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Mass of Rotating Black Holes in Gauged Supergravities

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ABSTRACT

The masses of several recently-constructed rotating black holes in gauged supergravities, including the general such solution in minimal gauged supergravity in five dimensions, have until now been calculated only by integrating the first law of thermodynamics. In some respects it is more satisfactory to have a calculation of the mass that is based directly upon the integration of a conserved quantity derived from a symmetry principal. In this paper, we evaluate the masses for the newly-discovered rotating black holes using the conformal definition of Ashtekar, Magnon and Das (AMD), and show that the results agree with the earlier thermodynamic calculations. We also consider the Abbott-Deser (AD) approach, and show that this yields an identical answer for the mass of the general rotating black hole in five-dimensional minimal gauged supergravity. In other cases we encounter discrepancies when applying the AD procedure. We attribute these to ambiguities or pathologies of the chosen decomposition into background AdS metric plus deviations when scalar fields are present. The AMD approach, involving no decomposition into background plus deviation, is not subject to such complications. Finally, we also calculate the Euclidean action for the five-dimensional solution in minimal gauged supergravity, showing that it is consistent with the quantum statistical relation.

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Contents

1 Introduction

With the discovery of the AdS/CFT correspondence, it has become of considerable interest to study the solutions of gauged supergravities in five and other dimensions. Amongst the most important such solutions are those that describe black holes. In recent times there has been much progress in constructing the black hole solutions of gauged supergravity, both supersymmetric and non-supersymmetric. For a variety of reasons, it is of particular interest to study the solutions describing charged rotating black holes.

The general solution describing a non-extremal charged rotating black hole in five-dimensional minimal gauged supergravity was obtained recently in [1]. It is characterised by four parameters, associated with the mass, the charge, and the two angular momenta in two orthogonal spatial 2-planes. It can also be viewed as a solution in $\mathcal{N}=2$ gauged supergravity coupled to two vector multiplets, in which the electric charges carried by the graviphoton and the two additional $U(1)$ gauge fields are set equal. A second recently-obtained solution of the $\mathcal{N}=2$ theory corresponds to a situation where the three charges are again non-zero, with two equal and the third related to these in a fixed ratio [2]. Another solution found in [2] corresponds to having only one non-vanishing charge, and one non-vanishing rotation parameter. Previously, solutions had been obtained in which the two rotation parameters were set equal, and with three equal [3] or three unequal [4] charges.

In this paper, we shall investigate some aspects of the thermodynamics of the solutions obtained in [1, 2]. One of the important quantities that one needs to know is the mass, or energy, of the solution. As was discussed in [5], the energy of a rotating black hole in an asymptotically AdS background must be calculated with considerable care, because of the complications associated with the absence of an asymptotically flat region at large distance, and because of the rotation. In particular, the absence of an asymptotically flat region means that one cannot use the standard ADM [6] procedure for defining the energy. Alternative approaches include that of Abbott and Deser [7], and the use of Komar integrals (see [8], for a discussion of this method in asymptotically AdS backgrounds). The Komar integral definition, involving the integration of $*dK$ over a spatial hypersurface at infinity, where $K = K_\mu dx^\mu$ and $K^\mu \partial_\mu$ is a timelike Killing vector, suffers from the problem that the integrand diverges at large radius. One therefore has to make a subtraction of a background AdS contribution in order to obtain a finite result, and finding a way to to this unambiguously can be somewhat problematical. In the Abbott and Deser (AD) definition one also makes decomposition of the metric in which a background AdS term is subtracted, and then integrates certain derivatives of the difference over a spatial hypersurface at infinity.

In [5], two relatively straightforward methods were employed, for calculating the energies of the uncharged rotating AdS black hole solutions that were found in $D = 4$ dimensions [9], $D = 5$ dimensions [10] and in all dimensions $D \geq 6$ [11, 12]. The first method involved evaluating all the other quantities that appear in the first law of thermodynamics

$$dE = TdS + \Omega_i dJ_i \,, \tag{1.1}$$

and then integrating (1.1) in order to obtain the energy E. The advantage of this method is that the Hawking temperature T, the entropy S, the angular velocities Ω_i and the angular momenta J_i are all easily calculated, with no complications associated with divergent integrals. The second method employed in [5] was to use the conformal mass definition of Ashtekar, Magnon and Das [13, 14]. This AMD definition expresses the mass in terms of an integral of certain components of the Weyl tensor over the spatial conformal boundary at infinity. Since the metric approaches AdS asymptotically, the integrand falls off and the integral is inherently well-defined. It was shown in [5] that the first-law calculation and the AMD calculation of the mass are in agreement for the uncharged rotating black holes. Analogous results for the AMD masses of the five-dimensional black holes with equal rotation parameters found in [3] were obtained in [15], giving agreement with an earlier thermodynamic calculation of the mass in [16]. Another calculation of the masses of the higher-dimensional uncharged rotating black holes was given in [17], using the Katz-Bičák-Lynden-Bell superpotential. (See also [18–20] for further discussions of mass in asymptotically AdS spacetimes.)

In [1, 2], the energies for the charged rotating black hole solutions were calculated using the method of integrating the first law of thermodynamics, which reads

$$dE = TdS + \Omega_i dJ_i + \Phi\, dQ \,, \tag{1.2}$$

where Φ is the electrostatic potential difference between the horizon and infinity, and Q is the conserved electric charge. In the present paper, we shall instead calculate the energies using the the Ashtekar-Magnon-Das approach. As we shall see, the results agree with the earlier calculations based on the integration of the first law of thermodynamics. Establishing this consistency is important, because it makes a direct connection between the mass and the integration of conserved quantities.

A further test of the thermodynamic properties of the black hole solutions is provided by calculating the Euclidean action I, since the partition function in a Gibbs ensemble at fixed temperature T, angular velocity Ω_i and electrostatic potential Φ should be given by

$$Z(T, \Omega_i, \Phi) = e^{-\beta\, \Phi_{\text{thermo}}} \,, \tag{1.3}$$

where $\beta = 1/T$ and $\Phi_{\rm thermo}$ denotes the thermodynamic potential. On the other hand, in the one-loop quantum gravity approximation the partition function is given by $Z = e^{-I}$, and so one has the *Quantum Statistical Relation*, or QSR, first proposed for quantum gravity in [24], that

$$\Phi_{\rm thermo} \equiv E - TS - \Omega_i J_i - \Phi Q = IT\,. \tag{1.4}$$

In this paper, we also calculate the Euclidean action for the general five-dimensional rotating black hole in minimal gauged supergravity, and verify that it is indeed consistent with the quantum statistical relation.

The AMD construction gives a conformal definition of a conserved quantity $Q[K]$ associated to any asymptotic Killing field K in an asymptotically AdS spacetime [13, 14]. We shall summarise the AMD method in the notation of [5]. We assume that asymptotically, the D-dimensional metric satisfies the Einstein equations

$$R_{\mu\nu} = -(D-1)\, l^{-2}\, g_{\mu\nu}\,, \tag{1.5}$$

where l is the length-scale of the asymptotically AdS metric. In canonical AdS coordinates, the metric therefore approaches

$$ds^2 = -(1 + y^2\, l^{-2})\, dt^2 + \frac{dy^2}{1 + y^2\, l^{-2}} + y^2\, d\Omega^2_{D-2} \tag{1.6}$$

at large distance y.

Consider an asymptotically AdS bulk spacetime $\{X, g\}$, equipped with a conformal boundary $\{\partial X, \bar h\}$. It admits a conformal compactification $\{\bar X, \bar g\}$ if $\bar X = \sqcup \partial X$ is the closure of X, and the metric $\bar g$ extends smoothly onto $\bar X$ where $\bar g = \Omega^2\, g$ for some function Ω with $\Omega > 0$ in X and $\Omega = 0$ on ∂X, with $d\Omega \ne 0$ on ∂X. One might, for example, take

$$\Omega = \frac{l}{y}\,. \tag{1.7}$$

Since Ω is determined only up to a factor, $\Omega \to f\,\Omega$, where the function f is non-zero on ∂X, the metric $\bar g$ on $\bar X$ and its restriction $\bar h = \bar g|_{\partial X}$ are defined only up to a non-singular conformal factor. The conformal equivalence class $\{\partial \bar X, \bar h\}$ is called the conformal boundary of X. If $\bar C^\mu{}_{\nu\rho\sigma}$ is the Weyl tensor of the conformally rescaled metric $\bar g_{\mu\nu} = \Omega^2\, g_{\mu\nu}$, and $\bar n_\mu \equiv \partial_\mu \Omega$, then in D dimensions one defines

$$\bar{\mathcal{E}}^\mu{}_\nu = l^2 \Omega^{D-3}\, \bar n^\rho\, \bar n^\sigma\, \bar C^\mu{}_{\rho\nu\sigma}\,. \tag{1.8}$$

This is the electric part of the Weyl tensor on the conformal boundary. The conserved charge $Q[K]$ associated to the asymptotic Killing vector K is then given by

$$Q[K] = \frac{l}{8\pi\,(D-3)} \oint_\Sigma \bar{\mathcal{E}}^\mu{}_\nu\, K^\nu\, d\bar\Sigma_\mu\,, \tag{1.9}$$

where $d\bar{\Sigma}_\mu$ is the area element of the $(D-2)$-sphere section of the conformal boundary.[1] Note that the expression (1.9) is invariant under the non-singular conformal transformations of the boundary metric that we discuss above. Thus, one may take for Ω any conformal factor that is related to (1.7) by a non-singular multiplicative factor.

In order to define the energy, one takes $K = \partial/\partial t$, where t is the time coordinate appearing in the asymptotic form (1.6) of the metric under investigation. The energy is then given by[2]

$$E = \frac{l}{8\pi\,(D-3)}\oint_\Sigma \bar{\mathcal{E}}^t{}_t\, d\bar{\Sigma}_t\,. \tag{1.11}$$

The organisation of the paper is as follows. In section 2, we use the AMD definition to calculate the mass of the recently-constructed general rotating black hole solution of five-dimensional minimal gauged supergravity, showing that it agrees with the earlier calculation of the mass in [1], where it was obtained by integrating the first law of thermodynamics. We also calculate the Euclidean action for the solution, and show that it is consistent with the quantum statistical relation (1.4).

