OPTIMAL BOUNDARY CONTROL OF THE STOKES FLUIDS WITH POINT VELOCITY OBSERVATIONS*

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Abstract. This paper studies constrained LQR problems in distributed boundary control systems governed by the Stokes equation with point velocity observations. Although the objective function is not well-defined, we are able to use hydrostatic potential theory and a variational inequality in a Banach space setting to derive a first order optimality condition and then a characterization formula of the optimal control. Since matrix-valued singularities appear in the optimal control, a singularity decomposition formula is also established, with which the nature of the singularities is clearly exhibited. It is found that in general, the optimal control is not defined at observation points. A necessary and sufficient condition that the optimal control is defined at observation points is then proved.

Key words. LQR, Stokes fluid, distributed boundary control, point observation, hydrostatic potential, BIE, singularity decomposition.

AMS subject classifications. 49N10,49J20,76D07,76D10,93C20,65N38

1. Introduction.

In this paper, we are concerned with the problems in boundary control of fluid flows. We consider the following constrained optimal boundary control problems in the systems governed by the Stokes equation with point velocity observations.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary Γ, Γ_1 an open subset of Γ and $\Gamma_0 = \Gamma \setminus \Gamma_1$.

$$\text{(LQR)} \quad \begin{cases} \min_{\vec{u}\in\mathcal{U}} J(\vec{u}) = \sum_{k=1}^{m} \mu_k |\vec{w}(P_k) - \vec{Z}_k|^2 + \gamma \int_{\Gamma_1} |\vec{u}(x)|^2 d\sigma_x, \\ \text{subject to (1.1)} \begin{cases} \nu\Delta\vec{w}(x) - \nabla p(x) = 0, & \text{in } \Omega, \\ \operatorname{div} \vec{w}(x) = 0, & \text{in } \Omega, \\ \vec{\tau}(\vec{w})(x) = \vec{g}(x), & \text{on } \Gamma_0, \\ \vec{\tau}(\vec{w})(x) = \vec{u}(x), & \text{on } \Gamma_1, \end{cases} \end{cases}$$

where

$\vec{w}(x)$ is the velocity vector of the fluid at $x \in \Omega$;
$p(x)$ is the pressure of the fluid at $x \in \Omega$;
$\vec{\tau}(\vec{w})(x)$ is the surface stress of the fluid along Γ defined by

$$\vec{\tau}(\vec{w})(x) = (\tau_1(\vec{w})(x), \tau_2(\vec{w})(x), \tau_3(\vec{w})(x))^T,$$

$$\tau_i(\vec{w})(x) = \sum_{k=1}^{3} \left[\frac{\partial w_i(x)}{\partial x_k} + \frac{\partial w_k(x)}{\partial x_i} \right] n_k(x) - p(x) n_i(x);$$

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$\vec{n}(x)$ is the unit outnormal vector of Γ at x;
$\vec{g}$ is a given (surface stress) Neumann boundary data (B.D.) on Γ_0;
$\vec{u}(x) \in \mathcal{U}$ is the (surface stress) Neumann boundary control on the surface Γ_1;
$\mathcal{U}$ is the admissible control set to be defined later for well-posedness of the problem and for applications;
$\gamma, \mu_k > 0, 1 \leq k \leq m$, are given weighting factors;
$P_k \in \Gamma, 1 \leq k \leq m$, are prescribed "observation points";
$Z_k \in \mathbb{R}^3, 1 \leq k \leq m$, are prescribed "target values" at P_k;
ν, a positive quantity, is the kinematic viscosity of the fluid. For simplicity, throughout this paper we assume that $\nu = 1$ and the density of the fluid is the constant one.

Let

$$M_0 = \{\vec{a} + \vec{b} \times \vec{x} \mid \vec{a}, \vec{b} \in \mathbb{R}^3\}, \tag{1.2}$$

which is the subspace of the rigid body motions in $\mathbb{R}^3$. Multiplying the Stokes equation by $\vec{a} + \vec{b} \times \vec{x} \in M_0$ and integration by parts yield the compatibility condition of the Stokes system, i.e.,

$$\int_\Gamma \vec{\tau}(\vec{w})(x) \cdot (\vec{a} + \vec{b} \times \vec{x}) d\sigma_x = 0,$$

or

$$\vec{\tau}(\vec{w}) - M_0.$$

For $q \geq 1$, let A be a subspace of $(L^q(\Gamma))^3$ and denote

$$(L^q(\Gamma))^3_{\perp A} = \{\vec{f} \in (L^q(\Gamma))^3 | \vec{f} - A\}.$$

The Stokes equation (1.1) describes the steady state of an incompressible viscous fluid with low velocity in $\mathbb{R}^3$. It is a frequently used model in fluid mechanics. It is also an interesting model in linear elastostatics due to its similarities. During the past years, considerable attention has been given to the problem of active control of fluid flows (see [1, 2, 7, 18, 19] and references therein). This interest is motivated by a number of potential applications such as control of separation, combustion, fluid-structure interaction, and super maneuverable aircraft. In the study of those control problems and Navier-Stokes equations, the Stokes equations, which describe the slow steady flow of a viscous fluid, play an important role because of the needs in stability analysis, iterative computation of numerical solutions, boundary control and etc.. The theoretical and numerical discussion of the Stokes equations on smooth or Lipschitz domains can be found from [14, 16, 17, 22, 25, 26, 27].

Our objective of this paper is to find the optimal surface stress $\vec{u}(x)$ on Γ_1, which yields a desired velocity distribution $\vec{w}(x)$, s.t. at observation points $P_k, 1 \leq k \leq m$, the observation values $\vec{w}(P_k)$ are as close as possible to the target values Z_k with a least possible control cost $\int_{\Gamma_1} |\vec{u}(x)|^2 d\sigma_x$, which arise from the contemporary fluid control problems in the fluid mechanics.

Notice that point observations are assumed in the problem setting, because they are much easier to be realized in applications than distributed observations. They can be used in modeling contemporary "smart sensors".

Sensors can be used in boundary control systems (BCS) governed by partial differential equations (PDE) to provide information on the state as a feedback to the systems. According to the space-measure of the data that sensors can detect, sensors can be divided into two types, point sensors and distributed sensors. Point sensors are much more realistic and easier to design than distributed sensors. In contemporary "smart materials", piezoelectric or fiber-optic sensors (called *smart sensors*) can be embedded to measure deformation, temperature, strain, pressure,...,etc. Each smart sensor detects only the *average* of the data in between the sensor and its size can be less than $10^{\perp 6}$m [29],[30],[24]. So in any sense, they should be treated as point sensors. As a matter of fact, so far *distributed sensors* have not been used in any real applications, to the best of our knowledge. However, once point observations on the boundary are used in a BCS, singularities will appear and very often the system becomes ill-posed. Mathematically and numerically, it becomes very tough to handle. On the other hand, when point observations are used in the problem setting, the state variable has to be continuous, so the regularity of the state variable stronger than the one in the case of distributed observations is required. The fact is that in the literature of related optimal control theory, starting from the classic book [23] by J.L. Lions until recent papers [3],[4] by E. Casas and others, distributed observations are always assumed and the optimal controls are characterized by an adjoint system. The system is then solved numerically by typically a finite-element method, which cannot efficiently tackle the singularity in the optimal control along the boundary.

On the other hand, since it is important in the optimal control theory to obtain a state-feedback characterization of the optimal control, with the bound constraints in the system, the Lagrange-Kuhn-Tucher approach is not desirable because theoretically it cannot provide us with a state-feedback characterization of the optimal control which is important in our regularity/singularity analysis of the optimal control and numerically it leads to a numerical algorithm to solve an optimization problem with a huge number of inequality constraints. A refinement of the boundary will double the number of the inequality constraints, so the numerical algorithm will be sensitive to the partition number of the boundary. Since the BCS is governed by a PDE system in $\mathbb{R}^3$, the partition number of the boundary can be very large, any numerical algorithm sensitive to the partition number of the boundary may fail to carry out numerical computation or provide reliable numerical solutions.

Recently in the study of a linear quadratic BCS governed by the Laplace equation with point observations, the potential theory and boundary integral equations (BIE) have been applied in [20],[10],[11], [12] to derive a characterization of the optimal control in terms of the optimal state directly and therefore bypass the adjoint system. This approach shows certain important advantages over others. It provides rather explicit information on the control and the state, and it is amenable to direct numerical computation through a boundary element method (BEM), which can efficiently tackle the singularities in the optimal control along the boundary.

In [10],[11],[9] several regularity results are obtained. The optimal control is characterized directly in terms of the optimal state. The exact nature of the singularities in the optimal control is exhibited through a decomposition formula. Based on the characterization formula, numerical algorithms are also developed to approximate the optimal control. Their insensitivity to the discretization of the boundary and fast uniform convergences are mathematically verified in [12],[31].

The case with the Stokes system is much more complicate than the one with the Laplace equation due to the fact that the fundamental solution of the Stokes system is matrix-valued and has rougher singular behaviors. In this paper, we assume that the control is active on a part of the surface and the control variable is bounded by two vector-valued functions. A Banach space setting has been used in our approach, we first prove a necessary and sufficient condition for a variational inequality problem (VIP) which leads to a first order optimality condition of our original optimization problem. A characterization of the optimal control and its singularity decomposition formula are then established. Our approach can be easily adopted to handle other cases and it shows the essence of the characterization of the optimal control, through which gradient related numerical algorithms can be designed to approximate the optimal control.

The organization of this paper is as follows: In the rest of Section 1, we introduce some basic definitions and known regularity results that are required in the later development; In Section 2, we first prove an existence theorem for an orthogonal projection, next we derive a characterization result for a variational inequality which serves as a first order optimality condition to our LQR problem; then a state-feedback characterization of the optimal control is established. Section 3 will be devoted to study regularity/singularity of the optimal control. Since the optimal control contains a singular term, we first derive a singularity decomposition formula for the optimal control, with which we find that in general the optimal control is not defined at observation points. A necessary and sufficient condition that the optimal control is defined at observation points is then established. Some other regularities of the optimal control will also be studied in this section. Based upon our characterization formulas a numerical algorithm, in a subsequent paper, we design a Conditioned Gradient Projection Method (CGPM)) to approximated the optimal control. Numerical analysis for its (uniform) convergence and (uniform) convergence rate are presented there. We show that CGPM converges uniformly sub-exponentially, i.e., faster than any integer power of $\frac{1}{n}$. Therefore CGPM is insensitive to discretization of the boundary. The insensitivity of our numerical algorithm to discretization of boundary is a significant advantage over other numerical algorithms. Since the fundamental solution of the Stokes system is matrix valued with a very rough singular behavior, numerical analysis is also much more complicated than the case with scalar-valued fundamental solution, e.g., the Laplacian equation.

