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# ANISOTROPIC EFFECTS IN GEOMETRICALLY ISOTROPIC LATTICES

by

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## I. Introduction

The study of the dielectric properties of a lattice composed of identical metallic or dielectric elements of various geometries has received considerable attention in recent years in connection with the practical application of such structures for polarizing devices, microwave lenses, and radome materials. It has been shown by the author[1] that, for spacings and element dimensions small with respect to wavelength, the dielectric constant of a completely general uniform lattice of identical elements may be represented by a tensor $(k_e)$ which may be written in functional form as

$$(k_e) = f[(T),(\delta)] \quad ,$$

where $(T)$ is the structural anisotropy tensor which is related to the geometry of the lattice and $(\delta)$ is the polarizability tensor of the elements of the lattice. $(\delta)$ describes element anisotropy which may be due to material, the shape of the element, or both. An example of a lattice element displaying only material anisotropy is a ferrite or gaseous element of spherical shape immersed in a magnetostatic field. An example of shape anisotropy is the case of metallic or dielectric objects of non-spherical shape. In general, then, for spacing and element dimensions small compared to wavelength, there can be three orders of anisotropy in a lattice--structural or lattice anisotropy, material anisotropy, and anisotropy because of element shape. Examples of each type are described and discussed in detail in the aforementioned reference. A fourth order of anisotropy, which is related to the granularity of the lattice and becomes important at higher frequencies will be described and investigated in this paper.

It is usually important that the artificial material be as close as possible in physical properties to a real dielectric. This requires that it should be isotropic; that is, the structure should exhibit the same properties for a plane wave propagating through it in any direction. This behavior requires that $(k_e)$ reduce to a scalar which, in turn, demands that $(T)$ and $(\delta)$ become scalars, except for the special case in which the structural anisotropy of the array is compensated for by the element anisotropy in which case $(k_e)$ is reduced to a scalar. The structural anisotropy tensor $(T)$ will reduce to a scalar only if the lattice is cubical, while $(\delta)$ becomes a scalar only if the geometry and material of the lattice elements are so restricted that the induced fields may be represented by a set of three mutually perpendicular static dipoles on the lattice points.[1] Isotropic behavior further requires that the moments of the resultant dipoles must be proportional to the inducing field. The proportionality factor is a scalar independent of direction. However, at shorter wavelengths the representation of the lattice elements by static dipoles will not be valid and the medium becomes anisotropic. The main objective of this paper is to evaluate the anisotropy produced by the finite ratio of wavelength to element spacing and to show that the Clausius-Mosotti relation so often used in predicting the properties of artificial lattice dielectrics is a satisfactory approximation only if the spacing is very small with respect to wavelength.

## II. Static Model of Triad Medium

The constitutive dielectric parameters for uniform space arrays of generalized structural geometry composed of similarly oriented elements of completely generalized material and shape have been derived by the author.[1] The theoretical procedure employed in evaluating these parameters is analogous to the classical method used in the study of the dielectric properties of nonpolar media, and assumes that the disturbing action of each element on a uniform static field can be allowed for if each generalized particle is replaced by a set of three mutually perpendicular static dipoles. This assumption implies the restriction that element size and spacing be small compared to wavelength.

The structure under consideration in this paper is a cubical lattice of arbitrary isotropic elements. Isotropic elements may best be simulated by objects whose geometry and material is so restricted that the induced fields may be represented by three mutually perpendicular static dipoles. Isotropy further requires that the moments of the resultant dipole must be proportional to the inducing field and that the proportionality factor is a scalar independent of direction. Therefore, in order to represent a collection of arbitrary isotropic elements, the lattice structure under consideration in this paper will consist of a triad of three mutually perpendicular elements and will, hereafter, be referred to as a triad medium. The material and geometry of these elements will be restricted so that only static dipoles will be induced. The moments of these dipoles are proportional to the components of the inducing field in the corresponding direction with a scalar proportionality constant which will be denoted by $\delta$ . Of course, the isotropic element could also have been represented by a sphere of diameter small compared to wavelength. The fundamental dielectric parameters for this lattice structure may be obtained from the results of the aforementioned article. The dielectric constant tensor $(k_e)$ when interaction between lattice elements is neglected, is given by

$$(k_e) = (1) + N(\delta)/\epsilon_0 , \qquad (1)$$

where $(\delta)$, the polarizability tensor, is defined by the relation

$$P = Np = N(\delta) \cdot E_0 ;$$

N = number of elements per unit volume, $\vec{p}_n = p_x \vec{a}_x + p_y \vec{a}_y + p_z \vec{a}_z$ denotes the resultant dipole moment of the triad at lattice points, $\epsilon_0$ = permittivity of free space, $\underline{E}_0$ = the incident electric vector, and $\underline{P}$ = the polarization vector. The derivation of Eq. (1) assumes that the field acting on each individual element in the presence of the others remains equal to the externally applied field. If the contribution of all elements in the exciting field of each element is taken into consideration, the dielectric constant is given by

$$(k_e) = (1) - (N/\epsilon_0)\,(T) , \qquad (2)$$

where $(T)$ is the diagonal tensor given by[1]

$$(T_1) = \begin{bmatrix} \dfrac{1}{(N/3\epsilon_0) - (1/\delta)} & 0 & 0 \\ 0 & \dfrac{1}{(N/3\epsilon_0) - (1/\delta)} & 0 \\ 0 & 0 & \dfrac{1}{(N/3\epsilon_0) - (1/\delta)} \end{bmatrix} .$$

## III. Dynamic Model of Triad Medium--Method 1.

Equations (1) and (2) are valid only if the spacing between the elements of the lattice remains small compared to the wavelength. This restriction can be removed by taking into consideration the total field scattered by the dipole elements rather than simply the static dipole contribution. As has already been emphasized, such a computation would be valuable in determining the approximation involved in lattice problems with specialized element geometries such as disks, strips, etc. where the static approximation has frequently been used by many authors.

In this cynamic model in which total scattered fields are considered, a semi-infinite region of the lattice is considered as shown in Fig. 1. A plane wave polarized in the y direction is incident from the left. The total field in the lattice medium is given by

$$E_y = E_{y_{inc}} + E_{y_s} \quad , \tag{3}$$

where $E_y$ is the y component of the exciting electric field at (X,Y,Z) and consists of the incident field $E_{y_{inc}}$ plus the contributions of the scattered fields $E_{y_s}$ from the remaining elements (see Fig. 1). The amplitude of the field scattered by a triad of mutually perpendicular dipoles is given by,

$$\vec{E} = \frac{1}{4\pi \epsilon_0} \nabla x \nabla x \left(\vec{p} \frac{e^{-jkr}}{\vec{r}}\right)$$

$$= \frac{1}{4\pi \epsilon_0} \nabla x \nabla x \left[ (\delta) \cdot \vec{E} \frac{e^{-jkr}}{\vec{r}} \right]$$

where $\vec{E}$ is the incident field on the triad and $\vec{r}$ is the distance from the center of the triad to the point of observation. Since all the elements in the plane $z = Z$ are influenced by the exciting field, in the same way, the exciting electric and magnetic fields are a function of the z coordinate only. The only restriction that will be imposed is that the geometry of the elements is such as to insure a predominantly dipole field; the higher multipole excitation should be small to insure isotropy.

The bulk constants, as used here, are such that the usual results derivable from Maxwell's equation in an ordinary dielectric medium hold in any macroscopic region of the loaded material. The idea of macroscopic field expresses the concept of an average field in a region large compared to the mean spacing between the loading particles.

Using the expression for the field scattered by a single triad of dipoles, the total scattered field at the point (X,Y,Z) equals

$$\vec{E}_s = \sum_{m_1=-\infty}^{\infty}{}' \sum_{m_2=-\infty}^{\infty}{}' \sum_{m_3=0}^{\infty}{}' \frac{1}{4\pi\epsilon_0}\left[\nabla(\nabla \cdot \vec{A}') + k^2\vec{A}'\right] ,$$

where

$$A' = \frac{e^{-jk|r-r_o|}}{|\vec{r} - \vec{r}_o|} \, (\delta) \cdot \vec{E}(m_3 d) ,$$

and

$$|\vec{r} - \vec{r}_o| = \left[(m_1 d - X)^2 + (m_2 d - Y)^2 + (m_3 d - Z)^2\right]^{1/2} .$$

