R_{21} &= \pm\Delta_{(p,p')}^{-1/2}\, e^{-\phi_0/2-\varphi_0/2\sqrt{3}}\left\{ e^{-\sigma_0}(p_2 - \chi_{10}p_3)^2 + e_0^{\sigma}\,[(p_2' + \rho_0 p_2) - \chi_{10}(p_3' + \rho_0 p_3)]^2\right\}^{1/2}, \\
R_{31} &= \pm\Delta_{(p,p')}^{-1/2}\, e^{\varphi_0/\sqrt{3}}\left[e^{-\sigma_0}p_3^2 + e_0^{\sigma}(p_3' + \rho_0 p_3)^2\right]^{1/2}.
\end{aligned}
\tag{80}
$$

The scalars σ and ρ parametrizing the coset $SL(2,R)/SO(2)$ are given by the following matrix equation

$$
\begin{aligned}
\mathcal{M}_2 &= A^{\sqrt{3/28}}(\rho)\,\Lambda_{20}\begin{pmatrix} A^{-\sqrt{3/7}}(\rho) & 0 \\ 0 & 1 \end{pmatrix}\Lambda_{20}^T, \\
&= A^{\sqrt{3/28}}(\rho)\,\nu_{20}\begin{pmatrix} C(\rho)\cos^2\alpha + 1 & C(\rho)\cos\alpha\sin\alpha \\ C(\rho)\cos\alpha\sin\alpha & C(\rho)\sin^2\alpha + 1 \end{pmatrix}\nu_{20}^T,
\end{aligned}
\tag{81}
$$

where $C(\rho) = A^{-\sqrt{3/7}}(\rho) - 1$, and the $\cos\alpha$ and $\sin\alpha$ are

$$
\begin{aligned}
\cos\alpha &= \pm\frac{e^{-\sigma_0/2}p_3}{[e^{-\sigma_0}p_3^2 + e_0^{\sigma}(p_3' + \rho_0 p_3)^2]^{1/2}}, \\
\sin\alpha &= \pm\frac{e^{\sigma_0/2}(p_3' + \rho_0 p_3)}{[e^{-\sigma_0}p_3^2 + e_0^{\sigma}(p_3' + \rho_0 p_3)^2]^{1/2}}.
\end{aligned}
\tag{82}
$$

In the above

$$
A(\rho) = \left(1 + \frac{Q_{(p,p')}}{\rho}\right)^{\sqrt{7/3}}.
\tag{83}
$$

We also expect as before that when any two of the three integers in either of the two integral charge triplets are relatively prime, then the 4-brane is stable. This completes the constructions of all the p-brane solutions in $D = 8$ type II string theory.

6 U-duality p-Branes

The previous sections along with the previous studies [9,11,12] lay out the ground for us to construct the general U-duality p-branes of various supergravity theories in diverse dimensions. To avoid possible complicaions, we limit ourselves to $10 \geq D \geq 4$. We also set the restriction that both $p \geq 0$ and $D - p - 4 \geq 0$, i.e., the spatial dimensions of both a p-brane and its magnetic dual $(D - p - 4)$-brane in D dimensions are greater than or equal to zero. The reason for this latter limitation is that in order to realize the classical Cremmer-Julia symmetries in $D \leq 7$ either at the level of action or at the level of equation of motion (EOM), every field strength should be dualized whenever this results in a field strength of a smaller degree, as pointed out in [16,15].

It apperas that we have two cases to study, depending on whether the Cremmer-Julia symmetry is realized naturally at the level of supergravity action or EOM. The latter consists of the possible dyonic objects, i.e., membranes in $D = 8$, strings in $D = 6$ and 0-branes in $D = 4$. Note that the dyonic solutions have some crucial differences from their non-dyonic counterparts. For example, the classical Cremmer-Julia symmetries associated with the dyonic objects break into the corresponding U-duality symmetries due to instanton effects rather than the charge quantizations as for the $SL(3, Z)$ strings and for all other U-duality non-dyonic p-branes. Furthermore, the 'electric'-charge carried by a dyonic object is in general not quantized integrally. A special example of the construction of U-duality dyonic membranes is given in [12]. But once a p-'magnetic' charge and a q-'electric' charge, which satisfy the corresponding dyonic quantization rule, are assigned to a dyonic object, we can employ the property of the maximal compact group of the corresponding Cremmer-Julia symmetry to determine the corresponding Δ-factor in terms of p and q and the vacuum moduli as we did for the $SL(3, R)$ strings in section 4. This in turn determines the corresponding central charge, ADM mass per unit p-brane volume and tension. Therefore, the construction of U-duality dyonic objects from a given solution is not much different from that of the non-dyonic p-branes. Actually, even for each of the dyonic cases, we can also realize the corresponding Cremmer-Julia symmetry formally at the level of action by introducing a second set of field strengths but with a constraint imposed at the level of EOM as discussed recently in [15]. These field strengths together with the original ones form a certain representation of the Cremmer-Julia symmetry. Such a formal action is useful for our unifying discussion of U-duality p-branes.

