On Instantons and Finite-Size D-Branes in String Theory

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Zusammenfassung

In dieser Dissertation untersuchen wir die Existenz von D-Branen mit endlicher Größe (im Gegensatz zu punktförmigen) in der Stringtheorie. Insbesondere betrachten wir gebundene Zustände von D-Instantonen und höherdimensionalen D-Branen und untersuchen die Möglichkeit die D-Instantonen aufzublasen.

Basierend auf der Äquivalenz zwischen getwisteten Bosonen auf einem Kreis und der Orbifold-Theorie am kritischen Radius erhalten wir eine bosonisierte Darstellung von Twistfeldern und somit eine freie Felddarstellung der letzteren. Dies erlaubt es, den Modulraum der marginalen Deformationen von gebundenen Zuständen von D-Branen zu untersuchen. Wir zeigen, dass das Aufblasen von D-Branen in der bosonischen Stringtheorie in zweiter Ordnung in der Größe obstruiert wird, sowohl aus Sicht der Weltfläche als auch der Stringfeldtheorie.

Wir erweitern die Analyse auf die Superstringtheorie, insbesondere auf den gebundenen Zustand von D-Instantonen und D3-Branen. Wir zeigen, dass die marginale Deformation, die das Aufblasen einer an einen D3-Hintergrund gebundenen D(-1)-Brane mit Größe null beschreibt, in der dritten Ordnung in der Größe der D(-1)-Brane obstruiert wird. Diese Obstruktion kann jedoch durch einen geeigneten Nullimpuls-Gluon-Hintergrund beseitigt werden. Diese Obstruktion ist auf Feinheiten in der Integration über ungerade Moduli im Supermodulraum zurückzuführen, die vom Standard-Weltflächen-Ansatz in Bezug auf Vertexoperatoren mit verschiedenen Picturedefiziten übersehen werden. Auf der anderen Seite bestätigt dies die Intuition, dass D-Branen ausschließlich die Rolle der effektiven Beschreibung der Weltfläche für Instantonen der Größe null spielen, was ein singulärer Punkt ihres Modulraums ist.

Da die Deformation in zweiter Ordnung in der Größe nicht obstruiert wird, ist es möglich, ein Instanton-Profil im Rahmen der Superstringtheorie zu definieren. Wir überprüfen diese Herleitung, verbinden sie mit den Yang-Mills-Instantonen und erweitern sie mit α'-Korrekturen.

Abstract

In this thesis we study the possibility of defining finite-size (as opposed to pointlike) D-branes in string theory. In particular, we consider bound states of D-instantons and higher dimensional D-branes and we explore the possibility of blowing up of the size of the D-instantons.

Based on the equivalence between twisted bosons on a circle and the orbifold theory at the critical radius, we provide a bosonized representation of boundary twist fields and thus a free field representation of the latter. This allows to explore the moduli space of marginal deformations of bound states of D-branes. We show that the blow-up of the size of D-branes in bosonic string theory is obstructed at second order in the size, both from the worldsheet and string field theory point of view.

We extend the analysis to superstring theory, in particular to the bound state of D-instantons and D3 branes. We show that the marginal deformation describing the blow-up of a zero-size D(−1) brane bound to a background of D3 branes is obstructed, at third order in the size of the D(−1) brane, by analyzing the equations of motion of superstring field theory at this order. However, this obstruction can be removed by an appropriate zero-momentum gluon background. This obstruction is due to subtleties in the integration over odd moduli in super-moduli space, which are missed by the standard worldsheet approach in terms of vertex operators of various pictures. On the other hand, this confirms the intuition that D-branes play the role of effective worldsheet description of zero size instantons, which is a singular point of their moduli space.

Since the deformation is not obstructed at second order in size, it is possible to define an instanton profile in the context of superstring theory. We review this derivation, connecting it to Yang-Mills instantons and extending it to include α'- corrections.

Contents

Chapter 1

Introduction

1.1 Motivation

The goal of theoretical physics is to achieve a mathematical description of reality, in particular of fundamental particles and forces. Progress in theoretical physics can be achieved in two ways: the first one is when the theory is able to describe some experimental facts that were previously inexplicable or predict new phenomena. For example, Einstein's theory of general relativity was able to correctly predict the perihelion precession of Mercury, which was previously unexplained. Furthermore, the theory predicted the precise deflection of light caused by gravity, and the existence of black holes and gravitational waves, which were all experimentally confirmed afterwards. The second type of progress theoretical physics can aim at is summarized by the concept of "unification". This is achieved whenever different phenomena can be explained by the same underlying theory, highlighting a connection that was previously hidden. For instance, Maxwell's theory unified the preexisting theories of electricity and magnetism, and the standard model of particle physics unified the theories of electromagnetic and nuclear interactions. Typically, the process of unification relies on the identification of few basic rules, that every theory should satisfy. After centuries of research, theoretical physicists have identified few fundamental principles: locality, unitarity and gauge invariance.

The easiest example of the importance of these principles is Maxwell's theory. Initially formulated in terms of the electric and magnetic field strengths $\vec{E}$ and $\vec{B}$, Maxwell's theory is local and unitary, and successfully describes many electromagnetic phenomena, including electromagnetic waves. Gauge invariance, however, appears only when the theory is reformulated in terms of a vector potential A_μ; this new formulation of Maxwell's theory is able to reproduce the electric and magnetic fields, but is subject to gauge redundancy: different vector potentials A_μ and A'_μ give rise to the same electromagnetic fields if they are related by a local gauge transformation $A'_\mu = A_\mu + \partial_\mu \Lambda$, where Λ is a scalar. Formally speaking, Maxwell's theory is invariant under local $U(1)$ transformations. This may seem an unnecessary complication; however, it turns out that this new mathematical description is simpler, more elegant and more powerful than the previous one. Furthermore, after the discovery of quantum properties of nature, such as the Aharonov-Bohm effect, it became clear that the vector potential should be regarded as a fundamental field, and not just as a mathematical tool.

The role of gauge invariance is evident also in the standard model of particle physics.

This theory is based on a generalization of Maxwell's theory, the so-called Yang-Mills theory, where the $U(1)$ gauge group is replaced by a bigger, non-abelian, gauge group, typically $SU(N)$. In particular, the standard model is characterized by a gauge group given by the product $U(1) \times SU(2) \times SU(3)$, with the addition of fermions and a Higgs sector. The standard model of particle physics has been extremely successful in describing and predicting various phenomena governed by the electromagnetic and nuclear forces; recently, the discovery of the Higgs boson at the LHC has confirmed the validity of the Higgs sector of the theory.

There are, however, some limitations in the predictive power of the standard model; the main one is due to perturbation theory. Typically, interactions are interpreted through the exchange of virtual particles; the total effect of an interaction has to be recovered order by order, in an expansion in the number of virtual particle states. Such a process is represented graphically in terms of Feynman diagrams, and of course is valid only in the framework of perturbation theory. As a consequence, results derived in this way are not exact, but are an approximation; the precision in the prediction depends on how small the coupling constant is, and how many orders in perturbation theory (or how many loops in Feynman diagrams) are considered. The perturbative approach fails when the theory is strongly coupled; this happens for quantum chromodynamics (QCD), since one cannot observe free quarks and gluons. This observation encouraged theoretical physicists to develop methods for finding exact non-perturbative solutions to quantum field theories. Such solutions do not have an interpretation in terms of particles, and give rise to concepts like domain walls, monopoles and instantons. In this thesis we are particularly interested in instantons, which are solutions of the classical equations of motion of a theory formulated in Euclidean space, which play an important role when studying quantum effects of the theory in Minkowski space. For Yang-Mills theory they are used to describe tunneling behaviors between different vacua of the theory, which are not captured by the perturbative approach.

Historically, quantum field theory was not the only approach to particle physics. A different proposal was the S-matrix theory, which focused on the properties of the S-matrix, which is supposed to connect states in the infinitely far past and future, without needing details on the intermediate steps. This approach is based on physical particle states, and the form of the S-matrix is restricted by a postulated set of symmetries. The application of the S-matrix approach to the strong interactions led to the development of string theory. In fact, one of the first compatible S-matrices that was found was the so-called dual resonance model. It turns out that this model can be interpreted as a theory where the fundamental objects are not particles, but strings. Even though the S-matrix theory was later abandoned in favor of QCD, string theory remained interesting to many physicists, since its spectrum includes massless spin 2 particle states. Consequently string theory was expected to describe interactions among gravitons. On the other hand, the spectrum of string theory contains massless spin 1 particles, able to describe gauge interactions; therefore, string theory was, and still is, considered as a promising candidate for a unified theory describing all interactions.

String theory is consistent only as a supersymmetric theory on the worldsheet, containing both bosons and fermions; a purely bosonic theory is, in fact, unstable due to the presence of a tachyonic state in the spectrum. Superstring theory is naturally defined in

10 spacetime dimensions, and is then related to lower dimensional quantum field theories by a process of compactification. Even though its prediction of spacetime supersymmetry at high energies has not been verified by the experiments so far, superstring theory provides a simpler and more elegant framework to study supersymmetric field theories; this is one of the biggest advantages of string theory.

The usual formulation of string theory is in the framework of "first quantization", where the fundamental degrees of freedom are the coordinates of the strings. The worldsheet of strings is defined in terms of a sigma model path integral with conformal two-dimensional gravity. Interactions among strings are then computed through a perturbative expansion in the string coupling constant, starting from a fixed background. This approach has been successful, reproducing the spectrum and the basic properties of gauge theories, as well as of general relativity. Furthermore, some non-perturbative effects are known in string theory, in particular D-branes; the latter are dynamical objects of the theory, analogous to instantons and monopoles in QFT. Since non-perturbative effects are usually the most complicated and counter-intuitive properties of a given theory, it is of great importance to establish a connection between such effects in different theories.

The goal of this thesis is to explore the connection between D-branes in string theory and instantons in Yang-Mills theory. In particular we want to investigate the possibility of defining finite-size (non pointlike) D-branes, in analogy to the presence of instanton of various sizes in QFT. An important comment has to be made here. The classical formulation of string theory is not completely satisfactory; as gauge theories have shown, a field theory, or "second quantization", approach is probably needed. We will follow both the worldsheet approach and the string field theory approach, and we will highlight when the first one fails, and the second one is necessary. Furthermore, we will start considering bosonic theories (pure Yang-Mills and bosonic string theory). We will then argue why the presence of fermions is required; therefore, we will later consider the supersymmetric extensions of string and Yang-Mills theory.

The rest of the introduction gives a short overview of instantons in Yang-Mills theory, string theory and D-branes. We will review these concepts in more detail in the following chapters.

1.2 Instantons in Yang-Mills theory

Yang-Mills theory is a generalization of Maxwell's theory with a non-abelian gauge group; in particular we consider $SU(N)$. The vector potential A_μ is matrix valued; precisely it belongs to the adjoint representation of the gauge group. The action of the theory is a simple generalization of the one characterizing Maxwell's theory of electromagnetism and reads

$$S_{YM} = \frac{1}{g^2} \int d^4x \; \mathrm{Tr}\left\{ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \right\}, \tag{1.1}$$

where g is the coupling constant and the field strength $F_{\mu\nu}$ is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. In the action (1.1) we have omitted the matrix indices of A_μ and $F_{\mu\nu}$.

Considering now the theory in Euclidean space, there is a class of exact solution of the equations of motion (2.1), which are characterized by a self-dual or anti-self-dual field strength. The presence of instantons distinguishes Yang-Mills theory from the abelian Maxwell's theory.

The simplest example of Yang-Mills instantons occurs for the gauge group is $SU(2)$. In this case we can use Pauli matrices τ^c ($c = 1, 2, 3$) as a basis, and write

$$A_\mu = A_\mu^c \frac{\tau^c}{2i} , \tag{1.2}$$

where $1/(2i)$ is the standard normalization. An instanton solution then reads

$$A_\mu^c(x) = 2\frac{\eta_{\mu\nu}^c (x - x_0)^\nu}{(x - x_0)^2 + \rho^2} , \tag{1.3}$$

where $\eta_{\mu\nu}^c$ are a set of numbers called 't Hooft symbols. The solution depends on five parameters: the position x_0^μ and the size of the instanton ρ. It can be thought of as a generalization of the electromagnetic vector potential due to a charge positioned at the point x_0^μ. The analogy, however, stops here, since the instanton (1.3) has a non-vanishing size ρ and, more importantly, is characterized by non-trivial topological properties, as we will see in chapter 2.

Furthermore, the instanton (1.3) is a non-perturbative solution of the equations of motion deriving from the action (1.1); in fact it is an exact solution of the theory and does not depend on the value of the coupling constant g.

1.3 String theory and D-branes

It is well-known in the literature that instantons are connected to certain objects in string theory, the so-called D-branes. Before giving the details in the following chapters, we want to highlight here some similarities and differences between instantons and D-branes.

The basic constituents of string theory, the strings, are a two-dimensional generalization of the concept of particles; they can either be closed or open. D-branes are other dynamical objects of the theory, and represent hyperplanes embedded in the spacetime, where open strings can attach (see figure 1.1).

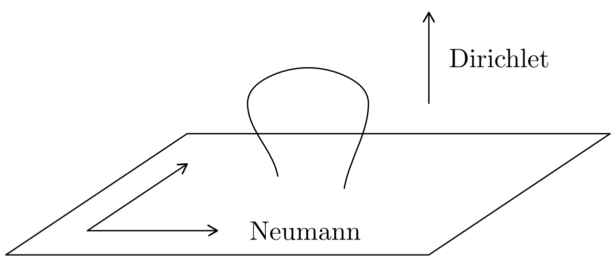

Figure 1.1: Open string with endpoints lying on a D-brane.

Like instantons, D-branes have a non-perturbative (in the string coupling constant g_s) nature.

Unlike for quantum field theories (e.g. Yang-Mills theory) the usual approach to string theory is in the framework of first quantization. The dynamical variables of the theory are the spacetime coordinates of the strings; when the theory involves open strings, certain boundary conditions (Neumann or Dirichlet) must be imposed at the endpoints. This means that the string theory action does not directly involve fields, like the gauge vector A_μ in Yang-Mills theory.

The D-brane setup that is directly connected to the four-dimensional instantons described above is a bound state of four-dimensional (D3) and instantonic (D(−1)) branes (see figure 1.2).

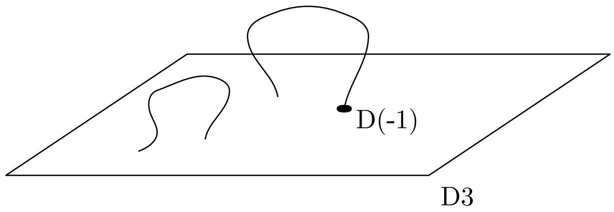

Figure 1.2: Open strings on the bound state of D3 and D(−1) branes, with different strings stretching between them.

The four-dimensional background represents the four-dimensional spacetime in which Yang-Mills theory lives, and the instantonic brane is analogous to the instanton solution.

We will see later that giving an expectation value to the strings stretching between branes of different types corresponds to switching on a gauge vector which, at large distances and in the limit where string effects disappear, looks like the instanton solution in singular gauge. One natural question to ask is whether the correspondence can be made complete, i.e. whether it is possible to recover the full instanton profile, not only at large distances. This would correspond to "blowing up" the size of the instantonic brane inside the D3 background, and it is the basic idea behind the project described in this thesis.

1.4 Content of the thesis

The content of this thesis is as follows. In chapter 2 we revisit instantons in Yang-Mills theory in more detail, and explain the idea of finding new solutions by deformation, in the contexts of field theory and string field theory. This will serve as a guiding line for the whole thesis.

In chapter 3 we review important background material. We discuss the conformal field theory of bosons and fermions in two dimensions, which will be the building blocks of string theory in the worldsheet approach. We discuss in particular the theories in the presence of boundaries, and the possibility of Dirichlet and Neumann boundary conditions; this will be useful when dealing with open strings and D-branes. Therefore, we study bosons and fermions both in the Neveu-Schwarz (NS) and Ramond (R) sectors. We review the process of bosonization for fermions, and finally we introduce twist fields, the operators characterizing the Ramond sector of a bosonic conformal field theory.

Chapter 4 is devoted to the discussion of the conformal field theory of bosons in the presence of twist fields. Using the connection to orbifold theories, we explore the possibility of a bosonization procedure for such operators. Furthermore we discuss in detail correlation functions involving up to four twist fields.

In chapter 5 we review the basics of bosonic string theory; we discuss the role of the ghosts b and c and we introduce the concept of D-branes. We highlight the connection between bound states of D-branes and bosons in NS and R sectors. We study some notable bound states, with difference of dimensions equal to 4, 8 and 16. For the D15-D(-1) system, in particular, we study the marginality of a deformation corresponding to increasing the size of the instantonic brane.

The study of bosonic string theory (and, correspondingly, pure Yang-Mills theory) turns out to be insufficient to describe the properties of instantons. Therefore the problem has to be studied in the context of superstring theory. Before doing that, chapter 6 contains a review of $\mathcal{N} = 4$ Super Yang-Mills theory, the maximally supersymmetric extension of the Yang-Mills theory discussed in this introduction. We present the theory both in Minkowski and Euclidean space, focusing then on instantonic solutions.

In chapter 7 we review the basics of superstring theory. In analogy to chapter 5 we introduce the β-γ ghost system, and we generalize the idea of D-branes. At the end of the chapter we focus on a particular bound state of D-branes, the D3-D(-1) system. We discuss all the vertex operators describing oscillations of the strings attaching to these branes, and we review the connection with $\mathcal{N} = 4$ super Yang-Mills theory and its instantons.

The study of the blow-up of the size of the D(-1) brane is presented in chapter 8. We do it in the framework of open superstring field theory (OSFT) and discuss the problems of the usual on-shell approach. In the process we review the derivation of the instanton profile, and extend it to include α'-corrections.

1.5 Published papers

Parts of this thesis are reproductions of the content of the author's publications. Some of the results presented here have been published in the following papers:

[1] **L. Mattiello** and I. Sachs, *$\mathbb{Z}_2$ boundary twist fields and the moduli space of D-branes*, *JHEP* **07** (2018) 099, `arXiv:1803.07500`.

[2] **L. Mattiello** and I. Sachs, *On Finite-Size D-Branes in Superstring Theory*, 2019, `arXiv:1902.10955` (preprint submitted to the Journal of High Energy Physics).

Chapter 2

Invitation: instantons, string field theory and deformations

In this chapter we review instantons in Yang-Mills theory, focusing at times on the gauge group $SU(2)$. We discuss the role of the moduli characterizing instantons, and the possibility of deforming such solutions in a consistent way. We then explore the same idea for D-branes in string theory; we introduce the second quantization approach, string field theory, and discuss possible solutions.

2.1 Instantons in Yang-Mills theory

The action of Yang-Mills theory for with gauge group $SU(N)$ is given in (1.1). The equations of motion deriving from this action are

$$D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0\,, \tag{2.1}$$

which are non-linear in the field A_μ. We did not specify yet which metric is used in order to contract spacetime indices in (1.1); the theory can be considered both in Minkowski and in Euclidean space.

Considering now the theory in Euclidean space, there is a class of exact solution of the equations of motion (2.1), the instantons, which are characterized by a self-dual or anti-self-dual field strength. In order to understand why they provide solutions, let us consider the quantity

$$\mathrm{Tr}\{(F_{\mu\nu} \pm \widetilde{F}_{\mu\nu})(F^{\mu\nu} \pm \widetilde{F}^{\mu\nu})\}\,, \tag{2.2}$$

where $\widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ is the dual field strength. In Euclidean space the quantity (2.2) is non-negative, from which it follows that

$$\mathrm{Tr}\{F_{\mu\nu}F^{\mu\nu}\} \geq \pm\mathrm{Tr}\{F_{\mu\nu}\widetilde{F}^{\mu\nu}\}\,. \tag{2.3}$$

Therefore, the action is minimized for a self-dual ($F_{\mu\nu} = \widetilde{F}_{\mu\nu}$) or anti-self-dual field strength ($F_{\mu\nu} = -\widetilde{F}_{\mu\nu}$).

The properties of instantons have been extensively discussed in the literature (see for example [3], [4] and [5]); we just recall that they are actually solutions extremizing the

Euclidean action in a particular topological sector. Every instanton is in fact characterized by the winding number (or Pontryagin class) k, defined as

$$k = -\frac{1}{16\pi^2} \int d^4x \ \mathrm{Tr} \left\{ F_{\mu\nu} \widetilde{F}^{\mu\nu} \right\} . \tag{2.4}$$

k is a topological quantity and gauge fields leading to a field strength with different winding numbers can not be related to each other by a gauge transformation.

2.1.1 $SU(2)$ instantons

For concreteness, we focus on a particular gauge group, and we choose the simplest non-trivial (and non-abelian) one, $SU(2)$. In the case of a different gauge group, instantons can be found starting with the $SU(2)$ ones, as we will explain later.

The gauge field A_μ belongs here to the adjoint representation of $SU(2)$; as done in chapter 1, let us write

$$A_\mu = A^c_\mu T^c \,, \tag{2.5}$$

where T^c are the generators of the $\mathfrak{su}(2)$ algebra; we will use also $T^c = \frac{\tau^c}{2i}$, where τ^c are the usual Pauli matrices (see appendix A). Instantonic solutions are well known in the literature (see for example [6] or [4]), and we will simply give the explicit solutions (for winding number $k = 1$) here. In the $SU(2)$ case, two sets of solutions are known; furthermore they are not independent, but they are related one to the other by means of a gauge transformation. The first gauge we analyze is the *regular gauge*: the gauge vector reads, as in (1.3),

$$A^c_\mu(x; x_0, \rho) = 2 \frac{\eta^c_{\mu\nu}(x - x_0)^\nu}{(x - x_0)^2 + \rho^2} . \tag{2.6}$$

Here $\eta^c_{\mu\nu}$ are the 't Hooft symbols, defined in appendix A. One can easily check that the corresponding field strength is given by

$$F^c_{\mu\nu} = -4\eta^c_{\mu\nu} \frac{\rho^2}{[(x - x_0)^2 + \rho^2]^2} ; \tag{2.7}$$

from this expression one can see that the field strength is self-dual and, using (2.4), that the winding number is $k = 1$. The anti-instanton solution can be found replacing $\eta^c_{\mu\nu}$ with $\bar{\eta}^c_{\mu\nu}$; in that case one has $k = -1$. The same solution can be written in another gauge, the *singular gauge*; the gauge vector then reads

$$A^c_\mu(x; x_0, \rho) = 2\bar{\eta}^c_{\mu\nu} \frac{\rho^2 (x - x_0)^\nu}{(x - x_0)^2 [(x - x_0)^2 + \rho^2]} . \tag{2.8}$$

Despite the presence of the anti-self-dual symbol $\bar{\eta}^c_{\mu\nu}$, this solution has a self-dual field strength and $k = 1$. This expression is singular at the point x_0, and it is the one we will use in the following when we will consider instantons from the point of view of string theory.

2.1.2 Generalizations

Other $SU(2)$ solutions can be found starting from the expressions (2.6) or (2.8). One can always act with a $SU(2)$ matrix on them and obtain a different solution:

$$A_\mu(x; x_0, \rho, \vec{\theta}) = U^{-1}(\vec{\theta})\,(\partial_\mu + A_\mu(x; x_0, \rho))\,U(\vec{\theta})\,. \tag{2.9}$$

The same procedure can be used to derive instantonic solutions in the case of a more general $SU(N)$ group: given a matrix $U \in SU(N)$ we can write the expression

$$A_\mu^{SU(N)} = U^\dagger \begin{pmatrix} 0 & 0 \\ 0 & A_\mu^{SU(2)} \end{pmatrix} U\,. \tag{2.10}$$

This represents an instanton for the Yang-Mills theory with gauge group $SU(N)$.

2.2 The instanton moduli space

Looking at the instanton solutions discussed above, we see that they are characterized by some arbitrary parameters, also called *collective coordinates* or *moduli*. In particular the solutions (2.6) and (2.8) are characterized by arbitrary size ρ and position of the center x_0^μ. Furthermore, other three arbitrary parameters (represented by $\vec{\theta}$) can be used to construct a unitary matrix U and find a new solution through (2.9). This gives a total of 8 independent collective coordinates for $SU(2)$ instanton at level $k = 1$. Generalizing, an $SU(N)$ instanton at level k has $4Nk$ collective coordinates [4, 5]. It turns out that the space spanned by this collective coordinates is actually a manifold, in particular a *hyper-Kähler manifold*. We will refer to this manifold as the *k-instanton moduli space*.

A natural question to ask is whether the moduli space describes all the instantonic solutions. One might try to start from a known instanton A_μ, deform it with a small deformation δA_μ and check if the sum preserves self-duality. It turns out that this is true only if the deformation δA_μ is self-dual itself, and satisfies the orthogonality condition [4, 5]

$$D_\mu \delta A^\mu = 0\,, \tag{2.11}$$

where the covariant derivative is defined by the original solution A_μ. This implies that the sum $A_\mu + \delta A_\mu$ has the same winding number k of the original solution, and therefore represents another point in the k-instanton moduli space.

2.2.1 ADHM construction

The moduli space of $SU(N)$ instantons at level k in four dimensions can be constructed in a systematic way thanks to the so-called *ADHM construction* [7, 5, 8]. This is a way of parametrizing the moduli space in a convenient way. The basic objects are the $(N + 2k) \times 2k$ and $2k \times (N + 2k)$ matrices

$$\Delta(x) = A + Bx\,, \qquad \bar{\Delta}(x) = \bar{A} + \bar{x}\bar{B}\,, \tag{2.12}$$

where $x_{\alpha\dot{\beta}} = x_\mu(\sigma^\mu)_{\alpha\dot{\beta}}$ and $\bar{x}^{\dot{\alpha}\beta} = x_\mu(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}$ describe the position of the center of the instanton, while the matrices A and $\bar{A}$ contain the remaining moduli, which can be written in the following form

$$A = \begin{pmatrix} w^{ui}_{\dot{\alpha}} \\ a^{li}_{\alpha\dot{\beta}} \end{pmatrix} , \qquad \bar{A} = \left(\bar{w}^{\dot{\alpha},iu},\, \bar{a}^{\dot{\alpha}\beta,il}\right) , \tag{2.13}$$

where a and $\bar{a}$ are defined in terms of a four vector A_μ analogously to x and $\bar{x}$. On the other hand, the matrices B and $\bar{B}$ can be conveniently chosen as

$$B = \begin{pmatrix} 0 \\ \mathbb{1}_{2k\times 2k} \end{pmatrix} , \qquad \bar{B} = (0,\, \mathbb{1}_{2k\times 2k}) . \tag{2.14}$$

These matrices have to satisfy the so-called *ADHM constraints*, which are

$$\bar{\Delta}\Delta = f^{-1}_{k\times k}\, \mathbb{1}_{2\times 2} , \tag{2.15}$$

where $f_{k\times k}$ is an invertible $k \times k$ matrix. In terms of the moduli, equation (2.15) explicitly reads

$$\bar{\eta}^{\mu\nu}_c \left([a_\mu + x_\mu,\, a_\nu + x_\nu] + \frac{1}{2}\bar{w}_{\dot{\alpha}}(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}} w_{\dot{\beta}} \right) = 0 . \tag{2.16}$$

Let us restrict to the case $N = 2$ and $k = 1$ for simplicity. Since $k = 1$, a_μ and x_μ are just numbers, therefore $[a_\mu{+}x_\mu, a_\nu{+}x_\nu] = 0$ and the constraint becomes $\bar{w}_{\dot{\alpha}}(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}} w_{\dot{\beta}} = 0$. The matrix $(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}}$ is symmetric, hence we can parametrize a generic solution as $\bar{w}_{\dot{\alpha}} w_{\dot{\beta}} = \rho^2 \epsilon_{\dot{\alpha}\dot{\beta}}$, where ρ is the size of the instanton.

2.2.2 Instantons as perturbative solutions

We have seen that instantons are intrinsically non-perturbative (in the coupling constant g) solutions of the equations of motion (2.1). Such equations are non-linear, and in fact the instanton solutions do not solve the linearized equations of motion, since

$$\partial_\mu \partial^\mu A_\nu - \partial^\mu \partial_\nu A_\mu \neq 0 . \tag{2.17}$$

It is however possible to "construct" the full instanton solution starting from a solution of the linearized equations of motion, and proceed perturbatively. The perturbation series is of course not related to the coupling constant g. Explicitly, if $A^{(1)}_\mu$ is a solution of the linearized EOM (2.17)

$$\partial_\mu \partial^\mu A^{(1)}_\nu - \partial^\mu \partial_\nu A^{(1)}_\mu = 0 , \tag{2.18}$$

it might be possible to construct a solution $A_\mu = \lambda A^{(1)}_\mu + \lambda^2 A^{(2)}_\mu + \ldots$ of the full non-linear equations of motion (2.1). If this is so, $A^{(2)}_\mu$ must be a solution of the second order equation

$$(\delta^\mu_\nu \Box - \partial^\mu \partial_\nu) A^{(2)}_\mu + \partial^\mu [A^{(1)}_\mu, A^{(1)}_\nu] + [A^{\mu(1)}, \partial_\mu A^{(1)}_\nu - \partial_\nu A^{(1)}_\mu] = 0 . \tag{2.19}$$

This equation will schematically be solved by

$$A^{(2)} = \mathcal{O}^{-1} F[A^{(1)}] , \tag{2.20}$$

where $F[A^{(1)}]$ is some functional of the first order solution $A^{(1)}_\mu$. Of course, in order for this equation to be solvable for $A^{(2)}_\mu$, the operator $\mathcal{O}^\mu_\nu = \delta^\mu_\nu \Box - \partial^\mu \partial_\nu$ must be invertible. This has to be done fixing a gauge for the gluon A_μ: we choose here Feynman gauge, to make calculations simpler. In principle there could be obstructions to this perturbative construction, which appear whenever the combination $\partial^\mu [A^{(1)}_\mu, A^{(1)}_\nu] + [A^{\mu(1)}, \partial_\mu A^{(1)}_\nu - \partial_\nu A^{(1)}_\mu]$ is such that equation (2.19) does not have solutions for $A^{(2)}_\mu$.

We will represent the obstructions in the following way; let us call P_0 the projector onto the space of solutions for which $\mathcal{O}$ is not invertible. Therefore, in order for (2.19) to be solvable, we have to require that $P_0 \left(F[A^{(1)}_\mu] \right) = 0$, where F is the functional defined above. An obstruction will be present whenever

$$P_0 \left(F[A^{(1)}_\mu] \right) \neq 0 \,. \tag{2.21}$$

If, instead, a solution $A^{(2)}_\mu$ can be found, the procedure has to be repeated order by order, checking that no obstructions appear. If this is the case, a solution of the full non-linear equations of motion $A_\mu = \sum_n \lambda^n A^{(n)}_\mu$ can be found.

This procedure can be successfully followed starting from the first order term in a large distance expansion of the full instanton solution. If we take, for example, the $k = 1$ $SU(2)$ instanton in the singular gauge (2.8) and expand it in $\lambda = \rho^2$, we arrive at

$$\begin{aligned} A_\mu &= \rho^2 A^{(1)}_\mu + \rho^4 A^{(2)}_\mu + \cdots = \\ &= \rho^2 \left(-\bar{\sigma}_{\mu\nu} \frac{(x-x_0)^\nu}{(x-x_0)^4} \right) + \rho^4 \left(\bar{\sigma}_{\mu\nu} \frac{(x-x_0)^\nu}{(x-x_0)^6} \right) + \ldots \end{aligned} \tag{2.22}$$

The perturbative construction discussed above will of course work starting from

$$A^{(1)}_\mu = -\bar{\sigma}_{\mu\nu} \frac{(x-x_0)^\nu}{(x-x_0)^4} \,, \tag{2.23}$$

at least for every point different from the origin. Notice that this is consistent with our choice of Feynman gauge, since $\partial^\mu A_\mu = 0$. On the other hand, we do not expect this construction to work if the starting point $A^{(1)}_\mu$ is a generic field configuration.

2.3 Open string field theory

As introduced in 1.3, we would like to understand if it is possible to blow-up the size of a D-brane in string theory, in order to reconstruct a moduli space corresponding to the one of instantons in Yang-Mills theory. The proper framework to discuss this question is, in analogy to Yang-Mills theory, a second quantization approach to string theory, the so-called *string field theory*. Although closed string field theory approaches are complicated, open string field theory (OSFT) is known (see [9, 10] for example).

String field theory is characterized by a space of states $\mathcal{H}$, with a non-degenerate inner product (Ψ_1, Ψ_2); the fields in $\mathcal{H}$ describe all possible string oscillations, including the one corresponding to the gauge vector A_μ in the field theory limit. The kinetic term of the action for a string field Ψ is given by

$$\frac{1}{2} \left(\Psi, Q\Psi \right) \,, \tag{2.24}$$

where Q is the open string BRST charge. In addition to the quadratic term, OSFT has a number of higher order interaction terms; to be precise, we have an infinite number for superstring field theory, while bosonic string field theory has only a cubic interaction. Let us write schematically

$$S(\Psi) = \frac{1}{2}(\Psi, Q\Psi) + \frac{1}{3}(\Psi, m_2(\Psi, \Psi)) + \frac{1}{4}(\Psi, m_3(\Psi, \Psi, \Psi)) + \cdots \tag{2.25}$$

The products m_2, m_3, ... are all associative and are defined in terms of the operator product expansion (OPE) of conformal fields.

The equations of motion deriving from this action are simply

$$Q\Psi + m_2(\Psi, \Psi) + m_3(\Psi, \Psi, \Psi) + \cdots = 0\,. \tag{2.26}$$

In analogy to the discussion of 2.2.2, we would like to understand if a solution $\Psi^{(1)}$ of the linearized equations of motion ($Q\Psi^{(1)} = 0$) can be lifted to a solution $\Psi = \lambda\Psi^{(1)} + \lambda^2\Psi^{(2)} + \ldots$ of the full non-linear equations of motion (2.26). In particular, our starting point will be a field $\Psi^{(1)}$ describing an oscillation of the D3-D(−1) bound state described above, and we will try to find a full solution, order by order, by solving the string field theory equations of motion perturbatively. We will have to check if any obstructions appear, analogously to 2.2.2. In particular, at second order, $\Psi^{(2)}$ needs to solve

$$Q\Psi^{(2)} + m_2(\Psi^{(1)}, \Psi^{(1)}) = 0\,, \tag{2.27}$$

which has solutions only if $P_0\left[m_2(\Psi^{(1)}, \Psi^{(1)})\right] = 0$, where P_0 is the projector on the space of fields for which Q is not invertible. If this is not the case, an obstruction is present.

Chapter 3

The conformal field theories of free bosons and fermions

In this chapter we introduce the conformal field theories of free bosons and free fermions, with a particular focus on boundary conditions in the presence of a boundary. These will serve as building blocks for the study of string theory, as we will see in chapter 5 and 7. We investigate the difference between Neveu-Schwarz and Ramond sector, and discuss the role of *spin fields* and *twist fields*. In this chapter, when discussing the bosonic theory, we use extensively the analogy to electrostatics in two dimensions.

3.1 Free boson

In this section we introduce the conformal field theory of a free boson X in one spacetime dimension. We consider the theory on the complex plane, possibly restricted to the upper part, with a boundary along the real line. The action is (in complex coordinates)

$$S[X] = \frac{1}{4\pi} \int dz d\bar{z}\, \partial X(z, \bar{z}) \bar{\partial} X(z, \bar{z})\,. \tag{3.1}$$

Let us split the field in its holomorphic and anti-holomorphic part as $X(z, \bar{z}) = X(z) + \bar{X}(\bar{z})$, focusing on the first one. The equation of motion deriving from the action (3.1) is $\partial\bar{\partial}X = 0$, which is the same equation characterizing an electrostatic potential in two dimensions. We will use this analogy in order to derive more properties of the CFT of a free boson. Out of the field X one constructs the current

$$j(z) = i\partial X(z)\,, \tag{3.2}$$

which is a primary field of conformal dimension 1 satisfying the operator product expansion (OPE)

$$j(z)j(w) = \frac{1}{(z-w)^2} + \dots\,. \tag{3.3}$$

Furthermore, let us assume that the domain of X is the upper half plane ($\mathrm{Im}\, z > 0$), and the boundary coincides with the real line. In this case one should always specify the boundary conditions satisfied by the fields. There are two obvious boundary conditions, which are consistent with the conformal symmetry. Neumann (N) boundary conditions are given by

$$(\partial - \bar{\partial})X(z,\bar{z})\big|_{z=\bar{z}} = 0\,, \tag{3.4}$$

while Dirichlet (D) boundary conditions are characterized by

$$(\partial + \bar{\partial})X(z,\bar{z})\big|_{z=\bar{z}} = 0\,, \qquad X(z,\bar{z})\big|_{z=\bar{z}} = x_0\,. \tag{3.5}$$

In the language of electrostatics, if X represents the electrostatic potential, Dirichlet boundary conditions correspond to the presence of a conductor along the boundary. On the other hand, Neumann boundary conditions correspond to having a fixed amount of charge along the boundary.

3.1.1 Boson in the Neveu-Schwarz sector

If the boundary condition is the same for the whole boundary, the boson is in the so-called Neveu-Schwarz (NS) sector. In this sector the chiral part of the boson is periodic under a rotation around the origin, i.e.

$$\partial X(e^{2\pi i}z) = \partial X(z)\,. \tag{3.6}$$

This can not be the case if the boundary condition changes along the boundary, as we will discuss in section 3.3. Let us study the boson in the NS sector by considering the Green's function for the Laplace operator, which is (using complex coordinates) $G(z,w) = \log(z-w)$ and satisfies $\triangle_z G(z,w) = 2\pi\delta^{(2)}(z-w)$ [11]. Some correlation functions among the fields can be written in terms of the Green's function and its derivatives, for example

$$\begin{aligned}\langle X(z)X(w)\rangle &= -G(z,w) = -\log(z-w)\,,\\ \langle j(z)j(w)\rangle &= \partial_z\partial_w G(z,w) = \frac{1}{(z-w)^2}\,.\end{aligned} \tag{3.7}$$

All the properties of the free boson can be encoded in a mode expansion for the two currents. In particular we can perform a Laurent expansion on the complex plane, namely

$$\begin{aligned}j(z) &= i\partial X(z) = \sum_{n\in\mathbb{Z}} j_n z^{-n-1}\,,\\ \bar{j}(\bar{z}) &= i\bar{\partial}\bar{X}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{j}_n \bar{z}^{-n-1}\,.\end{aligned} \tag{3.8}$$

where the modes satisfy the commutation relation $[j_n, j_m] = n\delta_{n+m}$, and analogously for $\bar{j}_m$. Notice that the expansion is over integer exponents, which enforces the periodicity condition in the NS sector (3.6). Neumann or Dirichlet boundary conditions are translated in conditions relating the modes j_n and $\bar{j}_n$. In particular

$$j_n = \bar{j}_n \quad \text{(Neumann)} \quad , \quad j_n = -\bar{j}_n \quad \text{(Dirichlet)} \tag{3.9}$$

Restricting on the chiral part of the theory, the modes generate a Hilbert space, starting from the vacuum $|0\rangle$, which is characterized by

$$\begin{aligned}j_n|0\rangle &= 0\,, \qquad n \geq 0\,,\\ \langle 0|j_n &= 0\,, \qquad n \leq 0\,.\end{aligned} \tag{3.10}$$

Therefore j_n behave as annihilation operators for $n \geq 1$ and as creation operators for $n \leq -1$. Using (3.10) one can easily derive some correlation functions, for example

$$\begin{aligned} \langle 0|j(z)|0\rangle &= 0\,, \\ \langle 0|j(z)j(w)|0\rangle &= \frac{1}{(z-w)^2}\,. \end{aligned} \tag{3.11}$$

The energy momentum tensor of the theory arises from the regular part of the OPE of two currents. In particular, let us define $N(jj)$ as

$$j(z)j(w) = \frac{1}{(z-w)^2} + N(jj)(w) + \ldots\,; \tag{3.12}$$

this quantity is related to the energy momentum tensor of the theory by $T(w) = \frac{1}{2}N(jj)(w)$. Here $N(\ )$ indicates the normal ordered product. There is another notion of normal ordering, which coincides with the prescription of having creation operators to the left of annihilation operators, which we indicate by : :. For the Neveu-Schwarz sector of the boson, the two notions of normal ordering coincide and we have

$$T(w) = \frac{1}{2}N(jj)(w) = \frac{1}{2} : jj : (w)\,. \tag{3.13}$$

Hence, the Laurent modes L_m of T are

$$L_m = \frac{1}{2}\sum_{k\geq 0} j_{m-k}\; j_k + \frac{1}{2}\sum_{k\leq -1} j_k\; j_{m-k}\,. \tag{3.14}$$

In addition

$$\widetilde{V}_\alpha(z) =: e^{i\alpha X} : (z)\,, \tag{3.15}$$

are primaries with conformal dimension $h_\alpha = \alpha^2/2$. The OPE among them takes the form

$$\widetilde{V}_\alpha(z)\widetilde{V}_\beta(w) = e^{-\alpha\beta\langle X(z)X(w)\rangle} : e^{i(\alpha+\beta)X} : (w) + \cdots = (z-w)^{\alpha\beta}\widetilde{V}_{\alpha+\beta}(w) + \ldots\,. \tag{3.16}$$

The OPE with the current j is given by

$$j(z)\widetilde{V}_\alpha(w) \sim \alpha\,\partial_z G(z,w)\,\widetilde{V}_\alpha(w) = \frac{\alpha}{(z-w)}\widetilde{V}_\alpha(w) + \ldots\,. \tag{3.17}$$

The correlation function of many primaries of the form $\widetilde{V}_\alpha$ is then given by

$$\langle \widetilde{V}_{\alpha_1}(z_1)\ldots\widetilde{V}_{\alpha_n}(z_n)\rangle = \exp\left(\sum_{i<j}\alpha_i\alpha_j G(z_i,z_j)\right)\delta\left(\sum_{i=1}^n \alpha_i\right) = \prod_{i<j}(z_{ij})^{\alpha_i\alpha_j}\;\delta\left(\sum_{i=1}^n \alpha_i\right)\,. \tag{3.18}$$

The delta function is a consequence of the integration over the zero modes.