$\sum'$ signifies that the field arising from the element (X,Y,Z) is to be omitted from the summation. Since the expression for $\vec{E}_s$ is slowly varying with respect to $m_1$, $m_2$, and $m_3$, the summation can be replaced by an integration by using the trapezoidal rule. Changing variables to $\alpha = m_1 d$, $\beta = m_2 d$, $\gamma = m_3 d$ the triple summation becomes

$$E_{ys} = \frac{1}{4\pi\epsilon_0 d^3}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\left(\frac{\partial^2 A_x}{\partial x\,\partial y} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial z\,\partial y} + k^2 A_y\right) d\alpha\, d\beta\, d\gamma - \int_{x-(d/2)}^{x+(d/2)}\int_{y-(d/2)}^{y+(d/2)}\int_{z-(d/2)}^{z+(d/2)}\left(\frac{\partial^2 A_x}{\partial x\,\partial y} + \frac{\partial^2 A_y}{\partial y^2} \quad \frac{\partial^2 A_z}{\partial z\,\partial y} + k^2 A_y\right)\right] d\alpha\, d\beta\, d\gamma , \qquad (4)$$

where

$$A = \frac{\exp\left\{-jk[(\alpha - X)^2 + (\beta - Y)^2 + (\gamma - Z)^2]^{1/2}\right\}}{[(\alpha - X)^2 + (\beta - Y)^2 + (\gamma - Z)^2]^{1/2}} (\delta)\cdot E(\gamma) .$$

The first integrand involves terms like

$$\frac{\partial^2}{\partial\alpha\,\partial\beta}\left[\frac{e^{-jk\,r-r_o}}{|\vec{r} - \vec{r}_o|}\ \delta\, E_x(\gamma)\right] ,$$

and these may be integrated by parts, the integrated part vanishing at both limits; the fourth term of the integrand integrated with respect to $\alpha$ and $\beta$ gives

$$k^2\delta\, E_y(\gamma)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-jk|r-r_o|}}{|\vec{r} - \vec{r}_o|}\, d\alpha\, d\beta$$

$$= -2j\pi\, k\delta\, e^{-jk|\gamma-z|}\, E_y(\gamma) . \qquad (5)$$

This integral is evaluated by performing a transformation to polar coordinate system centered at $(X,Y)$. The second integration around the point $(X,Y,Z)$ corresponds to the omission from the summation of the field of the particle at that point. If for this term the near field is considered, that is, the highest power of $1/(|\vec{r}-\vec{r}_o|)$ in the integrand, the second integral reduces to

$$\int_{X-(d/2)}^{X+(d/2)}\int_{Y-(d/2)}^{Y+(d/2)}\int_{Z-(d/2)}^{Z+(d/2)} E_y(Z)$$

$$\times\ \frac{\partial^2}{\partial\alpha^2}\left(\frac{1}{|\vec{r} - \vec{r}_o|}\right) d\alpha\, d\beta\, d\gamma = -\ (4\pi/3)\ E_y(Z) . \qquad (6)$$

Substitution of the results of Eqs. (5) and (6) in Eq. (3) gives

$$E_y(Z) = E_{y\,inc} + \frac{\delta}{3\epsilon_0\, d^3}\, E_y(Z) - \frac{j\,\delta\, k}{2\epsilon_0\, d^3} \int_0^\infty e^{-jk|\gamma - Z|}\, E_y(\gamma)\, d\gamma \, . \tag{7}$$

The solution of Eq. (7) for the exciting field is obtained by operating on both sides of the equation with $[(\partial^2/\partial Z^2) + k^2]$ . This gives

$$\left(\frac{\partial^2}{\partial Z^2} + k^2\right)\left(1 - \frac{\delta}{3\epsilon_0\, d^3}\right) E_y(Z) + \frac{j\,\delta k}{2\epsilon_0\, d^3}\left(\frac{\partial^2}{\partial Z^2} + k^2\right) \times \int_0^\infty E_y(\gamma)\, e^{-jk|\gamma - Z|}\, d\gamma = 0 \quad . \tag{8}$$

The expression under the integral sign yields

$$\int_0^\infty \left(\frac{d^2}{\partial Z^2} + k^2\right) e^{-jk|\gamma - Z|}\, E_y(\gamma)\, d\gamma = -\, 2jk\, E_y(Z) \quad . \tag{9}$$

Substituting this result in Eq. (8) yields the differential equation

$$\frac{d^2}{dZ^2} + k^2 + k^2\left[\frac{\delta/\epsilon_0 d^3}{1 - (\delta/3\epsilon_0\, d^3)}\right] E_y(Z) = 0 \quad , \tag{10}$$

the solution of which is

$$E_y(Z) = Be^{-j\kappa z}; \qquad \kappa^2 = k^2 + k^2\, \frac{\delta/\epsilon_0 d^3}{1 - (\delta/3\epsilon_0 d^3)} \quad . \tag{11}$$

Since the elements of the medium consist of electric dipoles, the permeability of the medium is unity and therefore

$$k_{zz} = \frac{\kappa^2}{k^2} = 1 + \frac{\delta/\epsilon_0 d^3}{1 - (\delta/3\epsilon_0 d^3)} . \tag{12}$$

It can be easily shown that

$$k_{xx} = k_{yy} = k_{zz} \, ,$$

and replacing $1/d^3$ by $N$ , number of elements per unit volume, the dielectric tensor is given by

$$(k_e) = (1) - (N/\epsilon_0)(T_1) \quad , \tag{13}$$

$(T_1)$ is defined by Eq. (2). This expression for the dielectric tensor $(k_e)$ is identical with the expression obtained by electrostatic considerations [see Eq. (2)] and represents the Clausius-Mosotti relations for an isotropic dielectric. This result clearly demonstrates that consideration of the retardation effects of the fields induced in the elements of the array, for practical purposes, can only give results identical to those obtained from simple electrostatic considerations. This degeneration of the exact solution is the result of the approximation used in evaluating the summations occurring in the formulation of the problem. The trapezoidal rule which was used to carry out this summation has been previously used by many authors in problems of this type.[2] Since in this paper simple dipole excitation only is considered, it has been possible to quantitatively explore the extent to which the use of the trapezoidal rule is valid in lattice problems of the type described. The foregoing analysis clearly indicates the limitations of the method. The application of the trapezoidal rule restricts the spacing to values small with respect to $\lambda$.

## IV. Dynamic Model of Triad Medium--Method 2.

In this section another formulation will be used to carry on a rigorous solution of the problem. The expression for the vector potential $\vec{A}$ may be expressed in terms of a Green's function and a source distribution function. The Green's function for the lattice structure being considered may be written as

$$G(\vec{r},\vec{r}_o) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \sum_{m_3=-\infty}^{\infty}$$

$$\frac{\exp(-jk|\vec{r}-m_1 d\vec{a}_{x1} - m_2 d\vec{a}x_2 - m_3 d\vec{a}x_3|)}{|\vec{r} - m_1 d\vec{a}x_1 - m_2 d\vec{a}x_2 - m_3 d\vec{a}x_3|} = \frac{e^{-jk|r - \vec{M}|}}{|\vec{r} - \vec{M}|}$$

where

$$\vec{M} = m_1 d\vec{a}x_1 + m_2 d\vec{a}x_2 + m_3 d\vec{a}x_3 \quad .$$

The definition of Green's function is similar to the conventional definition of a three-dimensional Green's function. The Floquet type phase variation between sources has been omitted in this definition and will be absorbed in the current distribution. Let $\vec{\kappa}$ represent a vector whose magnitude is equal to the propagation constant and whose sense is the direction of propagation. The current distribution on the $(m_1,m_2,m_3)$th dipole triad will be related by the factor

$$\exp\left[-j(\vec{\kappa} \cdot \vec{M})\right]$$

to the current distribution at the center of the coordinate system. The vector potential is expressible in this way in terms of the current on the elements of the infinite unbounded three-dimensional array. The exact amplitude of each mode depends, of course, on the specified excitation. For dipolar fields the current distribution is uniform and $\vec{A}$ may be expressed in terms of dipole moments as follows:

$$\vec{A} = \frac{1}{4\pi} \vec{i}\, G dv = \frac{j\omega}{4\pi} \sum_{m_1=-\infty}^{\infty}{}' \sum_{m_2=-\infty}^{\infty}{}' \sum_{m_3=-\infty}^{\infty}{}' G\vec{p} \exp\left[-j(\vec{\kappa} \cdot \vec{M})\right] \quad ,$$

$\vec{i}$ is the current distribution, dv is the volume element. The field at the point $\vec{r}$ is ,

$$E = \frac{p}{4\pi\,\epsilon_0} \sum_{m_1=-\infty}^{\infty}{}' \sum_{m_2=-\infty}^{\infty}{}' \sum_{m_3=-\infty}^{\infty}{}' \nabla \times \nabla \times$$

$$\times \left\{ G\vec{a}_p \exp[\,-j(\vec{\kappa}\cdot\vec{M})] \right\}, \qquad (14)$$

where $\vec{a}_p$ is a unit vector in the direction of the dipole moment vector $\vec{p}$. The vector operators are taken with respect to the field point $r$ but in view of the form of the operand, these operators may act upon the source point $\vec{M}$ without changing the value of the expression. If the observation point is chosen to be the origin of the coordinate system, the expression for the total field becomes

$$E = \frac{p}{4\pi\,\epsilon_0} \sum_{m_1=-\infty}^{\infty}{}' \sum_{m_2=-\infty}^{\infty}{}' \sum_{m_3=-\infty}^{\infty}{}' \vec{F}(m_1,m_2,m_3) \quad , \qquad (15)$$

where $\vec{F}(m_1,m_2,m_3)$ is defined by

$$\vec{F}(m_1,m_2,m_3) = \nabla \left\{ \nabla \cdot \left[ f(m_1,m_2,m_3)\vec{a}_p \right] \right\} - \nabla^2 \left[ f(m_1,m_2,m_3)\,\vec{a}_p \right]$$

and

$$f(m_1,m_2,m_3) = \frac{\exp\left[-j(kM + \vec{\kappa}\cdot\vec{M})\right]}{M} \quad .$$

