On the Blow-up of Solutions to Some Semilinear and Quasilinear Reaction-diffusion Systems

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Abstract. After a brief discussion of known global well-posedness results for semilinear systems, we introduce a class of quasilinear systems and obtain spatially local estimates which allow us to prove that if one component of the system blows up in finite time at a point x^* in space then at least one other component must also blow up at the same point. For a broad class of systems modelling one-step reversible chemical reactions, we show that blow-up in one component implies blow-up in *all* components at the same point in space and time.

1. Introduction

Considerable research has been done in the last decade on the problem of global well-posedness of semilinear parabolic systems of partial differential equations; *i.e.,* reaction-diffusion systems. See, *e.g.*, [1–7, 9–13]. A system is said to be globally well-posed if classical solutions continue for all time $t > 0$ given any nonnegative L^∞ initial data. Perhaps the greatest source of interesting problems in this area is the modelling of multi-species chemical reactions. For example, let us consider the following, seemingly simple, reversible reaction in which sulphur dioxide reacts with oxygen to form sulphur trioxide:

$$2\mathrm{SO}_2 + \mathrm{O}_2 \quad \rightleftharpoons \quad 2\mathrm{SO}_3. \tag{1.1}$$

If we set $A = [\mathrm{SO}_2]$, $B = [\mathrm{O}_2]$, and $C = [\mathrm{SO}_3]$, then this reaction, assuming mass action kinetics, may be modelled by the reaction-diffusion system:

$$\begin{aligned} A_t - d_1\Delta A &= 2(k_r C^2 - k_f A^2 B) \\ B_t - d_2\Delta B &= k_r C^2 - k_f A^2 B \\ C_t - d_3\Delta C &= 2(k_f A^2 B - k_r C^2) \end{aligned} \tag{1.2}$$

together with nonnegative L^∞ initial data and, say, homogeneous Neumann boundary conditions. Here the d_i are positive diffusivities and k_f, k_r are positive forward and reverse reaction rates, respectively, and we assume that the reaction takes place within a bounded domain Ω with smooth boundary. Even the most casual observer will note that the total concentration $\int_\Omega (A + B + C)dx$ remains bounded since the reaction functions sum to zero with appropriate positive scaling factors. However, *pointwise* bounds, which are necessary to prove continuability of classical solutions, are quite difficult to come by (unless $d_1 =$

$d_2 = d_3$) and, in fact, currently constitute an open question for this "simple" system if Ω has spatial dimension greater than two. It is, however, not difficult to show that if a pointwise bound were available for any one component of this system, then pointwise bounds for all other components would follow. This essentially says that if one component blows up at time T^*, then *all* components do likewise. In a subsequent section we will localize this result to show that if blow-up occurs, then it must occur in all components at the same point in space and time.

2. Global well-posedness of semilinear systems

Let us now consider the following, somewhat general, reaction-diffusion system:

$$\begin{aligned} \frac{\partial u_i}{\partial t} - d_i \Delta u_i &= f_i(u) && \text{in } \Omega \times \{t > 0\},\ i = 1, \ldots, m \\ \frac{\partial u_i}{\partial \mathrm{n}} &= \rho_i(\gamma_i - u_i) && \text{on } \partial\Omega \times \{t > 0\},\ i = 1, \ldots, m \\ u_i(\cdot, 0) &= u_{0_i}(\cdot) && \text{on } \Omega,\ i = 1, \ldots, m \end{aligned} \tag{2.1}$$

where $u = (u_i)_{i=1}^m$, the d_i are positive constants, the ρ_i, γ_i are nonnegative constants, and Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. (Here and in the remainder of this paper we mean by this that $\partial\Omega$ is an $n-1$ dimensional $C^{2+\alpha}$ manifold of which Ω lies locally on one side.) We assume that the initial data u_{0_i} are bounded, measurable, and nonnegative, that the reaction functions f_i are locally Lipschitz, and that f is *quasi-positive*; *i.e.,* for each $i = 1, \ldots, m$, we have $f_i(\xi) \geq 0$ for all $\xi \geq 0$ with $\xi_i = 0$. These conditions on f guarantee *local* existence of unique, nonnegative, classical solutions on a maximal time interval $0 \leq t < T^* \leq \infty$ [4, 10, 13]. An additional natural condition to place on f is that there are constants $\alpha_i > 0,\ i = 1, \ldots, m$, such that

$$\sum_{i=1}^{m} \alpha_i f_i(\xi) \leq 0 \quad \text{for all } \xi \geq 0 \tag{2.2}$$

This condition is tantamount to requiring conservation of total mass in the system. From a mathematical point of view, it allows one to obtain *a priori* bounds on solutions in the space $L^1(\Omega)$. To be specific we state the following lemma, which is a simple consequence of the divergence theorem.

Lemma 2.3. *If (2.2) holds, then*

$$\int_\Omega \sum_{i=1}^{m} u_i(\cdot, t) \leq \alpha^* (\int_\Omega \sum_{i=1}^{m} u_{0_i}(\cdot) + \beta t) \quad \text{for all } t \in [0, T^*),$$

where $\beta = |\partial\Omega| \sum_{i=1}^m d_i \rho_i \gamma_i$ *and* $\alpha^* = \max\{\alpha_i\}/\min\{\alpha_i\}$. *In particular, if* $\rho_i\gamma_i = 0$ *for each* $i = 1, \dots, m$, *then* $\beta = 0$.

We remark that in the case where each $\rho_i = \infty$, *i.e.*, each $u_i = \gamma_i \geq 0$ on $\partial\Omega \times \{t > 0\}$, one can obtain a similar bound on solutions in $L^1(\Omega \times (0, T))$ for all $T \leq T^*$. Moreover, if each $u_i = 0$ on $\partial\Omega \times \{t > 0\}$, one can obtain the result of Lemma 2.3 with $\beta = 0$. More general conditions than (2.2) on the f_i also allow similar results. See [4, 10, 11].

Another condition which is natural in the context of chemical reaction modelling (*cf.* (1.2)) is that

$$\text{each } |f_i(\xi)| \text{ is bounded above by a polynomial of degree } r_i. \tag{2.4}$$

While much progress has been made in recent years on the global well-posedness of such systems, it remains an open question as to whether the preceding conditions on f are enough to guarantee the global existence of nonnegative, classical solutions of (2.1).

