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SPARSITY AND DECOUPLING
IN LOAD POWER FLOW ANALYSIS

BY

KENNETH NAZIMEK

A THESIS

PRESENTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE

OF

MASTER OF SCIENCE IN ELECTRICAL ENGINEERING

AT

NEW JERSEY INSTITUTE OF TECHNOLOGY

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Newark, New Jersey

1977

ABSTRACT

This paper presents some recent ideas on and methods created by power engineers for the solution of the Load Flow Problem. Beginning with an analysis of the solution of large systems of linear equations, techniques employing elementary matrix operations are discussed to reduce the amount of calculations necessary to achieve repeated solutions. Recent interest in taking advantage of the characteristic sparsity of the admittance matrix is also examined. The problems of operating on and computer storage of large sparse matrices are investigated. Methods to efficiently order, store and work with large sparse matrices are discussed and demonstrated. The general acceptance and use of Newton's Method by power engineers for Load Flow Studies has resulted in many variations on it. Together with a review of the Load Flow Problem and Newton's Method, an analysis of three related methods is presented and a comparison of their characteristics is made to Newton's Method. This paper serves to keep the power engineer abreast of the many recent advances available for conducting Load Flow Studies.

APPROVAL OF THESIS

SPARSITY AND DECOUPLING

IN LOAD POWER FLOW ANALYSIS

BY

KENNETH NAZIMEK

FOR

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INTRODUCTION

One of the most important resources used by power engineers in operating large electric power systems is the Load Flow Study. Load Flow Studies are used in the planning and design of electric power systems, in their operation under optimal economic conditions, in studying the effects of future expansion to an existing system and also as a part of the system stability studies. In the not too distance past Load Flow Studies were carried out on a small scale model of the actual system constructed on an A-C Analyzer Board. The studies done in this manner resulted in data of sufficient accuracy to be useful, however the method was very time consuming. With the advent of the electronic digital computer, power engineer began to employ numerical techniques to perform their Load Flow Studies. Many numerical methods, applicable to the computer, were developed and continually refined by attempting to reduce the size of the computer memory necessary, while increasing the speed and accuracy of the solution. Many of these early Methods involved some type of iterative process due to the non-linear nature of the Load Flow Problem. The major concern was the amount of time necessary to achieve an accurate solution.

In the past twenty years, the American society has grown very fast and so has its electric power system which provides it with the energy it needs to exist and the potential for its growth. As a result, the modern electric power system has become enormously complex, larger than ever before and is continually expanding. Therefore in order to make efficient use of today's increasingly expensive energy resources, the physical equipment of the power system and the modern electronic digital computers, the power engineer must have at his or her command established methods to conduct Load Flow Studies that require a minimum amount of time but also a minimum amount computer memory to solve the large modern power system problem.

The purpose here is to present and evaluate a number of the more recent ideas on and methods used in Load Flow Studies. The methods to be considered will be those that have been presented to and possibly implemented by a large number of modern power system engineers. This presentation will begin with the development of a technique used for the solution of large systems of linear equation, that is, Optimal Ordered Triangular Factorization. The technique was introduced by a group of engineers and computer scientists led by W. F. Tinney at the Bonneville Power Administration. This method is an integral part of many procedures used in Load Flow Studies today. Also included are some new and unique ideas of other power engineers that attempt to reduce the amount of work required to perform Load Flow Studies on today's large power systems. It is hoped that this paper will be used to provide the young and inexperienced power engineer with a better understanding of the Load Flow Problem and acquaint him or her with a number of recent procedures used in its solution. And also to update the knowledge of the more experienced power engineers by keeping them abreast of the recent ideas and newly developed methods available to conduct more efficient Load Flow Studies.

ACKNOWLEDGMENT

I would like to thank Dr. Edwin Cohen for his wisdom and guidance, which shed light on my darkness and provided direction when I knew not where to turn.

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FORMS OF NETWORK EQUATIONS

In the analysis of electrical networks the equations describing the system can be written in one of the following general forms. One, by Kirchhoff's Voltage Law, the N equations of the system, where N is equal to the number of equations necessary to fully represent the network, are written by making use of the complex open-circuit driving point and transfer impedances.(1.1)

$$E = Z\ I \tag{1.1}$$

Here Z is a N x N matrix of the complex open-circuit driving point and transfer impedances, E is a column vector of known node or bys voltages and I is a column vector of unknown node currents. Alternatively, the N system equations could be formulated by Kirchhoff's Current Law, based on the complex short-circuit driving point and transfer admittances. (1.2)

$$I = Y\ E \tag{1.2}$$

With this approach, Y is the N x N matrix of the complex short-circuit driving point and transfer admittances, I is the column vector of known node currents and E is a column vector of node voltages to be found.

In equations (1.1) and (1.2), the Z and Y matrices will be referred to as the characteristic matrix of the particular system under investigation. After the equations have been written and for simplicity placed in matrix form, they must be solved in order to ascertain the response of the network to the given set of input values. In the next chapter, two methods of solution, applicable to equations in matrix form, will be given and the positive and negative aspects of each will be presented.

Chapter II

METHODS OF SOLUTION OF THE NETWORK EQUATIONS

Inverse Matrix

A simple and straightforward solution to either equations (1.1) or (1.2) is the computation of the inverse characteristic matrix and have it premultiply the corresponding E or I matrix of known quantities. (2.1),(2.2)

$$I = Z^{-1}E \qquad (2.1)$$

$$E = Y^{-1}I \qquad (2.2)$$

Formlation of an Inverse Matrix.[1] Let A be a non-singular, square matrix for which the inverse matrix A^{-1} is to be found. The conventional three step sequence for obtaining an inverse matrix is given below:
(1) Form the transpose of the A matrix by interchanging the rows and columns. (2) Replace every element in the transposed matrix of step (1) by the value of it's cofactor. The minor of the element a_{ij} is the (N-1) x (N-1) matrix formed by deleting row i and column j from the A matrix. cofactor of $a_{ij} = (-1)^{i+j}$ x (determinant of the (N-1) x (N-1) minor of a_{ij}. (3) Divide each element of the matrix formed in step (2) by the value of the determinant of A.

[1] John R. Neuenswander, Modern Power Systems (Scranton: International Textbook Company, 1971), p.400

<u>Example</u>: Find the inverse matrix A^{-1} given the following A matrix.

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 4 & 0 \\ -1 & 3 & 6 \end{bmatrix}$$

step (1): Form the transpose matrix A^T

$$A^T = \begin{bmatrix} 1 & 5 & -1 \\ 2 & 4 & 3 \\ -3 & 0 & 6 \end{bmatrix}$$

step (2): Replace each element of A^T by it's cofactor value.

$$\text{cofactor of } a_{11} = (-1)^{1+1} \begin{matrix} 4 & 3 \\ 0 & 6 \end{matrix} = 1 \times (24-0) = 24$$

$$\text{cofactor of } a_{21} = (-1)^{2+1} \begin{matrix} 5 & -1 \\ 0 & 6 \end{matrix} = -1 \times (30-0) = -30$$

Continuing in this manner for all other elements of A^T, the matrix produced in step (2) is,

$$\begin{bmatrix} 24 & -21 & 12 \\ -30 & 3 & -15 \\ 19 & -5 & -6 \end{bmatrix}$$

step (3): Divide the matrix of step (2) by the determinant of A.

$$\det A = 1 \begin{bmatrix} 4 & 0 \\ 3 & 6 \end{bmatrix} - 2 \begin{bmatrix} 5 & 0 \\ -1 & 6 \end{bmatrix} - 3 \begin{bmatrix} 5 & 4 \\ -1 & 3 \end{bmatrix}$$

$$\det A = 24 - 60 - 57 = -93$$

$$A^{-1} = \frac{-1}{93} \begin{bmatrix} 24 & -21 & 12 \\ -30 & 3 & -15 \\ 19 & -5 & -6 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{-8}{31} & \frac{-7}{31} & \frac{-4}{31} \\ \frac{10}{31} & \frac{-1}{31} & \frac{5}{31} \\ \frac{-19}{93} & \frac{5}{93} & \frac{2}{31} \end{bmatrix}$$

The resulting matrix of step (3) is the inverse matrix A^{-1}. This result can be verified by simply multiplying A by A^{-1}, if A^{-1} is the actual inverse the product will be the identity or unit matrix U.

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Several important facts about this method of solution are: (1) if the A matrix is symmetric step (1) can be omitted because the transpose of a symmetric matrix is equivalent to the original matrix.

$$A = A^T \quad \text{if } A \text{ is symmetric}$$

(2) if the value of the determinant of A is equal to zero,

$$\text{det of } A = 0$$

the divisions in step (3) are undefined and no inverse matrix exists, therefore the value of the deternimamt should be checked before employing this method.

Apart from this last hazard, matrix inversion is an acceptable method of manual solution provided that the characteristic matrix is not too large. As the size of this matrixincreases so does the number of calculations necessary to determine the inverse matrix. With this methid the manual computation of a large, $N > 4$, inverse matrix must give way to the speed and accuracy of digital computer, For the very large power system networks, which will be under consideration here, N is assumed to be on the order of a thousand equations. In this case even with the high speed of a modern computer, the formation of the inverse matrix requires an unacceptable amount of time and computer memory. Therefore in the analysis of today's power systems, an improvement in the method of the inverse matrix must be found or other methods of solution must be employed.

Gauss Elimination Method (G.E.M.). The Gauss Elimination Method is a procedure for solving a system of N simultaneous linear equations and it does not require an inverse matrix.

The N system equations are arranged in the manner outlined in (2.3)

$$\begin{array}{l}
a_{11}\, x_1 + a_{12}\, x_2 + a_{13}\, x_3 + \cdots\cdots + a_{1N}\, x_N = w_1 \\
a_{21}\, x_1 + a_{22}\, x_2 + a_{23}\, x_3 + \cdots\cdots + a_{2N}\, x_N = w_2 \\
a_{31}\, x_1 + a_{32}\, x_2 + a_{33}\, x_3 + \cdots\cdots + a_{3N}\, x_N = w_3 \\
\quad\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\qquad\qquad\quad \vdots \qquad \vdots \\
a_{N1}\, x_1 + a_{N2}\, x_2 + a_{N3}\, x_3 + \cdots\cdots + a_{NN}\, x_N = w_N
\end{array} \qquad (2.3)$$

Equation (2.3) can be written in a more convenient form using matrix notation.(2.4)

$$A\ x = W \qquad (2.4)$$

Where A is the N x N matrix of known system parameters or characteristic matrix, W is the column matrix os desired system reponses and x in the column matrix of unknown system excitations. The Gauss Elimination Method first trangularizes the matrix of known system parameters by either rows or columns and then with the application of back-substitution the solution of the unknown matrix is formed.

The steps necessary to triangularize a given A matrix by columns are presented below. To facilitate the process of back substitution the A matrix is augmented by the column matrix W before triangularization.(2.5)

$$
\begin{array}{cccccc}
a_{11} & a_{12} & a_{13} & a_{14} \cdots\cdots & a_{1N} & w_1 \\
a_{21} & a_{22} & a_{23} \cdots\cdots\cdots\cdots & & a_{2N} & w_2 \\
a_{31} & a_{32} \cdots\cdots\cdots\cdots\cdots\cdots & & & a_{3N} & w_3 \\
a_{41} \cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & a_{4N} & w_4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{N1} & a_{N2} & a_{N3} & a_{N4} & a_{NN} & w_N
\end{array}
\qquad (2.5)
$$

Step (1): Starting with row one, divide all the elements of the row by the value of it's diagonal element, a_{11}. The new elemental values for row one obtained in this step are denoted by superscript 1.(2.6)

$$
\begin{aligned}
a_{1j}^{(1)} &= a_{1j} / a_{11} \qquad j = 2,3,\ldots,N \\
w_1^{(1)} &= w_1 / a_{11}
\end{aligned}
\qquad (2.6)
$$

Note: If the value of the diagonal element in the i^{th} row, a_{ii}, is equal to zero the divisions in step (1) would produce undefined quantitites. In this case the entire i^{th} row must be interchanged with the nearest row below it that will permit a new value of a_{ii} different from zero.

Step (2): Multiply the new row ebtained in step (1) by the negative of the leading element in row two, a_{21}, and add the resulting row to row two. This step reduces element a_{21} to zero and produces a new row two.(2.7)

$$a_{2j}^{(1)} = a_{2j} - a_{21}\ a_{1j}^{(1)} \qquad j = 1,2,3,\ldots,N$$

$$w_2^{(1)} = w_2 - a_{21}\ w_1^{(1)} \qquad (2.7)$$

Step (3): step (2) is repeated for rows 3,4,5,...,N until all the elements below the diagonal element of column one are reduced to zero. If any of the elements in the column is equal to zero before step (2), the corresponding row is left unaltered.(2.8)

$$a_{ij}^{(1)} = a_{ij} - a_{i1}\ a_{1j}^{(1)}$$

$$i = 3,4,5,\ \ldots,\ N \qquad (2.8)$$

$$j = 1,2,3,\ \ldots,\ N$$

Step (4): The process consisting of steps (1), (2) and (3) is repeated for each column until all the elements of the matrix below the diagonal are eliminated and all diagonal elements are converted to one. After step (4) has been completed the now triangularized matrix is in the following form.(2.9)

$$\begin{matrix}
1 & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & \cdots\cdots & a_{1N}^{(1)} & w_1^{(1)} \\
0 & 1 & a_{23}^{(2)} & a_{24}^{(2)} & \cdots\cdots & a_{2N}^{(2)} & w_2^{(2)} \\
0 & 0 & 1 & a_{34}^{(3)} & \cdots\cdots & a_{3N}^{(3)} & w_3^{(3)} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots\cdots & 1 & w_N^{(N)}
\end{matrix} \qquad (2.9)$$

The solution of the unknown excitation matrix X is formed from the matrix of (2.9) by the procedure of back substitution. (2.10)

$$
\begin{aligned}
x_N &= w_N^{(N)} \\
x_{N-1} &= w_{N-1}^{(N-1)} - a_{N-1,N}^{(N-1)} x_N
\end{aligned}
\qquad (2.10)
$$

In general the solution for any x in the unknown excitation matrix X is given by (2.11)

$$x_i = w_i^{(i)} - \sum_{j=i+1}^{N} a_{ij}^{(i)} x_j \qquad (2.11)$$

<u>Example</u>: Solve the following set of simultaneous linear equations by the Gauss Elimination Method.

$$
\begin{aligned}
x_1 + 2x_2 - 5x_3 + 3x_4 &= 20 \\
-x_1 + 4x_2 + 4x_3 - 2x_4 &= 13 \\
2x_1 - 3x_2 + 5x_3 - 7x_4 &= 32 \\
5x_1 + x_2 - x_3 + x_4 &= 17
\end{aligned}
$$

In matrix form, the preceding set of equations are:

$$
\begin{bmatrix} 1 & 2 & -5 & 3 \\ -1 & 4 & 4 & -2 \\ 2 & -3 & 5 & -7 \\ 5 & 1 & -1 & 1 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
=
\begin{bmatrix} 20 \\ 13 \\ 32 \\ 17 \end{bmatrix}
$$