In section 3 we calculate the generalised AMD masses for some recently-obtained rotating black-hole solutions of maximal gauged five-dimensional supergravity, where there are three charges carried by fields in the $U(1)^3$ abelian subgroup of the $SO(6)$ gauge group. Again, we find that the AMD masses agree with the earlier results in [2], which were obtained by integrating the first law.

Section 4 contains similar calculations of the AMD masses for the known rotating black-hole solutions in four-dimensional and seven-dimensional gauged supergravities, and we find agreement with the earlier calculations based on the integration of the first law.

[1] The derivation of (1.9) is discussed in [13,14]. The key point is that by making use of Bianchi identities, one can show that

$$\bar{D}^\nu \bar{\mathcal{E}}_{\mu\nu} = 8\pi(D-3)\,\bar{T}_{\nu\rho}\,\bar{n}^\nu\,\bar{h}^\rho{}_\mu\,, \tag{1.10}$$

where $\bar{D}_\mu$ is the covariant derivative in the conformal boundary metric $\bar{h}_{\mu\nu}$, and $\bar{T}_{\mu\nu} = \Omega^{2-D}\,T_{\mu\nu}$, in the limit as the boundary is approached, where $T_{\mu\nu}$ is the energy-momentum tensor in the bulk.

[2] Note that some confusion in earlier literature arose when coordinate systems that were rotating asymptotically at infinity were used in an attempt to define the mass. As emphasised in [5], one should define the mass with respect to an asymptotically-static coordinate system. Especially, when considering the thermodynamics of the system, it is highly advantageous to avoid using an asymptotically rotating coordinate system whose rotation rate depends on the parameters of the black hole [23]. (For what appear to be largely historical reasons, rotating AdS black holes were often presented in such rotating coordinate systems.) Of course one could always readjust all calculations so as to refer them to the asymptotically-rotating frame, but describing the physics from such a parameter-dependent rigid rotating frame is an easily avoidable and unnecessary complication.

In section 5, we consider the calculation of the black hole masses using the methods of Abbott and Deser. The section begins with a brief summary of the AD procedure, in which we extend the standard discussion of a pure Einstein theory with cosmological constant to include to the case where there are matter fields, such as one has in a gauged supergravity. Especially, in gauged supergravity there is usually a scalar potential rather than a pure cosmological constant. We then use the AD procedure to calculate the mass of the general rotating black hole in five-dimensional minimal gauged supergravity, which was constructed in [1]. We find that the result agrees with our AMD calculation in section 2, and also therefore with the earlier calculation from the integration of the first law. We then consider the other rotating black holes in five, four and seven dimensions. We find in these four and five dimensional examples that the AD procedure is rather tricky to implement, because of ambiguities associated with the subtraction procedure when one separates the metric into an AdS background plus deviations. To study this more fully, we look also at the rather simple examples of multi-charge non-rotating black holes in five, four and seven-dimensional gauged supergravities. The AD procedure rather straightforwardly gives rise to the correct masses in the cases when the charges are set equal. However when the charges are unequal, implying that non-trivial scalar fields are present in the solution, the complications of the subtraction procedure again lead to difficulties in obtaining the correct mass in an unambiguous manner, in the five and four-dimensional cases. We then discuss two possible correction terms to the AD mass formula, incorporating additional contributions from the scalar fields, and we relate these to corrections discussed previously in the literature.

Finally, the paper ends with conclusions in section 6.

2 Rotating black holes in $D = 5$ minimal gauged supergravity

For our principal example, we consider the recently-discovered general rotating black holes in $D = 5$ minimal gauged supergravity. The Lagrangian for the bosonic sector of the theory is given by

$$\mathcal{L} = (R + 12g^2) {*\mathbb{1}} - \tfrac{1}{2} {*F} \wedge F + \frac{1}{3\sqrt{3}} F \wedge F \wedge A\,, \tag{2.1}$$

where $F = dA$, and the gauge-coupling g is assumed to be positive, without loss of generality. It is related to the asymptotic AdS radius l by $l = 1/g$. The rotating black hole with two independent rotating parameters is given by [1]

$$ds^2 = -\frac{\Delta_\theta\,[(1+g^2r^2)\rho^2 dt + 2q\nu]\,dt}{\Xi_a\,\Xi_b\,\rho^2} + \frac{2q\,\nu\omega}{\rho^2} + \frac{f}{\rho^4}\Big(\frac{\Delta_\theta\,dt}{\Xi_a \Xi_b} - \omega\Big)^2 + \frac{\rho^2 dr^2}{\Delta_r} + \frac{\rho^2 d\theta^2}{\Delta_\theta}$$

$$
\begin{aligned}
& + \frac{r^2+a^2}{\Xi_a} \sin^2\theta d\phi^2 + \frac{r^2+b^2}{\Xi_b}\cos^2\theta d\psi^2 \,, && (2.2)\\
A &= \frac{\sqrt{3}q}{\rho^2}\Big(\frac{\Delta_\theta\, dt}{\Xi_a\,\Xi_b} - \omega\Big)\,, && (2.3)
\end{aligned}
$$

where

$$
\begin{aligned}
\nu &= b\sin^2\theta d\phi + a\cos^2\theta d\psi\,, \qquad \omega = a\sin^2\theta\frac{d\phi}{\Xi_a} + b\cos^2\theta\frac{d\psi}{\Xi_b}\,,\\
\Delta_r &= \frac{(r^2+a^2)(r^2+b^2)(1+g^2r^2)+q^2+2abq}{r^2} - 2m\,,\\
\Delta_\theta &= 1 - a^2g^2\cos^2\theta - b^2g^2\sin^2\theta\,, \quad \rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta\,,\\
\Xi_a &= 1-a^2g^2\,, \quad \Xi_b = 1-b^2g^2\,, \qquad f = 2m\rho^2 - q^2 + 2abqg^2\rho^2\,. && (2.4)
\end{aligned}
$$

2.1 Conformal AMD mass

The metric (2.2) is written in an asymptotically non-rotating coordinate system. If one defines new coordinates $(y, \hat\theta)$ by

$$
\Xi_a\, y^2\, \sin^2\hat\theta = (r^2+a^2)\,\sin^2\theta\,, \qquad \Xi_b\, y^2\, \cos^2\hat\theta = (r^2+b^2)\,\cos^2\theta\,, \tag{2.5}
$$

then it can be seen to approach (1.6) at large y. As we discussed earlier, we could choose to define the boundary metric of the conformal compactification using the conformal factor $\Omega = l/y$ (1.7). In practice, however, it is simpler to take

$$
\Omega = \frac{l}{r} = \frac{1}{gr}\,, \tag{2.6}
$$

which is related to (1.7) by the non-singular scale factor $f = y/r$. With this choice, the relevant electric component of the Weyl tensor is given

$$
\bar{\mathcal{E}}^t{}_t = \frac{1}{g^2\Omega^2}\bar g^{\alpha r}\bar g^{\beta r}\bar n_r \bar n_r C^t{}_{\alpha t\beta} = \frac{1}{g^4r^4\Omega^6}(g^{rr})^2 C^t{}_{rtr}\,. \tag{2.7}
$$

We find that as $r\to\infty$, the leading order term of g^{rr} is g^2r^2, while asymptotically $C^t{}_{rtr}$ is given by

$$
\begin{aligned}
C^t{}_{rtr} &= \frac{2}{g^2\Xi_a\,\Xi_b\, r^6}\Big(3m - 3a^2g^2m + b^2g^2m - a^2b^2g^4m + 4abg^2q - 4a^3bg^4q && (2.8)\\
&\quad +4a^2g^2m\sin^2\theta - 4b^2g^2m\sin^2\theta + 4a^3bg^4q\sin^2\theta - 4ab^3g^4q\sin^2\theta\Big) + O(r^{-8})\,.
\end{aligned}
$$

Thus in the limit of large r we have

$$
\begin{aligned}
\bar{\mathcal{E}}^t{}_t &= \frac{2g^4}{\Xi_a\,\Xi_b}\Big(3m - 3a^2g^2m + b^2g^2m - a^2b^2g^4m + 4abg^2q - 4a^3bg^4q\\
&\quad +4a^2g^2m\sin^2\theta - 4b^2g^2m\sin^2\theta + 4a^3bg^4q\sin^2\theta - 4ab^3g^4q\sin^2\theta\Big) && (2.9)
\end{aligned}
$$

To perform the integral, we need to find the hypersurface normal to the Killing vector field. As $r \to \infty$, the conformally rescaled metric is

$$d\bar{s}^2 = -\frac{\Delta_\theta}{\Xi_a \Xi_b} dt^2 + \frac{1}{g^2 \Delta_\theta} d\theta^2 + \frac{1}{g^2 \Xi_a} \sin^2\theta d\phi^2 + \frac{1}{g^2 \Xi_b} \cos^2\theta d\psi^2 \tag{2.10}$$

The area element $d\Sigma_\mu$ is obtained from (2.10) as follows. First, we note that the 4-volume element for the boundary metric (2.10) is given by

$$\mathrm{Vol} = \frac{\sin\theta\, \cos\theta}{g^3 \Xi_a \Xi_b}\, dt \wedge d\theta \wedge d\phi \wedge d\psi\,. \tag{2.11}$$

Now, we define $d\Sigma_\mu \equiv \langle \partial_\mu, \mathrm{Vol}\rangle$, where the angle brackets indicate that one performs the contraction (inner product) between the vector and the form, using the rule $\langle \partial_\mu, dx^\nu\rangle = \delta_\mu^\nu$. Thus we shall, in particular, have

$$d\Sigma_t = \frac{\sin\theta\, \cos\theta}{g^3 \Xi_a \Xi_b}\, d\theta \wedge d\phi \wedge d\psi\,, \tag{2.12}$$

where "t" is the coordinate-frame time index. Performing the integration, we find

$$\begin{aligned} E &= \frac{1}{16\pi g^4 \Xi_a \Xi_b} \int_0^{2\pi} d\phi \int_0^{2\pi} d\psi \int_0^{\pi/2} d\theta\, \sin\theta \cos\theta\, \bar{\mathcal{E}}^t{}_t \\ &= \frac{m\pi(2\Xi_a + 2\Xi_b - \Xi_a \Xi_b) + 2\pi q a b g^2 (\Xi_a + \Xi_b)}{4\Xi_a^2 \Xi_b^2}\,. \end{aligned} \tag{2.13}$$

This result agrees precisely with the mass obtained in [1] by integrating the first law of thermodynamics.