Work in this area typically centers on establishing first an *a priori* bound on solutions in some L^p space with small p (usually an L^1 space) and bootstrapping the estimates into L^p with p sufficiently large to allow the use of condition (2.4) and classical parabolic regularity to obtain pointwise bounds. A pointwise bound allows one to contradict non-global existence via the standard result [13] that a classical solution either exists for all time or else blows up in the sup-norm in finite time.

Almost all progress in this area has required additional assumptions on the structure of f. In [4], Hollis, Martin, and Pierre considered (2.1) under assumptions (2.2), (2.4) with $m = 2$ components and proved global existence in any spatial dimension provided that an *a priori* L^∞ bound is available for one component. (Condition (2.2) was actually replaced with a similar one with a constant M majorizing $\sum f_i$.) This was sufficient to handle such models as the two-component "Brusselator," for example, in which f_1, f_2 in (2.1) take the form: $f_1(u) = -u_1u_2^2 + bu_2,\ f_2(u) = u_1u_2^2 - (b+1)u_2 + a$.

Subsequent work by Morgan [10, 11, 12] extended these results to handle m-component systems of the form (2.1) under condition (2.4) and with (2.2) replaced by the following "intermediate sums" condition.

$$\begin{aligned}
&\text{There exist } r \in [1, r_{\max}(n)),\ K \geq 0, \text{and, for each } i = 1, \dots, m,\\
&\text{nonnegative constants } \alpha_{i,j},\ j = 1, \dots, i, \text{ with } \alpha_{i,i} > 0 \text{ such that}\\
&\qquad \sum_{j=1}^{i} \alpha_{i,j} f_j(\xi) \ \leq\ K(\,1 + \sum_{j=1}^{m} \xi_j\,)^r \text{ for all } \xi \geq 0 \text{ in } \mathbb{R}^m.
\end{aligned} \tag{2.5}$$

The form of the upper bound $r_{\max}(n)$ on r depends upon the type of available *a priori* estimate, and in most cases one has $r_{\max}(n) \leq 2$, which restricts r, for the models under consideration here, to be 1, except for the case when $n = 1$, which allows $r = 2 < r_{\max}(1)$. Note that (2.5) does *not* require (except when $i = 1$) that each reaction function f_i be bounded above by a polynomial of degree less than $r_{\max}(n)$. It does, however, require some cancellation of higher order terms in the "intermediate sums". The reader should note the manner in which (2.5) is satisfied, with $r = 1$, by the aforementioned Brusselator. For more detail on the form of $r_{\max}(n)$ and its connection with the *a priori* estimate, we refer to Morgan [10]. More recent work of Morgan [12] has extended the results to $r = 2$ if $n = 2$.

Note also that if $d_i = d_j$ for all i, j, then condition (2.2) is sufficient for the global well-posedness of (2.1), since then $\sum \alpha_i u_i$ is bounded above by a solution of the heat equation.

All of the preceding remarks are also valid if the boundary conditions are nonnegative Dirichlet type (*i.e.*, each $\rho_i = \infty$ in (2.1)). It is important that either all of the ρ_i be ∞ or else all finite. Serious difficulties can arise from mixed boundary condition types, and, indeed, finite-time blow-up has been demonstrated for such a system; see Bebernes and Lacey [2]. It is interesting to note that the usual L^1 estimate does not follow from (2.2) if both Robin/Neumann and positive Dirichlet boundary conditions appear in (2.1).

Unfortunately, many models, while satisfying conditions (2.2) and (2.4) and posessing an intermediate sums-type structure, have intermediate sums of order $r \geq 2$; for example, (1.2), which posesses quadratic intermediate sums. (In fact, any f satisfying (2.4) trivially satisfies (2.5) with $r = \max\{r_i\}$.) Global existence for such systems remains, in general, an open problem. It is the purpose of this paper to discuss some interesting facets of these problems and prove some facts concerning problem (2.1) with *only* conditions (2.2) and (2.4).

We should make note of some results on related problems obtained without the intermediate sums condition. Kanel' [6, 7] has proved that solutions of (2.1) with $\Omega = \mathbb{R}^n$ exist globally provided that, in addition to (2.2) and (2.4), *each* f_i is at most quadratic if $n \geq 2$ and at most cubic if $n = 1$. This last result for cubic f_i's and $n = 1$ is also proved on the bounded domain $\Omega = (0, L)$ with each u_i satisfying Neumann conditions at the endpoints (*i.e.*, each $\rho_i = 0$ in (2.1)).

3. A question concerning linear, scalar equations

The work of Hollis, Martin, and Pierre [4], Morgan [10], and Hollis and Morgan [5] has relied upon a duality argument for bootstrapping *a priori* estimates into L^p estimates for all $p < \infty$. A simple modification of this duality argument shows a close connection between the question of global existence for (2.1) without Morgan's intermediate sums condition (*i.e.,* only (2.2), (2.4)) and a question on estimates for solutions of certain linear, scalar, parabolic equations.

Let us assume for simplicity that each $\rho_i = 0$ in (2.1). Consider the (local) solution u of (2.1), and define $w = \sum_{i=1}^m \alpha_i u_i$ and $\widetilde{w} = \sum_{i=1}^m d_i \alpha_i u_i$. Now note that due to (2.2) we have

$$\frac{\partial w}{\partial t} - \Delta \widetilde{w} \ \leq \ 0 \qquad \text{in } \Omega \times \{t > 0\} \tag{3.1}$$

$$\frac{\partial w}{\partial \mathrm{n}} \ = \ 0 \qquad \text{on } \partial\Omega \times \{t > 0\}. \tag{3.2}$$

Now if φ is a function in $C^{2,1}(\Omega \times [0,T])$ with $\varphi(\,\cdot\,,T) = 0$ and $\frac{\partial \varphi}{\partial \mathrm{n}} = 0$ on $\partial\Omega \times (0,T)$, then multiplying (3.1) by φ and integrating by parts result in

$$-\int_0^T \int_\Omega w \, (\frac{\partial \varphi}{\partial t} + \frac{\widetilde{w}}{w} \Delta \varphi) \ \leq \ \int_\Omega \varphi(x,0) w(x,0) dx. \tag{3.3}$$