From the energy momentum tensor on can derive the central charge of the theory. The OPE relation

$$T(z)T(w) = \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots \tag{3.19}$$

implies that the central charge of the bosonic theory is $c = 1$.

3.2 Free fermion

In this section we introduce the conformal field theory of a free fermion ψ in one spacetime dimension. Again, we consider the theory on the complex plane, possibly restricted to the upper part, with a boundary along the real line. The action is

$$S[\psi] = \frac{1}{4\pi}\int dz d\bar{z}\ \left(\psi(z)\bar{\partial}\psi(z) + \bar{\psi}(\bar{z})\partial\bar{\psi}(\bar{z})\right)\,. \tag{3.20}$$

The equations of motion imply that the fields ψ and $\bar{\psi}$ are chiral and anti-chiral respectively. Focusing on the first one, it turns out that it is a conformal dimension 1/2 primary field satisfying the operator product expansion

$$\psi(z)\psi(w) = \frac{1}{(z-w)} + \text{reg.}\,, \tag{3.21}$$

The chiral (and also the anti-chiral) part can be periodic (P) or antiperiodic (A) under a rotation around the origin:

$$\psi(e^{2\pi i}z) = \pm\psi(z)\,. \tag{3.22}$$

This is due to the OPE (3.21), which implies that the fermion lives naturally on a double cover of the complex plane. Periodic and antiperiodic boundary conditions correspond to different sectors for the fermion, Neveu-Schwarz and Ramond respectively.

If the theory is defined on the upper half plane, with boundary along the real line, the fermion can be subject to two kinds of boundary conditions, analogously to the boson:

$$\begin{aligned} \psi(z) &= \bar{\psi}(\bar{z})\big|_{z=\bar{z}} \qquad \text{(Neumann)} \\ \psi(z) &= -\bar{\psi}(\bar{z})\big|_{z=\bar{z}} \qquad \text{(Dirichlet)} \end{aligned} \tag{3.23}$$

Therefore, if the same boundary condition (Dirichlet or Neumann) is satisfied along the whole boundary, a fermion with periodic boundary conditions should be used. If the boundary condition changes from Dirichlet to Neumann at the origin, a fermion with antiperiodic boundary conditions should be considered. Let us focus here on the Neveu-Schwarz sector for the fermion; we will discuss the Ramond sector later.

3.2.1 Fermion in the Neveu-Schwarz sector

All the properties of the free fermion in the NS sector can be encoded in a mode expansion. In particular we can perform a Laurent expansion on the complex plane, namely

$$\psi(z) = \sum_{r\in\mathbb{Z}+1/2} \psi_r z^{-r-1/2}\,, \tag{3.24}$$

where the modes satisfy the anticommutation relation $\{\psi_r, \psi_s\} = \delta_{r+s}$. The antichiral part $\bar{\psi}$ can be expanded in a similar way.

In the NS sector there is no zero-mode for the fermion, and the other modes satisfy the conditions

$$\begin{aligned} \psi_r|0\rangle &= 0\,, \qquad r \geq 1/2\,, \\ \langle 0|\psi_r &= 0\,, \qquad r \leq -1/2\,. \end{aligned} \tag{3.25}$$

One can easily derive some correlation functions, for example

$$\langle 0|\psi(z)\psi(w)|0\rangle = \frac{1}{z-w}\,. \tag{3.26}$$

The energy momentum tensor of the theory arises from the regular part of the OPE of two fermions. In particular, the chiral part is

$$T(z) = \frac{1}{2}N(\psi\partial\psi)(z)\,. \tag{3.27}$$

As for the boson, in the NS sector the normal ordered product $N(\ )$ coincides with the prescription of having creation operators to the left of annihilation operators, which we indicate by $:\,:$. The Laurent modes L_m of T are then

$$L_m = \frac{1}{2}\sum_{s\geq -1/2}\left(s+\frac{1}{2}\right)\psi_{m-s}\;\psi_s - \frac{1}{2}\sum_{s\leq -3/2}\left(s+\frac{1}{2}\right)\psi_s\;\psi_{m-s}\,. \tag{3.28}$$

From the energy momentum tensor on can derive the central charge of the theory. The OPE relation

$$T(z)T(w) = \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots \tag{3.29}$$

implies that the central charge of the fermionic theory is $c = 1/2$.

3.3 Mixed boundary conditions and Ramond sector

In this section we consider free bosons and free fermions on the upper half plane with mixed boundary conditions along the real line. We consider in particular the case where the boundary $\mathrm{Im}\, z = 0$ is divided in two intervals, $\mathrm{Re}\, z > 0$ and $\mathrm{Re}\, z < 0$, with Dirichlet and Neumann boundary conditions respectively. The change in boundary condition appearing at the origin $z = 0$ implies that the chiral part of the bosonic current and the chiral part of the fermion must be antiperiodic under a rotation around the origin. In particular we require

$$\begin{aligned} \partial X(e^{2\pi i}z) &= -\partial X(z)\,, \\ \psi(e^{2\pi i}z) &= -\psi(z)\,. \end{aligned} \tag{3.30}$$

We discussed already the possibility of this antiperiodicity for the fermion, since it naturally lives on the double cover of the complex plane. Here we force the boson to behave in the same way; this may sound counterintuitive, but it is necessary in the presence of mixed boundary conditions. The conditions (3.30) imply that both the boson and the fermion are now in the Ramond sector.

3.3.1 Boson in the Ramond sector

Since the boson is characterized by mixed boundary conditions, the Green's function defined in section 3.1 is not valid. However, we can still use the electrostatic analogy in order to find the Green's function corresponding to these boundary conditions. We will do so using the methods of image charges [12]. The idea is to start with a boson defined on an infinite strip (parametrized by $0 < \operatorname{Im} w < \pi$), and then map it to the complex upper half plane $\operatorname{Im} z > 0$ via $z = e^w$. Let us impose different boundary conditions on the boundaries of the strip; here we choose Dirichlet boundary conditions at $\operatorname{Im} w = 0$ and Neumann boundary conditions at $\operatorname{Im} w = \pi$. Consequently, the upper half plane has Dirichlet and Neumann boundary conditions on the positive and negative real axis respectively (see figure 3.1).

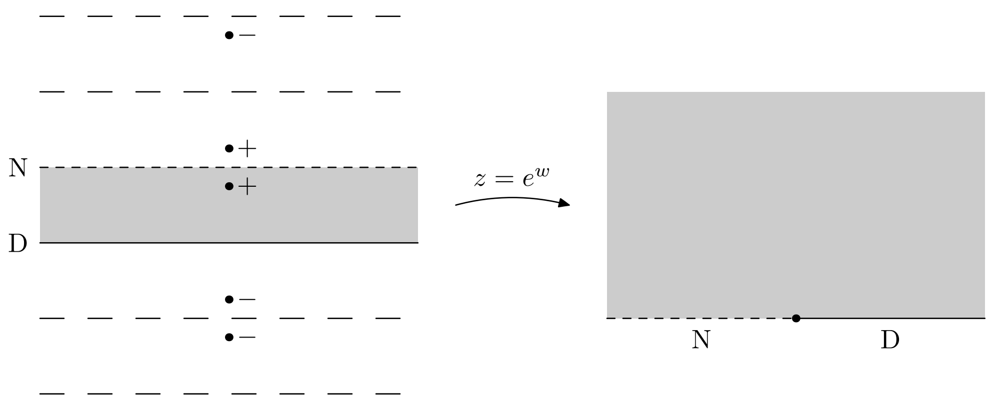

Figure 3.1: Green's function with the method of image charges.

The electrostatic potential due to a unit charge at position w on the strip can be easily determined if we introduce a set of image charges. We need one negative unit charge at position $\bar{w}$ in order to enforce Dirichlet boundary conditions at $\operatorname{Re} w = 0$ and a positive unit charge at position $\bar{w} + 2\pi i$ in order to enforce Neumann boundary conditions at $\operatorname{Re} w = \pi$. But this is not sufficient, because we have to insert both charges at the same time; therefore, in order to maintain the proper boundary conditions we need to introduce an infinite number of image charges. In particular we need alternating charges of value $(-1)^n$ at positions $w + 2\pi i n$ and $(-1)^{n+1}$ at positions $\bar{w} + 2\pi i n$, as in figure 3.1. In this way the boundary conditions are automatically satisfied at the boundaries of the strip. The electrostatic potential measured at some point w' is then given by [12]

$$\begin{aligned} G_\sigma(w, w') &= \\ &= \left(\sum_n (-1)^n \big\{ \log\left[w' - (w + 2\pi i n)\right] - \log\left[w' - (\bar{w} + 2\pi i n)\right] \big\} \right) + (w' \leftrightarrow \bar{w}') = \\ &= \left(\log\left[\frac{\prod_{n=2m} w' - (w + 2\pi i n)}{\prod_{n=2m+1} w' - (w + 2\pi i n)} \right] - (w \leftrightarrow \bar{w}) \right) + (w' \leftrightarrow \bar{w}')\,, \end{aligned} \tag{3.31}$$

where the subscript σ indicated the fact that we are considering mixed boundary conditions. Mapping now $z = e^w$ (and $z' = e^{w'}$) we obtain the Green's function on the upper half plane, which turns out to be (restricting ourselves to the chiral part)

$$G_\sigma(z, z') = \log\left[\frac{\prod_{n=2m} w' - (w + 2\pi i n)}{\prod_{n=2m+1} w' - (w + 2\pi i n)} \right] = \log\left[\frac{1 - e^{\frac{w'-w}{2}}}{1 + e^{\frac{w'-w}{2}}} \right] = \log\left[\frac{1 - \sqrt{\frac{z}{z'}}}{1 + \sqrt{\frac{z}{z'}}} \right]. \tag{3.32}$$

From the Green's function (3.32) one can derive some correlation functions and OPE's, in analogy to (3.7):

$$\begin{aligned}
\langle X(z)X(w)\rangle_\sigma &= -G_\sigma(z,w) = -\log\left(\frac{1-\sqrt{\frac{z}{w}}}{1+\sqrt{\frac{z}{w}}}\right) \\
\langle j(z)j(w)\rangle_\sigma &= \partial_z\partial_w G_\sigma(z,w) = \frac{1}{2(z-w)^2}\left(\sqrt{\frac{z}{w}}+\sqrt{\frac{w}{z}}\right) .
\end{aligned} \tag{3.33}$$

The presence of mixed boundary conditions can be encoded in a different mode expansion for the two currents, namely:

$$\begin{aligned}
j(z) = i\partial X(z) &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} j_r z^{-r-1} , \\
\bar{j}(\bar{z}) = i\bar{\partial}\bar{X}(\bar{z}) &= -\sum_{r\in\mathbb{Z}+\frac{1}{2}} j_r \bar{z}^{-r-1} ,
\end{aligned} \tag{3.34}$$

where the modes satisfy the commutation relation $[j_r, j_s] = r\delta_{r+s}$. The same set of modes appear in the expansion of both j and $\bar{j}$, in order to have Dirichlet boundary conditions on the positive real axis. A branch cut is present in the complex plane, extending from 0 to $-\infty$. We see that the expansion over half-integers implies that the boson is now, as expected, antiperiodic under a rotation around the origin:

$$\partial X(e^{2\pi i}z) = -\partial X(z) , \tag{3.35}$$

in contrast to section 3.1. Equation (3.34) defines a boson ∂X in the Ramond sector, instead of the usual integer mode expansion, which corresponds to the Neveu-Schwarz sector. The ground state of this sector is different for the vacuum $|0\rangle$ defined above. We will call it *twist vacuum* $|\sigma\rangle$: it can be defined (together with its dual) in such a way that the modes j_r are creation and annihilation operators:

$$\begin{aligned}
j_r|\sigma\rangle &= 0 , \qquad r \geq 1/2 , \\
\langle\sigma|j_r &= 0 , \qquad r \leq -1/2 .
\end{aligned} \tag{3.36}$$

The expansion (3.34) is not defined on the negative real axis; we can formally solve the problem introducing a new current $\mathsf{j}(z)$, defined on the whole complex plane as

$$\mathsf{j}(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} j_r z^{-r-1} . \tag{3.37}$$

This means that we are identifying $\mathsf{j}(z) = j(z)$ on the upper half plane, and $\mathsf{j}(z) = -\bar{j}(z)$ on the lower half plane. This new current is naturally defined on the two-fold branched cover of the complex plane. Therefore, if one wants to study correlation functions of $j(z)$ on the upper half plane with mixed boundary conditions, one should first study correlation functions of $\mathsf{j}(z)$ on the two-fold cover of the complex plane, and finally restrict the result to $\mathrm{Im}\, z > 0$.

Some correlation functions can be derived simply by means of the mode expansion (3.37), and using the property (3.36). For example one obtains:

$$\langle\sigma|\mathsf{j}(z)|\sigma\rangle = 0\,,$$
$$\langle\sigma|\mathsf{j}(z)\mathsf{j}(w)|\sigma\rangle = \frac{1}{2}\frac{\left(\sqrt{\frac{z}{w}}+\sqrt{\frac{w}{z}}\right)}{(z-w)^2}\,. \tag{3.38}$$

3.3.2 Normal ordering

In the case of a boson in the Ramond sector we have two useful definitions of normal ordering which do not coincide. Using the definitions of the previous section, the first one (indicated with $N(\ )$) arises from the operator product expansion, while the second one (indicated with : :) is a prescription on the order of annihilation and creation modes. Let us consider the OPE of two currents j; the normal ordered product $N(jj)$ is the finite term of the expansion, i.e.

$$j(z)j(w) = \frac{1}{(z-w)^2} + N(jj)(w) + \dots \tag{3.39}$$

This quantity is related to the energy momentum tensor of the theory by $T(w) = \frac{1}{2}N(jj)(w)$. Notice that since T is quadratic in j, there is no need to distinguish between j and j; the natural domain for T is simply the complex plane. Expanding as in 3.34 one obtains an explicit expression in terms of the creation-annihilation normal ordering, namely

$$T(z) = \frac{1}{2}N(jj)(z) = \frac{1}{2} : jj : (z) + \frac{1}{16z^2}\,. \tag{3.40}$$

Hence, the Laurent modes L_m of T are

$$L_m = \frac{1}{2}\sum_{r\in\mathbb{Z}+1/2} j_r\, j_{m-r}\,, \qquad (m\neq 0)$$
$$L_0 = \frac{1}{16} + \sum_{r=1/2}^{\infty} j_{-r}\, j_r\,. \tag{3.41}$$

Other primaries of this theory in the Ramond sector are defined by

$$\psi_\alpha(z) = N(e^{i\alpha X})(z) = \frac{:e^{i\alpha X}(z):}{(4z)^{\alpha^2/2}}\,. \tag{3.42}$$

The OPE of these primaries with the current $j = i\partial X$ is given by

$$j(z)\psi_\alpha(w) = -\alpha\,\partial_z G_\sigma(z,w)\,\psi_\alpha(w) = \frac{\alpha}{(z-w)}\sqrt{\frac{w}{z}}\psi_\alpha(w) + \dots\,. \tag{3.43}$$

The correlation function of many primaries ψ_α is slightly more complicated than the one in (3.18). As clarified in [13], the zero mode is absent but there is an extra contribution of the form

$$\exp\left(\sum_{i=1}^{n}\frac{\alpha_i^2}{2}\,S_0(z_i)\right)\,. \tag{3.44}$$

It can be interpreted as a renormalized electrostatic self-energy and it takes care of the difference in the two normal orderings (cfr. [14]). S_0 is defined in general in terms of the Green's function by

$$G(z,w) = \log(w-z) + S_0(z) + \mathcal{O}(w-z)\,. \tag{3.45}$$

The Green's function (3.32) gives $S_0(z) = \log(\frac{1}{4z})$, from which one can derive correlation functions like

$$\begin{aligned} \langle \psi_\alpha(z) \rangle_\sigma &= \frac{e^{i\alpha x_0}}{(4z)^{\alpha^2/2}}\,, \\ \langle \psi_\alpha(z)\psi_\beta(w) \rangle_\sigma &= \frac{e^{i(\alpha+\beta)x_0}}{(4z)^{\alpha^2/2}(4w)^{\beta^2/2}} \left(\frac{1-\sqrt{w/z}}{1+\sqrt{w/z}} \right)^{\alpha\beta}\,, \end{aligned} \tag{3.46}$$

where we highlighted the fact that the correlation functions have to be considered with respect to the vacuum $|\sigma\rangle$.

3.3.3 Fermion in the Ramond sector

For a fermion in the Ramond sector the mode expansion is

$$\psi(z) = \sum_{r\in\mathbb{Z}} \psi_r z^{-r-1/2} \quad (\text{R sector})\,; \tag{3.47}$$

the expansion over half-integer exponents implies that the antiperiodicity around the origin is automatically satisfied. Differently from the NS sector, in the R sector there is a zero-mode ψ_0, which is neither a creation nor an annihilation operator. It acts actually as a multiplicative factor, since it satisfies $\{\psi_0, \psi_0\} = 1/2$. Keeping this in mind, the Hilbert space can be constructed starting from a different vacuum $|S\rangle$, with the following properties:

$$\begin{aligned} \psi_r|S\rangle &= 0\,, \qquad r \geq 1\,, \\ \langle S|\psi_r &= 0\,, \qquad r \leq -1\,. \end{aligned} \tag{3.48}$$

The energy momentum tensor is again defined in terms of the OPE normal ordering, but in the Ramond sector it does not coincide with the creation-annihilation modes prescription. In particular we have

$$T(z) = \frac{1}{2}N(\psi\partial\psi)(z) = \frac{1}{2} :\psi\partial\psi(z) + \frac{1}{16z^2}\,. \tag{3.49}$$

Some correlation functions are different fro the NS sector, for example we have [15]

$$\langle \psi(z)\psi(w) \rangle_S = \frac{1}{2(z-w)} \left(\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}} \right)\,, \tag{3.50}$$

where the subscript S indicates the fact that we are in the Ramond sector, where the vacuum is $|S\rangle$.

It is possible to connect the two vacua of the two sectors $|S\rangle$ and $|0\rangle$ using the state-operator correspondence. We thus define an operator $S(z)$ such that

$$\lim_{z\to 0} S(z)|0\rangle = |S\rangle\,. \tag{3.51}$$

Since we can easily compute $\langle S|T(z)|S\rangle = 1/(16z^2)$ we conclude that this operator has conformal dimension 1/16. It is usually called *fermionic twist field*, but in this work we will reserve the name "twist field" for the field σ twisting the free boson (see 3.5). When fermions in higher dimensions are considered, the field S is directly connected to the *spin fields*.

3.4 Fermions in higher dimensions, spin fields and bosonization

Let us consider in this section d copies of a fermionic free theory; this will be useful later when dealing with superstring theory. Assume we have fermions $\psi^M(z)$ (and $\bar{\psi}^M(\bar{z})$) with $M = 1, \ldots, d$. If the dimension d is an even number, we can express the d fermions in terms of $d/2$ free bosons in the following way: we first define d complex fermions (in the so-called Cartan-Weyl basis) using

$$\Psi^{\pm i}(z) = \frac{1}{\sqrt{2}}\Big(\psi^{2i-1}(z) \pm i\psi^{2i}(z)\Big)\,, \qquad i = 1, \ldots d/2 \tag{3.52}$$

After that we can *bosonize* the d complex fermions, expressing them in terms of $d/2$ free bosons ϕ^i, in the following way:

$$\Psi^{\pm i}(z) =: e^{\pm i\phi^i} : (z)\,. \tag{3.53}$$

One can easily check that the bosonization is consistent with all the fundamental properties of fermions, in particular their operator product expansion. Furthermore, the total central charge is conserved, since $c = 1$ for bosons and $c = 1/2$ for fermions. This procedure allows for easier derivations of correlation functions involving many fermions, since equations like (3.18) can be used. To be precise, when many fermions are present some *cocycles* c_i must be added to the definition (3.53), in order to enforce the right anticommutation relations between different fermions [16].

Each copy of the fermionic theory comes in principle with its fermionic twist field, connecting the NS and R sector of the corresponding fermion. These can be bosonized in terms of exponential operators of the form

$$s^i(z) =: e^{\frac{1}{2}i\phi^i} : (z)\,, \qquad \bar{s}^i(z) =: e^{-\frac{1}{2}i\phi^i} : (z)\,, \tag{3.54}$$

which have both conformal dimension 1/16 as the fermionic twist fields. Out of these operators we can define the so called *spin fields* in the following way:

$$S^A = \prod_{i=1}^{d/2} : e^{iA_i\phi^i} : (z)\,, \tag{3.55}$$

where all the components of the vector A_i are $\pm 1/2$. The spin fields decompose into two different irreducible representations of $SO(d)$. We refer to them as positive and negative chirality, corresponding to an even and odd number of minus signs in the components of the vector A_i respectively. Furthermore, a spin field with positive chirality is usually indicated by S^A, while a spin field of negative chirality is indicated by $S^{\dot{A}}$. Again, as for the fermions, to be precise some cocycles must be added to the definition of the spin fields (3.55), in order to enforce the right (anti)commutation relations. If this is taken into account, formulas like (3.16) and (3.18) make it easy to derive OPE's and correlation functions involving spin fields and fermions. See for example [16] for an extensive discussion, and appendix B for some useful results in the case $d = 4$.

3.5 Twist fields

As for the fermion, we can connect the Ramond vacuum for the boson $|\sigma\rangle$ to the Neveu-Schwarz vacuum by means of the state operator correspondence. Let us define an operator $\sigma(z)$, the *(bosonic) twist field*, such that

$$\begin{aligned} |\sigma\rangle &= \lim_{z\to 0} \sigma(z)|0\rangle \,, \\ \langle\sigma| &= \lim_{z\to\infty} z^{1/8}\langle 0|\bar{\sigma}(z) \,. \end{aligned} \tag{3.56}$$

In the remainder of this work we will always use the name *twist fields* for bosonic twist fields, while we will always deal with fermionic twist fields using the spin fields defined above. The mode expansion for L_0 given in (3.41) implies that the operator $\sigma(z)$ has conformal dimension 1/16, thus the factor $z^{1/8}$ in the second line of (3.56). From the correlation functions above we deduce that the OPE among the current $j = i\partial X$ and the twist field contains a branch cut. We thus write (see for example [17] and [12])

$$\begin{aligned} i\partial X(z)\sigma(w) &= \frac{\sigma'(w)}{(z-w)^{1/2}} + \dots \\ i\partial X(z)\sigma'(w) &= \frac{\sigma(w)}{2(z-w)^{3/2}} + \frac{2\,\partial\sigma(w)}{(z-w)^{1/2}} + \dots \\ \bar{\sigma}(z)\sigma(w) &= \frac{1}{(z-w)^{1/8}} + \dots \end{aligned} \tag{3.57}$$

where $\sigma'(w)$ is another operator called *excited twist field*. In the rest of this thesis we take this OPE (3.57) as the defining property of twist fields. Similar relations to the first two hold for the conjugated fields $\bar{\sigma}$ and $\bar{\sigma}'$. σ (and its conjugated $\bar{\sigma}$) is a conformal primary of dimension 1/16, while σ' (and $\bar{\sigma}'$) has dimension 9/16. Notice that the square root branch cut implies that the field $X(z)$ changes sign when the point z is moved around the point where the twist field is inserted. In the following we will always insert twist fields at the boundary of the domain, i.e. on the real line ($z = \bar{z}$). Therefore the branch cut in the OPE changes the boundary condition from Neumann to Dirichlet (and vice versa), as expected.

3.5.1 Correlation functions with two twist fields

All the correlation functions of a boson in the Ramond sector defined above can be interpreted as correlation functions in the presence of two twist fields, one at the origin and one at infinity. Using conformal symmetry we can easily derive the corresponding correlation functions for the case where a pair of twist fields σ and $\bar{\sigma}$ is inserted at generic positions along the boundary. For example (3.38) is equivalent to

$$\begin{aligned} \langle\bar{\sigma}(z_1)j(z_2)\sigma(z_3)\rangle &= 0\,, \\ \langle\bar{\sigma}(z_1)j(z_2)j(z_3)\sigma(z_4)\rangle &= \frac{1}{2}\frac{1}{(z_{41})^{1/8}(z_{32})^2}\left(\sqrt{\frac{z_{31}z_{42}}{z_{21}z_{43}}}+\sqrt{\frac{z_{21}z_{43}}{z_{31}z_{42}}}\right)\,, \end{aligned} \tag{3.58}$$

where $z_{ij} = z_i - z_j$. Taking appropriate limits of the second correlation function, and using the OPE (3.57), we can derive correlation functions involving excited twist fields. For example

$$\langle\bar{\sigma}(z_1)j(z_2)\sigma'(z_4)\rangle = \lim_{z_3\to z_4}\sqrt{z_3-z_4}\langle\bar{\sigma}(z_1)j(z_2)j(z_3)\sigma(z_4)\rangle\,. \tag{3.59}$$

The result is

$$\begin{aligned} \langle\bar{\sigma}(z_1)j(z_2)\sigma'(z_3)\rangle &= \frac{z_{31}^{3/8}}{2z_{21}^{1/2}z_{32}^{3/2}}\,, \\ \langle\bar{\sigma}'(z_1)j(z_2)\sigma(z_3)\rangle &= \frac{z_{31}^{3/8}}{2z_{21}^{3/2}z_{32}^{1/2}}\,. \end{aligned} \tag{3.60}$$

On the other hand, from (3.46) we can derive the more general correlation functions

$$\begin{aligned} \langle\bar{\sigma}(z_1)\psi_\alpha(z_2)\sigma(z_3)\rangle &= \frac{e^{i\alpha x_0}}{(4z_{21}z_{32})^{\alpha^2/2}z_{31}^{1/8-\alpha^2/2}}\,, \\ \langle\bar{\sigma}(z_1)\psi_\alpha(z_2)\psi_\beta(z_3)\sigma(z_4)\rangle &= \frac{e^{i(\alpha+\beta)x_0}}{(4z_{21}z_{42})^{\alpha^2/2}(4z_{31}z_{43})^{\beta^2/2}z_{41}^{1/8-\alpha^2/2-\beta^2/2}}\left(\frac{1-\sqrt{\eta}}{1+\sqrt{\eta}}\right)^{\alpha\beta}\,, \end{aligned} \tag{3.61}$$

where η is the conformal ratio $\eta = \frac{z_{21}z_{43}}{z_{31}z_{42}}$. Again, taking appropriate limits we can derive correlation functions involving excited twist fields, in particular

$$\begin{aligned} \langle\bar{\sigma}(z_1)\psi_\alpha(z_2)\sigma'(z_3)\rangle &= \frac{-\alpha e^{i\alpha x_0}}{4^{\alpha^2/2}z_{21}^{\alpha^2/2-1/2}z_{32}^{\alpha^2/2+1/2}z_{31}^{5/8-\alpha^2/2}}\,, \\ \langle\bar{\sigma}'(z_1)\psi_\alpha(z_2)\sigma(z_3)\rangle &= \frac{\alpha e^{i\alpha x_0}}{4^{\alpha^2/2}z_{21}^{\alpha^2/2+1/2}z_{32}^{\alpha^2/2-1/2}z_{31}^{5/8-\alpha^2/2}}\,, \\ \langle\bar{\sigma}'(z_1)\psi_\alpha(z_2)\sigma'(z_3)\rangle &= \frac{-\alpha^2 e^{i\alpha x_0}}{(4z_{21}z_{32}^{\alpha^2/2})z_{31}^{9/8-\alpha^2/2}}\,. \end{aligned} \tag{3.62}$$

From the correlation functions of twist fields with operators ψ_α we can see that the operator product expansion of two twist fields must contain all these primaries. Therefore we can guess that

$$\bar{\sigma}(z)\sigma(w) = \int d\alpha\,\frac{e^{-i\alpha x_0}}{(z-w)^{1/8-\alpha^2/2}}\,\psi_\alpha(w)+\ldots\,, \tag{3.63}$$

where x_0 is the Dirichlet boundary condition of the interval between the insertions of the two twist fields. The rest of the OPE contains descendants of ψ_α, but can in principle contain also other primaries. In any case the most divergent term in (3.63) is the one corresponding to the identity, in agreement with (3.57).

3.6 Marginal operators and deformations

In this section we introduce the concept of marginal deformations in conformal field theory. This will be of great importance in the following, when discussing deformations of bound states of D-branes.

Conformal primeries can be categorized in three families, according to their conformal dimension. Considering chiral fields, we have:

- *relevant operators*, if the conformal dimension is smaller than 1;
- *irrelevant operators*, if the conformal dimension is bigger than 1;
- *marginal operators*, if the conformal dimension is equal to 1.

Marginal operators are related to the possible existence of deformations of a conformal field theory preserving the conformal symmetry and the central charge c [18]. This is because a deformation generated by an operator $V(z)$ results in an addition to the action of the form

$$\delta S \propto \int V(z)dz\,. \tag{3.64}$$

It is clear that only operators of conformal dimension 1 can preserve, at least at the classical level, the conformal invariance of the action. An example of that is of course the current ∂X of the bosonic theory. A perturbation generated by this operator results simply in an overall renormalization of the action. If the theory is compactified on a circle of radius R, this renormalization coincides with a change of the compactification radius.

It is not sufficient, however, to have marginal operators defining a deformation, in order for this to preserve the conformal symmetry. Such deformations are called *exactly marginal deformations*, in order to distinguish them from any deformation generated by operators of conformal dimension 1. There are many ways to check whether a deformation is exactly marginal; one of these tests is to make sure that the operator V defining the deformation does not change its own conformal dimension. At first order, this can not happen if the operator product expansion $V(z)V(w)$ contains V itself. Checking exact marginality at all orders is quite difficult; in some cases, however, it is sufficient to study the four-point function of operators V [19]. The situation is a bit more complicated for transformations generated by more than one marginal operator; we will see an example of that in 5.2.3.

The concept of exactly marginal deformations is the counterpart of the discussion of 2.2.2 and 2.3 in the context of conformal field theories. Since string theory, as we will see, is a conformal field theory on the two-dimensional worldsheet, well-defined deformations of string theory must be exactly marginal from the worldsheet point of view.

Chapter 4

The conformal field theory of twist fields

In this chapter we analyze the conformal field theory of twist fields. We first discuss the theory of a twisted boson, and highlight the connection with orbifolds. We focus on the theory with a boundary, in order to later consider its application to bound states of D-branes in chapter 5.

We explore the possibility of bosonizing the twist fields, analogously to what we did for spin fields in section 3.4. We introduce the bosonized twist fields and explain their relation with the usual twist fields; we argue that they describe an array of Dirichlet sectors. Furthermore we the study correlation functions on the upper half plane in the presence of two bosonized twist fields, and compare to the results of section 3.5. Afterwards we consider more twist field insertions, and we present some new explicit results for correlation functions in the presence of four or more twist fields. Finally we discuss ordering issues when considering the twist fields on the boundary, and modular invariance of bulk twist field correlation functions and their connection with partition functions on Riemann surfaces.

Twist fields are very important when studying solitons and other non-perturbative effects in string theory that can be described by bound states of D-branes [20, 21], for instance the worldsheet description of black holes [22, 23, 24] and the reconstruction of the instanton profile in terms of intersecting D-branes [25, 26, 27]. Generally, the role of twist fields is essential when considering open strings stretched between branes of different dimension, in such a way to have different boundary conditions on the two endpoints; scattering amplitudes contain vertex operators built using twist fields [12]. Other important applications of twist fields are in the context of entanglement entropy [28, 29] and in the context of intersecting D-branes at non-trivial angles [30, 31, 32]. In these cases twist fields allow transitions between many different kinds of boundary conditions. In this work we restrict to $\mathbb{Z}_2$ twist fields, since we are dealing with Neumann and Dirichlet boundary conditions only.

The content of this chapter is not directly relevant for the applications to string theory presented in this thesis, except for some results concerning array of D-branes in chapter 5. However, the complete understanding of the next chapters does not rely on the results presented here.

4.1 Orbifolds and bosonized twist fields

Let us start by considering the bulk CFT, where one can twist both the holomorphic and the anti-holomorphic part of the boson. Correspondingly we will have a pair of twist fields $\sigma(z)$ and $\sigma(\bar{z})$ satisfying the OPE (3.57) with $j = i\partial X$ and $\bar{j} = i\bar{\partial}\bar{X}(\bar{z})$ respectively. A bulk twist field, twisting the full boson $X(z, \bar{z})$, can be defined by $\sigma(z, \bar{z}) = \sigma(z)\sigma(\bar{z})$. In the previous chapter we have considered X in a non-compact space; if, instead, the boson is compactified on a circle of radius R, the insertion of twist fields creates the twisted sector of a symmetric orbifold [33]. Twist fields have the same local properties (3.57) independently of the radius of the orbifold, but correlation functions can be affected by the value of the radius. In general the calculation of correlation functions involving bosonic twist fields is a complicated task (see e.g. [17, 13]), since these fields are non-local with respect to the boson X. For fermions, ghosts and fermionic twist fields (spin fields) it is possible to use a bosonization procedure [16] in order to simplify the calculation of correlation functions, in analogy to section 3.4. In this chapter we will see that it is possible to apply the same procedure to bosonic twist fields; however this works only for a particular value of the radius of the orbifold.

To see how this comes about, we recall the classification of conformal field theories at $c = 1$ (see, for example, [15, 34]).

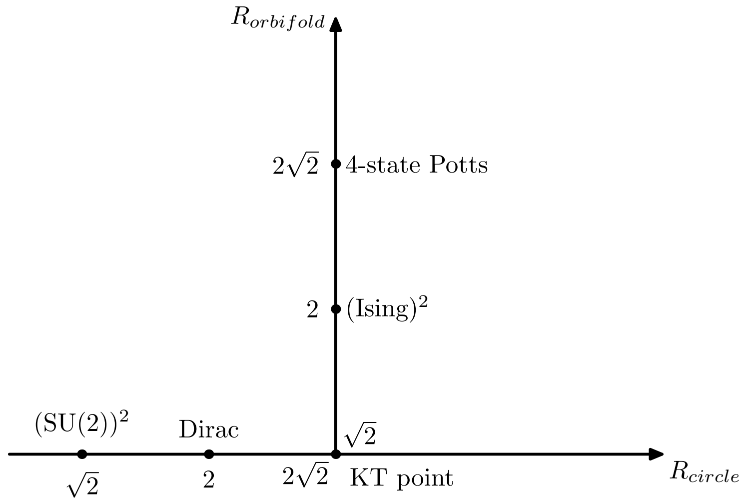

Figure 4.1: Classification of conformal field theories at central charge $c = 1$.

These theories can be divided in two families as in figure 4.1. One describing a boson compactified on a circle S_1, and one describing a boson compactified on an orbifold $S_1/\mathbb{Z}_2$. Both of these lines are parametrized by the radius of the circle. Some points on this graph correspond to particular models, for which it is possible to find a description in terms of a boson. It turns out that the two branches in the picture intersect, since the orbifold theory at $R = \sqrt{2}$ is equivalent to the circle theory at $R = 2\sqrt{2}$, which corresponds to the continuum limit of the XY-model at the Kosterlitz-Thouless point [18]. This duality is valid also at the level of boundary CFT; it has been shown that the two bulk CFT's admit the same boundary conditions and boundary operators or, in string theory language, the same set of D-branes [35]. This leads us to the idea of bosonizing boundary changing operators (the bosonized twist fields) in terms of another boson compactified on S^1.

4.1.1 $\mathfrak{su}(2)$ Kač-Moody algebra

Let us now return to boundary CFT; hence, let us restrict to the holomorphic part of the boson. Let us consider the primary operators $\widetilde{V}_\alpha(z)$, in particular the ones with $\alpha = \pm\sqrt{2}$. These are allowed operators when the boson is compactified on a circle with radius multiple of $1/\sqrt{2}$. Out of these two operators we can construct two other currents, namely:

$$j^1(z) = \frac{1}{\sqrt{2}}\left(\widetilde{V}_{\sqrt{2}}(z) + \widetilde{V}_{-\sqrt{2}}(z)\right), \qquad j^2(z) = \frac{i}{\sqrt{2}}\left(\widetilde{V}_{\sqrt{2}}(z) - \widetilde{V}_{-\sqrt{2}}(z)\right). \tag{4.1}$$

The three currents j^1, j^2 and $j^3 = j = i\partial X$ constitute a $\mathfrak{su}(2)$ Kač-Moody algebra, which means that they satisfy the following OPE's:

$$j^i(z)j^j(w) = \frac{\delta^{ij}}{(z-w)^2} - i\sqrt{2}\epsilon^{ijk}\frac{j^k(w)}{(z-w)} + \text{reg.}, \tag{4.2}$$

where ϵ^{ijk} is the completely antisymmetric tensor. If the currents are expanded as usual in Laurent modes, the OPE's are equivalent to the commutation relations

$$[j^i_m, j^j_n] = -i\sqrt{2}\epsilon^{ijk}j^k_{m+n} + m\delta^{ij}\delta_{m+n}. \tag{4.3}$$

To continue we perform a change of basis by introducing a new chiral boson $\Omega(z)$, satisfying the same OPE

$$\Omega(z)\Omega(w) \sim -\log(z-w), \tag{4.4}$$

as the chiral field $X(z)$. Out of the boson Ω we can construct three currents $J^i(z)$ $(i = 1, 2, 3)$, analogously to the currents j^i constructed out of X. We then express ∂X in terms of Ω, identifying

$$i\partial X(z) = j^3(z) \equiv J^2(z) = \frac{i}{\sqrt{2}}\left(: e^{i\sqrt{2}\Omega} : (z) - : e^{-i\sqrt{2}\Omega} : (z)\right). \tag{4.5}$$

This change of basis is equivalent to a rotation in the three-dimensional space generated by the three currents of the $\mathfrak{su}(2)$ Kač-Moody algebra (cfr. [15]). For consistency we also impose the identifications

$$\begin{gathered} i\partial\Omega(z) = J^3(z) \equiv j^1(z) = \frac{1}{\sqrt{2}}\left(: e^{i\sqrt{2}X} : (z) + : e^{-i\sqrt{2}X} : (z)\right), \\ \frac{1}{\sqrt{2}}\left(: e^{i\sqrt{2}\Omega} : (z) + : e^{-i\sqrt{2}\Omega} : (z)\right) = J^1(z) \equiv j^2(z) = \frac{i}{\sqrt{2}}\left(: e^{i\sqrt{2}X} : (z) - : e^{-i\sqrt{2}X} : (z)\right). \end{gathered} \tag{4.6}$$

Once we have done this rotation, we have a description of the CFT of a free boson in another basis. This may not seem convenient, since the conformal primaries $\widetilde{V}_\alpha =: e^{i\alpha X}:$ do not have a local description in terms of Ω in this new picture. However, this rotation allows us to identify new primaries which do not have a local description in terms of X.