The study of the $SL(3, Z)$ strings (3-branes) and the $SL(3, Z) \times SL(2, Z)$ 0-branes (4-branes) here alongwith the previous examples, i.e., the $SL(2, Z)$ strings [9] and fivebranes [11] as well as the $D = 8$ dyonic membranes [12], indicates that we really do not need to go through the whole procedure to construct these solutions as we did for the $SL(3, Z)$ strings in section 4. We can simply write down the these solutions in each case as we did for the $SL(3, Z) \times SL(2, Z)$ 0-branes (4-branes) in the previous section, if we know the scalar coset matrix $\mathcal{M}$, the transformation law of all the relevant field strengths (or gauge potentials) under the corresponding Cremmer-Julia symmetry, and a simple p-brane solution carrying a single charge associated with one of the field strengths (we always choose that field strength as the first component of the corresponding column vector). The other important quantity for the construction of solutions is the Δ-factor which can be deduced quite easily as it has to remain invariant under the corresponding U-duality symmetry. By employing the procedure just outlined we will construct here the U-duality p-brane solutions of various supergravity theories in diverse dimensions. Before we do so, we like to discuss certain general properties of various supergravity theories which will facilitate our constructions for these solutions.

For every supergravity theory in $D \leq 10$, there exist a non-compact global symmetry G realized non-linearly and a hidden compact local symmetry H of the theory [2,16]. The scalars in the theory are always described by the coset G/H. H, which is isomorphic to the maximal compact subgroup of G, is the automorphism symmetry of the algebra of supersymmetry. The tensor fields in the theory always form certain representations of G but inert under H. On the other hand, the spinors always transform under H wihle they are inert under G. Therefore, transformations of G preserve supersymmetry. This immediately implies that any new p-brane solution obtained by a transformation of G on a given p-brane solution will preserve the same number of unbroken supersymmetries as the original p-brane does. The central charge associated with these solutions should also be invariant under G. In Einstein frame, the metric is always a singlet of G. Therefore, the ADM mass per unit p-brane volume is also a singlet of G. We must conclude that the Δ-factor for any p-brane should also be invariant in general under G which is consistent with our observation for all the specific examples studied so far. To find a general static p-brane solution, the spinors as well as all the gauge potentials except the $(p+1)$-form ones in a supergravity are always set to zero. Under such a circumstance, we can always choose the scalar coset matrix $\mathcal{M}$ to transform in the same representation of the non-compact group G as the $(p+1)$-form gauge potentials or $(p+2)$-form field strengths do. This feature is useful for our subsequent discussions. It should be noted that given G as a symmetry of a supergravity theory, it is sufficient to consider the lowest order bosonic action involving the graviton, the scalars parametrizing the coset G/H, and the $(p+2)$-form field strengths to determine how the scalar coset matrix $\mathcal{M}$ and the column vector A_{p+1} of the $(p+1)$-form gauge potentials $A^{(i)}_{p+1}$ transform.

The lowest-order bosonic action associated with non-dyonic U-duality electric-like p-branes or magnetic-like $(D-p-4)$-branes in D dimensions can be cast in the Einstein frame as

$$S_D = \int d^D x \sqrt{-g}\left[R + \frac{1}{4}\,\mathrm{tr}\nabla_\mu \mathcal{M} \nabla^\mu \mathcal{M}^{-1} - \frac{1}{2(p+2)!}\tilde{F}^T_{(p+2)}\mathcal{M}\tilde{F}_{(p+2)}\right], \qquad (84)$$

where the column vector $\tilde{F}_{(p+2)}$ is defined as

$$\tilde{F}_{(p+2)} = \begin{pmatrix} dA^{(1)}_{(p+1)} \\ dA^{(2)}_{(p+1)} \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}. \qquad (85)$$

The above action is invariant under the following transformation

$$g_{\mu\nu} \to g_{\mu\nu},\ \mathcal{M} \to \Lambda \mathcal{M} \Lambda^T,\ F_{(p+2)} \to (\Lambda^T)^{-1} F_{(p+2)}, \qquad (86)$$

where Λ is a global Cremmer-Julia symmetry matrix.