The augmented matrix before triangularization:

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 20 \\ -1 & 4 & 4 & -2 & 13 \\ 2 & -3 & 5 & -7 & 32 \\ 5 & 1 & -1 & 1 & 17 \end{bmatrix}$$

Step (1): Since the first element in the first row of the augmented matrix happens to be 1, the method continues to Step (2). Step (2): Add the second row to first, Subtract the third row from 2x the first, Subtract the fourth row from 5x the first.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 20 \\ 0 & 6 & -1 & 1 & 33 \\ 0 & -7 & 15 & -13 & -8 \\ 0 & -9 & 24 & -14 & -83 \end{bmatrix}$$

Now operating on the second row and second column, divide the second row by 6 forming a new row 2, then add the third row to 7x the new second row and add row 4 to 9x the new second row.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 20 \\ 0 & 1 & -1/6 & 1/6 & 5.5 \\ 0 & 0 & 83/6 & -71/6 & 30.5 \\ 0 & 0 & 22.5 & -12.5 & -33.5 \end{bmatrix}$$

Continuing to the third row and column, multiply the third row by 6/83 forming a new third row. Then add -22.5x the new third row to row 4.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 20 \\ 0 & 1 & -1/6 & 1/6 & 5.5 \\ 0 & 0 & 1 & -71/83 & 2.2 \\ 0 & 0 & 0 & 6.75 & -83 \end{bmatrix}$$

Finally divide the fourth row by 6.75:

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 20 \\ 0 & 1 & -1/6 & 1/6 & 5.5 \\ 0 & 0 & 1 & -71/83 & 2.2 \\ 0 & 0 & 0 & 1 & -12.3 \end{bmatrix}$$

Rewriting the augmented matrix into original form:

$$\begin{bmatrix} 1 & 2 & -5 & 3 \\ 0 & 1 & -1/6 & 1/6 \\ 0 & 0 & 1 & -71/83 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 20 \\ 5.5 \\ 2.2 \\ -12.3 \end{bmatrix}$$

Writing out each equation it can be easily seen how the values of x_1, x_2, x_3 and x_4 are obtained through back-substitution.

$$\begin{aligned} x_1 + 2x_2 - 5x_3 + 3x_4 &= 20 \\ x_2 - 1/6x_3 + 1/6x_4 &= 5.5 \\ x_3 - 71/83x_4 &= 2.2 \\ x_4 &= -12.3 \end{aligned}$$

$x_4 = \underline{-12.3}$

$x_3 = 2.2 + 71/83\ x_4 = 2.2 + 71/83(-12.3) = \underline{-8.3}$

$x_2 = 5.5 + 1/6\ x_3 - 1/6\ x_4 = 5.5 + 1/6(-8.3) - 1/6(-12.3)$

$x_2 = \underline{6.2}$

$x_1 = 20 - 2\ x_2 + 5\ x_3 - 3\ x_4$

$x_1 = 20 - 2(6.2) + 5(-8.3) - 3(-12.3) = \underline{2.9}$

Substituting the calculated values above into the original equation as a check:

$$2.9 + 2(6.2) - 5(-8.3) + 3(-12.3) = 19.9 \quad : \quad 20$$

$$-2.9 + 4(6.2) + 4(-8.3) - 2(-12.3) = 13.3 \quad : \quad 13$$

$$2(2.9) - 3(6.2) + 5(-8.3) - 7(-12.3) = 31.8 \quad : \quad 32$$

$$5(2.9) + 6.2 - (-8.3) + (-12.3) = 16.7 \quad : \quad 17$$

Symmetry of the A Matrix.

When the A matrix is symmetric, as defined by (2.12), then as it is triangularized by the GEM, it can be shown that the resulting sub-matrix to the right of the last sub-column reduced to zero is symmetrical.

$$a_{ij} = a_{ji} \qquad i \neq j \tag{2.12}$$

For example, after column one of a symmetric A matrix has been eliminated by the GEM, the resulting sub-matrix is given in (2.13).

$$\begin{bmatrix} a_{22}^{(1)} & a_{23}^{(1)} & a_{24}^{(1)} & \cdots\cdots & a_{2N}^{(1)} \\ a_{32}^{(1)} & a_{33}^{(1)} & a_{34}^{(1)} & \cdots\cdots & a_{3N}^{(1)} \\ a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & \cdots\cdots & a_{4N}^{(1)} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{N2}^{(1)} & a_{N3}^{(1)} & a_{N4}^{(1)} & & a_{NN}^{(1)} \end{bmatrix} \qquad (2.13)$$

From the definition of symmetry in (2.12), the matrix of (2.13) is symmetric if (2.14) is satisfied.

$$a_{ij}^{(1)} = a_{ji}^{(1)} \quad i \neq j \quad i\ ,\ j = 2,3,4,\ \ldots\ N \qquad (2.14)$$

From the elimination process

$$a_{ij}^{(1)} = a_{ij} - a_{ix}\ (\ a_{xj}/\ a_{xx}\ )$$

$$(2.15)$$

$$a_{ji}^{(1)} = a_{ji} - a_{jx}\ (\ a_{xi}/\ a_{xx}\ )$$

where x is equal to the number of the last column eliminated. Here $x = 1$. Given that the original A matrix is symmetric, then all of the following must be true,

$$a_{ij} = a_{ji} \qquad\qquad a_{ix} = a_{xi}$$

$$(2.16)$$

$$a_{xj} = a_{jx} \qquad\qquad a_{ii} = a_{ii}$$

With the proper substitution of (2.16) into the two equations of (2.15) and with a small amount of algebraic manipulation it can be proved that (2.17) is true.

$$a_{ij}^{(1)} = a_{ji}^{(1)} \qquad j \neq i \qquad (2.17)$$

The fact that symmetry is maintained in a symmetric matrix provides an additional benefit to this method. It will prove useful in recording the triangularization operations for repeated solutions of system equations with symmetric characteristic matrices. A recording technique developed by W. F. Tinney and his group will be presented in the next section. This recording method will be applicable to unsymmetric matrices as well.

Chapter III

USE OF GAUSS ELIMINATION FOR REPEATED SOLUTIONS

In the operation of a large modern power system it is necessary to solve the same system problem with different input conditions repeatedly as the demand changes during the day. Therefore to save valuable computer time any method used to perform a Load Flow Study must be able to execute repeated solutions with a minimum of duplicate calculations. This was one of the goals set by a group of engineers and computer scientists headed by W. F, Tinney at the Bonneville Power Administration. This group realized that as power systems grew, the amount of calculations necessary for proper operation would also grow. They seem to have studied many aspects of a modern power system and decided to utilize an important characteristic of the power system's Y or admittance matrix, that being it's relative sparsity.

A matrix is characterized as sparse, whenever a significant percentage of it's elements are equal to zero. The admittance or Y matrix of a power system is relatively sparse, whereas the Z of impedance matrix of the same system is proportionately full, i.e. very few zero elements. Since the Gauss Elimination Method involves zeroing all lower off diagonal elements, if some of these are already zero apparently less calculations need be performed. This group decided to develop an analysis technique using the Y matrix to describe the system and the Gauss Elimination Method for solution, thereby, reducing the number of calculations involved and the overall solution time. Going one step further, they also developed a technique, which is presented and discussed in this section, to record the solution process so that for repeated solutions even fewer calculations need be performed. This technique is referred to as the method of triangular factorization. The technique was later refined by the development and application of a number of ordering schemes which rearrange the rows in the system Y matrix in the anticipation of fewer required calculations. These ordering schemes will be introduced and the benefit of each presented in the next section.

Decomposition by Elementary Matrices[1]

The Y matrix of short-circuit driving point and transfer admittances, when multiplied by the proper type of elementary matrices can be transformed into th identity or unit matrix U.(3.1)

$$U = (M_1 (M_2 (M_3 (\dots (M_p Y)))) \qquad (3.1)$$

The forms of the M_p matrices are outlined in (3.2) and (3.3)

1 N. Sato & W.F. Tinney,"Techniques for Exploiting the Sparsity of the Network Admittance Matrix",IEEE PAS Trans. Dec. 1963 p.944

$$M_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad i \neq j \qquad (3.2)$$

$$M_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad i = j \qquad (3.3)$$

Multiplication by an M_{ij} matrix reduces either an off-diagonal element to zero or a diagonal element to one. Since the multiplication of a matrix by its inverse results in the U matrix, the group of M_{ij} matrices in (3.1) premultipling Y must be equivalent to the inverse of Y namely Y^{-1}.(3.4)

$$Y^{-1} = M_1 \; M_2 \; M_3 \; .... \; M_p \qquad (3.4)$$

Therefore, the solution to (1.2) can now be expressed as,

$$E = (M_1 \; (M_2 \; (M_3 \; (\; .... \; (M_p \; I \;)))) \qquad (3.5)$$

The process of premultiplying Y by the different types of M_{ij} matrices is closely related to the triangularization and back substitution operations in the G. E. M. .

For the following analysis the M_{ij} matrices will be relabelled and defined by type and operation performed.

Matrices in the form of (3.2) that will eliminate a lower off-diagonal element in the position (i,j), where i and j are the row and column numbers respectively of the element to be eliminated, will be identified as F_{ij} (3.6) Forward Elimination.

$$F_{ij} = \begin{array}{cc} & \begin{matrix} & j & & \end{matrix} \\ \begin{matrix} \\ \\ i \\ \\ \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad i > j \qquad (3.6)$$

Matrices in the form of (3.3) which transform a diagonal element in position (i,i) to 1 are designated D_{ii} (3.7) Diagonal Reduction.

$$D_{ii} = \begin{array}{cc} & \begin{matrix} & & j & \end{matrix} \\ \begin{matrix} \\ \\ i \\ \\ \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad i = j \qquad (3.7)$$

And third, matrices of the form of (3.2) which eliminate an upper off-diagonal term will be denoted as B_{ij} (3.8) Back Substitution.

$$B_{ij} = \begin{array}{cc} \begin{matrix} & & j & \end{matrix} & \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{matrix} \\ i \\ \\ \\ \end{matrix} \end{array} \qquad j > i \qquad (3.8)$$

The proper order of these elementary matrices simply follows the steps outlined for the G. E. M. i. e., convert the diagonal element to one, zero all off-diagonal elements in that column and then repeat the process for the entire matrix. This is demonstrated with the use of the elementary matrices in (3.9) where T represents the triangularized matrix.

$$T = (D_{n,n} (F_{n,n-1} (D_{n-1,n-1} (F_{n-1,n-2} (D_{n-2,n-2} (F_{n,n-3} (\ldots\ldots (D_{2,2} (F_{n,1} \ldots\ldots (F_{3,1} (F_{2,1} (D_{1,1} Y)))) \qquad (3.9)$$

In each step of this forward triangularization operation, the major element, non-zero and non-unity, of the D_{ii} and F_{ij} matrices are equal to $1 / a_{ii}^{(i-1)}$ and $(- a_{ij}^{(j-1)} \; (i > j))$ respectively which are the values computed in the G. E. M.. The operations of back substitution are recorded in the B_{ij} matrices which, starting with the element in position (n-1,n), eliminate all off-diagonal elements above the diagonal column by column. The major element of the B_{ij} matrix is the $(- a_{ij}^{(i)})$ element of the triangularized matrix T.(3.9) The entire procedure of transforming Y to U by the use of the F_{ij}, D_{ii}, and B_{ij} matrices is expressed in (3.10).

$$U = \underbrace{\begin{array}{l} (B_{12} (B_{13} (B_{23} (B_{14} (B_{24} (B_{34} \cdots\cdots \\ (B_{1,n-1} (B_{2,n-1} \cdots\cdots (B_{n-3,n-1} (B_{n-2,n-1} \\ (B_{1,n} (B_{2,n} (B_{3,n} (B_{n-2,n} (B_{n-1,n} \end{array}}_{\text{Back Substitution}}$$

$$\underbrace{\begin{array}{l} (D_{nn} (F_{n,n-1} (D_{n-1,n-1} (F_{n,n-2} (F_{n-1,n-2} \\ (D_{n-2,n-2} \cdots\cdots (F_{n,1} (F_{n-1,1} (F_{n-2,1} \\ (F_{31} (F_{21} (D_{11} Y))))) \end{array}}_{\text{Forward Elimination}} \tag{3.10}$$

Combining equations (2.2) , (3.4) and (3.10) the solution to (1.2) can now be written as (3.11).

$$E = \underbrace{\begin{array}{l} (B_{12} (B_{13} (B_{23} (B_{14} (B_{24} (B_{34} \cdots\cdots \\ (B_{1,n-1} (B_{2,n-1} \cdots\cdots (B_{n-3,n-1} (B_{n-2,n-1} \\ (B_{1,n} (B_{2,n} \cdots\cdots (B_{n-2,n} (B_{n-1,n} \end{array}}_{\text{Back Substitution}}$$

$$\underbrace{\begin{array}{l} (D_{nn} (F_{n,n-1} (D_{n-1,n-1} (F_{n,n-2} (F_{n-1,n-2} \\ (D_{n-2,n-2} \cdots\cdots (F_{n,1} (F_{n-1,1} (F_{n-2,1} \\ (F_{31} (F_{21} (D_{11} I))))) \end{array}}_{\text{Forward Elimination}} \tag{3.11}$$

Since I is a column matrix and the F_{ij} , D_{ii} and B_{ij} matrices differ from the unit matrix U by only one element, each F_{ij} , D_{ii} and B_{ij} define the multilpication of one element or the multiplication of one element and the addition of this product to another element in I. The clue to which of the two above actions take place is provided by the ij subscript of each elementary matrix. The F_{ij} and B_{ij} matrices have a non-zero, non-unity element in position ij, when a N x N matrix of this type is multiplying a N x 1 column matrix the change in the column matrix is in the $(i,1)^{th}$ element to which has been added the $(j,1)^{th}$ element times the $(i,j)^{th}$ element of the F_{ij} or B_{ij} matrix.(3.12)

$$\begin{array}{cc} & \begin{array}{cccc} & & j & \end{array} \\ \begin{array}{c} \\ i \\ \\ \\ \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 + 3c \\ 3 \\ 4 \end{bmatrix} \qquad (3.12)$$

The effect of multiplying a N x 1 column matrix by either a B_{ij} or F_{ij} matrix.

The N x N D_{ii} matrix has its major element on the diagonal in the $(i,i)^{th}$ position. When multiplying a N x 1 column matrix the only change is in the $(i,1)^{th}$ element which is multiplied by the value of the D_{ii} major element.(3.13)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2c \\ 3 \\ 4 \end{bmatrix} \qquad (3.13)$$

The result of multipling a N x 1 column matrix by a D_{ii} matrix.