2.2 Euclidean action and the QSR

In order to verify that the Quantum Statistical Relation (1.4) is satisfied, it is necessary to calculate the Euclidean action, namely the integral of $\mathcal{L}$ given by (2.1), after Euclideanisation. This was evaluated in [10] for the case of the neutral Kerr-AdS solutions in four and five dimensions, and in [5] for the neutral Kerr-AdS solutions in arbitrary dimension. It was also evaluated for the rotating black hole in five-dimensional minimal gauged supergravity, in the case where the rotation parameters are equal, in [15].

The action has to be defined with care, since the naive integration over the volume of the Euclideanised metric gives infinity. To obtain a finite action, one cuts off the integration at some large radius $r = R$, and makes an appropriate subtraction for an AdS metric with the same boundary. Now, as R is sent to infinity, the subtracted action converges to a finite result. It should be noted that there is no need to include the usual Gibbons-Hawking boundary term involving the trace of the second fundamental form, because this is precisely removed when the AdS subtraction is performed.

The integration for the black hole metric is straightforward, albeit somewhat complicated. In particular, we integrate the radial variable r from the Euclideanised horizon at $r = r_+$ (i.e. the origin of coordinates in the Euclidean regime) to the chosen large radius R. To perform the AdS subtraction, we can use the metric obtained by setting the mass and charge to zero in the black-hole metric, since then the AdS metric will be expressed directly in an appropriately adapted coordinate system. There is one subtlety concerning the matching of the boundaries of the black-hole metric and the AdS metric at $r = R$. Namely, one must rescale the Euclidean time coordinate τ in the AdS metric so that the volume its $r = R$ boundary is the same as the volume of the $r = R$ boundary of the black-hole metric. If $\gamma_{\mu\nu}$ denotes the metric of the $r = R$ boundary in the black-hole metric, and $\bar{\gamma}_{\mu\nu}$ is the corresponding metric AdS boundary metric obtained by setting the mass and charge to zero, then we must choose a rescaled Euclidean time coordinate $\bar{\tau}$ for AdS such that

$$\int \sqrt{\gamma}\, d\theta d\phi d\psi d\tau = \int \sqrt{\bar{\gamma}}\, d\theta d\phi d\psi d\bar{\tau}\,, \tag{2.14}$$

where the integration is over the boundary at $r = R$. Thus we must define the rescaled Euclidean time coordinate $\bar{\tau}$ in the AdS background according to

$$\bar{\tau} = \tau\, \frac{\int \sqrt{\gamma}\, d\theta}{\int \sqrt{\bar{\gamma}}\, d\theta}\,. \tag{2.15}$$

In particular, with τ in the black-hole metric having period $\beta = 1/T$, where T is the Hawking temperature, it follows that $\bar{\tau}$ will have period

$$\bar{\beta} = \beta\, \frac{\int \sqrt{\gamma}\, d\theta}{\int \sqrt{\bar{\gamma}}\, d\theta}\,. \tag{2.16}$$

For the metric (2.2), we find that

$$\bar{\beta} = \beta \Big(1 - \frac{M}{g^2\, R^4}\Big) + \mathcal{O}(R^{-5})\,. \tag{2.17}$$

A further subtlety concerns the lower limit of the radial integration in the AdS subtraction. Expressed in terms of the y coordinate appearing in (1.6), one should integrate out from $y = 0$. Using (2.5), this translates into the statement that one should integrate out from a radius r_0, given by

$$r_0^2 = -\frac{a^2 \Xi_b\, \sin^2\theta + b^2\, \Xi_a\, \cos^2\theta}{\Xi_b\, \sin^2\theta + \Xi_a\, \cos^2\theta}\,. \tag{2.18}$$

(The fact that this defines an imaginary r_0 is merely an artefact of the coordinate system being used here.)

The Hawking temperature for the metric (2.2) is given by [1]

$$T = \frac{r_+^4[(1+g^2(2r_+^2+a^2+b^2)] - (ab+q)^2}{2\pi\, r_+\, [(r_+^2+a^2)(r_+^2+b^2)+abq]}\,. \tag{2.19}$$

After performing the steps described above, we then find that the Euclidean action for the black-hole metric (2.2) is given by

$$I_5 = \frac{\pi\beta}{4\Xi_a\Xi_b}\Big[m - g^2(r_+^2+a^2)(r_+^2+b^2) - \frac{q^2r_+^2}{(r_+^2+a^2)(r_+^2+b^2)+abq}\Big]\,, \tag{2.20}$$

The other relevant thermodynamic quantities were evaluated in [1], and are given by

$$\begin{aligned}
S &= \frac{\pi^2[(r_+^2+a^2)(r_+^2+b^2)+abq]}{2\Xi_a\Xi_b r_+}\,,\\
\Omega_a &= \frac{a(r_+^2+b^2)(1+g^2r_+^2)+bq}{(r_+^2+a^2)(r_+^2+b^2)+abq}\,, \qquad \Omega_b = \frac{b(r_+^2+a^2)(1+g^2r_+^2)+aq}{(r_+^2+a^2)(r_+^2+b^2)+abq}\,,\\
J_a &= \frac{\pi[2am+qb(1+a^2g^2)]}{4\Xi_a^2\,\Xi_b}\,, \qquad J_b = \frac{\pi[2bm+qa(1+b^2g^2)]}{4\Xi_b^2\,\Xi_a}\,,\\
\Phi &= \frac{\sqrt{3}qr_+^2}{(r_+^2+a^2)(r_+^2+b^2)+abq}\,, \qquad Q = \frac{\sqrt{3}\,\pi q}{4\Xi_a\,\Xi_b}\,.
\end{aligned} \tag{2.21}$$

It is now straightforward to substitute these and our expression (2.20) for the Euclidean action into (1.4), and to confirm that the Quantum Statistical Relation is indeed satisfied.

3 5-dimensional Black Holes in $U(1)^3$ Gauged Supergravity

The Lagrangian for the relevant bosonic sector of maximal gauged supergravity in five dimensions is given by

$$\mathcal{L} = R\,{*\mathbb{1}} - \tfrac12 {*d\varphi_i} \wedge d\varphi_i - \tfrac12 \sum_{i=1}^{3} X_i^{-2}\,{*F^i} \wedge F^i + 4g^2 \sum_{i=1}^{3} X_i^{-1}\,{*\mathbb{1}} + F^1\wedge F^2 \wedge A^3\,, \tag{3.1}$$

where

$$X_1 = e^{-\frac{1}{\sqrt6}\varphi_1 - \frac{1}{\sqrt2}\varphi_2}\,,\qquad X_2 = e^{-\frac{1}{\sqrt6}\varphi_1 + \frac{1}{\sqrt2}\varphi_2}\,,\qquad X_3 = e^{\frac{2}{\sqrt6}\varphi_1}\,. \tag{3.2}$$

In the following two subsections, we shall consider two recently-discovered rotating black-hole solutions in this theory, and use the AMD procedure to calculate the mass for each of them. We find that these masses agree with those derived previously by integration of the first law of thermodynamics.

3.1 A 3-charge rotating black hole

A rotating black hole solution with two independent rotation parameters was obtained [2]. The solution has three non-vanishing charges, with two of them set equal, and the third in a fixed ratio to the other two. Since the solution is rather complicated, we shall not present explicitly here, but we refer the reader to [2] for all the details.

In [2], the metric is given in a coordinate system that is asymptotically rotating at infinity. To obtain the mass of the black hole, it is necessary first to transform to an asymptotically non-rotating frame. Starting from the metric given in [2], this is achieved by making the redefinitions $\phi' = \phi + ag^2 t$ and $\psi' = \psi + bg^2 t$. We then take the conformal factor defining the conformally-compactified boundary metric to be given by $\Omega = 1/(g\,r)$. Following the same steps as before, we have $\bar{\mathcal{E}}^t{}_t$ given by (2.7), where at large r we find the component $C^t{}_{rtr}$ of the Weyl tensor takes the form

$$\begin{aligned} C^t{}_{rtr} \;=\;& \frac{2m}{3g^2(1-a^2g^2)(1-b^2g^2)\,r^6}\Big(3(3+4s^2)+b^4g^4s^2(1+4\sin^2\theta) \\ &+a^4g^4s^2(5-4\sin^2\theta+b^2g^2(7-8\sin^2\theta))-a^2g^2(9+3b^2g^2(1+2s^2) \\ &+12\sin^2\theta-s^2(17-16\sin^2\theta)-b^4g^4s^2(1-8\sin^2\theta)) \\ &+b^2g^2(3-12\sin^2\theta-s^2(1+16\sin^2\theta))\Big)+\mathcal{O}\Big(\frac{1}{r^7}\Big)\,. \end{aligned} \tag{3.3}$$

It follows that

$$\begin{aligned} \bar{\mathcal{E}}^t{}_t \;=\;& \frac{2g^4m}{3(1-a^2g^2)(1-b^2g^2)}\Big(3(3+4s^2)+b^4g^4s^2(1+4\sin^2\theta) \\ &+a^4g^4s^2(5-4\sin^2\theta+b^2g^2(7-8\sin^2\theta))-a^2g^2(9+3b^2g^2(1+2s^2) \\ &+12\sin^2\theta-s^2(17-16\sin^2\theta)-b^4g^4s^2(1-8\sin^2\theta)) \\ &+b^2g^2(3-12\sin^2\theta-s^2(1+16\sin^2\theta))\Big)\,. \end{aligned} \tag{3.4}$$

The metric on the conformal boundary is again given by (2.10). Integrating over the hypersurface normal to the Killing vector field $K = \partial/\partial t$, we then obtain the black hole mass

$$E = \frac{m\pi}{4\Xi_a^2\Xi_b^2}\Big(2\Xi_a+2\Xi_b-\Xi_a\Xi_b+(2\Xi_a^2+2\Xi_b^2+2\Xi_a\Xi_b-\Xi_a^2\Xi_b-\Xi_a\Xi_b^2)s^2\Big)\,, \tag{3.5}$$

where Ξ_a and Ξ_b are defined in (2.4). This is precisely the mass found in [2] by integrating the first law of thermodynamics.