Upon setting $\chi(\,\cdot\,,t) = \varphi(\,\cdot\,,T-t)$, (3.3) becomes

$$\int_0^T \int_\Omega w \, \vartheta \ \leq \ \int_\Omega \chi(x,T) w(x,0) dx, \tag{3.4}$$

where

$$\frac{\partial \chi}{\partial t} - \frac{\widetilde{w}}{w} \Delta \chi \ = \ \vartheta \qquad \text{in } \Omega \times (0,T). \tag{3.5}$$

Now, with the aid of duality, (3.4) would yield a bound on w in $L^{\frac{p}{p-1}}(\Omega \times (0,T))$ provided that one had an estimate on $\int_\Omega \chi(x,T) dx$ in the form $C\|\vartheta\|_{p,\Omega\times(0,T)}$. (Note that we may restrict ourselves to nonnegative ϑ, which implies $\chi \geq 0$). Moreover, one would think that this would be possible due to the facts that the coefficient $\widetilde{w}/w$ is continuous (so long as the solution u of (2.1) exists) and that it satisfies $d_{\min} \leq \widetilde{w}/w \leq d_{\max}$ where $d_{\max}, d_{\min}$ are the largest and smallest of the diffusion coefficients in (2.1). In summary, the question of global existence for solutions of (2.1), with only assumptions (2.2), (2.4), comes down to the following question:

Suppose that $\sigma \in C(\Omega \times (0,T))$, σ is uniformly continuous on $\Omega \times (0,\tau)$ for any $\tau \in (0,T)$, and $0 < c_0 \leq \sigma(x,t) \leq c_1 < \infty$ for all $x \in \Omega$ and $0 < t < T$. Suppose further that $1 < p < 2$, $\vartheta \in L^p_+(\Omega \times (0,T)$ and that χ is the solution in $W^{2,1}_p(\Omega \times (0,\tau))$ of

$$\frac{\partial \chi}{\partial t} - \sigma \, \Delta \chi \ = \ \vartheta \qquad \text{in } \Omega \times (0,\tau) \tag{3.6a}$$

$$\frac{\partial \chi}{\partial \mathrm{n}} \ = \ 0 \qquad \text{on } \partial\Omega \times (0,\tau) \tag{3.6b}$$

$$\chi(\,\cdot\,,0) \ = \ 0 \qquad \text{on } \Omega \tag{3.6c}$$

where $0 < \tau < T$. *Does there exist a constant* $C = C(p,T)$, *depending upon* c_0 *and* c_1 *but otherwise independent of* σ, *such that*

$$\int_\Omega \chi(x,\tau)dx \quad \leq \quad C\|\vartheta\|_{p,\Omega\times(0,T)} \tag{3.7}$$

for all $\tau \in (0,T)$*?*

At first glance, it would appear that an answer in the affirmative (and much more) follows from well-known estimates in Ladyženskaja, *et al.* [8; IV.9.1, II.3.3, II.3.4]; namely

$$\|\chi\|_{W_p^{2,1}(\Omega\times(0,T))} \;\leq\; C\|\vartheta\|_{p,\Omega\times(0,T)}, \tag{3.8}$$

from which follows

$$\|\chi(\,\cdot\,,T)\|_{p,\Omega} \;\leq\; C\|\vartheta\|_{p,\Omega\times(0,T)}. \tag{3.9}$$

However, the proof of the first of these estimates in [8] clearly requires that σ be *uniformly* continuous on $\Omega\times(0,T)$ (*i.e.,* C depends on the modulus of continuity of σ), in spite of the statement of the theorem, which indicates that σ need only be continuous on $\Omega\times(0,T)$. Note that we do not require this regularity of χ, only the estimate (3.7). We challenge those who are more talented than ourselves in this area to address the preceding question.

We should note, however, that (3.8) and (3.9) are valid for the solution of (3.6) provided that $c_1 - c_0$ is sufficiently small; that is, provided that σ is sufficiently close to a positive constant. To see this, suppose that $\sigma(x,t) = d^* + \varepsilon(x,t)$ for $(x,t) \in \Omega\times(0,T)$. Then rewrite (3.6a) as

$$\frac{\partial\chi}{\partial t} - d^*\Delta\chi \;=\; \vartheta + \varepsilon\Delta\chi \qquad \text{in } \Omega\times(0,T).$$

Now (3.8) formally yields

$$\|\chi\|_{W_p^{2,1}(\Omega\times(0,T))} \;\leq\; C\|\vartheta + \varepsilon\Delta\chi\|_{p,\Omega\times(0,T)},$$

from which the desired estimate follows by the triangle inequality provided that $C\|\varepsilon\|_{\infty,\Omega\times(0,T)} < 1$. (A simple iteration argument shows that the solution χ of this problem does indeed exist under the same condition on ε. See the proof of Lemma 4.7.) One simple upshot of this result is that if all of the diffusion coefficients in (2.1) lie within a sufficiently small interval, then global existence follows. We will also take advantage of this idea in the next section to obtain spatially local estimates for a class of quasilinear perturbations of (2.1).

4. A class of quasilinear systems

Let us now introduce the following quasilinear perturbation of (2.1):

$$\begin{aligned} \frac{\partial u_i}{\partial t} - \nabla\cdot(\delta_i(u_i)\nabla u_i) &= f_i(u) \qquad \text{in } \Omega\times\{t>0\},\ i=1,\dots,m \\ \delta_i(u_i)\frac{\partial u_i}{\partial \mathrm{n}} &= \rho_i(\gamma_i - u_i) \quad \text{on } \partial\Omega\times\{t>0\},\ i=1,\dots,m \\ u_i(\,\cdot\,,0) &= u_{0_i}(\,\cdot\,) \qquad \text{on } \Omega,\ i=1,\dots,m. \end{aligned} \tag{4.1}$$

Everything here is as in (2.1) with the exception of the nonlinear diffusivities $\delta_i(\cdot)$, which we will assume are members of $C^1([0,\infty);[a_i,b_i])$, where $0<a_i\le b_i<\infty$.

Local existence of classical, nonnegative solutions to (4.1) is well established; see *e.g.* [1]. We denote henceforth the maximal interval of existence by $[0,T^*)$ where $0<T^*\le\infty$. Moreover, it is easily verified that Lemma 2.3 remains valid for (4.1) if d_i in the expression for β is replaced with b_i.