The left hand side of Eq. (15) is a slowly converging series. A more tractable form can be obtained by using the Poisson summation formula.[3] The Poisson formula for an infinite triple sum may be written as

$$\sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \sum_{m_3=-\infty}^{\infty} g(m_1,m_2,m_3) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{n_3=-\infty}^{\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1,x_2,x_3)$$

$$\times \exp\left[-j2\pi(n_1 x_1 + n_2 x_2 + n_3 x_3)\right] dx_1 dx_2 dx_3 \quad . \qquad (16)$$

An essential restriction in the application of this formula is that continuity of the function is required. However, $f(m_1,m_2,m_3)$ is not continuous at $(m_1 = 0,\ m_2 = 0,\ m_3 = 0)$, but since this term is not included in the summation, the difficulty is avoided. Furthermore, the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |g(x_1,x_2,x_3)|\ dx_1dx_2dx_3$$

must be convergent. In the present application this integral has the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| \frac{\exp[-jk(y_1^2+y_2^2+y_3^2)]}{(y_1^2+y_2^2+y_3^2)^{1/2}} \right.$$

$$\left. \times \exp\left[-j(\kappa x_1y_1 + \kappa x_2y_2 + \kappa x_3y_3)\right] \right| dy_1dy_2dy_3 \ .$$

Assuming that $k$ equals $k_r - jk_i$, where $k_i$ is a small positive quantity, it can easily be shown that this integral converges. The assumption of an imaginary part for $k$ is a valid one, since the medium is always lossy. The integration can be performed by a transformation to polar coordinates (see Eq. 5), $k_i$ may be allowed to approach zero after integration.

Applying Eq. (16) and (A9) of the Appendix to Eq. (15) yields

$$\vec{E} = \frac{p}{4\pi\epsilon_0}\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\sum_{n_3=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\vec{F}(x_1,x_2,x_3)\exp[-j2\pi(n_1x_1+n_2x_2+n_3x_3)]$$

$$\times\ dx_1dx_2dx_3 - \frac{p}{4\pi\epsilon_0\ d^3}\left[\vec{L} + \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\vec{F}(x_1,x_2,x_3)dx_1dx_2dx_3\ + \right. \tag{17}$$

$$\left. +\ 8\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\vec{F}(x_1,x_2,x_3)\cos\ 2\pi n_1x_1\cos 2\pi n_2x_2\cos 2\pi n_3x_3dx_1dx_2dx_3\right]$$

In Eq. (17) the term in the brackets is the result of the application of Poisson's summation formula for a finite term [see Appendix, Eq. (A9)] to the term at ($m_1 = 0$, $m_2 = 0$, $m_3 = 0$). L represents the sum of the single term, single terms and double terms appearing in Eq. (A9). In this application L is simply the sum of a finite number of terms. The first integral in Eq. (17) is evaluated in the appendix [see appendix Eqs. (A1) to (A5)]. Substituting this value gives

$$\vec{E} = \frac{p}{\epsilon_0 d^3} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \frac{\vec{\kappa}(a_p \cdot \vec{\kappa}) + k^2 a_p}{|\vec{\kappa} - (2\pi m_1/d)ax_1 - (2\pi m_2/d)ax_2 - (2\pi m_3/d)ax_3|^2 - k^2}$$

$$- \frac{p}{4\pi\ \epsilon_0} \Bigg[ \vec{L} + \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \vec{F}(x_1,x_2,x_3)dx_1dx_2dx_3 + 8 \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty}$$

$$\times \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \vec{F}(x_1,x_2,x_3) \cos 2\pi n_1 x_1 \cos 2\pi n_2 x_2 \cos 2\pi n_3 x_3\ dx_1 dx_2 dx_3 \Bigg] \tag{18}$$

The electric moment of an element of the triad medium is given by

$$\vec{p} = (\delta) \cdot \vec{E} \quad . \tag{19}$$

Solving Eqs. (18) and (19) simultaneously gives

$$\vec{a}_p = (\delta) \cdot \Bigg\{ \frac{1}{\epsilon_0 d^3} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \frac{\vec{\kappa}(a_p \cdot \vec{\kappa}) + k^2 a_p}{|\vec{\kappa} - (2\pi m_1/d)\vec{a}x_1 - (2\pi m_2/d)\vec{a}x_2 - (2\pi m_3/d)\vec{a}x_3|^2 - k^2}$$

$$- \frac{1}{4\pi\epsilon_0 d^3} \Bigg[ \vec{L} + \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \vec{F}(x_1,x_2,x_3)dx_1dx_2dx_3 + 8 \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty}$$

$$\times \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \vec{F}(x_1,x_2,x_3) \cos 2\pi n_1 x_1 \cos 2\pi n_2 x_2 \cos 2\pi n_3 x_3\ dx_1 dx_2 dx_3 \Bigg] \Bigg\} \quad . \tag{20}$$

This is the most general expression for the propagation constant of the triad medium. Using Eq. (All) of the Appendix, Eq. (19) becomes

$$\vec{a}_p = (\delta) \cdot \left\{ \frac{1}{\epsilon_0 d^3} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \frac{\vec{\kappa}(a_p \cdot \vec{\kappa}) + k^2 \vec{a}_p}{|\vec{\kappa} - 2\pi n_1 \vec{b}_1 - 2\pi n_2 \vec{b}_2 - 2\pi n_3 \vec{b}_3|^2 - k^2} - \right.$$

$$- \frac{1}{4\pi\epsilon_0 d^3} \left[ \vec{L} + \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \vec{F}(x_1, x_2, x_3) dx_1 dx_2 dx_3 \right.$$

$$+ \frac{\vec{F}_{x_1^1 x_2^1 x_3^1}(1,1,1) - \vec{F}_{x_1^1 x_2^1 x_3^1}(-1,-1,-1)}{1728}$$

$$\left. \left. - \frac{\vec{F}_{x_1^3 x_2^1 x_3^1}(1,1,1) - \vec{F}_{x_1^3 x_2^1 x_3^1}(-1,-1,-1)}{26820} \cdots \right] \right\} . \qquad (21)$$

Equation (21) is expressed in terms of the reciprocal lattice vectors $\vec{b}_1$, $\vec{b}_2$ and $\vec{b}_3$.[4] The expression for $\kappa$ in terms of the derivatives of $\vec{F}(x_1,x_2,x_3)$ is more convenient since in this case it is easier to evaluate the derivatives than the integrals. It is seen that Eq. (21) is not defined when $\vec{\kappa}$ is so chosen that

$$|\vec{\kappa} - 2\pi n_1 \vec{b}_1 - 2\pi n_2 \vec{b}_2 - 2\pi n_3 \vec{b}_3|^2 = k^2 .$$

This is, however, precisely the condition for Bragg reflections in a lattice. Since in the case under consideration there exists coupling between the triad elements, $\vec{\kappa}$ can never assume values which describe Bragg reflections; hence the series can always be considered to converge for the purpose of this paper.

<u>Evaluation of the Propagation Constant.</u> After multiplying both sides of Eq. (21) by p, the summation term may be written as

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \frac{(1/k^2)\, \vec{\kappa}(\vec{p} \cdot \vec{\kappa}) + \vec{p}}{\left|(\vec{\kappa}/k) - (\lambda n_1/d)\vec{a}_{x_1} - (\lambda n_2/d)\vec{a}_{x_2} - (\lambda n_3/d)\vec{a}_{x_3}\right|^2 - 1} \tag{22}$$

This is a rapidly converging series so that a small number of terms yields reasonably accurate results.[5] The evaluation of $L$ involves a simple substitution for $m_1$, $m_2$, $m_3$ in $F(m_1,m_2,m_3)$ which is defined by Eq. (15). The third term in the right-hand side of Eq. (21) is the integral

$$I = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \vec{F}(x_1,x_2,x_3)\, dx_1 dx_2 dx_3 \quad . \tag{23}$$