Namely they are the primaries $V_\alpha(z) =: e^{i\alpha\Omega} : (z)$, and the bosonized twist fields will be among them.

Considering again the bulk theory, the same procedure can be done for the anti-holomorphic part of the boson, defining an anti-chiral field $\bar{\Omega}$. Notice that $\Omega(z,\bar{z}) = \Omega(z) + \bar{\Omega}(\bar{z})$, as a functional of $X(z,\bar{z}) = X(z) + \bar{X}(\bar{z})$, will be periodic under a shift of $2\pi\sqrt{2}n$ ($n \in \mathbb{Z}$), which means that also Ω is compactified on a circle at the self-dual radius. Let us now consider the boson $X(z,\bar{z})$ compactified on an orbifold at the self-dual radius, with the $\mathbb{Z}_2$ transformation defined by $X \to -X$. The boson $\Omega(z,\bar{z})$ should be unaffected by this transformation; this can be achieved if $\Omega(z,\bar{z})$ is compactified on a circle with half the radius, namely $R = 1/\sqrt{2}$. Notice that this is consistent with the identifications (4.5) and (4.6). Furthermore, T-duality implies that the circle theory at $R = 1/\sqrt{2}$ is in turn equivalent to a circle theory at radius $R' = 2/R = 2\sqrt{2}$, in accordance with what depicted in figure 4.1.

4.1.2 Boundary conditions and bosonized twist fields

Let us now consider a boundary CFT, with the chiral and anti-chiral part of the boson X related by appropriate boundary conditions. If we define the chiral and anti-chiral part of a boson Ω as in the previous subsection, we can deduce the boundary conditions in terms of Ω. For this we note that a change of sign in $\partial X(z)$, which one needs in order to interchange Dirichlet and Neumann boundary conditions, can be achieved by shifting the chiral field $\Omega(z)$ by $\pi/\sqrt{2}$; therefore Neumann and Dirichlet boundary conditions for X (on the real line $z = \bar{z}$) correspond to

$$\begin{aligned} \text{Neumann:} \qquad & \Omega(z) = \bar{\Omega}(\bar{z})\,, \\ \text{Dirichlet:} \qquad & \Omega(z) = \bar{\Omega}(\bar{z}) + \frac{\pi}{\sqrt{2}}\,. \end{aligned} \tag{4.7}$$

These boundary conditions may look unfamiliar from the point of view of the Ω boundary conformal field theory. However, it is not hard to see that they are conformal, as it must be since they correspond to the usual Neumann and Dirichlet boundary conditions for the boson X.

Among the primaries V_α, there are some that have the same local properties (3.57) as twist fields. Indeed, let us consider the two primaries $\sigma_B = V_{\sqrt{2}/4}$ and $\bar{\sigma}_B = V_{-\sqrt{2}/4}$, both with conformal dimension $1/16$. σ_B will be *called bosonized twist field*, and $\bar{\sigma}_B$ is its conjugated field. Moreover, we identify the *excited bosonized twist field* $\sigma'_B = \frac{1}{\sqrt{2}} V_{-3\sqrt{2}/4}$ and its conjugated $\bar{\sigma}'_B = \frac{1}{\sqrt{2}} V_{3\sqrt{2}/4}$. Given these definitions, using the relation (3.16) (which is valid for V_α as it was for $\widetilde{V}_\alpha$) we can derive that the bosonized twist fields satisfy the following OPE's with the current $j = i\partial X$:

$$\begin{aligned} i\partial X(z)\sigma_B(w) &= \frac{\sigma'_B(w)}{(z-w)^{1/2}} + \dots \\ i\partial X(z)\sigma'_B(w) &= \frac{\sigma_B(w)}{2(z-w)^{3/2}} + \frac{2\,\partial\sigma_B(w)}{(z-w)^{1/2}} + \dots\,, \end{aligned} \tag{4.8}$$

which are identical to (3.57), at least for the most divergent terms. The description in terms of Ω allows to treat the bosonized twist fields and the current ∂X in the same way, having a free field representation for all of them at the same time.

The way we interpret these (boundary) twist fields is the following: usually twist fields relate the Neumann sector $(\partial-\bar{\partial})X=0$ to the Dirichlet sector $X(z,\bar{z})=x_0$, where x_0 is some value for the boundary condition, and this information is encoded in the twist fields. The bosonized version $\sigma_B=V_{\sqrt{2}/4}$, however, cannot provide the information about x_0; furthermore, the periodicity property $X(z,\bar{z})\sim X(z,\bar{z})+2\pi\sqrt{2}n$ suggests that these twist fields describe the superposition of different Dirichlet sectors with boundary conditions $x_0^n=2\pi\sqrt{2}n$ $(n\in\mathbb{Z})$. We will give more evidence in favor of this interpretation in the following sections.

4.2 Single twist field insertion

In 3.5.1 we derived some correlation functions involving a pair of twist fields $\bar{\sigma}$-σ. Let us compare them to the corresponding ones with the bosonized version $\bar{\sigma}_B$-σ_B. The presence of at least two twist fields is needed, because one of them, say σ, changes the boundary condition from Dirichlet to Neumann, while $\bar{\sigma}$ changes from Neumann to Dirichlet. In the bosonized language, the presence of both a twist field and its conjugated is necessary, since the integration over the zero mode of Ω implies that the sum of all the exponents α_i in $\langle\prod_i V_{\alpha_i}\rangle$ has to be zero. Hence correlation functions with just one σ (or just one σ_B) would vanish.

4.2.1 Correlation functions with two bosonized twist fields

Correlation functions involving bosonized twist fields can be easily computed, since the current $j=i\partial X$ and the bosonized twist fields have both a local description in terms of the boson Ω. Correlation functions involving only j and bosonized twist fields are straightforward; for the explicit calculation we use

$$\langle V_{\alpha_1}(z_1)\dots V_{\alpha_n}(z_n)\rangle=\prod_{i<j}(z_{ij})^{\alpha_i\alpha_j}\;\delta\left(\sum_{i=1}^n\alpha_i\right)\,, \tag{4.9}$$

which is completely analogous to (3.18). From this one obtains

$$\begin{aligned}
\langle\bar{\sigma}_B(z_1)j(z_2)\sigma_B(z_3)\rangle&=0\,,\\
\langle\bar{\sigma}_B(z_1)j(z_2)j(z_3)\sigma_B(z_4)\rangle&=\frac{1}{2}\frac{1}{(z_{41})^{1/8}(z_{32})^2}\left(\sqrt{\frac{z_{31}z_{42}}{z_{21}z_{43}}}+\sqrt{\frac{z_{21}z_{43}}{z_{31}z_{42}}}\right)\,,\\
\langle\bar{\sigma}_B(z_1)j(z_2)\sigma'_B(z_3)\rangle&=\frac{z_{31}^{3/8}}{2z_{21}^{1/2}z_{32}^{3/2}}\,,\\
\langle\bar{\sigma}'_B(z_1)j(z_2)\sigma_B(z_3)\rangle&=\frac{z_{31}^{3/8}}{2z_{21}^{3/2}z_{32}^{1/2}}\,,
\end{aligned} \tag{4.10}$$

in agreement with (3.58) and (3.60). This is not surprising since the zero mode of X does not appear in the current $j=i\partial X$.

When fields like ψ_α are present, correlation functions depend explicitly on the particular boundary condition x_0. Therefore, we expect them to be different when the bosonized twist fields σ_B and $\bar{\sigma}_B$ are used instead of σ and $\bar{\sigma}$. As explained before, however, we want to interpret the result in terms of a superposition of different sectors with boundary conditions $x_0 = \sqrt{2}\pi n$ (considering only the holomorphic part of X). We thus assume, for example, that

$$\langle \bar{\sigma}_B(z_1)\psi_\alpha(z_2)\sigma_B(z_3)\rangle = \sum_{n\in\mathbb{Z}} \langle \bar{\sigma}(z_1)\psi_\alpha(z_2)\sigma(z_3)\rangle_{x_0=\sqrt{2}\pi n}\,, \tag{4.11}$$

and check this hypothesis. Since the sum is over an infinite number of Dirichlet sectors, the result must be normalized, in order to match with the normalization of the two-point function $\langle \bar{\sigma}(z_1)\sigma(z_3)\rangle = z_{13}^{-1/8}$. The right hand side of (4.11) is straightforward to compute, starting from (3.61), and involves the Dirac comb

$$\text{Ш}_T(t) = \frac{1}{T}\sum_{n\in\mathbb{Z}} e^{2\pi i n \frac{t}{T}} = \sum_{k\in\mathbb{Z}} \delta(t-kT)\,. \tag{4.12}$$

The final result for the right hand side of (4.11) is

$$\begin{aligned} &\frac{\sqrt{2}}{z_{13}^{1/8}}\left(\frac{z_{13}}{4z_{12}z_{23}}\right)^{\alpha^2/2}\text{Ш}_{\sqrt{2}}(\alpha) = \\ &\qquad =\frac{\sqrt{2}}{z_{13}^{1/8}}\left(\frac{z_{13}}{4z_{12}z_{23}}\right)^{\alpha^2/2}\left(\delta(\alpha)+\delta(\alpha+\sqrt{2})+\delta(\alpha-\sqrt{2})+\dots\right)\,. \end{aligned} \tag{4.13}$$

The calculation of the left hand side is more involved, since σ_B and $\bar{\sigma}_B$ are naturally written in terms of Ω, while ψ_α is not local with respect to it. However, we can rewrite the combination $\bar{\sigma}_B(z_1)\sigma_B(z_3)$ as

$$\bar{\sigma}_B(z_1)\sigma_B(z_3) = V_{-\sqrt{2}/4}(z_1)V_{\sqrt{2}/4}(z_3) = \frac{1}{z_{31}^{1/8}}\exp\left(\frac{\sqrt{2}}{4}\int_{z_3}^{z_1} i\partial\Omega(z)dz\right) \tag{4.14}$$

and express $\partial\Omega$ in terms of X using (4.6). This gives a path integral over X,

$$\langle \bar{\sigma}_B(z_1)\psi_\alpha(z_2)\sigma_B(z_3)\rangle = \frac{1}{Z}\int [dX]\,(\bar{\sigma}_B(z_1)\psi_\alpha(z_2)\sigma_B(z_3))\,e^{-S[X]}\,. \tag{4.15}$$

We can split the integral as $\int[dX] = \int dx_0 \int[dX_\perp]$, where x_0 represents the zero mode of X. The periodicity properties of $\partial\Omega$ imply that the $\int dx_0$ integral is of the form

$$\int dx_0\, e^{i\alpha x_0} f(x_0)\,, \tag{4.16}$$

where $e^{i\alpha x_0}$ accounts for the zero mode in ψ_α and f is a periodic function $f(x) = f(x+\sqrt{2}\pi)$. This integral can be rewritten as

$$\sum_{n\in\mathbb{Z}} e^{i\alpha n\sqrt{2}\pi}\int_0^{\sqrt{2}\pi} dx_0\, f(x_0)e^{i\alpha x_0}\,. \tag{4.17}$$

Using the definition of Dirac comb, we then notice that

$$\langle \bar{\sigma}_B(z_1)\psi_\alpha(z_2)\sigma_B(z_3)\rangle \propto \text{Ш}_{\sqrt{2}}(\alpha) = \Big(\delta(\alpha) + \delta(\alpha+\sqrt{2}) + \delta(\alpha-\sqrt{2}) + \dots\Big)\,, \qquad (4.18)$$

in agreement with (4.13). Furthermore, the dependence on the positions z_1, z_2 and z_3 is fixed by conformal invariance, hence it must coincide with the one in (4.13) (up to an overall normalization).

Finally we discuss the OPE of two bosonized twist fields. Let us consider the OPE of two normal twist fields (3.63): if the boson is compactified, the allowed values of α are restricted, and the integral becomes a sum. For example, in the case of bosonized twist fields, α must be a multiple of $\sqrt{2}$. This can be derived also in the Ω picture, since

$$\bar{\sigma}_B(z)\sigma_B(w) = \frac{1}{(z-w)^{1/8}}\left(1 - \frac{\sqrt{2}}{4}(z-w)i\partial\Omega(w) + \dots\right)\,, \qquad (4.19)$$

and $\partial\Omega$ is expressed in terms of exponential operators through (4.6).

4.3 More twist field insertions

With four or more twist fields the situation is more complicated, for three reasons. First, the operator formalism used in chapter 3 is not applicable. The second reason is that we have two or more cuts on the complex plane where the fields are defined; this means that the worldsheet is now effectively a hyperelliptic surface with genus $g > 0$ [36, 13]. Finally, using the electrostatic analogy for finding correlation functions on the upper half plane is still possible, but an explicit expression for the Green's function with appropriate boundary conditions is known only in integral form. Let us then review the connection between twist field insertions and hyperelliptic surfaces, in particular in the case of four twist fields.

4.3.1 Twist fields and hyperelliptic surfaces

Let us consider $2n$ twist fields at positions z_i on the real line. We assume that the fields σ are at position z_i with i even, and the fields $\bar{\sigma}$ correspond to odd i. The current j has Neumann boundary conditions on the intervals $[z_{2i-1}, z_{2i}]$, and Dirichlet boundary conditions on the intervals $[z_{2i}, z_{2i+1}]$. The real line has to be considered as compactified, therefore there are Dirichlet boundary conditions also on the interval $[z_{2n}, z_1]$, containing the point at infinity. The complex plane (described by a coordinate z) has cuts along the real axis, in correspondence to the intervals with Neumann boundary conditions. The associated hyperelliptic surface, which has genus $g = n-1$, is described by the equation

$$w^2 = P(z) := \prod_{i=1}^{2n}(z - z_i)\,. \qquad (4.20)$$

Let us define some useful quantities; first of all we consider a canonical homology class $\{A_k, B_k\}$, where A_k and B_k are the A and B cycles of the hyperelliptic surface. In the description on the complex plane, these cycles surround two neighboring ramification points. There is a basis for holomorphic 1-forms on this hyperelliptic surface, given by

$$\omega_k = \frac{z^{k-1}\,\mathrm{d}z}{\sqrt{P(z)}} \qquad \text{for} \quad k = 1, \ldots, g\,. \tag{4.21}$$

Denoting with A_k the A-cycle surrounding the two ramification points z_{2k-1} and z_{2k}, the period of the 1-form ω_l along A_k is defined as

$$\Omega_{kl} = \oint_{A_k} \omega_l = \oint_{A_k} \frac{z^{l-1}\,\mathrm{d}z}{\sqrt{P(z)}}\,. \tag{4.22}$$

There is also a dual basis for holomorphic 1-forms ζ_l, satisfying

$$\oint_{A_k} \zeta_l = \delta_{kl}\,. \tag{4.23}$$

The period matrix of the hyperelliptic surface is defined in the following way:

$$\tau_{kl} = \oint_{B_k} \zeta_l\,, \tag{4.24}$$

where B_k is the B-cycle surrounding the two ramification points z_{2k} and z_{2k+1}.

4.3.2 Four twist fields and the associated torus

When we have only four twist fields, the genus of the surface is $g = 1$. This means that we are dealing with a torus, whose A and B cycles are shown in figure 4.2.

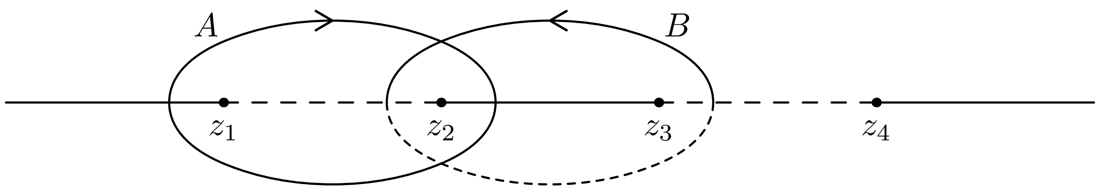

Figure 4.2: A and B cycles for a complex plane with 4 twist fields insertions.

We have only one holomorphic 1-form

$$\omega = \frac{\mathrm{d}z}{\sqrt{P(z)}}\,, \tag{4.25}$$

and its dual ζ given by

$$\zeta = \frac{\omega}{\Omega} = \omega \Big/ \oint_A \omega\ . \tag{4.26}$$

Therefore the period τ is simply

$$\tau = \oint_B \omega \Big/ \oint_A \omega\ . \tag{4.27}$$

It can be useful to distinguish the two periods of the torus as $\tau_1 = \oint_A \omega$ and $\tau_2 = \oint_B \omega$, τ being the ratio of the two. Notice that, given the definition (4.20) and assuming that the twist fields are inserted on the real line, the quantity τ_2 is real, while $\Omega = \tau_1$ and τ are purely imaginary. Introducing the conformal cross ratio $\eta = z_{43}z_{21}/(z_{42}z_{31})$, where $z_{ij} = z_i - z_j$, the period can be written as

$$\tau = i\frac{K(1-\eta)}{K(\eta)}\,, \tag{4.28}$$

where K is the complete elliptic integral of the first kind. This relation can be inverted using Jacobi theta functions, namely

$$\eta = \left(\frac{\vartheta_2(0;\tau))}{\vartheta_3(0;\tau)}\right)^4\,. \tag{4.29}$$

A positive, purely imaginary τ corresponds to $0 < \eta < 1$; the modular transformation $\tau \to -1/\tau$ corresponds to the map $\eta \to 1-\eta$. We will also use the so-called *uniformized coordinates*, defined by

$$x(z) = \frac{1}{\Omega}\int_{z_1}^{z} \omega = \frac{1}{\Omega}\int_{z_1}^{z} \frac{dw}{\sqrt{P(w)}}\,. \tag{4.30}$$

In these coordinates the torus is flat, and we identify points on the complex plane via $x \equiv x + m + n\tau$, where $m, n \in \mathbb{Z}$. The four points z_1, z_2, z_3 and z_4 are mapped to 0, $\tau/2$, $(\tau+1)/2$ and $1/2$ respectively. The torus can thus be described as the quotient

$$\mathbb{T}_2 = \frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}\,. \tag{4.31}$$

The fundamental domain is shown in figure 4.3.

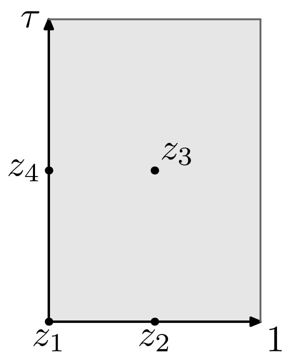

Figure 4.3: Fundamental domain of a torus in uniformized coordinates.

4.3.3 Correlation functions with four twist fields

Let us now consider correlation functions involving four twist fields; many of them are already known and have been derived solving systems of differential equations, similar to the Knizhnik-Zamolodchikov equations (see [17, 13, 37] and also [38, 39] for parallel results in the context of D-branes at angles). We review here some of these results, and we extend them to the corresponding correlation functions involving the bosonized version of the twist fields. Let us consider first of all the correlation function of four twist fields, namely

$$\langle \bar{\sigma}(z_1)\sigma(z_2)\bar{\sigma}(z_3)\sigma(z_4)\rangle \,. \tag{4.32}$$

This correlation function is well known in the literature, and was computed for example in [17]; the detailed derivation can be found in appendix C. It is important to notice that in the presence of four twist fields there are two Dirichlet intervals on the boundary. The boson $X(z,\bar{z})$ can in principle have different boundary conditions $X = x_0^i$ $(i = 1, 2)$ on the two intervals. Adapting the result of [17] to our notations, the correlation function is

$$\langle \bar{\sigma}(z_1)\sigma(z_2)\bar{\sigma}(z_3)\sigma(z_4)\rangle = \left(\frac{z_{31}z_{42}}{z_{21}z_{41}z_{32}z_{43}}\right)^{1/8} \sqrt{\frac{\pi}{2K(\eta)}} \exp\left(\frac{i}{8\pi}(x_0^1 - x_0^2)^2\tau\right) \,, \tag{4.33}$$

where the conformal ratio η is given by $\eta = z_{43}z_{21}/(z_{42}z_{31})$, and $K(\eta)$ is the complete elliptic integral of the first kind. In order to derive this four-point function one encounters other correlation functions, namely $\langle j(w)\bar{\sigma}(z_1)\sigma(z_2)\bar{\sigma}(z_3)\sigma(z_4)\rangle$, $\langle \bar{\sigma}(z_1)\sigma'(z_2)\bar{\sigma}(z_3)\sigma(z_4)\rangle$ and $\langle j(w)\bar{\sigma}(z_1)\sigma'(z_2)\bar{\sigma}(z_3)\sigma(z_4)\rangle$; explicit expressions are given in appendix C.

Another important correlation function that can be computed is the one involving two currents j and four twist fields. The result is known (see [33, 37]) when the difference of the two boundary conditions $\delta = x_0^1 - x_0^2$ is zero. In appendix D we generalize to the case $\delta \neq 0$, the result being

$$\begin{aligned}
&\langle j(z)j(w)\bar{\sigma}(z_1)\sigma(z_2)\bar{\sigma}(z_3)\sigma(z_4)\rangle = \\
&\quad = \frac{G(z_i)}{2(z-w)^2}\left[\sqrt{\frac{(z-z_1)(w-z_2)(z-z_3)(w-z_4)}{(w-z_1)(z-z_2)(w-z_3)(z-z_4)}} + (z \leftrightarrow w)\right] + \\
&\quad + \frac{\sqrt{2\pi}}{\sqrt{P(z)P(w)}}\left(\frac{z_{31}z_{42}}{z_{21}z_{41}z_{32}z_{43}}\right)^{-7/8} \partial_\eta\left[\frac{1}{\sqrt{K(\eta)}}\exp\left(\frac{i\delta^2}{8\pi}\tau(\eta)\right)\right] \,,
\end{aligned} \tag{4.34}$$

where $G(z_i)$ is the four-point function $G(z_i) = \langle \bar{\sigma}(z_1)\sigma(z_2)\bar{\sigma}(z_3)\sigma(z_4)\rangle$.

Starting with the correlation function with two currents one can easily obtain other correlation functions involving excited twist fields. It is sufficient to consider the limit when one of the currents approaches a twist fields, and use the corresponding OPE, as done in section 4.2. In particular, we derive explicit results for $\langle \bar{\sigma}'(z_1)\sigma'(z_2)\bar{\sigma}(z_3)\sigma(z_4)\rangle$ and $\langle \bar{\sigma}'(z_1)\sigma(z_2)\bar{\sigma}'(z_3)\sigma(z_4)\rangle$ in appendix D. Similar and other correlation functions involving excited twist fields can be found in [40], and in [41] for the case of twist fields connecting D-branes at different angles.

4.3.4 Correlation functions with four bosonized twist fields

We now compare the above results to the case with four bosonized twist fields. The calculation of the four twist correlator is straightforward using (4.9):

$$\langle \bar{\sigma}_B(z_1)\sigma_B(z_2)\bar{\sigma}_B(z_3)\sigma_B(z_4)\rangle = \left(\frac{z_{31}z_{42}}{z_{21}z_{41}z_{32}z_{43}}\right)^{1/8} \,. \tag{4.35}$$

This correlation function should represent a double array of Dirichlet sectors, whose boundary conditions are separated by $2\sqrt{2}\pi$. This is because each pair $\bar{\sigma}_B$-σ_B connects the Neumann sector to the array; therefore a sum over the array has to be performed for both pairs. Thus we should compare (4.35) with the quantity

$$\sum_{a,b\in\mathbb{Z}}\left(\frac{z_{31}z_{42}}{z_{21}z_{41}z_{32}z_{43}}\right)^{1/8}\sqrt{\frac{\pi}{2K(\eta)}}\exp\left(\frac{i}{8\pi}(2\sqrt{2}\pi(a-b))^2\tau\right). \tag{4.36}$$

The sum is infinite but will give a finite result after dividing by the two-point function $\sum_{n\in\mathbb{Z}}\langle\bar{\sigma}(z_1)\sigma(z_2)\rangle_{x_0=2\sqrt{2}\pi n}$, which is the correct normalization of correlation functions. Using the Jacobi theta function ϑ_3, which satisfies

$$\vartheta_3(0;\tau)=\sum_{n\in\mathbb{Z}}e^{i\pi\tau n^2}=\sqrt{\frac{2K(\eta)}{\pi}}, \tag{4.37}$$

we notice that (4.36) is equal to (4.35).

We can proceed in an analogous way for the correlation function with two currents and four twist fields. Using the bosonized expression of j one easily derives

$$\begin{aligned}&\langle j(z)j(w)\bar{\sigma}_B(z_1)\sigma(z_2)_B\bar{\sigma}_B(z_3)\sigma_B(z_4)\rangle=\\&=\frac{1}{2(z-w)^2}\left(\frac{z_{31}z_{42}}{z_{21}z_{41}z_{32}z_{43}}\right)^{1/8}\left[\sqrt{\frac{(z-z_1)(w-z_2)(z-z_3)(w-z_4)}{(w-z_1)(z-z_2)(w-z_3)(z-z_4)}}+(z\leftrightarrow w)\right].\end{aligned} \tag{4.38}$$

Let us now compare (4.38) to the sum of (4.34) over the array. Summing the first term gives simply

$$\frac{G_B(z_i)}{2(z-w)^2}\left[\sqrt{\frac{(z-z_1)(w-z_2)(z-z_3)(w-z_4)}{(w-z_1)(z-z_2)(w-z_3)(z-z_4)}}+(z\leftrightarrow w)\right], \tag{4.39}$$

where $G_B(z_i)=\langle\bar{\sigma}_B(z_1)\sigma_B(z_2)\bar{\sigma}_B(z_3)\sigma_B(z_4)\rangle$. The second term in (4.34) is proportional to

$$\partial_\eta\left[\frac{1}{\sqrt{K(\eta)}}\exp\left(\frac{i\delta^2}{8\pi}\tau(\eta)\right)\right]. \tag{4.40}$$

Summing over $\delta=2\sqrt{2}\pi n$ we get $\partial_\eta\sqrt{\frac{2}{\pi}}=0$. Putting all together we notice that (4.38) is recovered.

Correlation functions involving excited twist fields, in particular $\langle\bar{\sigma}'_B\sigma'_B\bar{\sigma}_B\sigma_B\rangle$ and $\langle\bar{\sigma}'_B\sigma_B\bar{\sigma}'_B\sigma_B\rangle$ are computed in appendix D, and agree with the sum over the array of $\langle\bar{\sigma}'\sigma'\bar{\sigma}\sigma\rangle$ and $\langle\bar{\sigma}'\sigma\bar{\sigma}'\sigma\rangle$ respectively. The results of this section give further support to our claim that bosonized twist fields describe an array of Dirichlet sectors.

4.3.5 Correlation functions with more than four bosonized twist fields

The calculation of correlation functions becomes increasingly more complicated when the number of twist fields is more than four. Some results have been derived through the electrostatic analogy (see [13] for $\mathbb{Z}_2$ twist fields and [42, 43, 32] for generic angle twist fields); this procedure is however quite formal, since an explicit expression for the Green's function is known only in integral form. Furthermore, the generalization of the methods described in appendices C and D is problematic.

In the bosonized case, however, it is still possible to compute correlation functions involving bosonized twist fields and, possibly, the current ∂X. For example the $2n$-point function of twist fields is given by

$$\langle \bar{\sigma}_B(z_1)\sigma_B(z_2)\ldots\bar{\sigma}_B(z_{2n-1})\sigma_B(z_{2n})\rangle = \prod_{\substack{i>j\\(i-j)\in 2\mathbb{N}}} z_{ij}^{1/8} \prod_{\substack{i>j\\(i-j)\in 2\mathbb{N}+1}} z_{ij}^{-1/8} = \prod_{i>j} z_{ij}^{\frac{(-1)^{i-j}}{8}} \,. \tag{4.41}$$

The correlator with two currents reads

$$\langle j(z)j(w)\bar{\sigma}_B(z_1)\ldots\sigma_B(z_{2n})\rangle = \frac{G_B^{2n}(z_i)}{2(z-w)^2}\left(\prod_{i\text{ odd}}\sqrt{\frac{z-z_i}{w-z_i}}\prod_{i\text{ even}}\sqrt{\frac{w-z_i}{z-z_i}} + (z\leftrightarrow w)\right)\,, \tag{4.42}$$

where $G_B^{2n}(z_i)$ is the $2n$-point function (4.41). Correlators involving excited twist fields can also be considered; for example the correlation function of two excited and four normal twist fields is

$$\langle \bar{\sigma}'_B(z_1)\sigma'_B(z_2)\bar{\sigma}_B(z_3)\sigma_B(z_4)\bar{\sigma}_B(z_5)\sigma_B(z_6)\rangle = \frac{1}{2z_{21}^{9/8}}\left(\frac{z_{53}z_{64}}{z_{43}z_{63}z_{54}z_{65}}\right)^{1/8}\left(\frac{z_{31}z_{51}z_{42}z_{62}}{z_{32}z_{52}z_{41}z_{61}}\right)^{3/8}. \tag{4.43}$$

Through the bosonization procedure one might easily see if a correlation function vanishes; this happens whenever the sum of all the exponents of operators V_α can not give zero. For example, a correlator with one excited and $2n-1$ normal twist fields is always zero:

$$\langle \bar{\sigma}'_B(z_1)\sigma_B(z_2)\ldots\bar{\sigma}_B(z_{2n-1})\sigma_B(z_{2n})\rangle = 0\,. \tag{4.44}$$

The same is true for a correlator of m excited and $2n-m$ normal twist fields, when m is odd. Analogously, a correlator involving two excited twist fields vanishes if they are both conjugated (or both non-conjugated). For example

$$\langle \bar{\sigma}'_B(z_1)\sigma_B(z_2)\bar{\sigma}'_B(z_3)\ldots\bar{\sigma}_B(z_{2n-1})\sigma_B(z_{2n})\rangle = 0\,. \tag{4.45}$$

Furthermore, every correlation function with an odd number of currents and $2n$ normal twist fields is zero:

$$\langle j(w_1)\ldots j(w_{2m+1})\bar{\sigma}_B(z_1)\ldots\sigma_B(z_{2n})\rangle = 0\,. \tag{4.46}$$

We have seen in appendices C and D that these correlation functions (with $n=2$) are in general non-vanishing for normal (non-bosonized) twist fields. The setup with the array of Dirichlet sectors is special, and makes many correlation functions vanish.

4.4 Ordering of boundary twist fields

In the previous sections, when computing correlation functions with twist fields on the boundary, we have always implicitly assumed a particular ordering of the twist fields.

This is because a twist field connects a CFT to a different one (corresponding to different boundary condition for the boson), and it must be followed by a conjugated twist field, in such a way that the correlation function is computed with respect to the vacuum of the original CFT. For concreteness, let us consider the two-point function of twist fields $\langle\bar\sigma(z)\sigma(w)\rangle$. The two twist fields connect the boundary conformal field theory of a boson with Dirichlet boundary condition (BCFT_D) to the boundary conformal field theory of a boson with Neumann boundary condition (BCFT_N); more precisely, reading the correlation function from left to right, $\bar\sigma$ connects BCFT_D to BCFT_N and σ connects BCFT_N to BCFT_D. To make things clear, we will indicate explicitly with N or D the vacuum of the reference CFT, which is the BCFT_D in this case. Therefore, the correlation function has to be interpreted as

$$\langle\bar\sigma(z)\sigma(w)\rangle_D = \frac{\langle \mathbb{1}\rangle_D}{(z-w)^{1/8}} = \frac{Z_D}{(z-w)^{1/8}}\,, \tag{4.47}$$

where we have used the OPE (3.57), and Z_D is the partition function in BCFT_D. If we now want to consider the opposite ordering, we have to consider the BCFT_N as reference theory. Assuming that the OPE is

$$\sigma(z)\bar\sigma(w) = \frac{\alpha}{(z-w)^{1/8}}\,, \tag{4.48}$$

we will have $\langle\sigma(z)\bar\sigma(w)\rangle_N = \alpha(z-w)^{-1/8}\langle\mathbb{1}\rangle_N = \alpha(z-w)^{-1/8}Z_N$, where Z_N is the partition function in BCFT_N. Since we are considering the twist fields inserted on the boundary of a disk (or, alternatively, on the compactified real line), cyclicity implies that $\langle\bar\sigma(z)\sigma(w)\rangle_D = \langle\sigma(w)\bar\sigma(z)\rangle_N$, from which we conclude that

$$\alpha = Z_D/Z_N\,. \tag{4.49}$$

On the other hand, the number α can be computed using the four-point function. In fact

$$\lim_{z_2\to z_3}(z_2-z_3)^{1/8}\langle\bar\sigma(z_1)\sigma(z_2)\bar\sigma(z_3)\sigma(z_4)\rangle_D = \alpha\langle\bar\sigma(z_1)\sigma(z_4)\rangle_D = \frac{Z_D^2/Z_N}{(z_1-z_4)^{1/8}}\,. \tag{4.50}$$

In the previous section we have chosen to normalize $Z_D = 1$. The explicit form of the four-point function gives the result

$$\lim_{z_2\to z_3}(z_2-z_3)^{1/8}\langle\bar\sigma(z_1)\sigma(z_2)\bar\sigma(z_3)\sigma(z_4)\rangle_D = 0\,, \tag{4.51}$$

which means that the partition function Z_N is divergent. However, Z_N can be regularized, for example compactifying the boson on a circle of radius R, which would give $Z_N = 1/R$ (see[44]).

The cyclical property we used for the two-point function generalizes to more complicated correlation functions of twist fields, provided that the appropriate reference CFT is taken into account. For example, for the four-point function,

$$\langle\bar\sigma(z_1)\sigma(z_2)\bar\sigma(z_3)\sigma(z_4)\rangle_D = \langle\sigma(z_2)\bar\sigma(z_3)\sigma(z_4)\bar\sigma(z_1)\rangle_N\,. \tag{4.52}$$

With this rule, every correlation function with an even number of boundary twist fields, and with alternating σ's and $\bar\sigma$'s has a precise and unambiguous meaning.

4.5 Bulk twist fields and modular invariance

In the previous section we reviewed how the cuts created by the presence of twist fields have the effect of transforming the worldsheet into a higher genus Riemann surface. It is thus natural to think that the correlation function of twist fields, without any other operator, is associated to the partition function of a twisted boson on this surface. This was examined for example in [34] and [45]. In order to connect to this result, we have to consider bulk twist fields, twisting both the chiral (X) and anti-chiral ($\bar{X}$) parts of the boson. The bulk twist fields are given by the product of a chiral and an anti-chiral twist fields. Effectively, a correlation function of bulk twist fields is given by the square of the correlation function of chiral twist fields. An important observation that we have to make is that the Riemann surface is not sensible to which points the cuts are connecting and to which twist fields are conjugated and which are not. For concreteness, if we indicate (12)(34) the correlation function $\langle\bar{\sigma}(z_1)\sigma(z_2)\bar{\sigma}(z_3)\sigma(z_4)\rangle$, where the cuts are connecting z_1 to z_2 and z_3 to z_4, we see that the combination (34)(12), (21)(43) and (43)(21) describe the same situation. In total there are 6 independent ways of partitioning the points z_i in two non-ordered pairs, which correspond to different conformal ratios and, correspondingly, to different periods of the associated torus (see table 4.1).

Partition	Conformal ratio	Period
(12)(34)	η	τ
(14)(32)	$1-\eta$	$-\frac{1}{\tau}$
(12)(43)	$\frac{\eta}{\eta-1}$	$\tau+1$
(13)(42)	$\frac{1}{1-\eta}$	$-\frac{1}{\tau+1}$
(13)(24)	$\frac{1}{\eta}$	$\frac{\tau}{1-\tau}$
(14)(23)	$\frac{\eta-1}{\eta}$	$-\frac{1}{\tau}+1$

Table 4.1: Partition of four points and modular transformations.

In order to recover the partition function on the torus, one should sum over all these independent partitions. Looking at the associated periods, we notice that the different partitions generate modular transformations on the period τ. To be precise, the modular group can be generated by only two transformations:

$$\begin{aligned} S: \qquad & \tau \to -\frac{1}{\tau}\,, \qquad && \eta \to 1-\eta\,, \\ T: \qquad & \tau \to \tau+1\,, \qquad && \eta \to \frac{\eta}{\eta-1}\,. \end{aligned} \tag{4.53}$$

We refer to [34] for the proof that the sum

$$Z \propto |\langle\bar{\sigma}(z_1)\sigma(z_2)\bar{\sigma}(z_3)\sigma(z_4)\rangle|^2 + \text{permutations.} \tag{4.54}$$

is indeed the partition function of a twisted boson on a torus. Here we just notice that the result (4.54) is manifestly modular invariant.

The same discussion can be done for bosonized twist fields; the resulting modular invariant partition function will be the one corresponding to a twisted boson on an orbifold of radius $R = \sqrt{2}$. This partition function and the one obtained by normal twist fields are related. The "quantum" part of the partition function (which depends only on the local property of twist fields) is the same, while the "classical" part, which depends on the topology of the surface, is different. In order to obtain the classical part, one has to sum over all the classical solutions in the different winding sectors around the circle (see e.g. [34]):

$$Z^{cl}(R) = \sum_{(p,\bar{p})} \exp\left[i\pi(p \cdot \tau \cdot p - \bar{p} \cdot \bar{\tau} \cdot \bar{p})\right] , \tag{4.55}$$

where p and $\bar{p}$ are the allowed momenta running through the loops of the hyperelliptic surface. It was noticed in [17] that when the radius of the compactification is exactly $\sqrt{2}$, the total partition function simplifies, and can be expressed in terms of correlation functions of operators of the form $: e^{i\alpha\phi(z)} :$, where ϕ is a scalar field. The bosonization introduced in this chapter makes it clear that this scalar field is not the boson X, but it is the dual boson Ω, and that the operators $: e^{i\alpha\phi(z)} :$ are the bosonized twist fields.

If one wants to insert twist fields on the boundary, and interpret them as boundary changing operators, not all the partitions of table 4.1 are allowed. As we discussed in section 4.4, only the partitions (12)(34) and (14)(32) are well defined. This means that summing over the allowed partitions would give a result which is invariant only under the subgroup of the modular group generated by the S transformation. This is consistent with the fact that, if the four twist fields are inserted on the real line, the associated period is purely imaginary, and a T transformation would spoil this property.

Chapter 5

Finite-size D-branes in bosonic string theory

In this chapter we introduce bosonic string theory, stressing its connection to the conformal field theory of a free boson discussed in chapter 3. We introduce the theory, both for closed and open strings, and discuss the importance of ghosts for the quantization. Furthermore, we discuss scattering amplitudes and vertex operators. Focusing on open strings, we introduce D-branes, which are dynamical objects of the theory, non-perturbative by nature. Bound states of D-branes make it necessary to consider bosons in the Ramond sector; we discuss relevant correlation functions including twist fields and the possibility of having a marginal deformation corresponding to the blow-up of the size of a D-brane.

5.1 Bosonic String Theory and D-branes

A fundamental string is the one-dimensional generalization of the concept of particle. Each point of a string in d dimension is characterized by its position $X^M(\sigma^0 = \tau, \sigma^1 = \sigma)$, where τ and σ are the worldsheet coordinates. σ indicates the position of the point along the string and τ is a time coordinate. The action for a bosonic string is given by the so-called *Polyakov action*, which is [46]

$$S_P = \frac{1}{4\pi\alpha'} \int d^2\sigma \, \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N \eta_{MN} \, , \tag{5.1}$$

where $g_{\alpha\beta}$ is the worldsheet metric, which is an auxiliary field, and $g = \det g_{\alpha\beta}$; σ^α ($\alpha = 0, 1$) are the worldsheet coordinates. The indices M, N are contracted using a *target space metric* η_{MN}. The quantity $T = 1/(2\pi\alpha')$ is the so-called *tension* of the string, and is the generalization of the concept of mass for particles. The quantity α' is called *universal Regge slope*; it has dimension (length)^2, and is sometimes written as $\alpha' = l_s^2$, where l_s is the *string scale*.

The Polyakov action is characterized by some symmetries, namely

- Poincaré invariance of the target space $X^M \to \Lambda^M{}_N X^N$.
- Reparametrization invariance of the worldsheet coordinates σ^α.