Here we present only U-duality p-brane solutions preserving half of the spacetime supersymmetry. U-duality p-brane solutions preserving less than half of the spacetime supersymmetry as well as U-duality black p-branes can be written down in exactly the same fashion. As discussed earlier, the number of unbroken supersymmetries associated with a U-duality p-brane is completely determined by that of the initial NSNS p-brane configuration.

The NSNS $(d-1)$-brane or $(\tilde{d}-1)$-brane configuartions in diverse dimensions have been obtained some time ago in [6]. For a $(d-1)$-brane carrying electric-like charge $Q(d)$, we have, for zero asymptotic value of the dilaton,

$$ds^2 = \left(1 + \frac{Q(d)}{\tilde{d}\,\rho^{\tilde{d}}}\right)^{-\tilde{d}/(d+\tilde{d})} \left[-(dt)^2 + (dx^i)^2\right] + \left(1 + \frac{Q(d)}{\tilde{d}\,\rho^{\tilde{d}}}\right)^{d/(d+\tilde{d})} \left[(d\rho)^2 + \rho^2 d\Omega^2_{\tilde{d}+1}\right],$$

$$e^{-2\Phi} = \left(1 + \frac{Q(d)}{\tilde{d}\,\rho^{\tilde{d}}}\right)^{\alpha(d)} = A_d(\rho),\ \ \tilde{F}^{(1)}_{d+1} = Q(d)\, A_d^{-\alpha(d)/2}(\rho)\ * \epsilon_{\tilde{d}+1}, \qquad (87)$$

where $d = p+1, \tilde{d} = D-p-3$, $d\Omega_n^2$ is the metric on the unit n-sphere and ϵ_n is the corresponding volume form. The magnetic dual of the above $(d-1)$-brane, i.e., the $(\tilde{d}-1)$-brane, can be obtained from the above by the following replacements: $e^{-\alpha(d)\Phi} * \tilde{F}^{(1)}_{d+1} \to \tilde{F}^{(1)}_{\tilde{d}+1}$, $\Phi \to -\Phi$, $d \leftrightarrow \tilde{d}$. They are solutions of the following action

$$S_D = \int d^D x \sqrt{-g}\left[R - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2(d+1)!}\, e^{-\alpha(d)\Phi}\, (\tilde{F}^{(1)}_{d+1})^2\right]. \qquad (88)$$

In the above, the parameter $\alpha(d)$ is given by

$$\alpha(d) = \sqrt{4 - \frac{2d\tilde{d}}{d+\tilde{d}}}. \tag{89}$$

If we denote the electric-like integral charge column vector associated with a general U-duality $(d-1)$-brane as q and the magnetic-like integral charge column vector associated with a general U-duality $(\tilde{d}-1)$-brane as p, the U-duality invariant Δ-factor is, for the $(d-1)$-brane,

$$\Delta_q(d) = q^T \mathcal{M}_0^{-1} q, \tag{90}$$

and, for the $(\tilde{d}-1)$-brane

$$\Delta_p(\tilde{d}) = p^T \mathcal{M}_0\, p. \tag{91}$$

The ADM mass per unit $(d-1)$-brane volume $M_q(d)$, the central charge $Q_q(d)$ and the $(d-1)$-brane tension $T_q(d)$ measured in Einstein metric are

$$M_q(d) = Q_q(d) = T_q(d) = \Delta_q^{1/2} Q_0(d), \tag{92}$$

where $Q_0(d)$ is the unit charge which can be taken as the fundamental $(d-1)$-brane tension. The same relations for $(\tilde{d}-1)$-brane can be obtained from the above by the following replacements: $q \to p$, $d \to \tilde{d}$. The field strength column vector $\tilde{F}_{d+1}$ for the $(d-1)$-brane is now

$$\tilde{F}_{d+1} = \mathcal{M}_0^{-1}\, q\, Q_0(d)\, A_d^{-\alpha(d)/2}(\rho)\, * \epsilon_{\tilde{d}+1}, \tag{93}$$

while the field strength column vector $\tilde{F}_{d+1}$ for the $(\tilde{d}-1)$-brane is

$$\tilde{F}_{d+1} = p\, Q_0(\tilde{d}) \epsilon_{\tilde{d}+1}. \tag{94}$$

The metric for the $(d-1)$-brane is still given by the one in Eq.(87) but now with $Q(d) = Q_q(d)$. The same is true for the $(\tilde{d}-1)$-brane but now with $Q(\tilde{d}) = Q_p(\tilde{d})$.