We assume further the following property for each $\delta_i,\ i=1,\dots,m$:

$$\lim_{w\to\infty} w^{-1}\int_0^w \delta_i(\xi)d\xi = d_i. \tag{4.2}$$

Note that this condition does not imply that δ_i is asymptotically constant; *e.g.,* it is satisfied by $\delta_i(w)=a_i+(b_i-a_i)(1+\cos w)/2$ with $d_i=(a_i+b_i)/2$. The reason for this condition is that in what follows we shall make use of auxilliary functions $\sigma_i:\Omega\times(0,T^*)\to\mathbb{R}_+$ defined by

$$\sigma_i(x,t) = (u_i(x,t)+K)^{-1}\Big(\int_0^{u_i(x,t)}\delta_i(\xi)d\xi + d_iK\Big) \tag{4.3}$$

where $K>0$. The important properties of these functions are given in the following technical lemma.

Lemma 4.4. *The functions σ_i defined by (4.3) satisfy $\nabla((u_i+K)\sigma_i)=\delta_i(u_i)\nabla u_i$, and, because of (4.2), for any $\epsilon>0$ there exists $K>0$ such that $|\sigma_i(x,t)-d_i|\le\epsilon$ for all $(x,t)\in\Omega\times(0,T^*)$.*

Proof. The first assertion concerning the gradient of $(u_i+K)\sigma_i$ is a routine calculation. To see the second assertion, let $\epsilon>0$ and consider the function $\zeta:\mathbb{R}_+^2\to\mathbb{R}$ defined by

$$\zeta(u,K) = (u+K)^{-1}(\int_0^u \delta_i(\xi)d\xi + d_iK)$$

and note that for all $u,K>0$

$$|\zeta(u,K)-d_i| = \frac{u}{u+K}\Big|\frac{1}{u}\int_0^u \delta_i(\xi)d\xi - d_i\Big|,$$

from which it is easily seen that there is some $\tilde{u} > 0$, independent of K, such that $|\zeta(u, K) - d_i| < \epsilon$ for all $K > 0$ and $u \geq \tilde{u}$. Now suppose that $0 < u < \tilde{u}$. Then

$$|\zeta(u, K) - d_i| = \frac{1}{u + K}\Big|\int_0^u \delta_i(\xi)d\xi - d_i u\Big| \leq \frac{1}{K}(\int_0^{\tilde{u}} \delta(\xi)d\xi + d_i\tilde{u}),$$

and thus $|\zeta(u, K) - d_i| < \epsilon$ for all $u > 0$ if $K > \frac{1}{\epsilon}(\int_0^{\tilde{u}} \delta(\xi)d\xi + d_i\tilde{u})$. The second assertion of the lemma follows. $\square$

In this section we will prove for the system (4.1) under conditions (2.2), (2.4), and (4.2) that if one component of the solution blows up in finite time at a point x^*, then at least one other component of the solution must also blow up *at the same point*. Moreover, the same is true for infinite time blow-up if the boundary conditions are homogeneous. More precisely, we will prove the following theorem.

Theorem 4.5. *Let conditions (2.2), (2.4), and (4.2) be satisfied and let* u *be the solution of (4.1) on* $\Omega \times (0, T^*)$*. Suppose that* $T^* < \infty$ *so that there exist* $i \in \{1, \dots, m\}$ *and a sequence* $\{(x_k, t_k)\}_{k=1}^{\infty} \subset \Omega \times (0, T^*)$ *with* $\lim x_k = x^* \in \overline{\Omega}$ *and* $\lim t_k = T^*$ *such that* $\lim u_i(x_k, t_k) = \infty$*. Then there is at least one* $j \in \{1, \dots, m\}$*,* $j \neq i$*, and a corresponding sequence* $\{(\tilde{x}_k, \tilde{t}_k)\}_{k=1}^{\infty} \subset \Omega \times (0, T^*)$ *converging to the same limit* (x^*, T^*) *such that* $\lim u_j(\tilde{x}_k, \tilde{t}_k) = \infty$*. If* $T^* = \infty$ *and such an* i *and a sequence* $\{(x_k, t_k)\}_{k=1}^{\infty}$ *exist, then the same conclusion is true provided that* $\rho_j\gamma_j = 0$ *for each* $j \in \{1, \dots, m\}$*.*

We remark that what is new here (in the semilinear case) is the fact that simultaneous blow-up of two components must occur at the same point x^*. It has been known for some time (and this follows from the methods in Hollis, Martin, and Pierre [4]) that, for the semilinear system (2.1) satisfying (2.2) and (2.3), blow-up in the sup-norm of one component in time $T^* \leq \infty$ implies blow-up in the sup-norm of at least one other component in time T^*.

A central role in our proofs is played by the solution of the scalar equation

$$\begin{aligned} \frac{\partial \chi}{\partial t} - \sigma\,\Delta\chi &= \vartheta && \text{in } G \times (0, T) \\ \sigma\frac{\partial \chi}{\partial \mathrm{n}} + \beta\chi &= 0 && \text{on } \partial G \times (0, T) \\ \chi(\,\cdot\,, 0) &= 0 && \text{on } G \end{aligned} \tag{4.6}$$

where G is a bounded domain in $\mathbb{R}^n$ with smooth boundary ∂G, β is a nonnegative constant, and $\sigma : G \times (0, T) \to [a, b] \subset (0, \infty)$. We now state some well-known L^p regularity results for (4.6).