- Weyl invariance, which corresponds to a rescaling of the worldsheet metric $g_{\alpha\beta}(\sigma,\tau) \to \Omega^2(\sigma,\tau)g_{\alpha\beta}(\sigma,\tau)$.

These symmetries imply that string theory is a two dimensional conformal field theory on the worldsheet. Furthermore, one can use these symmetries in order to fix three degrees of freedom. The most convenient choice is to work in the so-called *conformal gauge*, where the worldsheet metric is flat and the action reads

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\, \partial_\alpha X^M \partial^\alpha X_M\,. \tag{5.2}$$

The equations of motion for the worldsheet metric, on the other hand, imply that string theory can describe only oscillations transverse to the string. The action (5.2) can describe both closed and open strings. For closed strings we assume that the σ coordinates is compactified, identifying $X^M(\sigma,\tau) = X^M(\sigma+2\pi,\tau)$. For open strings we consider σ restricted to the interval $\sigma \in [0,\pi]$, with $\sigma = 0$ and $\sigma = \pi$ representing the two endpoints of the string. The worldsheets for closed and open strings are represented in figure 5.1.

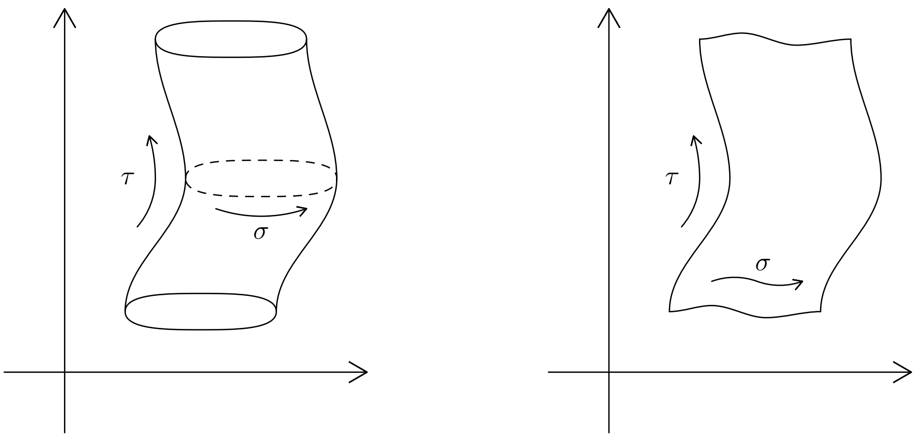

Figure 5.1: Worldsheets for closed (left) and open (right) strings.

We can use complex coordinates for describing the worldsheet, i.e.

$$w = \tau + i\sigma\ , \qquad \bar{w} = \tau - i\sigma\,. \tag{5.3}$$

In this way the worldsheet is mapped to a strip on the complex plane. Defining now $z = e^w$ we can map the strip to the complex plane, analogously to what we have done in section 3.3. In these coordinates the action (5.2) becomes

$$S = \frac{1}{4\pi}\int d^2z\, \left(\frac{2}{\alpha'}\partial X^M \bar{\partial} X_M\right)\,, \tag{5.4}$$

which is nothing else than d copies of the free bosonic theory described in chapter 3. For open strings, the endpoints $\sigma = 0, \pi$ are mapped to the positive and negative real line respectively, therefore the boundary is the same one considered in chapter 3.

5.1.1 The b-c ghost system

The symmetries of the Polyakov action described above can be considered as gauge invariance of the theory. This gauge invariance plays a fundamental role when we want to quantize the theory. We do not go into details here, but if we quantize the theory using the Faddeev-Popov method, as one would do for other gauge theories, it turns out that we have to introduce new non-physical fields to the theory, the so-called ghosts. In the case of bosonic string theory we need to introduce two anticommuting fields $b(z)$ and $c(z)$ (together with their anti-holomorphic partners), with action given in complex coordinates by [46, 47]

$$S_{b,c} = \frac{1}{4\pi} \int d^2 z \ \left[b(z)\bar{\partial} c(z) + \bar{b}(\bar{z})\partial \bar{c}(\bar{z}) \right] \,. \tag{5.5}$$

This is a conformal model, independent on the dimension of the spacetime d, that can be studied with the same methods we used for the free boson and the free fermion in chapter 3. In particular b and c are primaries of conformal dimension 2 and -1 respectively, and they satisfy the OPE

$$b(z)c(w) \sim c(z)b(w) = \frac{1}{z-w} + \dots \,, \tag{5.6}$$

and the energy momentum tensor is given by

$$T_{b,c}(z) = 2N(\partial c\, b)(z) + N(c\, \partial b)(z) \,. \tag{5.7}$$

We refer to appendix B for other properties of these fields, in particular for their bosonization. What is important for us now is the central charge of the theory, which can be computed through the OPE $T(z)T(w)$. The explicit calculation gives $c_{b,c} = -26$. The fact that the central charge is negative is another sign that the theory describes non-physical fields.

The central charge of the theory is related to the anomaly at quantum level for the Weyl symmetry. This means that a conformal theory is anomalous if the central charge does not vanish. If this is the case, the spectrum of the quantized theory turns out not to be Lorentz-invariant [47]. The theories of a free boson and a free fermion are manifestly anomalous, since their central charge is 1 and 1/2 respectively. For string theory, however, the presence of the ghost system can solve the Weyl anomaly. Considering bosonic string theory for the moment, where no fermions are present, the total central charge in d dimensions is given by

$$c_{\text{total}} = d \cdot c_{\text{boson}} + c_{b,c} = d - 26 \,, \tag{5.8}$$

which vanishes in 26 dimensions. This is the so-called *critical dimension*, and from now on we consider bosonic string theory in $d = 26$, where the Weyl anomaly is absent.

5.1.2 Vertex operators and tree-level scattering amplitudes

When bosonic string theory is quantized, one can construct states corresponding to string excitations, which will form a Hilbert space generated from a vacuum state $|0\rangle$ applying a given number of creation operators. These states will correspond, through the state-operator correspondence, to operators of the conformal field theory describing the string worldsheet. For example there will be states corresponding to the operators $\widetilde{V}_\alpha =: e^{i\alpha X}$:

defined in chapter 3. Since bosonic string theory is defined in $d = 26$, the correct generalization of these operators is

$$\widetilde{V}_k(z) =: e^{ik_M X^M} : (z)\,, \tag{5.9}$$

where k_M is a d-dimensional vector, describing the momentum of the corresponding state. We focus here only on the chiral part of the vertex operators; this is enough when dealing with open strings, while the treatment of closed strings involves also the anti-chiral part of vertex operators. States like that can be physical or not: physical states turn out to be the ones for which the conformal dimension of the operator $\widetilde{V}_k$ is equal to 1. Furthermore, physical vertex operators can be expressed in two versions [47, 48]:

- Unintegrated version. A c-ghost is added to the vertex operator; the resulting operator has combined conformal dimension equal to zero. Example: $c(z)\widetilde{V}_k(z)$.
- Integrated version. The vertex operator of conformal dimension 1 is integrated over the position of the insertion z. Example: $\int dz\, \widetilde{V}_k(z)$.

Like any other physical theory, it is not sufficient to know which states arise from string oscillations; one would also like to understand how these strings (or some particular string states) interact among themselves. We do not enter in details here, but we just highlight the procedure one should follow. The S-matrix is constructed considering amplitudes corresponding to particular external asymptotic states. Thanks to conformal symmetry, such amplitudes reduce to the calculation of correlation functions of vertex operators over certain Riemann surfaces with genus g. The analogous of the loop expansion in quantum field theory is then an expansion over all types of topologies. Tree-level amplitudes are the ones for which the genus is 0: correspondingly, the amplitude is calculated on a sphere for closed strings or a disk for open strings. In the latter case appropriate boundary conditions have to be imposed at the border of the disk.

All vertex operators are inserted inside amplitudes with appropriate powers of g_s, the string coupling constant; the coupling constant for closed strings is $g_{\text{closed}} = g_s$, while for open strings we have $g_{\text{open}} = \sqrt{g_s}$. This is not a fundamental constant of the theory, since it does not appear in the action, and can be changed simply rescaling all vertex operators. Amplitudes are normalized with a certain power of g_s as well; for closed strings the weight depends on the genus g of the Riemann surface, in particular it is given by $(g_s^2)^{g-1}$. The sphere, for example, comes with a prefactor g_s^{-2}. For open strings the normalization is the square root of the corresponding one for closed string; for the disk, for example, it is g_s^{-1}. Notice that the "loop" expansion, or expansion over Riemann surfaces with different genus, corresponds to a perturbative expansion with parameter g_s (or $\sqrt{g_s}$ for open strings).

Furthermore, the integration over the moduli space has to be taken into account: it turns out that this corresponds to integrating over the positions of all the vertex operators, except three of them, which can be inserted at a fixed position in their unintegrated form, with a c-ghost.

5.1.3 D-branes

Let us consider now the (bosonic) theory of open strings in d dimensions. In complex coordinates it corresponds to d copies of the theory of a free boson on the upper half plane, with boundary along the real line. We have seen in chapter 3 that in order to have a well-defined theory we need to specify the boundary conditions along the boundary. Since the real line corresponds to the endpoints of open strings, this corresponds to impose certain boundary conditions on such points. In terms of the coordinates σ and τ, Neumann and Dirichlet boundary conditions correspond to:

$$\begin{aligned} \partial_\sigma X^M &= 0 \qquad (\text{Neumann})\,, \\ \partial_\tau X^M &= 0 \qquad (\text{Dirichlet})\,. \end{aligned} \tag{5.10}$$

These conditions must be imposed on the two endpoints of the open string. In principle it is possible to have different boundary conditions on the two endpoints; for the moment we restrict to the case when the condition is the same at the two extremities. In such a case, the theory in complex coordinates is characterized by homogeneous boundary conditions along the whole boundary, which means that the bosons X^M are all in the NS sector.

The physical meaning of the boundary conditions is the following: for Dirichlet conditions the endpoints of the string are forced to lie at some fixed positions along the directions x^M. On the contrary, for Neumann conditions the extremities of the string can move freely. It is possible to choose Neumann boundary conditions for some coordinates ($M = 0, \ldots, p$), and Dirichlet conditions for the others ($M = p+1, \ldots, d-1$); this is what defines a *Dp brane* [49, 21, 50], where D stands for "Dirichlet" and p indicates the number of spatial dimensions. Such an object can be seen as an hypersurface where endpoints of open strings can lie (see picture 5.2).

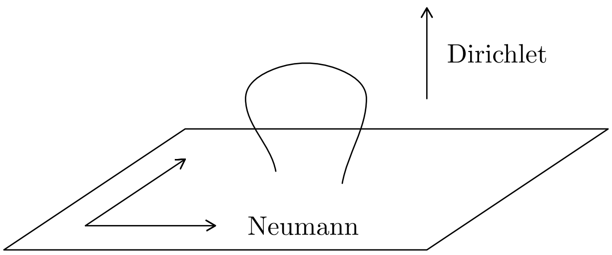

Figure 5.2: Open string with endpoints lying on a D-brane.

The presence of a Dp-brane has the effect to break the Lorentz group of the target space into

$$SO(1, d-1) \longrightarrow SO(1,p) \times SO(d-p-1)\,. \tag{5.11}$$

A particular case is a D(-1) brane, or *D-instanton*, for which the strings satisfy Dirichlet boundary conditions for all M, including the time direction.

It turns out that D-branes should be considered as dynamical objects in string theory; their action is the generalization of the string action, i.e.

$$S_{Dp} = T_p \int d^{p+1}\xi \sqrt{-\text{det}\gamma}\,, \tag{5.12}$$

where T_p is the *tension* of the brane, ξ^a ($a = 0, \ldots, p$) are coordinates of the $(p+1)$-dimensional worldvolume, and γ_{ab} is the pullback of the spacetime metric given by

$$\gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} \eta_{\mu\nu}\,. \tag{5.13}$$

For D-branes the action is proportional to the area of the branes itself, in analogy to the action of the string, which is proportional to its length.

There are however fundamental differences between strings and D-branes. As we have seen, the string action is characterized by the conformal symmetry, which makes the theory easy to study and quantize. An important consequence is the fact that the spectrum of string oscillation is discrete. For D-branes, however, this is not the case; the action is not Weyl invariant and thus the spectrum is continuous. Intuitively, this can be explained by the fact that a higher dimensional D-brane is allowed to change its shape without changing its area, and therefore without changing its energy. This fact shows that a quantized D-branes can not have an interpretation in terms of particle states, but it should rather describe multi-particle states [46].

On the other hand, string theory allows to explicitly compute the D-brane tension T_p appearing in (5.12); the result is proportional to g_s^{-1} [47, 48]. This means that D-branes can not be properly considered in a perturbative approach to string theory, since the latter relies on a perturbative expansion in powers of g_s. Therefore D-branes should be considered as non-perturbative objects, analogously to monopoles and instantons in quantum field theory.

5.2 D-branes bound states in bosonic string theory

As mentioned above, in principle it is possible to have different boundary conditions on the two endpoints of an open string, for some particular direction x^M. This means that the corresponding coordinate X^M will behave as a boson in the Ramond sector, as explained in chapter 3. The physical meaning of this, is that one endpoint of the string lies on some Dn brane, while the other one lies on a Dm brane, a D-brane which extends on a different number of spatial dimensions. The theory thus contains a *bound states* of D-branes. For example, let us consider a bound state of a Dn and a Dm brane in bosonic string theory, with $n < m$. An open string stretching between the two different branes has mixed boundary conditions along the directions X^M with $(n < M \leq m)$. Therefore the coordinates X^M will behave as bosons in the NS sector for $M \leq n$ and $M > m$, and in the R sector for $n < M \leq m$. Along these directions twist fields need to be taken into account, in order to properly describe the properties of the string.

5.2.1 Boundary changing operators

In the following we will focus on the bound state of a D(-1) and a D$(n-1)$ brane; other bound states with a difference of dimension equal to n can be related to this one. Let us study then the n “mixed” directions together. The corresponding bosons X^M are all in the Ramond sector, and the corresponding vacuum state is related to twist fields, thanks to the operator-state correspondence. Let us define the boundary changing operators $\Delta(z)$ (and $\bar{\Delta}$) as the product of these twist fields, i.e.

$$\Delta(z) = \prod_{\mu=0}^{n-1} \sigma^\mu(z) , \qquad \bar{\Delta}(z) = \prod_{\mu=0}^{n-1} \bar{\sigma}^\mu(z) . \tag{5.14}$$

Boundary changing operators are primaries of conformal dimension $n/16$, and satisfy

$$\bar{\Delta}(z)\Delta(w) = \frac{1}{(z-w)^{n/8}} + \dots \,. \tag{5.15}$$

Correlation functions involving boundary changing operators can be derived from correlation functions with twist fields. When the difference of dimension equals some particular values, correlation functions can become quite simple. This is the case when n is a multiple of four. We start considering co-dimension 8 and 16, and comment on the $n = 4$ case at the end, due to its importance in superstring theory.

5.2.2 D7-D(−1) system

Let us now consider the difference of dimension to be 8, in particular the bound state of a D7 and a D(−1) brane. The four-point function of boundary changing operator has a simple form, namely

$$\langle \bar{\Delta}(z_1)\Delta(z_2)\bar{\Delta}(z_3)\Delta(z_4)\rangle = \left(\frac{z_{31}z_{42}}{z_{21}z_{41}z_{32}z_{43}}\right)\left(\frac{\pi}{2K(\eta)}\right)^4 . \tag{5.16}$$

The bosonized version of twist fields, as described in chapter 4, can also be used, but it describes a different setup. The periodicity properties of the boson Ω can be used for describing a set of D(−1) branes positioned on a lattice with period $2\sqrt{2}\pi$. A pair of boundary changing operators connects the D7 brane to one of these D(−1) branes; the four-point function can then depend on the positions of two different branes. If the difference of the two positions is given by the vector $\vec{\delta}$, the four-point function is

$$\langle \bar{\Delta}(z_1)\Delta(z_2)\bar{\Delta}(z_3)\Delta(z_4)\rangle = \left(\frac{z_{31}z_{42}}{z_{21}z_{41}z_{32}z_{43}}\right)\left(\frac{\pi}{2K(\eta)}\right)^4 \exp\left(\frac{i|\vec{\delta}|^2\tau(\eta)}{8\pi}\right) . \tag{5.17}$$

The position of every brane on the lattice can be described by a vector of four integer numbers n^μ, i.e. $x^\mu = 2\sqrt{2}\pi n^\mu$. The correlation function of bosonized twist fields is then given by the superposition of four-point function corresponding to single branes, the result being

$$\langle \bar{\Delta}_B(z_1)\Delta_B(z_2)\bar{\Delta}_B(z_3)\Delta_B(z_4)\rangle = \frac{z_{31}z_{42}}{z_{21}z_{41}z_{32}z_{43}} . \tag{5.18}$$

Notice that in this case (co-dimension $n = 8$) the boundary changing operator Δ has conformal dimension 1/2. We want now to argue that, at least in the bosonized case, it behaves effectively as a fermion. From the eight bosons Ω^μ, we can construct the normalized boson Ω_{CM} as

$$\Omega_{CM} = \frac{1}{\sqrt{8}}\sum_{\mu=1}^{8}\Omega^\mu . \tag{5.19}$$

Given this definition, the boundary changing operator can be written as

$$\Delta_B(z) = \exp\left(i\frac{\sqrt{2}}{4}\sum_{\mu=1}^{8}\Omega_i(z)\right) = e^{i\Omega_{CM}}(z) . \tag{5.20}$$

We notice that this expression represents a complex fermion in its bosonized representation (cfr. section 3.4).

5.2.3 D15-D(−1) system

Another notable situation is when the difference of dimension is 16. The four-point function of boundary changing operators is simply (allowing D(−1) branes at different positions)

$$\langle\bar{\Delta}(z_1)\Delta(z_2)\bar{\Delta}(z_3)\Delta(z_4)\rangle = \left(\frac{z_{31}z_{42}}{z_{21}z_{41}z_{32}z_{43}}\right)^2 \left(\frac{\pi}{2K(\eta)}\right)^8 \exp\left(\frac{i|\vec{\delta}|^2\tau(\eta)}{8\pi}\right) , \tag{5.21}$$

while in the bosonized case the result is

$$\langle\bar{\Delta}_B(z_1)\Delta_B(z_2)\bar{\Delta}_B(z_3)\Delta_B(z_4)\rangle = \left(\frac{z_{31}z_{42}}{z_{21}z_{41}z_{32}z_{43}}\right)^2 . \tag{5.22}$$

The boundary changing operator has conformal dimension 1, and can be written (in the bosonized case), as

$$\Delta_B(z) = \exp\left(i\frac{\sqrt{2}}{4}\sum_{\mu=1}^{16}\Omega_i(z)\right) = e^{i\sqrt{2}\Omega_{CM}}(z) =: J^+_{CM}(z)\,, \tag{5.23}$$

where $\Omega_{CM} = \sum\Omega^\mu/\sqrt{16}$ and $J^+_{CM}(z)$ is a generator of the current algebra described in section 4.1. Following the discussion of section 3.6, a natural question to ask is whether this dimension 1 operator can generate an exactly marginal deformation of the boundary conformal field theory. We have to remember, however, that a twist field must always appear together with its conjugate

$$\bar{\Delta}_B(z) = e^{-i\sqrt{2}\Omega_{CM}}(z) =: J^-_{CM}(z)\,. \tag{5.24}$$

This means that the deformation of the boundary CFT is given by

$$\exp\left(\lambda^2\int J^+_{CM}(z)dz\int J^-_{CM}(w)dw\right) , \tag{5.25}$$

where λ is the modulus of the deformation. As discussed in [51], a set of dimension 1 boundary operators produces a marginal deformation only if these operators are mutually local, meaning that the OPE among them must not contain single poles. A similar result for bulk deformations states that a set of operators of the form $J_i(z)\bar{J}_i(\bar{z})$ generates an exactly marginal deformation of the theory if and only if these currents form an abelian subalgebra (see e.g. [52, 53]). In our case, however, we have

$$\bar{\Delta}_B(z)\Delta_B(w) = J^-_{CM}(z)J^+_{CM}(w) = \frac{1}{(z-w)^2} - \frac{i\sqrt{2}\partial\Omega_{CM}}{z-w} + \dots\,, \tag{5.26}$$

which means that $\bar{\Delta}_B$ and Δ_B are not mutually local. Equivalently, J^+_{CM} and J^-_{CM} do not constitute a subalgebra of the $\mathfrak{su}(2)$ Kač-Moody, since $[J^+_{CM}, J^-_{CM}] \sim \partial\Omega_{CM}$. In conclusion, even if the boundary changing operator has conformal dimension 1, it does not generate an exactly marginal deformation of the bosonic conformal theory. Geometrically, the deformation generated by the twist field $\Delta_B(z)$ (which is the massless excitation of the

(−1)/15 string) corresponds to blowing up the point-like D(−1) branes inside the D15 brane. We then conclude that this blow-up mode is not a modulus in the lattice.

One may wonder if this obstruction is an artifact of compactification. Recalling the OPE (3.63) of the original twist field we see that a simple pole will be present whenever the compactification radius is a multiple of $\sqrt{2}$, and also if the boson is not compactified. So, we expect the obstruction to persist if this condition is met. A possible interpretation for the lifting of this modulus from string theory is that the the constituents of the array feel each other through the exchange of a massless primary.

5.2.4 D3-D(−1) system

Let us consider now a difference of dimension equal to 4. The D3-D(−1) state is very important in superstring theory, as we will see in chapter 7; here we focus on the bosonic content of the spectrum. As we will see later, in superstring theory the full boundary changing vertex operators contains also spin fields. Due to picture changing, one also encounters the "excited" bosonic boundary changing operator, which consists of the product of one excited twist field and 3 normal ones. More specifically we define

$$\tau^\mu = \sigma'^\mu(z) \prod_{\substack{\nu=0 \\ \nu\neq\mu}}^{3} \sigma^\nu(z)\,, \qquad \bar{\tau}^\mu = \bar{\sigma}'^\mu(z) \prod_{\substack{\nu=0 \\ \nu\neq\mu}}^{3} \bar{\sigma}^\nu(z)\,. \tag{5.27}$$

Excited boundary changing operators are primaries of conformal dimension 3/2; the operator product expansions can be easily derived from the ones defining the twist fields, for example

$$\begin{aligned} i\partial X^\mu(z)\Delta(w) &= \frac{\tau^\mu(w)}{(z-w)^{1/2}} + \dots\,, \\ i\partial X^\mu(z)\tau^\mu(w) &= \frac{\Delta(w)}{2(z-w)^{3/2}} + \frac{2\,\partial\Delta(w)}{(z-w)^{1/2}} + \dots\,, \end{aligned} \tag{5.28}$$

where we are not summing over the index μ in the second expression. Furthermore we have

$$\bar{\tau}^\mu(z)\tau^\nu(w) = \frac{\eta^{\mu\nu}}{2(z-w)^{n/8+1}}\,, \tag{5.29}$$

where $\eta^{\mu\nu}$ is the metric of the target space. The calculation of four-point correlation functions is straightforward, and gives

$$\langle \bar{\Delta}(z_1)\Delta(z_2)\bar{\Delta}(z_3)\Delta(z_4)\rangle = \left(\frac{z_{31}z_{42}}{z_{21}z_{41}z_{32}z_{43}}\right)^{1/2} \left(\frac{\pi}{2K(\eta)}\right)^2 \exp\left(\frac{i\vec{\delta}^2\tau(\eta)}{8\pi}\right), \tag{5.30}$$

$$
\begin{aligned}
&\langle \bar{\tau}^\mu(z_1)\tau^\nu(z_2)\bar{\Delta}(z_3)\Delta(z_4)\rangle = \\
&\qquad \frac{\eta^{\mu\nu}}{z_{21}^{3/2} z_{43}^{1/2}} \left(\frac{\pi}{2K(\eta)}\right)^2 \frac{1}{1-\eta} \left(\frac{E(\eta)}{2K(\eta)} - \frac{\vec{\delta}^2}{16K(\eta)^2}\right) \exp\left(\frac{i\vec{\delta}^2\tau(\eta)}{8\pi}\right), \\
&\langle \bar{\tau}^\mu(z_1)\Delta(z_2)\bar{\tau}^\nu(z_3)\Delta(z_4)\rangle = \\
&\quad = \frac{\eta^{\mu\nu} z_{42}^{1/2}}{z_{31}^{1/2} z_{43} z_{21}} \left(\frac{\pi}{2K(\eta)}\right)^2 \frac{1}{1-\eta} \left(\frac{1-\eta}{2} - \frac{E(\eta)}{2K(\eta)} + \frac{\vec{\delta}^2}{16K(\eta)^2}\right) \exp\left(\frac{i\vec{\delta}^2\tau(\eta)}{8\pi}\right).
\end{aligned}
\tag{5.31}
$$

where $K(\eta)$ and $E(\eta)$ are the complete elliptic integrals of the first and second kind respectively. Correlation functions of bosonized twist fields are then given by the superposition of four-point functions corresponding to single branes, the results being

$$
\begin{aligned}
\langle \bar{\Delta}_B(z_1)\Delta_B(z_2)\bar{\Delta}_B(z_3)\Delta_B(z_4)\rangle &= \left(\frac{z_{31}z_{42}}{z_{21}z_{41}z_{32}z_{43}}\right)^{1/2}, \\
\langle \bar{\tau}_B^\mu(z_1)\tau_B^\nu(z_2)\bar{\Delta}_B(z_3)\Delta_B(z_4)\rangle &= \frac{\eta^{\mu\nu}}{2z_{21}^{3/2} z_{43}^{1/2}}, \\
\langle \bar{\tau}_B^\mu(z_1)\Delta_B(z_2)\bar{\tau}_B^\nu(z_3)\Delta_B(z_4)\rangle &= 0\,.
\end{aligned}
\tag{5.32}
$$

These correlation functions can also be derived in a straightforward way by expressing the boundary changing operators in the Ω picture.

5.3 String field theory approach

In this section we reformulate the question of whether it is possible to find a marginal deformation corresponding to twist fields in the language of open string field theory, as discussed in chapter 1. The BRST charge for bosonic string theory is given by [48]

$$
Q = \oint \frac{dz}{2\pi i} \left(cT^{X,\psi,\beta,\gamma} + c(\partial c)b\right)(z)\,. \tag{5.33}
$$

The space of physical states of string field theory is then given by the cohomology of Q. This means that it is spanned by states Ψ annihilated by Q ($Q\Psi = 0$), with the equivalence relation $\Psi \sim \Psi + Q\Phi$. For states with ghost number 1, physical states are the ones at conformal dimension 0. These are exactly the states corresponding to unintegrated vertex operators discussed in 5.1.2.

Since the ghost c has conformal dimension -1, we need a matter vertex operator of conformal dimension 1. Let us then consider the boundary changing operators Δ and $\bar{\Delta}$ of the D15-D(-1) system, and arrange them in the matrix

$$
\Psi^{(1)} = c(z) \begin{pmatrix} 0 & \Delta \\ \bar{\Delta} & 0 \end{pmatrix}(z)\,, \tag{5.34}
$$

where the different entries represent 15/15, 15/(-1), (-1)/15 and (-1)/(-1) string oscillations. We would like then to explore the possibility of extending the solution $\Psi^{(1)}$ of the linearized equations of motion of OSFT to a solution of the full non-linear equation. As found in (2.27), at second order we would have

$$Q\Psi^{(2)} + m_2(\Psi^{(1)}, \Psi^{(1)}) = 0\,, \tag{5.35}$$

provided that no obstruction is present. An obstruction appears whenever Q can not be inverted, and this happens if $m_2(\Psi^{(1)}, \Psi^{(1)})$ contains states of conformal dimension 0. This is completely analogous to the requirement of mutual locality discussed above in 5.2.3. In particular for the bosonized version we have

$$P_0 m_2(\Psi^{(1)}, \Psi^{(1)}) = -i\sqrt{2}\partial\Omega_{CM}\,\mathrm{Tr}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = -15i\sqrt{2}\partial\Omega_{CM} \neq 0\,, \tag{5.36}$$

which shows the presence of an obstruction. Since mixed strings oscillations (Δ and $\bar{\Delta}$) would correspond to switching on the size of the D(−1) branes, (5.36) shows that the blow-up of these branes is obstructed at second order in the size. This is true not only for bosonized boundary changing, but also for normal boundary changing operators, since the OPE of normal twist fields also contains the field $\partial\Omega$. As a consequence, the blow-up mode is obstructed also when there is only one D(−1) brane, instead of an array.

The presence of this obstruction has also another consequence; the impossibility of defining a second-order solution $\Psi^{(2)}$ means that it is also impossible to associate a profile to the field A_μ living on the D3 brane. As a consequence, a connection to Yang-Mills instantons can not be found in bosonic string theory. For this reason, from now on we consider superstring theory; this was in any case necessary, due to the instability of bosonic string theory related to the presence of a tachyon in the spectrum. In chapter 8 we will see that the obstruction (5.36) is resolved in superstring theory, and this will allow us to properly define a profile for the field A_μ, and establish a connection to Yang-Mills instantons.

Chapter 6

Supersymmetric Yang-Mills theory

Up to this point in this thesis we have worked only with bosonic string theory. This might seem reasonable, since we are ultimately interested in the connection to instantons in pure Yang-Mills theory, which contains only the vector boson A_μ in its spectrum. However, bosonic string theory is not sufficient for mainly two reason. First of all, its spectrum contains a tachyonic state, indicating instability in the theory. Secondly, bound states of D-branes in string theory fail to represent instantons in Yang-Mills theory. Both of these problems can be solved by introducing supersymmetry in the worldsheet theory. This will give the so-called superstring theory, which we will discuss in chapter 7. In this chapter we discuss the supersymmetric generalization of the pure Yang-Mills theory presented in chapters 1 and 2; in particular we focus on the maximally supersymmetric $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory. We review the action and the spectrum both in Minkowski and Euclidean space, and see how instanton solutions presented in chapter 2 are still valid.

6.1 Minkowskian $\mathcal{N}$=4 SYM theory

In order to discuss instantons in this theory, we need to consider the Euclidean version of the theory; however, before doing that, we review the theory in Minkowski space. The $\mathcal{N}$=4 Super Yang-Mills theory is well known in Minkowski space, and its action and properties have been extensively studied in the literature. Using the conventions of [4], we write the action as:

$$S_{SYM} = \frac{1}{g^2}\int d^4x \ \mathrm{Tr}\Big\{\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - i\bar{\lambda}^{\dot{\alpha}}_A \bar{\not{D}}_{\dot{\alpha}\beta}\lambda^{\beta A} - i\lambda^A_\alpha \not{D}^{\alpha\dot{\beta}}\bar{\lambda}_{A\dot{\beta}} + \frac{1}{2}(D_\mu\bar{\phi}_{AB})(D^\mu\phi^{AB}) - \sqrt{2}\bar{\phi}_{AB}\{\lambda^{\alpha A},\lambda^B_\alpha\} - \sqrt{2}\phi^{AB}\{\bar{\lambda}^{\dot{\alpha}}_A,\bar{\lambda}_{\dot{\alpha}B}\} + \frac{1}{8}[\phi^{AB},\phi^{CD}][\bar{\phi}_{AB},\bar{\phi}_{CD}]\Big\} \tag{6.1}$$

The indices $\mu,\nu = 1,\ldots,4$ are space-time indices, while $\alpha,\dot{\alpha} = 1,2$ are chiral and anti-chiral spinor indices. The indices $A, B = 1,\ldots,4$ take into account the R-symmetry group $SU(4)$. The fields described by this action are:

- A gauge field A_μ, with field strength $F_{\mu\nu}$ (This field appears in the action also through the covariant derivative D_μ);

- 4 pairs of Weil spinors (gauginos) $\lambda^{\alpha A}$ and $\bar{\lambda}_{\dot{\alpha}A}$, respectively right- and left-handed;
- 6 scalars $\phi^{AB} = -\phi^{BA}$. Their duals are not independent, and are defined by $\bar{\phi}_{AB} = \frac{1}{2}\epsilon_{ABCD}\phi^{CD}$.

All these fields belong to the adjoint representation of the gauge group (typically $SU(N)$); in the action (6.1) we have omitted the corresponding indices. The trace is taken precisely in this adjoint representation. The reality conditions on the fields are the Majorana conditions $(\lambda^{\alpha A})^* = -\bar{\lambda}^{\dot{\alpha}}_A$. Furthermore, the scalars satisfy the reality condition $(\phi^{AB})^* = \bar{\phi}_{AB}$. Some properties of the σ^μ matrices used to define the operator $\not{D}$ are collected in appendix A.

The action (6.1) is invariant under the following supersymmetry transformations, with parameters ζ^A_α and $\bar{\zeta}_{\dot{\alpha}A}$:

$$\begin{aligned} \delta_\zeta A_\mu &= -i\bar{\zeta}^{\dot{\alpha}}_A(\bar{\sigma}_\mu)_{\dot{\alpha}\beta}\lambda^{\beta A} + i\bar{\lambda}_{\dot{\beta}A}(\sigma_\mu)^{\alpha\dot{\beta}}\zeta^A_\alpha\,; \\ \delta_\zeta\phi^{AB} &= \sqrt{2}\left(\zeta^{\alpha A}\lambda^B_\alpha - \zeta^{\alpha B}\lambda^A_\alpha + \epsilon^{ABCD}\bar{\zeta}^{\dot{\alpha}}_C\bar{\lambda}_{\dot{\alpha}D}\right)\,; \\ \delta_\zeta\lambda^{\alpha A} &= -\frac{1}{2}(\sigma^{\mu\nu})^\alpha{}_\beta F_{\mu\nu}\zeta^{\beta A} - i\sqrt{2}\bar{\zeta}_{\dot{\alpha}B}\not{D}^{\alpha\dot{\alpha}}\phi^{AB} + [\phi^{AB},\bar{\phi}_{BC}]\zeta^{\alpha C}\,. \end{aligned} \tag{6.2}$$

We now turn to the analysis of $\mathcal{N}$=4 SYM theory in Euclidean space, since we are ultimately interested in instantonic solutions.

6.2 Euclidean $\mathcal{N}$=4 SYM theory

The definition of the SYM theory in 4-dimensional Euclidean space is not easy, given to the fact that a real representation of Dirac matrices in such a space does not exist. Consequently, one can not define easily Majorana spinors. However we can proceed in another way; one can derive the action of the $\mathcal{N}$=4 SYM in four dimensional Euclidean space starting from the $\mathcal{N}$=1 SYM theory in 10-dimensional (9+1) Minkowski space-time. The action of this theory is extremely simple, and reads:

$$S_{10} = \frac{1}{g^2_{10}}\int d^{10}x\ \mathrm{Tr}\left\{\frac{1}{2}F_{MN}F^{MN} + \bar{\Psi}\Gamma^M D_M\Psi\right\}\,, \tag{6.3}$$

where the index M runs from 0 to 9, and Ψ is a Majorana-Weyl spinor. In order to derive the $\mathcal{N}$=4 SYM theory in four Euclidean dimensions we have to compactify over a torus with one time and five space coordinates. We refer to [4] for details, and here we just state the results.

The action in Euclidean space turns out to be formally identical to the one in Minkowski space given in (6.1); anyway the matrices used to define the $\not{D}$ operator are different from the Minkowskian case; details can be found in appendix A. Furthermore, now the fields satisfy different relations; the usual Majorana condition is replaced by the so-called symplectic Majorana condition:

$$\begin{aligned} (\lambda^{\alpha A})^* &= -\widetilde{\epsilon}_{AB}\epsilon_{\alpha\beta}\lambda^{\beta B} \\ (\bar{\lambda}_{\dot{\alpha}A})^* &= -\widetilde{\epsilon}^{AB}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\lambda}_{\dot{\beta}B}\,, \end{aligned} \tag{6.4}$$

where $\widetilde{\epsilon}_{AB}$ is an antisymmetric tensor whose non-zero components are $\widetilde{\epsilon}_{14} = -\widetilde{\epsilon}_{41} = \widetilde{\epsilon}_{23} = -\widetilde{\epsilon}_{32} = 1$. Furthermore, the scalar fields are constrained by the relation

$$(\phi^{AB})^* = \widetilde{\epsilon}_{AC}\phi^{CD}\widetilde{\epsilon}_{DB}\,. \tag{6.5}$$

The action is invariant under supersymmetry transformations, which are formally identical to the ones in (6.2), provided that also the fermionic parameters ζ and $\bar{\zeta}$ satisfy the symplectic Majorana condition (6.4). From the action (6.1) one can derive the equations of motion of the theory, which are [54]:

$$\begin{aligned}
&D^\nu F_{\mu\nu} - i\left\{\bar{\lambda}^{\dot{\alpha}}_A(\bar{\sigma}_\mu)_{\dot{\alpha}\beta}, \lambda^{\beta A}\right\} - \frac{1}{2}\left[\bar{\phi}_{AB}, D_\mu\phi^{AB}\right] = 0\,,\\
&D^2\phi^{AB} + \sqrt{2}\left\{\lambda^{\alpha A}, \lambda^B_\alpha\right\} + \frac{1}{\sqrt{2}}\epsilon^{ABCD}\left\{\bar{\lambda}^{\dot{\alpha}}_C, \bar{\lambda}_{\dot{\alpha}D}\right\} - \frac{1}{2}\left[\bar{\phi}_{CD}, \left[\phi^{AB}, \phi^{CD}\right]\right] = 0\,,\\
&\bar{\not{D}}_{\dot{\alpha}\beta}\lambda^{\beta A} + i\sqrt{2}\left[\phi^{AB}, \bar{\lambda}_{\dot{\alpha}B}\right] = 0\,,\\
&\not{D}^{\alpha\dot{\beta}}\bar{\lambda}_{\dot{\beta}A} - i\sqrt{2}\left[\bar{\phi}_{AB}, \lambda^{\alpha B}\right] = 0\,.
\end{aligned} \tag{6.6}$$

This is a complicated system of coupled differential equations, and in general it is a difficult task to find solutions of these equations. Anyway, simple solutions can be found by setting the gauginos and the scalars to zero. If this is the case, the equations of motion reduce to

$$D^\nu F_{\mu\nu} = 0\,, \tag{6.7}$$

which are exactly the equations of motion of the pure Yang-Mills theory (2.1). This means that all the instanton solutions discussed in chapter 1 are still valid, once vanishing gauginos and scalars are added.

The equations of motion (6.6) have also more complicated solutions involving fermions and scalars, and not only the gauge field. In the case of $\mathcal{N} = 4$ SYM theory, we have non trivial fermion-scalar interactions. This means that, once we have a non trivial fermion, the complete solution has also non trivial scalar fields; this generalization of the instanton solution is called *superinstanton*. However, the set of equations of motion (6.6) is too complicated to be solved in full generality; a super-instanton solution is not known in closed form. As explained in [54] and [5] we can only recover it iteratively, starting from the pure gauge instanton solution and applying suitable supersymmetry transformations. The discussion of super-instantons is beyond the purpose of this thesis, hence we focus in the following on pure instantons.

We conclude this section by writing the action in 4 euclidean dimensions in an equivalent version, which will be useful later:

$$\begin{aligned}
\mathcal{S}_{SYM} =& \frac{1}{g^2_{YM}}\int d^4x\, \mathrm{Tr}\bigg\{\frac{1}{2}F^2_{\mu\nu} - 2\bar{\Lambda}_{\dot{\alpha}A}\bar{\not{D}}^{\dot{\alpha}\beta}\Lambda_\beta{}^A + (D_\mu\varphi_a)^2 - \frac{1}{2}[\varphi_a, \varphi_b]^2\\
&- i(\Sigma^a)^{AB}\bar{\Lambda}_{\dot{\alpha}A}[\varphi_a, \bar{\Lambda}^{\dot{\alpha}}{}_B] - i(\bar{\Sigma}^a)_{AB}\Lambda^{\alpha A}[\varphi_a, \Lambda_\alpha{}^B]\bigg\}\,,
\end{aligned} \tag{6.8}$$

where the six scalars are now labeled by φ_a, with $a = 1, \ldots, 6$.

Chapter 7

Superstring theory and the D3-D(−1) bound state

In this chapter we review some basic concepts in superstring theory. We present the action of the theory, and highlight its connection to the conformal theories of free bosons and fermions discussed in chapter 3. We generalize the discussion of chapter 5 and introduce the superconformal ghosts. We study vertex operators in superstring theory, in particular the ones describing massless oscillations of open strings on a D3-D(-1) brane system. We discuss the connection of this model to the Super Yang-Mills theory presented in chapter 6.