Using these facts, it is only necessary to record the major non-zero, non-unity element, it's position and type of matrix it is in F_{ij} , B_{ij} or D_{ii} and not all of the N x N F_{ij} , B_{ij} and D_{ii} matrices. With this information recorded and equation (3.11), equation (1.2) can be solved for many different column matrices of I without repeating the G.E.M.. In large problems this technique could result in significant savings of time and computer memory.

<u>Symmetry of the A Matrix</u>.

When the Y matrix is symmetrical, it was previously shown that its symmetry is preserved in its resulting sub-matrix as it undergoes the G.E.M.. This fact will now be used to relate the F_{ij} and B_{ij} matrices thus reducing the size of (3.11) and the time required for solution. Each matrix F_{ij} is related to the transpose of matrix B_{ij}.(3.14)

$$F_{ij} = D_{jj}\, B^{T}_{ji}\, D^{-1}_{jj} \qquad (3.14)$$

$$F_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -a_{ij} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad D_{jj} = \begin{bmatrix} 1/a_{jj}^{(j-1)} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_{jj}^{-1} = \begin{bmatrix} a_{jj}^{(j-1)} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad B_{ji} = \begin{bmatrix} 1 & a_{ji}^{(j)} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$a_{ji}^{(j)} = a_{ji}^{(j-1)} \;/\; a_{jj}^{(j-1)} \tag{3.15}$$

The major element of the B_{ji} matrix is $a_{ji}^{(j)}$, where $i > j$, obtained from the last expression in (3.15). The B_{ji} matrix differs from the unit matrix by one element, namely the ji^{th} element which is equal to $a_{ji}^{(j)}$. The transpose of B_{ji}, B_{ji}^{T}, differs from the unit matrix in the ij^{th} element which is equal to $a_{ji}^{(j)}$.(3.16)

$$B_{ji}^{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_{ji}^{(j)} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.16}$$

The B_{ji}^{T} matrix can be expressed as:

$$B_{ji}^{T} = U + 0_{ji}^{T} \tag{3.17}$$

where U is the unit matrix and 0_{ji} is the null matrix with it's ji^{th} element replaced by $a_{ji}^{(j)}$.

$$0_{ji} = \begin{bmatrix} 0 & a_{ji}^{(j)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (3.17) \qquad 0_{ji}^{T} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_{ji}^{(j)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore:

$$\begin{aligned} D_{jj}\, B_{ji}^{T}\, D_{jj}^{-1} &= D_{jj}\, (\, U + 0_{ji}^{T}\,)\, D_{jj}^{-1} \\ &= (\, D_{jj} + D_{jj}\, 0_{ji}^{T}\,)\, D_{jj}^{-1} \\ &= U + D_{jj}\, 0_{ji}^{T}\, D_{jj}^{-1} \end{aligned} \qquad (3.18)$$

From (3.15), the jj^{th} element of D_{jj} is ($1 / a_{jj}^{(j-1)}$) and the jj^{th} element of D_{jj}^{-1} is ($a_{jj}^{(j-1)}$). Consequently, $D_{jj}\, B_{ji}^{T}\, D_{jj}^{-1}$ is an elementary matrix differing from a unit matrix with its ij^{th} element equal to $a_{ji}^{(j-1)}$.

In (3.10), after the premultiplication by $F_{n,\, j-1}$ is performed, the subnatrix obtained by this elimination of the rows and columns of index less than j is symmetric,

$$a_{ij}^{(j-1)} = a_{ji}^{(j-1)}$$

for j = 2 refer to expression (2.13). Thus the elementary matrix formed by $D_{jj}\, B_{ji}^{T}\, D_{jj}^{-1}$ is equivalent to a matrix different from a unit matrix with its ij^{th} element equal to $a_{ij}^{(j-1)}$, which is the F_{ij} matrix.

Now substituting (3.14) into (3.11), the equation for the solution of (1.2) with a symmetric Y matrix is (3.19)

$$E = (B_{12} (B_{13} (B_{23} (B_{14} (B_{24} (B_{34} \cdots\cdots$$
$$(B_{1,n-1} (B_{2,n-1} \cdots\cdots (B_{n-2,n-1} (B_{1n} (B_{2n}$$
$$(\cdots\cdots (B_{n-2,n} (B_{n-1,n} (D_{nn} (D_{n-1,n-1} B^T_{n-1,n}$$
$$D^{-1}_{n-1,n-1} (D_{n-1,n-1} (D_{n-2,n-2} B^T_{n-2,n} D^{-1}_{n-2,n-2}$$
$$(D_{n-2,n-2} B^T_{n-2,n-1} D^{-1}_{n-2,n-2} (D_{n-2,n-2} (\cdots\cdots$$
$$(D_{11} B^T_{1,n} D^{-1}_{11} (D_{11} B^T_{1,n-1} D^{-1}_{11} (D_{11} B^T_{1,n-2} D^{-1}_{11} (\cdot\cdot$$
$$(D_{11} B^T_{13} D^{-1}_{11} (D_{11} B^T_{12} D^{-1}_{11} (D_{11} I)))) \qquad (3.19)$$

Canceling the quantities (D_{jj} D^{-1}_{jj}) in (3.19) reduces it to the form (3.20).

$$E = (B_{12} (B_{13} (B_{23} (B_{14} (B_{24} (B_{34} \cdots\cdots$$
$$(B_{1,n-1} (B_{2,n-1} \cdots\cdots (B_{n-2,n-1} (B_{1n} (B_{2n}$$
$$(\cdots\cdots (B_{n-2,n} (B_{n-1,n} (D_{nn} (D_{n-1,n-1} (B^T_{n-1,n}$$
$$(D_{n-2,n-2} (B^T_{n-2,n} (B^T_{n-2,n-1} (D_{n-3,n-3} (B^T_{n-3,n}$$
$$(B^T_{n-3,n-1} (B^T_{n-3,n-2} (\cdots\cdots (D_{11} (B^T_{1,n} (B^T_{1,n-1}$$
$$(B^T_{1,n-2} \cdots\cdots (B^T_{13} (B^T_{12} I)))) \qquad (3.20)$$

Equation (3.20) provides the form of the solution to (1.2) if the Y matrix is symmetrical. It also indicates that the F_{ij} matrices, corresponding to the forward elimination in the G.E.M., are not necessary. This reduces the required amount of computer memory needed to solve the problem since the major elements of the F_{ij} matrices need not be recorded.

Combining Similar Types of Elementary Matrices.[2]

An examination of equation (3.11) reveals that matrices in the form (3.21) are grouped together.

$$F_{ij} \qquad i = j+1,\ j+2,\ \ldots\ldots,\ N \qquad (3.21)$$

This is a result of the Gauss Elimination Method (G.E.M.), because the Y matrix was triangularized column by column. The matrices in (3.21) differ from the unit matrix in the $(i.j)^{th}$ element and from each other by one element in the i^{th} row. By understanding their individual effects on a column vector, as demonstrated in (3.12), these matrices can be combined into a matrix that differs from the unit matrix in the j^{th} column.(3.22)

$$C_j : (\text{ group of } F_{ij} \text{ matrices with } i = j+1,\ j+2,\ \ldots,\ N \qquad (3.22)$$

$j = 2$

$$C_2 = \begin{bmatrix} 1 & 0 & 0 \ldots\ldots & 0 \\ 0 & 1 & 0 \ldots\ldots & 0 \\ 0 & f_{32} & 1 \ldots\ldots & 0 \\ 0 & f_{42} & 0 \ldots\ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & f_{N2} & 0 \ldots\ldots & 1 \end{bmatrix} \qquad (3.23)$$

The f_{ij} elements in (3.23) are the major elements in the

2 W.F. Tinney & J.W. Walker."Direct Solutions of Sparse Network Equation by Optimally Ordered Triangular Factorization" Proceeding of the IEEE, Nov. 1967 p. 1801

individual F_{ij} matrices. Also in (3.11) the B_{ij} matrices in the form of (3.24) are multiplied together.

$$B_{ij} \qquad i = j+1,\ j-2,\ \ldots\ldots,\ 1 \qquad (3.24)$$

These matrices correspond to the back elimination process. Again by understanding the effects of these matrices, they can be grouped into a single matrix differing from the unit matrix only in the j^{th} column.

$$R_j : (\text{ group of } B_{ij} \text{ matrices } i = j-1,\ j-2,\ \ldots\ldots,\ 1\ ) \qquad (3.25)$$

$j = 3$

$$R_3 = \begin{bmatrix} 1 & 0 & b_{13} & 0 & \ldots\ldots & 0 \\ 0 & 1 & b_{23} & 0 & \ldots\ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots\ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \ldots\ldots & 1 \end{bmatrix} \qquad (3.26)$$

The b_{ij} are the major elements in the individual B_{ij} matrices.

Since the C_j and R_j matrices have only a single subscript matrices of the D_{ii} type will now be relabelled as D_i and will remain unchanged in form. Using equations (3.26) and (3.23), equation (3.11) can be written in the following form.(3.27)

$$E = (\ R_1\ (\ R_2\ (\ R_3\ \ldots\ (\ R_{n-2}\ (\ R_{n-1}\ (\ D_n\ (\ C_{n-1}\ (\ D_{n-1}\ \ldots\ (\ C_2\ (\ D_2\ (\ C_1\ (\ D_1\ I\ )))) \qquad (3.27)$$

When the Y matrix is symmetrical, the relationship between the F_{ij} and B_{ij} matrices can be extended to the C_j and R_j matrices.(3.28)

$$C_j = D_j \; R_j^T \; D_j^{-1} \qquad (3.28)$$

Substituting (3.28) into (3.27), the solution to (1.2) becomes (3.29).

$$E = (R_1 (R_2 \ldots (R_{n-2} (R_{n-1} (D_n (D_{n-1} (R_{n-1}^T \ldots (D_2 (R_2^T (D_1 (R_1^T \; I)))) \qquad (3.29)$$

Table of Factors[3]

The final step in this recording technique, developed by Tinney's group, is the formation of a Table of Factors. Elements for the Table of Factors are created as the Gauss Elimination Method is performed on the characteristic matrix of the system. Each element in the N x N field is the major element in the corresponding elementary matrix.(3.30)

$$\begin{vmatrix} d_{11} & b_{12} & b_{13} & \cdots\cdots & b_{1N} \\ f_{21} & d_{22} & b_{23} & \cdots\cdots & b_{2N} \\ f_{31} & f_{32} & d_{33} & \cdots\cdots & b_{3N} \\ \vdots & \vdots & \vdots & & \vdots \\ f_{N1} & f_{N2} & f_{N3} & & d_{NN} \end{vmatrix} \qquad (3.30)$$

3 W.F. Tinney & J.W. Walker."Direct Solutions of Sparse Network Equation by Optimally Ordered Triangular Factorization" Proceeding of the IEEE, Nov. 1967 p. 1801

The Table of Factors contains all the information required to compute the response of the system to any set of input conditions. Also, as will be demonstrated in a following example, the Table of Factors retains approximately the same degree of sparsity as the original Y matrix. This added benefit of sparsity makes this triangular factorization method superior to most others used in the past. Additional savings in calculations and storage space can be realized when triangular factorization is used in conjunction with the matrix ordering and storage schemes which will be developed in the next section.

It is important to realize that the magnitude of the decrease in storage space and calculations must be related to the system under investigation. Those systems which are larger and more closely interconnected and therefore less sparse might not be able to achieve as significant a savings in storage space and calculations as a much sparser network might.

Although most of today's power engineers would gladly accept a reduction in storage space and calculations, the cost of implementing a new method of solution must be carefully weighed against the actual savings afforded by the new method and its reliability. The reliability of the triangular factorization method is very good, testimony to this fact can be found in many power systems where it has been used for many years. However the economic feacibility of replacing a presently used method with triangular factorization must be evaluated on a system by system basis.

<u>Example</u>: This is an example of the Gauss Elimination Method and the techniques for recording the G.E.M. for repetitive solution for the six bus network[3] of figure one.

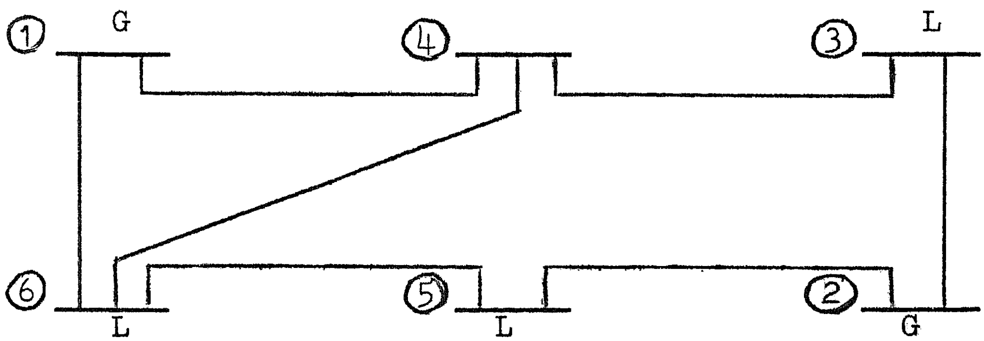


Figure 1

The busses are numbered in arbitrary order and are encircled.

Busses 1 and 2 are generating busses G.

Busses 2, 3, 5, and 6 are load busses L.

Equation (1.2) will be solved given the following I and Y matrices for the system.

$$I = \begin{bmatrix} 1.0 & 0^\circ \\ 1.0 & 0^\circ \\ 1.0 & 0^\circ \\ 1.0 & 0^\circ \\ 1.0 & 0^\circ \\ 1.0 & 0^\circ \end{bmatrix}$$

3 J.B. Ward & H.W. Hale,"Digital Computer Solution of Power Flow Problems",AIEE Trans. Power,June 1956, p. 398

$$Y = \begin{bmatrix} 1.0-j4.4 & 0 & 0 & -0.6+j2.6 & 0 & -0.4+j1.8 \\ 0 & 1.0-j1.9 & -0.4+j0.6 & 0 & -0.6+j1.3 & 0 \\ 0 & -0.4+j0.6 & 0.9-j8.2 & 0 & 0 & 0 \\ -0.6+j2.6 & 0 & 0 & 1.1-j13.9 & 0 & -0.5+j2.3 \\ 0 & -0.6+j1.3 & 0 & 0 & 1.6-j4.6 & j3.4 \\ -0.4+j1.8 & 0 & 0 & -0.5+j2.3 & j3.4 & 0.9-j7.6 \end{bmatrix}$$

% of sparsity = 100 x (number of elements equal to zero/ total number of elements).

% of sparsity of Y = 100 x (17 / 36) = 47.2 %

The problem will be solved in the following manner:

Part 1 The Y matrix will be triangularized by the Gauss Elimination Method and a table of factors will be formed.

Part 2 Neglecting the symmetry of the Y matrix, an equation using the B_{ij}, F_{ij} and D_{ii} matrices will be developed.

Part 3 Substituting the proper values from the table of factors into the corresponding elementary matrices, the equation developed in Part 2 will be solved for the E matrix by using the properties of these matrices outlined in equations (3.12) and (3.13).

Part 4 An equation using the elementary B_{ij}, F_{ij} and D_{ii} matrices will be written taking into account the symmetry of the Y matrix.