Let us now briefly recall some hydrostatic potential theory, BEM and some known regularity results. Throughout of this paper, for a sequence of elements in $\mathbb{R}^n$, we use superscript to denote sequential index and subscript to denote components, e.g., $\{x^k\} \subset \mathbb{R}^n$ and $x^k = (x_1^k, \cdots, x_n^k)$. We may also use $\vec{x}^k$ to emphasize that x^k is a vector. We may write $\vec{w}(x, \vec{u})$ to indicate that the velocity $\vec{w}$ depends also on $\vec{u}$. Unless stated otherwise, we assume $p > 2, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$ and $\|\cdot\|$ is the norm in $(L^h(\Gamma))^n (h \geq 1)$.

Let $\{E(x,\xi), \vec{e}(x,\xi)\} = \{[E_{ij}(x,\xi)]_{3\times 3}, [e_i(x,\xi)]_{3\times 1}\}$ be the fundamental solution of the Stokes systems, i.e.

$$\begin{cases} \Delta_x E(x,\xi) - \nabla_x \vec{e}(x,\xi) = -\delta(x-\xi) I_3, \\ \mathrm{div}_x E(x,\xi) = 0 \end{cases} \tag{1.3}$$

where $\delta(x-\xi)$ is the unit Dirac delta function at $x = \xi$ and I_3 is the 3×3 identity matrix. It is known [22] that

$$E_{ij}(x,\xi) = \frac{1}{8\pi}\left(\frac{\delta_{ij}}{|x-\xi|} + \frac{(x_i - \xi_i)(x_j - \xi_j)}{|x-\xi|^3}\right), \quad 1 \leq i, j \leq 3,$$

$$e_i(x,\xi) = \frac{1}{4\pi}\frac{x_i - \xi_i}{|x-\xi|^3},$$

where $\delta_{i,j}$ is the Kronecker symbol.

REMARK 1. The significant difference between the case with point observations and the case with distributed observations is as follows: for a given vector $\vec{V} \in \mathbb{R}^3$ the function

$$x \to \sum_{k=1}^{m} \mu_k E(P_k, x)\vec{V} \tag{1.4}$$

has a singularity of order $O(\frac{1}{|x \perp P_k|})$ at $x = P_k$ and however it may oscillate between $-\infty$ and $+\infty$ as $x \to P_k$, so it is very tough to deal with. Whereas the function

$$x \to \int_{\Gamma_0} E(\xi, x)\vec{V} d\sigma_\xi \tag{1.5}$$

is well-defined and continuous.

On the other hand, if $E(P_k, x)$ in (1.4) and (1.5) is replaced by the fundamental solution of the Laplace equation, in this case, $E(P_k, x)$ becomes scalar-valued, then (1.4) has the same order $O(\frac{1}{|x \perp P_k|})$ of singularity at $x = P_k$, but the limit as $x \to P_k$ exists (including $-\infty$ or $+\infty$). So the singularity can be easily handled.

It is then known that the solution $(\vec{w}, p)$ of the Stokes equation (1.1) has a simple-layer representation

$$\vec{w}(x) = \int_\Gamma E(x,\xi)\vec{\eta}(\xi) d\sigma_\xi + \vec{a} + \vec{b} \times \vec{x} \quad \forall\, x \in \overline{\Omega}, \tag{1.6}$$

$$p(x) = \int_\Gamma \vec{e}(x,\xi) \cdot \vec{\eta}(\xi) d\sigma_\xi + a \quad \forall\, x \in \Omega, \tag{1.7}$$

for some constants $\vec{a}, \vec{b} \in \mathbb{R}^3$ and $a \in \mathbb{R}$. $\vec{\eta}$ is called the layer density and $\vec{a} + \vec{b} \times \vec{x}$ represents a rigid body motion. By the jump property of the layer potentials, we obtain the boundary integral equation

$$\begin{aligned} \vec{\tau}(\vec{w})(x) &= \frac{1}{2}\vec{\eta}(x) + \text{p.v.} \int_\Gamma T(x,\xi)\vec{\eta}(\xi) d\sigma_\xi \quad \forall\, x \in \Gamma, \\ &= \frac{1}{2}\vec{\eta}(x) + \lim_{\varepsilon \to 0^+} \int_{\Gamma \setminus B(x,\varepsilon)} T(x,\xi)\vec{\eta}(\xi) d\sigma_\xi \quad \forall\, x \in \Gamma, \end{aligned} \tag{1.8}$$

where

$$T(x,\xi) = [\vec{\tau}_x(E_1)(x,\xi), \vec{\tau}_x(E_2)(x,\xi), \vec{\tau}_x(E_3)(x,\xi)] = [T_{ij}(x,\xi)]_{3\times 3},$$

$$T_{ij}(x,\xi) = -\frac{3}{4\pi}\frac{(x_i - \xi_i)(x_j - \xi_j)}{|x-\xi|^5}(x-\xi) \cdot \vec{n}_x.$$

With a given Neumann B.D., the layer density $\vec{\eta}$ can be solved from the above BIE (1.8). Once the layer density is known, the solution $(\vec{w}(x), p(x))$ can be computed from (1.6) and (1.7). The velocity solution $\vec{w}(x)$ is unique only up to a rigid body motion and the pressure solution $p(x)$ is unique up to a constant.

In BEM, the boundary $\Gamma = \Gamma_1 \bigcup \Gamma_0$ is divided into N elements with nodal points x_i. Assume that the layer density $\vec{\eta}(x)$ is piecewise smooth, e.g. piecewise constant, piecewise linear,$\cdots$, etc., then the BIE (1.8) becomes a linear algebraic system. This system can be solved for $\vec{\eta}(x_i)$ and then $(\vec{w}(x), p(x))$ can be computed from a discretized version of (1.6) and (1.7) for any $x \in \overline{\Omega}$.

For each $\vec{f} \in (L^2(\Gamma))^3$ and $x \in \mathbb{R}^3$, we define the simple layer potential of velocity $\mathcal{S}_v(\vec{f})$ by

$$\mathcal{S}_v(\vec{f})(x) = \int_\Gamma E(x,\xi)\vec{f}(\xi)d\sigma_\xi.$$

For each $\vec{f} \in (L^2(\Gamma))^3$ and $x \in \Gamma$, we define the boundary operators $\mathcal{K}$ and $\mathcal{K}^*$ by

$$\begin{aligned}\mathcal{K}(\vec{f})(x) &= \text{p.v.} \int_\Gamma Q(x,\xi)\vec{f}(\xi)d\sigma_\xi \\ &= \lim_{\varepsilon \to o^+} \int_{\Gamma \setminus B(x,\varepsilon)} Q(x,\xi)\vec{f}(\xi)d\sigma_\xi, \\ \mathcal{K}^*(\vec{f})(x) &= \text{p.v.} \int_\Gamma T(x,\xi)\vec{f}(\xi)d\sigma_\xi \\ &= \lim_{\varepsilon \to o^+} \int_{\Gamma \setminus B(x,\varepsilon)} T(x,\xi)\vec{f}(\xi)d\sigma_\xi,\end{aligned}$$

where

$$Q(x,\xi) = [\vec{\tau}_\xi(E_1)(x,\xi), \vec{\tau}_\xi(E_2)(x,\xi), \vec{\tau}_\xi(E_3)(x,\xi)] = [Q_{ij}(x,\xi)]_{3\times 3},$$

$$Q_{ij}(x,\xi) = \frac{3}{4\pi}\frac{(x_i-\xi_i)(x_j-\xi_j)}{|x-\xi|^5}(x-\xi)\cdot\vec{n}_\xi.$$

Next we collect some regularity results on $\mathcal{S}_v, \mathcal{K}$ and $\mathcal{K}^*$ into a lemma. Let

$$N = \ker\,(\frac{1}{2}I + \mathcal{K}^*),$$

which represents the set of all layer densities corresponding to the zero Neumann B.D., with which the Stokes system has only a rigid body motion. Hence we have

$$M_0 = \mathcal{S}_v(N) = \ker\,(\frac{1}{2}I + \mathcal{K}). \tag{1.9}$$

LEMMA 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain with smooth boundary Γ.*

(a) $\mathcal{S}_v : (L^p(\Gamma))^3 \mapsto (C^{0,\alpha}(\mathbb{R}^3))^3$ is a bounded linear operator for $p > 2$ and $0 < \alpha < \frac{p\perp 2}{p}$;

(b) For any $1 \le p < +\infty, \mathcal{K}\ (\mathcal{K}^*) : (L^p(\Gamma))^3 \mapsto (L^p(\Gamma))^3$ is a bounded linear operator and $\mathcal{K}\ (\mathcal{K}^*)$ is the adjoint of $\mathcal{K}^*\ (\mathcal{K})$;

(c) For $p > 2$ and $0 < \alpha < \frac{p\perp 2}{p}$, $\mathcal{K} : (L^p(\Gamma))^3 \mapsto (C^{0,\alpha}(\Gamma))^3$ is a bounded linear operator;

(d) For $1 < p < \infty$

(1) $(\frac{1}{2}I + \mathcal{K}^*) : (L^p(\Gamma))^3_{\perp M_0} \mapsto (L^p(\Gamma))^3_{\perp M_0}$ is invertible,

(2) $(\frac{1}{2}I + \mathcal{K}) : (L^p(\Gamma))^3_{\perp M_0} \mapsto (L^p(\Gamma))^3_{\perp N}$ is invertible.

(e) *For* $1 < q < 2$ *and* $s < \frac{2q}{2-q}$, $\mathcal{K} : (L^q(\Gamma))^3 \mapsto (L^s(\Gamma))^3$ *is a bounded linear operator. Therefore* $\mathcal{K} \circ \mathcal{K} : (L^q(\Gamma))^3 \mapsto (C^{0,\alpha}(\Gamma))^3$ *for every* $q > 1$ *and* $0 < \alpha < \frac{q-1}{q}$*;*

(f) $(\frac{1}{2}I + \mathcal{K}) : (C(\Gamma))^3_{\perp M_0} \mapsto (C(\Gamma))^3_{\perp N}$ *is invertible.*

Proof. (a)–(d) can be found from [5],[8], [13], [14] and [22].