The scalars for either the $(d-1)$-brane or the $(\tilde{d}-1)$-brane can be determined uniquely by the matrix equation

$$\mathcal{M} = \Lambda_0\, \mathcal{M}_{\text{initial}}\, \Lambda_0^T, \tag{95}$$

where $\mathcal{M}_{\text{initial}}$ is the scalar coset matrix describing the initial NSNS $(d-1)$-brane or $(\tilde{d}-1)$-brane configuration. We expect that $\mathcal{M}_{\text{initial}}$ approaches unity as $\rho \to \infty$. It is described by the function $A_d(\rho)$ for the $(d-1)$-brane or the $A_{\tilde{d}}(\rho)$ for $(\tilde{d}-1)$-brane and has only non-vanishing diagonal elements. As for the case of $SL(3,Z)$ strings, the above Cremmer-Julia symmetry matrix Λ_0 is only partially determined by the equation $\mathcal{M}_0 = \Lambda_0 \Lambda_0^T$. Nevertheless, the scalar coset matrix $\mathcal{M}$ is completely determined in terms of the asymptoptic values of the scalars, the charge q (or p) and the function $A_d(\rho)$ (or the function $A_{\tilde{d}}(\rho)$) as we have seen in our previous examples. Also note that $\mathcal{M}$ should approach $\mathcal{M}_0$ as $\rho \to \infty$.

We now come to discuss the dyonic U-duality p-branes. For each of the dyonic p-branes with $p = 0, 1, 2$ ($D = 2p + 4 = 4, 6, 8$), the $(p+1)$-form gauge potentials appear in the corresponding action only through their $(p+2)$-form field strengths. These field strengths $\tilde{F}^{(i)}_{p+2}$ by themselves do not form a representation of the corresponding Cremmer-Julia symmetry. However, if we introduce an equal number of $p+2$-form field strengths $\tilde{G}^{(i)}_{p+2}$, then the column vector $\tilde{H}_{p+2}$

$$\tilde{H}_{p+2} = \begin{pmatrix} \tilde{F}_{p+2} \\ \tilde{G}_{p+2} \end{pmatrix}, \tag{96}$$

does form a fundamental representation of the corresponding Cremmer-Julia symmetry group G. The lowest-order bosonic action in Einstein frame can now be expressed formally as

$$S_{2p+4} = \int d^{2p+4}x\sqrt{-g}\left[R + \frac{1}{4}\,\mathrm{tr}\,\nabla_\mu \mathcal{M} \nabla^\mu \mathcal{M}^{-1} - \frac{1}{4(p+2)!}\,\tilde{H}^T_{p+2}\mathcal{M}\tilde{H}_{p+2}\right], \tag{97}$$

which is invariant under the following transformations

$$g_{\mu\nu} \to g_{\mu\nu},\ \ \mathcal{M} \to \Lambda \mathcal{M} \Lambda^T,\ \ \tilde{H}_{p+2} \to (\Lambda^T)^{-1}\tilde{H}_{p+2} \tag{98}$$

with Λ the Cremmer-Julia symmetry group matrix. The equations of motion from the above action reduce to the original ones only as the covariant constraint relation $\tilde{H}_{p+2} = \Omega\,\mathcal{M} * \tilde{H}_{p+2}$ is imposed at the level of EOM as discussed recently in [15] with Ω the invariant matrix of the corresponding Cremmer-Julia symmetry group. It is given by

$$\Omega = \begin{pmatrix} 0 & (-1)^{p+1} I \\ I & 0 \end{pmatrix}, \tag{99}$$

where I is the unit matrix. With the constraint imposed, the Bianchi identity $d\tilde{G}_{p+2} = 0$ is actually the original equation of motion for the field strength $\tilde{F}_{p+2}$. So $d\tilde{H}_{p+2} = 0$ gives rise to a charge vector $\mathcal{Z}$, i.e.,