Part 5 The C_j, R_j and D_j matrices will be identified and equations for the solution of the problem using this method will be formulated with and without considering the symmetry of the Y matrix.

Part 1

Elimination of column 1

$$\begin{bmatrix}
1.0 & 0 & 0 & 0.60\ 0.2^{\circ} & 0 & 0.41\ 0.3^{\circ} \\
0 & 2.15\ -62.2^{\circ} & 0.72\ -56.3^{\circ} & 0 & 1.43\ -65.2^{\circ} & 0 \\
0 & 0.72\ -56.3^{\circ} & 8.24\ -83.7^{\circ} & 0 & 0 & 0 \\
0 & 0 & 0 & 15.5\ -84.6^{\circ} & 0 & 1.26\ 101.5^{\circ} \\
0 & 1.43\ -65.2^{\circ} & 0 & 0 & 4.87\ -70.8^{\circ} & 3.4\ 90^{\circ} \\
0 & 0 & 0 & 1.26\ 101.5^{\circ} & 3.4\ 90^{\circ} & 8.4\ -82.7^{\circ}
\end{bmatrix}$$

Elimination of column 2

$$\begin{bmatrix}
1.0 & 0 & 0 & 0.60\ 0.2^{\circ} & 0 & 0.41\ 0.3^{\circ} \\
0 & 1.0 & 0.33\ 5.9^{\circ} & 0 & 0.66\ -3.0^{\circ} & 0 \\
0 & 0 & 8.22\ -84.7^{\circ} & 0 & 0.47\ 120.7^{\circ} & 0 \\
0 & 0 & 0 & 15.5\ -84.6^{\circ} & 0 & 1.26\ 101.5^{\circ} \\
0 & 0 & 0.47\ 120.7^{\circ} & 0 & 3.93\ -71.4^{\circ} & 3.4\ 90^{\circ} \\
0 & 0 & 0 & 1.26\ 101.5^{\circ} & 3.4\ 90^{\circ} & 8.4\ -82.7^{\circ}
\end{bmatrix}$$

Elimination of column 3

$$\begin{bmatrix}
1.0 & 0 & 0 & 0.60\ 0.2^{\circ} & 0 & 0.41\ 0.3^{\circ} \\
0 & 1.0 & 0.33\ 5.9^{\circ} & 0 & 0.66\ -3.0^{\circ} & 0 \\
0 & 0 & 1.0 & 0 & 0.06\ 205.4^{\circ} & 0 \\
0 & 0 & 0 & 15.5\ -84.6^{\circ} & 0 & 1.26\ 101.5^{\circ} \\
0 & 0 & 0 & 0 & 3.9\ -71.7^{\circ} & 3.4\ 90^{\circ} \\
0 & 0 & 0 & 1.26\ 101.5^{\circ} & 3.4\ 90^{\circ} & 8.4\ -82.7^{\circ}
\end{bmatrix}$$

Elimination of column 4

$$\begin{bmatrix} 1.0 & 0 & 0 & 0.60\ 0.2^\circ & 0 & 0.41\ 0.3^\circ \\ 0 & 1.0 & 0.33\ 5.9^\circ & 0 & 0.66\ -3.0^\circ & 0 \\ 0 & 0 & 1.0 & 0 & 0.06\ 205.4^\circ & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0.08\ 186.1^\circ \\ 0 & 0 & 0 & 0 & 3.9\ -71.7^\circ & 3.4\ 90^\circ \\ 0 & 0 & 0 & 0 & 3.4\ 90^\circ & 8.3\ -82.8^\circ \end{bmatrix}$$

Elimination of column 5

$$\begin{bmatrix} 1.0 & 0 & 0 & 0.60\ 0.2^\circ & 0 & 0.41\ 0.3^\circ \\ 0 & 1.0 & 0.33\ 5.9^\circ & 0 & 0.66\ -3.0 & 0 \\ 0 & 0 & 1.0 & 0 & 0.06\ 205.4^\circ & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0.08\ 186.1^\circ \\ 0 & 0 & 0 & 0 & 1.0 & 0.87\ 161.7^\circ \\ 0 & 0 & 0 & 0 & 0 & 5.7\ -70.0^\circ \end{bmatrix}$$

Elimination of column 6

$$\begin{bmatrix} 1.0 & 0 & 0 & 0.60\ 0.2^\circ & 0 & 0.41\ 0.3^\circ \\ 0 & 1.0 & 0.33\ 5.9^\circ & 0 & 0.66\ -3.0^\circ & 0 \\ 0 & 0 & 1.0 & 0 & 0.06\ 205.4^\circ & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0.08\ 186.1^\circ \\ 0 & 0 & 0 & 0 & 1.0 & 0.87\ 161.7^\circ \\ 0 & 0 & 0 & 0 & 0 & 1.0 \end{bmatrix}$$

Back Substitution Process

Column 6

$$\begin{bmatrix} 1.0 & 0 & 0 & 0.6\ 0.2^{\circ} & 0 & 0 \\ 0 & 1.0 & 0.33\ 5.9^{\circ} & 0 & 0.66\ -3.0^{\circ} & 0 \\ 0 & 0 & 1.0 & 0 & 0.06\ 205.4^{\circ} & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \end{bmatrix}$$

Column 5

$$\begin{bmatrix} 1.0 & 0 & 0 & 0.6\ 0.2^{\circ} & 0 & 0 \\ 0 & 1.0 & 0.33\ 5.9^{\circ} & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \end{bmatrix}$$

Column 4

$$\begin{bmatrix} 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0.33\ 5.9^{\circ} & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \end{bmatrix}$$

Column 3

$$\begin{bmatrix} 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \end{bmatrix}$$

Table of Factors

$$\left| \begin{array}{llllll} 0.22\ 77.2^\circ & 0 & 0 & 0.6\ -179.8^\circ & 0 & 0.41\ 180.3^\circ \\ 0 & 0.46\ 62.2^\circ & 0.33\ -174.1^\circ & 0 & 0.66\ 177^\circ & 0 \\ 0 & 0.72\ 123.7^\circ & 0.12\ 95.3^\circ & 0 & 0.06\ 25.4 & 0 \\ 2.67\ 103^\circ & 0 & 0 & 0.06\ 84.6 & 0 & 0.08\ -6.1^\circ \\ 0 & 1.43\ 114.8^\circ & 0.47\ -59.3^\circ & 0 & 0.25\ 71.7^\circ & 0.87\ -18.3^\circ \\ 1.84\ 102.5^\circ & 0 & 0 & 1.26\ 101.5^\circ & 3.4\ -90^\circ & 0.17\ 70.9^\circ \end{array} \right.$$

% of sparsity in Table of Factors = 16/36 x 100 = 44.4 %

% of sparsity in original Y matrix = 47.2 %

A significant percentage of the original Y matrix sparsity has been preserved in the Table of Factors.

Part 2

From (3.11), the solution to (1.2) employing the B_{ij} , F_{ij} and D_{ii} elementary matrices is:

$$E = (B_{23} (B_{14} (B_{25} (B_{35} (B_{16} (B_{46} (B_{56}$$

$$(D_{66} (F_{65} (D_{55} (F_{64} (D_{44} (F_{53} (D_{33}$$

$$(F_{52} (F_{32} (D_{22} (F_{61} (F_{41} (D_{11} I)))$$

Part 3

I	F_{41}	F_{61}	D_{22}	F_{32}
$1.00\angle 0.0^\circ$	$0.22\angle 77.2^\circ$	$0.22\angle 77.2^\circ$	$0.22\angle 77.2^\circ$	$0.22\angle 77.2^\circ$
$1.00\angle 0.0^\circ$	$1.00\angle 0.0^\circ$	$1.00\angle 0.0^\circ$	$0.46\angle 62.2^\circ$	$0.46\angle 62.2^\circ$
$1.00\angle 0.0^\circ$	$1.00\angle 0.0^\circ$	$1.00\angle 0.0^\circ$	$1.00\angle 0.0^\circ$	$0.67\angle -2.6^\circ$
$1.00\angle 0.0^\circ$	$0.41\angle 0.0^\circ$	$0.41\angle 0.0^\circ$	$0.41\angle 0.0^\circ$	$0.41\angle 0.0^\circ$
$1.00\angle 0.0^\circ$	$1.00\angle 0.0^\circ$	$1.00\angle 0.0^\circ$	$1.00\angle 0.0^\circ$	$1.00\angle 0.0^\circ$
$1.00\angle 0.0^\circ$	$1.00\angle 0.0^\circ$	$0.60\angle 0.0^\circ$	$0.60\angle 0.0^\circ$	$0.60\angle 0.0^\circ$

F_{52}	D_{33}	F_{53}	D_{44}	F_{64}
$0.22\angle 77.2^\circ$	$0.22\angle 77.2^\circ$	$0.22\angle 77.2^\circ$	$0.22\angle 77.2^\circ$	$0.22\angle 77.2^\circ$
$0.46\angle 62.2^\circ$	$0.46\angle 62.2^\circ$	$0.46\angle 62.2^\circ$	$0.46\angle 62.2^\circ$	$0.46\angle 62.2^\circ$
$0.67\angle -2.6^\circ$	$0.08\angle 92.7^\circ$	$0.08\angle 92.7^\circ$	$0.08\angle 92.7^\circ$	$0.08\angle 92.7^\circ$
$0.41\angle 0.0^\circ$	$0.41\angle 0.0^\circ$	$0.41\angle 0.0^\circ$	$0.02\angle 84.6^\circ$	$0.02\angle 84.6^\circ$
$0.34\angle 5.0^\circ$	$0.34\angle 5.0^\circ$	$0.37\angle 7.7^\circ$	$0.37\angle 7.7^\circ$	$0.37\angle 7.7^\circ$
$0.60\angle 0.0^\circ$	$0.60\angle 0.0^\circ$	$0.60\angle 0.0^\circ$	$0.60\angle 0.0^\circ$	$0.60\angle 0.0^\circ$

$$
\begin{array}{ccccc}
D_{55} & F_{65} & D_{66} & B_{56} & \mathrm{B46} \\
\begin{bmatrix} 0.22\underline{/77.2^\circ} \\ 0.46\underline{/62.2^\circ} \\ 0.08\underline{/92.7^\circ} \\ 0.02\underline{/84.6^\circ} \\ 0.09\underline{/79.4^\circ} \\ 0.57\underline{/\ 0^\circ} \end{bmatrix} &
\begin{bmatrix} 0.22\underline{/77.2^\circ} \\ 0.46\underline{/62.2^\circ} \\ 0.08\underline{/92.7^\circ} \\ 0.02\underline{/84.6^\circ} \\ 0.09\underline{/79.4^\circ} \\ 0.87\underline{/-3.7^\circ} \end{bmatrix} &
\begin{bmatrix} 0.22\underline{/77.2^\circ} \\ 0.46\underline{/62.2^\circ} \\ 0.08\underline{/92.7^\circ} \\ 0.02\underline{/84.6^\circ} \\ 0.09\underline{/79.4^\circ} \\ 0.15\underline{/67.2^\circ} \end{bmatrix} &
\begin{bmatrix} 0.22\underline{/77.2^\circ} \\ 0.46\underline{/62.2^\circ} \\ 0.08\underline{/92.7^\circ} \\ 0.02\underline{/84.6^\circ} \\ 0.21\underline{/61.2^\circ} \\ 0.15\underline{/67.2^\circ} \end{bmatrix} &
\begin{bmatrix} 0.22\underline{/77.2^\circ} \\ 0.46\underline{/62.2^\circ} \\ 0.08\underline{/92.7^\circ} \\ 0.03\underline{/76.8^\circ} \\ 0.21\underline{/61.2^\circ} \\ 0.15\underline{/67.2^\circ} \end{bmatrix}
\end{array}
$$

$$
\begin{array}{ccccc}
B_{16} & B_{35} & B_{25} & B_{14} & B_{23} \\
\begin{bmatrix} 0.16\underline{/81.0^\circ} \\ 0.46\underline{/62.2^\circ} \\ 0.08\underline{/92.7^\circ} \\ 0.03\underline{/76.8^\circ} \\ 0.21\underline{/61.2^\circ} \\ 0.15\underline{/67.2^\circ} \end{bmatrix} &
\begin{bmatrix} 0.16\underline{/81.0^\circ} \\ 0.46\underline{/62.2^\circ} \\ 0.09\underline{/91.8^\circ} \\ 0.03\underline{/76.8^\circ} \\ 0.21\underline{/61.2^\circ} \\ 0.15\underline{/67.2^\circ} \end{bmatrix} &
\begin{bmatrix} 0.16\underline{/81.0^\circ} \\ 0.32\underline{/64.0^\circ} \\ 0.09\underline{/91.8^\circ} \\ 0.03\underline{/76.8^\circ} \\ 0.21\underline{/61.2^\circ} \\ 0.15\underline{/67.2^\circ} \end{bmatrix} &
\begin{bmatrix} 0.14\underline{/81.5^\circ} \\ 0.32\underline{/64.0^\circ} \\ 0.09\underline{/91.8^\circ} \\ 0.03\underline{/76.8^\circ} \\ 0.21\underline{/61.2^\circ} \\ 0.15\underline{/67.2^\circ} \end{bmatrix} &
\begin{bmatrix} 0.14\underline{/81.5^\circ} \\ 0.29\underline{/61.0^\circ} \\ 0.09\underline{/91.8^\circ} \\ 0.03\underline{/76.8^\circ} \\ 0.21\underline{/61.2^\circ} \\ 0.15\underline{/67.2^\circ} \end{bmatrix}
\end{array}
$$

$$
\begin{array}{c}
E \\
\begin{bmatrix} 0.14\underline{/81.5^\circ} \\ 0.29\underline{/61.0^\circ} \\ 0.09\underline{/91.8^\circ} \\ 0.03\underline{/76.8^\circ} \\ 0.21\underline{/61.2^\circ} \\ 0.15\underline{/67.2^\circ} \end{bmatrix}
\end{array}
$$

Part 4

An alternate solution could be performed by taking into consideration the symmetry of the Y matrix in Part 2 and using equation (3.20). The solution of (1.2) then becomes:

$$E = (B_{23} (B_{14} (B_{25} (B_{35} (B_{16} (B_{46} (B_{56}$$
$$(D_{66} (D_{55} (B^T_{56} (D_{44} (B^T_{46} (D_{33} (B^T_{35}$$
$$(D_{22} (B^T_{25} (B^T_{23} (D_{11} (B^T_{16} (B^T_{14} \; I))))$$

The solution to the above equation is executed in the same manner as Part 3 using the properties of these elementary matrices demonstrated in (3.12) and (3.13).