Lemma 4.7. *Let $1 < p < \infty$ and suppose that $\vartheta \in L^p_+(G \times (0,T))$. There exists an $\epsilon > 0$ such that if $0 < d - \epsilon \leq \sigma(x,t) \leq d + \epsilon$ for all $(x,t) \in G \times (0,T)$, where d is a positive constant, then (4.6) has a unique solution $\chi \in W^{2,1}_p(G \times (0,T))$ with $\chi \geq 0$. If $\|\vartheta\|_{p,G\times(0,T)} = 1$, then there exists a constant $C = C(p,T)$, independent of ϑ, such that $\|\chi\|_{W^{2,1}_p(G\times(0,T))} \leq C$. Furthermore, C can be chosen so that:*

(i) *if $p > 1$, then $\|\chi(\cdot,T)\|_{p,G} \leq C$;*

(ii) *if $p > n+2$, then $\||\nabla\chi|\|_{\infty,G\times(0,T)} \leq C$;*

(iii) *if $p > (n+2)/2$, then $\|\chi\|_{\infty,G\times(0,T)} \leq C$;*

(iv) *if $1 < p < n+1$ and $p \leq q \leq \frac{np}{n+1-p}$, then $\|\chi(\cdot,T)\|_{q,G} \leq C$;*

(v) *if $1 < p < n+2$ and $p \leq q \leq \frac{(n+2)p}{n+2-p}$, then $\|\chi\|_{W^{1,0}_q(G\times(0,T))} \leq C$;*

(vi) *if $p > 1$, then $\|\chi\|_{W^{1,0}_p(\partial G\times(0,T))} \leq C$.*

Proof. Let us assume first that $\sigma(x,t) = d > 0$ for all $(x,t) \in G \times (0,T)$. In this case we refer to section IV.9 of Ladyženskaja *et al.* [8] for the proof of the existence of the solution $\chi \in W^{2,1}_p(G \times (0,T))$ of

$$\begin{aligned} \frac{\partial\chi}{\partial t} - \sigma\,\Delta\chi &= \vartheta && \text{in } G\times(0,T) \\ \sigma\frac{\partial\chi}{\partial \mathrm{n}} + \beta\chi &= \Phi && \text{on } \partial G\times(0,T) \\ \chi(\,\cdot\,,0) &= 0 && \text{on } G \end{aligned}$$

satisfying the estimate

$$\|\chi\|_{W^{2,1}_p(G\times(0,T))} \leq C\Big(\|\vartheta\|_{p,G\times(0,T)} + \|\Phi\|_{W^{\ell,\ell/2}_p(\partial G\times(0,T))}\Big) \tag{4.8}$$

where $\ell = (2p-1)/p$. Now we assume that $0 < d - \epsilon \leq \sigma(x,t) \leq d + \epsilon$ for all $(x,t) \in G\times(0,T)$. Define the sequence $\{\chi_k\}_{k=0}^{\infty} \subset W^{2,1}_p(G\times(0,T))$ by $\chi_0 = 0$ and for $k = 0,1,2\ldots$

$$\begin{aligned} \frac{\partial\chi_{k+1}}{\partial t} - d\,\Delta\chi_{k+1} &= \vartheta + (\sigma - d)\chi_k && \text{in } G\times(0,T) \\ \frac{\partial\chi_{k+1}}{\partial \mathrm{n}} + \frac{\beta}{d}\chi_{k+1} &= (\frac{\beta}{d} - \frac{\beta}{\sigma})\chi_k && \text{on } \partial G\times(0,T) \\ \chi_{k+1}(\,\cdot\,,0) &= 0 && \text{on } G. \end{aligned}$$

With $\psi_{k+1} = \chi_{k+1} - \chi_k$, it is easily seen by applying (4.8) and the inequality [8; Lemma II.3.4]

$$\|w\|_{W^{\ell,\ell/2}_p(\partial G\times(0,T))} \leq c_0\|w\|_{W^{2,1}_p(G\times(0,T))}$$

that

$$\|\psi_{k+1}\|_{W^{2,1}_p(G\times(0,T))} \leq \tilde{c}\|\psi_k\|_{W^{2,1}_p(G\times(0,T))}$$

where

$$\tilde{c} = C \max\{\|d-\sigma\|_{\infty,G\times(0,T)},\ c_0\|\beta/d-\beta/\sigma\|_{\infty,G\times(0,T)}\}.$$

If $\tilde{c} < 1$, which clearly may be achieved by choosing ϵ sufficiently small, then it follows that the sequence $\{\chi_k\}_{k=0}^{\infty}$ converges to a limit $\chi \in W_p^{2,1}(G \times (0,T))$ satisfying the estimate

$$\|\chi\|_{W_p^{2,1}(G\times(0,T))} \le \frac{C}{1-\tilde{c}}\|\vartheta\|_{p,G\times(0,T)}.$$

Assertions (i) – (vi) now follow via embedding theorems in section II.3 of Ladyženskaja *et al.* [8]. Finally, the nonnegativity of χ is a consequence of the maximum principle. $\square$

The localized estimates needed for the proof of Theorem 4.5 are provided by the following lemmas. The proofs are essentially variations on methods applied in more general settings in Hollis and Morgan [4] and in Morgan [12], where global existence is established for such systems under an intermediate sums condition that allows for the first time quadratic intermediate sums when $n = 2$. The first of these lemmas is concerned with boundedness of solutions on the time interval $(0,T)$ with $T < \infty$.

Lemma 4.9. *Suppose that $\Omega_0 \subset \Omega_1 \subseteq \Omega$, where each Ω_k is a subdomain with smooth boundary $\partial\Omega_k$ and dist$(\Omega_0, \Omega\backslash\Omega_1) > 0$. Let u be a solution of (4.1) on $\Omega \times (0,T)$, $T < \infty$, subject to conditions (2.2), (2.4), (4.2). If there exist a constant $K_1(T)$ and an $i \in \{1,\dots,m\}$ such that $\sum_{j\neq i} u_j \le K_1(T)$ on $\Omega_1 \times (0,T)$, then there exists a constant $K_0(T)$ such that $u_i \le K_0(T)$ on $\Omega_0 \times (0,T)$. The constant $K_0(T)$ tends to infinity as dist$(\Omega_0, \Omega\backslash\Omega_1) \to 0$ but is bounded as diam$(\Omega_0) \to 0$ if dist$(\Omega_0, \Omega\backslash\Omega_1)$ is bounded away from 0.*