To prove (e), since $\Gamma \subset \mathbb{R}^3$ is a compact set, it suffices to prove (e) for $q < s < \frac{2q}{2-q}$. Then we have $\frac{1}{q} > \frac{1}{s} > \frac{1}{q} - \frac{1}{2} = \frac{1}{2} + \frac{1}{q} - 1$. There exists an $\varepsilon \in (0,1)$, s.t. $\frac{1}{s} = \frac{1}{2-\varepsilon} + \frac{1}{q} - 1$. Let $r = 2 - \varepsilon$, $\alpha = \frac{r'}{s'}$, $\beta = \frac{q'}{s'}$, where r', q', s' are the conjugates of r, q, s, respectively. It can be verified that $1 < r < 2$ and

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1, \quad (1 - \frac{q}{s})s' = \frac{q}{\alpha}, \quad (1 - \frac{r}{s})s' = \frac{r}{\beta}, \quad \frac{1}{\alpha} \cdot \frac{s}{s'} = \frac{s-q}{q}, \quad \frac{1}{\beta} \cdot \frac{s}{s'} = \frac{s-r}{r}.$$

Note

$$|Q_{ij}(x,\xi)| \le \frac{C}{|x-\xi|}, \quad 1 \le i,j \le 3 \tag{1.10}$$

and

$$\left(\int_\Gamma \frac{1}{|x-\xi|^r} \, d\sigma_\xi \right) < M < \infty, \quad \forall x \in \Gamma$$

where M is a constant independent of $x \in \Gamma$. Let $h(x) = \mathcal{K}(\vec{f})(x)$. Applying Hölder's inequality twice, we get

$$\begin{aligned}
|h(x)|^s &\le C^s \left(\int_\Gamma \frac{1}{|x-\xi|} |\vec{f}(\xi)| \, d\sigma_\xi \right)^s \\
&\le C^s \left(\int_\Gamma (\frac{1}{|x-\xi|})^{\frac{r}{s}} |\vec{f}(\xi)|^{\frac{q}{s}} (\frac{1}{|x-\xi|})^{1-\frac{r}{s}} |\vec{f}(\xi)|^{1-\frac{q}{s}} \, d\sigma_\xi \right)^s \\
&\le C^s \left(\int_\Gamma \frac{1}{|x-\xi|^r} |\vec{f}(\xi)|^q \, d\sigma_\xi \right) \left(\int_\Gamma (\frac{1}{|x-\xi|})^{\frac{r}{\beta}} |\vec{f}(\xi)|^{\frac{q}{\alpha}} \, d\sigma_\xi \right)^{\frac{s}{s'}} \\
&\le C^s \left(\int_\Gamma \frac{1}{|x-\xi|^r} |\vec{f}(\xi)|^q \, d\sigma_\xi \right) \left(\int_\Gamma \frac{1}{|x-\xi|^r} \, d\sigma_\xi \right)^{\frac{s-r}{r}} \left(\int_\Gamma |\vec{f}(\xi)|^q \, d\sigma_\xi \right)^{\frac{s-q}{q}} \\
&\le C^s M^{s-r} \left(\int_\Gamma \frac{1}{|x-\xi|^r} |\vec{f}(\xi)|^q \, d\sigma_\xi \right) \cdot \|\vec{f}\|_q^{s-q}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|h\|_{L^s(\Gamma)} &= \left(\int_\Gamma |h(x)|^s \, d\sigma_x \right)^{\frac{1}{s}} \\
&\le C M^{\frac{s-r}{s}} \left(\int_\Gamma \int_\Gamma \frac{1}{|x-\xi|^r} |\vec{f}(\xi)|^q \, d\sigma_\xi d\sigma_x \right)^{\frac{1}{s}} \cdot \|\vec{f}\|_q^{\frac{s-q}{s}} \\
&\le C M \|\vec{f}\|_q.
\end{aligned}$$

This proves the first part of (e). The second part follows from (c).

To prove (f), by (1.10), $Q_{ij}(x,\xi)$ is weakly singular for $1 \le i,j \le 3$. Thus $\mathcal{K}$ is an integral operator with weakly singular kernel. By Theorem 2.22 in [21], $\mathcal{K}$ is a compact operator from $(C(\Gamma))^3$ to $(C(\Gamma))^3$. The rest follows from the Fredholm alternative (see [21], p.44). ∎

For a given Neumann B.D. $\vec{g} \in (L^p(\Gamma_0))^3$, we extend our control bound constraints $Bl, Bu \in (L^p(\Gamma_1))^3$ to the entire boundary Γ by

$$Bl(x) = \begin{cases} Bl(x) & x \in \Gamma_1 \\ \vec{g}(x) & x \in \Gamma_0 \end{cases} \quad \text{and} \quad Bu(x) = \begin{cases} Bu(x) & x \in \Gamma_1 \\ \vec{g}(x) & x \in \Gamma_0 \end{cases},$$

with

$$Bl(x) \le -\vec{B} < \vec{B} \le Bu(x) \quad \forall\, x \in \Gamma_1,$$

where $\vec{B} > 0$ is a constant vector depending on $\vec{g}$ and will be specified later. Define the feasible control set

$$\mathcal{U} = \left\{ \vec{u} \in (L^p(\Gamma))^3 \mid Bl(x) \le \vec{u}(x) \le Bu(x),\ \forall\, x \in \Gamma \text{ and } \vec{u} - M_0 \right\}, \tag{1.11}$$

where $\vec{u} - M_0$ stands for the compatibility condition of the Neumann B.D. in the Stokes system (1.1). It is clear that $\mathcal{U}$ is a closed bounded convex set in $(L^p(\Gamma))^3$.

According to Lemma 1.1 (a), for each given Neumann B.D. $\vec{u} \in \mathcal{U}$, the Stokes system (1.1) has a solution $\vec{w}$ in $(C(\overline{\Omega}))^3$ unique up to a vector $\vec{a} + \vec{b} \times \vec{x} \in M_0$, i.e.,

$$\vec{w}(x, \vec{u}) = \mathcal{S}_v \circ (\frac{1}{2} I + \mathcal{K}^*)^{\perp 1}(\vec{u})(x) + \vec{a} + \vec{b} \times \vec{x}, \quad x \in \Omega, \tag{1.12}$$

$$= \vec{w}_0(x, \vec{u}) + \vec{a} + \vec{b} \times \vec{x}, \quad x \in \Omega, \tag{1.13}$$

where

$$\vec{w}_0(x, \vec{u}) = \mathcal{S}_v \circ (\frac{1}{2} I + \mathcal{K}^*)^{\perp 1}(\vec{u})(x). \tag{1.14}$$

That is, for each given $\vec{u}$, the velocity state variable $\vec{w}$ is multiple-valued, so the objective function $J(\vec{u})$ is not well-defined. However among all these velocity solutions, there is a unique solution $\vec{w}$ s.t.

$$\sum_{k=1}^{m} \mu_k |\vec{w}(P_k) - \vec{Z}_k|^2 = \min_{\vec{h} \in M_0} \sum_{k=1}^{m} \mu_k |\vec{w}_0(P_k) + \vec{h}(P_k) - \vec{Z}_k|^2. \tag{1.15}$$

A direct calculation yields that $\vec{w}(x) = \vec{w}_0(x) + \vec{a} + \vec{b} \times \vec{x}$ must satisfy

$$\begin{cases} \sum_{k=1}^{m} \mu_k(\vec{w}_0(P_k) + \vec{a} + \vec{b} \times \vec{P}_k - \vec{Z}_k) & = & 0, \\ \sum_{k=1}^{m} \mu_k(\vec{w}_0(P_k) + \vec{a} + \vec{b} \times \vec{P}_k - \vec{Z}_k) \times \vec{P}_k & = & 0. \end{cases} \tag{1.16}$$

Since such a $\vec{w}$ is unique and continuous, the point observations $\vec{w}(P_k)$ in our LQR problem setting make sense and the LQR problem is well-posed.

From (1.14) and Lemma 1.1, we know

$$|\vec{w}(x, \vec{u}) - \vec{a}_u - \vec{b}_u \times \vec{x}| = |\vec{w}_0(x, \vec{u})| \le C \|\vec{u}\|_{L^p(\Gamma))^3} \tag{1.17}$$

where C is a constant depending only on Γ. Let us observe (1.16). If we notice that $\vec{w}_0(x, \vec{u})$ is linear in $\vec{u}$, then we have

LEMMA 1.2. *Let $\vec{a}_0, \vec{b}_0 \in \mathbb{R}^3$ be the unique solution to*

$$\begin{cases} \vec{a}_0 (\sum_{k=1}^{m} \mu_k) + \vec{b}_0 \times (\sum_{k=1}^{n} \mu_k \vec{P}_k) & = & \sum_{k=1}^{m} \mu_k \vec{Z}_k \\ \vec{a}_0 \times (\sum_{k=1}^{m} \mu_k \vec{P}_k) + \sum_{k=1}^{m} \mu_k (\vec{b}_0 \times \vec{P}_k) \times \vec{P}_k & = & \sum_{k=1}^{m} \mu_k \vec{Z}_k \times \vec{P}_k. \end{cases}$$

Then for $\vec{u}_1, \vec{u}_2 \in \mathcal{U}$ *and* $t_1, t_2 \in \mathbb{R}$,

$$\vec{w}(x, t_1\vec{u}_1 + t_2\vec{u}_2) = t_1\vec{w}(x,\vec{u}_1) + t_2\vec{w}(x,\vec{u}_2) + (1 - t_1 - t_2)(\vec{a}_0 + \vec{b}_0 \times \vec{x}) \tag{1.18}$$

and

$$|\vec{w}(x,\vec{u}_1) - \vec{w}(x,\vec{u}_2)| \le C\|\vec{u}_1 - \vec{u}_2\|_{(L^p(\Gamma))^3}, \tag{1.19}$$

where C *is a constant depending only on* Γ.

2. Characterization of the Optimal Control.

We establish an optimality condition of the LQR problem through a variational inequality problem (VIP). The characterization of the optimal control is then derived from the optimality condition.