Part 5

Another method of solution would use the C_i , R_i and D_i matrices. With (3.27), which does not take into account the symmetry of the Y matrix, the equation for the solution is:

$$E = (R_1 (R_2 (R_3 (R_4 (R_5 (D_6 (C_5 (D_5 (C_4$$
$$(D_4 (C_3 (D_3 (C_2 (D_2 (C_1 (D_1 \; I))))$$

Considering the symmetry with (3.29), the above equation becomes:

$$E = (R_1 (R_2 (R_3 (R_4 (R_5 (D_6 (D_5 (R^T_5 (D_4$$
$$(R^T_4 (D_3 (R^T_3 (D_2 (R^T_2 (D_1 (R^T_1 \; I))))$$

In either of the last two equations the R_i and C_i matrices, which were created from the information available in the Table of Factors, are as follows:

$$C_1 = \begin{bmatrix} 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 & 0 \\ 2.67\underline{/103^\circ} & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \\ 1.84\underline{/102.5^\circ} & 0 & 0 & 0 & 0 & 1.0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0.72\underline{/123.7^\circ} & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 1.43\underline{/114.8^\circ} & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0.47\underline{/-59.3^\circ} & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \end{bmatrix}$$

$$C_4 = \begin{bmatrix} 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.26\underline{/101.5^\circ} & 0 & 1.0 \end{bmatrix}$$

$$C_5 = \begin{bmatrix} 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 3.4\underline{/90^\circ} & 1.0 \end{bmatrix}$$

$$R_3 = \begin{bmatrix} 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0.33\underline{/-174.1^\circ} & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \end{bmatrix}$$

$$R_4 = \begin{bmatrix} 1.0 & 0 & 0 & 0.6\underline{/-179.8^\circ} & 0 & 0 \\ 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \end{bmatrix}$$

$$R_5 = \begin{bmatrix} 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 & 0.66\underline{/177^\circ} & 0 \\ 0 & 0 & 1.0 & 0 & 0.06\underline{/25.4^\circ} & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \end{bmatrix}$$

$$R_6 = \begin{bmatrix} 1.0 & 0 & 0 & 0 & 0 & 0.41\underline{/180.3^\circ} \\ 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0.08\underline{/-6.1^\circ} \\ 0 & 0 & 0 & 0 & 1.0 & 0.97\underline{/-18.3^\circ} \\ 0 & 0 & 0 & 0 & 0 & 1.0 \end{bmatrix}$$

The D_i matrices differ from the unit matrix in the d_{ii} element.

$$D_1 : d_{11} = 0.22 \, \underline{/77.2^\circ}$$

$$D_2 : d_{22} = 0.46 \, \underline{/62.2^\circ}$$

$$D_3 : d_{33} = 0.12 \, \underline{/95.3^\circ}$$

$$D_4 : d_{44} = 0.06 \, \underline{/84.6^\circ}$$

$$D_5 : d_{55} = 0.25 \, \underline{/71.7^\circ}$$

$$D_6 : d_{66} = 0.17 \, \underline{/70.9^\circ}$$

Chapter IV

SPARSE MATRIX ORDERING AND COMPUTER STORAGE

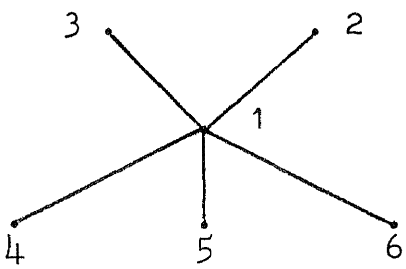
In an electric power system the matrix of complex short-circuit driving point and transfer admittances or the Y matrix is relatively sparse, i.e., there are relatively few non-zero off-diagonal terms. Alternatively, the matrix of complex open-circuit driving point and transfer impedances or Z matrix of the same system is full, i.e., there are very few off-diagonal elements equal to zero. Since the Gauss Elimination Method reduces or zeroes all lower off-diagonal elements, the Y matrix could conceivably require fewer operations to triangularize it than the corresponding Z matrix. The reason for the sparsity in the Y matrix is the diversity in the interconnections between the buses of the system. The purpose behind sparse matrix ordering is to arrange the Y matrix, so that during the triangular factorization process a minimum number of operations need be performed and as much of the original matrix sparsity is preserved. In this section, the efforts of Tinney's group to further improve the method of triangular factorization through the application of sparse matrix ordering will be presented and discussed.

Matrix Ordering[1,2,3,4]

The need for and benefit of matrix ordering can be easily demonstrated by using a few simple example networks. All example networks will be represented by their equivalent flowgraphs. A flowgraph, which was introduced by B. Stott in his investigation of ordering methods, is a graph of the network with all buses included and the transfer admittances replaced by single lines. Using this approach the results of matrix ordering can be seen more clearly, without the visual complications and distractions of the actual admittance values.

Example 1

In the five branch star network of figure two, the nodes have been labelled or "ordered" in a random manner. This ordering was then used to form an equivalent Y matrix.



$$Y_{eq} = \begin{bmatrix} X & X & X & X & X & X \\ X & X & 0 & 0 & 0 & 0 \\ X & 0 & X & 0 & 0 & 0 \\ X & 0 & 0 & X & 0 & 0 \\ X & 0 & 0 & 0 & X & 0 \\ X & 0 & 0 & 0 & 0 & X \end{bmatrix}$$

Figure 2

1 N. Sato & W.F. Tinney, "Techniques for Exploiting the Sparsity of the Network Admittance Matrix", PAS Trans. Dec. 1963 p. 944

2 W.F. Tinney & C.E. Hart, "Power Flow Solution by Newton's Method", IEEE PAS Trans. Nov. 1967 p. 1449

3 W.F. Tinney & J.W. Walker, "Direct Solutions of Sparse Network Equations by Optimally Ordered Triangular Factorization", Proceedings of the IEEE, Nov. 1967 p. 1801

4 B. Stott & E. Hobson, "Solution of Large Power-System Networks by Ordered Elimination: A Comparison of Ordering Schemes". Proceedings of the IEE, Jan. 1971 p. 125

As the Y_{eq} matrix undergoes the Gauss Elimination Method, for solution, the lack of sparsity in the first row will cause the upper triangular portion ofthe matrix to be completely filled in, thus destroying all of the original matrix sparsity.

Y_{eq} matrix of example 1 after Gauss Elimination;

$$Y'_{eq} = \begin{bmatrix} X & X & X & X & X & X \\ 0 & X & X & X & X & X \\ 0 & 0 & X & X & X & X \\ 0 & 0 & 0 & X & X & X \\ 0 & 0 & 0 & 0 & X & X \\ 0 & 0 & 0 & 0 & 0 & X \end{bmatrix}$$

As a measurement of the effectiveness of an ordering method, define the following relation as the percentage of sparsity preserved:

Percentage of Sparsity Preserved = 100 X

Number of upper triangular elements equal to zero after elimination / Number of upper triangular elements equal to zero before elimination

Percentage of Sparsity Preserved by the random ordering in example 1 =(0 / 10) X 100 = 0 %

With 100 % considered as the optimal ordering, this random ordering of example 1 is the most inefficient from the sparsity viewpoint.

In an effort to preserve more of the sparsity in the Y_{eq} matrix of example 1 another ordering of the system flowgraph must be used.

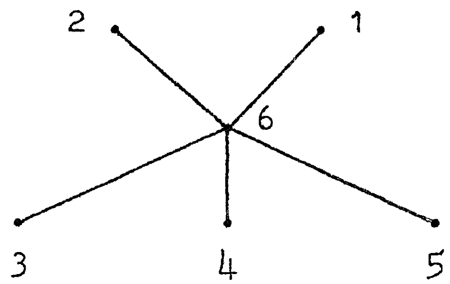


Figure 3

$$Y_{eq} = \begin{bmatrix} X & 0 & 0 & 0 & 0 & X \\ 0 & X & 0 & 0 & 0 & X \\ 0 & 0 & X & 0 & 0 & X \\ 0 & 0 & 0 & X & 0 & X \\ 0 & 0 & 0 & 0 & X & X \\ X & X & X & X & X & X \end{bmatrix}$$

As this Y_{eq} matrix is solved by the Gauss Elimination Method, there will be no further fill-in of the upper triangular portion and hence all of the original matrix sparsity has been saved.

Y_{eq} matrix after Elimination

$$Y_{eq} = \begin{bmatrix} X & 0 & 0 & 0 & 0 & X \\ 0 & X & 0 & 0 & 0 & X \\ 0 & 0 & X & 0 & 0 & X \\ 0 & 0 & 0 & X & 0 & X \\ 0 & 0 & 0 & 0 & X & X \\ 0 & 0 & 0 & 0 & 0 & X \end{bmatrix}$$

Percentage of Sparsity Preserved by new ordering of example 1

$= (10 / 10) \times 100 = 100\%$

Therefore, this ordering is the optimal ordering for this simple five branch star. However as networks get larger and more complicated, an optimal ordering will be much harder to obtain and in some cases may not exist.

In comparing the two orderings of example 1 two important observations are made:

1. The use of the flowgraph and equivalent Y matrix allow quick visualization of the results of a network ordering.
2. By numbering those nodes in the system which have the fewest number of connections first (the ends of the star branches) and those with the most connections last (the center node in thestar)more sparsity can be preserved.

This last and most significant observation will be put to use in the next example.

<u>Example 2</u>

In this second example, the network flowgraph has been ordered again in a random manner and the equivalent Y matrix has been formed.

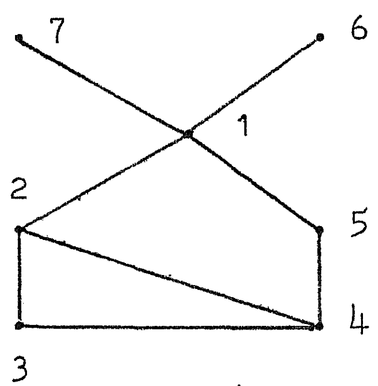


Figure 4

$$Y_{eq} = \begin{bmatrix} X & X & 0 & 0 & X & X & X \\ X & X & X & X & 0 & 0 & 0 \\ 0 & X & X & X & 0 & 0 & 0 \\ 0 & X & X & X & X & 0 & 0 \\ X & 0 & 0 & X & X & 0 & 0 \\ X & 0 & 0 & 0 & 0 & X & 0 \\ X & 0 & 0 & 0 & 0 & 0 & X \end{bmatrix}$$

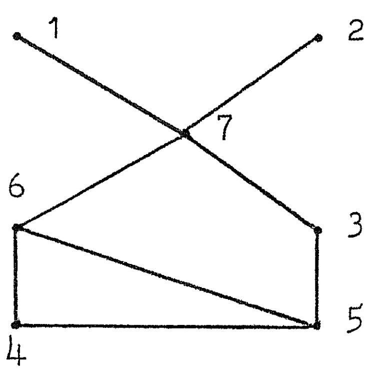
As the Y_{eq} matrix is processed by the Gauss Elimination Method most of the sparsity will be lost due to the last three non-zero terms in the first row.

Y_{eq} matrix of example 2 after elimination.

$$Y_{eq} = \begin{bmatrix} X & X & 0 & 0 & X & X & X \\ 0 & X & X & X & X & X & X \\ 0 & 0 & X & X & X & X & X \\ 0 & 0 & 0 & X & X & X & X \\ 0 & 0 & 0 & 0 & X & X & X \\ 0 & 0 & 0 & 0 & 0 & X & X \\ 0 & 0 & 0 & 0 & 0 & 0 & X \end{bmatrix}$$

Percentage of Sparsity Preserved = (2 / 13) X 100 = 15 %

Using the last observation made in the first example, this network can be reordered to save more of the system's sparsity. The new flowgraph ordering is:



$$Y_{eq} = \begin{bmatrix} X & 0 & 0 & 0 & 0 & 0 & X \\ 0 & X & 0 & 0 & 0 & 0 & X \\ 0 & 0 & X & 0 & X & 0 & X \\ 0 & 0 & 0 & X & X & X & 0 \\ 0 & 0 & X & X & X & X & 0 \\ 0 & 0 & 0 & X & X & X & X \\ X & X & X & 0 & 0 & X & X \end{bmatrix}$$

Figure 5

After elimination:

$$Y_{eq} = \begin{bmatrix} X & 0 & 0 & 0 & 0 & 0 & X \\ 0 & X & 0 & 0 & 0 & 0 & X \\ 0 & 0 & X & 0 & X & 0 & X \\ 0 & 0 & 0 & X & X & X & 0 \\ 0 & 0 & 0 & 0 & X & X & X \\ 0 & 0 & 0 & 0 & 0 & X & X \\ 0 & 0 & 0 & 0 & 0 & 0 & X \end{bmatrix}$$

Percentage of Sparsity Preserved with new ordering of example 2 = (12 / 13) X 100 = 92 %

Therefore, by simply reordering the buses in the network according to a certain scheme based on past observations a significant amount of sparsity can be preserved. These last two examples, although not representative of the actual size and high percentage (> 90 %) of sparsity in power system networks, serve to demonstrate that significant savings in time and storage space are attainable with the assistance of matrix ordering.

After investigating many power system networks, Tinney's group at the Bonneville Power Administration developed three "schemes" for the near-optimal ordering of sparse power system matrices. A scheme for the optimal ordering of a sparse power system is considered by Tinney's group and others in this field, including this author, to be much too complicated and time consuming to be economically justifiable.

Near Optimal Schemes for Sparse Matrix Ordering.

1) Number the buses by starting with the one that has the fewest connected lines or branches and ending with that bus having the most connected lines or branches.

2) Order th buses so that at each step of the elimination process the next bus to be eliminated is the one having the fewest connected lines or branches, i.e. the fewest non-zero off-diagonal terms.

3) Number the busses such that at each step of the elimination process the next row to be eliminated is the one that will introduce the fewest new non-zero off-diagonal terms.

In any of these three schemes, if more than one bus meets the particular criterion of that scheme select any one.

Notes on Each Scheme. Scheme 1 does not take into account anything that happens during the G. E. M. but it is a fast scheme to execute and simple to program. The only data required is a list of the number of branches connected to each bus in the systme. For large power systems this scheme is a trade-off between quick Y matrix formulation in return for more computations due to less sparsity preservation.

Scheme 2 requires a simulation of the elimination process to take into account the accumulation of non-zero off-diagonal terms. The data required is a list by rows of the column numbers of the non-zero off-diagonal terms. This scheme requires more time and information to execute than scheme 1 however it results in a higher percentage of system sparsity being preserved.

Scheme 3 is the most complex. It involves the trial simulation of every feasible alternative bus at each step during the entire elimination process. The data required is the same as scheme 2. Although superior to both schemes 1 and 2 in most cases, this scheme requires such a large amount of time and energy to use that it become uneconomical to use on very large systems.

After testing numerous power systems of various sizes and configurations, Tinney's group has concluded that although scheme 2 is more complicated than 1, it's advantages outweigh the added work required to impliment it in Load Flow Studies. They have also discovered that there exists many networks for which none of these three schemes will produce a near-optimal ordering. It is assumed that these special networks will never be found in any normal power system.

Situations Affecting Ordering. In some cases it maybe necessary to modify the ordering scheme being used by numbering certain rows or busses last to satisfy a particular condition in the system problem. Here are a few possible reasons:

1) It is known before ordering that changes will be made in a certain row later. By placing this row last, only that part of the G.E.M. used to reduce this last row need be repeated to modify the table of factors.