7.1 Superstring theory

A complete discussion of superstring theory is far beyond the purpose of this work; here we will just review some concepts we will use in the following. Further details can be easily found in the literature, for example in [55], [48] or [56].

The starting point is the superstring action, which differs from (5.2) because of the presence of fermions. The action reads, in the so-called *superconformal gauge*,

$$S = \frac{1}{8\pi}\int d^2\sigma \, \left(\frac{2}{\alpha'}\partial_\alpha X^M \partial^\alpha X_M + 2i\bar{\psi}^M \rho^\alpha \partial_\alpha \psi_M\right) \,, \tag{7.1}$$

where ρ^α are matrices satisfying the Clifford algebra in two dimensions. The invariance under supersymmetry transformations has been used to write the action in the simple form (7.1). As in the case of bosonic string theory, this action can describe both closed and open strings.

It is useful to express the theory in terms of complex coordinates, as was done for bosonic string theory. The action (7.1) can thus be written in the following form:

$$S = \frac{1}{4\pi}\int d^2z \, \left(\frac{2}{\alpha'}\partial X^M \bar{\partial} X_M + \psi^M \bar{\partial}\psi_M + \bar{\psi}^M \partial \bar{\psi}_M\right) \,. \tag{7.2}$$

From this we see that superstring theory consists of d copies of a free bosonic theory coupled to d copies of a free fermionic theory. The action is invariant under the following

$\mathcal{N} = 1$ supersymmetry transformations with (anticommuting) parameter η:

$$\begin{aligned}\sqrt{\frac{2}{\alpha'}}\delta_\eta X^\mu(z,\bar{z}) &= \eta(z)\psi(z) + \eta(z)^*\bar{\psi}(\bar{z})\,,\\ \delta_\eta\psi^\mu(z) &= -\eta(z)\sqrt{\frac{\alpha'}{2}}\partial X^\mu(z)\,,\\ \delta_\eta\bar{\psi}^\mu(z) &= -\eta(z)^*\sqrt{\frac{\alpha'}{2}}\bar{\partial} X^\mu(\bar{z})\,.\end{aligned} \tag{7.3}$$

7.1.1 The β-γ ghost system

In analogy to what discussed in chapter 5 for the bosonic string, the quantization of a theory subject to gauge invariance implies the presence of ghost fields. In addition to the b-c ghost system, superstring theory is characterized by another ghost system, with action given by

$$S_{\beta,\gamma} = \frac{1}{4\pi}\int d^2z\ \left[\beta(z)\bar{\partial}\gamma(z) + \bar{\beta}(\bar{z})\partial\bar{\gamma}(\bar{z})\right]\,. \tag{7.4}$$

β and γ are commuting fields, with conformal dimension $3/2$ and $-1/2$ respectively, and are the superpartners of b and c. They satisfy the OPE

$$\beta(z)\gamma(w) \sim -\gamma(z)\beta(w) = -\frac{1}{z-w} + \dots\,, \tag{7.5}$$

and the energy momentum tensor is given by

$$T_{\beta,\gamma}(z) = 2N(\partial\gamma\,\beta)(z) + N(\gamma\,\partial\beta)(z)\,. \tag{7.6}$$

The bosonization of these ghosts is slightly more complicated than the one of the b-c ghosts: we refer to B for details. The central charge of the β-γ system is $c_{\beta,\gamma} = 11$. This means that the total central charge of superstring theory is

$$c_{\text{total}} = d\cdot(c_{\text{boson}} + c_{\text{fermion}}) + c_{b,c} + c_{\beta,\gamma} = d\cdot\left(1+\frac{1}{2}\right) - 26 + 11\,, \tag{7.7}$$

which vanishes in $d = 10$ dimensions. Therefore the Weyl anomaly is absent in $d = 10$, which is the critical dimension for superstring theory.

7.1.2 Quantization of the theory and spectrum

There are different ways of quantizing superstring theory; roughly speaking, one has to deal both with bosonic and fermionic oscillators, and impose appropriate quantization conditions. The physical states of the theory are constructed applying bosonic and fermionic creation operators to a vacuum state; at the end one gets an infinite tower of states, corresponding to particles with increasing mass. The first physical particles are the massless ones, which are the ones we are interested in. We focus only on open superstrings; the spectrum of the theory is divided in the two sectors (NS and R):

- NS sector: the vacuum state $|0\rangle$ corresponds to a tachyon, which is not physical. Applying a fermionic creation operator to the vacuum one gets the first excited state $|A^M\rangle$, which corresponds to a massless vector boson in $d = 10$.
- R sector: there are two different ground states, corresponding to spin fields of even and odd chirality respectively, which create two massless spinors of opposite chirality, which we indicate by $|S^{\mathcal{A}}\rangle$ and $|S^{\dot{\mathcal{A}}}\rangle$. We denote with $\mathcal{A}, \dot{\mathcal{A}} = 1, \dots, 16$ chiral and antichiral indices in 10 dimensions.

There are many other physical states in the theory, but they are all massive, with masses proportional to α'^{-1}; in the field theory limit $\alpha' \to 0$ they can be integrated out. For the purposes of this work it is sufficient to deal with massless states, i.e. only with $|A^M\rangle$, $|S^{\mathcal{A}}\rangle$ and $|S^{\dot{\mathcal{A}}}\rangle$.

7.1.3 Vertex operators

Given the conformal character of string theory, the calculation of scattering amplitudes can be simplified a lot. As for the bosonic theory, it turns out that the contribution of every external state can be reduced to the insertion of a vertex operator on the worldsheet. The latter is topologically equivalent to a sphere or a disk, in the case of closed and open strings respectively. The calculation of an amplitude is performed computing the correlation function of the vertex operators corresponding to each external leg.

We can express the vertex operators using the bosonization procedure explained in section 3.4, in terms of five bosons ϕ^i; the explicit form for the vector boson and the two massless spinor is ([48]):

$$\begin{aligned}
&|A^M\rangle \longrightarrow V_A(z) = \psi^M(z) =: e^{i\lambda_A \cdot \phi} : && \lambda_A = (0, \dots, \pm 1, \dots, 0)\,, \\
&|S^{\mathcal{A},\dot{\mathcal{A}}}\rangle \longrightarrow V_S(z) = S^{\mathcal{A},\dot{\mathcal{A}}}(z) =: e^{i\lambda_{\mathcal{A},\dot{\mathcal{A}}} \cdot \phi} : && \lambda_{\mathcal{A},\dot{\mathcal{A}}} = \left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)\,.
\end{aligned} \tag{7.8}$$

The vector λ_A has only one entry equal to ± 1, while the remaining four are 0. The vectors $\lambda_{\mathcal{A}}$ and $\lambda_{\dot{\mathcal{A}}}$ have an even and an odd number of minus signs respectively, for even and odd chirality. The vertex operators are primary operators; their conformal dimensions is 1/2 and 5/8 respectively.

In superstring theory, the operators (7.8) are not sufficient to describe the massless states. First of all one has to take care of the ingoing momentum k_M (with $k^2 = 0$) carried by the massless particle. Furthermore, as for bosonic string theory, the vertex operators corresponding to physical massless particles must have conformal dimensions 1 (or 0 if the c-ghost is added). The correct operators are of the form([48]):

$$V_{\lambda,q}(z;k) =: e^{i\lambda_i \phi^i(z)} e^{q\phi(z)} e^{ik\cdot X(z)} :\,, \tag{7.9}$$

where $\phi(z)$ comes from the bosonization of the superghosts (see appendix B), and q is called superghost charge. Such a field has conformal dimension equal to

$$h = \frac{1}{2}\lambda^2 - \frac{1}{2}q^2 - q\,. \tag{7.10}$$

Therefore, the complete vertex operators for the vector boson and the two fermions are given by:

$$\begin{aligned} |A^M\rangle &\longrightarrow V^{(-1)}_{\lambda_A}(z;k) = \epsilon_M \psi^M(z) e^{-\phi(z)} e^{ik\cdot X(z)} \,, \\ |S^{\mathcal{A}}\rangle &\longrightarrow V^{(-1/2)}_{\lambda_{\mathcal{A}}}(z;k) = u_{\mathcal{A}} S^{\mathcal{A}}(z) e^{-\frac{1}{2}\phi(z)} e^{ik\cdot X(z)} \,, \end{aligned} \tag{7.11}$$

and similarly for $|S^{\dot{\mathcal{A}}}\rangle$. We have also added a possible polarization vector or spinor which, for on-shell states, must satisfy

$$k \cdot \epsilon = 0 \,, \qquad (\not{k} u)_{\mathcal{A}} = 0 \,. \tag{7.12}$$

The vertex operators in (7.11) are in the so-called *canonical picture*. There are other equivalent pictures that can be used, with different superghost charge.

7.1.4 Superstring scattering amplitudes

We discussed already in 5.1.2 scattering amplitudes in bosonic string theory. We generalize here the discussion to superstring theory. The general idea is the same: the calculation of the S-matrix reduces to the computation of scattering amplitudes on Riemann surfaces, with the insertion of vertex operators. Again, conformal invariance implies that three vertex operators have to be inserted in their unintegrated version (with a c-ghost) while the others have to be integrated over all possible positions of the insertions. The difference resides in the integration over the supermoduli. In superstring theory all vertex operators have to be inserted in their canonical picture, and the integration over the supermoduli results in the insertion of a number of *picture changing operators* such that the total superghost charge is -2. If all the states in consideration are on-shell, these vertex operators can be moved in such a way to act on vertex operators. When this is the case the amplitude can be simply computed considering vertex operators in various pictures, in such a way that the total picture is -2. The result does not depend on where the picture changing operators are inserted, as long as the total picture is conserved [48, 55].

However, this is not valid for off-shell amplitudes, and the integration over the supermoduli is more delicate. This will be important later, when considering the theory in Euclidean space, since a state with non-vanishing momentum is necessarily off-shell ($k^2 \neq 0$).

7.2 The D3-D(−1) system

In this section we study the string theory setup we will use in order to describe instantons in $\mathcal{N} = 4$ SYM theory in 4 dimensions. As explained in [57], [26] and [25], such a setup consists of a bound state of N D3 branes and k D(−1) branes. This configuration can describe instantons with winding number k in a theory with gauge group $SU(N)$. The bosonic coordinates X^M and ψ^M ($M = 0, \dots, 9$) obey different boundary conditions depending on the type of boundary: on the D(−1) branes all the coordinates satisfy Dirichlet boundary conditions, while on the D3 branes the first coordinates X^μ and ψ^μ ($\mu = 0, \dots, 3$) satisfy Neumann conditions and the remaining X^a and ψ^a ($a = 4, \dots, 9$)

satisfy Dirichlet conditions. Furthermore the presence of the D3 branes breaks $SO(10)$ to $SO(4)\times SO(6)$ (we consider the euclidean theory); therefore the spin fields $S^{\mathcal{A}}$ and $S^{\dot{\mathcal{A}}}$ can be expressed in terms of the spin fields in 4 and 6 dimensions as follows ([27]):

$$\begin{aligned} S^{\mathcal{A}} &\longrightarrow (S_\alpha S^A, S^{\dot\alpha} S_A)\,,\\ S^{\dot{\mathcal{A}}} &\longrightarrow (S_\alpha S_A, S^{\dot\alpha} S^A)\,, \end{aligned} \tag{7.13}$$

where S_α and $S^{\dot\alpha}$ are $SO(4)$ spin fields of even and odd chirality respectively, and S^A and S_A are $SO(6)$ spin fields of even and odd chirality respectively. The explicit bosonized form for these spin fields is analogous to the one in (7.8); in $d=4$ we have only few possibilities:

$$\begin{aligned} \lambda_\alpha &= \left(\frac{1}{2},\frac{1}{2}\right) \quad \text{or} \quad \left(-\frac{1}{2},-\frac{1}{2}\right)\,,\\ \lambda_{\dot\alpha} &= \left(\frac{1}{2},-\frac{1}{2}\right) \quad \text{or} \quad \left(-\frac{1}{2},\frac{1}{2}\right)\,. \end{aligned} \tag{7.14}$$

Other details on the spin fields can be found in [21] and [27].

In the system we are considering, there are four types of open strings: those stretching between two D3 branes (3/3 strings), those stretching between two D(−1) branes ((−1)/(−1) strings) and finally those with one endpoint on a D3 brane and the other one on a D(−1) brane (3/(−1) and (−1)/3 strings). We have to consider each type of string separately, as each one has its own spectrum and properties. Let us consider first of all the 3/3 strings. The massless NS state, corresponding to A^M in (7.11), can be divided in a four-vector A^μ and six scalars φ^a; the corresponding (unintegrated) vertex operators in the canonical (−1) picture are:

$$\begin{aligned} V_A(z;k) &= A_\mu c(z)\psi^\mu(z)e^{-\phi(z)}e^{ik\cdot X(z)}\,,\\ V_\varphi(z;k) &= \varphi_a c(z)\psi^a(z)e^{-\phi(z)}e^{ik\cdot X(z)}\,, \end{aligned} \tag{7.15}$$

where the momentum k^μ is ingoing and the polarizations have to satisfy $A_\mu k^\mu = 0$. Furthermore we consider momentum flowing only along the 4 directions parallel to the D3 brane. The on-shell condition is $k^2=0$; notice that this condition can be met only in Minkowski space. In Euclidean space these vertex operators are necessarily off-shell, if the momentum k^μ is not zero. In the R sector, the massless excitations, corresponding to $S^{\mathcal{A}}$ and $S^{\dot{\mathcal{A}}}$ in (7.11), are the gauginos $\lambda^{\alpha A}$ and $\bar\lambda_{\dot\alpha A}$, with vertex operators (in the canonical (−1/2) picture):

$$\begin{aligned} V_\lambda(z;k) &= \lambda^{\alpha A} c(z)S_\alpha(z)S_A(z)e^{-\frac{1}{2}\phi(z)}e^{ik\cdot X(z)}\,,\\ V_{\bar\lambda}(z;k) &= \bar\lambda_{\dot\alpha A} c(z)S^{\dot\alpha}(z)S^A(z)e^{-\frac{1}{2}\phi(z)}e^{ik\cdot X(z)}\,, \end{aligned} \tag{7.16}$$

where the polarizations satisfy $\not{k}\lambda=0$ and $\bar\lambda\not{k}=0$, where $\not{k}=k_\mu\gamma^\mu$ is defined through the gamma matrices in $d=10$. We can immediately recognize that the spectrum of the 3/3 strings reproduces exactly the fields of $\mathcal{N}$=4 SYM in four dimensions, as described in chapter 6. If N is greater than 1, these vertex operators must be multiplied by a $N\times N$ Chan-Paton factor $(T^I)^{uv}$, in order to take care of all the possible D3 branes on which the endpoints of the strings can lie. Here I is a $SU(N)$ color index; therefore all the polarizations will transform in the adjoint representation of $SU(N)$, as expected. In what follows we will assign Chan-Paton indices directly to the polarizations if needed (for example we will write A_μ^{uv}).

Let us now consider $(-1)/(-1)$ strings; the situation is different from the 3/3 case, because now there are no longitudinal Neumann direction. Therefore, the states corresponding to oscillations of these strings do not carry momentum, and have to be considered as moduli rather than dynamical fields. Among the 10 scalars of the NS sector, it is convenient to divide the ones corresponding to the longitudinal directions of the D3 branes from the others; their vertex operators are (in the canonical (-1) picture):

$$\begin{aligned} V_a(z;k) &= a_\mu c(z)\psi^\mu(z)e^{-\phi(z)}\,, \\ V_\chi(z;k) &= \chi_a c(z)\psi^a(z)e^{-\phi(z)}\,. \end{aligned} \tag{7.17}$$

In the R sector we have sixteen fermionic moduli, indicated by $M^{\alpha A}$ and $\bar{M}_{\dot\alpha A}$; their vertex operators in the canonical $(-1/2)$ picture are:

$$\begin{aligned} V_M(z;k) &= M^{\alpha A}c(z)S_\alpha(z)S_A(z)e^{-\frac{1}{2}\phi(z)}\,, \\ V_{\bar{M}}(z;k) &= \bar{M}_{\dot\alpha A}c(z)S^{\dot\alpha}(z)S^A(z)e^{-\frac{1}{2}\phi(z)}\,. \end{aligned} \tag{7.18}$$

Again, we have not written explicitly the indices labeling between which of the k D(-1) branes the string is stretching; we should add to all the vertex operators a $k\times k$ Chan-Paton matrix $(t^U)^{ij}$ with indices $i,j=1,\dots,k$. Here U is a $SU(k)$ color index.

Finally we consider $3/(-1)$ and $(-1)/3$ strings. In this case the four directions $\mu = 0,\dots,4$ are characterized by mixed boundary conditions, again, the fields corresponding to these strings do not carry momentum. A closer analysis of these strings (see [12]) shows that in the NS sector one has two bosonic Weyl spinors of SO(4), w and $\bar{w}$, with vertex operators given, in the canonical (-1) picture, by

$$\begin{aligned} V_w &= w_{\dot\alpha}c(z)\Delta(z)S^{\dot\alpha}(z)e^{-\phi(z)}\,, \\ V_{\bar{w}} &= \bar{w}_{\dot\alpha}c(z)S^{\dot\alpha}(z)\bar{\Delta}(z)e^{-\phi(z)}\,. \end{aligned} \tag{7.19}$$

Notice that only the anti-chiral spin fields $S^{\dot\alpha}$ appear in these vertex operators; the analogous vertex operators of opposite chirality, containing S^α, can be considered instead. In technical terms, the choice of chirality is called *GSO projections*, and is needed to eliminate tachyons from the spectrum of the theory [55, 56]. Δ and $\bar{\Delta}$ are the bosonic twist and anti-twist fields; they have conformal dimension 1/4, and they change the boundary conditions of the four X^μ coordinates. They can be expressed as product of four twist fields corresponding to each direction longitudinal to the D3 branes, as we did in chapter 5:

$$\Delta(z) = \sigma^0(z)\sigma^1(z)\sigma^2(z)\sigma^3(z)\,. \tag{7.20}$$

In the R sector we find two fermionic spinors of SO(6) (μ and $\bar{\mu}$), with vertex operators given, in the canonical $(-1/2)$ picture by:

$$\begin{aligned} V_\mu^{(-1/2)} &= \mu^A\Delta(z)S_A(z)e^{-\frac{1}{2}\phi(z)}\,, \\ V_{\bar{\mu}}^{(-1/2)} &= \bar{\mu}^A S_A(z)\bar{\Delta}(z)e^{-\frac{1}{2}\phi(z)}\,. \end{aligned} \tag{7.21}$$

Again, the two spinors have definite chirality. We also have to add to all the vertex operators in (7.19) and (7.21) a matrix ζ_{ui} or $\bar{\zeta}^{ui}$ with $N \times k$ entries, corresponding to all the possible pairs of D3 and D(−1) branes.

7.2.1 Tree-level amplitudes, effective actions and ADHM constraints

From the setup consisting of N D3 branes and k D(−1) branes it is possible to derive the corresponding effective action in the following way. One should first compute all string scattering amplitudes involving massless string states, using the vertex operators defined above. One should then find an effective Lagrangian able to reproduce these amplitudes. Since the string tension α' is the only dimensionful constant, an expansion of the action in the number of derivatives corresponds to an expansion in powers of $\sqrt{\alpha'}$. The low-energy effective action is the one resulting from the field theory limit $\alpha' \to 0$.

When dealing with string scattering amplitudes in the presence of two different sets of D-branes, it is important to specify what kind of correlation function one is considering. For example, a scattering amplitude involving only 3/3 strings must be normalized with the disk amplitude with the boundary conditions of a D3 brane (see [44, 1] and section 4.4 for related discussions). For example, the scattering amplitude of a gauge vector and two gauginos is given by (see [27])

$$\langle\langle V_{\bar{\Lambda}} V_A V_{\Lambda} \rangle\rangle_{D3} = C_4 \langle V_{\bar{\Lambda}} V_A V_{\Lambda} \rangle \,, \tag{7.22}$$

where $\langle V_{\bar{\Lambda}} V_A V_{\Lambda} \rangle$ is the pure CFT correlation function and $C_4 = \langle\langle \mathbb{1} \rangle\rangle_{D3}$. Correlation functions of $(-1)/(-1)$ strings, on the other hand, must be normalized with the prefactor $C_0 = \langle\langle \mathbb{1} \rangle\rangle_{D(-1)}$. The values of C_4 and C_0 can be computed using unitarity methods, the results being (see *e.g.* [58])

$$C_4 \propto \frac{1}{g_{YM}^2 \alpha'^2} \,, \qquad C_0 \propto \frac{1}{g_{YM}^2} \,, \tag{7.23}$$

where g_{YM} is the (adimensional) gauge coupling constant of the four-dimensional Euclidean theory. In fact, the full low-energy effective field theory corresponding to the massless 3/3 interactions is

$$\begin{aligned} \mathcal{S}_{SYM} = & \frac{1}{g_{YM}^2} \int d^4x \, \mathrm{Tr} \Big\{ \frac{1}{2} F_{\mu\nu}^2 - 2 \bar{\Lambda}_{\dot{\alpha}A} \bar{\not{D}}^{\dot{\alpha}\beta} \Lambda_\beta{}^A + (D_\mu \varphi_a)^2 - \frac{1}{2} [\varphi_a, \varphi_b]^2 \\ & - i (\Sigma^a)^{AB} \bar{\Lambda}_{\dot{\alpha}A} [\varphi_a, \bar{\Lambda}^{\dot{\alpha}}{}_B] - i (\bar{\Sigma}^a)_{AB} \Lambda^{\alpha A} [\varphi_a, \Lambda_\alpha{}^B] \Big\} \,, \end{aligned} \tag{7.24}$$

which is exactly the action of the four-dimensional $\mathcal{N} = 4$ SYM theory (6.8), with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. An alternative choice is to rescale all the vertex operators by g_{YM}; the corresponding effective action would be the same as (6.8), but without the prefactor g_{YM}^{-2}, and with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g_{YM}[A_\mu, A_\nu]$. The same procedure can be repeated for the $(-1)/(-1)$ strings and the mixed strings. Since the normalization of the corresponding amplitudes is C_0, the resulting effective action will be of the form

$$\mathcal{S}_{\text{moduli}} = \frac{1}{g_0^2} \mathrm{tr} \Big\{ - \frac{1}{4} [a_\mu, a_\nu]^2 + \dots \Big\} \,, \tag{7.25}$$

where we have written explicitly only one term, and the trace is over $SU(k)$ and not $SU(N)$ as before. We have highlighted one particular term of the action, in order to discuss the role of the prefactor g_0^{-2}, where g_0 is the coupling of a zero-dimensional SYM theory. Unlike the gauge coupling g_{YM}, however, g_0 is a dimensionful constant, that can be expressed as $g_0 \propto g_{YM}/\alpha'$. Therefore the field theory limit $\alpha' \to 0$ is problematic. The solution to this issue is to rescale some of the moduli with a prefactor g_0, as explained in [27], in order to obtain a well defined low-energy action for the moduli. For the mixed strings, for example, the vertex operators one should use are

$$\begin{aligned} V_a &\sim g_0\sqrt{\alpha'} a_\mu c\psi^\mu e^{-\phi} \sim \frac{g_{YM}}{\sqrt{\alpha'}} a_\mu c\psi^\mu e^{-\phi}\,, \\ V_w &\sim g_0\sqrt{\alpha'} w_{\dot\alpha} c\Delta S^{\dot\alpha} e^{-\phi} \sim \frac{g_{YM}}{\sqrt{\alpha'}} w_{\dot\alpha} c\Delta S^{\dot\alpha} e^{-\phi}\,, \\ V_{\bar w} &\sim g_0\sqrt{\alpha'} \bar w_{\dot\alpha} cS^{\dot\alpha}\bar\Delta e^{-\phi} \sim \frac{g_{YM}}{\sqrt{\alpha'}} \bar w_{\dot\alpha} cS^{\dot\alpha}\bar\Delta e^{-\phi}\,. \end{aligned} \tag{7.26}$$

Since the vertex operators for the moduli should be dimensionless, this means that the polarizations of the vertex operators are dimensionful. In this case a, w and $\bar w$ have dimension (length)^1, and are associated to the position and size of the instanton. After this rescaling, the limit $\alpha' \to 0$ (with g_{YM} held fixed) is well defined. This also means that we are considering the limit $\alpha' \to 0$ with the size of the instanton kept constant. The final result for the moduli action is given in [27], in terms of some auxiliary fields. The equations of motion of these auxiliary fields give rise to some constraints on the moduli. In particular we have

$$\bar\eta_c^{\mu\nu}\left([a_\mu, a_\nu] + \frac{1}{2}\bar w_{\dot\alpha}(\bar\sigma_{\mu\nu})^{\dot\alpha\dot\beta} w_{\dot\beta}\right) = 0\,, \tag{7.27}$$

which is the bosonic ADHM constraint, as seen in 2.2.1 (here $x_\mu = 0$). Let us restrict to the case $N = 2$ and $k = 1$ for simplicity. Since $k = 1$, a_μ are just numbers, therefore $[a_\mu, a_\nu] = 0$ and the constraint becomes $\bar w_{\dot\alpha}(\bar\sigma_{\mu\nu})^{\dot\alpha\dot\beta} w_{\dot\beta} = 0$. The matrix $(\bar\sigma_{\mu\nu})^{\dot\alpha\dot\beta}$ is symmetric, hence we can parametrize a generic solution as $\bar w_{\dot\alpha} w_{\dot\beta} = \rho^2\epsilon_{\dot\alpha\dot\beta}$, where ρ has dimension (length)^1, and corresponds to the size of the instanton, as we will see later. For $SU(2)$ an explicit solution to the constraint is given by

$$\bar w_{\dot 1} = (\rho, 0)\,, \quad \bar w_{\dot 2} = (0, \rho)\,, \quad w_{\dot 1} = \begin{pmatrix} 0 \\ -\rho \end{pmatrix}\,, \quad w_{\dot 2} = \begin{pmatrix} \rho \\ 0 \end{pmatrix}\,. \tag{7.28}$$

7.2.2 Marginal vertex operators

Out of all the vertex operators introduced in the previous section, we can identify some which are marginal, i.e. vertex operators of conformal dimension 0 (or 1 if the c-ghost is not taken into account). In the NS sector they correspond to the moduli w, $\bar w$, a and χ, and to the zero-momentum A and ϕ. We will focus on the four mixed directions, thus neglecting ϕ and χ. We can join the remaining vertex operators into a matrix, taking into account all possible strings. This matrix has $(N + k) \times (N + k)$ entries, and is of the form (eventually rescaling the polarizations)

$$V(z) = c(z)\begin{pmatrix} V_A & V_w \\ V_{\bar w} & V_a \end{pmatrix}(z) = \frac{g_{YM}}{\sqrt{\alpha'}} c(z)\begin{pmatrix} A_\mu^{uv}\psi^\mu & w_{\dot\alpha}^{uj}\Delta S^{\dot\alpha} \\ \bar w_{\dot\alpha}^{iv} S^{\dot\alpha}\bar\Delta & a_\mu^{ij}\psi^\mu \end{pmatrix}(z) e^{-\phi}(z)\,, \tag{7.29}$$

where we have explicitly written the Chan-Paton indices. This vertex operator can be expressed in the canonical −1 picture, as in (7.29), or in picture 0. In order to change the picture of this vertex operator we notice that it has the form $V = c\mathbb{V}_{1/2}e^{-\phi}$, where $\mathbb{V}_{1/2}$ is a Grassmann-odd superconformal matter primary of weight 1/2. Changing the picture on such a vertex operator is simple, and was discussed in [59]. Here we choose a slightly different notation: we call X the picture changing operator, and we express it in terms of the BRST charge and the ξ ghost (which comes from the bosonization of the β and γ ghosts, see appendix B) as

$$X = \{Q, \xi\}\,. \tag{7.30}$$

The BRST charge for superstring theory, in turn, is given explicitly by $Q = Q_0 + Q_1 + Q_2$ with (see [48, 60])

$$\begin{aligned} Q_0 &= \oint \frac{dz}{2\pi i} \left(cT^{X,\psi,\beta,\gamma} + c(\partial c)b\right)(z)\,, \\ Q_1 &= -\oint \frac{dz}{2\pi i}\,\gamma T_F(z)\,, \\ Q_2 &= -\frac{1}{4}\oint \frac{dz}{2\pi i}\,b\gamma^2(z)\,, \end{aligned} \tag{7.31}$$

where T_F is the matter supercurrent $T_F(z) = \frac{i}{\sqrt{2\alpha'}}\psi_\mu\partial X^\mu$. This is the proper generalization of (5.33). We then have

$$X\,V = -c\mathbb{V}_1 + \frac{1}{4}\gamma\mathbb{V}_{1/2}\,, \tag{7.32}$$

where $\mathbb{V}_1$ is a Grassmann-even superconformal primary of weight one defined by

$$T_F(z)\mathbb{V}_{1/2}(0) = \frac{\mathbb{V}_1(0)}{z} + \text{regular}\,. \tag{7.33}$$

The explicit calculation for the vertex operator (7.29) gives

$$\begin{aligned} XV(z) = -\frac{g_{YM}}{\sqrt{\alpha'}}c &\begin{pmatrix} \frac{i}{\sqrt{2\alpha'}}A^{uv}_\mu\partial X^\mu & -\frac{1}{2\sqrt{2}}w^{uj}_{\dot\alpha}\tau_\mu(\bar\sigma^\mu)^{\dot\alpha}{}_\beta S^\beta \\ -\frac{1}{2\sqrt{2}}\bar w^{iv}_{\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha}{}_\beta S^\beta\bar\tau_\mu & \frac{i}{\sqrt{2\alpha'}}a^{ij}_\mu\partial X^\mu \end{pmatrix}(z) + \\ &+\frac{1}{4}\frac{g_{YM}}{\sqrt{\alpha'}}\gamma\begin{pmatrix} A^{uv}_\mu\psi^\mu & w^{uj}_{\dot\alpha}\Delta S^{\dot\alpha} \\ \bar w^{iv}_{\dot\alpha}S^{\dot\alpha}\bar\Delta & a^{ij}_\mu\psi^\mu \end{pmatrix}(z)\,, \end{aligned} \tag{7.34}$$

where τ_μ (and analogously $\bar\tau_\mu$) is a combination of an excited twist field along the μ direction and three normal twist fields along the other directions [1] (see also 5.2.4 for some relevant correlation functions).

Chapter 8

Finite-size D-branes in superstring theory

In this chapter we explore the possibility of a marginal deformation corresponding to the blow-up of the size of a D-brane in superstring theory. For concreteness, we consider the bound state of D3-D(-1) branes introduced in chapter 7. We discuss the importance of string field theory for this problem, and study marginality up to third order in size. In the process we review the derivation of the instanton profile from superstring theory.

8.1 Second order deformation

In this section we start analyzing exact marginality of the blow-up of the size of a D(-1) brane in a D3 background. We do this in the framework of open superstring field theory (OSFT); this is necessary for two reasons. As we shall see shortly, we will encounter amplitudes with on-shell internal states as well as off-shell external states. Off-shell states, for example, appear whenever we consider a vector boson vertex operator V_A with non-vanishing momentum in euclidean space, as in (7.15). Furthermore, the analysis discussed in 5.2.3 is not easily generalizable, since in superstring theory one has to deal with vertex operators in different pictures.

The Yang-Mills action (6.8) arises, in the limit $\alpha' \to 0$, from open superstring field theory after integrating out massive and auxiliary fields [61, 59]. We will not need all of the technical material that goes into its construction. Let us instead begin by reviewing some details relevant for this thesis.

8.1.1 Open superstring field theory

The NS sector of open superstring field theory is defined perturbatively on the space of states $\mathcal{H}$ of the worldsheet SCFT of $(-1)/(-1)$, $(-1)/3$, $3/(-1)$ and $3/3$ strings with a non-degenerate BPZ inner product

$$(\Psi_1, \Psi_2) = \lim_{z\to 0} \mathrm{Tr}\,\langle (I^*\mathcal{O}_{\Psi_1})(z)\mathcal{O}_{\Psi_2}(z)\rangle_H \ , \tag{8.1}$$

where the trace is over NN and DD boundary conditions, $\langle \dots \rangle_H$ is the correlator on the upper half plane and $I(z) = -1/z$ while $I^*\mathcal{O}$ denotes the conformal transformation of $\mathcal{O}$ with respect to I. The BPZ inner product is graded symmetric due to the 3 ghost insertions originating from the $SL(2,\mathbb{R})$ isometry group of the disk. With this, the kinetic term is given by

$$\frac{1}{2}(\Psi, Q\Psi) \,, \tag{8.2}$$

where Q is the open string BRST charge (7.31) of ghost number one and Ψ is an arbitrary state in the state space of the matter plus ghost SCFT.

In addition to the quadratic term, OSFT has an infinite number of higher order interaction terms, in analogy to bosonic string field theory:

$$S(\Psi) = \frac{1}{2}(\Psi, Q\Psi) + \frac{1}{3}(\Psi, M_2(\Psi,\Psi)) + \frac{1}{4}(\Psi, M_3(\Psi,\Psi,\Psi)) + \cdots \tag{8.3}$$

All of these vertices are contact terms, meaning that they do not involve integrals over even directions in moduli space. However, they do involve integrals over the odd directions, which are implemented by the insertion of a BPZ-even picture changing operator X [62]. Let us focus on M_2 at present. Ignoring picture changing for the moment, M_2 reduces to an associative product

$$m_2 \;: \mathcal{H} \times \mathcal{H} \to \mathcal{H} \,, \tag{8.4}$$

which can be described by the three-point correlator

$$(\Psi_1, m_2(\Psi_2, \Psi_3)) := \mathrm{Tr}\,\langle (f_\infty^* \mathcal{O}_{\Psi_1})(0)(f_1^* \mathcal{O}_{\Psi_2})(0)(f_0^* \mathcal{O}_{\Psi_3})(0)\rangle \,, \tag{8.5}$$

where $f_w(z)$ is a family of conformal maps from the half disk to the upper half plane such that $f_w(0) = w$. For now we will not need any further information on $f_w(z)$ since we will consider mostly on-shell operators (except in section 8.2). Equivalently, m_2 is defined in terms of the operator product expansion (OPE) of conformal fields. This will sometimes be more convenient for our use below.

In order to implement the integration over the odd moduli we define the picture changing operators [62]

$$X = \frac{1}{2\pi i}\oint \frac{dx}{z} X(z) \;, \quad \xi = \frac{1}{2\pi i}\oint \frac{dx}{z}\xi(z) \tag{8.6}$$

around each puncture, with $X(z) = \{Q, \xi(z)\}$ where $\xi(z)$ is defined in (B.11). The product M_2 can then be expressed as

$$M_2(A,B) = \frac{1}{3}\Big(X m_2(A,B) + m_2(XA,B) + m_2(A,XB)\Big) \,, \tag{8.7}$$

where $Xm_2(A,B)$ can be evaluated with the help of the BPZ inner product, using

$$(C, Xm_2(A,B)) = (XC, m_2(A,B)) \,, \tag{8.8}$$

where $|C|$ is the ghost number of C. Note that M_2 so defined is associative only up to homotopy, that is a Q-exact term, due to the fact that X does not commute with the m_2 operation. Consequently the algebraic structure of OSFT is that of a homotopy associative (or A_∞) algebra. This structure then uniquely determines the higher order products, up to field redefinitions. This is how the A_∞-OSFT of [62] is constructed. However, we will not need any details of this construction other than the fact that M_3 cancels the non-associativity of M_2 and that M_2 itself is exact in the *large Hilbert space*, that is,

$$M_2 = \{Q, \mu_2\}\,, \qquad \mu_2(A,B) = \frac{1}{3}\left(\xi m_2(A,B) + m_2(\xi A, B) + (-1)^{|A|} m_2(A, \xi B)\right)\,. \quad (8.9)$$

The large Hilbert space includes the totality of all operators. We consider also the so-called *small Hilbert space*, which consists of states that do not depend on the zero mode of ξ [60]. Alternatively, the small Hilbert space is defined as the kernel of the zero-mode of the ghost field η (defined in appendix B). In the following we consider massless fields Ψ of the small Hilbert space at ghost number 1 and picture -1. This defines the so-called *Siegel gauge*; formally we write

$$\eta_0 \Psi = 0\,, \qquad b_0 \Psi = 0\,, \qquad pict[\Psi] = -1\,. \quad (8.10)$$

Nonetheless, it will be useful to perform some calculations in the large Hilbert space.