In optimal control theory it is important to obtain a state-feedback characterization of the optimal control, i.e., the optimal control is stated as an explicit function of the optimal state. So the optimal control can be determined by a physical measurement of the optimal state. Our efforts are devoted to derive such a result.

For each $\vec{f} \in (L^1(\Gamma))^3$, we define the vector-valued truncation function

$$\left[\vec{f}\right]_{Bl}^{Bu} = \left\{[f_i(x)]_{Bl_i(x)}^{Bu_i(x)} = \left\{ \begin{array}{lll} Bu_i(x) & \text{if} & f_i(x) \ge Bu_i(x) \\ f_i(x) & \text{if} & Bl_i(x) < f_i(x) < Bu_i(x) \\ Bl_i(x) & \text{if} & f_i(x) \le Bl_i(x) \end{array} \right. \right\}.$$

Let $\langle \cdot,\cdot \rangle$ be the pairing on $((L^q(\Gamma))^3, (L^p(\Gamma))^3)$. Since our feasible control set $\mathcal{U}$ defined in (1.11) is a convex closed bounded set in $(L^p(\Gamma))^3$, it is known that $\vec{u}^*$ is an optimal control of the LQR problem if

$$\langle \nabla J(\vec{u}^*), \vec{u} - \vec{u}^* \rangle \ge 0, \quad \forall\, \vec{u} \in \mathcal{U}. \tag{2.1}$$

For any $\alpha > 0$, (2.1) is equivalent to

$$\langle \vec{u}^* - (\vec{u}^* - \alpha \nabla J(\vec{u}^*)), \vec{u} - \vec{u}^* \rangle \ge 0, \quad \forall\, \vec{u} \in \mathcal{U}. \tag{2.2}$$

To derive an optimality condition, we need to find a characterization of a solution to the above variational inequality.

THEOREM 2.1. *For each* $f \in (L^q(\Gamma))^3$, u^f *is a solution to the variational inequality*

$$\langle u^f - f, u - u^f \rangle \ge 0 \quad \forall\, u \in \mathcal{U} \tag{VIP}$$

if and only if

$$u^f = [f + z^f]_{Bl}^{Bu} \tag{2.3}$$

where $z^f \in M_0$ *such that* $[f + z^f]_{Bl}^{Bu} - M_0$ *(refer Theorem 2.2 for the existence of such a* z^f*).*

Moreover, (2.3) is well-defined in the sense that if z^1 *and* z^2 *are two vectors in* M_0 *s.t.*

$$[f + z^1]_{Bl}^{Bu} - M_0 \quad \text{and} \quad [f + z^2]_{Bl}^{Bu} - M_0,$$

then

$$[f(x) + z^1(x)]_{Bl}^{Bu} = [f(x) + z^2(x)]_{Bl}^{Bu} \quad \text{a.e. } x \in \Gamma. \tag{2.4}$$

Proof. By Theorem 2.2, there exists $z^f \in M_0$ s.t. $[f+z^f]_{Bl}^{Bu} - M_0$. Let $u^f = [f+z^f]_{Bl}^{Bu}$. We have for each $u \in \mathcal{U}$,

$$\begin{aligned}
&\langle u^f - f, u - u^f\rangle \\
&= \langle u^f - (f+z^f), u - u^f\rangle \\
&= \sum_{i=1}^{3} \int_\Gamma \left\{ [f_i(x) + z_i^f(x)]_{Bl_i}^{Bu_i} - (f_i(x) + z_i^f(x)) \right\} \left\{ u_i(x) - [f_i(x) + z_i^f(x)]_{Bl_i}^{Bu_i} \right\} d\sigma_x \\
&\geq 0,
\end{aligned}$$

where the last inequality holds since each integrand, the product of two terms, is nonnegative.

Next we assume that u^f is a solution to the VIP, i.e.,

$$\langle u^f - f, u - u^f\rangle \geq 0 \quad \forall\, u \in \mathcal{U}.$$

Take $u = [f+z^f]_{Bl}^{Bu}$, which is in $\mathcal{U}$, we obtain

$$\langle u^f - f, [f+z^f]_{Bl}^{Bu} - u^f\rangle \geq 0. \tag{2.5}$$

By the first part, we have

$$\langle [f+z^f]_{Bl}^{Bu} - f, u - [f+z^f]_{Bl}^{Bu}\rangle \geq 0 \quad \forall\, u \in \mathcal{U}. \tag{2.6}$$

Taking $u = u^f$ in (2.6) yields

$$\langle [f+z^f]_{Bl}^{Bu} - f, u^f - [f+z^f]_{Bl}^{Bu}\rangle \geq 0. \tag{2.7}$$

Combining (2.5) with (2.7) gives us

$$\langle u^f - [f+z^f]_{Bl}^{Bu}, u^f - [f+z^f]_{Bl}^{Bu}\rangle \leq 0. \tag{2.8}$$

Thus

$$u^f = [f+z^f]_{Bl}^{Bu}.$$

The proof of the second part of the theorem follows directly from taking $z^f = z^1$ and $u^f = [f+z^2]_{Bl}^{Bu}$ in (2.8). ∎

In a Hilbert space setting, the above theorem is called a characterization of projection. When $\mathcal{U}$ is a convex closed subset of a Hilbert space H, for each $f \in H$, u_f is a solution to the VIP if and only if

$$u_f = P_{\mathcal{U}}(f),$$

i.e., u_f is the projection of f on $\mathcal{U}$. This characterization is used to derive a first order optimality condition for convex inequality constrained optimal control problems. However, this result is not valid in general Banach spaces. Instead we prove a characterization of truncation, which is a special case of a projection. Note that in a Hilbert space setting, a projection maps a point in the space into a subset of the same space. However our truncation is a projection that maps a point in $(L^q(\Gamma))^3$ into a subset of $(L^p(\Gamma))^3$, $(p > 2, \frac{1}{p} + \frac{1}{q} = 1)$. It crosses spaces. This characterization gives a connection between the truncation and the solution to VIP, in our case, an optimality condition in terms of the gradient. That is, by our characterization of truncation, $\vec{u}^* \in \mathcal{U}$ is a solution to the VIP (2.2) if and only if

$$\vec{u}^* = [\vec{u}^* - \alpha\nabla J(\vec{u}^*) + \vec{z}^*]_{Bl}^{Bu}, \tag{2.9}$$

where $\vec{z}^* \in M_0$ is defined in Theorem 2.2 s.t.

$$[\vec{u}^* - \alpha\nabla J(\vec{u}^*) + \vec{z}^*]_{Bl}^{Bu} - M_0.$$

To prove the existence of a rigid body motion z^f in (2.3), we establish the following existence theorem for an orthogonal projection, which is given in a very general case and plays a key role in establishing the optimality condition. It can be used to solve LQR problems governed by PDE's, e.g., the Laplacian, the Stokes, the linear elastostatics, ...,etc. where the PDE has multiple solutions for a given a Neumann type boundary data satisfying certain orthogonality condition.

THEOREM 2.2. *Let Γ be a bounded closed set in $\mathbb{R}^n$ and $\Gamma_0 \subset \Gamma$ be a subset s.t. meas$(\Gamma_1) > 0$ where $\Gamma_1 = \Gamma \setminus \Gamma_0$. Let $\vec{g} \in (L^p(\Gamma_0))^n$ and $\vec{B}l, \vec{B}u \in (L^p(\Gamma))^n$ $(p \geq 2)$ be given s.t.*

$$\vec{B}l(x) < -\vec{B} < \vec{B} < \vec{B}u(x) \quad \text{(a.e.)} \quad \forall\, x \in \Gamma_1$$

where $\vec{B} = (B, \cdots, B)$ is given by (2.17) and

$$\vec{B}l(x) = \vec{g}(x) = \vec{B}u(x) \quad \forall\, x \in \Gamma_0.$$

Assume that M_0 is an m-dimensional subspace in $(L^q(\Gamma))^n$ $(q \leq 2, \frac{1}{p} + \frac{1}{q} = 1)$ and $M_1 = \{\vec{z}|_{\Gamma_1} \mid \vec{z} \in M_0\}$, then a necessary and sufficient condition that for each $\vec{f} \in (L^1(\Gamma))^n$ there exists $\vec{z}_f \in M_0$ s.t.

$$\Big[\vec{f}(x) + \vec{z}_f(x)\Big]_{Bl}^{Bu} - M_0. \tag{2.10}$$

is that

$$\vec{g} - M_1^c = \{\vec{z}|_{\Gamma_0} \mid \vec{z} \in M_0, \vec{z}|_{\Gamma_1} = 0\}. \tag{2.11}$$

Moreover the set of all solutions $\vec{z}_f$ in (2.10) is locally uniformly bounded in the sense that for each given $\vec{f} \in (L^1(\Gamma))^n$ there exist $r_0 > 0$ and $b > 0$ s.t. for any $\vec{h} \in (L^1(\Gamma))^n$ with $\|\vec{f} - \vec{h}\| \leq r_0$ and for any $\vec{z}_h \in M_0$ with.

$$\Big[\vec{h}(x) + \vec{z}_h(x)\Big]_{Bl}^{Bu} - M_0$$

we have

$$\|\vec{z}_h\| \leq b. \tag{2.12}$$

Proof. Case 1: $\dim(M_1) = \dim(M_0)$, i.e., $M_1^c = \{0\}$. Let $y = (\vec{y}^1, \cdots, \vec{y}^m)$ be an orthonormal basis in M_1 (in M_0 as well). To prove the first part of the theorem, we have to show that for each $\vec{f} \in (L^1(\Gamma))^n$, there exists $C^f = (c_1^f, \cdots, c_m^f) \in \mathbb{R}^m$ s.t.

$$\langle \Big[\vec{f}(x) + \sum_{i=1}^m c_i^f \vec{y}^i(x)\Big]_{Bl}^{Bu}, \vec{y}^j\rangle_\Gamma = 0, \quad \forall\, j = 1, \cdots, m.$$

For each $\vec{f} \in (L^1(\Gamma))^n$, we define a map $T_f : \mathbb{R}^m \to \mathbb{R}^m$, for $C = (c_1, \cdots, c_m) \in \mathbb{R}^m$, by

$$T_f(C) = \left\{ \langle \left[\vec{f}(x) + \sum_{i=1}^m c_i \vec{y}^i(x) \right]_{Bl}^{Bu}, \vec{y}^j \rangle_\Gamma \right\}_{j=1,\cdots,m}. \tag{2.13}$$