Proof. For simplicity we assume that each $\alpha_i = 1$ in (2.2); trivial modifications of what follows handle the more general case. Now let us assume the hypotheses of the lemma and let $\{G_k\}_{k\in\mathbb{N}}$ be a nested sequence of smooth subdomains of Ω_1 satisfying $\Omega_0 \subset G_k \subset G_{k-1} \subset \Omega_1$ and $\text{dist}(\Omega_0, \Omega\backslash G_{k-1})$, $\text{dist}(G_k, \Omega\backslash G_{k-1}) > 0$ for each $k \ge 2$. Then construct, for each natural number $k \ge 2$, a smooth function $g_k : \mathbb{R}^n \to [0,1]$ such that $g_k \equiv 0$ on $\Omega\backslash G_{k-1}$, $g_k \equiv 1$ on G_k, and $\frac{\partial g_k}{\partial \mathrm{n}} = 0$ on $\partial\Omega \cap \partial G_{k-1}$ if $\partial\Omega \cap \partial G_{k-1} \neq \emptyset$. Now with φ a nonnegative member of $W_p^{2,1}(\Omega_1 \times (0,T))$ that satisfies $\sigma_i \frac{\partial\varphi}{\partial \mathrm{n}} + \rho_i\varphi = 0$ on $\partial\Omega_1 \times (0,T)$, an elementary but tedious integration by parts exercise gives

$$
\begin{aligned}
-\int_0^T\int_{G_{k-1}}\sum_{j=1}^m(u_j+K)(\varphi_t+\sigma_i\Delta\varphi)g_k \quad\le\quad & \int_{G_{k-1}}\varphi_0 g_k\sum_{j=1}^m(u_{0_j}+K)\\
&-\int_0^T\int_{G_{k-1}}\Delta\varphi g_k\sum_{j=1}^m(\sigma_i-\sigma_j)(u_j+K)\\
&+\int_0^T\int_{G_{k-1}}(\varphi\Delta g_k+2\nabla\varphi\cdot\nabla g_k)\sum_{j=1}^m\sigma_j(u_j+K) \qquad (4.10)\\
&+\int_0^T\int_{\partial\Omega\cap\partial G_{k-1}}\Bigg(\varphi\sum_{j=1}^m\rho_j\gamma_j-\frac{\partial\varphi}{\partial \mathrm{n}}\Big(mK\sigma_i\\
&\qquad\qquad+\sum_{j\neq i}(2\sigma_i-\sigma_j)(u_j+K)\Big)\Bigg)g_k.
\end{aligned}
$$

Now if we set $\varphi_t+\sigma_i\Delta\varphi=-\vartheta\le 0$ where $\|\vartheta\|_{p,\Omega_1\times(0,T)}=1$, $1<p<\infty$, and assume that $\varphi(\cdot,T)=0$ on Ω_1, then Hölder's inequality, the nonnegativity of u and ϑ, and Lemma 4.7 with $\chi(\cdot,t)=\varphi(\cdot,T-t)$ imply that

$$
\begin{aligned}
\int_0^T\int_{G_k}\sum_{j=1}^m(u_j+K)\vartheta\le C(T)\Big(&\Big\|\sum_{j=1}^m u_{0_j}\Big\|_{\infty,G_{k-1}}+\Big\|\sum_{j\neq i}u_j\Big\|_{\infty,G_{k-1}\times(0,T)}\\
&+\Big\|\sum_{j=1}^m u_j\Big\|_{a,G_{k-1}\times(0,T)}+\sum_{j=1}^m\rho_j\gamma_j+1\Big)
\end{aligned}
\qquad (4.11)
$$

holds provided that K is sufficiently large (see (4.3) and Lemma 4.4) and that either $p>n+2$ with $a=1$ (*cf.* (ii) and (iii) in Lemma 4.7) or else $1<p<n+2$ with $p\le\frac{a}{a-1}\le\frac{(n+2)p}{n+2-p}$ (*cf.* (v) in Lemma 4.7). Thus if $p>n+2$ and $a=1$, then (4.11), together with the L^1 bound given by Lemma 2.3, puts $u_i\in L^{\frac{p}{p-1}}(G_2\times(0,T))$ for all $\frac{p}{p-1}\in(1,\frac{n+2}{n+1})$. Now, we can take $\frac{p}{p-1}\in[\frac{n+2}{n+1},\frac{n+2}{n})$ and $a=\frac{(n+2)p}{(p-1)(n+2)+p}$ so that $\frac{n+2}{2}<p<n+2$, $\frac{a}{a-1}=\frac{(n+2)p}{n+2-p}$, and $1<a<\frac{n+2}{n+1}$ and thus deduce from (4.11) and Lemma 4.7 that $u_i\in L^{\frac{p}{p-1}}(G_3\times(0,T))$ for all $\frac{p}{p-1}\in[\frac{n+2}{n+1},\frac{n+2}{n})$. In a similar manner, one can now proceed by induction to show that $u_i\in L^{((n+1)/n)^k}(G_{k+2}\times(0,T))$ for all $k\in\mathbb{N}$ by taking $a=(\frac{n+1}{n})^k$ and $\frac{np}{n+1-p}=\frac{a}{a-1}$ so that $\frac{p}{p-1}=a\frac{n+1}{n}=(\frac{n+1}{n})^{k+1}$. Note here that if $k\ge 2$ then $\frac{p}{p-1}>\frac{n+2}{n}$ and $1<p<\frac{n+2}{2}$, so $\frac{np}{n+1-p}<\frac{(n+2)p}{n+2-p}$. Now with $(\frac{n+1}{n})^k$ sufficiently large, classical interior estimates [8; Chapter III, §8] and condition (2.4) put $u_i\in L^\infty(\Omega_0\times(t_0,T))$ for any $t_0>0$. Finally, close inspection of the dependence of $C(T)$ upon the functions g_k reveals the nature of the dependence of the bounding constant $K_0(T)$ upon $\operatorname{dist}(\Omega_0,\Omega\backslash\Omega_1)$. □

The next lemma is concerned with boundedness of solutions on the time interval $(0,\infty)$.