2) The system matrix is slightly non-symmetrical. By numbering the rows or busses so the unsymmetrical part is at the end, the advantage of symmetry maybe used up to that point.

Computer Storage of Sparse Matrices[5]

In the normal power system network there are an average of three to four lines or branches connected to each bus. This, as was previously mentioned, is the reason for the high degree of sparsity in the system's admittance matrix Y. If for example, the network has a thousand buses, the size of the Y matrix will be 1000 X 1000. It will contain a total of 10^6 elements but of these approximately 4×10^3 are non-zero. Computer storage of the entire Y matrix would require a prohibitively large amount of storage space and would be extremely inefficient. However the storage of the approximately 4×10^3 essential non-zero elements is acceptable. The problem then is how to store only the non-zero elements of the Y matrix without losing the inherent column and row addressing of each element in the matrix. This will not be a tutorial section on how to write a computer program to store sparse matrices because many computer scientists, engineers and mathematicians, such as W.F.Tinney's group at B.P.A., interested in sparse matrices have written and made available many such programs. What will be done here, is a demonstration of how an idea of compressed storage of sparse matrices, developed by F. Gustavson of the IBM T.J. Watson Research Center, can be used to efficiently store sparse matrices in a computer.

5 J.K. Reid, Large Sparse Sets of Linear Equations(New York: Academic Press Inc.,1971)

It is presented to enlighten power engineers as to one of the many possible ways significant amounts of computer storage can be saved when dealing with sparse matrices as in Load Flow Studies.

Let A be the following sparse matrix:

$$A = \begin{bmatrix} 7 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 5 & 2 & 0 & 0 & 6 & 0 \\ 0 & 8 & 3 & 0 & 9 & 1 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 & 6 \\ 8 & 0 & 0 & 9 & 4 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 7 & 12 \\ 4 & 0 & 10 & 0 & 8 & 0 & 11 \end{bmatrix}$$

The problem is:

1) Store the A matrix with the minimum amount of space in the computer.

2) And since this matrix will later be operated on by a row-by-row elimination process, each row should be easily recreated from the compressed storage state.

Gustavson's idea is perfectly applicable to this problem. All information present in the A matrix such as, values of the non-zero elements and their individual row and column addresses are stored in three one-dimensional arrays. The first array which will be labelled VALUE, begins with row one and proceeds row by row storing values of all non-zero elements in order of appearance.

For the A matrix:

VALUE = 7 4 5 2 6 8 3 9 1 10 6 8 9 4 3 7 12 4 10 8 11

The dimension of VALUE is equal to the total number of non-zero terms in the matrix. The second array is labelled COLUMN, it stores, with a one-to-one correspondence, the column number of each element in the array VALUE.

For the A matrix:

COLUMN = 1 5 2 3 6 2 3 5 6 4 7 1 4 5 2 6 7 1 3 5 7

The dimension of COLUMN is equal to the dimension of VALUE. The last array will be labelled TOTAL, the first element in this array is always equal to one. TOTAL stores a running total of the number of non-zero terms in each row.

TOTAL (1) = 1

TOTAL (2) = TOTAL (1) + number of non-zero terms in row 1.

TOTAL (3) = TOTAL (2) + number of non-zero terms in row 2.

and so on.

For the A matrix:

TOTAL = 1 3 6 10 12 15 18 22

The number of non-zero terms in the i^{th} row is:

TOTAL (i + 1) - TOTAL (i)

The following procedure is used to recreate any row from compressed storage in the three arrays.

To recreate the i^{th} row:

Let the value of TOTAL (i) = a and TOTAL (i+1) = b .

There are (b - a) non-zero terms in the i^{th} row.

The i^{th} row begins at VALUE (a) in column COLUMN (a), and ends at VALUE (b-1) in column COLUMN (b-1).

These are the column numbers and values of the first and last non-zero elements in the i^{th} row. If the value of COLUMN (a) $\neq$ 1 zeroes are simply filled in all positions until COLUMN (a) is reached. The next non-zero term in the i^{th} row is VALUE (a+1) in column (a+1). This process continues until the entire i^{th} row is recreated in usable form.

Example 3

Recreate the 4^{th} row of the A matrix from it's compressed form in the three arrays VALUE, COLUMN and TOTAL.

The number of non-zero elements in row 4 is:

$$\text{TOTAL (5) - TOTAL (4)}$$

From the TOTAL array, 12 - 10 = 2

The first non-zero terms in row 4 is equal to ,

$$\text{VALUE (TOTAL (4)) = VALUE (10) = 10}$$

it is in column position equal to COLUMN (10) = 4

The first part of this row is:

0 0 0 10 __ __ __

The next term is = VALUE (11) = 6 ,
in column position = COLUMN (11) = 7 .
Since there are only 2 significant elements in row 4 this process is completed.

0 0 0 10 0 0 6

Acheck with the original matrix confirms the result.

The A matrix could also have been stored by columns for a column elimination process. In this case, the array VALUE would store all non-zero terms column by column in order of appearance. Array COLUMN could be relabelled array ROW recording the row each element in array VALUE is in. And finally array TOTAL would keep a running total of the number of non-zero elements in each column. If the matrix is symmetrical, there is no need to alter the storage pattern since each row and column would contain the same terms.

Gustavson's method, which is one of many invented by people who deal with sparse matrices, is quick, reliable and simple to program. It allows the power engineer to take full advantage of matrix sparsity and yields an important saving in computer storage space over the conventional 2-dimensional array method of computer matrix storage.

Chapter V

THE LOAD FLOW PROBLEM

Formation of the Bus Power Equations

The network of an electric power system can be represented by a set of nodal current equations.(5.1)

$$I_k = \sum_{m=1}^{N} Y_{km} E_m \qquad (5.1)$$

Each of the three quantities in (5.1), I_k the current into bus k, E_m the voltage at bus m and Y_{km} the admittance between buses k and m are complex numbers and as such can be expressed in either rectangular or polar form.(5.2)

$$\begin{aligned} I_k &= |I_k| \; e^{j\alpha k} = a_k + j\ b_k \\ E_m &= |E_m| \; e^{j\delta m} = e_m + j\ f_m \qquad (5.2) \\ Y_{km} &= |Y_{km}| \; e^{j\theta km} = G_{km} + j\ B_{km} \end{aligned}$$

If the expression for I_k is conjugated and then multiplied by that of E_k , the following power flow equations are obtained.(5.3)

$$P_k + j\ Q_k = \sum_{m=1}^{N} Y_{km}^{*} E_m^{*} E_k$$

$$P_k + j\ Q_k = \sum_{m=1}^{N} |Y_{km}| \, |E_m| \, |E_k| \; e^{j(\delta k - \delta m - \theta km)} \qquad (5.3)$$

$$P_k + j\ Q_k = (e_k + j\ f_k)\left(\sum_{m=1}^{N} (G_{km} - j\ B_{km})(e_m - j\ f_m)\right)$$

The solution of equation (5.3) will determine the magnitude and direction of the real and reactive power entering the network at the k^{th} bus.

Types of Busses

The Load Flow Study begins by labellong all busses or nodes in the system as one of three types. At each bus there are four variables, P the real power , Q the reactive power, $|E|$ the magnitude of the bus voltage and δ the angle of the bus voltage. The first type of bus is referred to as the "slack bus" or "swing bus". The slack bus is a generating bus which is designated to supply the difference between the specified or scheduled real power and the total system output plus losses. At this bus only the voltage magnitude $|E|$ and angle δ are specified. The second type of bus is a generator bus, all generator busses are classified as such except for the swing bus. For every generator bus the real power output P and the bus voltage magnitude $|E|$ are specified. All other busses are classified as the third type namely a load bus. At each load bus, the real power P and reactive power Q consumed by the load are specified.

With two of the four variables given at every bus, the set of equations (5.3) is a nonlinear set with two unknowns in each subscript. In general, if the correct voltage magnitudes and angles for each bus were known, the real and reactive powers could easily be computed with equation(5.3).

In the next section several methods for the solution of equation (5.3) will be presented.

Chapter VI

SOLUTION METHODS FOR THE LOAD FLOW PROBLEM

In this chapter, several methods for the solution of the Load Flow equations will be presented and discussed. Many of these methods linearize the non-linear Load Flow Problem in an effort to simplify the solution process. Most involve some type of iterative process and as such arrive at an approximate numerical solution. In order to minimize the number of iterations and still compute an acceptable solution, a degree of accuracy is determined beforehand by the power system engineer. Whenever the magnitude of the difference between the calculated powers and specified powers is less than this index of precision, the iteration procedure is terminated and the final solution has been reached. A more accurate solution can be obtained by increasing the degree of accuracy, however only at the expense of more iterations. It is here that the power system engineer can exercise his or her judgement, based on experience and intuition, as to which method of solution to use and how precise a solution is needed. In deciding which method to use, the power engineer must take into consideration the following points.

1) Select a method that can be implemented on the computational equipment available. Can the engineer afford a method that sacrifices computer storage space for a faster

but slightly less accurate solution? Or is the computer large enough to employ a method that requires a rather large amount of computer memory in return for a slower but more accurate solution.

2) All methods sre not suitable for all systems. The engineer must select a procedure that will not react adversely to the characteristics of the network, i.e. diverge, and produce incorrect results if any.

In the determination of the degree of accuracy or an index of precision in the final solution, the system engineer must take into account the following:

1) A high degree of accuracy is desirable but may take an excessive amount of time and therefore prevent the engineer from reacting swiftly enough to the changing demand. This could possibly result in severe voltage fluctuations throughout the system.

2) A low degree of accuracy would yield a quicker but less accurate solution. The resulting possibilities are an excessive system power loss costing the company money and the engineer his or her job and/or an overall shortage in power production.

3) The stability of the power system as a whole should also be taken into account.

Newton's Method[1]

This method is based on the principle that any function can be written as the summation of a Taylor Series.(6.1)

$$y = f(x) = f(a) + f'(a)(x-a) + f''(a)((x-a)^2/2!) + \ldots$$

$$\ldots . + f^n(a)(x-a)/n! \qquad (6.1)$$

For values of x near a this series is known to converge rapidly. In order to reduce the size and complexity of this series, it is normally assumed that it will converge after the second term. This results in the truncated series of equation (6.2).

$$y = f(x) \doteq f(a) + f'(a)(x-a) \qquad (6.2)$$

Equation (6.2) is modified by the following substitution to mske it more adaptable to solution by an iterative process. Replace a by x and x by $(x + \Delta x)$. With this substitution (6.2) becomes (6.3).

$$y = f(x+\Delta x) \doteq f(x) + f'(x)(\Delta x)$$

or (6.3)

$$y - f(x) = f'(x)(\Delta x)$$

In (6.3) all the terms except Δx are known or can be computed given a value for x. Equation (6.3) reveals that

1 John R. Neunswander, Modern Power Systems (Scranton: International Textbook Company, 1971), p. 271

Newton's Method has reduced any function of x into a linear equation in Δx. The solution of (6.3) is obtained by the following iterative process.

Lit the first approximation of $x = x^0$

and the second approximation of $x = x^1$

and so on.

A) Choose a value of x^0 and compute $f(x^0)$ and $f'(x^0)$

B) Using equation (6.4) below, solve for (Δx)

$$\Delta x = (y - f(x^0)) \,/\, f'(x^0) \tag{6.4}$$

C) Use the value of Δx found in B to determine the next approximate value of x (6.5).

$$x^1 = x^0 + \Delta x \tag{6.5}$$

Newton's Method can be applied to a set of N nonlinear equations in N unknowns (x_1, x_2, x_3,........, x_N) by extending equation (6.1) and assuming the series will converge after the first derivative terms (6.6).

$$\begin{array}{l} y_1 = f_1(x_1,x_2,\ldots,x_N) + \Delta x_1(df_1/dx_1) + \ldots + \Delta x_N(df_1/dx_N) \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ y_N = f_N(x_1,x_2,\ldots,x_N) + \Delta x_1(df_N/dx_1) + \ldots + \Delta x_N(df_N/dx_N) \end{array} \tag{6.6}$$

As before let the first approximation to the unknowns be represented by x_1^0, x_2^0, x_3^0,......., x_N^0. With the first iteration, the second approximation is obtained (6.7).

$$
\begin{aligned}
y_1 &= f_1(x_1^0, x_2^0, \ldots, x_N^0) + \Delta x_1^0 (df_1/dx_1)\big|_0 + \ldots + \Delta x_N^0 (df_1/dx_N)\big|_0 \\
&\vdots \\
y_N &= f_N(x_1^0, x_2^0, \ldots, x_N^0) + \Delta x_1^0 (df_N/dx_1)\big|_0 + \ldots + \Delta x_N^0 (df_N/dx_N)\big|_0
\end{aligned}
\tag{6.7}
$$

Define $f_k(x_1^0, x_2^0, \ldots, x_N^0) = f_k^0 \quad k = 1,2,3,\ldots\ldots, N$

The values of $y_1, y_2, \ldots\ldots, y_N$ are normally specified. The partials can be calculated as numerical quantities as can be $f_1^0, f_2^0, \ldots, f_N^0$ given a set of values for the unknowns. Therefore the unknown quantities in (6.7) are the Δx^0. Rewriting (6.7) using the above substitution results in (6.8).

$$
\begin{aligned}
y_1 - f_1^0 &= \Delta x_1^0 (df_1/dx_1)\big|_0 + \ldots\ldots\ldots + \Delta x_N^0 (df_1/dx_N)\big|_0 \\
&\vdots \\
y_N - f_N^0 &= \Delta x_1^0 (df_N/dx_1)\big|_0 + \ldots\ldots\ldots + \Delta x_N^0 (df_N/dx_N)\big|_0
\end{aligned}
\tag{6.8}
$$

Again the original equations have been reduced to linear equations in $\Delta x_1^0, \Delta x_2^0, \ldots, \Delta x_N^0$. After solving for these incremental unknowns, the new x approximations for the next iteration will be,

$$
x_1^1 = x_1^0 + \Delta x_1^0 \quad , \quad x_2^1 = x_2^0 + \Delta x_2^0 , \ldots \; x_N^1 = x_N^0 + \Delta x_N^0 \tag{6.9}
$$

The iteration process is then repeated until the Δx values are small enough to give satisfactory results. Equation (6.8) can be expressed in matrix form.(6.10)

$$\begin{bmatrix} y_1 - f_1^0 \\ y_2 - f_2^0 \\ \vdots \\ y_N - f_N^0 \end{bmatrix} = \begin{bmatrix} (df_1/dx_1)_0 & \cdots\cdots & (df_1/dx_N)_0 \\ (df_2/dx_1)_0 & \cdots\cdots & (df_2/dx_N)_0 \\ \vdots & & \vdots \\ (df_N/dx_1)_0 & \cdots\cdots & (df_N/dx_N)_0 \end{bmatrix} \begin{bmatrix} \Delta x_1^0 \\ \Delta x_2^0 \\ \vdots \\ \Delta x_N^0 \end{bmatrix} \qquad (6.10)$$

The matrix of partial derivatives coefficients is referred to as the Jacobian Matrix. Any determination of the unknown Δx array would involve some sort of a reduction or inversion process on the Jacobian Matrix. Of particular interest here is the application of Optimally Ordered Triangular Factorization technique to the Jacobian Matrix.