The integrand of (23) can be written as

$$F(x_1,x_2,x_3) = \left[p_{x_1} \frac{\partial^2 f}{\partial x_1^2} + p_{x_2} \frac{\partial^2 f}{\partial x_1 \partial x_2} + p_{x_3} \frac{\partial^2 f}{\partial x_1 \partial x_3} + k^2\, p_{x_1} f\right] \vec{a}_{x_1}$$

$$\times \left[p_{x_1} \frac{\partial^2 f}{\partial x_1 \partial x_2} + p_{x_2} \frac{\partial^2 f}{\partial x_2^2} + p_{x_3} \frac{\partial^2 f}{\partial x_2 \partial x_3} + k^2\, p_{x_2} f\right] \vec{a}_{x_2}$$

$$\times \left[p_{x_1} \frac{\partial^2 f}{\partial x_1 \partial x_3} + p_{x_2} \frac{\partial^2 f}{\partial x_2 \partial x_3} + p_{x_3} \frac{\partial^2 f}{\partial x_3} + k^2\, p_{x_3} f\right] \vec{a}_{x_3} \,, \tag{24}$$

where

$$f = f(x_1,x_2,x_3) = \frac{\exp\left[-jk(x_1^2 + x_2^2 + x_3^2)^{1/2} + x_1\kappa_{x_1} + x_2\kappa_{x_2} + x_3\kappa_{x_3}\right]}{R}$$

where $R = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ . This expression involves terms like

$$\frac{\partial^2 f}{\partial x_1^2} = \left[ \frac{3x_1^2 - R^2}{R^4} + jkd \frac{3x_1^2 - R^2}{R^3} + \frac{jx_1}{R^2} (2\,\kappa x_1 d + jk^2 d^2 x_1^2) - \right.$$

$$\left. \kappa x_1 d \frac{\kappa x_1 d\, R + 2kdx_1}{R} \right] f(x_1, x_2, x_3) \quad . \tag{25}$$

The convergence of Eq. (23) can be demonstrated by considering the term involving the highest power of $1/R$ ,

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \frac{3x^2 - R^2}{R^5}\, dx_1 dx_2 dx_3 = -\frac{4\pi}{3} \quad ,$$

which shows that the integral of Eq. (23) is convergent. Engineering approximations may be used to reduce the number of terms to be integrated in Eq. (24). These approximations will depend on the magnitudes of the ratio of spacing to wavelength and the degree of accuracy desired. For $d/\lambda < 1$ the consideration of the terms with the highest powers of $1/R$ is sufficiently accurate.

The expressions commencing with the fourth term in Eq. (21) are terms involving derivatives. The values of the derivatives at the limits (-1,-1,-1) and (1,1,1) do not increase as the order of the derivatives increases. Therefore the evaluation of the first few terms will be sufficient for an accurate determination of $\kappa$ .

Simplification of Eq. (22) after the aforementioned evaluation results in an equation of the form

$$\begin{bmatrix} Q_{x_1x_1} & Q_{x_1x_2} & Q_{x_1x_3} \\ Q_{x_2x_1} & Q_{x_2x_2} & Q_{x_2x_3} \\ Q_{x_3x_1} & Q_{x_3x_2} & Q_{x_3x_3} \end{bmatrix} \begin{bmatrix} p_{x_1} \\ p_{x_2} \\ p_{x_3} \end{bmatrix} = 0 \quad , \tag{26}$$

where $Q_{ij}$ is a function of $\kappa_{x_1}$ , $\kappa_{x_2}$ , $\kappa_{x_3}$ and $k$ . (Q) will be referred to as the high frequency anisotropy tensor. In order that Eq. (26) have a solution other than the trivial solution $p_{x_1} = p_{x_2} = p_{x_3} = 0$ , it is necessary and sufficient that the determinant

$$\begin{bmatrix} Q_{x_1x_1} & Q_{x_1x_2} & Q_{x_1x_3} \\ Q_{x_2x_1} & Q_{x_2x_2} & Q_{x_2x_3} \\ Q_{x_3x_1} & Q_{x_3x_2} & Q_{x_3x_3} \end{bmatrix} = 0 \quad . \qquad (27)$$

Equation (27) is a function of $\kappa_{x_1}$, $\kappa_{x_2}$, $\kappa_{x_3}$ and $\kappa$ . In order to find the propagation constant in a given direction $\phi$ and $\theta$ with respect to a spherical coordinate system three more equations may be written

$$\frac{\kappa_{x_1}}{(\kappa_{x_1}^2 + \kappa_{x_2}^2)^{1/2}} = \cos\phi, \qquad \frac{\kappa_{x_3}}{\kappa} = \cos\theta$$

$$\kappa = (\kappa_{x_1}^2 + \kappa_{x_2}^2 + \kappa_{x_3}^2)^{1/2} \quad . \qquad (28)$$

The simultaneous solution of Eq. (27) and Eq. (28) yields the propagation constant to any desired direction.

To illustrate the suggested technique for the evaluation of the general results given in Eq. (22), the propagation constant is computed for values of $kd < 1$ and $\kappa_{x_1}d < 1$ . In this frequency range, the expression for $f(x_1,x_2,x_3)$ may be expressed by a finite number of terms of its series representation, making possible an exact evaluation of the propagation constant.

It is assumed that the propagation is along the direction $x_1$, in which case Eqs. (27) and (28) become

$$\kappa_{x_2} = \kappa_{x_3} = 0 , \quad \kappa_{x_1} = \kappa$$

and

$$\text{Det } Q = 0 .$$

The simultaneous solution of these equations gives a relation between $\kappa_{x_1} d$ and kd, which is plotted in Fig. 2. (In Fig. 2, K represents $\kappa_{x_1}$.) The details of the solution are given in Appendix B.

An examination of the elements of the high-frequency anisotropy tensor (Q) given in Eq. (B4) in Appendix B, shows that the magnitude of the off diagonal elements are small in comparison with the diagonal terms for the range of frequencies considered in this computation. The computation in Appendix B clearly indicates that the magnitude of anisotropy will depend on the relative importance of the off diagonal terms. Therefore for this case the anisotropy introduced is small. However, at shorter wavelengths, the magnitude of the off diagonal elements of (Q) becomes important, introducing large anisotropies the magnitude of which is a function of frequency.

A comparison of the curves in Fig. 3 shows that for the larger value of (d/a), corresponding to the case of small dipole moments or small element density (see Appendix B for the definition of a), the dependence of $\kappa_{x_1}/k$ on frequency is not as significant as for dipoles of large dipole moment or large element density. The dielectric constant $k_{x_1 x_1}$ evaluated from the slope of the curve for d/a in the neighborhood of $\kappa_{x_1} d = 0$, $kd = 0$ gives

$$k_{x_1 x_1} = \left(\frac{\kappa_{x_1} d}{kd}\right)^{1/2} = 1.53 .$$

For the same value of d/a, the Clausius Mosotti relation gives $k_{x_1 x_1} = 1.55$ . This is in excellent agreement.

For spacings small compared with wavelength, the resultant dipole moment may be considered perpendicular to the propagation constants (see Sec. I and II). In this particular case, for the first Brillouin zone $n_1 = n_2 = n_3 = 0$ , Eq. (21) reduces to

$$\vec{a}_p = (\delta) \cdot \frac{k^2}{\epsilon_0 d^3} \frac{\vec{a}_p}{\kappa^2 - k^2} - \frac{\delta}{4\pi \epsilon_0}$$

$$\vec{L} + \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \vec{F}(x_1,x_2,x_3)\, dx_1 dx_2 dx_3 \quad . \tag{29}$$

To compare the expression in Eq. (29) with that obtained in Sec. II, the same approximation used in that section for the evaluation of the near fields will be used here. Therefore, in Eq. (29) representation of $\vec{F}(x_1,x_2,x_3)$, defined by Eq. (15), by the first term of its series expression yields

$$1 = \frac{1}{\epsilon_0 d^3} \frac{k^2}{\kappa^2 - k^2} + \frac{1}{4\pi\epsilon_0 d^3}$$

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \frac{\partial^2}{\partial x_1^2} \frac{1}{(x_1^2 + x_2^2 + x_3^2)^{1/2}}\, dx_1 dx_2 dx_3 \; , \tag{30}$$

where $\vec{a}_p$ is taken parallel to the $x_1$ axis; L defined by Eq. (A10) in the appendix, reduces to 0 . Carrying on the integration yields

$$k_{xx} - 1 = \frac{N\delta/\epsilon_0}{1 - N\delta/3\epsilon_0}$$

with the notation of Sec. I, the dielectric tensor may be written as

$$(k_e) = (1) - (N/\epsilon_0)(T) \quad . \tag{31}$$

This is identical with the results of Sec. I and II and very clearly indicates the limitations of the method given in those sections. Considering the expression in (21), it is seen that for large values of $\lambda/d$, Eq. (31) is a very good approximation since the contribution of terms $n_1, n_2$, and $n_3 \neq 0$ may be considered negligible. On the other hand, for $d$ of the order of $\lambda$, the contribution of the terms when $n_1$, $n_2$, and $n_3 \neq 0$ is important and must be considered. Therefore it can be seen that the Clausius-Mosotti relation is a very good approximation for large values of $\lambda/d$. For general values of the ratio $\lambda/d$, on the other hand, the complete expression derived in this section must be used. The result expressed in Eq. (21) is very general and quantitatively predicts that part of the anisotropy of an artificial dielectric which is due to the finite ratio of wavelength to element spacing. Since the design of artificial dielectrics to simulate real dielectrics is an important practical problem, these results are useful in calculating how far short of this goal the artiricial structures fall with respect to the simulation of truly isotropic characteristics.