8.1.2 Second order deformation

A marginal deformation in the worldsheet CFT is exactly marginal if the corresponding solution of the linearized equation of motion of OSFT can be integrated to a solution of the nonlinear equation of motion. So let us start by writing down the equation of motion following from (8.3):

$$Q\Psi + M_2(\Psi, \Psi) + M_3(\Psi, \Psi, \Psi) + \cdots = 0\,. \quad (8.11)$$

We then expand the field in a perturbation series, as in 2.3 and 5.3, with dimensionless parameter $\lambda = \frac{\rho}{\sqrt{\alpha'}}$:

$$\Psi = \frac{\rho}{\sqrt{\alpha'}}\Psi^{(1)} + \left(\frac{\rho}{\sqrt{\alpha'}}\right)^2 \Psi^{(2)} + \left(\frac{\rho}{\sqrt{\alpha'}}\right)^3 \Psi^{(3)} + \ldots\,, \quad (8.12)$$

where

$$\frac{\rho}{\sqrt{\alpha'}}\Psi^{(1)} = V\,, \quad (8.13)$$

which, for V as in (7.29), is a solution of the linearized equation of motion $QV = 0$ that describes an infinitesimal blow-up of the D(-1) brane, and $(\frac{\rho}{\sqrt{\alpha'}})^2\Psi^{(2)} + \ldots$ are the higher order correction to $(\frac{\rho}{\sqrt{\alpha'}})\Psi^{(1)}$. To first order in the non-linearity (second order in $\rho/\sqrt{\alpha'}$) we then have

$$Q\Psi^{(2)} + M_2(\Psi^{(1)}, \Psi^{(1)}) = 0\,, \quad (8.14)$$

which is solved by

$$\Psi^{(2)} = -Q^{-1} M_2(\Psi^{(1)}, \Psi^{(1)}) + \psi_2\,, \quad (8.15)$$

with ψ_2 a solution to the homogeneous equation, $Q\psi_2 = 0$. Equation (8.15) is well defined provided that Q has an inverse. For this we need to choose a gauge fixing. Here we will work in Siegel gauge, $b_0\Psi = 0$, with $Q^{-1} = \frac{b_0}{L_0}$. Then

$$Q\,Q^{-1} + Q^{-1}\,Q = 1 - P_0\,, \quad (8.16)$$

where P_0 is the projector on the cohomology $H(Q) \subset \ker(L_0)$, satisfying

$$QP_0 = P_0 Q = 0\,. \tag{8.17}$$

To see if $\Psi^{(2)}$ in (8.15) is well defined we then compute

$$Q\Psi^{(2)} = -Q\,Q^{-1}M_2(\Psi^{(1)},\Psi^{(1)}) = (Q^{-1}Q + P_0 - 1)M_2(\Psi^{(1)},\Psi^{(1)})\,. \tag{8.18}$$

The first term on the r.h.s vanishes using $[Q, M_2] = 0$ (which, in turn, follows from the fact that $[Q, X] = 0$) and that Q is a derivation of the star product m_2 defined through (8.5). Thus (8.15) is meaningful provided that

$$P_0 M_2(\Psi^{(1)},\Psi^{(1)}) = 0. \tag{8.19}$$

To prove that (8.19) holds we show that $P_0 M_2(V,V) = 0$ for a vertex operator of the form $V(z) = c\mathbb{V}_{1/2}\, e^{-\phi}(z)$, where $\mathbb{V}_{1/2}$ is a matter vertex operator of conformal dimension 1/2; this is the prototype of vertex operator we are interested in (cfr. (7.29)). Let us first consider the contribution $P_0[m_2(V, XV) + m_2(XV, V)]$ in (8.7). Since all operators involved have total conformal weight zero we can evaluate this expression using the OPE. That is, using (7.32),

$$\begin{aligned}
&P_0[m_2(V, XV) + m_2(XV, V)] = \lim_{z\to 0} P_0\left[V(z)XV(0) + XV(0)V(z)\right] = \\
&= \lim_{z\to 0} P_0\left[c\mathbb{V}_{1/2}e^{-\phi}(z)\left(-c\mathbb{V}_1 + \frac{1}{4}\gamma\mathbb{V}_{1/2}\right)(0) + \left(-c\mathbb{V}_1 + \frac{1}{4}\gamma\mathbb{V}_{1/2}\right)(z)c\mathbb{V}_{1/2}e^{-\phi}(0)\right]\,,
\end{aligned} \tag{8.20}$$

where $\mathbb{V}_1$ is the matter operator of conformal dimension 1 defined in (7.33). Since the OPE $\mathbb{V}_{1/2}(z)\mathbb{V}_{1/2}(0)$ contains a single pole, while the OPE $\mathbb{V}_{1/2}(z)\mathbb{V}_1(0)$ does not, we conclude that

$$P_0[m_2(V, XV) + m_2(XV, V)] = \lim_{z\to 0} P_0\left[\frac{1}{4} z\, c\eta(0)\left(\mathbb{V}_{1/2}(z)\mathbb{V}_{1/2}(0) - \mathbb{V}_{1/2}(z)\mathbb{V}_{1/2}(0)\right)\right] = 0, \tag{8.21}$$

where we used the fact that $\mathbb{V}_{1/2}$ and c are fermionic operators. Let us then consider the remaining term $P_0[Xm_2(V,V)] = XP_0[m_2(V,V)]$ in (8.7). Recalling (8.16) we restrict the OPE to the kernel of L_0,

$$m_2(V,V)|_{\ker(L_0)} = \lim_{z\to 0}(c\mathbb{V}_{1/2}\, e^{-\phi})(z)(c\mathbb{V}_{1/2}\, e^{-\phi})(0) = \partial(c\partial c e^{-2\phi}\mathbb{V}'_0) + c\partial c\mathbb{V}'_1 e^{-2\phi}\,, \tag{8.22}$$

where $\mathbb{V}'_0$ and $\mathbb{V}'_1$ are matter vertex operators of conformal weight 0 (thus proportional to the identity) and 1 respectively. It is not hard to see that the first term in (8.22) is Q-exact, i.e.

$$\partial(c\partial c e^{-2\phi}\mathbb{V}'_0) = Q\left(\partial c e^{-2\phi}\mathbb{V}'_0\right)\,, \tag{8.23}$$

and therefore it is annihilated by P_0, since $P_0 Q = 0$, leaving

$$P_0 m_2(V,V) = c\partial c\mathbb{V}'_1 e^{-2\phi}\,. \tag{8.24}$$

For the second term in (8.22) we proceed using the the identity $X = \{Q, \xi\}$ in the large Hilbert space. Since V is on-shell,

$$P_0 M_2(V,V) = Q\left(\xi c\partial c \mathbb{V}_1' e^{-2\phi}\right) = (Q_0 + Q_1 + Q_2)\left(\xi c\partial c \mathbb{V}_1' e^{-2\phi}\right), \tag{8.25}$$

where we used the explicit definition (7.31). Bosonizing the ghosts as in appendix B we can compute each term in (8.25). The first and last term clearly vanish, while the second term extracts the double pole of the OPE

$$T_F(z)\mathbb{V}_1'(0). \tag{8.26}$$

In our case, however, the operator $\mathbb{V}_1'$ is proportional to $\psi^{\mu\nu}$, as we will see in section 8.3. The OPE with the supercurrent is then given by

$$T_F(z)\psi^{\mu\nu}(0) \sim \frac{1}{z}\left(\partial X^\mu \psi^\nu - \partial X^\nu \psi^\mu\right)(0) + \dots \tag{8.27}$$

Therefore we conclude that (8.25) vanishes, thus establishing that $P_0 M_2(V,V) = 0$. Hence, the first order correction $\Psi^{(2)}$ in (8.15) is well defined, even without specifying the ADHM constraints. Notice that this is an improvement with respect to bosonic string theory, where an obstruction at second order in size was found (see equation (5.36)).

8.2 Instanton profile

The fact that no obstruction is present at second order in the deformation implies that we can compute the first order correction to the instanton profile, that is the projection of $\Psi^{(2)}$ to a gluon state. This should correspond to computing the instanton profile at second order in $\rho/\sqrt{\alpha'}$, valid for $\rho^2 \ll \alpha'$. For simplicity we set $a_\mu = 0$; a different value for a_μ would correspond to moving the position of the instanton. Concretely we consider

$$A_\mu^{c(1)} = \left(\frac{\rho}{\sqrt{\alpha'}}\right)^2 (\mathcal{V}_{A_\mu^c}, \Psi^{(2)}) = -\left(\frac{\rho}{\sqrt{\alpha'}}\right)^2 (\mathcal{V}_{A_\mu^c}, Q^{-1} M_2(\Psi^{(1)}, \Psi^{(1)})). \tag{8.28}$$

Since $\mathcal{V}_{A_\mu^c}$ is a vertex operator in the 3/3 sector, this matrix element projects the 3/3 component of $M_2(\Psi^{(1)}, \Psi^{(1)})$. Thus

$$A_\mu^{c(1)} = -(\mathcal{V}_{A_\mu^c}, Q^{-1} M_2(V_w, V_{\bar{w}})). \tag{8.29}$$

where we used the same symbol M_2 for the matrix components of M_2. The latter involves picture changing operators on the inputs as well as on the output of the product. However, since X is a conformal scalar we can pull it through the propagator Q^{-1} onto $\mathcal{V}_{A_\mu^c}$. Furthermore, since none of the vertex operators involves the η ghost, we can move X from either input to $\mathcal{V}_{A_\mu^c}$ in spite of $\mathcal{V}_{A_\mu^c}$ being off-shell. Consequently we can take $\mathcal{V}_{A_\mu^c}$ in picture zero while the boundary changing vertex operators are in picture -1.

The calculation of this quantity can be done in two steps. Using the definition (8.5), we first need to compute the correlator

$$\mathrm{Tr}\langle\langle (f_\infty^* V_{\bar{w}}^{(-1)})(0)(f_1^* \mathcal{V}_{A_\mu^c}^{(0)})(0;-k)(f_0^* V_w^{(-1)})(0)\rangle\rangle_{D(-1)}, \tag{8.30}$$

where $V_{\bar{w}}$ and V_w are boundary changing operators, and $\mathcal{V}_{A^c_\mu}$ is the gluon vertex operator with outgoing momentum, with a free Lorentz and color index. Notice that the topological normalization is the one of the lowest brane [27]. Then we act with Q^{-1}, which, in Siegel gauge, results in multiplication by $1/k^2$. The calculation of (8.28) is sketched in figure 8.1.

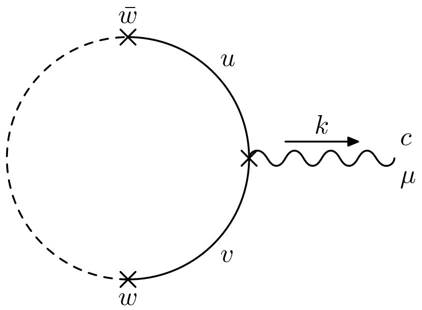

Figure 8.1: First order contribution to the instanton profile. The solid line represents the D3 branes, while the dashed one represents the D(−1) brane; the indices $u, v = 1, 2$ label the particular D3 branes. The vector A^c_μ (with outgoing momentum k^ν) comes from a 3/3 string, and the corresponding vertex operator has to be inserted in the middle of the solid line. The curly line represents the presence of a gluon propagator.

Explicitly, the boundary changing operators (in picture −1) in (8.30) are the ones given in (7.19) with the rescaling (7.26), while $\mathcal{V}_{A_\mu}$ is given (in picture −1) by

$$\mathcal{V}^{(-1)uv}_{A^c_\mu}(z; -k) = \sqrt{\frac{\alpha'}{2}} \frac{(\tau^c)^{uv}}{2} c(z)\psi_\mu(z) e^{-\phi(z)} e^{-ik\cdot X(z)} \,, \tag{8.31}$$

where we have used the Chan-Paton factor $(T^c)^{uv} = (\tau^c)^{uv}/2i$. Applying the picture changing operator to (8.31) we get

$$\mathcal{V}^{(0)uv}_{A^c_\mu}(z; -k) = \frac{(\tau^c)^{vu}}{2}(i\partial X_\mu - \frac{\alpha'}{2} k \cdot \psi\psi_\mu) e^{-ik\cdot X(z)} \,. \tag{8.32}$$

In contrast to (7.34), there is an extra contribution due to the non-vanishing momentum k^μ. Furthermore, only the term with a c-ghost (and not the one with a γ-ghost) can contribute to the correlation function (8.30). We note here that the action of the maps f_z reduces for primary operator $A(w)$ to

$$f^*_z A(w) = f'_z(w)^h A(f_z(w)) \,, \tag{8.33}$$

where h is the conformal dimension. In our case, since two operators are on-shell, we need only one map f_1, for the gauge vector. Therefore the correlation function reduces to

$$A^{c(1)}_\mu(k) = C_0 f'_1(0)^{\alpha' k^2/2} \langle V^{(-1)u}_{\bar{w}}(\infty) \mathcal{V}^{(0)uv}_{A_\mu}(1; k) V^{(-1)v}_w(0)\rangle \,, \tag{8.34}$$

where we made the $SU(2)$ indices explicit. The detailed calculation is done in appendix E. Here we state the final result in momentum space, that is

$$A^{c(1)}_\mu(k) = \left(\frac{f'_1(0)}{4}\right)^{\alpha' k^2/2} i\rho^2 k^\nu \bar{\eta}^c_{\nu\mu} e^{-ik\cdot x_0} \,, \tag{8.35}$$

where the factor $(1/4)^{\alpha' k^2/2}$ takes into account the proper normal ordering on a twisted background (see (3.44) [63, 14]). Notice that this result depends on α' and on the choice of the map f_1 (different maps correspond to different field redefinitions); (8.35) differs from the result in [27], where the off-shell amplitude (8.30) was computed within the on-shell formalism. Let us now perform a Fourier transform, in order to have a result in configuration space; as explained above, the propagator in Siegel gauge has to be added. The result is (see appendix E for the detailed calculation)

$$A_\mu^{c\,(1)}(x) = 2\rho^2 \bar{\eta}^c_{\mu\nu} \frac{(x-x_0)^\nu}{(x-x_0)^4} \left[1 + e^{(x-x_0)^2/(2\alpha' L_1)} \left(1 - \frac{(x-x_0)^2}{2\alpha' L_1}\right)\right] , \tag{8.36}$$

where $L_1 = \log(f_1'(0)/4) < 0$ since $f_1'(0) < 1$. In the field theory limit $\alpha' k^2 \ll 1$ the dependence on $f_1'(0)$ drops out. Since we also assumed from the beginning that $\rho^2 \ll \alpha'$, the field theory limit will also correspond to a large distance (compared to the size ρ) limit. In this limit the profile in position space is (see appendix E)

$$A_\mu^{c\,(1)}(x) = 2\rho^2 \bar{\eta}^c_{\mu\nu} \frac{(x-x_0)^\nu}{(x-x_0)^4} , \tag{8.37}$$

which is exactly the leading term in a large distance expansion ($\rho^2 \ll (x-x_0)^2$) of the full $SU(2)$ instanton solution (2.8), as previously found in [27].

In closing this section we note that a zero momentum gluon, appearing in vertex operator in (7.29), does not source a non-linear correction to (8.36). This is because the correction would be proportional to the three point function

$$\langle V_A^{(-1)}(\infty;0) \mathcal{V}_A^{(0)}(1;k) V_A^{(-1)}(0;0)\rangle , \tag{8.38}$$

which vanishes, since two of the vertex operators have vanishing momentum. Thus, the complete profile up to order ρ^2 is given by

$$A_\mu^c(x) = A_\mu^{c\,(0)} + A_\mu^{c\,(1)}(x) , \tag{8.39}$$

where $A_\mu^{c\,(0)}$ is constant in position space and $A_\mu^{c\,(1)}(x)$ given by (8.36). For the same reason this zero momentum gluon does not source a deformation in the 3/(-1), (-1)/3 or (-1)/(-1) sectors.

8.3 Third Order Deformation

At second order in the deformation (third order in $\rho/\sqrt{\alpha'}$) the equation of motion (8.11) reads

$$\begin{aligned} &Q\Psi^{(3)} - M_2(Q^{-1}M_2(\Psi^{(1)},\Psi^{(1)}) - \psi_2, \Psi^{(1)}) + \\ &\quad - M_2(\Psi^{(1)}, Q^{-1}M_2(\Psi^{(1)},\Psi^{(1)}) - \psi_2) + M_3(\Psi^{(1)},\Psi^{(1)},\Psi^{(1)}) = 0 , \end{aligned} \tag{8.40}$$

where we used the solution for Ψ_1 given in (8.15). The obstruction to inverting Q is given by

$$\begin{aligned} &(Q^{-1}Q + P_0)\Big[M_2(Q^{-1}M_2(\Psi^{(1)},\Psi^{(1)}) - \psi_2, \Psi^{(1)}) + \\ &\quad + M_2(\Psi^{(1)}, Q^{-1}M_2(\Psi^{(1)},\Psi^{(1)}) - \psi_2) - M_3(\Psi^{(1)},\Psi^{(1)},\Psi^{(1)})\Big] . \end{aligned} \tag{8.41}$$

Let us first consider the terms involving $Q^{-1}Q$. They add up to (using also $Q\psi_1 = 0$)

$$Q^{-1}\left[M_2(M_2(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)}) - M_2(\Psi^{(1)},M_2(\Psi^{(1)},\Psi^{(1)})) - QM_3(\Psi^{(1)},\Psi^{(1)},\Psi^{(1)})\right], \tag{8.42}$$

which vanishes by the A_∞ relations (see e.g [62, 64]). The remaining obstruction is then

$$\begin{aligned} P_0\Big[&M_2(Q^{-1}M_2(\Psi^{(1)},\Psi^{(1)}) - \psi_2, \Psi^{(1)}) + \\ &+ M_2(\Psi^{(1)}, Q^{-1}M_2(\Psi^{(1)},\Psi^{(1)}) - \psi_2) - M_3(\Psi^{(1)},\Psi^{(1)},\Psi^{(1)})\Big]. \end{aligned} \tag{8.43}$$

We note, in passing, that (8.43) is just the minimal model map to fourth order of the underlying A_∞ algebra which extracts the S-matrix elements of string field theory. This does not come as a surprise, since S-matrix elements are known to be given by the obstructions of a linearized solution to give rise to a non-linear solution (e.g. [65]).

In order to analyze this obstruction we first note that $P_0M_2(\psi_1,\Psi^{(1)})$ and $P_0M_2(\Psi^{(1)},\psi_1)$ vanish. The proof of this is completely analogous to that given above for $P_0M_2(\Psi^{(1)},\Psi^{(1)})$. Next, we consider $P_0M_2(Q^{-1}M_2(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)})$, which we write as

$$\sum_i e^i\langle e_i, M_2(Q^{-1}M_2(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)})\rangle\,, \tag{8.44}$$

where e_i (e^i) is a basis (and its dual) of the image of P_0 with $\langle e^i, e_j\rangle = \delta^i_j$. To continue we use (8.9) to write

$$\begin{aligned} \langle e_i, M_2(Q^{-1}M_2(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)})\rangle = &-\frac{1}{2}\langle e_i, \xi\, M_2(Q^{-1}\{Q,\mu_2\}(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)})\rangle_L + \\ &-\frac{1}{2}\langle e_i, \xi\, \{Q,\mu_2\}(Q^{-1}M_2(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)})\rangle_L\,, \end{aligned} \tag{8.45}$$

where, since (8.9) holds only in the large Hilbert space H_L, we now use the BPZ inner product in H_L with an extra insertion of ξ to saturate the extra zero mode in H_L. The second term in (8.43) is treated analogously. Next we commute Q^{-1} across Q and use the fact that Q commutes with M_2 and annihilates $\Psi^{(1)}$. In doing so we pick up the contributions

$$\begin{aligned} &\frac{1}{2}\langle e_i, \xi\, M_2(P_0\mu_2(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)})\rangle_L + \frac{1}{2}\langle e_i, X\, M_2(Q^{-1}\mu_2(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)})\rangle_L + \\ &-\frac{1}{2}\langle e_i, X\, \mu_2(Q^{-1}M_2(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)})\rangle_L - \frac{1}{2}\langle e_i, \xi\, \mu_2(P_0M_2(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)})\rangle_L\,. \end{aligned} \tag{8.46}$$

The last term above vanishes for the same reason as in subsection 8.1.2. In the two terms above involving X, the ξ zero-mode must be provided by μ_2, so that, for instance,

$$-\frac{1}{2}\langle e_i, X\, \mu_2(Q^{-1}M_2(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)})\rangle_L = -\frac{1}{2}\langle e_i, X\xi\, m_2(Q^{-1}M_2(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)})\rangle_L \tag{8.47}$$

and similarly for the second term in (8.46). In what follows, we will neglect the terms that originate from the identity in (8.16) since, as shown in [62], these cancel against M_3

in (8.43). Applying (8.9) to (8.47) we get

$$-\frac{1}{2}\langle e_i, X\,\mu_2(Q^{-1}M_2(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)})\rangle_L = -\frac{1}{2}\langle e_i, X\xi\, m_2(Q^{-1}\{Q,\mu_2\}(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)})\rangle_L\,. \tag{8.48}$$

We then commute Q^{-1} across Q and use again that Q annihilates $\Psi^{(1)}$. Therefore (8.48) gives a contribution

$$\frac{1}{2}\langle e_i, X\xi\, m_2(P_0\,\mu_2(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)})\rangle_L - \frac{1}{2}\langle e_i, X^2\, m_2(Q^{-1}\,\mu_2(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)})\rangle_L\,. \tag{8.49}$$

The objective of deriving the expressions (8.46) and (8.49) was to isolate the contact terms that originate in the integration over odd moduli in the super moduli space (encoded in the super string products M_2 and M_3). This procedure can be applied in complete analogy to the remaining terms in (8.43). More details on this derivation can be found in appendix F. The result is a sum of two contributions, one involving the projector P_0 and the other involving the propagator Q^{-1}. The first contribution reads

$$\begin{aligned}A = &-\frac{1}{3}\langle P_0\mu_2(\Psi^{(1)},\Psi^{(1)}), 4m_2(\Psi^{(1)},\xi X e_i) - 4m_2(\xi X e_i,\Psi^{(1)})\rangle && (8.50)\\ &-\frac{1}{3}\langle P_0\mu_2(\Psi^{(1)} m_2(X\Psi^{(1)},\xi e_i) - m_2(\xi e_i, X\Psi^{(1)})\rangle_L &&\\ &-\frac{1}{3}\langle P_0 m_2(\Psi^{(1)},\Psi^{(1)}), \xi m_2(\Psi^{(1)},\xi X e_i) - \xi m_2(\xi X e_i,\Psi^{(1)})\rangle && (8.51)\\ &-\frac{1}{3}\langle P_0 m_2(\Psi^{(1)},\Psi^{(1)}), m_2(\xi\Psi^{(1)},\xi X e_i) - m_2(\xi X e_i,\xi\Psi^{(1)})\rangle_L &&\\ &-\frac{1}{3}\langle X P_0\mu_2(\Psi^{(1)},\Psi^{(1)}), m_2(\Psi^{(1)},\xi e_i) - m_2(\xi e_i,\Psi^{(1)})\rangle_L\,, && (8.52)\end{aligned}$$

where we used the cyclic properties of the string products m_2 and μ_2 (see Appendix F) as well as $X\xi\Psi^{(1)} = \xi X\Psi^{(1)}$ (and similarly for e_i). The second term, involving the propagator is given by

$$B = -2\langle X\circ X\, e_i, m_2(Q^{-1}m_2(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)}) + m_2(\Psi^{(1)}, Q^{-1}m_2(\Psi^{(1)},\Psi^{(1)}))\rangle\,. \tag{8.53}$$

In the next two subsections we will evaluate these two terms separately.

8.3.1 Evaluation of A

To continue we evaluate the terms appearing in (8.50). In principle there are anomalous contributions due to the fact that $\xi X e_i$ contains the operator $:\xi\eta:$ which is not primary; we are going to discuss this problem in appendix G, where we show that all anomalous contributions cancel. For the moment we proceed as we would do if all the vertex operators were primaries; let us start with $P_0 m_2(\Psi^{(1)},\Psi^{(1)})$. Using the OPE relations given in appendix B with $\frac{\rho}{\sqrt{\alpha'}}\Psi^{(1)} = V$, we find (for a single D(-1) brane, i.e. $k=1$ and assuming $a_\mu = 0$ for simplicity)

$$\left(\frac{\rho}{\sqrt{\alpha'}}\right)^2 P_0 m_2(\Psi^{(1)},\Psi^{(1)}) = P_0 m_2(V,V) = \frac{1}{2}\frac{g_{YM}^2}{\alpha'}c\partial c e^{-2\phi}\psi^{\mu\nu}M_{\mu\nu}\,, \tag{8.54}$$

with

$$M_{\mu\nu} = \begin{pmatrix} [A_\mu, A_\nu] + \frac{1}{2} w_{\dot\alpha} (\bar\sigma_{\mu\nu})^{\dot\alpha\dot\beta} \bar w_{\dot\beta} & 0 \\ 0 & \frac{1}{2} \bar w_{\dot\alpha} (\bar\sigma_{\mu\nu})^{\dot\alpha\dot\beta} w_{\dot\beta} \end{pmatrix} , \tag{8.55}$$

where we have projected out the Q-exact piece, in analogy to the one in (8.22). On the other hand, $P_0 \mu_2(V,V)$ is given by

$$\left(\frac{\rho}{\sqrt{\alpha'}}\right)^2 P_0 \mu_2(\Psi^{(1)}, \Psi^{(1)}) = P_0 \mu_2(V,V) = \frac{1}{2} \frac{g_{YM}^2}{\alpha'} \xi c \partial c e^{-2\phi} \psi^{\mu\nu} M_{\mu\nu} + \frac{g_{YM}^2}{3\alpha'} \partial \xi c \partial c e^{-2\phi} U , \tag{8.56}$$

where the last term is proportional to the identity in the matter sector and

$$U = \begin{pmatrix} A_\mu A_\nu \delta^{\mu\nu} + w_{\dot\alpha} \epsilon^{\dot\alpha\dot\beta} \bar w_{\dot\beta} & 0 \\ 0 & \bar w_{\dot\alpha} \epsilon^{\dot\alpha\dot\beta} w_{\dot\beta} \end{pmatrix} . \tag{8.57}$$

To continue note that, without restricting the generality, we may parametrize a generic zero-momentum Siegel gauge state e_i in physical subspace H_{phys} by

$$e_i(z) = \frac{g_{YM}}{\sqrt{\alpha'}} c(z) \begin{pmatrix} B_\mu \psi^\mu & v_{\dot\alpha} \Delta S^{\dot\alpha} \\ \bar v_{\dot\alpha} S^{\dot\alpha} \bar\Delta & b_\mu \psi^\mu \end{pmatrix} (z) e^{-\phi}(z) , \tag{8.58}$$

which is basically the same as (7.29), but with a different generic polarizations B_μ, $v_{\dot\alpha}$, $\bar v_{\dot\alpha}$ and b_μ. In order to evaluate A We need the explicit expressions of ξe_i and $\xi X e_i$, given by

$$\xi e_i = \frac{g_{YM}}{\sqrt{\alpha'}} \xi c \begin{pmatrix} B_\mu \psi^\mu & v_{\dot\alpha} \Delta S^{\dot\alpha} \\ \bar v_{\dot\alpha} S^{\dot\alpha} \bar\Delta & b_\mu \psi^\mu \end{pmatrix} e^{-\phi} , \qquad \xi X e_i = \frac{g_{YM}}{4\sqrt{\alpha'}} : \xi\eta : e^{\phi} \begin{pmatrix} B_\mu \psi^\mu & v_{\dot\alpha} \Delta S^{\dot\alpha} \\ \bar v_{\dot\alpha} S^{\dot\alpha} \bar\Delta & b_\mu \psi^\mu \end{pmatrix} , \tag{8.59}$$

where we used (8.6) and for $X e_i$ we kept only the term with the γ-ghost in (7.32). This is because all the terms involving $\xi X e_i$ in (8.50) already have three c-ghost insertions, therefore only the term with a γ ghost in $\xi X e_i$ can contribute to the correlation functions. Using the OPE's in appendix B we can check that

$$P_0[m_2(V, \xi X e_i) - m_2(\xi X e_i, V)] = P_0[V, \xi X e_i]_{m_2} = \frac{g_{YM}^2}{4\alpha'} c : \xi\eta : W , \tag{8.60}$$

with

$$W = \begin{pmatrix} A_\mu B_\nu \delta^{\mu\nu} + w_{\dot\alpha} \epsilon^{\dot\alpha\dot\beta} \bar v_{\dot\beta} & 0 \\ 0 & \bar w_{\dot\alpha} \epsilon^{\dot\alpha\dot\beta} v_{\dot\beta} , \end{pmatrix} - \begin{pmatrix} B_\mu A_\nu \delta^{\mu\nu} + v_{\dot\alpha} \epsilon^{\dot\alpha\dot\beta} \bar w_{\dot\beta} & 0 \\ 0 & \bar v_{\dot\alpha} \epsilon^{\dot\alpha\dot\beta} w_{\dot\beta} , \end{pmatrix} \equiv W' - W'' \tag{8.61}$$

while

$$P_0 \xi m_2(V, \xi X e_i) = P_0 \xi m_2(\xi X e_i, V) = 0 . \tag{8.62}$$

The terms in (8.60), when coupled to (8.54), cannot contribute to (8.50), since they produce the one-point function $\langle \psi^{\mu\nu} \rangle$ in the matter sector, which vanishes. However, we get a non-vanishing contribution from the remaining terms in (8.50). In particular, we can compute

$$
\begin{aligned}
P_0[XV,\xi e_i]_{m_2} &= \frac{g_{YM}^2}{4\alpha'}c\psi^{\rho\sigma}N_{\rho\sigma} - \frac{g_{YM}^2}{4\alpha'}c:\xi\eta:W + \frac{g_{YM}^2}{4\alpha'}c\partial\phi W + \frac{g_{YM}^2}{4\alpha'}\partial c W''\,,\\
P_0[\xi V,\xi X e_i]_{m_2} &= -\frac{g_{YM}^2}{4\alpha'}\xi_0 c\psi^{\rho\sigma}N_{\rho\sigma} + \frac{g_{YM}^2}{4\alpha'}\xi_0 c\partial\phi W - \frac{g_{YM}^2}{4\alpha'}\xi_0\partial c W' - \frac{g_{YM}^2}{4\alpha'}\partial\xi c W''\,,\\
P_0[V,\xi e_i]_{m_2} &= -\frac{g_{YM}^2}{\alpha'}\xi_0 c\partial c e^{-2\phi}\psi^{\rho\sigma}N_{\rho\sigma} - \frac{g_{YM}^2}{\alpha'}\partial\xi c\partial c e^{-2\phi}W''\,,
\end{aligned}
\tag{8.63}
$$

where $[\,\cdot,\cdot\,]_{m_2}$ denotes the graded commutator with respect to Witten's star product m_2,

$$
N_{\rho\sigma} = \begin{pmatrix} [A_\rho,B_\sigma]+\frac{1}{4}\left(w_{\dot\gamma}(\bar\sigma_{\rho\sigma})^{\dot\gamma\dot\delta}\bar v_{\dot\delta}+v_{\dot\gamma}(\bar\sigma_{\rho\sigma})^{\dot\gamma\dot\delta}\bar w_{\dot\delta}\right) & 0\\ 0 & \frac{1}{4}\left(\bar v_{\dot\gamma}(\bar\sigma_{\rho\sigma})^{\dot\gamma\dot\delta}w_{\dot\delta}+v_{\dot\gamma}(\bar\sigma_{\rho\sigma})^{\dot\gamma\dot\delta}\bar w_{\dot\delta}\right)\end{pmatrix}\,,
\tag{8.64}
$$

and W, W' and W'' were defined above. Here ξ_0 is the zero mode of ξ. In the last line of (8.63) we furthermore used that the combination

$$
\frac{g_{YM}^2}{\alpha'}\xi c e^{-2\phi}\left(\partial c\partial\phi - \frac{1}{2}\partial^2 c\right)(W'+W'')
\tag{8.65}
$$

contributing to $[V,\xi e_i]_{m_2}$ is Q-exact and thus annihilated by the projector P_0. Indeed, $Q(\partial c e^{-2\phi}) = c\partial^2 c e^{-2\phi} - 2\partial\phi c\partial c e^{-2\phi}$. Again, only the term with a γ ghost in XV can contribute to (8.50), because it is inserted inside correlation functions with already three c-ghost insertions. Let us now contract the terms in (8.63) with (8.54) and (8.56) respectively. Focussing first on the terms containing the matter operator $\psi_{\mu\nu}$, and using the known correlation functions

$$
\begin{aligned}
\langle\xi c\partial c e^{-2\phi}(z)c(w)\rangle_L &= -(z-w)^2\,,\\
\langle\psi^{\mu\nu}(z)\psi^{\rho\sigma}(w)\rangle_L &= \frac{-\delta^{\mu\rho}\delta^{\nu\sigma}+\delta^{\mu\sigma}\delta^{\nu\rho}}{(z-w)^2}\,,
\end{aligned}
\tag{8.66}
$$

we conclude that the first two lines of (8.50) exactly cancel for the state e_i. First, there is a precise cancellation of the terms proportional to $\psi^{\mu\nu}$ in (8.63). The terms proportional to the identity, on the other hand, give rise to a contribution proportional to

$$
\mathrm{Tr}\left[A_\mu A_\mu(A_\nu B_\nu - B_\nu A_\nu)\right]\,.
\tag{8.67}
$$

While this is in general non-zero, it vanishes for the $SU(2)$ gauge group[1].

Concerning the last line of (8.50), the first term in (8.56) can be treated in the large Hilbert space,

$$
P_0 X\xi m_2(V,V) = P_0\xi Q\xi m_2(V,V) = \xi P_0 X m_2(V,V) = 0\,,
\tag{8.68}
$$

where we have used the fact that V is on-shell and the last step was proven in subsection 8.1.2. The second term in (8.56), on the other hand, does not contain any zero mode of ξ. This means that the zero mode has to come from the first term in $P_0[m_2(V,\xi e_i) - m_2(\xi e_i,V)]$ (see the third line of (8.63)); however, this would give rise, in the matter sector, to the one-point function $\langle\psi^{\rho\sigma}\rangle$, which is zero. Therefore the last line in (8.50) vanishes as well; this concludes the proof that $A=0$.

[1] More generally, these terms are absent if one uses a symmetric OPE as in [66]. These two prescriptions are related by a field redefinition.

8.3.2 Evaluation of B

Let us now analyze the terms involving the propagator Q^{-1}, that is

$$B = -2\langle X \circ X\, e_i, m_2(Q^{-1}m_2(\Psi^{(1)}, \Psi^{(1)}), \Psi^{(1)}) + m_2(\Psi^{(1)}, Q^{-1}m_2(\Psi^{(1)}, \Psi^{(1)}))\rangle\,. \quad (8.69)$$

The field e_i is of the form $e_i = c\widetilde{\mathbb{V}}_{1/2}e^{-\phi}$, where $\widetilde{\mathbb{V}}_{1/2}$ is a matter primary operator of conformal dimension 1/2; using the picture changing we get

$$Xe_i = -c\widetilde{\mathbb{V}}_1 + \frac{1}{4}\gamma\widetilde{\mathbb{V}}_{1/2}\,; \quad (8.70)$$

we now apply another picture changing operator. We consider only terms with a c-ghost in the final result, since they are the only ones contributing to correlation functions. For the first term in (8.70) only $Q_1(-\xi c\widetilde{\mathbb{V}}_1)$ maintains the c-ghost; for the second term we get a contribution from $Q_0(:\xi\eta: e^{\phi}\widetilde{\mathbb{V}}_{1/2})$, due to the fact that $:\xi\eta:$ is not a primary field (see appendix B for details). We thus have, up to terms that do not contribute to the correlators,

$$\begin{aligned} X \circ Xe_i &= Q_1(-\xi c\widetilde{\mathbb{V}}_1) + Q_0(:\xi\eta: e^{\phi}\widetilde{\mathbb{V}}_{1/2}) + \dots = \\ &= \oint \frac{dz}{2\pi i}(\eta e^{\phi}T_F)(z)(\xi c\widetilde{\mathbb{V}}_1)(0) + \oint \frac{dz}{2\pi i}(cT)(z)\left(\frac{1}{4}:\xi\eta: e^{\phi}\widetilde{\mathbb{V}}_{1/2}\right)(0) + \dots \end{aligned} \quad (8.71)$$

For the explicit calculation we notice that the supercurrent satisfies

$$\begin{aligned} T_F(z)\widetilde{\mathbb{V}}_{1/2}(0) &= \frac{1}{z}\widetilde{\mathbb{V}}_1(0) + \dots\,, \\ T_F(z)\widetilde{\mathbb{V}}_1(0) &= \frac{1}{4z^2}\widetilde{\mathbb{V}}_{1/2}(z) + \mathcal{O}(z^0)\,. \end{aligned} \quad (8.72)$$

The OPE relations (8.72) imply that the supercurrent can always be written as normal ordered product of the spacetime fermion and boson appearing in $\widetilde{\mathbb{V}}_{1/2}$ and $\widetilde{\mathbb{V}}_1$ respectively. In particular this is obvious for the gluon vertex operator, for which the spacetime fermion and boson are proportional to ψ_μ and $i\partial X^\mu$, but it is also true in the case of boundary changing operators, since we can write

$$T_F \propto \psi_\mu \partial X^\mu = \frac{1}{\sqrt{2}} : \Delta S^{\dot{\alpha}}(\bar{\sigma}_\mu)_{\dot{\alpha}\beta}S^\beta \tau^\mu := \frac{1}{\sqrt{2}} : \bar{\Delta} S^{\dot{\alpha}}(\bar{\sigma}_\mu)_{\dot{\alpha}\beta}S^\beta \bar{\tau}^\mu : \,. \quad (8.73)$$

Therefore (8.71) becomes

$$\begin{aligned} X \circ Xe_i =& c(0)\oint \frac{dz}{2\pi i}e^{\phi}(z)\widetilde{\mathbb{V}}_{1/2}(z)\left(\frac{1}{z} + :\eta\xi: + z:\partial\eta\,\xi: + \dots\right)\left(\frac{1}{4z^2} + :\widetilde{\mathbb{V}}_1\widetilde{\mathbb{V}}_1: + \dots\right) + \\ &+ \oint \frac{dz}{2\pi i}\frac{1}{4}c(z)\left(-\frac{e^{\phi}\widetilde{\mathbb{V}}_{1/2}}{z^3} + \frac{\partial(:\xi\eta: e^{\phi}\widetilde{\mathbb{V}}_{1/2})}{z}\right) + \dots = \\ &= \frac{1}{8}c\partial^2\left(e^{\phi}\widetilde{\mathbb{V}}_{1/2}\right) - \frac{1}{8}(\partial^2 c)e^{\phi}\widetilde{\mathbb{V}}_{1/2} + ce^{\phi}\widetilde{\mathbb{V}}_{1/2} :\widetilde{\mathbb{V}}_1\widetilde{\mathbb{V}}_1: + \frac{1}{4}c :\partial\xi\,\eta: e^{\phi}\widetilde{\mathbb{V}}_{1/2} + \dots\,, \end{aligned} \quad (8.74)$$

where the $1/z^3$ term comes from the anomalous OPE (B.15) between the energy momentum tensor and $:\xi\eta:$ and $\dots$ indicates terms without a c-ghost. We notice that $ce^{\phi}\widetilde{\mathbb{V}}_{1/2} : \widetilde{\mathbb{V}}_1\widetilde{\mathbb{V}}_1 :$ and $c : \partial\xi\,\eta : e^{\phi}\widetilde{\mathbb{V}}_{1/2}$ are not primary, since the OPE with the energy-momentum tensor gives

$$\begin{aligned} T(z)\ ce^{\phi}\widetilde{\mathbb{V}}_{1/2} : \widetilde{\mathbb{V}}_1\widetilde{\mathbb{V}}_1 : (0) &= \frac{1}{4z^4} ce^{\phi}\widetilde{\mathbb{V}}_{1/2} + \dots \\ T(z)\ c : \partial\xi\,\eta : e^{\phi}\widetilde{\mathbb{V}}_{1/2}(0) &= -\frac{1}{z^4} ce^{\phi}\widetilde{\mathbb{V}}_{1/2} + \dots \end{aligned} \tag{8.75}$$

From these equations, however, we can see that the combination

$$ce^{\phi}\widetilde{\mathbb{V}}_{1/2} : \widetilde{\mathbb{V}}_1\widetilde{\mathbb{V}}_1 : + \frac{1}{4} c : \partial\xi\,\eta : e^{\phi}\widetilde{\mathbb{V}}_{1/2} \tag{8.76}$$

is a primary field, and thus behaves regularly inside the BPZ product (8.69).

In the absence of twist field insertions these two terms will not contribute, since they give rise to one-point functions of normal ordered products. In particular, the term proportional to $:\widetilde{\mathbb{V}}_1\widetilde{\mathbb{V}}_1:$ contributes, in the matter sector, a correlator of the form

$$\begin{aligned} &\langle \widetilde{\mathbb{V}}_{1/2} : \widetilde{\mathbb{V}}_1\widetilde{\mathbb{V}}_1 : (z_1) \mathbb{V}_{1/2}(z_2) \mathbb{V}_{1/2}(z_3) \mathbb{V}_{1/2}(z_4) \rangle = \\ &= \langle : \widetilde{\mathbb{V}}_1\widetilde{\mathbb{V}}_1 : (z_1) \rangle \langle \widetilde{\mathbb{V}}_{1/2}(z_1) \mathbb{V}_{1/2}(z_2) \mathbb{V}_{1/2}(z_3) \mathbb{V}_{1/2}(z_4) \rangle = 0\,, \end{aligned} \tag{8.77}$$

where the first factor is evaluated in the un-twisted vacuum.