The direct calculation of the Euclidean action is rather intricate in this example. As we saw in the previous case of the rotating black hole in the minimal gauged supergravity, one always needs to perform the subtraction of a fiducial action for a pure AdS background with a matching boundary at large distance, in order to obtain a finite result. In the present case the process is rather more involved, presumably because of the presence of scalar fields in the solution. We shall not present a direct calculation of the Euclidean action here.

However, since it is useful for some purposes to know the expression for the action, we shall present the result here obtained by substitution of the thermodynamic quantities, which were derived in [2], into the quantum statistical relation (1.4). We then find that the Euclidean action is given by

$$\begin{aligned} I_5 &= \frac{\pi\beta}{4\Xi_a\Xi_b}\Big[m - g^2(r_+^2+a^2)(r_+^2+b^2) \\ &\quad - \frac{g^2q\Big((r_+^2+a^2)(r_+^2+b^2)(4r_+^2+a^2+b^2)+6qr_+^2(2r_+^2+a^2+b^2)+8q^2r_+^2\Big)}{(r_+^2+a^2)(r_+^2+b^2)+2qr_+^2}\Big]\,. \end{aligned} \tag{3.6}$$

3.2 Single-charge rotating black hole

The solution is presented in equation (23) of [2], and owing to its complexity, we shall not repeat it here. To achieve an asymptotic non-rotating frame, we make the coordinate redefinition $\phi' = \phi + ag^2cwt$. We define the conformally rescaled metric $\bar{g}_{ab} = \Omega^2 g_{ab}$ where $\Omega = 1/(gr)$. With $n = d\Omega = -1/(gr^2)\,dr$, the relevant electric Weyl tensor component is

$$\bar{\mathcal{E}}^t{}_t = \frac{1}{g^2\Omega^2}\bar{g}^{\alpha r}\bar{g}^{\beta r}\bar{n}_r\bar{n}_r C^t{}_{\alpha t\beta} = \frac{1}{g^2\Omega^6}g^{rr}g^{rr}(-\frac{1}{gr^2})^2 C^t{}_{rtr}\,. \tag{3.7}$$

As $r\to\infty$, we find that the leading order behaviour of $C^t{}_{rtr}$ is given by

$$C^t{}_{rtr} = \frac{2m}{3g^2r^6\Xi}\Big(4\sin^2\theta\,(1-\Xi)(2wc^2-s^2\Xi) - \Xi(s^2-9wc^2+2s^2\Xi\Big) + \mathcal{O}\Big(\frac{1}{r^7}\Big)\,. \tag{3.8}$$

It follows that

$$\bar{\mathcal{E}}^t{}_t = \frac{2g^4m}{3\Xi}\Big(4\sin^2\theta\,(1-\Xi)(2wc^2-s^2\Xi) - \Xi(s^2-9wc^2+2s^2\Xi\Big)\,. \tag{3.9}$$

The metric on the conformal boundary is given by

$$d\bar{s}^2 = -\frac{\Delta_\theta}{\Xi}dt^2 + \frac{1}{g^2\Delta_\theta}d\theta^2 + \frac{1}{g^2\Xi}\sin^2\theta d\phi^2 + \frac{1}{g^2}\cos^2\theta d\psi^2 \tag{3.10}$$

We integrate over the hypersurface normal to the Killing vector field and obtain the black hole mass

$$E = \frac{m\pi}{4\Xi^2 w(\Xi-w)}[\Xi - w(2+\Xi) + w^2\Xi(1+\Xi)]\,, \tag{3.11}$$

in agreement with the thermodynamic calculation in [2].

The thermodynamic quantities for this black hole were obtained in [2]. Using these, we can derive the Euclidean action using the quantum statistical relation (1.4). We obtain the result that

$$I_5 = \frac{\pi\beta}{4\Xi}\Big[m - g^2r_+^2(r_+^2+a^2) - \frac{m(1-w)(2g^2(1+w)r_+^2+1-w\Xi)}{w(w-\Xi)}\Big]\,. \tag{3.12}$$

3.3 3-charge black hole with equal rotation parameters

The solution with three independendent charges, and with the two rotation parameters set equal, was obtained in [4]. We shall work with the solution in the variables that were used in section 3.1 of [26]. We find that the relevant component of the Weyl tensor has the asymptotic form

$$C^t{}_{rtr} = \frac{2m}{g^2 r^6}\Big(3 + a^2 g^2 + 2\sum_i s_i^2\Big) + \mathcal{O}\Big(\frac{1}{r^7}\Big)\,, \tag{3.13}$$

where $s_i \equiv \sinh\delta_i$ and the δ_i are the charge (boost) parameters. Accordingly, we find that the electric component in the conformal boundary metric is given by

$$\bar{\mathcal{E}}^t{}_t = 2mg^4\,\Big(3 + a^2 g^2 + 2\sum_i s_i^2\Big)\,. \tag{3.14}$$

From this, we find after performing the integration in (1.11) that the mass is given by

$$E = \tfrac{1}{4} m\pi \Big(3 + a^2 g^2 + 2\sum_i s_i^2\Big)\,, \tag{3.15}$$

which precisely reproduces the result obtained in [26] by integrating the first law of thermodynamics.

4 Rotating Black Holes in $D = 4$ and $D = 7$ Gauged Supergravities

4.1 $D = 4$ $SO(4)$ gauged supergravity

The general solution for rotating black holes in $D = 4$, $\mathcal{N} = 4$, $SO(4)$ gauged supergravity were obtained in [25]. These carry two charges, associated with the gauged fields in the $U(1)\times U(1)$ Cartan subalgebra of $SO(4)$. First it is convenient to rescale the azimuthal coordinate ϕ in [25] by a factor of Ξ^{-1}, so that it has the canonical period 2π. Then to achieve a non-rotating coordinate system at infinity, we define a azimuthal angle $\phi' = \phi + ag^2 t$. In the new coordinate system, the relevant electric Weyl tensor component is given by

$$\bar{\mathcal{E}}^t{}_t = \frac{1}{g^2\Omega}\bar{g}^{\alpha r}\bar{g}^{\beta r}\bar{n}_r\bar{n}_r C^t{}_{\alpha t\beta} = \frac{1}{g^4 r^4 \Omega^5}(g^{rr})^2 C^t{}_{rtr}\,, \tag{4.1}$$

for the conformally scaled metric $\bar{g}_{ab} = \Omega^2 g_{ab}$, where we take $\Omega = 1/(gr)$.

We find that the leading order behaviour of $C^t{}_{rtr}$ at large r is given by

$$\begin{aligned} C^t{}_{rtr} \;\;=\;\; & \frac{m}{g^2(1-a^2g^2)\,r^5}\Big(2 - 2a^2g^2 + 2s_1^2 - 2a^2g^2s_1^2 + 2s_2^2 - 2a^2g^2s_2^2 \\ & +3a^2g^2\sin^2\theta + 3a^2g^2s_1^2\sin^2\theta + 3a^2g^2s_2^2\sin^2\theta\Big) + \mathcal{O}\Big(\frac{1}{r^6}\Big)\,, \end{aligned} \tag{4.2}$$

and hence we obtain

$$\bar{\mathcal{E}}^t{}_t = \frac{g^3 m}{1-a^2g^2}\Big(2-2a^2g^2+2s_1^2-2a^2g^2s_1^2+2s_2^2-2a^2g^2s_2^2 \\ +3a^2g^2\sin^2\theta+3a^2g^2s_1^2\sin^2\theta+3a^2g^2s_2^2\sin^2\theta\Big)\,. \tag{4.3}$$

The metric on the conformal boundary is given by

$$ds_4^2 = -\frac{\Delta_\theta}{\Xi}dt^2+\frac{1}{g^2\Delta_\theta}d\theta^2+\frac{\sin^2\theta}{g^2\Xi}d\phi'^2\,. \tag{4.4}$$

Integrating over the hypersurface normal to the Killing vector field $K=\partial/\partial t$, i.e.

$$\frac{1}{g^2\Xi}\int_0^{2\pi}d\phi'\int_0^{\pi}d\theta\sin\theta\,, \tag{4.5}$$

we obtain the black hole mass

$$E=\frac{m(1+s_1^2+s_2^2)}{\Xi^2}\,. \tag{4.6}$$

This expression for the mass agrees with the one obtained in [26] by integrating the first law of thermodynamics.

The thermodynamic quantities for this black hole were obtained in [26]. Using these, we can derive the Euclidean action using the quantum statistical relation (1.4); it is given by

$$I_4=\frac{\beta}{2\Xi}\Big[M-g^2r(r^2+a^2)-\frac{4a^2q_1q_2}{r(r_1r_2+a^2)}-g^2a^2(q_1+q_2)-\frac{g^2(r_1^2r_2^2-r^4)}{r}\Big]\Big|_{r=r_+}\,, \tag{4.7}$$

where $q_i=ms_i^2$.