<u>Application of Newton's Method to Load Flow Studies.</u>[2]

At this point Newton's Method will be used to solve the general set of Load Flow equations derived in Chapter V.

$$P_k + j\,Q_k = (|E_k|\, e^{j\delta_k})(\sum_{m=1}^{N} |E_m|\, e^{-j\delta_m} |Y_{km}|\, e^{-j\theta_{km}}) \qquad (6.11)$$

The method will divide each equation into two equations, one

2 J.E. Van Ness,"Iteration Methods for Digital Load Flow Studies", AIEE Trans. Power, August 1959 p.583

for real power P, and the other for reactive power Q. The unknowns in each new equation will be $\Delta\delta$ and $\Delta|E|$, i.e. small variations in voltage magnitude and angle.

Every load bus, since neither $|E|$ nor δ are known, will have two equations one for real power and another for reactive power. Each generator bus will require only an equation for real power because $|E|$ has been previously specified. The swing bus has no equations associated with it, because the $|E|$ and δ are known. The real and reactive power produced by the swing bus can be computed once all other bus powers are computed. The equations for real and reactive powers will be in the form of equation (6.8). For the Load Flow Problem, the variables in (6.8) are defined as:

$$x_1 = \delta \qquad\qquad x_2 = |E|$$

y_1 = the specified real power

y_2 = the specified reactive power

f_1^0 = the first approximation of real power

f_2^0 = the first approximation of reactive power

$$P = y_1 - f_1^0 \qquad\qquad Q = y_2 - f_2^0$$

The partial derivative coefficients of $\Delta\delta$ and $\Delta|E|$ are formulated by taking partial derivatives of equation (6.11) with respect to $\Delta\delta$ and $\Delta|E|$. Taking the partial derivative of (6.11) with respect to one value of δ_m other than δ_k .

$$\frac{dP_k}{d\delta_m} + j\,\frac{dQ_k}{d\delta_m} = -j\,(\,|E_k|\,e^{j\delta_k})(\,|E_m|\,e^{-j\delta_m})(\,|Y_{km}|\,e^{-j\theta_{km}}) \tag{6.12}$$

The last two terms in (6.12) represent an expression for the current. Rewriting (6.12) in rectangular form.

$$\frac{dP_k}{d\delta_m} + j\,\frac{dQ_k}{d\delta_m} = -j\,(\,e_k + j\,f_k\,)(\,a_{km} - j\,b_{km}\,) \tag{6.13}$$

Separate the real and imaginary parts of (6.13) and redefine them as,

$$\frac{dP_k}{d\delta_m} = H_{km} = a_{km}\,f_k - b_{km}\,e_k \tag{6.14}$$

$$\frac{dQ_k}{d\delta_m} = J_{km} = -(\,a_{km}\,e_k + b_{km}\,f_k\,) \tag{6.15}$$

Take the partial derivatives of with respect to $|E_m|$ with $k \neq m$.

$$\frac{dP_k}{d|E_m|} + j\,\frac{dQ_k}{d|E_m|} = (\,|E_k|\,e^{j\delta_k})(\,e^{-j\delta_m})(\,|Y_{km}|\,e^{-j\theta_{km}}\,) \tag{6.16}$$

Multiply the right side by $|E_m|\,/\,|E_m|$.

$$\frac{dP_k}{d|E_m|} + j\,\frac{dQ_k}{d|E_m|} = \frac{1}{|E_m|}\,(\,|E_k|\,e^{j\delta_k})(\,|E_m|\,e^{-j\delta_m}\,)(\,|Y_{km}|\,e^{-j\theta_{km}}) \tag{6.17}$$

Substitute the expression for the current for the last two terms in (6.17) and rewrite in rectangular form.(6.18)

$$\frac{dP_k}{d|E_m|} + j\frac{dQ_k}{d|E_m|} = \frac{1}{|E_m|}(e_k + j f_k)(a_{km} - j b_{km}) \quad (6.18)$$

Separate the real and imaginary parts of (6.18) and redefine them as,

$$\frac{dP_k}{d|E_m|} + j\frac{dQ_k}{d|E_m|} = N_{km} = (a_{km} e_k + b_{km} f_k) / |E_m| \quad (6.19)$$

$$\frac{dP_k}{d|E_m|} + j\frac{dQ_k}{d|E_m|} = L_{km} = (a_{km} f_k - b_{km} e_k) / |E_m| \quad (6.20)$$

Take the partial derivative of (6.11) with respect to δ_k where k = m, here terms in the summation where m = k must be considered.

$$\frac{dP_k}{d\delta_k} + j\frac{dQ_k}{d\delta_k} = j(|E_k| e^{j\delta k})(\sum_{m=1}^{N} |E_m| e^{-j\delta m} |Y_{km}| e^{-j\theta km}) - j(|E_k| e^{j\delta k})(|E_k| e^{-j\delta k})(|Y_{kk}| e^{-j\theta kk}) \quad (6.21)$$

Collecting terms and rewriting,

$$\frac{dP_k}{d\delta_k} + j\frac{dQ_k}{d\delta_k} = j(P_k + j Q_k) - j|E_k^2|(G_{kk} - j B_{kk}) \quad (6.22)$$

Again separate the real and imaginary parts and redefine them as,

$$\frac{dP_k}{d\delta_k} = H_{kk} = - Q_k - B_{kk}|E_k|^2 \qquad (6.23)$$

$$\frac{dQ_k}{d\delta_k} = J_{kk} = P_k - G_{kk}|E_k|^2 \qquad (6.24)$$

Take partial derivatives of (6.11) with respect to $|E_k|$ with $k = m$.

$$\frac{dP_k}{d|E_k|} + \frac{dQ_k}{d|E_k|} = \frac{1}{|E_k|} (P_k + j Q_k) + |E_k| (G_{kk} - j B_{kk}) \qquad (6.25)$$

Equating real and imaginary parts and redefining,

$$\frac{dP_k}{d|E_k|} = N_{kk} = \frac{P_k}{|E_k|} + G_{kk}|E_k| \qquad (6.26)$$

$$\frac{dQ_k}{d|E_k|} = L_{kk} = \frac{Q_k}{|E_k|} - B_{kk}|E_k| \qquad (6.27)$$

With all the substitutions and modifications previously formulated and defined, the general equations of Newton's Method for Load Flow Studies become,

$$\begin{aligned} \Delta P_k &= \sum_{m=1}^{N} H_{km} \Delta\delta_m + \sum_{m=1}^{N} N_{km} \Delta |E_m| \\ \Delta Q_k &= \sum_{m=1}^{N} J_{km} \Delta\delta_m + \sum_{m=1}^{N} L_{km} \Delta |E_m| \end{aligned} \qquad (6.28)$$

In matrix form,

$$\begin{bmatrix} \Delta P_1 \\ \Delta P_2 \\ \vdots \\ \Delta P_N \\ \hdashline \Delta Q_1 \\ \Delta Q_2 \\ \vdots \\ \Delta Q_N \end{bmatrix} = \left[\begin{array}{c:c} H & N \\ \hdashline J & L \end{array}\right] \begin{bmatrix} \Delta \delta_1 \\ \Delta \delta_2 \\ \vdots \\ \Delta \delta_N \\ \hdashline \Delta |E_1| \\ \Delta |E_2| \\ \vdots \\ \Delta |E_N| \end{bmatrix} \tag{6.29}$$

Note: Due to the fact that the generator busses have only one equation associated with them, that being for real power flow, and that the load busses have two equations corresponding to real and reactive power flow related to them, the number of equations for ΔP and ΔQ in (6.29) will not be equal. The number of equations for reactive power changes in (6.29) will be greater than the number for real power changes.

<u>Procedure for Solution</u>. The procedure for the solution of the Load Flow Problem described in equation (6.29) by Newton's Method is as follows.

1.) Set the voltage magnitudes, where given, to their given values and set the other voltage magnitudes equal to that of the slack node. All angles are set equal to the slack bus angle. This is referred to as a flat voltage start.

2.) Using equation (6.11) and the voltages of step 1 compute the real and reactive power for each load bus and the real power for each generator bus. This is the first approximation to the specified or scheduled powers.

3.) Compute all ΔP and ΔQ needed for equation (6.29).

4.) Form the Jacobian matrix and augment it with the column of ΔP and ΔQ.

5.) The voltage and angular corrections are solved by Gaussian elimination and back substitution.

6.) The corrections to voltage magnitudes and angles are added to the first approximations of their values.

7.) Repeat step 2

8.) Repeat step 3

9.) The residuals ΔP and ΔQ are checked. If they are within a predetermined degree of accuracy the problem is solved. If not the process repeated from step 4.

<u>Application of Optimally Ordered Elimination</u>. The partial derivative coefficients in the Jacobian matrix of (6.29) were formulated from the bus power equations (6.11). These in turn were derived from the original set of nodal current equations (5.1) used to describe the system. Since the Y matrix of the power system exhibits a large degree of sparsity, then the coefficients of the Jacobian matrix will

reflect this sparsity. Therefore by applying one of the optimal ordering schemes to the system Y matrix before equation (5.1) is written, Newton's Method can be made to take advantage of this sparsity which will provide a faster solution because of less computation.

Notes on Newton's Method. Newton's Method is an accurate, quick and reliable method for the solution of the Load Flow Problem. It has been successfully used for several years in many power systems throughout the country. The method is applicable to many networks which were considered ill-conditioned and insolvable by previous methods. It is the industry standard to which all newly developed methods are compared.

Decoupled Newton's Method

This method of solution of the Load Flow Problem was developed by Brian Stott in Great Britain. It is a modification of the standard Newton's Method, in an effort to reduce the total amount of time and storage required for a solution. This modification is based on the following assumptions:

A.) The magnitude of real power flowing at any bus is more sensitive to a variation in the bus voltage angle than in voltage magnitude.

B.) The magnitude of reactive power flowing at any bus is more sensitive to a variation in the bus voltage magnitude than in voltage angle.

It is assumed that the real power P_k at bus k is a function of δ_k only and the reactive power Q_k at bus k is a function of $|E_k|$ only. In the following example a set of real and reactive power flow equations, for a simple system, will be formulated and used to demonstrate the basis of these assumptions.

Example 1[3]

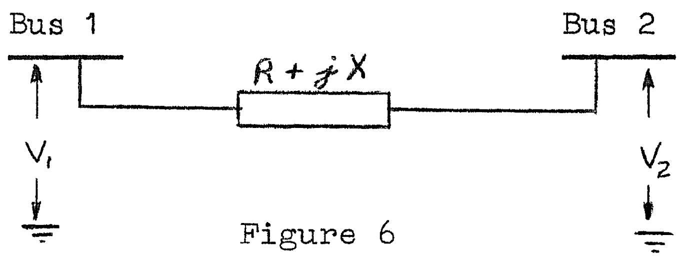


Figure 6

$$V_1 = |V_1| \angle \delta 1 \qquad V_2 = |V_2| \angle \delta 2$$

$$I = \frac{|V_1| \angle \delta 1 - |V_2| \angle \delta 2}{R + j X} \qquad (6.30)$$

$$P + j Q = V I^* \qquad (6.31)$$

The power flow from bus 1 to bus 2 is:

$$P_{12} + j Q_{12} = \frac{V_1 (V_1^* - V_2^*)}{(R + jX)^*} = \frac{|V_1|^2 - |V_1||V_2| e^{j(\delta_1 - \delta_2)}}{(R + jX)^*} \qquad (6.32)$$

Define:

$$\alpha = \delta_1 - \delta_2 \qquad (6.33)$$

Separating the real and imaginary parts:

3 Olle I. Elgerd,"Electric Energy Systems Theory: An Introduction",(New York:McGraw-Hill Book Co.,1971),p. 201

$$P_{12} = \frac{1}{R^2 + X^2} \left(R|V_1|^2 - R|V_1||V_2| \cos\alpha + X|V_1||V_2| \sin\alpha \right) \tag{6.34}$$

$$Q_{12} = \frac{1}{R^2 + X^2} \left(X|V_1|^2 - X|V_1||V_2| \cos\alpha + R|V_1||V_2| \sin\alpha \right) \tag{6.35}$$

By making use of a generally accepted approximation about power system transmission lines, that is

$$X \gg R \tag{6.36}$$

The above equations can be reduced to a much simpler form.

$$P_{12} = \frac{|V_1||V_2| \sin\alpha}{X} = |V_1| \frac{|V_2| \sin\alpha}{X} \tag{6.37}$$

$$Q_{12} = \frac{|V_1|^2 - |V_1||V_2| \cos\alpha}{X} = |V_1| \frac{|V_1| - |V_2| \cos\alpha}{X} \tag{6.38}$$

In the interval of 0° to 45^0, the value of the $\sin\alpha$ increases much more rapidly than the corresponding decrease in the value of $\cos\alpha$.

α	0°	10°	20°	30°	40°	45°
$\sin\alpha$	0.000	0.173	0.342	0.500	0.643	0.707
$\cos\alpha$	1.000	0.985	0.940	0.866	0.766	0.707

$$\sin 45^\circ - \sin 0^\circ = 0.707 - 0.000 = 0.707$$

$$\cos 45^\circ - \cos 0^\circ = 0.707 - 1.000 = -0.293$$

Therefore with all other quantities in equations (6.37) and (6.38) being constant, a change in α or in bus voltage angle will produce a more significant change in P_{12} than Q_{12}. In equation (6.38), the presence of the $|V_1|^2$ term reveals that any variation in $|V_1|$ alone will have a more pronounced affect on the value of Q_{12} than on P_{12} because of the subtraction involved in Q_{12}.

With these approximations, the equations for the real power P and reactive power Q in the conventional Newton's Method have been "decoupled" into 2 independent equations. In matrix form the equations of the Decoupled Newton's Method are:

$$\left[\begin{array}{c} \Delta P \\ \hline \Delta Q \end{array}\right] = \left[\begin{array}{c|c} H & 0 \\ \hline 0 & L \end{array}\right] \left[\begin{array}{c} \Delta \delta \\ \hline \Delta |E| \end{array}\right] \qquad (6.39)$$

In the standard Jacobian, all elements of the N and J submatrices which represent the ($dP / d|E|$) and the ($dQ / d\delta$), respectively, are equal to zero. The equations for solution are:

$$\Delta P_k = \sum_{m=1}^{N} H_{km} \Delta\delta_m \qquad (6.40)$$

$$\Delta Q_k = \sum_{m=1}^{N} L_{km} \Delta|E_m| \qquad (6.41)$$

Here each equation could be iterated several times independently during the solution process.