$$\mathcal{Z} = \begin{pmatrix} p \\ q \end{pmatrix}, \tag{100}$$

where p and q correspond respectively to the magnetic and electric charge column vectors associated with the dyonic p-brane, i.e.,

$$p = \frac{1}{V_{p+2}} \int_{S^{p+2}} \tilde{F}_{p+2}, \quad q = \frac{1}{V_{p+2}} \int_{S^{p+2}} \tilde{G}_{p+2}, \tag{101}$$

with S^{p+2} the asymptotic $(p+2)$-sphere, and V_{p+2} the volume of unit $(p+2)$-sphere. Two of such charge vectors $\mathcal{Z}$ and $\mathcal{Z}'$ obey the dyonic quantization rule

$$\mathcal{Z}^T \Omega \mathcal{Z}' = q^T p' + (-1)^{p+1} p^T q' \in Z. \tag{102}$$

We like to emphasize that the above formal action only serves us the purpose to identify the scalar coset matrix, to deduce the transformations as given above, and to draw analogy with the non-dyonic cases discussed above. For the construction of the dyonic U-duality p-branes, we only employ those transformation relations but not the action. However, the constraint relation is always imposed. In other words, $d\tilde{G}_{p+2} = 0$ is the equation of motion for the field strength $\tilde{F}_{p+2}$.

Starting with a NSNS p-brane configuration carrying a pure magnetic charge $Q(p+1)$ in $D = 2p+4$ as described right after Eq.(87), we can obtain a general U-duality dyonic p-brane in a similar way as we did for the U-duality non-dyonic p-branes carrying magnetic charges. The corresponding Δ-factor is now

$$\Delta_{\mathcal{Z}}(p+1) = \mathcal{Z}^T \mathcal{M}_0 \mathcal{Z}, \tag{103}$$

which is invariant under the corresponding U-duality transformation. Then the ADM mass per unit dyonic p-brane volume $M_{\mathcal{Z}}(p+1)$ and the central charge $Q_{\mathcal{Z}}(p+1)$ are

$$M_{\mathcal{Z}}(p+1) = Q_{\mathcal{Z}}(p+1) = \Delta_{\mathcal{Z}}^{1/2}\, Q_0(p+1), \tag{104}$$

where $Q_0(p+1)$ is again the unit of charge which can be taken as the fundamental NSNS p-brane tension. The metric remains the same as that for the initial NSNS p-brane but now with $Q(p+1) = Q_{\mathcal{Z}}(p+1)$. For $D = 2p+4$, the form of metric is much simpler since $d = \tilde{d}$. The scalars can be obtained exactly the same way as for non-dyonic p-branes carrying magnetic charges. But there is an important difference in determining the field strengths for the U-duality dyonic p-brane. We start with $\tilde{F}^{(1)}_{p+2} = Q_{\mathcal{Z}}(p+1)\, \epsilon_{p+2}$. If we impose the constraint $\tilde{H}_{p+2} = \Omega\, \mathcal{M} * \tilde{H}_{p+2}$ to obtain $\tilde{G}^{(1)}_{p+2}$ from the outset, then the constraint will be automatically satisfied for the U-duality dyonic p-brane. Therefore, taking $d\tilde{G}^{(1)}_{p+2} = 0$

as just the equation of motion for $\tilde{F}^{(1)}_{p+2}$, we have $\tilde{G}^{(1)}_{p+2} = Q_{\mathcal{Z}}(p+1)\, A_{p+1}^{-\sqrt{3-p}/2}(\rho)\ * \epsilon_{p+2}$ with

$$A_{p+1}(\rho) = \left(1 + \frac{Q_{\mathcal{Z}}(p+1)}{(p+1)\,\rho^{p+1}}\right)^{\sqrt{3-p}}. \tag{105}$$

With the above $\tilde{F}^{(1)}_{p+2}$ and $\tilde{G}^{(1)}_{p+2}$, we have the $\tilde{H}_{p+2}$ for the U-duality dyonic p-brane as

$$\begin{aligned} \tilde{H}_{p+2} &= \begin{pmatrix} \tilde{F}_{p+2} \\ \tilde{G}_{p+2} \end{pmatrix} \\ &= \left[\begin{pmatrix} p \\ q \end{pmatrix} \epsilon_{p+2} + \Omega \mathcal{M}_0 \begin{pmatrix} p \\ q \end{pmatrix} A_{p+1}^{-\sqrt{3-p}/2}(\rho)\ * \epsilon_{p+2}\right] Q_0(p+1). \end{aligned} \tag{106}$$