Then to prove the first part, it suffices to show that for each $\vec{f} \in (L^1(\Gamma))^n$, there exists $C_f \in \mathbb{R}^m$ s.t.

$$T_f(C_f) = 0.$$

It is easy to check that for any $\vec{f}, \vec{h} \in (L^1(\Gamma))^n$ and $C_1, C_2 \in \mathbb{R}^m$, there exist two constants γ_1, γ_2 depending only on Γ and the basis y s.t.

$$|T_f(C_1) - T_h(C_2)| \le \gamma_1 |\vec{f} - \vec{h}|_{L^1} + \gamma_2 |C_1 - C_2|. \tag{2.14}$$

So $C \to T_f(C)$ is a bounded (depends on Bl and Bu) Lipschitz continuous map. To show that T_f has a zero, we prove that there exists a constant $R > 0$ s.t. when $C \in \mathbb{R}^m$ and $|C| > R$, we have

$$T_f(C) \cdot C > 0. \tag{2.15}$$

Once (2.15) is verified, we have

$$\begin{aligned} |C - T_f(C)|^2 &= |C|^2 - 2T_f(C) \cdot C + |T_f(C)|^2 \\ &< |C|^2 + |T_f(C)|^2 \qquad\qquad \forall\, C \in \mathbb{R}^m, |C| > R. \end{aligned}$$

By Altman's fixed point theorem [15], the map $C \to C - T_f(C)$ has a fixed point $C^f \in B_R$ (B_R is the ball of radius R at the origin), i.e.,

$$T_f(C^f) = 0.$$

So it remains to verify (2.15). Define

$$D = \left\{ C = (c_1, \cdots, c_m) \in \mathbb{R}^m \mid \sum_{i=1}^m c_i^2 = 1 \right\}.$$

It suffices to show that there exists $R > 0$ s.t. for $t > R$,

$$T_f(tC) \cdot C > 0 \quad \forall C \in D.$$

In the following, we prove that for each given $\vec{f} \in (L^1(\Gamma))^n$ and $C \in D$, there exist $r_0 > 0$ and $R > 0$ s.t. when $t > R$, for any $\vec{h} \in (L^1(\Gamma))^n$ with $\|\vec{f} - \vec{h}\|_{L^1} \le r_0$, we have

$$T_h(tC) \cdot C > 0 \quad \forall C \in D.$$

So the second part of the theorem also follows. For each $C \in D$, we denote

$$\vec{y}^C(x) = \sum_{i=1}^m c_i \vec{y}^i(x).$$

It is obvious that

$$\int_{\Gamma_1} |\vec{y}^C(x)| d\sigma_x$$

is continuous in C and positive on the compact set D, hence

$$m_y = \min_{C\in D}\{\int_{\Gamma_1} |\vec{y}^C(x)| d\sigma_x\} > 0 \tag{2.16}$$

and we set

$$B = \frac{\max_{C\in D} \int_{\Gamma_0} |\vec{g}(x)\cdot\vec{y}^C(x)| d\sigma_x}{m_y}. \tag{2.17}$$

For any given $\varepsilon > 0$, we assume

$$Bl_i(x) \le -B-\varepsilon, \quad Bu_i(x) \ge B+\varepsilon \quad \forall\, x\in\Gamma_1,\ \ i=1,\cdots,n.$$

For each $C\in D, t>0$,

$$\begin{aligned}
T_f(tC)\cdot C &= \sum_{j=1}^m \Big(\int_\Gamma \Big[\vec{f}(x)+\sum_{i=1}^m tc_i\vec{y}^i(x)\Big]_{Bl}^{Bu}\cdot\vec{y}^j(x)d\sigma_x\Big)c_j \\
&= \int_\Gamma \Big[\vec{f}(x)+t\vec{y}^C(x)\Big]_{Bl}^{Bu}\cdot\vec{y}^C(x)d\sigma_x \\
&= \int_{\Gamma_1} \Big[\vec{f}(x)+t\vec{y}^C(x)\Big]_{Bl}^{Bu}\cdot(\vec{y}^C(x))d\sigma_x + \int_{\Gamma_0}\vec{g}(x)\cdot\vec{y}^C(x)d\sigma_x \\
&= \sum_{i=1}^n I_i^C(t) + \int_{\Gamma_0}\vec{g}(x)\cdot\vec{y}^C(x)d\sigma_x
\end{aligned}$$

where for $i=1,\cdots,n$,

$$I_i^C(t) = \int_\Gamma [f_i(x)+ty_i^C(x)]_{Bl_i(x)}^{Bu_i(x)} y_i^C(x)d\sigma_x.$$

Let

$$\Gamma_i^{C+} = \{x\in\Gamma_1 \mid y_i^C(x)>0\} \quad \text{and} \quad \Gamma_i^{C\perp} = \{x\in\Gamma_1 \mid y_i^C(x)<0\}.$$

We have

$$\begin{aligned}
\lim_{t\to+\infty} I_i^C &= \int_{\Gamma_i^{C+}} Bu_i(x)\cdot y_i^C(x)d\sigma_x + \int_{\Gamma_i^{C-}} Bl_i(x)\cdot y_i^C(x)d\sigma_x \\
&\ge (B+\varepsilon)\int_{\Gamma_1}|y_i^C(x)|d\sigma_x.
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{t\to+\infty} T_f(tC)\cdot C &\ge (B+\varepsilon)\sum_{i=1}^n\int_{\Gamma_1}|y_i^C(x)|d\sigma_x + \int_{\Gamma_0}\vec{g}(x)\cdot\vec{y}^C(x)d\sigma_x \\
&\ge (B+\varepsilon)\int_{\Gamma_1}|y^C(x)|d\sigma_x + \int_{\Gamma_0}\vec{g}(x)\cdot\vec{y}^C(x)d\sigma_x \\
&\ge \varepsilon m_y,
\end{aligned}$$

where m_y given by (2.16) is independent of C. From (2.14), we see that $T_f(C)\cdot C$ is continuous in both $\vec{f}$ and C, therefore there exist $R^C>0$, r_C and $\delta_C>0$, as $t>R^C$, $\|\vec{h}-\vec{f}\|_{L^1}\le r^C$ and $|C'-C|<\delta_C$, we have

$$T_h(tC')\cdot C'\ge \frac{1}{2}\varepsilon m_y>0.$$

Since D is compact, there exist $C_1,\cdots,C_s\in D$ and $\delta_1,\cdots,\delta_s$ s.t.

$$D\subset\cup_{k=1}^s B_{\delta_k}(C_k).$$

Let

$$R^0=\max\{R^{C_1},\cdots,R^{C_s}\}\quad\text{and}\quad r_0=\min\{r^{C_1},\cdots,r^{C_s}\}.$$

When $t>R^0$, for all $\vec{h}\in(L^1(\Gamma))^n$ with $\|\vec{h}-\vec{f}\|_{L^1}\le r_0$, we have

$$T_h(tC)\cdot C\ge\frac{1}{2}\varepsilon m_y>0\quad\forall\, C\in D.$$

So we only need to take

$$\vec{B}=(B,\cdots,B)$$

and

$$\vec{B}l<-\vec{B}<\vec{B}<\vec{B}u,\quad\text{a.e. on }\Gamma_1.$$

Case 2: $m_1=\dim(M_1)<\dim(M_0)=m$. Let $y=(\vec{y}^1,\cdots,\vec{y}^m)$ be an orthonormal basis in M_0, where $(\vec{y}^1,\cdots,\vec{y}^{m_1})$ is a basis in M_1 with

$$\vec{y}^i|_{\Gamma_0}=0,\ (i=1,\cdots,m_1)\quad\text{and}\quad \vec{y}^j|_{\Gamma_1}=0,\ (j=m_1+1,\cdots,m). \tag{2.18}$$

By the proof in Case 1, for each $\vec{f}\in(L^1(\Gamma))^n$, there exists $C^f=(c_1^f,\cdots,c_{m_1}^f)\in\mathbb{R}^{m_1}$ s.t.

$$\langle\Big[\vec{f}(x)+\sum_{i=1}^{m_1}c_i^f\vec{y}^i(x)\Big]_{Bl}^{Bu},\vec{y}^j\rangle_{\Gamma_1}=0,\quad\forall\, j=1,\cdots,m_1.$$

Then for any $c_{m_1+1}^f,\cdots,c_m^f\in\mathbb{R}$, by (2.18), we have

$$\begin{aligned}\langle\Big[\vec{f}(x)+\sum_{i=1}^{m}c_i^f\vec{y}^i(x)\Big]_{Bl}^{Bu},\vec{y}^j\rangle_{\Gamma}&=\langle\vec{g}(x),\vec{y}^j\rangle_{\Gamma_0}+\langle\Big[\vec{f}(x)+\sum_{i=1}^{m_1}c_i^f\vec{y}^i(x)\Big]_{Bl}^{Bu},\vec{y}^j\rangle_{\Gamma_1}\\&=0,\quad\forall j=1,\cdots,m_1.\end{aligned}$$

On the other hand, when $j>m_1$, for any $c_1,\cdots,c_m\in\mathbb{R}$, by (2.18), we have

$$\langle\Big[\vec{f}(x)+\sum_{i=1}^{m}c_i\vec{y}^i(x)\Big]_{Bl}^{Bu},\vec{y}^j\rangle_{\Gamma}=\langle\vec{g}(x),\vec{y}^j\rangle_{\Gamma_0}.$$

Therefore

$$\langle\Big[\vec{f}(x)+\sum_{i=1}^{m}c_i\vec{y}^i(x)\Big]_{Bl}^{Bu},\vec{y}^j\rangle_{\Gamma}=0,\quad j>m_1,$$

if and only if

$$\langle \vec{g}(x), \vec{y}^j \rangle_{\Gamma_0} = 0, \quad j > m_1,$$

i.e., (2.11) is satisfied. The proof is complete. ∎

REMARK 2. In the above theorem,

(1) when rigid body motion is considered,

$$M_0 = \{\vec{a} + \vec{b} \times \vec{x} \mid \vec{a}, \vec{b} \in \mathbb{R}^3\},$$

we have $\dim(M_0) = \dim(M_1) = 6$, so all the conditions in the theorem are satisfied. So for each $\vec{f} \in (L^1(\Gamma))^3$ there is $\vec{a}_f + \vec{b}_f \times \vec{x} \in M_0$ such that

$$\left[\vec{f} + \vec{a}_f + \vec{b}_f \times \vec{x}\right]_{Bl}^{Bu} - M_0;$$

(2) if

$$Bl(x) \equiv -\infty \quad \text{or} \quad Bu(x) \equiv +\infty \quad \text{on } \Gamma_1$$

the conclusion still holds for each $\vec{f} \in (L^l(\Gamma))^n$ $(l \geq 1)$ and M_0 an m-dimensional subspace of $(L^q(\Gamma))^n$ where $q \geq 1$, $\frac{1}{h} + \frac{1}{q} = 1$ and $h = \min\{l, p\}$. When $h = 1$, $q = +\infty$;

(3) the vector C in (2.13) represents the rigid body motion in our case. From the above theorem, we can see that the solution C_f such that $T_f(C_f) = 0$ is not unique.