The expression given by Eq. (31) shows that the dielectric constant of the lattice may be less than or greater than unity for negative or positive polarizabilities respectively. The elements of the polarizability tensor will be positive if the dipoles are operated at a frequency below their resonant frequency and will be negative if operated above.

To obtain a dielectric constant of unity, the lattice medium must be embedded in a dielectric binder of appropriate dielectric constant. The expression for the dielectric constant corresponding to Eq. (31) for the case in which the lattice is embedded in a uniform dielectric with dielectric constant $k_m$, is given by

$$(k_e) - (k_m) = -(N/\epsilon_0)(T_1) \quad . \tag{32}$$

For the composite dielectric tensor to be a unity tensor, the following relation must be satisfied

$$(1) - (k_m) = - (N/\epsilon) (T_1) \quad . \tag{33}$$

The corresponding scalar equation is

$$1 - k_{mx_1x_1} = (n\delta/\epsilon_0)/(1 - N\delta/3\epsilon_0) \quad ,$$

with similar equations of $k_{mx_2x_2}$ and $k_{mx_3x_3}$. This relation is plotted in Fig. 2. Since ordinary embedding dielectric materials have a dielectric constant greater than one, only the portion of the curve to the right of $x = 1$ is of interest. There are two distinct regions of importance. For the first region, $1 < k_m < 4$, the polarizability is negative and therefore the medium has to be operated at a frequency above the resonant frequency of the triad elements. For the region $k_m > 4$, the polarizability is positive and therefore the medium has to be operated at a frequency below the resonant frequency.

In certain applications of isotropic artificial dielectrics such as in radome design, it is important to obtain dielectrics with an index of refraction of unity. In order for this to be true, not only must the dielectric constant be unity, but the relative permeability must also be equal to 1. The relative permeability may be made equal to unity by properly restricting the geometry of the elements so that magnetic dipole fields are not induced. Consequently in order to have a reflectionless and also isotropic dielectric, the geometry of the cubical lattice must be so restricted that the anisotropy caused by the finite ratio of wavelength to spacing is negligible. Furthermore, the material and shape of elements must be so restricted that neither higher order multipoles (electric or magnetic) nor magnetic dipoles are excited. Work is continuing on the realizability of the triad elements for such arrays.

## V. Conclusion.

Three different approaches have been used to evaluate the expression for the dielectric parameters of a cubical lattice with isotropic elements.

a) A molecular analogy with a consideration of static dipole interaction leading to the Clausius-Mosotti relations.

b) An analysis based on the summation of scattered time-varying fields in which it is demonstrated that the use of the trapezoidal summation approximation which is so frequently used in these problems, completely removes retardation effects. The results obtained in this case could much more easily have been obtained directly from static considerations.

c) An exact solution valid for all values of spacing to wavelength ratio. In this case, the importance of the retardation effect in the general result for the propagation constant of the composite medium is clearly seen. This exact solution also brings to light a type of anisotropy existing in artificial structures which has usually been ignored in previous investigations. This is an anisotropic effect which exists even in arrays composed of isotropic elements arranged in structurally isotropic patterns; it is an isotropy caused by the granularity of the artificial structure that becomes increasingly important at higher frequencies when the interelement spacing becomes appreciable in terms of wavelength. This effect is calculated in the paper in terms of the high-frequency anisotropy tensor, Q .

This investigation also has direct application in the design of isotropic materials for the control and direction of microwave energy. By varying the frequency of operation above and below the resonant frequency of the triad elements, the effective dielectric constant can be made less than or greater than unity, respectively. If the triad medium is embedded in a uniform dielectric, it is possible to simulate materials with dielectric constant of unity. Such material may be useful in applications in which a reflectionless dielectric is required.

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## APPENDIX A

### (a) Evaluation of the Expression S

In order to evaluate

$$S = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \vec{F}(x_1,x_2,x_3)\ \exp\left[-j2\pi(n_1x_1+n_2x_2+n_3x_3)\right]dx_1dx_2dx_3$$

the change of variables $x_1d = y_1$, $x_2d = y_2$, and $x_3d = y_3$ is made. Hence the expression for S becomes

$$S = \frac{1}{d^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\nabla\nabla\cdot\left[\frac{\exp\left[-jk(y_1^2+y_2^2+y_3^2)^{1/2}\right]}{\sqrt{y_1^2+y_2^2+y_3^2}}\exp\left[-j\vec{\kappa}\cdot(y_1\vec{a}_x+y_2\vec{a}_y+y_3\vec{a}_z)\right]\vec{a}_p\right]\right.$$

$$\left.-\nabla^2\left[\frac{\exp\left[-jk(y_1^2+y_2^2+y_3^2)^{1/2}\right]}{\sqrt{y_1^2+y_2^2+y_3^2}}\ \exp\left[-j\vec{\kappa}\cdot(y_1\vec{a}_x+y_2\vec{a}_y+y_3\vec{a}_z\right]\vec{a}_p\right]\right\}$$

$$\times \exp\left[-j(2\pi/d)(n_1y_1+n_2y_2+n_3y_3)\right]dy_1dy_2dy_3\ . \qquad (A1)$$

Consider the expression

$$\frac{\exp\left[-jk(y_1^2+y_2^2+y_3^2)^{\frac{1}{2}}\right]}{(y_1^2+y_2^2+y_3^2)^{1/2}},$$

the Fourier transform of this expression with respect to $y_1$ is

$$\int_{-\infty}^{\infty}\frac{\exp\left[-jk(y_1^2+y_2^2+y_3^2)^{1/2}\right]}{(y_1^2+y_2^2+y_3^2)^{1/2}}\exp(-j\kappa_{x1}y_1)dy_1 = -\pi H_0^{(2)}\left[(k^2-\kappa_{x1}^2)^{1/2}\right.$$

$$\left.\times\ (y_2^2+y_3^2)^{1/2}\right]\ . \qquad (A2)$$

Taking the Fourier transform of the right-hand side of Eq. (A2) with respect to $y_2$ gives

$$\int_{-\infty}^{\infty} -\pi jH_0^{(2)}\left[(k^2-\kappa_{x1}^2)^{\frac{1}{2}}(y_2^2+y_3^2)^{\frac{1}{2}}\right]\exp(-j\kappa_{x2}y_2)dy_2 = 2\pi\ \frac{\exp\left[(\kappa_{x1}^2+\kappa_{x2}^2-k^2)^{\frac{1}{2}}|y_3|\right]}{(\kappa_{x1}^2+\kappa_{x2}^2-k^2)^{1/2}}$$

$$(A3)$$

The Fourier transform expression on the right-hand side of (A3) with respect to $y_3$ yields

$$\int_{-\infty}^{\infty} 2\pi \frac{\exp[(\kappa x_1^2+\kappa x_2^2-k^2)^{\frac{1}{2}}|y_3|]}{(\kappa x_1^2+\kappa x_2^2-k^2)^{1/2}} \exp(-j\kappa x_3 y_3)dy_3 = 4\pi \frac{1}{\kappa x_1^2+\kappa x_2^2+\kappa x_3^3-k^2} \,. \quad (A4)$$

With the use of well-known relations of operational calculus the expression for S reduces to

$$S = \frac{4\pi}{d^3} \frac{\vec{\kappa}(\vec{a}_p \cdot \vec{\kappa}) + k^2\vec{a}_p}{[|\vec{\kappa}-(2\pi n_1/d)\vec{a}x_1-(2\pi n_2/d)\vec{a}x_2-(2\pi n_3/d)\vec{a}x_3|^2 - k^2]} \,. \quad (A5)$$

(b) Poisson Summation Formula for a Finite Sum

A relation similar to Eq. (16) will be desired for a finite sum. For simplicity, the derivation will be carried through in one dimension. The derivation for the three-dimensional case is similar and the result only will be stated.