Lemma 4.12. *Suppose that $\Omega_0 \subset \Omega_1 \subseteq \Omega$, where each Ω_k is a subdomain with smooth boundary $\partial\Omega_k$ and dist$(\Omega_0, \Omega\backslash\Omega_1) > 0$. Let u be a solution of (4.1) on $\Omega \times (0,\infty)$ subject to conditions (2.2), (2.4), and (4.2) and suppose further that $\rho_j\gamma_j = 0$ for each $j \in \{1,\ldots,m\}$. If there exist a constant K_1 and an $i \in \{1,\ldots,m\}$ such that $\sum_{j\neq i} u_j \le K_1$ on $\Omega_1 \times (0,\infty)$, then there exists a constant K_0 such that $u_i \le K_0$ on $\Omega_0 \times (0,\infty)$. The constant K_0 tends to infinity as dist$(\Omega_0, \Omega\backslash\Omega_1) \to 0$ but is bounded as diam$(\Omega_0) \to 0$ if dist$(\Omega_0, \Omega\backslash\Omega_1)$ is bounded away from 0.*

Proof. Assume the hypotheses of the lemma and let $\{G_k\}_{k\in\mathbb{N}}$ and φ be as in the proof of Lemma 4.9. A slight modification of the argument there produces

$$\begin{aligned}\int_{t_\nu}^{t_\nu+2}\int_{G_k}\sum_{j=1}^m (u_j+K)\vartheta \le C\Big(1 + \Big\|\sum_{j=1}^m u_j(\cdot,t_\nu)\Big\|_{a,G_{k-1}} \\ + \Big\|\sum_{j\neq i} u_j\Big\|_{\infty, G_{k-1}\times(t_\nu,t_\nu+2)} + \Big\|\sum_{j=1}^m u_j\Big\|_{a,G_{k-1}\times(t_\nu,t_\nu+2)}\Big)\end{aligned} \tag{4.13}$$

for any sequence $\{t_\nu\}_{\nu=1}^\infty \subset [0,\infty)$, where C is independent of ν, provided that either $p > n+2$ with $a=1$ or else $\frac{n+2}{2} < p < n+2$ with $p \le \frac{a}{a-1} \le \frac{(n+2)p}{n+2-p}$. Taking $a=1$ and $t_\nu = \nu$, we thus obtain a t-independent bound on u_i in $L^{\frac{p}{p-1}}(G_2 \times (t,t+1))$ for $1 < \frac{p}{p-1} < \frac{n+2}{n+1}$. (Note that the second and last terms in the parentheses on the right side of (4.13) are bounded independent of ν when $a=1$ because boundary conditions are homogeneous.) Now by choosing $\frac{n+2}{2} < p < n+2$ and $a = \frac{(n+2)p}{(p-1)(n+2)+p}$ we have that $\frac{a}{a-1} = \frac{(n+2)p}{n+2-p} < \frac{pn}{n+1-p}$ and $1 < a < \frac{n+2}{n+1}$. So if we choose the sequence $\{t_\nu\}_{\nu=1}^\infty$ such that $\nu - 1 < t_\nu < \nu$ and $\|\sum_{j=1}^m u_j(\cdot,t_\nu)\|_{a,G_2} \le \|\sum_{j=1}^m u_j\|_{a,G_2\times(\nu-1,\nu)}$ and bring into play item (iv) of Lemma 4.7, we obtain a t-independent bound on u_i in $L^{\frac{p}{p-1}}(G_3 \times (t,t+1))$ for $1 < \frac{p}{p-1} < \frac{n+2}{n}$. Proceeding inductively as in the proof of Lemma 4.9, we eventually obtain via classical interior estimates a t-independent bound on u_i in $L^\infty(\Omega_1 \times (t,t+1))$. ◻

In conclusion of this section, we now give the

Proof of Theorem 4.5. Assume the hypotheses of the theorem and, for the sake of contradiction, that there are a constant $K_1(T^*)$ and an open ball $B_\rho(x^*)$, with radius ρ and centered at x^*, such that $\sum_{j\neq i} u_j \le K_1(T^*)$ on $(B_\rho(x^*) \cap \Omega) \times (0,T^*)$. Now choose smooth subdomains Ω_0 and Ω_1 so that Ω_0 contains a tail of the sequence $\{x_k\}_{k=1}^\infty$, $\Omega_0 \subseteq (B_{\rho/2}(x^*) \cap \Omega) \subset \Omega_1 \subseteq (B_\rho(x^*) \cap \Omega)$, and $\mathrm{dist}(\Omega_0, \Omega\backslash\Omega_1) > \rho/4$. By Lemma 4.9 if $T^* < \infty$ and by Lemma 4.12 if $T^* = \infty$, it follows that u_i is also bounded on $\Omega_0 \times (0,T^*)$, contradicting the assumption that $\lim_{k\to\infty} u_i(x_k,t_k) = \infty$. ◻

Remark. For the sake of emphasis, we point out that in each of Lemmas 4.9 and 4.12 it is possible for Ω_0, Ω_1 and Ω to share a portion of their boundaries – it is only required that $\operatorname{dist}(\Omega_0, \Omega\backslash\Omega_1) > 0$. Consequently, the blow-up point x^* in Theorem 4.5 is indeed allowed to lie on $\partial\Omega$. This does not conflict with the boundary point blow-up of a single component of a two-component system demonstrated in [2], which resulted from imposing both Neumann and Dirichlet boundary condition types within the system. This situation is precluded here by the assumption of compatible boundary condition types throughout the system.

5. One-step reversible reactions

In this section we consider a general reaction mechanism of the form

$$\mu_1 R_1 + \mu_2 R_2 + \cdots + \mu_k R_k \quad \rightleftharpoons \quad \nu_1 P_1 + \nu_2 P_2 + \cdots + \nu_\ell P_\ell, \tag{5.1}$$

of which (1.1) is a special case. Here the R_i and P_i represent reactant and product species, respectively, and $\mu_i, \nu_i \in \mathbb{N}$ for each i. Now, if we set $u_i = [R_i]$ and $v_i = [P_i]$ and let k_f, k_r be the (nonnegative) forward and reverse reaction rates, respectively, then we may model the process with the following reaction-diffusion system (*cf.* (1.2)):

$$\frac{\partial u_i}{\partial t} - \nabla\cdot(\delta_i(u_i)\nabla u_i) = \mu_i\Big(k_r \prod_{j=1}^{\ell} v_j^{\nu_j} - k_f \prod_{j=1}^{m} u_j^{\mu_j}\Big), \quad i = 1, \ldots, m \tag{5.2a}$$

$$\frac{\partial v_i}{\partial t} - \nabla\cdot(\delta_{m+i}(v_i)\nabla v_i) = \nu_i\Big(k_f \prod_{j=1}^{m} u_j^{\mu_j} - k_r \prod_{j=1}^{\ell} v_j^{\nu_j}\Big), \quad i = 1, \ldots, \ell. \tag{5.2b}$$

We assume that the reaction takes place in a bounded domain Ω with smooth boundary $\partial\Omega$, that the δ_i are as in the preceding section, and that boundary and initial conditions of the form in (2.1) are satisfied by all components u_i, v_i.