The following error estimate contains an uniqueness result, which will also be used in proving the uniform convergence rate in a subsequent paper.

THEOREM 2.3. *Let us maintain all the assumptions in Theorem 2.2. Let $\vec{f}, \vec{h}$ be given in $(L^1(\Gamma))^n$, C_f, C_h be respectively two zeros of T_f and T_h defined by (2.13). If*

$$meas\,(\Gamma_{C_f}) + meas\,(\Gamma_{C_h}) > 0$$

where

$$meas\,(\Gamma_{C_f}) = \sum_{i=1}^{n} meas\,\{x \in \Gamma \mid Bl_i(x) < f_i(x) + y_i^{C_f}(x) < Bu_i(x)\},$$

$$meas\,(\Gamma_{C_h}) = \sum_{i=1}^{n} meas\,\{x \in \Gamma \mid Bl_i(x) < h_i(x) + y_i^{C_h}(x) < Bu_i(x)\},$$

$$y^{C_f}(x) = \sum_{i=1}^{m} c_i^f y^i(x) \quad \text{and} \quad y^{C_h} = \sum_{i=1}^{m} c_i^h y^i(x),$$

then

$$|C_f - C_h| \leq \gamma \|\vec{f} - \vec{h}\|_{(L^1(\Gamma))^n} \tag{2.19}$$

where the constant γ is independent of C_f and C_h.

Proof. We may assume that

$$\text{meas}\,(\Gamma_{C_f}) > 0.$$

For $T_f(C)$, we denote

$$\Gamma_C^k = \{x \in \Gamma \mid Bl_k(x) < f_k(x) + y_k^C(x) < Bu_k(x)\},$$

where

$$y_k^C(x) = \sum_{i=1}^{m} c_i y_k^i(x).$$

Write

$$\text{meas}\,(\Gamma_C) = \sum_{k=1}^{n} \text{meas}\,(\Gamma_C^k).$$

Since $T_f(C)$ is Lipschitz continuous in C, a direct calculation leads to the Frechet derivative

$$T_f'(C) = \left[\sum_{k=1}^{n} \langle y_i^k, y_j^k\rangle_{\Gamma_C^k}\right]_{m\times m} \quad \text{a.e. } C \in \mathbb{R}^m,$$

a Gram-matrix, which is symmetric positive semi-definite, i.e., for any nonzero vector $b = (b_1, \cdots, b_m) \in \mathbb{R}^m$,

$$(b_1, \cdots, b_m) T_f'(C)(b_1, \cdots, b_m)^T = \sum_{k=1}^{n} \langle \sum_{i=1}^{m} b_i y_i^k, \sum_{i=1}^{m} b_i y_i^k\rangle_{\Gamma_C^k} \geq 0,$$

where "$>$" holds strictly if

$$\text{meas}\,(\Gamma_C) > 0,$$

because $\{\vec{y}_1, \cdots, \vec{y}_m\}$ is linearly independent.

On the other hand, we have

$$\left[\sum_{k=1}^{n} \langle y_i^k, y_j^k\rangle_{\Gamma_C^k}\right]_{m\times m} + \left[\sum_{k=1}^{n} \langle y_i^k, y_j^k\rangle_{\Gamma\setminus\Gamma_C^k}\right]_{m\times m} = \left[\sum_{k=1}^{n} \langle y_i^k, y_j^k\rangle_{\Gamma}\right]_{m\times m} = I_{m\times m},$$

where the Gram-matrix

$$\left[\sum_{k=1}^{n} \langle y_i^k, y_j^k\rangle_{\Gamma\setminus\Gamma_C^k}\right]_{m\times m}$$

is also symmetric positive semi-definite. Therefore

$$0 \leq |T_f'(C)| \leq 1 \quad \text{a.e. } C \in \mathbb{R}^m,$$

where "$<$" holds strictly in the first inequality if meas $(\Gamma_C) > 0$ and "$<$" holds strictly in the second inequality if meas $(\Gamma \setminus \Gamma_C) > 0$. Next for given f, h in $(L^1(\Gamma))^n$ and two zeros C_f, C_h of T_f and T_h, respectively, we let

$$C_t = tC_h + (1-t)C_f, \quad t \in (0,1).$$

Since $T_f(C)$ is Lipschitz continuous in C, once meas $(\Gamma_{C_f}) > 0$, there exists $\varepsilon > 0$ s.t.

$$\text{meas}\,(\Gamma_{C_t}) > 0 \quad \forall\, 0 < t < \varepsilon.$$

It follows that $T'_f(C_t)$ is a symmetric positive definite matrix with

$$0 < |T'_f(C_t)| \le 1, \quad \text{a.e. } 0 < t < \varepsilon.$$

Therefore $\int_0^1 T'_f(C_t)dt$ defines a symmetric positive definite matrix with

$$0 < \left|\int_0^1 T'_f(C_t)dt\right| \le 1.$$

For any $0 < \mu < 1$, we have

$$0 < \left|I - \mu\int_0^1 T'_f(C_t)dt\right| = (1-\lambda_f) < 1.$$

for some $0 < \lambda_f < 1$. Take

$$C_f - \mu T_f(C_f) = C_f \quad \text{and} \quad C_h - \mu T_h(C_h) = C_h$$

into account, we arrive at

$$\begin{aligned} |C_f - C_h| &= |C_f - C_h - \mu(T_f(C_f) - T_h(C_h))| \\ &= |C_f - C_h - \mu(T_f(C_f) - T_f(C_h) + T_f(C_h) - T_h(C_h))| \quad \text{(use (2.14))} \\ &\le \left|I - \mu\int_0^1 T'_f(C_t)dt\right| |C_f - C_h| + \mu\gamma_1\|f-h\|_1 \\ &= (1-\lambda_f)|C_f - C_h| + \mu\gamma_1\|f-h\|_1. \end{aligned}$$

Consequently, we have

$$|C_f - C_h| \le \frac{\gamma_1\mu}{\lambda_f}\|f-h\|_{(L^1(\Gamma))^n},$$

and the proof is complete. ∎

As a direct consequence of Theorem 2.3, we obtain the following uniqueness result.

COROLLARY 2.4. *Let us maintain all the assumptions in Theorem 2.2. For given $\vec{f} \in (L^1(\Gamma))^n$, if C_f is a zero of T_f with*

$$\textit{meas}\,(\Gamma_{C_f}) > 0,$$

then C_f is the unique zero of T_f. ∎

Now we present a state-feedback characterization of the optimal control.

THEOREM 2.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary Γ. The LQR problem has a unique optimal control $\vec{u}^* \in \mathcal{U}$ and a unique optimal velocity state $\vec{w}^* \in (C(\Gamma))^3$ s.t.*

$$\left\{ \begin{array}{rcl} \sum_{k=1}^M \mu_k(\vec{w}^*(P_k) - \vec{Z}_k) &=& 0, \\ \sum_{k=1}^M \mu_k(\vec{w}^*(P_k) - \vec{Z}_k) \times \vec{P}_k &=& 0. \end{array} \right. \tag{2.20}$$

and

$$\vec{u}^*(x) = \left[-\frac{1}{\gamma}(\frac{1}{2}I + \mathcal{K})^{\perp 1}\Big(\sum_{k=1}^{m} \mu_k E(P_k, \cdot)(\vec{w}^*(P_k) - \vec{Z}_k)\Big)(x) + \vec{a} + \vec{b} \times \vec{x} \right]_{Bl}^{Bu}, \tag{2.21}$$

$$\forall x \in \Gamma,$$

where $\vec{a} + \vec{b} \times \vec{x}$ is defined in Theorem 2.2 s.t. $\vec{u}^* - M_0$ and M_0 is given in (1.2).

Proof. Let $X = (L^p(\Gamma))^3_{\perp M_0}$. Since our objective function $J(\vec{u})$ is strictly convex and differentiable, and the feasible control set $\mathcal{U}$ is a closed bounded convex subset in the reflexive Banach space X, the existence and uniqueness of the optimal control are well-established. Equation (2.20) is just a copy of (1.16). By our characterization of truncation, Theorem 2.1 with $\alpha = \frac{1}{2\gamma}$,

$$\vec{u}^*(x) = \left[\vec{u}^*(x) - \frac{1}{2\gamma}\nabla J(\vec{u})(x) + \vec{a} + \vec{b} \times \vec{x} \right]_{Bl}^{Bu}, \quad \forall\, x \in \Gamma$$

where $\vec{a} + \vec{b} \times \vec{x} \in M_0$ is defined in Theorem 2.2 s.t.

$$\left[\vec{u}^* - \frac{1}{2\gamma}\nabla J(\vec{u}) + \vec{a} + \vec{b} \times \vec{x} \right]_{Bl}^{Bu} - M_0.$$

To prove (2.21), we only need to show

$$\nabla J(\vec{u}) = 2\Big\{(\frac{1}{2}I + \mathcal{K})^{\perp 1} \sum_{k=1}^{m} \mu_k E(P_k, \cdot)(\vec{w}(P_k, \vec{u}) - \vec{Z}_k) + \gamma \vec{u}\Big\} \tag{2.22}$$