We then rewrite the two remaining terms in (8.74) as

$$X \circ X\, e_i = \frac{1}{8} c\partial^2 \left(e^{\phi}\widetilde{\mathbb{V}}_{1/2} \right) - \frac{1}{8}\partial^2 c (e^{\phi}\widetilde{\mathbb{V}}_{1/2}) + \dots = \frac{1}{8} Q \left(\partial (e^{\phi}\widetilde{\mathbb{V}}_{1/2}) \right) =: Q\Phi + \dots\,, \tag{8.78}$$

up to terms that do not not contribute to the correlation function. Since this is a Q-exact quantity we can compute the propagator term (8.69), which becomes

$$\begin{aligned} B =& -2\langle Q\Phi, m_2(Q^{-1}m_2(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)}) + m_2(\Psi^{(1)}, Q^{-1}m_2(\Psi^{(1)},\Psi^{(1)}))\rangle = \\ =& -2\langle \Phi, m_2((1-P_0)m_2(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)}) - m_2(\Psi^{(1)},(1-P_0)m_2(\Psi^{(1)},\Psi^{(1)}))\rangle = \\ =& 2\langle \Phi, m_2(P_0 m_2(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)}) - m_2(\Psi^{(1)}, P_0 m_2(\Psi^{(1)},\Psi^{(1)}))\rangle\,, \end{aligned} \tag{8.79}$$

where the terms with the identity cancel, due to the associativity of the m_2 product. This can be written as

$$\begin{aligned} B &= 2\langle P_0 m_2(\Psi^{(1)},\Psi^{(1)}), m_2(\Psi^{(1)},\Phi) - m_2(\Phi,\Psi^{(1)})\rangle = \\ &= \frac{1}{4}\langle P_0 m_2(\Psi^{(1)},\Psi^{(1)}), m_2(\Psi^{(1)},\partial(e^{\phi}\widetilde{\mathbb{V}}_{1/2})) - m_2(\partial(e^{\phi}\widetilde{\mathbb{V}}_{1/2}),\Psi^{(1)})\rangle\,. \end{aligned} \tag{8.80}$$

The operator $\partial(e^{\phi}\widetilde{\mathbb{V}}_{1/2})$ is not primary, therefore there are anomalous contributions analogous to the ones appearing in (8.50). We refer to appendix G for the proof that all anomalies cancel. In the meantime we proceed as if all vertex operators were primaries, so that the product m_2 can be evaluated simply as the OPE. We have already computed $P_0 m_2(\Psi^{(1)},\Psi^{(1)})$; in fact (8.54) gives

$$\left(\frac{\rho}{\sqrt{\alpha'}}\right)^2 P_0 m_2(\Psi^{(1)},\Psi^{(1)}) = P_0 m_2(V,V) = \frac{1}{2}\frac{g_{YM}^2}{\alpha'} c\partial c e^{-2\phi}\psi^{\mu\nu}M_{\mu\nu}\,, \tag{8.81}$$

with $M_{\mu\nu}$ given in (8.55). On the other hand we have

$$P_0\left[m_2(V,\,\partial(e^{\phi}\widetilde{\mathbb{V}}_{1/2}))-m_2(\partial(e^{\phi}\widetilde{\mathbb{V}}_{1/2}),V)\right]=P_0\left[V,\,\partial(e^{\phi}\widetilde{\mathbb{V}}_{1/2})\right]_{m_2}$$
$$=\lim_{z\to w}\Big(\partial_w[c\mathbb{V}_{1/2}e^{-\phi}(z)e^{\phi}\widetilde{\mathbb{V}}_{1/2}(w)]-\partial_z[e^{\phi}\widetilde{\mathbb{V}}_{1/2}(z)c\mathbb{V}_{1/2}e^{-\phi}(w)]\Big)=-\frac{g_{YM}^2}{\alpha'}c\psi^{\rho\sigma}N_{\rho\sigma}\,, \tag{8.82}$$

with $N_{\rho\sigma}$ as in (8.64). Putting all together we get

$$\left(\frac{\rho}{\sqrt{\alpha'}}\right)^3 B=\frac{1}{4}\mathrm{Tr}\left[M_{\mu\nu}N^{\mu\nu}\right]\,, \tag{8.83}$$

or, explicitly, assuming the ADHM constraints,

$$\frac{g_{YM}^4}{8\alpha'^2}\mathrm{Tr}\left[\left([A_\mu,A_\nu]+\frac{1}{2}w_{\dot\alpha}(\bar\sigma_{\mu\nu})^{\dot\alpha\dot\beta}\bar w_{\dot\beta}\right)\left([A^\mu,B^\nu]+\frac{1}{4}w_{\dot\gamma}(\bar\sigma^{\mu\nu})^{\dot\gamma\dot\delta}\bar v_{\dot\delta}+\frac{1}{4}v_{\dot\gamma}(\bar\sigma^{\mu\nu})^{\dot\gamma\dot\delta}\bar w_{\dot\delta}\right)\right]\,. \tag{8.84}$$

In the absence of twist fields ($w_{\dot\alpha}=0$) this gives the correct equation of motion for a zero-momentum gluon field, in agreement with the 4-gluon vertex in Yang-Mills theory. For non-vanishing $w_{\dot\alpha}$, while there is a choice, as we will see later, of a zero-momentum gluon such that the anti-self-dual part of the commutator $[A_\mu,A_\nu]$ cancels the combination $\frac{1}{2}w_{\dot\alpha}(\bar\sigma_{\mu\nu})^{\dot\alpha\dot\beta}\bar w_{\dot\beta}$, that still leaves us with the self-dual part of $[A_\mu,A_\nu]$ so that full matrix

$$[A_\mu,A_\nu]+\frac{1}{2}w_{\dot\alpha}(\bar\sigma_{\mu\nu})^{\dot\alpha\dot\beta}\bar w_{\dot\beta} \tag{8.85}$$

does not vanish all together, indicating an obstruction to the blow-up mode at this order. The loop-hole in this argument is that the first term in (8.76), being normal ordered w.r.t. the untwisted vacuum, may still be give a non-vanishing contribution in the twisted vacuum. It turns out that the contribution form this term is rather cumbersome to evaluate explicitly due to the presence of branch-cuts in the integrand. This difficulty can be circumvented by evaluating (8.69) in a different manner, making use of the fact that the world-sheet CFT has an $SO(4)$-invariance acting exclusively on the world-sheet fermions ψ^μ, $\mu=1,\cdots,4$ (e.g. [67]) and on the spin fields. As advocated in [59, 68], but with a slight difference due to the opposite choice of chirality for the twisted vertex operators, a convenient basis is

$$\psi_1^\pm=\frac{1}{\sqrt2}\left(\psi^1\pm i\psi^2\right)\quad,\quad\psi_2^\pm=\frac{1}{\sqrt2}\left(\psi^4\pm i\psi^3\right) \tag{8.86}$$

in which only a $U(2)$ invariance is manifest. As a consequence of the $SO(4)$-invariance just described the $U(1)$-charge

$$J=-\frac{1}{2\pi i}\oint\sum_{i=1}^{2}:\psi_i^+\psi_i^-:dz=\frac{i}{2\pi i}\oint\left(\psi^{12}-\psi^{34}\right)dz \tag{8.87}$$

is conserved, with

$$[J,\psi_i^+]=\psi_i^+\quad\text{and}\quad[J,\psi_i^-]=-\psi_i^-\qquad(i=1,2)\,, \tag{8.88}$$

while the spin fields have $U(1)$-charge

$$[J, S^{\dot 1}] = S^{\dot 1}, \quad [J, S^{\dot 2}] = -S^{\dot 2} \quad \text{and} \quad [J, S^{\alpha}] = 0. \tag{8.89}$$

With our choice of chirality for the vertex operators only the spin fields with non-vanishing $U(1)$-eigenvalues will enter in the fields $\Psi^{(1)}$ and e_i. Consequently, $\Psi^{(1)}$ decomposes into eigenstates of the $U(1)$-charge, i.e. $\Psi^{(1)} \mapsto Psi^{(1)+} + Psi^{(1)-}$, in particular

$$\frac{\rho}{\sqrt{\alpha'}}\Psi^{(1)} = V = V^+ + V^- = c\mathbb{V}^+_{1/2}e^{-\phi} + c\mathbb{V}^-_{1/2}e^{-\phi}\,. \tag{8.90}$$

An analogous decomposition holds for e_i, while

$$\frac{\rho}{\sqrt{\alpha'}}X\Psi^{(1)} = XV = -c\mathbb{V}_1 + \frac{1}{4}\gamma\mathbb{V}^+_{1/2} + \frac{1}{4}\gamma\mathbb{V}^-_{1/2}\,; \tag{8.91}$$

(and analogously for Xe_i), where $\mathbb{V}_1$ is uncharged ($[J, \mathbb{V}_1] = 0$), both for the twisted and untwisted sector. Upon substitution of this decomposition into (8.69) we get

$$\begin{aligned}
\langle X \circ X\, e_i, m_2(Q^{-1}m_2(\Psi^{(1)}, \Psi^{(1)}), \xi_0\Psi^{(1)})\rangle = &\langle X \circ X\, e_i, m_2(Q^{-1}m_2(\Psi^{(1)+}, \Psi^{(1)+}), \xi_0\Psi^{(1)-})\rangle_L \\
&+ \langle X \circ X\, e_i, m_2(Q^{-1}m_2(\Psi^{(1)-}, \Psi^{(1)-}), \xi_0\Psi^{(1)+})\rangle_L \\
&- \langle X \circ X\, e_i, m_2(Q^{-1}m_2(\Psi^{(1)+}, \xi_0\Psi^{(1)-}), \Psi^{(1)+})\rangle_L \\
&+ \langle X \circ X\, e_i, m_2(Q^{-1}m_2(\xi_0\Psi^{(1)+}, \Psi^{(1)-}), \Psi^{(1)-})\rangle_L \\
&+ \langle X \circ X\, e_i, m_2(Q^{-1}m_2(\xi_0\Psi^{(1)-}, \Psi^{(1)+}), \Psi^{(1)+})\rangle_L \\
&- \langle X \circ X\, e_i, m_2(Q^{-1}m_2(\Psi^{(1)-}, \xi_0\Psi^{(1)+}), \Psi^{(1)-})\rangle_L
\end{aligned} \tag{8.92}$$

and analogously for the second term in (8.69). Here we have used the conservation of J and that, while the J-charge of $X \circ X\, e_i$ can take all values form -3 to 3, in order to saturate the ghost zero-modes only the he J-charge ± 1 part of $X \circ X\, e_i$ can contribute to the correlator. In addition the r.h.s. of (8.92) is expressed in the large Hilbert space. The position of the ξ-zero mode is correlated with relative sign of each term. Next we write $X \circ X\, e_i = Q\xi \circ X\, e_i$ and bring the BRST charge Q to the other side through BPZ-conjugation. The only contribution comes from the commutator $\{Q, Q^{-1}\}$ since, whenever Q hits a ξ, the J-charge does not add up to zero or the ghost zero-modes are not saturated. Adding in the second term on the r.h.s. of (8.69) we are left with

$$\begin{aligned}
-\frac{1}{2}B =&\langle \xi \circ X\, e_i, m_2(P_0 m_2(\Psi^{(1)+}, \Psi^{(1)+}), \xi\Psi^{(1)-}) - m_2(\Psi^{(1)+}, P_0 m_2(\Psi^{(1)+}, \xi\Psi^{(1)-}))\rangle_L \\
&+ \langle \xi \circ X\, e_i, m_2(P_0 m_2(\Psi^{(1)-}, \Psi^{(1)-}), \xi\Psi^{(1)+}) - m_2(\Psi^{(1)-}, P_0 m_2(\Psi^{(1)-}, \xi\Psi^{(1)+}))\rangle_L \\
&- \langle \xi \circ X\, e_i, m_2(P_0 m_2(\Psi^{(1)+}, \xi\Psi^{(1)-}), \Psi^{(1)+}) + m_2(\Psi^{(1)+}, P_0 m_2(\xi\Psi^{(1)-}, \Psi^{(1)+}))\rangle_L \\
&+ \langle \xi \circ X\, e_i, m_2(P_0 m_2(\xi\Psi^{(1)+}, \Psi^{(1)-}), \Psi^{(1)-}) - m_2(\xi\Psi^{(1)+}, P_0 m_2(\Psi^{(1)-}, \Psi^{(1)-}))\rangle_L \\
&+ \langle \xi \circ X\, e_i, m_2(P_0 m_2(\xi\Psi^{(1)-}, \Psi^{(1)+}), \Psi^{(1)+}) - m_2(\xi\Psi^{(1)-}, P_0 m_2(\Psi^{(1)+}, \Psi^{(1)+}))\rangle_L \\
&- \langle \xi \circ X\, e_i, m_2(P_0 m_2(\Psi^{(1)-}, \xi\Psi^{(1)+}), \Psi^{(1)-}) + m_2(\Psi^{(1)-}, P_0 m_2(\xi\Psi^{(1)+}, \Psi^{(1)-}))\rangle_L\,,
\end{aligned} \tag{8.93}$$

where we have used the associativity of m_2. With the help of the cyclic property (F.11) of m_2 this can be recast into

$$-\frac{1}{2}B =\langle P_0 m_2(\Psi^{(1)+},\Psi^{(1)+}),[\xi\Psi^{(1)-},\xi X e_i]_{m_2}\rangle_L + \langle P_0 m_2(\Psi^{(1)-},\Psi^{(1)-}),[\xi\Psi^{(1)+},\xi X e_i]_{m_2}\rangle_L \\ -\langle P_0[\Psi^{(1)+},\xi\Psi^{(1)-}]_{m_2},[\Psi^{(1)+},\xi X e_i]_{m_2}\rangle_L + \langle P_0[\xi\Psi^{(1)+},\Psi^{(1)-}]_{m_2},[\Psi^{(1)-},\xi X e_i]_{m_2}\rangle_L \,. \tag{8.94}$$

The four contributions to the r.h.s. of (8.94) can be read-off from eqns. (8.54-8.64). Explicitly we have

$$\begin{aligned}
\left(\frac{\rho}{\sqrt{\alpha'}}\right)^2 P_0 m_2(\Psi^{(1)\pm},\Psi^{(1)\pm}) &= P_0 m_2(V^\pm,V^\pm) = -\frac{1}{4}\frac{g_{YM}^2}{\alpha'}c\partial c e^{-2\phi}\bar\eta_\mp^{\mu\nu}M_{\mu\nu}\psi_{12}^{\pm\pm}\,,\\
\left(\frac{\rho}{\sqrt{\alpha'}}\right) P_0[\xi\Psi^{(1)\pm},\xi X e_i^\pm]_{m_2} &= P_0[\xi V^\pm,\xi X e_i^\pm]_{m_2} = \frac{1}{8}\frac{g_{YM}^2}{\alpha'}\xi_0 c\bar\eta_\mp^{\rho\sigma}N_{\rho\sigma}\psi_{12}^{\pm\pm}+\dots\,,\\
\left(\frac{\rho}{\sqrt{\alpha'}}\right)^2 P_0[\Psi^{(1)\pm},\xi\Psi^{(1)\mp}]_{m_2} &= P_0[V^\pm,\xi V^\mp]_{m_2} = \pm\frac{i}{4}\frac{g_{YM}^2}{\alpha'}\partial\xi c\partial c e^{-2\phi}\bar\eta_3^{\mu\nu}M_{\mu\nu}+\dots\,,\\
\left(\frac{\rho}{\sqrt{\alpha'}}\right) P_0[\Psi^{(1)\pm},\xi X e_i^\mp]_{m_2} &= P_0[V^\pm,\xi X e_i^\mp]_{m_2} = \mp\frac{i}{8}c:\xi\eta:\bar\eta_3^{\rho\sigma}M_{\rho\sigma}\,,
\end{aligned} \tag{8.95}$$

where $\bar\eta_\pm^{\mu\nu}=\bar\eta_1^{\mu\nu}\pm i\bar\eta_2^{\mu\nu}$ are defined in terms of the 't Hooft symbols, and $M_{\mu\nu}$ and $N_{\rho\sigma}$ are matrices defined above. The $\dots$ denote terms that vanish upon insertion in the inner product in (8.94). Putting all together we end up with

$$\left(\frac{\rho}{\sqrt{\alpha'}}\right)^3 B = \frac{1}{8}\mathrm{Tr}\,[M^a N^a]\,, \tag{8.96}$$

where the matrices M^a and N^a are as

$$\begin{aligned}
M^a &= \bar\eta^a_{\mu\nu}\left([A^\mu,A^\nu]+\frac{1}{2}w_{\dot\alpha}(\bar\sigma^{\mu\nu})^{\dot\alpha\dot\beta}\bar w_{\dot\beta}\right)\,,\\
N^a &= \bar\eta^a_{\mu\nu}\left([A^\mu,B^\nu]+\frac{1}{4}w_{\dot\gamma}(\bar\sigma^{\mu\nu})^{\dot\gamma\dot\delta}\bar v_{\dot\delta}+\frac{1}{4}v_{\dot\gamma}(\bar\sigma^{\mu\nu})^{\dot\gamma\dot\delta}\bar w_{\dot\delta}\right)\,.
\end{aligned} \tag{8.97}$$

Notice that this reproduces (8.83), however, with the important difference that $\mathrm{Tr}\,(2M_{\mu\nu}N^{\mu\nu})$ is replaced by $\mathrm{Tr}\,(M^aN^a)$. This can be seen clearly if we rewrite, assuming the ADHM constraints,

$$\begin{aligned}
\mathrm{Tr}(M_{\mu\nu}N^{\mu\nu}) =&\frac{1}{2}\mathrm{Tr}\,(M^aN^a) - \mathrm{Tr}\left[[A_\mu,A_\nu]\left(\frac{1}{4}w_{\dot\gamma}(\bar\sigma^{\mu\nu})^{\dot\gamma\dot\delta}\bar v_{\dot\delta}+\frac{1}{4}v_{\dot\gamma}(\bar\sigma^{\mu\nu})^{\dot\gamma\dot\delta}\bar w_{\dot\delta}\right)\right]+\\
&-\mathrm{Tr}\left([A_\mu,B_\nu]\,\frac{1}{2}w_{\dot\alpha}(\bar\sigma_{\mu\nu})^{\dot\alpha\dot\beta}\bar w_{\dot\beta}\right)\,.
\end{aligned} \tag{8.98}$$

This means that the contributions coming from (8.77) in the twisted sector have the effect of exactly cancelling all the terms in $\mathrm{Tr}[M_{\mu\nu}N^{\mu\nu}]$ that are not anti-self-dual, leaving only terms proportional to $\mathrm{Tr}\,(M^a N^a)$. It is then possible, in agreement with [68, 66], to set M^a to zero assuming the ADHM constraints (7.27)

$$\bar{\eta}_a^{\mu\nu}\left([a_\mu, a_\nu] + \frac{1}{2}\bar{w}_{\dot{\alpha}}(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}}w_{\dot{\beta}}\right) = 0\,, \tag{8.99}$$

and with a suitable choice of the matrices A_μ, that is

$$A_\mu = \frac{\rho}{\sqrt{2}}\sigma_\mu = \frac{\rho}{\sqrt{2}}(\,\mathbb{1}, -i\vec{\tau})\,. \tag{8.100}$$

As discussed in section 8.2, this zero momentum gluon contributes to the instanton profile at order ρ (see (8.39)) but not at order ρ^2. Furthermore, it is in principle possible to compute all contributions to the instanton profile at order $\mathcal{O}(\rho^3)$, inverting (8.40). The explicit calculation is, however, quite involved.

Conclusions

The main goal of this thesis has been to explore the moduli space of bound states of D-branes, both in bosonic and superstring theory, and to understand its relation to the moduli space of instantons in Yang-Mills theories. Bound states of D-branes were considered, in particular systems involving instantonic D(-1) branes. Given the fact that string theory is a conformal field theory on the worldsheet, this results in the study of exactly marginal deformations of a given background.

We studied in detail the conformal field theory of twist fields, and provided a bosonization for them, thanks to their relation to orbifold theories. The OPE of twist fields allowed to study the bound state of D-instantons and D15 branes in bosonic string theory. We found that the blow-up of the size of D-instantons is obstructed at second order. As a consequence, the profile of four dimensional Yang-Mills instanton can not be recovered from bosonic string theory.

Turning then to superstring theory, a marginal operator corresponding to the blow-up of the size of a D-instanton is present only for a bound state of branes with codimension 4, in particular for the D3-D(-1) system. The spectrum of open strings populating this bound state is, in the low-energy limit, equivalent to the maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory in four dimensions, and field configurations of pointlike instantons can be recovered in this limit.

We studied the possibility to extend this connection to finite-size (not pointlike) D(-1) branes inside a D3 background, constructing them as marginal deformations of the worldsheet theory of pointlike D(-1) branes. The standard worldsheet approach can not be applied here for two reasons. First, the computation of the instanton profile is an off-shell problem in string theory; second, there are subtleties with the integration over odd moduli in super moduli space which are not captured by the worldsheet description. We dealt with these problems by working with a second quantization approach, the A_∞ superstring field theory. We studied the deformation corresponding to the blow-up mode of the size of a D(-1) brane inside a D3 background. This deformation was found to be marginal at second order in size, unlike for bosonic string theory, while at third order in addition to the ADHM constraints an addition zero-momentum gluon is required for marginality. Marginality at second order allowed us to derive the instanton profile, including α'-corrections.

An interesting question to explore is whether it is possible to find more generic hermitian string fields that are solutions to the equations of motion not satisfying the ADHM constraints. We were not able to find any such solutions but cannot exclude them on general grounds at this point.

While we considered the specific case of the D(−1)-D3 brane bound state in this paper, our approach applies equally well to generic Dp-D($p+4$) brane bound states. Furthermore, in this work we focused on open strings and D-branes, hence we had to deal with boundary twist operators. It would be interesting to extend the analysis to bulk twist fields, which are naturally related to closed strings and orbifolds; an interesting extension to our work would concern the blow-up of orbifold singularities in closed superstring field theory [71, 72, 73, 74].

Appendix A

Notation and conventions

Notation for indices

In this work we use many indices with different meanings. The most used ones are the following:

- $d = 4$ vector indices: $\mu, \nu = 0, \ldots, 3$;
- $d = 6$ vector indices: $a, b = 4, \ldots, 9$;
- Chiral and anti-chiral spinor indices in $d = 4$: α and $\dot{\alpha}$;
- Spinor indices in $d = 6$: ^{A} and ${}_{A}$ in the fundamental and anti-fundamental of $SU(4) \simeq SO(6)$;
- D3 indices: $u, v = 1, \ldots, N$;
- D(-1) indices: $i, j = 1, \ldots, k$;
- $SU(2)$ color indices: $c, d = 1, 2, 3$.

d=4 Clifford algebra and spinors

In $d = 4$ we can either deal with the Euclidean (SO(4)) or Minkowskian (SO(1,3)) Lorentz group; its Clifford algebra is defined by $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,\mathbb{1}$, where the metric η has signature $(+,+,+,+)$ or $(-,+,+,+)$ respectively. Let us consider the Pauli matrices τ^c

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1}$$

Gamma matrices in four dimensions can be expressed in terms of the matrices $(\sigma^\mu)_{\alpha\dot{\beta}}$ and $(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}$ in the following way:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \tag{A.2}$$

where σ^μ and $\bar\sigma^\mu$ are defined in terms of the Pauli matrices, but in a different way for Euclidean and Minkowski space:

$$\begin{aligned} \sigma^\mu &= (\mathbb{1}, -i\vec\tau) \quad \text{and} \quad \bar\sigma^\mu = (\mathbb{1}, i\vec\tau) \qquad \text{(Euclidean)} \\ \sigma^\mu &= (\mathbb{1}, \vec\tau) \quad \text{and} \quad \bar\sigma^\mu = (-\mathbb{1}, \vec\tau) \qquad \text{(Minkowski)} \end{aligned} \tag{A.3}$$

They satisfy the appropriate Clifford algebra $\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu = 2\eta^{\mu\nu}\,\mathbb{1}$.

It is convenient to divide every Dirac spinor into its two Weyl components as follows:

$$\psi = \begin{pmatrix} \psi_\alpha \\ \psi^{\dot\alpha} \end{pmatrix} \tag{A.4}$$

We raise and lower spinor indices contracting always with the second index of the anti-symmetric ε tensor:

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta \quad , \quad \psi_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\psi^{\dot\beta}\,, \tag{A.5}$$

with $\varepsilon^{12} = \varepsilon_{12} = -\varepsilon^{\dot1\dot2} = -\varepsilon_{\dot1\dot2} = 1$. Therefore, we have also

$$\psi_\alpha = \psi^\beta\varepsilon_{\beta\alpha} \quad , \quad \psi^{\dot\alpha} = \psi_{\dot\beta}\varepsilon^{\dot\beta\dot\alpha}\,. \tag{A.6}$$

Depending on the metric η, the σ matrices behave differently under complex conjugation. In Euclidean space one has

$$(\sigma^\mu)^*_{\alpha\dot\beta} = -(\sigma^\mu)^{\alpha\dot\beta} \qquad \text{and} \qquad (\bar\sigma^\mu)^{\dot\alpha\beta *} = -(\bar\sigma^\mu)_{\dot\alpha\beta}\,, \tag{A.7}$$

while in Minkowski space one type of index gets changed into the other:

$$(\sigma^\mu)^*_{\alpha\dot\beta} = (\sigma^\mu)_{\beta\dot\alpha}\,. \tag{A.8}$$

Both in Minkowskian and in Euclidean case we have the following important relation:

$$(\sigma^\mu)_{\alpha\dot\beta} = (\bar\sigma^\mu)_{\dot\beta\alpha}\,. \tag{A.9}$$

Euclidean d=4 Clifford algebra and 't Hooft symbols

In the following we focus only on the Euclidean case, because it is the one we are interested in when dealing with instantons. The SO(4) generators are defined in terms of σ matrices in the following way:

$$\sigma_{\mu\nu} = \frac{1}{2}(\sigma_\mu\bar\sigma_\nu - \sigma_\nu\bar\sigma_\mu) \quad , \quad \bar\sigma_{\mu\nu} = \frac{1}{2}(\bar\sigma_\mu\sigma_\nu - \bar\sigma_\nu\sigma_\mu)\,; \tag{A.10}$$

These matrices satisfy self-duality or anti-self-duality conditions respectively, in particular:

$$\sigma_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\sigma_{\rho\sigma} \quad , \quad \bar\sigma_{\mu\nu} = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\bar\sigma_{\rho\sigma} \tag{A.11}$$

The mapping between a self-dual (or anti-self-dual) SO(4) tensor into the corresponding adjoint representation of SU(2) is given in terms of the 't Hooft symbols as follows:

$$(\sigma_{\mu\nu})_\alpha{}^\beta = i\eta^c_{\mu\nu}(\tau^c)_\alpha{}^\beta \quad , \quad (\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = i\bar\eta^c_{\mu\nu}(\tau^c)^{\dot\alpha}{}_{\dot\beta}\,. \tag{A.12}$$

An explicit representations of these symbols is given by:

$$\begin{aligned} \eta^c_{\mu\nu} &= \bar{\eta}^c_{\mu\nu} = \varepsilon_{c\mu\nu} \,, \qquad \mu,\nu \in \{1,2,3\} \\ \eta^c_{0\nu} &= -\bar{\eta}^c_{0\nu} = \delta^c_\nu \,, \\ \eta^c_{\mu\nu} &= -\eta^c_{\nu\mu} \,, \\ \bar{\eta}^c_{\mu\nu} &= -\bar{\eta}^c_{\nu\mu} \,. \end{aligned} \tag{A.13}$$

Many properties of these symbols can be found in the literature. In particular the symbols $\eta^c_{\mu\nu}$ and $\bar{\eta}^c_{\mu\nu}$ are self-dual and anti-self-dual respectively.

Appendix B

Relevant operators, OPE's and bosonization

In the calculation of amplitudes in a conformal field theory it is important to know the operator product expansion (OPE) of primary fields $\mathcal{O}(z_1)\mathcal{O}(z_2)$. In this appendix we focus on the properties of the operators appearing in superstring theory. First of all, let us consider the primary fields $\partial X_\mu(z)$, with conformal weight 1. The OPE of two of them is

$$\partial X^\mu(z)\partial X^\nu(w) = -\frac{\alpha'}{2}\frac{\delta^{\mu\nu}}{(z-w)^2} + \dots\,, \tag{B.1}$$

where $\dots$ indicate regular terms. For the spinors ψ^μ the OPE is given by

$$\psi^\mu(z)\psi^\nu(w) = \frac{\delta^{\mu\nu}}{z-w} + \psi^{\mu\nu} + \dots\,. \tag{B.2}$$

The presence of D3 branes breaks $SO(10)$ to $SO(4)\times SO(6)$ (we consider the euclidean theory); therefore the ten dimensional spin fields $S^{\mathcal{A}}$ and $S^{\dot{\mathcal{A}}}$ can be expressed in terms of the spin fields in 4 and 6 dimensions as follows:

$$\begin{aligned} S^{\mathcal{A}} &\longrightarrow (S_\alpha S^A, S^{\dot\alpha}S_A)\,,\\ S^{\dot{\mathcal{A}}} &\longrightarrow (S_\alpha S_A, S^{\dot\alpha}S^A)\,, \end{aligned} \tag{B.3}$$

where S_α and $S^{\dot\alpha}$ are $SO(4)$ spin fields of even and odd chirality respectively, and S^A and S_A are $SO(6)$ spin fields of even and odd chirality respectively. Spin fields in $d=4$ can be bosonized with exponents:

$$\begin{aligned} \lambda_\alpha &= \left(\frac{1}{2},\frac{1}{2}\right) \quad \text{or} \quad \left(-\frac{1}{2},-\frac{1}{2}\right)\,,\\ \lambda_{\dot\alpha} &= \left(\frac{1}{2},-\frac{1}{2}\right) \quad \text{or} \quad \left(-\frac{1}{2},\frac{1}{2}\right)\,. \end{aligned} \tag{B.4}$$

Their OPE contains branch cuts; explicitly we have

$$
\begin{aligned}
S^{\dot\alpha}(z)S_\beta(w) &= \frac{1}{\sqrt{2}}(\bar\sigma^\mu)^{\dot\alpha}{}_\beta\psi_\mu(w)+\dots\,,\\
S^{\dot\alpha}(z)S^{\dot\beta}(w) &= -\frac{\varepsilon^{\dot\alpha\dot\beta}}{(z-w)^{1/2}}+\frac{1}{4}(z-w)^{1/2}(\bar\sigma_{\mu\nu})^{\dot\alpha\dot\beta}\psi^{\mu\nu}+\dots\,,\\
S_\alpha(z)S_\beta(w) &= \frac{\varepsilon_{\alpha\beta}}{(z-w)^{1/2}}-\frac{1}{4}(z-w)^{1/2}(\sigma_{\mu\nu})_{\alpha\beta}\psi^{\mu\nu}+\dots\,.
\end{aligned}
\tag{B.5}
$$

All these expressions can be derived using the bosonization of the spin fields; furthermore, one can also derive the following OPE involving spinors and spin fields:

$$
\begin{aligned}
\psi_\mu(z)S^{\dot\alpha}(w) &= \frac{1}{\sqrt{2}}\frac{(\bar\sigma_\mu)^{\dot\alpha\beta}S_\beta(w)}{(z-w)^{1/2}}+\dots\,,\\
\psi_{\mu\nu}(z)S^{\dot\alpha}(w) &= -\frac{1}{2}\frac{(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}S^{\dot\beta}(w)}{z-w}+\dots\,.
\end{aligned}
\tag{B.6}
$$

From these OPE one can easily compute some three-point functions, for example:

$$
\begin{aligned}
\langle S^{\dot\alpha}(z_1)\psi_\mu(z_2)S_\beta(z_3)\rangle &= \frac{1}{\sqrt{2}}\frac{(\bar\sigma_\mu)^{\dot\alpha}{}_\beta}{z_{12}^{1/2}z_{23}^{1/2}}\,,\\
\langle S^{\dot\alpha}(z_1)\psi_{\mu\nu}(z_2)S^{\dot\beta}(z_3)\rangle &= -\frac{1}{2}(\bar\sigma_{\mu\nu})^{\dot\alpha\dot\beta}\frac{z_{13}^{1/2}}{z_{12}z_{23}}\,,
\end{aligned}
\tag{B.7}
$$

where we have introduced the notation $z_{ij}=z_i-z_j$. Other details on the spin fields and their bosonization can be found in [16, 21].

Regarding the twist operators, we have to deal with non trivial OPE with the fields ∂X^μ and $e^{ik\cdot X}$. Remembering that the field $\Delta(z)$ is made of four twist operators ($\Delta(z)=\sigma^0\sigma^1\sigma^2\sigma^3(z)$), it is sufficient to know the behavior of one field $\sigma^\mu(z)$, which has conformal dimension $1/16$. We have the following relevant OPE, involving also the so-called excited twist field $\sigma'^\mu(z)$, with conformal dimension $9/16$ [1]:

$$
\begin{aligned}
\sigma^\mu(z)\bar\sigma^\nu(w) &= \frac{\delta^{\mu\nu}}{(z-w)^{1/8}}+\dots\,,\\
\sqrt{\frac{2}{\alpha'}}i\partial X^\mu(z)\sigma^\nu(w) &= \frac{\delta^{\mu\nu}\sigma'^\nu(w)}{(z-w)^{1/2}}+\dots\,,\\
\sqrt{\frac{2}{\alpha'}}i\partial X^\mu(z)\sigma'^\nu(w) &= \frac{1}{2}\frac{\delta^{\mu\nu}\sigma^\nu(w)}{(z-w)^{3/2}}+\frac{2\delta^{\mu\nu}\partial\sigma^\nu(w)}{(z-w)^{1/2}}+\dots
\end{aligned}
\tag{B.8}
$$

where we do not sum over equal indices. From these OPE one can derive the three-point function

$$
\langle\bar\Delta(z_1)e^{-ik\cdot X(z_2)}\Delta(z_3)\rangle = \frac{e^{-ik\cdot x_0}}{(z_{13})^{1/2-\alpha' k^2/2}(4z_{12}z_{23})^{\alpha' k^2/2}}\,,
\tag{B.9}
$$

where x_0^μ is the zero-mode of the field $X^\mu(z)$. More properties of these twist operators can be found for example in [1, 17].

In superstring theory one has also to deal with ghosts and superghosts, which are characterized by the OPE relations

$$\begin{aligned}
&b(z)c(w) \sim c(z)b(w) = \frac{1}{z-w} + \dots\,, \\
&c(z)c(w) = -(z-w)c\partial c(w) - \frac{1}{2}(z-w)^2 c\partial^2 c(w) + \dots\,, \\
&\beta(z)\gamma(w) \sim -\gamma(z)\beta(w) = -\frac{1}{z-w} + \dots\,.
\end{aligned} \tag{B.10}$$

These ghosts can be bosonized in the following way

$$\begin{aligned}
&b = e^{-\sigma}\,, \qquad c = e^{\sigma}\,, \\
&\beta = e^{-\phi}\partial\xi = e^{-\phi}e^{\chi}\partial\chi\,, \qquad \gamma = \eta e^{\phi} = e^{-\chi}e^{\phi}\,,
\end{aligned} \tag{B.11}$$

with the following OPE relations

$$\begin{aligned}
&\sigma(z)\sigma(w) = \log(z-w) + \dots\,, \\
&\phi(z)\phi(w) = -\log(z-w) + \dots\,, \\
&\chi(z)\chi(w) = \log(z-w) + \dots\,, \\
&\xi(z)\eta(w) = \eta(z)\xi(w) \sim \frac{1}{z-w} + \dots\,, \\
&e^{-\phi}(z)e^{\phi}(w) \sim e^{\phi}(z)e^{-\phi}(w) = (z-w) + \dots\,, \\
&e^{-\phi}(z)e^{-\phi}(w) = \frac{1}{z-w}e^{-2\phi}(w) - \partial\phi e^{-2\phi}(w) + \dots\,.
\end{aligned} \tag{B.12}$$

The relevant two- and three-point functions used in this work are the following ones:

$$\begin{aligned}
&\langle c(z_1)c(z_2)c(z_3)\rangle = z_{12}z_{23}z_{13}\,, \\
&\langle e^{-\phi(z_1)}e^{-\phi(z_2)}\rangle = \frac{1}{z_{12}}\,, \\
&\langle c\partial c e^{-2\phi}(z)c(w)\rangle = -(z-w)^2\,.
\end{aligned} \tag{B.13}$$

Non-Primary Operators

In chapter 8 we have to deal with some operators that are not primary. In particular we encounter $:\xi\eta: e^{\phi}\widetilde{\mathbb{V}}_{1/2}$ and $\partial(e^{\phi}\widetilde{\mathbb{V}}_{1/2})$. The normal ordered product is defined in terms of the OPE as

$$:\xi\eta:(w) = \oint \frac{dx}{2\pi i}\frac{\xi(x)\eta(w)}{x-w}; \tag{B.14}$$

ξ and η are primaries, thus we can compute

$$T(z):\xi\eta:(0) = T(z)\oint \frac{dx}{2\pi i}\frac{\xi(x)\eta(0)}{x} = -\frac{1}{z^3} + \frac{:\xi\eta:(0)}{z^2} + \frac{\partial:\xi\eta:(0)}{z} + \dots \tag{B.15}$$

The presence of a cubic pole shows that $:\xi\eta:$ is not a primary operator; from this we derive

$$T(z):\xi\eta: e^{\phi}\widetilde{\mathbb{V}}_{1/2}(0) = T(z)\oint \frac{dx}{2\pi i}\frac{\xi(x)\eta(0)}{x} = -\frac{e^{\phi}\widetilde{\mathbb{V}}_{1/2}(0)}{z^3} + \frac{\partial(:\xi\eta: e^{\phi}\widetilde{\mathbb{V}}_{1/2})(0)}{z} + \dots \tag{B.16}$$

Similarly, for $\partial(e^{\phi}\widetilde{\mathbb{V}}_{1/2})$ we get

$$T(z)\partial(e^{\phi}\widetilde{\mathbb{V}}_{1/2})(0) = -2\frac{e^{\phi}\widetilde{\mathbb{V}}_{1/2}(0)}{z^3} + \frac{\partial^2(e^{\phi}\widetilde{\mathbb{V}}_{1/2})(0)}{z} + \dots \tag{B.17}$$

Appendix C

Four-point function of twist fields

In this appendix we follow the procedure of [17] in order to compute the four-point function of twist fields

$$G(z_i) = \langle \bar{\sigma}(z_1)\sigma(z_2)\bar{\sigma}(z_3)\sigma(z_4)\rangle \,. \tag{C.1}$$

As in appendix 4.3.2, we have two Dirichlet intervals $[z_4, z_1]$ (which includes the point at infinity) and $[z_2, z_3]$. We consider the closed cycle C, that encircles the point z_1 and z_2. We assume furthermore that the cycle is symmetric with respect to the real axis. We have that

$$\oint_C dz\, \mathsf{j}(z) = \int_{C^>} dz\, j(z) - \int_{C_<} dz\, \bar{j}(z) = \int_{C^>} dz\, i(\partial + \bar{\partial})X(z,\bar{z}) = i\delta x_0 \,, \tag{C.2}$$

where δx_0 is the difference between the zero modes of $X(z,\bar{z})$ on the two Dirichlet intervals. Consider now a new correlation function

$$\Gamma(w, z_i) = \langle \mathsf{j}(w)\bar{\sigma}(z_1)\sigma(z_2)\bar{\sigma}(z_3)\sigma(z_4)\rangle \,. \tag{C.3}$$

Integrating around the circle C we get the so-called block condition

$$\oint \frac{dw}{2\pi i}\, \Gamma(w, z_i) = pG(z_i) \,, \tag{C.4}$$

where $p = \delta x_0/2\pi$. Considering the OPE defining the twist fields, we can use the following Ansatz for Γ:

$$\Gamma(w, z_i) = [(w-z_1)(w-z_2)(w-z_3)(w-z_4)]^{-1/2} A(z_i) \,, \tag{C.5}$$

where $A(z_i)$ does not depend on w. Performing now the limit $w \to z_2$, and using again the OPE, we find

$$\lim_{z\to z_2} \Gamma(w, z_i) = \frac{1}{(w-z_2)^{1/2}} G^{(2)}(z_i) + \dots \,, \tag{C.6}$$

where $G^{(2)}(z_i) = \langle \bar{\sigma}(z_1)\sigma'(z_2)\bar{\sigma}(z_3)\sigma(z_4)\rangle$ and $\dots$ represent terms of order $(z-w)^{1/2}$. On the other hand (C.5) implies

$$\lim_{z\to z_2} \Gamma(w, z_i) = \frac{1}{(w-z_2)^{1/2}} \frac{A(z_i)}{\sqrt{z_{21}z_{32}z_{42}}} + \dots \,. \tag{C.7}$$

Comparing the two equations gives $A(z_i) = \sqrt{z_{21}z_{32}z_{42}}\, G^{(2)}(z_i)$. Consider now another correlation function, namely

$$\Gamma^{(2)}(w, z_i) = \langle \mathsf{j}(w)\bar{\sigma}(z_1)\sigma'(z_2)\bar{\sigma}(z_3)\sigma(z_4)\rangle \,. \tag{C.8}$$

Integrating over w around the cycle C we obtain another block condition, that reads

$$\oint \frac{dw}{2\pi i}\,\Gamma^{(2)}(w, z_i) = pG^{(2)}(z_i)\,, \tag{C.9}$$

Considering now the local properties when w approaches the insertion points z_i, the proper Ansatz for $\Gamma^{(2)}$ is

$$\Gamma^{(2)}(w, z_i) = [(w-z_1)(w-z_2)(w-z_3)(w-z_4)]^{-1/2}\left(\frac{B(z_i)}{w-z_2} + C(z_i)\right) . \tag{C.10}$$

Expanding this for $w \to z_2$ we find

$$\begin{aligned}&\lim_{z\to z_2}\Gamma^{(2)}(w, z_i) = \\ &\quad = \frac{1}{\sqrt{(w-z_2)z_{21}z_{32}z_{42}}}\left(\frac{B(z_i)}{w-z_2} + C(z_i) - \frac{1}{2}B(z_i)\left(\frac{1}{z_{21}} + \frac{1}{z_{23}} + \frac{1}{z_{24}}\right)\right) + \ldots\end{aligned} \tag{C.11}$$

On the other hand, the OPE implies that

$$\lim_{z\to z_2}\Gamma^{(2)}(w, z_i) = \frac{1}{2(w-z_2)^{3/2}}G(z_i) + \frac{2}{(w-z_2)^{1/2}}\partial_{z_2}G(z_i) + \ldots \tag{C.12}$$

Comparing the last two equations we find closed expression for $B(z_i)$ and $C(z_i)$:

$$\begin{aligned}B(z_i) &= \frac{1}{2}\sqrt{z_{21}z_{32}z_{42}}\, G(z_i)\,,\\ C(z_i) &= \sqrt{z_{21}z_{32}z_{42}}\left(\frac{1}{4}\left(\frac{1}{z_{21}} + \frac{1}{z_{23}} + \frac{1}{z_{24}}\right) + 2\frac{\partial}{\partial x_2}\right) G(z_i)\,.\end{aligned} \tag{C.13}$$

Finally we use the relation

$$\mathcal{K}(z_i) = \oint_C dw[(w-z_1)(w-z_2)(w-z_3)(w-z_4)]^{-1/2} = \frac{4i}{\sqrt{z_{31}z_{42}}}K(\eta)\,, \tag{C.14}$$

where $\eta = z_{43}z_{21}/(z_{42}z_{31})$, and $K(\eta)$ is the complete elliptic integral of the first kind. Using this we can rewrite the two block conditions as

$$\begin{aligned}&A(z_i)\mathcal{K}(z_i) = 2\pi i p G(z_i)\,,\\ &\left(C(z_i) + 2B(z_i)\frac{\partial}{\partial x_2}\right)\mathcal{K}(z_i) = 2\pi i p G^{(2)}(z_i)\,.\end{aligned} \tag{C.15}$$

Inserting the relations we found for A, B and C we finally find a differential equation for the original correlation function:

$$\mathcal{K}^{3/2}(z_i)\frac{\partial}{\partial x_2}\left[(z_{21}z_{32}z_{42})^{1/8}\mathcal{K}^{1/2}(z_i)G(z_i)\right] = -2\pi^2p^2(z_{21}z_{32}z_{42})^{-7/8}G(z_i)\,, \tag{C.16}$$

whose solution is

$$G(z_i) \propto \left(\frac{z_{31}z_{42}}{z_{21}z_{41}z_{32}z_{43}}\right)^{1/8} \frac{1}{\sqrt{K(\eta)}} \exp\left(\frac{i(\delta x_0)^2}{8\pi}\tau(\eta)\right) . \tag{C.17}$$

Here $\tau(\eta)$ is given by $\tau(\eta) = iK(1-\eta)/K(\eta)$. The overall normalization factor can be fixed using the OPE of twist fields. Knowing that $\bar{\sigma}(z)\sigma(w) \sim (z-w)^{-1/8} + \dots$, we have to require that

$$\lim_{z_1 \to z_2} G(z_i)(z_1 - z_2)^{1/8} = (z_3 - z_4)^{-1/8}. \tag{C.18}$$

This fixes the overall factor to be $\sqrt{\frac{\pi}{2}}$. We summarize here the result for the four-point function of twist fields and the other correlators introduced for the derivation:

$$\begin{aligned}
G(z_i) =& \langle\bar{\sigma}(z_1)\sigma(z_2)\bar{\sigma}(z_3)\sigma(z_4)\rangle = \left(\frac{z_{31}z_{42}}{z_{21}z_{41}z_{32}z_{43}}\right)^{1/8} \sqrt{\frac{\pi}{2K(\eta)}} \exp\left(\frac{i(\delta x_0)^2}{8\pi}\tau(\eta)\right) ,\\
\Gamma(w, z_i) =& \langle \mathsf{j}(w)\bar{\sigma}(z_1)\sigma(z_2)\bar{\sigma}(z_3)\sigma(z_4)\rangle = \\
=& \frac{1}{4}\sqrt{\frac{\pi}{2P(w)}} \frac{(z_{31}z_{42})^{5/8}}{(z_{21}z_{41}z_{32}z_{43})^{1/8}} \frac{\delta x_0}{K(\eta)^{3/2}} \exp\left(\frac{i(\delta x_0)^2}{8\pi}\tau(\eta)\right) ,\\
G^{(2)}(z_i) =& \langle\bar{\sigma}(z_1)\sigma'(z_2)\bar{\sigma}(z_3)\sigma(z_4)\rangle = \\
=& \frac{1}{4}\sqrt{\frac{\pi}{2}} \left(\frac{z_{31}}{z_{21}z_{32}}\right)^{5/8} \left(\frac{z_{42}}{z_{41}z_{43}}\right)^{1/8} \frac{\delta x_0}{K(\eta)^{3/2}} \exp\left(\frac{i(\delta x_0)^2}{8\pi}\tau(\eta)\right) ,\\
\Gamma^{(2)}(w, z_i) =& \langle \mathsf{j}(w)\bar{\sigma}(z_1)\sigma'(z_2)\bar{\sigma}(z_3)\sigma(z_4)\rangle = \\
=& \sqrt{\frac{\pi}{2P(w)K(\eta)}} z_{31}^{9/8} \left(\frac{z_{42}}{z_{21}z_{32}}\right)^{5/8} \left(\frac{1}{z_{41}z_{43}}\right)^{1/8} \delta x_0 \cdot \\
& \cdot \left(\frac{w-z_1}{2(w-z_2)}\frac{z_{32}}{z_{31}} + \frac{E(\eta)}{2K(\eta)} - \frac{\delta^2}{16K(\eta)^2}\right) \exp\left(\frac{i(\delta x_0)^2}{8\pi}\tau(\eta)\right) .
\end{aligned} \tag{C.19}$$

Here $P(w)$ indicates the product $P(w) = (w-z_1)(w-z_2)(w-z_3)(w-z_4)$. Notice that the three correlation functions $\Gamma(w, z_i)$, $G^{(2)}(z_i)$ and $\Gamma^{(2)}(w, z_i)$ are proportional to the difference δx_0; therefore, when summed over the array, they give vanishing results. This means that the bosonized version of these correlation functions are zero, as one could derive by direct calculation in the Ω picture.