4.2 $D=7$ gauged supergravity

The gauge group of $D=7$ gauged maximal supergravity is $SO(5)$, which has $U(1)\times U(1)$ as its Cartan subalgebra. Rotating black holes charged under these two $U(1)$ gauge fields, with three equal angular momenta, were constructed in [27]. In order to achieve an asymptotic non-rotating frame, we make a coordinate transformation $\psi'=\psi+\frac{g}{\Xi_-}t$ (where $\Xi_-=1-ag$, as defined in [27]), starting from the metric given in [27]. Furthermore, it is necessary to scale the time coordinate according to $t\to\Xi\,t$ so that it matches with the canonical time coordinate of AdS$_7$ at infinity, as defined by (1.6). Having done this, we find that the relevant electric component of the Weyl tensor is given by

$$\bar{\mathcal{E}}^t{}_t=\frac{1}{g^2\Omega^4}\bar{g}^{\alpha r}\bar{g}^{\beta r}\bar{n}_r\bar{n}_rC^t{}_{\alpha t\beta}=\frac{1}{g^2\Omega^8}g^{rr}g^{rr}(-\frac{1}{g\,r^2})^2C^t{}_{rtr} \tag{4.8}$$

in the conformally rescaled metric $\bar{g}_{ab} = \Omega^2 g_{ab}$ where $\Omega = 1/(g\,r)$, with $n = d\Omega = -\frac{l}{r^2}dr$. Asymptotically as $r \to \infty$, the leading order behaviour of g^{rr} is $g^{rr} \sim g^2 r^2$, and that for $C^t{}_{rtr}$ is

$$\begin{aligned} C^t{}_{rtr} &= \frac{m}{g^2(1-a^2g^2)\,r^8}\Big(12(-1+4a^2g^2+4a^3g^3+a^4g^4) \\ &\quad -2c_1c_2a^2g^2(-30-16ag+51a^2g^2+64a^3g^3+21a^4g^4) \\ &\quad +(c_1^2+c_2^2)(16-52a^2g^2-40a^3g^3+45a^4g^4+64a^5g^5+21a^6g^6)\Big)\,. \end{aligned} \tag{4.9}$$

Therefore, we have

$$\begin{aligned} \bar{\mathcal{E}}^t{}_t &= \frac{g^6 m}{1-a^2g^2}\Big(12(-1+4a^2g^2+4a^3g^3+a^4g^4) \\ &\quad -2c_1c_2a^2g^2(-30-16ag+51a^2g^2+64a^3g^3+21a^4g^4) \\ &\quad +(c_1^2+c_2^2)(16-52a^2g^2-40a^3g^3+45a^4g^4+64a^5g^5+21a^6g^6)\Big)\,. \end{aligned}$$

The metric on the conformal boundary is given by

$$ds_7^2 = -\frac{1}{\Xi}dt^2 + \frac{1}{g^2\Xi}d\Omega_5^2 \tag{4.10}$$

where $d\Omega_5^2$ is the standard metric on the unit 5-sphere. Integrating over the hypersurface normal to the Killing vector field $K = \partial/\partial t$, we obtain the black hole mass

$$\begin{aligned} E &= \frac{m\pi^2}{32\Xi^4}\Big[(12(-1+4a^2g^2+4a^3g^3+a^4g^4) \\ &\quad -2c_1c_2a^2g^2(-30-16ag+51a^2g^2+64a^3g^3+21a^4g^4) \\ &\quad +(c_1^2+c_2^2)(16-52a^2g^2-40a^3g^3+45a^4g^4+64a^5g^5+21a^6g^6)\Big]\,. \end{aligned} \tag{4.11}$$

This agrees with the mass that was calculated in [26] by integrating the first law of thermodynamics.

Again, we can present an expression for the Euclidean action for this rotating seven-dimensional black hole, by substitution of the thermodynamic quantities, which were calculated in [26], into the quantum statistical relation (1.4). The expression we obtain is rather complicated in the general case when the two charges are unequal, and so here we shall just present the result when the charges are set equal. We then define a charge parameter q by setting $c_1 = c_2 = \sqrt{1+q/m}$, and find that the Euclidean action is given by

$$I = \frac{\beta\,\pi^2}{8\Xi^3}\left(m - g^2R_+^6 - g^2\,q\,(4R_+^2 - a^2) - \frac{4gq^2[gR_+^4 + a^2g(1+ag)R_+^2 + 2gq - a^3(1+ag)^2]}{R_+^6 + 2qR_+^2 - 2a^2(1+ag)}\right), \tag{4.12}$$

where R_+ is the radius of the outer horizon.

5 Abbott-Deser Mass for the Rotating Black Holes

5.1 The Abbott-Deser mass in gauged supergravity

In the Abbott-Deser AD construction [7], one splits the asymptotically-AdS metric $g_{\mu\nu}$ in the form

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,, \tag{5.1}$$

where $\bar{g}_{\mu\nu}$ is the AdS metric. We shall summarise the procedure here, including the extension needed for qconsidering asymptotically-AdS spacetimes as solutions of gauged supergravities, where there is a scalar potential with a stationary point rather than a pure cosmological constant.

Consider a D-dimensional theory whose bosonic Lagrangian is

$$\mathcal{L} = \sqrt{-g}\,[R - V(\phi)] + \mathcal{L}_{\rm kin}\,, \tag{5.2}$$

where ϕ represents the scalar fields, with potential $V(\phi)$, and $\mathcal{L}_{\rm kin}$ denotes the kinetic terms for the scalars and the other matter fields in the theory. We assume that $V(\phi)$ has a stationary point at $\phi = 0$, and that there exists a pure AdS background solution with $g_{\mu\nu} = \bar{g}_{\mu\nu}$ and $\phi = 0$, with all other fields vanishing too, where

$$\bar{R}_{\mu\nu} - \tfrac{1}{2}\bar{R}\,\bar{g}_{\mu\nu} + \tfrac{1}{2}V(0)\,\bar{g}_{\mu\nu} = 0\,. \tag{5.3}$$

The extension of the AD prescription involves taking the full Einstein equation,

$$R_{\mu\nu} - \tfrac{1}{2}R\,g_{\mu\nu} + \tfrac{1}{2}V(\phi)\,g_{\mu\nu} = T^{\rm matter}_{\mu\nu}\,, \tag{5.4}$$

where $T^{\rm matter}_{\mu\nu}$ represents the energy-momentum tensor for the other matter fields and the remaining contribution from the scalars. Substituting $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ into (5.4), one then separates the terms linear in $h_{\mu\nu}$ from the remainder, which will acquire the interpretation of an effective energy-momentum tensor for gravity plus the other fields. The appropriate integral involving the effective energy-momentum tensor will then yield the mass. Collecting the terms linear in $h_{\mu\nu}$ on the left-hand side, we shall have

$$R_L^{\mu\nu} - \tfrac{1}{2}R_L\,\bar{g}^{\mu\nu} + \frac{1}{D-2}\,V(0)\,h^{\mu\nu} = \frac{1}{8\pi\sqrt{-\bar{g}}}\,T^{\mu\nu}\,, \tag{5.5}$$

where $R_L^{\mu\nu}$ and R_L denote the linearised Ricci tensor and Ricci scalar, and $T^{\mu\nu}$ is the effective energy-momentum tensor density, including the contribution from gravity as well as from the matter fields. Note that the contribution from the scalar fields on the right-hand side is of the form

$$\frac{1}{\sqrt{-\bar{g}}}\,T^{\mu\nu}_{\rm scal} = \frac{1}{\sqrt{-\bar{g}}}T^{\mu\nu}_{\rm kinetic} - \tfrac{1}{2}V(\phi)\,g_{\mu\nu} + \tfrac{1}{2}V(0)\,\bar{g}_{\mu\nu}\,, \tag{5.6}$$

since the effective cosmological constant $\frac{1}{2}V(0)$ in the background AdS metric $\bar{g}_{\mu\nu}$ has been included on the left-hand side of (5.5).

As in [7], one defines

$$
\begin{aligned}
H^{\mu\nu} &= h^{\mu\nu} - \tfrac{1}{2}\bar{g}^{\mu\nu}\, h^\rho{}_\rho\,, \\
K^{\mu\nu\rho\sigma} &= \tfrac{1}{2}(\bar{g}^{\mu\sigma}\, H^{\rho\nu} + \bar{g}^{\rho\nu}\, H^{\mu\sigma} - \bar{g}^{\mu\rho}\, H^{\nu\sigma} - \bar{g}^{\nu\sigma}\, H^{\mu\rho})\,,
\end{aligned} \tag{5.7}
$$

where here, and in what follows, all indices are raised and lowered using the background AdS metric $\bar{g}_{\mu\nu}$. It follows that the left-hand side in (5.5) is given by

$$
\begin{aligned}
& R_L^{\mu\nu} - \tfrac{1}{2}R_L\, \bar{g}^{\mu\nu} + \frac{1}{D-2}\, V(0)\, h^{\mu\nu} \\
&= \tfrac{1}{2}(\bar{\nabla}_\lambda \bar{\nabla}^\mu H^{\lambda\nu} + \bar{\nabla}_\lambda \bar{\nabla}^\nu H^{\lambda\mu} - \bar{\Box} H^{\mu\nu} - \bar{\nabla}_\alpha \bar{\nabla}_\beta H^{\alpha\beta}\, \bar{g}^{\mu\nu}) - \frac{V(0)}{D-2}\, H^{\mu\nu}\,, \\
&= \bar{\nabla}_\alpha \bar{\nabla}_\beta K^{\mu\alpha\nu\beta} + \tfrac{1}{2}\bar{R}^\mu{}_{\alpha\beta}{}^\nu\, H^{\alpha\beta} - \frac{V(0)}{2(D-2)}\, H^{\mu\nu}\,.
\end{aligned} \tag{5.8}
$$

A straightforward calculation shows that the divergence of this quantity, with respect to the background covariant derivative $\bar{\nabla}_\mu$, vanishes identically, upon the use of the background Einstein equation (5.3).