Applying (1.9), i.e., $M_0 = \mathcal{S}_v(N)$ and (2.20), we get

$$\sum_{k=1}^{m} \mu_k E(P_k, \cdot)(\vec{w}(P_k, \vec{u}) - \vec{Z}_k) \in (L^q(\Gamma))^3_{\perp N}, \tag{2.23}$$

and then

$$(\frac{1}{2}I + \mathcal{K})^{\perp 1}\Big\{ \sum_{k=1}^{m} \mu_k E(P_k, \cdot)(\vec{w}(P_k, \vec{u}) - \vec{Z}_k)\Big\} \in (L^q(\Gamma))^3_{\perp M_0}. \tag{2.24}$$

Since $\nabla J(\vec{u})$ defines a bounded linear functional on X, for any $\vec{h} \in X$, take (1.12) into account, we have

$$\begin{aligned}
&\langle \nabla J(\vec{u}), \vec{h} \rangle \\
&= 2\sum_{k=1}^{m} \mu_k (\vec{w}(P_k, \vec{u}) - \vec{Z}_k)\mathcal{S}_v((\frac{1}{2}I + \mathcal{K}^*)^{\perp 1}\vec{h})(P_k) + 2\gamma\langle \vec{u}, \vec{h}\rangle \\
&= 2\sum_{k=1}^{m} \mu_k (\vec{w}(P_k, \vec{u}) - \vec{Z}_k)\int_{\Gamma} E(P_k, \xi)[(\frac{1}{2}I + \mathcal{K}^*)^{\perp 1}\vec{h}](\xi)d\sigma_\xi + 2\gamma\langle \vec{u}, \vec{h}\rangle \\
&= 2\int_{\Gamma}(\frac{1}{2}I + \mathcal{K})^{\perp 1}\Big[\sum_{k=1}^{m} \mu_k E(P_k, \cdot)(\vec{w}(P_k, \vec{u}) - \vec{Z}_k)\Big](\xi) \cdot \vec{h}(\xi)d\sigma_\xi + 2\gamma\langle \vec{u}, \vec{h}\rangle \\
&= 2\langle \Big[(\frac{1}{2}I + \mathcal{K})^{\perp 1}\sum_{k=1}^{m} \mu_k E(P_k, \cdot)(\vec{w}(P_k, \vec{u}) - \vec{Z}_k)\Big] + \gamma\vec{u}(\cdot), \vec{h}(\cdot)\rangle.
\end{aligned}$$

So (2.22) is verified and the proof is complete. ∎

3. Regularities of the Optimal Control.

It is clear that (2.21) is a feedback characterization of the optimal control. To obtain such a characterization, $\alpha = \frac{1}{2\gamma}$ in (2.9) is crucial. Later on we will see that $\alpha = \frac{1}{2\gamma}$ is also crucial in proving the uniform convergence of our numerical algorithms in a subsequent paper. Observe that when $Bl = -\infty$ and $Bu = +\infty$, it corresponds to the LQR problem without constraints on the control variable, the optimal solution, if it exists, becomes

$$\vec{u}^*(x) = -\frac{1}{\gamma}(\frac{1}{2}I + \mathcal{K})^{\perp 1}\Big(\sum_{k=1}^{m} \mu_k E(P_k, \cdot)(\vec{w}^*(P_k) - \vec{Z}_k)\Big)(x) + \vec{a} + \vec{b} \times \vec{x}, \quad \forall\, x \in \Gamma,$$

where $\vec{a} + \vec{b} \times \vec{x}$ is defined in Theorem 2.2 s.t. $\vec{u}^* - M_0$ (see Remark 2). But according to Lemma 1.1(d) such a solution $\vec{u}^*$ is only in $(L^q(\Gamma))^3$ $(q < 2)$, since $E(P_k, \cdot)$ is only in $(L^q(\Gamma))^3$. So it is reasonable to apply bound constraints Bl and Bu on the control variable $\vec{u}$. However we notice that the optimal control still contains a singular term

$$(\frac{1}{2}I + \mathcal{K})^{\perp 1} \sum_{k=1}^{m} \mu_k E(P_k, \cdot)(\vec{w}(P_k, \vec{u}) - \vec{Z}_k)(x),$$

which is not computable at $x = P_k$. In order to carry out the truncation by Bl and Bu, we have to know the sign of this singular term. Hence we derive a singularity decomposition formula of (2.21), in which the singular term is expressed as continuous bounded terms plus a simple dominant singular term and a lower order singular term. With the simple dominant singular term, the nature of the singularity is clearly exposed.

THEOREM 3.1. *For the optimal control $\vec{u}^*$ given in (2.21), let*

$$\vec{f}^*(x) = \sum_{k=1}^{m} \mu_k E(P_k, x)(\vec{w}^*(P_k) - \vec{Z}_k).$$

Then

$$(3.1)\quad (\frac{1}{2}I + \mathcal{K})^{\perp 1}\vec{f}^*(x) = 2\vec{f}^*(x) - 4\mathcal{K}\vec{f}^*(x) + 4(\frac{1}{2}I + \mathcal{K})^{\perp 1} \circ \mathcal{K} \circ \mathcal{K}\vec{f}^*(x) + \vec{a}_* + \vec{b}_* \times \vec{x},$$

where in the singular part, the second term $4\mathcal{K}\vec{f}^*(x)$ is dominated by the first term $2\vec{f}^*(x)$ whose nature of singularity can be determined at each P_k and the regular term $4(\frac{1}{2}I + \mathcal{K})^{\perp 1} \circ \mathcal{K} \circ \mathcal{K}\vec{f}^*(x)$ is continuous on Γ.

Proof. For given $\vec{g} \in (L^q(\Gamma))^3_{\perp N}$ with $q > 2 - \varepsilon(\Gamma)$, we have

$$(3.2)\qquad (\frac{1}{2}I + \mathcal{K})^{\perp 1}\vec{g} = 2\vec{g} - 4\mathcal{K}\vec{g} + 4(\frac{1}{2}I + \mathcal{K})^{\perp 1} \circ \mathcal{K} \circ \mathcal{K}\vec{g} + \vec{a}_g + \vec{b}_g \times \vec{x}.$$

Let

$$\vec{f}^*(x) = \sum_{k=1}^{m} \mu_k E(P_k, x)(\vec{w}^*(P_k) - \vec{Z}_k).$$

By (2.23), $\vec{f}^* \in (L^q(\Gamma))^3_{\perp N}$ for every $q < 2$, thus (3.1) follows. The first part of Lemma 1.1 (e) states that the singularity in $2\vec{f}^*$ dominates the one in $4\mathcal{K}\vec{f}^*$. While the second part of Lemma 1.1 (e) and (f) imply that $(\frac{1}{2}I+\mathcal{K})^{\perp 1}\circ\mathcal{K}\circ\mathcal{K}\vec{f}^*$ is continuous. ☐

The above singularity decomposition formula plays an important role in our singularity analysis and also in our numerical computation. It is used to prove the uniform convergence and to estimate the uniform convergence rate of our numerical algorithms in a subsequent paper.

Note that the fundamental velocity solution

$$E(\xi,x)=\Big\{\frac{1}{8\pi}(\frac{\delta_{ij}}{|x-\xi|}+\frac{(x_i-\xi_i)(x_j-\xi_j)}{|x-\xi|^3})\Big\},\quad 1\le i,j\le 3,$$

is not defined when $\xi=P_k$ and $x\to P_k$, in the sense that when $x\to P_k$, some of the entries may oscillate between $-\infty$ and $+\infty$. So if we look at the simple dominant singular term in the singularity decomposition formula of the optimal control, we can see that in general, the optimal control $\vec{u}^*(x)$ is not defined at P_k even with the truncation by Bl and Bu. This is a significant difference between systems with scalar valued fundamental solution and with matrix-valued fundamental solution. For the formal case, e.g., the Laplacian, the optimal control is continuous at every point where Bl and Bu are continuous. Of course, if $Bl(P_k)=Bu(P_k)=\vec{g}(P_k)$, i.e., $P_k\in\Gamma_0$, which means the control is not active at P_k, then trivially $\vec{u}^*(P_k)=\vec{g}(P_k)$, a prescribed value. This is the case when a sensor is placed at P_k, then a control device can not be put at the same point P_k. However, in general point observation case, the control may still be active at P_k. The above analysis then states that the optimal control is not defined at P_k unless some other conditions are posed. This is the nature of point observations. Notice that a distributed parameter control is assumed in our problem setting, theoretically the values of the control variable at finite points will not affect the system. But, in numerical computation we can only evaluate the optimal control $\vec{u}^*$ at finite number of points. The observation points P_k's usually are of the most interest. On the other hand, the optimal velocity state $\vec{w}^*$ is well-defined and continuous at P_k, no matter $\vec{u}^*(P_k)$ is defined or not. So if one does want the optimal control $\vec{u}^*$ to be defined at P_k, when $Bl(P_k)=Bu(P_k),\ k=1,...,m$, it is clear that $\vec{u}^*(P_k)$ is defined at each P_k. When $Bl(P_k)>Bu(P_k)$ for some $k=1,...,m$, then we have the following necessary and sufficient condition.