Appendix D

Correlation function with four twist fields and two currents

In this appendix we consider the Green's function in the presence of four twist fields. In particular, following [33] and [37], we compute (assuming that Im$z > 0$ and Im$w > 0$)

$$g(z, w, z_i) = \partial_z \partial_w G(z, w) = \frac{\langle j(z) j(w) \bar{\sigma}(z_1) \sigma(z_2) \bar{\sigma}(z_3) \sigma(z_4) \rangle}{\langle \bar{\sigma}(z_1) \sigma(z_2) \bar{\sigma}(z_3) \sigma(z_4) \rangle} . \tag{D.1}$$

Taking in consideration the OPE among j and the twist fields, we can make an Ansatz for g, which reads

$$g(z,w, z_i) = \\ = \frac{1}{2(z-w)^2} \left[\sqrt{\frac{(z-z_1)(w-z_2)(z-z_3)(w-z_4)}{(w-z_1)(z-z_2)(w-z_3)(z-z_4)}} + (z \leftrightarrow w) \right] + \frac{A(z_i)}{\sqrt{P(z)P(w)}} , \tag{D.2}$$

where $P(z) = (z-z_1)(z-z_2)(z-z_3)(z-z_4)$. We now use the definition of energy-momentum tensor

$$T(z) = \frac{1}{2} N(jj)(z) = \frac{1}{2} \left(\lim_{w \to z} j(z) j(w) - \frac{1}{(z-w)^2} \right) ; \tag{D.3}$$

this implies that

$$\frac{1}{2} \left(\lim_{w \to z} j(z) j(w) - \frac{1}{(z-w)^2} \right) = \frac{\langle T(z) \bar{\sigma}(z_1) \sigma(z_2) \bar{\sigma}(z_3) \sigma(z_4) \rangle}{G(z_i)} . \tag{D.4}$$

The direct calculation gives

$$\frac{\langle T(z) \bar{\sigma}(z_1) \sigma(z_2) \bar{\sigma}(z_3) \sigma(z_4) \rangle}{G(z_i)} = \frac{1}{2} \frac{A(z_i)}{P(z_i)} + \frac{1}{16} \left(\frac{1}{z-z_1} - \frac{1}{z-z_2} + \frac{1}{z-z_3} - \frac{1}{z-z_4} \right)^2 . \tag{D.5}$$

We can now use the OPE of T with $\sigma(z_2)$, in order to get the condition

$$\lim_{z \to z_2} \langle T(z) \bar{\sigma}(z_1) \sigma(z_2) \bar{\sigma}(z_3) \sigma(z_4) \rangle = \frac{G(z_i)}{16(z-z_2)^2} + \frac{\partial_{z_2} G(z_i)}{z-z_2} . \tag{D.6}$$

Performing the limit on (D.5) gives the equation

$$\partial_{z_2}\log(G(z_i)) = \frac{A(z_i)}{2z_{21}z_{23}z_{24}} - \frac{1}{8}\left(\frac{1}{z_{21}} + \frac{1}{z_{23}} - \frac{1}{z_{24}}\right), \tag{D.7}$$

from which we find the function $A(x)$:

$$\begin{aligned} &A(z_i) = \\ &= 2z_{21}z_{23}z_{24}\frac{\partial}{\partial_{z_2}}\left[\log\left((z_{21}z_{23}/z_{24})^{1/8}G(z_i)\right)\right] = z_{42}z_{31}\left(\frac{1-\eta}{2} - \frac{E(\eta)}{2K(\eta)} + \frac{\delta^2}{16K(\eta)^2}\right), \end{aligned} \tag{D.8}$$

where $\delta = \delta x_0$ and $K(x)$ and $E(x)$ are the complete elliptic integrals of the first and second kind respectively. Analogous equations can be found for z_1, z_3 and z_4. We can finally join the equations together, using the property (for any function $f(z_1, z_2, z_3, z_4)$)

$$\eta(1-\eta)\partial_\eta f(z_i) = \frac{1}{z_{42}z_{31}}\left(z_{12}z_{13}z_{14}\partial_{z_1} + z_{21}z_{23}z_{24}\partial_{z_2} + z_{31}z_{32}z_{34}\partial_{z_3} + z_{41}z_{42}z_{43}\partial_{z_4}\right) f(z_i)\,. \tag{D.9}$$

The final compact expression for $A(z_i)$ is

$$A(z_i) = 2z_{42}z_{31}\eta(1-\eta)\partial_\eta \log\left[\frac{1}{\sqrt{K(\eta)}}\exp\left(\frac{i\delta^2}{8\pi}\tau(\eta)\right)\right], \tag{D.10}$$

which was found in [33], in the case $\delta = 0$. Multiplying (D.2) by $G(z_i)$ we find the correlator

$$\begin{aligned} &\langle j(z)j(w)\bar{\sigma}(z_1)\sigma(z_2)\bar{\sigma}(z_3)\sigma(z_4)\rangle = \\ &\quad = \frac{G(z_i)}{2(z-w)^2}\left[\sqrt{\frac{(z-z_1)(w-z_2)(z-z_3)(w-z_4)}{(w-z_1)(z-z_2)(w-z_3)(z-z_4)}} + (z \leftrightarrow w)\right] + \\ &\quad + \frac{\sqrt{2\pi}}{\sqrt{P(z)P(w)}}\left(\frac{z_{31}z_{42}}{z_{21}z_{41}z_{32}z_{43}}\right)^{-7/8}\partial_\eta\left[\frac{1}{\sqrt{K(\eta)}}\exp\left(\frac{i\delta^2}{8\pi}\tau(\eta)\right)\right]. \end{aligned} \tag{D.11}$$

Taking appropriate limits one can derive correlation functions involving excited twist fields. For example, we can recover the correlator $\langle j(w)\bar{\sigma}(z_1)\sigma'(z_2)\bar{\sigma}(z_3)\sigma(z_4)\rangle$, which was already computed in appendix C. Considering the limit when both of the currents collide with a twist field we find correlators with two excited twist fields. For example

$$\begin{aligned} &\langle\bar{\sigma}'(z_1)\sigma'(z_2)\bar{\sigma}(z_3)\sigma(z_4)\rangle = \\ &\quad = \frac{1}{z_{21}^{9/8}z_{43}^{1/8}}\left(\frac{z_{31}z_{42}}{z_{41}z_{32}}\right)^{5/8}\left(\frac{E(\eta)}{2K(\eta)} - \frac{\delta^2}{16K^2(\eta)}\right)\sqrt{\frac{\pi}{2K(\eta)}}\exp\left(\frac{i\delta^2}{8\pi}\tau(\eta)\right). \end{aligned} \tag{D.12}$$

When the two excited twist fields are not adjacent we get

$$\begin{aligned} &\langle\bar{\sigma}'(z_1)\sigma(z_2)\bar{\sigma}'(z_3)\sigma(z_4)\rangle = \\ &\quad = \frac{z_{31}^{1/8}z_{42}^{9/8}}{(z_{43}z_{41}z_{32}z_{21})^{5/8}}\left(\frac{1-\eta}{2} - \frac{E(\eta)}{2K(\eta)} + \frac{\delta^2}{16K^2(\eta)}\right)\sqrt{\frac{\pi}{2K(\eta)}}\exp\left(\frac{i\delta^2}{8\pi}\tau(\eta)\right). \end{aligned} \tag{D.13}$$

Correlation functions involving excited twist fields are easily computed summing (D.12) and (D.13) over the array of Dirichlet sectors, or simply using the bosonized expressions of these fields. The results are

$$\begin{aligned}\langle \bar{\sigma}'_B(z_1)\sigma'_B(z_2)\bar{\sigma}_B(z_3)\sigma_B(z_4)\rangle &= \frac{1}{2z_{21}^{9/8}z_{43}^{1/8}}\left(\frac{z_{41}z_{32}}{z_{42}z_{31}}\right)^{3/8}, \\ \langle \bar{\sigma}'_B(z_1)\sigma_B(z_2)\bar{\sigma}'_B(z_3)\sigma_B(z_4)\rangle &= 0\,.\end{aligned} \tag{D.14}$$

Appendix E

Details on the calculation of the instanton profile

In this appendix we discuss in detail the calculation of the instanton profile sketched in section 8.2. We start from

$$A^{c(1)}_\mu(k) = C_0 f_1'(0)^{\alpha' k^2/2} \langle V^{(-1)u}_{\bar{w}}(\infty) \mathcal{V}^{(0)uv}_{A_\mu}(1;-k) V^{(-1)v}_{w}(0)\rangle \,. \tag{E.1}$$

The boundary changing operators (in picture -1) are the ones given in (7.19) with the rescaling (7.26), while $\mathcal{V}_{A_\mu}$ is given (in picture 0) by (8.32). We also compute the correlation function at generic positions z_1, z_2 and z_3, and then consider the particular case $z_1 \to \infty$, $z_2 = 1$ and $z_3 = 0$. We can split the amplitude (E.1) in four sub-amplitudes, which are independent from each other because they contain fields belonging to different CFT's:

$$\begin{aligned} A^{c(1)}_\mu(k) \sim\ & \bar{w}^u_{\dot\alpha}(\tau^c)^{vu} w^v_{\dot\beta} k^\nu \langle c(z_1)c(z_2)c(z_3)\rangle \langle e^{-\phi(z_1)} e^{-\phi(z_3)}\rangle \cdot \\ & \cdot \langle \bar\Delta(z_1) e^{-ik\cdot X}(z_2)\Delta(z_3)\rangle \langle S^{\dot\alpha}(z_1)\psi_\nu\psi_\mu(z_2) S^{\dot\beta}(z_3)\rangle \,. \end{aligned} \tag{E.2}$$

The first term in (8.32) has not been taken into account: its contribution would be proportional to k_μ; anyway, the polarization A^μ of the vector is subjected to the constraint $A\cdot k = 0$. Notice that all factors of α' (except the exponent of $f_1'(0)$) and g_{YM} disappear, thanks to the rescaling (7.26). All the correlation functions appearing in (E.2) are well known (see appendix B); we can thus write

$$\begin{aligned} A^{c(1)}_\mu(k) \sim\ & f_1'(0)^{\alpha' k^2/2} \bar{w}^u_{\dot\alpha}(\tau^c)^{vu} w^v_{\dot\beta} k^\nu \left(z_{12}z_{23}z_{13}\right)\left(\frac{1}{z_{13}}\right) \cdot \\ & \cdot \left(\frac{e^{-ik\cdot x_0}}{z_{13}^{(1-\alpha' k^2)/2}(4z_{12}z_{23})^{\alpha' k^2/2}}\right)\left(-\frac{1}{2}(\bar\sigma_{\nu\mu})^{\dot\alpha\dot\beta}\frac{z_{13}^{1/2}}{z_{12}z_{23}}\right) \,, \end{aligned} \tag{E.3}$$

where the 4-vector x_0^μ denotes the position of the D(-1) brane inside the D3 brane. Simplifying the result we are left with

$$A^{c(1)}_\mu(k) \sim \left(\frac{f_1'(0) z_{13}}{4 z_{12} z_{23}}\right)^{\alpha' k^2/2} \frac{1}{2} \bar{w}^u_{\dot\alpha}(\bar\sigma_{\nu\mu})^{\dot\alpha}{}_{\dot\beta} w^{v\dot\beta}(\tau^c)^{vu} k^\nu \, e^{-ik\cdot x_0} \,. \tag{E.4}$$

It is convenient now to use the 't Hooft symbols (see (A.12)); we then obtain

$$A_\mu^{c(1)}(k) \sim \left(\frac{f_1'(0)z_{13}}{4z_{12}z_{23}}\right)^{\alpha' k^2/2} \frac{i}{2}\bar{\eta}^d_{\nu\mu} \left(\bar{w}^u_{\dot\alpha}(\tau^d)^{\dot\alpha}{}_{\dot\beta} w^{v\dot\beta}\right)(\tau^c)^{vu} k^\nu \, e^{-ik\cdot x_0} \,. \tag{E.5}$$

Using the solution to the ADHM constraint

$$w^{v\dot\beta}\bar{w}^v_{\dot\alpha} = \rho^2 \delta^{\dot\beta}_{\dot\alpha} \,, \tag{E.6}$$

we can see that the $N \times N$ matrices

$$(t^d)^{uv} = \frac{1}{\rho^2}\left(\bar{w}^u_{\dot\alpha}(\tau^d)^{\dot\alpha}{}_{\dot\beta} w^{v\dot\beta}\right) \tag{E.7}$$

satisfy the relation $[t^d, t^e] = 2i\epsilon^{def} t^f$. If the solution (7.28) is considered, we can identify them with the Pauli matrices $(t^d)^{uv} = (\tau^d)^{uv}$. We can then conclude that (rescaling the size ρ if necessary, and setting the three points to ∞, 1 and 0 respectively)

$$A_\mu^{c(1)}(k) = \left(\frac{f_1'(0)}{4}\right)^{\alpha' k^2/2} \frac{i\rho^2}{2} k^\nu \bar{\eta}^d_{\nu\mu} e^{-ik\cdot x_0} \mathrm{Tr}(\tau^d\tau^c) = f_1'(0)^{\alpha' k^2/2} i\rho^2 k^\nu \bar{\eta}^c_{\nu\mu} e^{-ik\cdot x_0} \,. \tag{E.8}$$

As discussed in section 8.2, we now insert the gluon propagator and perform a Fourier transform, in order to obtain the result in position space. First of all we do it in the field theory limit $\alpha' k^2 \to 0$. In this case we have

$$A_\mu^{c(1)}(x; \alpha' k^2 \to 0) = \int \frac{d^4k}{(2\pi)^2} A_\mu^{c(1)}(k; \alpha' k^2 \to 0) \frac{1}{k^2} e^{ik\cdot x} = \rho^2 \bar{\eta}^c_{\nu\mu} \int \frac{d^4k}{(2\pi)^2} \frac{ik^\nu}{k^2} e^{ik\cdot(x-x_0)} \,. \tag{E.9}$$

We remember that the scalar massless propagator in configuration space is

$$G(x - x_0) = \int \frac{d^4k}{(2\pi)^2} \frac{1}{k^2} e^{ik\cdot(x-x_0)} = \frac{1}{(x-x_0)^2} \,. \tag{E.10}$$

Deriving it with respect to x_ν we have

$$\partial^\nu G(x - x_0) = \int \frac{d^4k}{(2\pi)^2} \frac{ik^\nu}{k^2} e^{ik\cdot(x-x_0)} = -2\frac{(x-x_0)^\nu}{(x-x_0)^4} \,; \tag{E.11}$$

going back to (E.9) we can conclude that

$$A_\mu^{c(1)}(x; \alpha' k^2 \to 0) = 2\rho^2 \bar{\eta}^c_{\mu\nu} \frac{(x-x_0)^\nu}{(x-x_0)^4} \,, \tag{E.12}$$

which is exactly the leading term of the full instanton solution (with size ρ) in an SU(2) gauge theory (2.8). It is also possible to compute α'-correction to the profile (still in the limit $\rho \ll \sqrt{\alpha'}$): it is sufficient to perform the Fourier transform of (E.8), adding a the propagator in Siegel gauge. Therefore

$$A_\mu^{c(1)}(x) = \int \frac{d^4k}{(2\pi)^2} A_\mu^{c(1)}(k) \frac{1}{k^2} e^{ik\cdot x} = \rho^2 \bar{\eta}^c_{\nu\mu} \int \frac{d^4k}{(2\pi)^2} \frac{ik^\nu}{k^2} e^{ik\cdot(x-x_0)} e^{\alpha k^2/2} \,, \tag{E.13}$$

where $\alpha = \alpha' L_1 = \alpha' \log(f_1'(0)/4)$. It can be checked that the quantity α is always negative, since $f_1'(0) < 1$ (see below). We have

$$\frac{d}{d\alpha} A_\mu^{c\,(1)}(x) = \frac{1}{2}\partial^\nu \left[\rho^2 \bar{\eta}^c_{\mu\nu} \int \frac{d^4k}{(2\pi)^2} e^{ik\cdot(x-x_0)} e^{\alpha k^2/2}\right] . \tag{E.14}$$

The gaussian integral is easily computed, and we are left with

$$\frac{d}{d\alpha} A_\mu^{c\,(1)}(x) = \frac{1}{2}\rho^2 \bar{\eta}^c_{\mu\nu} \partial^\nu \left[\frac{e^{(x-x_0)^2/(2\alpha' L_1)}}{(\alpha' L_1)^2}\right] , \tag{E.15}$$

which integrates to

$$A_\mu^{c\,(1)}(x) = A_\mu^{c\,(1)}(x; \alpha' k^2 \to 0) + \frac{1}{2}\rho^2 \bar{\eta}^c_{\mu\nu} \partial^\nu \left[-2\frac{e^{(x-x_0)^2/(2\alpha' L_1)}}{(x-x_0)^2}\right] , \tag{E.16}$$

where the integration constant is given by the result in the field theory limit (E.12). Altogether the final result is

$$A_\mu^{c\,(1)}(x) = 2\rho^2 \bar{\eta}^c_{\mu\nu} \frac{(x-x_0)^\nu}{(x-x_0)^4} \left[1 + e^{(x-x_0)^2/(2\alpha' L_1)} \left(1 - \frac{(x-x_0)^2}{2\alpha' L_1}\right)\right] . \tag{E.17}$$

Notice that, as expected, the correction to (E.12) disappears in the limit $\alpha'/(x-x_0)^2 \to 0$.

Conformal map

The function f_1 defined above can be obtained in two steps. First of all the unit semicircle around the origin can be mapped to a third of a disk with unit radius, with the origin mapped to the point 1. This corresponds to the map g_1 of figure E.1, which is given by

$$g_1(z) = \left(\frac{i-z}{i+z}\right)^{2/3} . \tag{E.18}$$

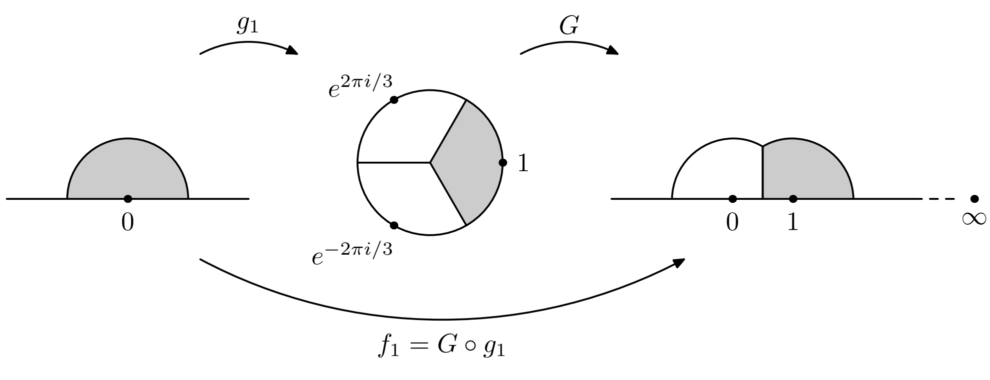

Figure E.1: Conformal map from the disk around the origin, to a disk divided in three sectors, and back to the upper half plane.

The second step is to map this unit disk to the upper half plane. We choose a conformal map such that the insertion points $e^{-2\pi i/3}$, 1 and $e^{2\pi i/3}$ are mapped to 0, 1 and ∞ respectively. Such a map is given by

$$G(w) = \frac{\left(1 - e^{2\pi i/3}\right)\left(w - e^{-2\pi i/3}\right)}{\left(1 - e^{-2\pi i/3}\right)\left(w - e^{2\pi i/3}\right)} \,. \tag{E.19}$$

The map f_1 is then given by the composition $f_1(z) = G(g_1(z))$, and its derivative at the origin is $f_1'(0) = \frac{4}{3\sqrt{3}} < 1$.

It would also be possible to choose three different points z_1, z_2 and z_3 in (E.4); in that case the map $G(w)$ would have to be a different $SL(2,\mathbb{C})$ function, such that $G(e^{-2\pi i/3}) = z_1$, $G(1) = z_2$ and $G(e^{2\pi i/3}) = z_3$. In this way the derivative $f'(0)$ can change its value, but it turns out that the combination $f_1'(0)z_{13}/(z_{12}z_{23})$ appearing in the prefactor of (E.4) has always norm equal to $\frac{4}{3\sqrt{3}} < 1$, independently on the particular choice of z_1, z_2 and z_3.

Appendix F

Derivation of the contact terms

In order to keep track of all the signs that arise in the graded algebra of the compositions of string vertices it is convenient to work with a shifted vector space (or a suspension). That is, we define the degree of x as $\deg(x) = |x| - 1$, where $|x|$ is the ghost number of x. We denote the shifted vector space by $A[1]$ and a shift operator s as $A[1] = sA$, or

$$(sA)_i = A_{i-1}. \tag{F.1}$$

where the subscript denotes the degree. With this convention, Q, ξ and the products

$$\begin{aligned} \hat{m}_2 &:= s \circ m_2 \circ (s^{-1} \otimes s^{-1}) \,, \\ \hat{M}_2 &:= s \circ M_2 \circ (s^{-1} \otimes s^{-1}) \,, \\ \hat{M}_3 &:= s \circ M_3 \circ (s^{-1} \otimes s^{-1} \otimes s^{-1}) \end{aligned} \tag{F.2}$$

all have degree one while X and $\Psi^{(1)}$ have degree zero. In addition we define

$$Q \circ \hat{m}_2 := Q\hat{m}_2 \quad \text{and} \quad \hat{m}_2 \circ Q := \hat{m}_2 \circ (Q \otimes \mathbb{1}) + \hat{m}_2 \circ (\mathbb{1} \otimes Q) \tag{F.3}$$

and similarly for $\hat{M}_2$, $\hat{M}_3$, $\hat{\mu}_2$ etc. Then (8.7) and (8.9) become simply

$$\hat{M}_2 = \frac{1}{3}\{X, \hat{m}_2\}_\circ = [Q, \hat{\mu}_2]_\circ \,, \quad \text{with} \quad \hat{\mu}_2 = \frac{1}{3}\{\xi, \hat{m}_2\}_\circ \,. \tag{F.4}$$

We are now ready to extract the contact terms in (8.43) which, when expressed in terms of the maps on the shifted vector space, take the simple form

$$\begin{aligned} (8.43) &= P_0 \left[\hat{M}_2 \circ Q^{-1} \circ \hat{M}_2 - \hat{M}_3\right] \left(s\Psi^{(1)} \otimes s\Psi^{(1)} \otimes s\Psi^{(1)}\right) = \\ &= \sum_i se^i \langle se_i, \left[\xi \circ \hat{M}_2 \circ Q^{-1} \circ \hat{M}_2 - \xi \circ \hat{M}_3\right] \left(s\Psi^{(1)} \otimes s\Psi^{(1)} \otimes s\Psi^{(1)}\right)\rangle_L \,. \end{aligned} \tag{F.5}$$

Then, using (F.4) we have

$$\begin{aligned}
&\langle se_i, \Big[\xi \circ \hat{M}_2 \circ Q^{-1} \circ \hat{M}_2 - \xi \circ \hat{M}_3\Big] \big(s\Psi^{(1)} \otimes s\Psi^{(1)} \otimes s\Psi^{(1)}\big)\rangle_L = \\
&= \frac{1}{2}\langle se_i, \Big[\xi \circ \hat{M}_2 \circ Q^{-1} \circ [Q, \hat{\mu}_2]_\circ - \xi \circ \hat{M}_3\Big] \big(s\Psi^{(1)} \otimes s\Psi^{(1)} \otimes s\Psi^{(1)}\big)\rangle_L + \\
&\quad + \frac{1}{2}\langle se_i, \Big[\xi \circ [Q, \hat{\mu}_2]_\circ \circ Q^{-1} \circ \hat{M}_2 - \xi \circ \hat{M}_3\Big] \big(s\Psi^{(1)} \otimes s\Psi^{(1)} \otimes s\Psi^{(1)}\big)\rangle_L = \\
&= \frac{1}{2}\langle se_i, \Big[\xi \circ \hat{M}_2 \circ Q^{-1} \circ Q \circ \hat{\mu}_2 - \xi \circ \hat{\mu}_2 \circ Q \circ Q^{-1} \circ \hat{M}_2\Big] \big(s\Psi^{(1)} \otimes s\Psi^{(1)} \otimes s\Psi^{(1)}\big)\rangle_L + \\
&\quad + \frac{1}{2}\langle se_i, \Big[X \circ \hat{\mu}_2 \circ Q^{-1} \circ \hat{M}_2 - 2\xi \circ \hat{M}_3\Big] \big(s\Psi^{(1)} \otimes s\Psi^{(1)} \otimes s\Psi^{(1)}\big)\rangle_L \,,
\end{aligned} \tag{F.6}$$

where we used $\{Q, \xi\}_\circ = X$ and $Qe_i = 0$ in the last identity. So far this calculation is identical to the calculation of the four-point scattering amplitude in [62]. To continue we commute Q^{-1} through Q using (8.16). This is again identical to [62] apart form the presence of P_0 in (8.16).[1] This leaves us with

$$\begin{aligned}
&\langle se_i, \Big[-\xi \circ \hat{M}_2 \circ P_0 \circ \hat{\mu}_2 + \xi \circ \hat{\mu}_2 \circ P_0 \circ \hat{M}_2\Big] \big(s\Psi^{(1)} \otimes s\Psi^{(1)} \otimes s\Psi^{(1)}\big)\rangle_L + \\
&\quad + \langle se_i, \Big[X \circ \xi \circ \hat{M}_2 \circ Q^{-1} \circ \hat{m}_2 + X \circ \xi \circ \hat{m}_2 \circ Q^{-1} \circ \hat{M}_2\Big] \big(s\Psi^{(1)} \otimes s\Psi^{(1)} \otimes s\Psi^{(1)}\big)\rangle_L \,,
\end{aligned} \tag{F.7}$$

where we used $\{Q, \xi\}_\circ = X$ and $Qe_i = 0$ one more time and furthermore, that in the second line, the ξ zero-mode has to be provided by $\hat{\mu}_2$. Applying (8.9) once more to the second line of (F.7) we are left with

$$\begin{aligned}
&\langle se_i, \Big[-\xi \circ \hat{M}_2 \circ P_0 \circ \hat{\mu}_2 + \xi \circ \hat{\mu}_2 \circ P_0 \circ \hat{M}_2\Big] \big(s\Psi^{(1)} \otimes s\Psi^{(1)} \otimes s\Psi^{(1)}\big)\rangle_L + \\
&\quad + \langle se_i, \big[X \circ \xi \circ \hat{\mu}_2 \circ P_0 \circ \hat{m}_2 - X \circ \xi \circ \hat{m}_2 \circ P_0 \circ \hat{\mu}_2\big] \big(s\Psi^{(1)} \otimes s\Psi^{(1)} \otimes s\Psi^{(1)}\big)\rangle_L + \\
&\quad + \langle se_i, \big[X \circ X \circ \hat{\mu}_2 \circ Q^{-1} \circ \hat{m}_2 + X \circ X \circ \hat{m}_2 \circ Q^{-1} \circ \hat{\mu}_2\big] \big(s\Psi^{(1)} \otimes s\Psi^{(1)} \otimes s\Psi^{(1)}\big)\rangle_L \,.
\end{aligned} \tag{F.8}$$

Before continuing, we note that the second term in the first line vanishes since in section 8.1.2 we showed that $P_0 M_2(\Psi^{(1)}, \Psi^{(1)}) = 0$.

To express the contribution containing P_0 in terms of elementary operator products we undo the shift (F.2) and use (8.7) as well as (8.9). This gives, for example,

$$\begin{aligned}
&3\cdot\langle se_i, \xi \circ \hat{M}_2 \circ P_0 \circ \hat{\mu}_2 \big(s\Psi^{(1)} \otimes s\Psi^{(1)} \otimes s\Psi^{(1)}\big)\rangle_L = \\
&= \langle X\xi e_i, m_2(P_0\mu_2(\Psi^{(1)}, \Psi^{(1)}), \Psi^{(1)})\rangle_L + \langle \xi e_i, m_2(XP_0\mu_2(\Psi^{(1)}, \Psi^{(1)}), \Psi^{(1)})\rangle_L + \\
&\quad + \langle \xi e_i, m_2(P_0\mu_2(\Psi^{(1)}, \Psi^{(1)}), X\Psi^{(1)})\rangle_L + \langle X\xi e_i, m_2(\Psi^{(1)}, P_0\mu_2(\Psi^{(1)}, \Psi^{(1)}))\rangle_L + \\
&\quad + \langle \xi e_i, m_2(X\Psi^{(1)}, P_0\mu_2(\Psi^{(1)}, \Psi^{(1)}))\rangle_L + \langle \xi e_i, m_2(\Psi^{(1)}, XP_0\mu_2(\Psi^{(1)}, \Psi^{(1)}))\rangle_L \,,
\end{aligned} \tag{F.9}$$

[1] In [62] P_0 did not contribute due to kinematics for scattering states with finite momentum.

where we used the fact that ξ and X are both BPZ even. The remaining terms in the second line of (F.8) give in turn

$$\begin{aligned}&-\langle X\xi e_i, m_2(P_0\mu_2(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)}) + m_2(\Psi^{(1)}, P_0\mu_2(\Psi^{(1)},\Psi^{(1)}))\rangle_L + \\ &-\langle X\xi e_i, \mu_2(P_0 m_2(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)}) - \mu_2(\Psi^{(1)}, P_0 m_2(\Psi^{(1)},\Psi^{(1)}))\rangle_L \,.\end{aligned} \tag{F.10}$$

Making use of the cyclic properties of m_2 and μ_2 (e.g. [64])

$$\begin{aligned}\langle a, m_2(b,c)\rangle &= (-1)^{|a|(|b|+|c|)}\langle b, m_2(c,a)\rangle\,,\\ \langle a, \mu_2(b,c)\rangle &= (-1)^{|a|+|b|+|a|(|b|+|c|)}\langle b, \mu_2(c,a)\rangle\end{aligned} \tag{F.11}$$

we can recast (F.9) and (F.10) into

$$\begin{aligned}&-\frac{1}{3}\langle P_0\mu_2(\Psi^{(1)},\Psi^{(1)}), 4m_2(\Psi^{(1)},\xi X e_i) - 4m_2(\xi X e_i,\Psi^{(1)}) + m_2(X\Psi^{(1)},\xi e_i) - m_2(\xi e_i, X\Psi^{(1)})\rangle_L + \\ &-\frac{1}{3}\langle XP_0\mu_2(\Psi^{(1)},\Psi^{(1)}), m_2(\Psi^{(1)},\xi e_i) - m_2(\xi e_i,\Psi^{(1)})\rangle_L + \\ &-\langle P_0 m_2(\Psi^{(1)},\Psi^{(1)}), \mu_2(\Psi^{(1)},\xi X e_i) - \mu_2(\xi X e_i,\Psi^{(1)})\rangle_L\,,\end{aligned} \tag{F.12}$$

where we have furthermore used that $X\xi A = \xi X A$ for $A = V, e_i$. Finally, we can use the definition of the product μ_2 in the last line, and arrive at the result that we quote in (8.50).

This leaves us with the terms containing the propagator Q^{-1} (F.8). Here, the ξ zero-mode has to be provided by μ_2. Thus, the last line in (F.8) can be written in the small Hilbert space as

$$-2\langle X\circ X\, e_i, m_2(Q^{-1}m_2(\Psi^{(1)},\Psi^{(1)}),\Psi^{(1)}) + m_2(\Psi^{(1)}, Q^{-1}m_2(\Psi^{(1)},\Psi^{(1)}))\rangle\,. \tag{F.13}$$

Appendix G

Anomalous Contributions due to Non-Primary Fields

The explicit calculation of the product m_2 and of the BPZ inner product requires a set of conformal transformation that map each vertex operator to the upper half plane. In this way the product m_2 can be expressed in terms of the operator product expansion of operators in CFT, and the BPZ inner product is equivalent to a correlation function on the upper half plane. If all the vertex operators are primaries of conformal dimension 0 this does not pose any problem. In chapter 8, however, we are dealing with some non-primaries operators, hence we should consider anomalous contributions due to these conformal transformations. Let us consider an operator W of scaling dimension $h = 0$, but with anomalous OPE with the energy-momentum tensor given by

$$T(z)W(0) = \frac{\alpha}{z^3} + \frac{\partial W(0)}{z} + \dots \tag{G.1}$$

Considering now an infinitesimal transformation $z \to z + \epsilon(z)$, the operator W transforms according to

$$\delta_\epsilon W(w) = \frac{1}{2\pi i}\oint dz[T(z)\epsilon(z), W(w)] = \alpha\partial^2\epsilon(w)W(w) + \epsilon(w)\partial W(w)\,. \tag{G.2}$$

The last term in (G.2) would be present even if the operator W was primary, while the first term $\epsilon''(z)W(z)$ is an anomalous contribution.

In chapter 8 we have encountered two non-primary operators, namely $\frac{1}{4} : \xi\eta : e^{\phi}\widetilde{\mathbb{V}}_{1/2}$ and $\partial(e^{\phi}\widetilde{\mathbb{V}}_{1/2})$ (see appendix B). The first one appearns in $\xi X e_i$ and gives anomalous contributions to (8.50). This anomalous contribution will thus be equal, using (B.15), to

$$-\frac{6}{3}\frac{1}{4}(\partial^2\epsilon_3(0) - \partial^2\epsilon_4(0))\langle \xi P_0 m_2(\Psi^{(1)}, \Psi^{(1)}), m_2(\Psi^{(1)}, e^{\phi}\widetilde{\mathbb{V}}_{1/2}) - m_2(e^{\phi}\widetilde{\mathbb{V}}_{1/2}, \Psi^{(1)})\rangle_L\,, \tag{G.3}$$

where $\epsilon_{3,4}$ represent the infinitesimal part of the two conformal transformation that have to be done in order map the BPZ product to a correlation function on the upper half plane. On the other hand, $\partial(e^{\phi}\widetilde{\mathbb{V}}_{1/2})$ gives anomalous contributions to (8.80). Using (B.17) we find that the anomaly is given by

$$\frac{1}{4}(2\partial^2\epsilon_3(0) - 2\partial^2\epsilon_4(0))\langle P_0 m_2(\Psi^{(1)}, \Psi^{(1)}), m_2(\Psi^{(1)}, e^{\phi}\widetilde{\mathbb{V}}_{1/2}) - m_2(e^{\phi}\widetilde{\mathbb{V}}_{1/2}, \Psi^{(1)})\rangle\,, \tag{G.4}$$

which exactly cancels the other anomalous contribution (G.3).

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