Taking $\bar{\xi}^\mu \partial_\mu = \partial/\partial t$ as the canonically-normalised timelike Killing vector, the generalised AD mass is then given by

$$
\begin{aligned}
E &= -\frac{1}{8\pi} \oint d^{D-1}x\, T^{t\nu}\, \bar{\xi}_\nu\,, \\
&= \frac{1}{8\pi} \oint dS_i \mathcal{M}^i\,, \\
\mathcal{M}^i &= -\sqrt{-\bar{g}}\, \Big[\bar{\xi}_\nu\, \bar{\nabla}_\mu K^{ti\nu\mu} - K^{tj\nu i}\, \bar{\nabla}_j \bar{\xi}_\nu\Big]\,,
\end{aligned} \tag{5.9}
$$

where dS_i is the area element of the spatial surface at large radius, and the t superscript denotes a coordinate index in the time direction.[3] Note that the index t denotes the time coordinate index, Greek indices range over all spacetime directions, and Latin indices range over the spatial directions. Eventually, one sends the radius to infinity. It should be emphasised that one must choose a coordinate frame with respect to which the deviation $h_{\mu\nu}$ of the full metric $g_{\mu\nu}$ from the background AdS metric $\bar{g}_{\mu\nu}$ tends to zero appropriately at infinity.

The AD definition was used recently in [22] to calculate the masses of the higher-dimensional neutral Kerr-AdS black holes constructed in [11, 12].

[3] The original four-dimensional Abbott-Deser construction was generalised to arbitrary spacetime dimensions in [21]. We adopt the normalisation given in (5.9) for all dimensions (rather than the one chosen in [21]), since this accords with the conventions for the definition of mass appearing in most of the earlier literature, and, in particular, the definitions in [5].

5.2 Rotating black hole in five-dimensional minimal gauged supergravity

We first apply the AD procedure described above to the case of the general rotating black hole in five-dimensional minimal gauged supergravity. Note that there are no scalar fields in the minimal gauged supergravity theory, and one has just a cosmological constant, as give in (2.1). The calculation is a purely mechanical one, although of such a complexity that it is most easily carried out with the aid of a computer. We find that at large distance, the relevant integrand in the expression (5.9) for the AD mass takes the form

$$\begin{aligned}&\sqrt{-\bar{g}}\,(\bar{\xi}_\nu\bar{\nabla}_\mu K^{tr\nu\mu} - K^{tj\nu r}\,\bar{\nabla}_j\bar{\xi}_\nu) \qquad (5.10)\\ &=-\frac{\sin\theta\cos\theta}{\Xi_a^2\Xi_b^2}\,[\Xi_b(3m + a^2g^2m + 4abqg^2) - 4(a^2-b^2)g^2(m + abqg^2)\cos^2\theta] + \mathcal{O}\Big(\frac{1}{r}\Big)\,.\end{aligned}$$

After integration over the angular coordinates, it follows from (5.9) that the mass is given by

$$\begin{aligned}E &= -\frac{1}{8\pi}\int_0^{2\pi}d\phi\int_0^{2\pi}d\psi\int_0^{\pi/2}d\theta\,\sqrt{-\bar{g}}\,(\bar{\xi}_\nu\bar{\nabla}_\mu K^{tr\nu\mu} - K^{tj\nu r}\,\bar{\nabla}_j\bar{\xi}_\nu)\\ &= \frac{\pi m(2\Xi_a + 2\Xi_b - \Xi_a\Xi_b) + 2\pi qabg^2(\Xi_a+\Xi_b)}{4\Xi_a^2\Xi_b^2}\,. \qquad (5.11)\end{aligned}$$

This result agrees precisely with the mass obtained in [1] by integrating the first law of thermodynamics, and that we obtained in section 2.1 by applying the AMD procedure.

5.3 AD masses for the other rotating black holes

In this subsection, we apply the AD procedure to the other examples of rotating black holes discussed previously in this paper. We find that in the case of the other five-dimensional black holes, and the four-dimensional black holes, the answers do not agree with the AMD and thermodynamic results. By contrast, the mass of the seven-dimensional black hole obtained by the AD procedure does agree with the AMD and thermodynamic calculations.

5.3.1 3-charge rotating black hole

First, we consider the five-dimensional 3-charge rotating black hole discussed in section 3.1, with two equal charges and a third in a fixed ratio to these [2]. This solution involves non-trivial scalar fields. In the coordinate frame used in [2], the $h_{\mu\nu}$ components already fall off at large r. We find that $\mathcal{M}^i$ in (5.9) is given by

$$\begin{aligned}\mathcal{M}^r &= \frac{m\sin\theta\,\cos\theta}{3\Xi_a^2\Xi_b^2}\Big(\Xi_b(3(4-\Xi_a) + (6\Xi_a + 12\Xi_b + \Xi_a^2 - 7\Xi_a\Xi_b)s^2) \qquad (5.12)\\ &\quad +4\cos^2\theta(\Xi_a-\Xi_b)(3 + s^2(3\Xi_a + 3\Xi_b - 2\Xi_a\Xi_b)\Big) - \frac{4g^2m^2s^4\sin\theta\,\cos\theta}{3\Xi_a\Xi_b} + \mathcal{O}\Big(\frac{1}{r}\Big)\,,\end{aligned}$$

which leads to the mass

$$E' = E - \frac{g^2 m^2 \pi s^4}{3\Xi_a \Xi_b}\,, \tag{5.13}$$

where E is the mass found in [2] by integrating the first law, and which we reproduced in (3.5) by using the AMD procedure.

5.3.2 Single-charge rotating black hole

Next, we consider the single-charge rotating black hole in five dimensional, whose AMD mass we calculated in 3.2. To apply the AD procedure in this case, it is necessary first to make the coordinate transformation

$$r \to r(1 - \frac{m s^2}{3r^2})\,, \tag{5.14}$$

to ensure that $h_{\mu\nu}$ falls off at large distance. Then we find that

$$\begin{aligned} \mathcal{M}^r &= \frac{m \sin\theta\, \cos\theta}{3\Xi^2}\left[4\cos^2\theta(\Xi-1)(\Xi s^2 - 3(1+s^2)w) - \Xi(s^2(1+2\Xi) - 9w(1+s^2))\right] \\ &\quad -\frac{4g^2 m^2 s^4 \sin\theta\, \cos\theta}{3\Xi} + \mathcal{O}\Big(\frac{1}{r}\Big)\,. \end{aligned} \tag{5.15}$$

Thus performing the integral as in (5.9), we obtain the mass

$$E' = E - \frac{g^2 m^2 \pi s^4}{3\Xi}\,, \tag{5.16}$$

where E is the mass obtained in [2] by integrating the first law, and reproduced in (3.11) by applying the AMD procedure.

5.3.3 3-charge black hole with equal rotation parameters

For the solution obtained in [4], which has three unequal charges and the two rotation parameters set equal, we use the notation and conventions of section 3.1 of [26]. First, it is necessary to redefine the radial coordinate r according to

$$r \longrightarrow r - \frac{m}{3r}\sum_i s_i^2\,, \tag{5.17}$$

in order that $h_{\mu\nu}$ in the decomposition (5.1) fall off at infinity. Then, we find that $\mathcal{M}^r$, defined by (5.9), is given by

$$\mathcal{M}^r = \tfrac{1}{8} m \sin\theta\Big[3 + a^2 g^2 + 2\sum_i s_i^2 - \tfrac{4}{3} m g^2\Big(\sum_i s_i^4 - \sum_{i<j} s_i^2 s_j^2\Big)\Big] + \mathcal{O}\Big(\frac{1}{r}\Big)\,. \tag{5.18}$$

After performing the surface integration as in (5.9), we obtain the expresssion

$$E' = E - \tfrac{1}{3}\pi m^2 g^2 \Big(\sum_i s_i^4 - \sum_{i<j} s_i^2 s_j^2\Big) \tag{5.19}$$

for the AD mass, where E is the mass obtained in [26] by integrating the first law, and that we reproduced in this paper using the AMD procedure in (3.15).

5.3.4 Four-dimensional black hole

To apply the AD procedure to calculate the mass for the four-dimensional gauged supergravity black holes discussed in section 4.1, we first need to make the coordinate transformation

$$r \to r - m(s_1^2 + s_2^2) + \frac{m^2(s_1^2 - s_2^2)^2}{2r}\,, \tag{5.20}$$

in order to ensure that $h_{\mu\nu}$ falls off at large distance. We then find

$$\mathcal{M}^r = \frac{m^2 g^2 (s_1^2 - s_2^2)^2 r \sin\theta}{1 - a^2 g^2} + \frac{m(1 + s_1^2 + s_2^2)[-2 + a^2 g^2(-1 + 3\cos^2\theta)]\sin\theta}{(1 - a^2 g^2)^2} + \mathcal{O}\Big(\frac{1}{r}\Big)\,, \tag{5.21}$$

which actually diverges as r is sent to infinity.

5.3.5 Seven-dimensional black hole

The implementation of the AD procedure for calculating the mass of the seven-dimensional gauged supergravity black holes that we discussed in section 4.2 is rather straightforward. In the coordinate frame we are using, the components $h_{\mu\nu}$ already tend to zero at large distance. We then find that the AD mass calculated from (5.9) agrees precisely with the expression (4.11) which was obtained in [26]) by integrating the first law, and that we reproduced in this paper by applying the AMD procedure. (We have also checked the AD calculation of the mass for the non-rotating black hole in six-dimensional gauged supergravity found in [28], and found that it agrees with the mass calculated using the AMD method.)

In the following subsection, we shall discuss in more detail the problems we encountered above in calculating the AD masses of the five and four-dimensional black hole examples. Before moving on to this, it is perhaps worth pointing out that in the four cases where we have encountered difficulties in determining the mass by the AD procedure, summarised by equations (5.13), (5.16), (5.19) and (5.21), one gets the correct answer if follows the "rule of thumb" of retaining only the terms linear in the mass or charge, and discarding terms of higher order in the mass or charge.

5.4 Subtleties in the AD procedure

It has been remarked upon previously in the literature that the background-subtraction prescription inherent in the AD definition of the mass can lead to ambiguities associated with coordinate reparameterisations of the metric and the background. (See, for example, [14, 29].) On the other hand, it has been applied successfully to calculate the masses of the rotating AdS black holes in arbitrary dimension [22], and we have applied it successfully in this paper in the case of the five-dimensional charged rotating black holes of five-dimensional minimal gauged supergravity, and charged solutions of seven-dimensional gauged supergravity. All the cases that we tried where it failed involve solutions with non-trivial scalar fields, and as we shall discuss below, these scalar fields can have a quite significant contribution in the calculation of the energy. Although the seven-dimensional black holes, for which we obtained the correct mass by the AD approach, also involve non-trivial scalars, we find that in this case the scalars make a less significant contribution to the energy, in a way that we shall elaborate on below.