We have now completed our constructions of both the dyonic and the non-dyonic U-duality p-brane solutions in diverse dimensions. In order to see how various quantities depend on the asymptotic values of the scalars, we have to give an explicit parametrization of the coset matrix $\mathcal{M}$ in terms of these scalars. This can be done without much difficulty based on various known supergravity theories in diverse dimensions. Further, to see how these quantities depend on the string coupling constant and the asymptotic values for various scalars and axions, we have to follow the route, described in section 3, to construct the coset matrix $\mathcal{M}$. In general, this must be very tedious. But in principle, it can always be done. Without the explicit form for $\mathcal{M}$, we can still, for example, give the criteria for the stability of the U-duality p-branes. For a non-dyonic p-brane carrying either an electric-like or a magnetic-like integral charge column vector, this p-brane is absolutely stable if any two integers in the corresponding charge column vector are relatively prime. For a dyonic p-brane, the magnetic-like charge column vector can in general be integral but the electric-like charge column vector cannot as discused in [12]. Nevertheless the dyonic p-brane is still stable if any two integers in the magnetic-like integral charge column vector are relatively prime. As for the case of $SL(3,Z)$-string, the general U-duality p-brane solution contains all the information about the corresponding U-duality p-brane multiplets. Other similar discussions can also be made here as we did for the $SL(3,Z)$ strings. In what follows, we will give a brief discussion about possible U-duality p-branes in each of the $10 \geq D \geq 4$ maximal supergravity theories.

- In $D = 10$ type IIB supergravity, there is a well-known Cremmer-Julia type symmetry $SL(2,R)$. The corresponding U-duality symmetry is conjectured to be $SL(2,Z)$. In this theory, there are two 2-form potentials forming a doublet of $SL(2,R)$, one 4-form potential which is a singlet of $SL(2,R)$ and whose field strength is self-dual, and two scalars parametrizing the coset $SL(2,R)/SO(2)$. We therefore expect $SL(2,Z)$ superstrings [9] and $SL(2,Z)$ superfivebranes [11] both of which were constructed recently. The self-dual 3-brane was found sometime ago [8,25].

- Maximal supergravity in $D = 9$ has a Cremmer-Julia symmetry $GL(2,R) \simeq SL(2,R) \times SO(1,1)$. The corresponding conjectured U-duality symmetry is $SL(2,Z) \times Z_2$ with $SL(2,Z)$ a strong-weak duality symmetry and Z_2 a T-duality symmetry. In this theory, there are a dilaton inherited from $D = 10$ and an axion parametrizing the coset $SL(2,R)/SO(2)$, a second dilatonic scalar which is a singlet of the $SL(2,R)$, one 3-form gauge potential which is also a singlet of the $SL(2,R)$, two 2-form gauge potentials forming a doublet of the $SL(2,R)$, and three 1-form gauge potentials two of which forms a doublet of the $SL(2,R)$ while the other is a singlet. The $SO(1,1)$ symmetry transforms the second dilatonic scalar by a constant shift and rescales all the other gauge potentials but leaves the $D = 10$ dilaton and the axion inert. That is why the $SO(1,1)$ is merely a classical T-duality symmetry. This symmetry will not be useful in generating new solutions. The bosonic action exhibiting the $SL(2,R) \times SO(1,1)$ symmetry has been given explicitly, for example, in [26]. The scalar matrix $\mathcal{M}$ parametrizing the coset $SL(2,R)/SO(2)$ is a familiar one. The $SL(2,Z)$ strings carrying electric-like charges and $SL(2,Z)$ 4-branes carrying magnetic-like charges can be read off readily from our general formulae. There are also $SL(2,Z)$ 0-branes and $SL(2,Z)$ 5-branes. There are also solutions which are inert under the $SL(2,Z)$, namely, 2-brane and 3-brane as well as 0-brane and 5-brane found some time ago in [6].

- The various U-duality p-branes in $D = 8$ maximal supergravity have all been studied in this paper and in [12]. As discussed in detail in the previous sections, we have $SL(2,Z)$ dyonic membranes, $SL(3,Z)$ strings and 3-branes and $SL(3,Z) \times SL(2,Z)$ 0-branes and 4-branes.