If in the interval $m \leq x \leq m+1$, $f(y)$ meets the requirements for representation as a Fourier series, the following equations give the value of that representation both in the open interval and at the end points of the interval $m \leq y \leq m+1$

$$f(y) = \sum_{n=-\infty}^{\infty} \int_m^{m+1} f(x)e^{-j2\pi n(y-x)} \qquad m<y<m+1 \,,$$

and

$$\frac{f(m)+f(m+1)}{2} = \sum_{n=-\infty}^{\infty} \int_m^{m+1} f(x)\, e^{-j2\pi nx}\, dx \qquad y = m,\ m+1 \,. \quad (A6)$$

If both sides are summed up from $m = a+1$ to $m = b-1$ and if $-[f(a)+f(b)]/2$ is added to both sides, the Poisson transformation for a finite sum is obtained.

$$\sum_{m=a+1}^{b-1} f(m) = -\frac{f(a)+f(b)}{2} + \sum_{n=-\infty}^{\infty} \int_a^b f(x) e^{-j2\pi nx}\, dx$$

$$= -\frac{f(a)+f(b)}{2} + \int_a^b f(x)dx + 2\sum_{n=1}^{\infty} \int_a^b f(x) \cos 2\pi nx dx \,. \qquad \text{(A7)}$$

The first two terms on the right-hand side of Eq.(A7) are the results of approximating a summation by an integration using the trapezoidal rule; hence the last term of Eq.(A7) can be considered as the correction to the trapezoidal rule. In some cases this trapezoidal part is quite accurate in summing the series, thus transferring difficulty from summing $f(m)$ to integrating the same function. Integrating the last integral in Eq. (A7) $P$ times yields

$$\sum_{a+1}^{b-1} f(m) = -\frac{f(a)+f(b)}{2} + \int_a^b f(x)dx + 2\sum_{r=1}^{P} (-1)^r \frac{\zeta(2r)}{(2\pi)^{2r}}$$

$$\times \left[f^{(2r-1)}(a) - f^{(2r-1)}(b)\right] + \frac{2(-1)^{r+1}}{(2\pi)^{2r}} \sum_{n=1}^{\infty} \frac{1}{(n)^{2r}} \int_a^b f(x) \cos 2\pi nx dx, \qquad \text{(A8)}$$

where

$$\zeta(2r) = \sum_{n=1}^{\infty} \frac{1}{n^{2r}}$$

is the Riemann Zeta function. This derivation can be extended to the three-dimensional case, giving

$$\sum_{a_1+1}^{b_1-1} \sum_{a_2+1}^{b_2-1} \sum_{a_3+1}^{b_3-1} f(m_1,m_2,m_3)$$

$$= -\frac{1}{8}\left[f(a_1,a_2,a_3)+ f(b_1,a_2,a_3)+ f(b_1,a_2,b_3)+ f(b_1,a_2,a_3)+ f(a_1,b_2,a_3)\right.$$

$$\left.+ f(a_1,b_2,b_3)+ f(b_1,b_2,a_3)+ f(b_1,b_2,b_3)\right]$$

$$-\frac{1}{4}\sum_{m_1=a_1+1}^{b_1-1}\left[f(m_1,a_2,a_3)+ f(m_1,a_2,b_3)+ f(m_1,b_2,a_3)+ f(m_1,b_2,b_3)\right]$$

$$-\frac{1}{4}\sum_{m_2=a_2+1}^{b_2-1}\left[f(a_1,m_2,a_3)+ f(b_1,m_2,a_3)+ f(a_1,m_2,b_3)+ f(b_1,m_2,b_3)\right]$$

$$-\frac{1}{4}\sum_{m_3=a_3+1}^{b_3-1}\left[f(a_1,a_2,m_3)+ f(b_1,a_2,b_3)+ f(a_1,b_2,m_3)+ f(b_1,b_2,m_3)\right]$$

$$-\frac{1}{2}\sum_{m_1=a_1+1}^{b_1-1}\sum_{m_2=a_2+1}^{b_2-1}\left[f(m_1,m_2,a_3)+ f(m_1,m_2,b_3)\right]$$

$$-\frac{1}{2}\sum_{a_2+1}^{b_2-1}\sum_{a_3+1}^{b_3-1}\left[f(a_1,m_2,m_3) + f(b_1,m_2,m_3)\right]$$

$$-\frac{1}{2}\sum_{a_1+1}^{b_1-1}\sum_{a_3+1}^{b_1-1}\left[f(m_1,a_2,m_3)+ f(m_1,b_2,m_3)\right] + \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(x_1,x_2,x_3)\,dx_1dx_2dx_3$$

$$+ 8\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(x_1,x_2,x_3)\cos 2\pi m_1x_1 \cos 2\pi m_2x_2 \cos 2\pi m_3x_3$$

$$\times\, dx_1dx_2dx_3\ . \qquad \text{(A9)}$$

Changing Eq. (A9) into a form similar to Eq.(A8) results in

$$\sum_{a_1+1}^{b_1-1} \sum_{a_2+1}^{b_2-1} \sum_{a_3+1}^{b_3-1} f(m_1,m_2,m_3)$$

$$= L + \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(x_1,x_2,x_3)dx_1dx_2dx_3 + \sum_{r=1}^{N} \sum_{s=1}^{P} \sum_{t=1}^{Q} \frac{(-1)^{r+s+t}}{(2\pi m_2)^{2r}(2\pi m_2)^{2s}(2\pi m_3)^{2t}}$$

$$\times \left[ f_{x_1^{2r-1}\, x_2^{2s-1}\, x_3^{2t-1}}(a_1,a_2,a_3) - f_{x_1^{2r-1}\, x_2^{2s-1}\, x_3^{2t-1}}(b_1,b_2,b_3) \right]$$

$$+ \underset{x_1x_2x_3}{P} \sum_{r=1}^{N} \sum_{s=1}^{P} \frac{(-1)^{r+s}\,(-1)^{t+1}}{(2\pi m_1)^{2r}(2\pi m_2)^{2s}(2\pi m_3)^{2t}}$$

$$\times \int_{a_3}^{b_3} \left[ f_{x_1^{2r-1}\, x_2^{2s-1}\, x_3^{2t}}(a_1,a_2,x_3) - f_{x_1^{2r-1}\, x_2^{2s-1}\, x_3^{2t}}(b_1,b_2,b_3) \right]$$

$$\times \cos 2\pi m_3x_3dx_3 + \underset{x_1x_2x_3}{P} \sum_{r=1}^{N} \frac{(-1)^{r}\,(-1)^{s+1}\,(-1)^{t+1}}{(2\pi m_1)^{2r}(2\pi m_2)^{2s}(2\pi m_3)^{2t}}$$

$$\times \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left[ f_{x_1^{2r-1}\, x_2^{2s}\, x_3^{2t}}(a_1,x_2,x_3) - f_{x_1^{2r-1}\, x_2^{2s}\, x_3^{2t}}(b_1,x_2,x_3) \right]$$

$$\times \cos 2\pi m_2x_2 \cos 2\pi m_3x_3\, dx_1dx_2dx_3$$

$$+ \frac{(-1)^{r+1}\,(-1)^{s+1}\,(-1)^{t+1}}{(2\pi m_1)^{2r}(2\pi m_2)^{2s}(2\pi m_3)^{2t}} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f_{x_1^{2r}\, x_2^{2s}\, x_3^{2t}}(x_1,x_2,x_3)$$

$$\times \cos 2\pi m_1x_1 \cos 2\pi m_2x_2 \cos 2\pi m_3x_3\, dx_1dx_2dx_3 \;, \qquad \text{(A10)}$$

where L represents the sum of the single terms, single sums and double sums.

In Eq. (A10) $\overset{P}{x_1x_2x_3}$ indicates circular permutation of $(x,y,z)$. Unless the derivatives increase very rapidly the transformed series in Eq. (A10) may be written as

$$\sum_{a_1+1}^{b_1-1}\sum_{a_2+1}^{b_2-1}\sum_{a_3+1}^{b_3-1} f(m_1,m_2,m_3) = L + \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(x_1,x_2,x_3)\,dx_1dx_2dx_3$$

$$- \frac{f_{x_1^1\,x_2^1\,x_3^1}(a_1,a_2,a_3) - f_{x_1^1\,x_2^1\,x_3^1}(b_1,b_2,b_3)}{1728}$$

$$+ \frac{f_{x_1^3\,x_2^1\,x_3^1}(a_1,a_2,a_3) - f_{x_1^3\,x_2^1\,x_3^1}(b_1,b_2,b_3)}{26820}$$

$$+ \frac{f_{x_1^1\,x_2^3\,x_3^1}(a_1,a_2,a_3) - f_{x_1^1\,x_2^3\,x_2^1}(b_1,b_2,b_3)}{26820}$$

$$+ \frac{f_{x_1^1\,x_2^1\,x_3^3}(a_1,a_2,a_3) - f_{x_1^1\,x_2^1\,x_3^3}(b_1,b_2,b_3)}{26820} \cdots \quad . \qquad \text{(A11)}$$

## APPENDIX B

### Solution of Equation Det $Q = 0$

The evaluation of $L$ in Eq. (22) gives

$$\begin{aligned}\vec{L} = &\ [4.48(kd)^4 + 1.49(kd)^2(\kappa_{x_1}d)^2 + 3.2(kd)^2 + 3.07(\kappa_{x_1}d)^2]px_1\,\vec{a}x_1 \\ + &\ [4.48(kd)^4 + 1.49(kd)^2(\kappa_{x_1}d)^2 - 3.2(kd)^2 - 0.59(\kappa_{x_1}d)^2]px_2\,\vec{a}x_2 \\ + &\ [4.48(kd)^4 + 1.49(kd)^2(\kappa_{x_1}d)^2 + 3.2(kd)^2 - 0.59(\kappa_{x_1}d)^2]px_3\,\vec{a}x_3 \quad .\end{aligned} \qquad \text{(B1)}$$