THEOREM 3.2. *Let $Bl(P_k)>Bu(P_k)$ for some $k=1,...,m$, then the optimal control $\vec{u}^*$ is well-defined at the observation points P_k if and only if*

$$|(\vec{w}(P_k,\vec{u}^*)-\vec{Z}_k)_i|\le 2|(\vec{w}(P_k,\vec{u}^*)-\vec{Z}_k)_j|,\quad 1\le i\ne j\le 3, \tag{3.3}$$

where for each fixed k and i, the equality holds for at most one $j\ne i$ unless

$$\vec{w}(P_k,\vec{u}^*)=\vec{Z}_k.$$

When $\vec{u}^*$ is well-defined at P_k, we have

$$(\vec{u}^*(P_k))_i=\begin{cases} Bl_i(P_k) & \textit{if}\quad (\vec{w}(P_k,\vec{u}^*)-\vec{Z}_k)_i<0,\\ Bu_i(P_k) & \textit{if}\quad (\vec{w}(P_k,\vec{u}^*)-\vec{Z}_k)_i>0.\end{cases} \tag{3.4}$$

Proof. If we observe the fundamental velocity solution, we can see that the proof follows from the following argument. For $x=(x_1,x_2,x_3)$ and $1\le i,j,k\le 3$,

$$\begin{aligned}\lim_{x\to 0}\bar{e}_i(x)&=\lim_{x\to 0}\Big(c_i\Big(\frac{1}{|x|}+\frac{x_i^2}{|x|^3}\Big)+c_j\frac{x_ix_j}{|x|^3}+c_k\frac{x_ix_k}{|x|^3}\Big)\\&=\lim_{x\to 0}\frac{1}{|x|^3}\Big((c_ix_i^2+c_jx_ix_j+c_ix_j^2)+(c_ix_i^2+c_kx_ix_k+c_ix_k^2)\Big)\end{aligned}$$

exists (including $\pm\infty$) if and only if

$$c_j^2 - 4c_i^2 \le 0 \quad \text{and} \quad c_k^2 - 4c_i^2 \le 0 \tag{3.5}$$

where at most one equality can hold unless $c_i = c_j = c_k = 0$. Notice that when (3.5) holds, $c_i = 0$ leads to $c_j = c_k = 0$. So if $c_i \neq 0$ and two equalities hold in (3.5), then

$$\bar{e}_i(x) = \frac{c_i}{|x|^3}\Big((x_i \pm x_j)^2 + (x_i \pm x_k)^2\Big).$$

We can make the limit either equal to zero by taking $x_i = \mp x_j = \mp x_k \to 0$ or equal to sign $(c_i)\infty$ by taking $x_i \neq \mp x_j$ or $x_i \neq \mp x_k$ and $x \to 0$. So the limit will not exist. When $\lim_{x\to 0} \bar{e}_i(x)$ exists and $c = (c_1, c_2, c_3) \neq 0$, we have

$$\lim_{x\to 0} \bar{e}_i(x) = \text{sign } (c_i)\infty.$$

□

With the above result and the singularity decomposition formula for the optimal control, the following continuous result can be easily verified.

THEOREM 3.3. *Let Bu and Bl be continuous on Γ_1. If for each $k = 1, \cdots, m$ either $Bl(P_k) = Bu(P_k)$ or the condition (3.3) holds strictly with $(\vec{w}(P_k, \vec{u}_p^0) - \vec{Z}_k)_i \neq 0$, then the optimal control $\vec{u}^*$ is continuous on Γ_1. So the equality in (2.4) holds for every point on Γ.*

From the state-feedback characterization (2.20), the control can be determined by a physical measurement of the state at finite number of observation points $P_k, k = 1, .., m$. The question is then asked, will a small error in the measurement of the state cause a large deviation in the controlΓ Due to the appearance of the singular term in (2.20), in general the answer is yes, i.e., the state-feedback system is not stable. However under certain conditions, we can prove that the state-feedback system is uniformly stable.

THEOREM 3.4. *Let $\vec{w}(P_k)$ be the exact velocity state at observation points and $\vec{u}_p$ be the control determined from (2.20) in terms of $\vec{w}(P_k)$. If for each $k = 1, \cdots, m$, either Bl and Bu are continuous and equal at P_k or Bu and Bl are locally bounded at P_k, the condition (3.3) holds strictly with $(\vec{w}(P_k, \vec{u}_p^0) - \vec{Z}_k)_i \neq 0$, then the state-feedback system (2.20) is uniformly stable in the sense that for any $\varepsilon > 0$, there is $\delta > 0$ such that for any measurement $\vec{w}'(P_k)$ of $\vec{w}(P_k)$,*

$$|\vec{u}'(x) - \vec{u}(x)| < \varepsilon, \quad \forall x \in \Gamma \qquad \textit{whenever } |\vec{w}'(P_k) - \vec{w}(P_k)| < \delta,$$

where $\vec{u}'$ is the control determined from (2.20) in terms of $\vec{w}'(P_k)$.

Proof. For each $\varepsilon > 0$. For each fixed $k = 1, \cdots, m$, if Bl and Bu are continuous and equal at P_k, there is $d_k' > 0$ such that

$$Bu(x) - Bl(x) < \varepsilon, \quad \forall x \in \Gamma_1,\ |x - P_k| \le d_k'.$$

Since the control variable is bounded by Bl and Bu,

$$|\vec{u}'(x) - \vec{u}(x)| < \varepsilon, \quad \forall x \in \Gamma_1,\ |x - P_k| \le d_k'.$$

If instead the condition (3.3) holds strictly with $(\vec{w}(P_k, \vec{u}_p^0) - \vec{Z}_k)_i \neq 0$, let $\delta_1 > 0$ be chosen so that when $|\vec{w}'(P_k) - \vec{w}(P_k)| < \delta_1$, condition (3.3) still holds strictly with $(\vec{w}'(P_k, \vec{u}_p^0) - \vec{Z}_k)_i \neq 0$. Due to the singular term in (2.20) and since Bu and Bl are locally bounded at P_k, there is $d_k > 0$ such that when $x \in \Gamma$ and $|x - P_k| < d_k$ for some $k = 1, ..., m$, we have

$$\left(-\frac{1}{\gamma}(\frac{1}{2}I + \mathcal{K})^{\perp 1}\left[\sum_{k=1}^{m} \mu_k E(P_k, \cdot)(\vec{w}'(P_k) - \vec{Z}_k))\right](x) + \vec{a}' + \vec{b}' \times x\right)_i$$

either $> Bu(x)_i$

or $< Bl(x)_i$.

After the truncation by Bu and Bl, it follows that

$$\vec{u}'(x)_i = \vec{u}(x)_i = \text{ either } Bu(x)_i \text{ or } Bl(x)_i, \quad \forall x \in \Gamma_1, |x - P_k| < d_k.$$

So if we define

$$\Gamma_+ = \{x \in \Gamma |\ |x - P_k| < \min\{d'_k, d_k, k = 1, \cdots, m\} \text{ for some } k = 1, ..., m\},$$

then in either case we have

$$|\vec{u}'(x) - \vec{u}(x)| < \varepsilon, \quad \forall x \in \Gamma_+.$$

Denote

$$\vec{F}(x) = -\frac{1}{\gamma}(\frac{1}{2}I + \mathcal{K})^{\perp 1}\Big(\sum_{k=1}^{m} \mu_k E(P_k, \cdot)(\vec{w}(P_k) - \vec{Z}_k))\Big)(x),$$
$$\vec{F}'(x) = -\frac{1}{\gamma}(\frac{1}{2}I + \mathcal{K})^{\perp 1}\Big(\sum_{k=1}^{m} \mu_k E(P_k, \cdot)(\vec{w}'(P_k) - \vec{Z}_k))\Big)(x)$$

and

$$\text{meas}\,(\Gamma_{C_F}) = \sum_{i=1}^{3} \text{meas}\,\{x \in \Gamma \mid Bl_i(x) < (\vec{F}(x) + \vec{a} + \vec{b} \times x)_i < Bu_i(x)\},$$

$$\text{meas}\,(\Gamma_{C_{F'}}) = \sum_{i=1}^{3} \text{meas}\,\{x \in \Gamma \mid Bl_i(x) < (\vec{F}'(x) + \vec{a}' + \vec{b}' \times x)_i < Bu_i(x)\}.$$

Since $\text{meas}\,(\Gamma_{C_F}) + \text{meas}\,(\Gamma_{C_{F'}}) = 0$ implies that

$$\vec{u}'(x)_i = \vec{u}(x)_i = \text{ either } Bu(x)_i \text{ or } Bl(x)_i, \quad \forall x \in \Gamma,$$

there is nothing to prove. So we assume that $\text{meas}\,(\Gamma_{C_F}) + \text{meas}\,(\Gamma_{C_{F'}}) > 0$, then Theorem 2.3 can be applied. For $x \in \Gamma_\perp = \Gamma \setminus \Gamma_+$, a compact set, by using (2.20) and triangle inequality, we obtain

$$\begin{aligned}
|\vec{u}'(x) - \vec{u}(x)| &= \left|\Big[\vec{F}(x) + \vec{a} + \vec{b} \times x\Big]_{Bl}^{Bu} - \Big[\vec{F}'(x) + \vec{a}' + \vec{b}' \times x\Big]_{Bl}^{Bu}\right| \\
&\leq \left|-\frac{1}{\gamma}(\frac{1}{2}I + \mathcal{K})^{\perp 1}\left[\sum_{k=1}^{m} \mu_k E(P_k, \cdot)(\vec{w}'(P_k) - \vec{w}(P_k))\right](x)\right| \\
&\qquad + |\vec{a}' + \vec{b}' \times x - (\vec{a} + \vec{b} \times x)| \\
&\equiv |I_1(x)| + |I_2(x)|.
\end{aligned}$$

Since the operator $(\frac{1}{2}I + \mathcal{K})^{\perp 1}$ is linear and bounded, and the function $E(P_k, \cdot)$ is continuous and bounded on the compact set $\Gamma_\perp$, there is $\delta_2 > 0$ such that

$$|I_1(x)| < \frac{1}{2}\varepsilon \quad \forall x \in \Gamma_\perp, \quad \text{when } |\vec{w}'(P_k) - \vec{w}(P_k)| < \delta_2.$$

As for $I_2(x)$, Theorem 2.3 yields

$$|(\vec{a}', \vec{b}') - (\vec{a}, \vec{b})| \le \gamma \|\vec{F}' - \vec{F}\|_{L^1(\Gamma))^3},$$

where the constant γ depends only on Γ. Since there is constant C_0 independent of $\vec{w}'(P_k)$ such that

$$\|\vec{F}' - \vec{F}\|_{L^1(\Gamma))^3} \le C_0 |\vec{w}'(P_k) - \vec{w}(P_k)|, \quad k = 1, ..., m$$

there is $\delta_3 > 0$ such that

$$|I_2(x)| = |\vec{a}' + \vec{b}' \times x - (\vec{a} + \vec{b} \times x)| < \frac{1}{2}\varepsilon, \quad \forall x \in \Gamma_\perp, \quad \text{whenever } |\vec{w}'(P_k) - \vec{w}(P_k)| < \delta_3.$$

Finally for $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, we have

$$|\vec{u}'(x) - \vec{u}(x)| < \varepsilon, \quad \forall x \in \Gamma, \quad \text{whenever } |\vec{w}'(P_k) - \vec{w}(P_k)| < \delta \quad \text{for } k = 1, ..., m.$$

The proof is complete. □

As a final comment, it is worth while indicating that though in the problem setting, the governing differential equation, the Stokes, is linear, the bound constraint on the control variable introduces a nontrivial nonlinearity into the system. This can be clearly seen in Theorem 2.2. Also our approach can be adopted to deal with certain nonlinear boundary control problems.

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