- The Cremmer-Julia symmetry in $D = 7$ maximal supergravity is $SL(5,R)$. The 14 scalars in this theory parametrize the coset $SL(5,R)/SO(5)$. The conjectured U-duality symmetry is $SL(5,Z)$. There are five 2-form gauge potentials forming 5-dimensional fundamental representation of $SL(5,R)$ and ten 1-form gauge potentials forming a 10-dimensional irreducible representation of $SL(5,R)$. We have therefore $SL(5,Z)$ strings and membranes for which the 5×5 scalar coset matrix $\mathcal{M}$ can be found in [2,15]. We also have $SL(5,Z)$ 0-branes and 3-branes for which the 10×10 scalar coset matrix $\mathcal{M}$ can be parametrized explicitly based on the known $D = 7$ supergravity [27].

- Maximal supergravity in $D = 6$ has a Cremmer-Julia symmetry $SO(5,5)$. The 25 scalars appearing in this theory parametrize the coset $SO(5,5)/SO(5) \times SO(5)$. The conjectured U-duality symmetry is $SO(5,5;Z)$. There are five 2-form gauge potentials which appear in the action only through their 3-form field strengths. This is the dimension for which dyonic string solutions appear. These five 3-form field strengths do not form a representation of $SO(5,5)$ but as discussed at length before, this symmetry can be realized at the level of EOM through interchanging the Bianchi identities and the equations of motion for the 3-form field strengths. In other words, the five equations and five Bianchi identities do form a 10-dimentional fundamental representation of $SO(5,5)$. We therfore have $SO(5,5;Z)$ dyonic strings for which the 10×10 scalar coset matrix $\mathcal{M}$ has been constructed, for example, in [28,15]. There are also sixteen 1-form gauge potentials which in this case form a 16-dimensional spinor representation of $SO(5,5)$. We have also $SO(5,5;Z)$ 0-branes and membranes for which the 16×16 scalar coset matrix $\mathcal{M}$ can be parametrized explicitly based on the already known $D = 6$ supergravity theory [29].

- The Cremmer-Julia symmetry in $D = 5$ is the non-compact $E_{6(+6)}$. There are 42 scalars in this theory which parametrize the coset $E_6/USp(8)$. The conjectured U-duality symmetry is $E_6(Z)$. In this theory, there are only twenty seven 1-form gauge potentials which form a 27-dimensional fundamental representation of E_6. Therefore, we have $E_6(Z)$ 0-branes and strings for which the 27×27 scalar coset matrix $\mathcal{M}$ can be parametrized explicitly based on the well-studied $D = 5$ supergravity theory [30].

- Maximal supergravity in $D = 4$ has a Cremmer-Julia symmetry $E_{7(+7)}$. The seventy scalars in this theory parametrize the coset $E_{7(+7)}/SU(8)$. The conjectured U-duality symmetry is $E_7(Z)$. This is the dimension for which we have dyonic 0-branes. There are only twenty eight 1-form gauge potentials which appear in the action only through their 2-form field strengths. Similar to the cases in $D = 6$ and $D = 8$ for dyonic strings and dyonic membranes, these twenty eight 2-form field strengths combined with the other twenty eight 2-form field strengths whose Bianchi identities give the 28 equations of motion form a 56-dimensional fundamental representation of $E_{7(+7)}$. We have therefore $E_7(Z)$ dyonic 0-branes for which the 56×56 scalar coset matrix $\mathcal{M}$ is given in [1,16].

Appendix

In this appendix, we will present the field strengths without tildes in $D = 8$ maximal supergravity (i.e., those discussed in section 3), obtained directly by a T^2 compactification of $D = 10$ type IIA supergravity theory. In our notations, we have