The integral in Eq. (22) gives

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \vec{F}(x_1,x_2,x_3)\ dx_1 dx_2 dx_3$$

$$= \left[-\frac{4\pi}{3} + 6.49\ (kd)^2 - 4.12(\kappa_{x_1}d)^2 - 3.86(kd)^4 - 1.91(kd)^2(\kappa_{x_1}d)^2\right] p_{x_1}\ \vec{a}_{x_1}$$

$$+ \left[-\frac{4\pi}{3} + 6.49(kd)^2 + 0.47(\kappa_{x_1}d)^2 - 3.86(kd)^4 - 1.91(kd)^2(\kappa_{x_1}d)^2\right] p_{x_1}\ \vec{a}_{x_2}$$

$$+ \left[-\frac{4\pi}{3} + 6.49(kd)^2 + 0.47(\kappa_{x_1}d)^2 - 3.86(kd)^4 - 1.91(kd)^2(\kappa_{x_1}d)^2\right] p_{x_3}\ \vec{a}_{x_3}$$

$$+ \text{ higher order terms in } (kd) \text{ and } (\kappa_{x_1}d)\ . \qquad \text{(B2)}$$

The expressions in (B2) are evaluated by considering the first three terms of the series expansion of $f(x_1,x_2,x_3)$ . The expressions involving the derivatives yield

$$\frac{F_{x_1^1 x_2^1 x_3^1}(1,1,1) - F_{x_1^1 x_2^1 x_3^1}(-1,-1,-1)}{1728}$$

$$= \frac{\vec{a}x_1}{1728}\Big\{px_1\left[-0.658(\kappa_{x_1}d)^2 - 0.064(kd)^2(\kappa_{x_1}d)^2 - 0.192(kd)^4 - 0.43(kd)^2\right]$$

$$px_2\left[-0.4(\kappa_{x_1}d)^2 + 0.21(kd)^2 - 2.14\right]$$

$$px_3\left[-0.4(\kappa_{x_1}d)^2 + 0.21(kd)^2 - 2.14\right]\Big\}$$

$$+ \frac{\vec{a}x_2}{1728}\Big\{px_1\left[-0.4(\kappa_{x_1}d)^2 + 0.21(kd)^2 - 2.14\right]$$

$$px_2\left[0.416(\kappa_{x_1}d)^2 - 0.064(\kappa_{x_1}d)^2(kd)^2 - 0.192(kd)^2 - 0.43(kd)^2\right]$$

$$px_3\left[0.38(\kappa_{x_1}d)^2 + 0.21(kd)^2 - 2.14\right]\Big\}$$

$$+ \frac{\vec{a}_{x_3}}{1728} \Big\{ p_{x_1} \left[-0.4(\kappa_{x_1}d)^2 + 0.21(kd)^2 - 2.14\right]$$

$$p_{x_2} \left[0.38(\kappa_{x_1}d)^2 + 0.21(kd)^2 - 2.14\right]$$

$$p_{x_3} \left[0.416(\kappa_{x_1}d)^2 - 0.064(\kappa_{x_1}d)^2(kd)^2 - 0.192(kd)^4 - 0.43(kd)^2\right]\Big\} . \qquad (B3)$$

It is assumed that the polarizability $\delta$ of the lattice elements is equivalent to that of a spherically shaped conducting scatterer. Therefore $\delta$ will be substituted by $4\pi\epsilon_0 a^3$, where $a$ is the equivalent radius of the triad elements when replaced by conducting spheres.

The substitution of the expressions given by (B1), (B2), (B3) and $\delta = 4\pi\epsilon_0 a^3$ in Eq. (22) yields for the elements of Det Q

$$Q_{x_1x_1} = -\frac{d^3}{a^3} + 4\pi \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{n_3=-\infty}^{\infty}$$

$$\times \frac{[(\kappa_{x_1}d)^2/(kd)^2] + 1}{[(\kappa_{x_1}d/kd) - (2\pi n_1/kd)]^2 + [(2\pi n_2/kd)^2] + [(2\pi n_3/kd)^2]}$$

$$- \left[-4.2 + 0.62(kd)^4 - 0.42(kd)^2(\kappa_{x_1}d)^2 + 9.69(kd)^2 - 1.05(\kappa_{x_1}d)^2\right]$$

$$- \frac{1}{1728}\left[-0.658(\kappa_{x_1}d)^2 - 0.064(kd)^2(\kappa_{x_1}d)^2 - 0.192(kd)^4 - 0.43(kd)^2\right]$$

$$Q_{x_1x_2} = Q_{x_2x_1} = Q_{x_1x_3} = Q_{x_3x_1} = -\frac{1}{1728}\left[-0.4(\kappa_{x_1}d)^2 + 0.21(kd)^2 - 2.14\right]$$

$$Q_{x_2x_3} = Q_{x_3x_2} = \frac{-1}{1728}\left[0.38(\kappa_{x_1}d)^2 + 0.21(kd)^2 - 2.14\right]$$

$$Q_{x_2x_2} = Q_{x_3x_3} = -\frac{d^3}{a^3} + 4\pi \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{n_3=-\infty}^{\infty}$$

$$\times \frac{1}{[(\kappa_{x_1}d/kd) - (2\pi n_1/kd)]^2 + [(2\pi n_2/kd)^2] + [(2\pi n_3/kd)^2]}$$

$$-[-4.2 + 0.62(kd)^4 - 0.42(kd)^2(\kappa x_1 d)^2 + 9.69(kd)^2 - 0.12(\kappa x_1 d)^2]$$

$$-\frac{1}{1728}\ [0.416(\kappa x_1 d)^2 - 0.064(kd)^2(\kappa x_1 d)^2 - 0.192(\kappa x_1 d)^4 - 0.43(kd)^2]\ .$$

(B4)

Substitution of the previous expressions in

$$\text{Det } Q = 0 \tag{B5}$$

gives a relation between $(\kappa x_1 d)$ and $(kd)$, which is plotted in Fig. 2. It should be pointed out that because of the rapid convergence of the series appearing in the expressions for the elements of Det Q, only a very small number of terms was necessary to plot Fig. 2.

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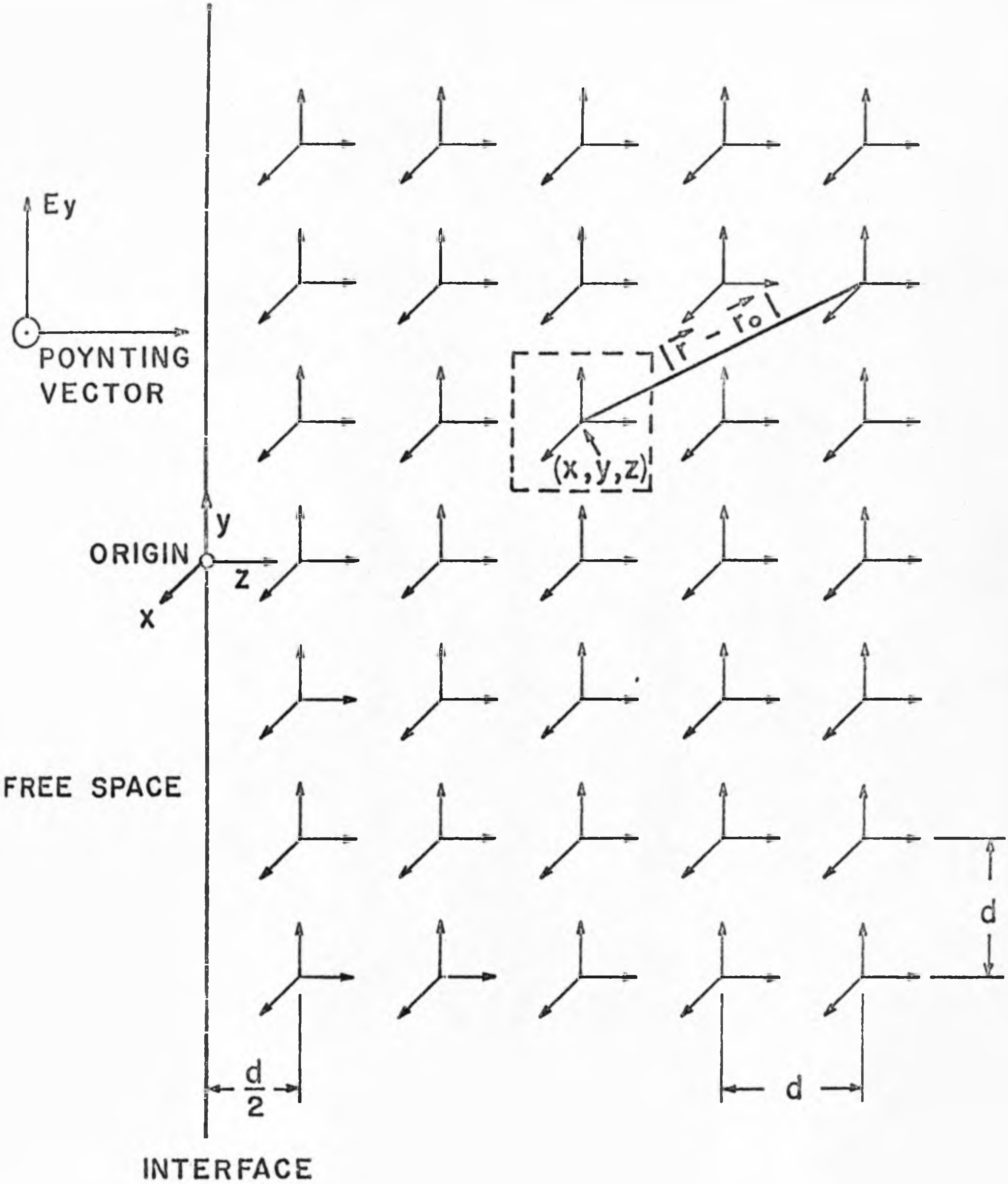