It might be natural to suppose that the difficulties we have encountered are ultimately related to some ambiguities in the decomposition of the metric into AdS background plus deviations, and that these ambiguities become more acute in the cases where the scalar fields play a rôle.

5.4.1 Scalar fields and the AD mass formula

Although we successfully applied the AD procedure above to calculate the mass of the rotating black hole in five-dimensional minimal gauged supergravity, we find that in more complicated situations we encounter problems in extracting results by using the AD method. In fact the difficulties can already be illustrated if we consider the example of 3-charge non-rotating black holes in five-dimensional maximal gauged supergravity, for which the relevant bosonic Lagrangian is given by (3.1).

The 3-charge non-rotating black hole in five-dimensional gauged supergravity is given by [30]

$$\begin{aligned} ds^2 &= -(H_1 H_2 H_3)^{-2/3}\, f\, dt^2 + (H_1 H_2 H_3)^{1/3} \Big(\frac{dr^2}{f} + r^2 d\Omega_3^2\Big)\,, \\ X_i &= H_i^{-1}\, (H_1 H_2 H_3)^{1/3}\,, \qquad A^i = (1 - H_i^{-1})\, \coth\delta_i\, dt\,, \end{aligned} \tag{5.22}$$

where

$$f = 1 - \frac{2m}{r^2} + g^2 r^2\, H_1 H_2 H_3\,, \qquad H_i = 1 + \frac{2m\, \sinh^2\delta_i}{r^2}\,. \tag{5.23}$$

The mass of the black hole is given by [30]

$$E = \frac{\pi\, m}{4}\,\Big(3 + 2\sum_i s_i^2\Big)\,, \tag{5.24}$$

where $s_i \equiv \sinh\delta_i$.

In the AD calculation of the mass, the background AdS metric satisfying (5.3) is most easily obtained by setting the mass parameter m and charge parameters δ_i to zero in (5.22). However, in the coordinate frame used in (5.22), one finds that the components of $h_{\mu\nu}$ defined by (5.1) do not fall off at large r. This can be remedied by performing the coordinate transformation $r \to \rho$, where

$$r^2 = \rho^2 - \tfrac{2}{3}m\,(s_1^2 + s_2^2 + s_3^2)\,. \tag{5.25}$$

Substituting into (5.9), we find that at large ρ, the integral gives the expression

$$\begin{aligned} E' &= -\frac{1}{8\pi}\oint d\Sigma_i\,\sqrt{-\bar g}\,\Big[(\bar\xi_\nu\bar\nabla_\mu K^{ti\nu\mu} - K^{tj\nu i}\,\bar\nabla_j\bar\xi_\nu\Big] \\ &= \frac{\pi\, m}{4}\,\Big(3 + 2\sum_i s_i^2\Big) - \tfrac{1}{3}g^2 m^2 (s_1^4 + s_2^4 + s_3^4 - s_1^2 s_2^2 - s_2^2 s_3^2 - s_3^2 s_1^2)\,. \end{aligned} \tag{5.26}$$

This disagrees with the standard result (5.24), unless one sets the three charges equal.

The disagreement appears to be associated with the presence of the non-trivial scalar fields. In fact, if we calculate the contribution of potential energy term in (5.6) for this solution, we find that this contributes

$$-\frac{1}{8\pi\sqrt{-\bar g}}\,T^{t\nu}_{\rm pot}\,\bar\xi_\nu = \frac{g^2 m^2}{48\pi\,\rho}\,\sum_{i<j}(s_i^2 - s_j^2)^2\,\sin\theta\cos\theta + \mathcal{O}(1/\rho^3) \tag{5.27}$$

to the energy density. When integrated over the spatial 4-volume, this would give rise to a logarithmic divergence at large distance. In fact the scalar kinetic terms contribute an equal and opposite divergence,

$$-\frac{1}{8\pi\sqrt{-\bar g}}\,T^{t\nu}_{\rm kinetic}\,\bar\xi_\nu = -\frac{g^2 m^2}{48\pi\,\rho}\,\sum_{i<j}(s_i^2 - s_j^2)^2\,\sin\theta\cos\theta + \mathcal{O}(1/\rho^3) \tag{5.28}$$

(Of course it is the $(\partial\phi)^2\, g^{\mu\nu}$ term, and not the $\partial^\mu\phi\,\partial^\nu\phi$ term in the scalar energy-momentum tensor that contributes here, since the solution is time-independent.) Although the scalar energy density therefore integrates to a finite total, it is possibly significant that the potential and kinetic energies are separately divergent.

By contrast, if we calculate the analogous scalar energy contributions in the case of 2-charge black holes in seven-dimensional gauged supergravity (see, for example, [31] for details of these solutions), we find that both the potential and kinetic energy densities integrate to give separately finite energy contributions. Significantly, we find in this case that (as also for the rotating seven-dimensional black holes discussed in section 5.3.5) the AD calculation of the mass agrees with the thermodynamic calculation and the AMD calculation.

The situation is even more striking in four dimensions. Let us consider the non-rotating 4-charge black holes of maximal gauged supergravity [32, 33], in the notation given in [31] (but with the gauge coupling rescaled according to $g \to g/2$, so that $1/g = l$, the AdS radius). Again, we perform a radial coordinate redefinition,

$$r = \rho - \tfrac{1}{2} m \sum_i s_i^2 + \frac{m^2}{3\rho} \sum_{i<j} (s_i^2 - s_j^2)^2 \,, \tag{5.29}$$

so that the components $h_{\mu\nu}$ fall off at large distance. We find that the scalar potential energy and kinetic energy both diverge (with linear and logarithmic divergences at large distance), but now, the total scalar energy also diverges (with a linear, but no logarithmic, divergence):

$$\begin{aligned}
-\frac{1}{8\pi\sqrt{-\bar g}}\, T^{t\nu}_{\rm pot}\, \bar\xi_\nu &= -\tfrac{1}{4\pi} g^2 m^2 \sum_{i<j} (s_i^2 - s_j^2)^2 \, \sin\theta \\
&+ \frac{3g^2 m^3 (s_1^2 + s_2^2 - s_3^2 - s_4^2)(s_1^2 - s_2^2 + s_3^2 - s_4^2)(s_1^2 - s_2^2 - s_3^2 + s_4^2)}{4\pi\, \rho} \sin\theta + \mathcal{O}(1/\rho^2) \,, \\
-\frac{1}{8\pi\sqrt{-\bar g}}\, T^{t\nu}_{\rm kinetic}\, \bar\xi_\nu &= \tfrac{1}{8\pi} g^2 m^2 \sum_{i<j} (s_i^2 - s_j^2)^2 \, \sin\theta \\
&- \frac{3g^2 m^3 (s_1^2 + s_2^2 - s_3^2 - s_4^2)(s_1^2 - s_2^2 + s_3^2 - s_4^2)(s_1^2 - s_2^2 - s_3^2 + s_4^2)}{4\pi\, \rho} \sin\theta + \mathcal{O}(1/\rho^2) \,.
\end{aligned} \tag{5.30}$$

To conclude this section, we shall consider two possible options for modifying the AD prescription, so as to obtain the proper expressions for the mass in the presence of scalar fields. In fact, both have featured in earlier discussions in the literature.

Option A:

The divergence in the volume integral for the total energy of the scalar fields in four dimensions can in fact be removed, if one makes an integration by parts in the kinetic energy contribution for each scalar, of the form

$$\begin{aligned}
-\frac{1}{8\pi}\int_V d^3x T^{t\nu}_{\rm kinetic}\, \bar\xi_\nu &= \frac{1}{32\pi}\int_V \sqrt{-\bar g}\, (\partial\phi)^2 \, d^3x \\
&= -\frac{1}{32\pi}\int_V \sqrt{-\bar g}\, \phi \bar\Box \phi \, d^3x + \frac{1}{32\pi}\int_V \partial^\mu (\sqrt{-\bar g}\, \phi \partial^\mu \phi)\, d^3x \\
&= -\frac{1}{32\pi}\int_V \sqrt{-\bar g}\, \phi \bar\Box \phi \, d^3x + \frac{1}{32\pi}\oint_{\partial V} dS_i \sqrt{-\bar g}\, \phi \partial^i \phi \,,
\end{aligned} \tag{5.31}$$

and then uses $-\phi\bar{\Box}\phi$ rather than $(\partial\phi)^2$ in the definition of the bulk energy-momentum tensor for each scalar field. This suggests therefore that one could define a "corrected" AD mass in situations where there are scalar fields, in which one adds an extra scalar boundary term to the expression given in (5.9), so that

$$E = \frac{1}{8\pi} \oint dS_i (\mathcal{M}^i + \mathcal{N}^i)\,, \tag{5.32}$$

where $\mathcal{M}^i$ is still as given in (5.9), and the extra term $\mathcal{N}^i$ is given by

$$\mathcal{N}^i = -\tfrac14 \sqrt{-\bar{g}}\, \bar{g}^{ij}\, \mathcal{G}_{IJ}(\phi)\, \phi^I\, \partial_j \phi^J\,. \tag{5.33}$$

Note that here, we are allowing for the general situation of scalar fields ϕ^I with a non-linear sigma-model kinetic Lagrangian given by

$$\mathcal{L}^{\rm kinetic} = -\tfrac12 \mathcal{G}_{IJ}(\phi)\, \partial_\mu \phi^I\, \partial^\mu \phi^J\,. \tag{5.34}$$

Remarkably, we find that with the inclusion of the $\mathcal{N}^i$ term in the AD formula, we now obtain the correct results for the mass of the 3-charge black holes (5.22) in five dimensions, and also for the rotating black holes in five dimensions that were described in sections 5.3.1, 5.3.2 and 5.3.3, and for the rotating black holes that were described in section 5.3.4, where we previously obtained incorrect masses using the AD prescription. It also leaves unaffected the already-correct result in seven dimensions. In fact we have only found one example where (5.32) with (5.33) fails to give the correct result for the mass, and that, ironically enough, is the very example that we used for motivating the introduction of the correction term, namely the four-dimensional non-rotating black hole with 4 unequal charges. (If the charges are set pairwise equal, which corresponds to the non-rotating limit of the rotating black-hole solution of [25], then (5.32) gives the correct mass.)