$$\begin{aligned}
F_4 &= \tilde{F}_4 - \tilde{F}_3^{(1)} \wedge \mathcal{A}_1^{(1)} - \tilde{F}_3^{(2)} \wedge \mathcal{A}_1^{(2)} - \tilde{F}_3^{(3)} \wedge \mathcal{A}_1^{(3)} \\
&\quad -\chi_1 \tilde{F}_3^{(2)} \wedge \mathcal{A}_1^{(3)} - (\chi_2 + \chi_1\chi_3)\, \tilde{F}_3^{(1)} \wedge \mathcal{A}_1^{(3)} - \chi_3 \tilde{F}_3^{(1)} \wedge \mathcal{A}_1^{(2)} \\
&\quad +\tilde{F}_2^{(3)} \wedge \mathcal{A}_1^{(1)} \wedge \mathcal{A}_1^{(2)} - \tilde{F}_2^{(2)} \wedge \mathcal{A}_1^{(1)} \wedge \mathcal{A}_1^{(3)} + \tilde{F}_2^{(1)} \wedge \mathcal{A}_1^{(2)} \wedge \mathcal{A}_1^{(3)} \\
&\quad +\chi_1 \tilde{F}_2^{(3)} \wedge \mathcal{A}_1^{(1)} \wedge \mathcal{A}_1^{(3)} + \chi_2 \tilde{F}_2^{(3)} \wedge \mathcal{A}_1^{(3)} \wedge \mathcal{A}_1^{(2)} - \chi_3 \tilde{F}_2^{(2)} \wedge \mathcal{A}_1^{(2)} \wedge \mathcal{A}_1^{(3)} \\
&\quad -d\rho \wedge \mathcal{A}_1^{(1)} \wedge \mathcal{A}_1^{(2)} \wedge \mathcal{A}_1^{(3)}.
\end{aligned} \tag{107}$$

$$\begin{aligned}
F_3^{(1)} &= \tilde{F}_3^{(1)} + \tilde{F}_2^{(3)} \wedge \mathcal{A}_1^{(2)} - \tilde{F}_2^{(2)} \wedge \mathcal{A}_1^{(3)} + \chi_1 \tilde{F}_2^{(3)} \wedge \mathcal{A}_1^{(3)} + d\rho \wedge \mathcal{A}_1^{(2)} \wedge \mathcal{A}_1^{(2)}, \\
F_3^{(2)} &= \tilde{F}_3^{(2)} + \chi_3 \tilde{F}_3^{(2)} - \tilde{F}_2^{(3)} \wedge \mathcal{A}_1^{(1)} + \tilde{F}_2^{(1)} \wedge \mathcal{A}_1^{(3)} - \chi_3 \tilde{F}_2^{(2)} \wedge \mathcal{A}_1^{(3)} - \tilde{F}_2^{(3)} \wedge \mathcal{A}_1^{(3)} \\
&\quad -d\rho \wedge \mathcal{A}_1^{(1)} \wedge \mathcal{A}_1^{(3)}, \\
F_3^{(3)} &= \tilde{F}_3^{(3)} + \chi_1 \tilde{F}_3^{(2)} + (\chi_2 + \chi_1\chi_3)\, \tilde{F}_3^{(1)} + \tilde{F}_2^{(2)} \wedge \mathcal{A}_1^{(1)} - \tilde{F}_2^{(1)} \wedge \mathcal{A}_1^{(2)} \\
&\quad -\chi_1 \tilde{F}_2^{(3)} \wedge \mathcal{A}_1^{(1)} + \chi_2 \tilde{F}_2^{(3)} \wedge \mathcal{A}_1^{(2)} + \chi_3 \tilde{F}_2^{(2)} \wedge \mathcal{A}_1^{(2)} \\
&\quad +d\rho \wedge \mathcal{A}_1^{(1)} \wedge \mathcal{A}_1^{(2)},
\end{aligned} \tag{108}$$

$$\begin{aligned}
F_2^{(1)} &= \tilde{F}_2^{(1)} - \chi_3 \tilde{F}_2^{(2)} - \chi_2 \tilde{F}_2^{(3)} - d\rho \wedge \mathcal{A}_1^{(1)}, \\
F_2^{(2)} &= \tilde{F}_2^{(2)} - \chi_1 \tilde{F}_2^{(3)} - d\rho \wedge \mathcal{A}_1^{(2)}, \\
F_2^{(3)} &= \tilde{F}_2^{(3)} - d\rho \wedge \mathcal{A}_1^{(3)}.
\end{aligned} \tag{109}$$

$$\begin{aligned}
\mathcal{F}_2^{(1)} &= \tilde{\mathcal{F}}_2^{(1)} + d\chi_3 \wedge \mathcal{A}_1^{(2)} + (d\chi_2 + \chi_1 d\chi_3) \wedge \mathcal{A}_1^{(3)}, \\
\mathcal{F}_2^{(2)} &= \tilde{F}_2^{(2)} + d\chi_1 \wedge \mathcal{A}_1^{(3)}, \\
\mathcal{F}_2^{(3)} &= \tilde{\mathcal{F}}_2^{(3)}.
\end{aligned} \tag{110}$$

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