Fig. 1 Plane Polarized Wave Incident on Region $z > 0$ Containing Triad Elements in Cubic Lattice of Side $d$.

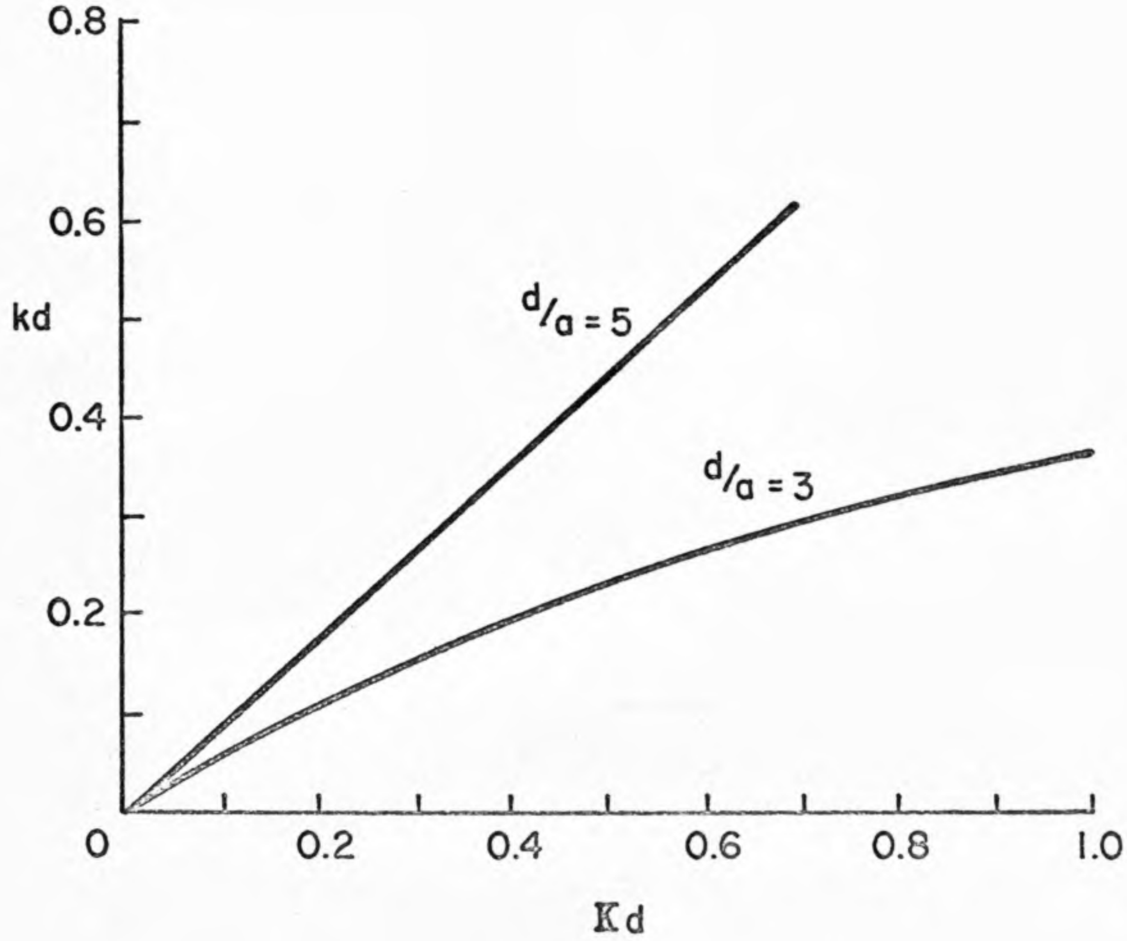


Fig. 2. Propagation constants for various ratios d/a .

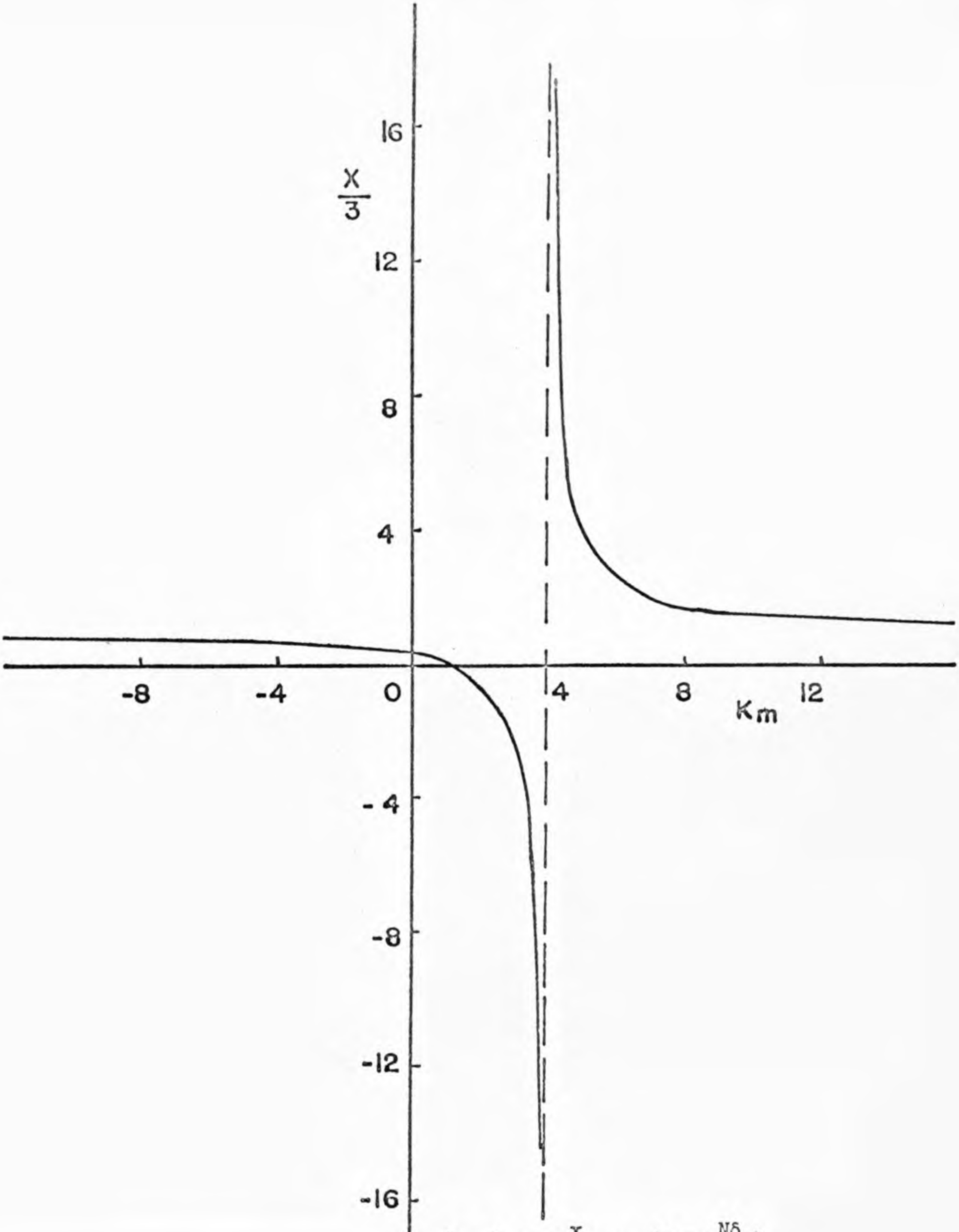


Fig.3 Graphical Representation of $1 - k_m = \frac{x}{1 - x/3}$ $(x = \frac{N\delta}{3\epsilon_o})$