At this point, let us make some routine observations concerning the global well-posedness of this system in the semilinear case; *e.g.,* where each $\delta_i(\cdot) \equiv d_i > 0$. First, note that condition (2.2) is satisfied, and as a result Lemma 2.3 provides a bound on the quantity $\int_\Omega(\frac{1}{m}\sum_{i=1}^m \frac{u_i}{\mu_i} + \frac{1}{\ell}\sum_{i=1}^\ell \frac{v_i}{\nu_i})dx$. If the reaction is irreversible, *i.e.,* $k_r = 0$, then the u_i are bounded *a priori* in $L^\infty(\Omega)$ by the maximum principle, and the methods of [4] then put each v_i in $L^p(\Omega \times (0, T^*))$ for all $p < \infty$ if $T^* < \infty$. Consequently, Sobolev embedding then puts each v_i in $L^\infty(\Omega \times (0, T^*))$, thus implying global existence. However, in the reversible case, the order of the intermediate sums in (2.5) is $r = \min\{\sum_{i=1}^m \mu_i,\ \sum_{i=1}^\ell \nu_i\}$, and hence (2.5) is satisfied if $r < r_{\max}(n)$. Consequently, the results of Morgan [10, 12] refered to in section 2 yield global existence for the system provided that $r = 1$ or that

$r \leq 2$ with $n \leq 2$. The results of Kanel' [7] imply global existence when $n = 1$ provided that $\max\{\sum_{i=1}^{m} \mu_i,\ \sum_{i=1}^{\ell} \nu_i\} \leq 3$. This is indicative of the very restrictive nature of the intermediate sums condition and the limited scope of the known results in this area.

A recent result of Fitzgibbon, Hollis, and Morgan [3] shows, in the case where all components satisfy homogeneous Neumann boundary conditions, that the zero equilibrium point is locally stable for (5.2). This shows that solutions exist globally, provided that the initial data are sufficiently near zero in $L^\infty(\Omega)$. Moreover, this new local stability result applies also to (2.1) (if $f(0) = 0$) assuming only (2.2) and that the f_i are locally Lipschitz; *i.e.*, without the usual polynomial growth assumptions. Local (asymptotic) stability follows from standard linearized stability theory if the boundary conditions are homogeneous Dirichlet or Robin type, but it is quite non-trivial in the Neumann case.

We now return our attention to the quasilinear system (5.2) subject to condition (4.2). As previously noted, our primary interest here is in what can be said about these systems when the only conditions placed on the reaction functions are (2.2), (2.4) and quasipositivity. Hence, in what follows, we make no further assumptions on the coefficients μ_i, ν_i in (5.2). Theorem 4.5 states that if one component of the solution to (5.2) blows up in time $T^* \leq \infty$ at a point x^*, then so does at least one other component. We will show, moreover, that due to the special structure of (5.2), if one component of the solution blows up in time $T^* \leq \infty$ at a point x^*, then so does *every* component. Toward this end, we need the following lemma.

Lemma 5.3. *Assume that conditon (4.2) holds and let (u, v) be a solution of (5.2) for $0 < t < T \leq \infty$. Let $x_0 \in \overline{\Omega}$ and suppose that there exist $\varepsilon > 0$, some $i \in \{1, \ldots, m\}$, and a constant $C_1(T)$ such that $u_i \leq C_1(T)$ on $(B_\varepsilon(x_0) \cap \Omega) \times (0, T)$, where $B_\varepsilon(x_0)$ is the open ball of radius ε centered at x_0. If $T = \infty$, suppose further that the boundary condition for each u_j and v_j is homogeneous. Then there is a constant $C_2(T)$ such that $\sum_{j=1}^{m} u_j + \sum_{j=1}^{\ell} v_j \leq C_2(T)$ on $(B_{\varepsilon/2}(x_0) \cap \Omega) \times (0, T)$. The same result is true if u_i is replaced with v_i for some $i \in \{1, \ldots, \ell\}$.*

Proof. First consider the case $T < \infty$. Suppose that $i \in \{1, \ldots, m\}$ and $u_i \leq C_1(T)$ on $(B_\varepsilon(x_0) \cap \Omega) \times (0, T)$. For any $j \in \{1, \ldots, \ell\}$, we have

$$\frac{\partial}{\partial t}(\nu_j u_i + \mu_i v_j) - \nabla \cdot (\nu_j \delta_i(u_i) \nabla u_i + \mu_i \delta_{m+j}(v_j) \nabla v_j) = 0.$$

Hence, arguments similar to those in the proof of Lemma 4.9 show that $v_j \in L^p((B_{5\varepsilon/6}(x_0) \cap \Omega) \times (0, T))$ for all $p < \infty$. Now by the same argument, all of the u_i's are in $L^p((B_{2\varepsilon/3}(x_0) \cap \Omega) \times (0, T))$ for all $p < \infty$. Hence the same is true for each of the reaction functions on the right side of (5.2). Classical interior estimates for parabolic equations [8] now put each u_i and v_i in $L^\infty((B_{\varepsilon/2}(x_0) \cap \Omega) \times (0, T))$. For the case $T = \infty$, proceeding in a manner similar to that in the proof of Lemma 4.12 produces t-independent bounds on each of the u_i and v_i in $L^\infty((B_{\varepsilon/2}(x_0) \cap \Omega) \times (t, t+1))$. $\square$

From this lemma follows

Theorem 5.4. *Suppose that* (u, v) *is a solution of (5.2) for* $0 < t < T^* < \infty$ *and that there exists a sequence* $\{(x_k, t_k)\}_{k=1}^{\infty} \subset \Omega \times (0, T^*)$ *with* $\lim x_k = x^* \in \overline{\Omega}$ *and* $\lim t_k = T^*$ *along which some component of the solution tends to infinity. Then every component of the solution tends to infinity along some sequence in* $\Omega \times (0, T^*)$ *that converges to the same point* (x^*, T^*)*. The same is true with* $T^* = \infty$ *provided that all boundary conditions are homogeneous.*

We should remark here that we know of no one who believes that it is possible for solutions of these systems to blow up in finite time. Indeed, we believe that conditions (2.2), (2.4) should imply global existence of solutions in the semilinear case. The results of this section should be taken as evidence of the extreme pathology inherent in finite time blow-up for such systems rather than indications of possible behavior.

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