The extra term involving $\mathcal{N}^i$ that we have added to the AD calculation of the mass is strikingly similar to the surface term introduced by Hertog, Horowitz and Maeda in a discussion of negative-energy solutions in five-dimensional maximal gauged supergravity [34]. There, a surface term of the form $\oint dS_\mu\, \phi\, \partial^\mu \phi$ was added to the action for a scalar field ϕ, leading to a corresponding correction to the Hamiltonian and hence to the mass.[4]

Option B:

A detailed discussion of the definition of mass in asymptotically AdS backgrounds with scalar fields has been given in [35, 36]. The focus in these papers was on cases where the

[4] We thank Gary Gibbons for drawing our attention to [34] after this paper was completed.

scalars have masses that saturate the Breitenlöhner-Freedman stability bound, namely

$$m^2 = -\frac{(D-1)^2}{4l^2} \tag{5.35}$$

in D dimensions. The reason for considering these limiting cases was that the possibility then arises of a less rapid fall-off for the scalar fields fields, with a logarithmic dependence on the asymptotic radial coordinate. However, even in the absence of this logarithmic behaviour, it was shown in [35, 36] that the scalar fields can provide a contribution to the total energy, if one makes a decomposition of the full metric as in (5.1). Translated into the notation that we are using in this paper, we find that the scalar surface-integral modifications of the type considered in [36] can be expressed as a different modification of the AD mass formula (5.9), analogous to the modification in (5.32), except that now we have

$$E = \frac{1}{8\pi} \oint dS_i(\mathcal{M}^i + \widetilde{\mathcal{N}}^i)\,, \tag{5.36}$$

with $\widetilde{\mathcal{N}}^i$ given by

$$\widetilde{\mathcal{N}}^r = \frac{\sqrt{-\bar{g}}\, r}{4(D-1)} \Big(\phi^I\, \phi^J\, \frac{\partial^2 V(0)}{\partial\phi^I\, \partial\phi^J} + \mathcal{G}_{IJ}(\phi)\, g^{ij}\, \partial_i\phi^I\, \partial_j\phi^J\Big)\,. \tag{5.37}$$

Note that the first term in the large parentheses is just the contribution of the squared masses of the scalar fields, since a scalar field ϕ has mass squared given by

$$m^2 = -\frac{\partial^2 V(0)}{\partial\phi^2}\,. \tag{5.38}$$

A more compact way to write the correction term in this AD mass formula is

$$\widetilde{\mathcal{N}}^r = -\frac{r}{D-1}\, (T_{\rm scal})^t{}_t\,, \tag{5.39}$$

where $T^{\mu\nu}_{\rm scal}$ is the effective energy-momentum tensor density for the scalar fields, as defined in equation (5.6).

We find that using this modification, the AD mass formula (5.36) with (5.37) or (5.39) then gives expressions that agree in all cases with those that we obtained by using the AMD mass formula. This includes not only all the rotating black-hole solutions, but also the non-rotating four-dimensional solution with four unequal charges, for which a discrepancy still remained if we used the modification (5.32) with (5.33).

It is worth recording that except in five fimensions, the scalar fields participating in the black-hole solutions in the various gauged supergravities that we have been considering have (mass)^2 values that exceed the Breitenlöhner-Freedman bound. Specifically, the relevant scalars arising in the theories in $D = 4, 5, 6$ and 7 have $m^2 = (-2, -4, -6, -8)l^{-2}$, which can be contrasted with the corresponding Breitenlöhner-Freedman masses $m^2 = (-9/4, -4, -25/4, -9)l^{-2}$. Of course in all cases, the scalar fields we are using have apparent mass terms that imply "masslessness" in the appropriate sense of being members of massless supermultiplets.

Finally, it should be emphasised that if instead we use the AMD procedure to calculate the masses for the general non-rotating black holes in five, seven, six and four dimensions, and the various known rotating black holes, we always get the correct mass without the need for including any scalar modification terms. In other words, the AMD procedure yields the correct results for the masses by making reference only to the metric.

6 Discussion and Conclusions

In this paper, we have principally been concerned with calculating the masses of the various recently-discovered rotating black hole solutions in gauged supergravities in five, four and seven dimensions. Until now, the masses for the examples we have considered in this paper had been calculated only by integrating the first law of thermodynamics. This has proved to be a reliable and straightforward procedure for calculating the mass, which avoids some of the ambiguities inherent in certain other approaches. On the other hand, the thermodynamic calculation is somewhat indirect, and does not emphasise the explicit relation between the energy and a conservation law. For this reason, it is of considerable interest to perform alternative calculations of the masses of the rotating black holes, that more directly relate the answer to conservation laws.

We have focused on two such approaches in this paper, each of which comes with its attendant advantages and disadvantages. We first considered the Ashtekar-Magnon-Das procedure for calculating the mass of an asymptotically AdS spacetime. The AMD procedure involves integrating a certain electric component of the Weyl tensor over the spatial section of the compactified conformal boundary. It is inherently well-defined, without the need for any subtraction, since the Weyl tensor falls off suitably rapidly at large distance as the metric approaches AdS spacetime. The calculation is insensitive to coordinate choices, and we have encountered no ambiguities at all when calculating the masses for the various rotating black holes in gauged supergravities that are known. Furthermore, the masses that we have calculated using the AMD prescription agree with the those obtained previously by integrating the first law of thermodynamics.

The only difficulties that one encounters when following the AMD procedure are of a purely calculational nature, in that one has to evaluate a certain component of the Weyl tensor of the full black-hole metric (at least to leading order in an expansion in powers of the inverse distance). By contrast, the Abbott-Deser approach would be computationally somewhat simpler, since one need only take first derivatives of the deviation $h_{\mu\nu}$ and the timelike Killing vector, and furthermore the derivatives are covariant just with respect to the background AdS metric. In practice the calculations for rotating asymptotically AdS black holes are sufficiently complicated that in either approach it is highly advantageous to use a computer, and so the greater complexity of the AMD approach does not necessarily represent a severe obstacle.

In certain cases the AD approach is relatively easy to implement, for example in the uncharged rotating AdS metrics as discussed in [22], and in the general charged rotating solutions of five-dimensional minimal gauged supergravity, which we analysed in section 5.2. The idea is to write the asymptotically AdS metric $g_{\mu\nu}$ as the sum of a background AdS metric $\bar{g}_{\mu\nu}$ plus deviation terms $h_{\mu\nu}$, and work in a frame where $h_{\mu\nu}$ falls of appropriately at infinity. However, we encountered difficulties when trying to apply the AD procedure to the five and four-dimensional rotating black hole solutions in which scalar fields play a non-trivial rôle. We considered two ways to modify the original AD definition of the mass, to incorporate the effects of the scalar fields. One of these, given by (5.32) with (5.33), is related to a modification introduced in [34]. We found that it then led to agreement with the AMD and thermodynamic masses for all the known rotating AdS black-holes solutions, but a discrepancy remained in the case of four-dimensional AdS black holes with four unequal charges. We then considered a different modification to the AD mass formula, generalising one introduced in [36], and we found that this, given by (5.36) with (5.37) or (5.39), led to agreement with the AMD and thermodynamic masses for all the black hole examples, including the four-dimensional black hole with four unequal charges.

Since the AMD approach gives reliable results for the masses of all the black holes, which agree with the thermodynamic calculations without the need for any scalar modifications, it would seem to be a more "robust" prescription than the AD approach. It yields expressions for the masses by making reference only to the metric iteslf, and not to the scalar fields. Furthermore, the AMD approach does not involve the potentially hazardous process of decomposing the black hole metric as a deviation $h_{\mu\nu}$ from a background AdS metric $\bar{g}_{\mu\nu}$. The hazards of this decomposition are highlighted in the example of the non-rotating four-dimensional gauged supergravity black hole with four unequal charges, where the integral of the background-subtracted scalar energy density is actually divergent at large distance. This raises questions about the validity of the assumption that the deviation $h_{\mu\nu}$ is sufficiently small asymptotically. More generally, the whole question of how one should split the solution into background plus deviation in the AD approach is somewhat unclear, and the results might be affected by choices of field variable or coordinate reparameterisations beyond those that we have considered.

There are other methods that have also been used in order to calculate the mass of asymptotically AdS spacetimes. For example, as we mentioned in the introduction, one could use a Komar integral, although there are complications associated with the need to regularise the divergent result by performing a background AdS subtraction. (However, see [37] for a recent discussion of this approach, and its relation to the AMD method.) Another approach to calculating the mass of asymptotically AdS spacetimes is via the holographic stress tensor, introduced in the context of the AdS/CFT correspondence in string theory [38–43]. As was shown in [43] for the uncharged rotating AdS black holes, this yields bulk masses that are in agreement with those obtained previously from the integration of the first law and from the AMD approach. It also yields Casimir contributions, which are not relevant in a classical discussion of the energy of a black hole, but which do play a rôle in the AdS/CFT correspondence and the map to the boundary theory. As was demonstrated explicitly in [44], for the most natural choice of conformal boundary metric the Casimir energy is a pure constant, independent of the parameters of the black hole, and thus its inclusion need not complicate the discussion of the thermodynamics.

Finally, we also carried out a check of the consistency of the quantum statistical relation (1.4) for the general charged rotating black holes in minimal five-dimensional gauged supergravity. This involved calculating the Euclidean action for the solution, and comparing it with the thermodynamic potential. This calculation also is rather subject to subtraction ambiguities, since one has to subtract the action of a pure AdS spacetime with the same boundary as that of the black-hole metric in order to obtain a finite Euclidean action. We did not carry through this procedure for the other, more complicated, examples of charged rotating black holes, leaving this for future investigation.

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