The application of these two equations to a variety of power system networks has revealed that the equation ΔQ_k is very unstable in areas far away from the solution. To correct this situation Stott formulated an alternative to equation (6.41), which preserves the decoupled nature of the method and has much better convergence characteristics. The new equation is based on the mismatch of the imaginary part of the bus current in terms of changes in bus voltage magnitude. Equation (6.42), below, is the expression for the real and reactive power balance at bus k.

$$P_k^{sp.} + j\, Q_k^{sp.} = E_k \sum_{m=1}^{N} E_m^{*}\, Y_{km}^{*} \qquad (6.42)$$

The superscript sp. refers to the specified bus powers. Here all bus voltages in the system are at the proper values to match the specified total bus power consumption. Under this condition, the total current into bus k is exactly equal to the total current output to the load at bus k, (Kirchhoff's Current Law). Any change in the load at bus k or elsewhere in the system will adversely affect this current balance. The expression for this current balance is:

$$\Delta I_k = \frac{P_k^{sp} + j\, Q_k^{sp}}{E_k^{*}} - \sum_{m=1}^{N} E_m\, Y_{km} \qquad (6.43)$$

where,

$$E_k = |E_k| \underline{/\delta k} \quad ; \quad E_m = |E_m| \underline{/\delta m} \quad ; \quad Y_{km} = G_{km} + j\, B_{km} \qquad (6.44)$$

The amount of reactive power flowing in the network can be monitored by observing variations in the imaginary part of the bus currents. The imaginary part of (6.43) is:

$$\Delta I_{k(imag)} = \frac{1}{|E_k|} (P_k^{sp} \sin\delta_k - Q_k^{sp} \cos\delta_k) - \sum_{m=1}^{N} (G_{km} \sin\delta_m + B_{km} \cos\delta_m) |E_m| \quad (6.45)$$

Equation (6.45) is used in place of (6.41) in the Decoupled Newton's Method, it performs basically the same function as (6.41) and, more important, it has proven to be a much more stable equation. Placing (6.45) in a form suitable for solution by a Newton's Method, as a function of $|E|$, bus voltage magnitude, involves the formation of the truncated Taylor Series.

$m \neq k$

$$\frac{d \Delta I_{k(imag)}}{d|E_m|} \doteq D_{km} = - (G_{km} \sin\delta_m + B_{km} \cos\delta_m) \quad (6.46)$$

$m = k$

$$\frac{d \Delta I_{k(imag)}}{d|E_m|} \doteq D_{kk} = - ((G_{kk} \sin\delta_k + B_{kk} \cos\delta_k) + \frac{1}{|E_k|^2}(P_k^{sp} \sin\delta_k - Q_k^{sp} \cos\delta_k)) \quad (6.47)$$

The final equations in the modified Decoupled Newton's Method are:

$$\Delta P_k = \sum_{m=1}^{N} H_{km} \Delta\delta_m \quad (6.40)$$

$$\Delta I_{k(imag)} = \sum_{m=1}^{N} D_{km} \Delta |E_m| \qquad (6.48)$$

Or in matrix form:

$$\left[\Delta P_k\right] = \left[H_{km}\right]\left[\Delta \delta_m\right] \qquad (6.49)$$

$$\left[\Delta I_{k(imag)}\right] = \left[D_{km}\right]\left[\Delta |E_m|\right] \qquad (6.50)$$

Procedure for Solution. The approvimations used in deriving the Decoupled Newton's Method allow the power system engineer to exercise a greater degree of his or her intuition, intelligence and judgement based on past experience with the power system. To demonstrate the amount of flexibility available in this method, two possible procedures for solution will be outlined. It is the decision of the power system engineer to select either procedure or to create a unique variation of both to solve the Load Flow Problem for his or her particular system.

Procedure A.

1.) Unless a better approximation is available, set all unknown voltage magnitudes and angles equal to values for the system's slack bus.

2.) Using the equation for real and reactive bus powers, (6.11), compute the real power for each load bus and the real power for each generator bus. This is the first approximation to the specified bus power.

3.) Compute all ΔP_k quantities for equation (6.40)

$$\Delta P_k = P_k^{sp} - P_k^{1}$$

($P_k^1 = 1^{st}$ approximation)

4.) Check the residuals ΔP_k's . If they are each within the predetermined degree of accuracy proceed to step 8., if not continue to step 5 .

5.) Form the H matrix and augment it with the column of ΔP_k's .

6.) Solve for the angular corrections $\Delta \delta_k$ by the Gauss Elimination Method.

7.) Update the voltage angle approximations by adding the $\Delta \delta_k$ of step 6 to first approximation.

8.) Compute reactive power for each load bus. Use the updated voltage angles of step 7 and the first approximation of voltage magnitudes.

9.) Compute all ΔQ_k quantities for equation (6.41)

10.) Check the residuals ΔQ_k . If they are individually within the predetermined degree of accuracy and the ΔP_k residuals are also within the acceptable range, the problem is solved. If ΔQ_k is not within the acceptable limits continue to step 11. If ΔQ_k is within limits repeat steps 2 - 7 until ΔP_k is also within limits.

11.) Form the column matrix of ΔI_k , use the most recent values of voltage angles and magnitudes.

12.) Form the D matrix using the same data as in step 11 and augment it with the column ΔI_k .

13.) Solve for the voltage magnitude corrections $\Delta|E_k|$

14.) Update the voltage magnitude approximations by adding the $\Delta|E_k|$ of step 13 to the first approximations.

15.) Return to step 2.

Procedure B

1.) Unless a better approximation is available, set all unknown voltage magnitudes and angles equal to the values for the system's slack bus.

2.) Compute the real power for every load and generator bus except the slack bus.

3.) Compute all ΔP_k quantities for equation (6.40).

4.) Check the residuals ΔP_k . If they are within the predetermined degree of accuracy proceed to step 9, if not continue.

5.) Form the H matrix and augment it with the column of ΔP_k .

6.) Solve for the angular corrections $\Delta\delta_k$ by the Gauss Elimination Method.

7.) Update the voltage angle approximations by adding the $\Delta\delta_k$ to the most recent values of δ_k.

8.) Repeat steps 2 - 7 , however in step 5 do not compute a new H matrix, simply augment the original H matrix with a new column of ΔP_k , until:

a) the process is given a new direction in step 4 or

b) the magnitude of the angular corrections $\Delta\delta_k$ computed in step 6 produce an insignificant change in the approximation for the δ_k.

9.) Compute the reactive power for each load bus using the most recent values of voltage magnitudes and angles.

10.) Compute all ΔQ_k quantities for equation (6.41).

11.) Check the residual ΔQ_k. If they and the ΔP_k are within the predetermined degree of accuracy, the problem is solved. If ΔQ_k is not within the acceptable limits, continue. If ΔQ_k is and ΔP_k is not within the preset limits repeat steps 2 - 8 until the ΔP_k values are acceptable.

12.) Form the column vector ΔI_k, using the most recent values of voltage angles and magnitudes.

13.) Form the D matrix and augment it with the column ΔI_k

14.) Solve for the voltage magnitude corrections $\Delta|E_k|$

15.) Update the voltage magnitude approximations

16.) Repeat steps 9 - 15 , however in step 13 do not compute a new D matrix, simply augment the original D matrix with a new column of ΔI_k , until:

a) the process is given a new direction in step 11

or

b) the magnitude of the voltage magnitude corrections $\Delta|E_k|$ computed in step 14 produce an insignificant change in the approximations for $|E_k|$.

<u>Notes on the Suggested Procedures</u>. In procedure A each equation (6.40) and (6.48) is iterated individually and the newest updated values are used in the next iteration of the other equation. It is hoped, due to the many assumptions

involved, that use of these updated values of voltage magnitude and angle will allow the Decoupled Newton's Method to arrive at an approximate solution much faster than procedure B. Procedure A sacrifices the accuracy of solution for savings in the time required for solution. With procedure B each equation is iterated several times to achieve the best possible correction in voltage magnitude or angle before iterating the other. Here use of the best possible updated values should permit the Decoupled Newton's Method to achieve a more accurate solution. Procedure B forfeits speed in an effort to reach a more accurate solution than procedure A. The speed of procedure B can be increased by making use of the triangular factorization process during solution.

Notes on the Decoupled Newton's Method. In a comparison with the conventional Newton's Method, the Decoupled Newton's Method is faster, requires less storage space, but due to the many assumptions made in its derivation, it produces a less accurate solution. It is faster due to the fewer number of calculations necessary and it requires less storage because only half of the original Jacobian storage space is used. As in Newton's Method, the Decoupled Newton's Method can take advantage of network sparsity provided the system Y matrix is properly ordered before any calculations are started. The method's characteristics of quickly arriving at a good approximate solution make it an ideal method for use in studies not requiring a very accurate solution, such as planning future systme expansions. When employing the Decoupled Newton's Method for a Load Flow Study, the power engineer should remember that it will at best produce an approximate solution. Therefore wiht this in mind, the engineer should select a lower degree of accuracy for the P and Q residuals than the one chosen for use with the standard Newton's Method. The Decoupled Newton's Method can be effectively utilized in conjunction with the conventional Newton's Method to produce fast accurate solutions. The number of time-consuming iterations in Newton's Method depend on, among other things, the "closeness" of the initial estimate to the actual solution. Since the Decoupled Newton's Method converges very rapidly to the area of solution, the use of it's solution as the initial approximation for Newton's Method should produce a more accurate solution with a significant savings in total time for solution.

The Hessian Load Flow Method.[4]

The Load Flow Problem has been investigated by another group of power engineers led by A. M. Sasson at the Instituto Technologico de Monterrey in Monterrey, Mexico. They have developed a method by applying the Taylor Series expansion

4 A.M. Sasson, F. Viloria and F. Aboytes, "Optimal Load Flow Solution Using The Hessian Matirx", IEEE PAS Trans., Jan. 1973 p. 31

concept to the derivative of the square of the Load Flow equations. Published reports of Load Flow Studies conducted withe this method have shown it to be slightly inferior to the conventional Newton's Method. However these same reports indicate that in solution of the Optimal Load Flow Problem, which is the basic problem considered here plus additional cost functions, the Hessian Load Flow Method is superior to any procedure employing the conventional Newton's Method.

Recalling the derivation of the conventional Newton's Method, any function f(x) can be expanded in a Taylor Series.

$$f(x) = f(a) + f'(a)(x-a) + \tfrac{1}{2} f''(a)(x-a)^2 + \ldots \quad (6.51)$$

Substituting x_o for a and $(x+\Delta x)$ for x in (6.51),

$$f(x) = f(x_o) + f'(x_o)\Delta x + \tfrac{1}{2} f''(x_o)(\Delta x)^2 + \ldots \quad (6.52)$$

Rewriting (6.52),

$$f(x) - f(x_o) = f'(x_o)\Delta x + \tfrac{1}{2} f''(x_o)(\Delta x)^2 + \ldots \quad (6.53)$$

In equation (6.53), f(x) represents the specified total bus power and $f(x_o)$ represents the calculated total bus power based on the approximate values for the unknown voltage magnitudes and angles. The procedure of Newton's Method is to truncate the Taylor Series after the second term and minimize the value of the resulting function,

$$f(x) - f(x_o) = f'(x_o)\Delta x \qquad (6.54)$$

The method of Sasson's group, The Hessian Load Flow Method, is based on the concept that the minimum value of $f^2(x)$ is at a point where its first derivative is equal to zero.

Let

$$F(x) = f^2(x) \tag{6.55}$$

$$g(x) = \frac{dF(x)}{dx} \tag{6.56}$$

Expand g(x) using the first two terms of the Taylor Series,

$$g(x) = g(x_o) + g'(x_o)\Delta x \tag{6.57}$$

To find the minimum of F(x) set the first derivative in (6.57) equal to zero,

$$0 = g(x_o) + g'(x_o)\Delta x \tag{6.58}$$

Rewriting (6.58),

$$g(x_o) = - g'(x_o)\Delta x \tag{6.59}$$

In (6.59), the term on the lefthandside is the gradient of F(x) evaluated at x_o.

$$g(x_o) = \frac{d}{d\,x}\left(F(x) \right)\bigg|_{x=x_o} \tag{6.60}$$

On the righthandside the term, $g'(x_o)$, is referred to as the Hessian, it represents the second derivative of F(x) evaluated at x_o.

$$g'(x_o) = \frac{d^2}{dx^2}\left(F(x) \right)\bigg|_{x=x_o} \tag{6.61}$$

The Hessian is denoted by the letter H, not to be confused with the submatrix H in the conventional Newton's Method.

$$H = g'(x_o) \tag{6.62}$$

The general equation for the solution by this method is,

$$g(x_o) = - H \Delta x \tag{6.63}$$

General Solution Procedure.

1.) Choose a value for $x_o = x_o^{(1)}$. This is the first approximation to x.

2.) Compute the values of the gradient of F(x) and the Hessian of F(x) at x_o, and solve for Δx with (6.63).

3.) Update the value of x_o with Δx, in general

$$x_o^{N+1} = x_o^N + \Delta x^N \tag{6.64}$$

4.) Compute the value of F(x) using the most recent value of x_o and check if F(x) is within the predetermined index of precision. If not continue.

5.) Repeat steps 2, 3, 4 until F(x) is minimized.

Application of the Hessian Load Flow Method. The general set of Load Flow equations were derived in Chapter V and are repeated below,

$$P_k + j\ Q_k = (|E_k|\ e^{j\delta_k})(\sum_{m=1}^{N} |E_m|\ e^{-j\delta_m} |Y_{km}|\ e^{-j\theta_{km}}) \tag{6.65}$$

The equation to be minimized by the Hessian Method is the following,

$$F(\delta, |E|) = \sum_{i=1}^{N} (P_i(\delta, |E|) - P_i^{sp})^2 + \sum_{i=1}^{N} (Q_i(\delta, |E|) - Q_i^{sp})^2 \tag{6.66}$$

The superscript sp stands for the specified values. The method requires expressions for the gradient and all second partial derivatives for the Hessian of (6.66). There are two expressions for the gradient of (6.66),

$$\frac{dF}{d\delta_m} = 2 \left(\sum_{k=1}^{N} (P_k(\delta, |E|) - P_k^{sp})\left(\frac{d}{d\delta_m} (P_k(\delta, |E|))\right)\right) + 2 \left(\sum_{k=1}^{N} (Q_k(\delta, |E|) - Q_k^{sp})\left(\frac{d}{d\delta_m} (Q_k(\delta, |E|))\right)\right) \tag{6.67}$$

Substituting known quantities and rewriting (6.67),

$$\frac{dF}{d\delta_m} = 2 \left(\sum_{k=1}^{N} \Delta P_k H_{km} + \sum_{k=1}^{N} \Delta Q_k J_{km} \right) \tag{6.68}$$

$$\frac{dF}{d|E_m|} = 2\left(\sum_{k=1}^{N} (P_k(\delta, |E|) - P_k^{sp})\left(\frac{d}{d|E_m|} (P_k(\delta, |E|))\right)\right) + 2\left(\sum_{k=1}^{N} (Q_k(\delta, |E|) - Q_k^{sp})\left(\frac{d}{d|E_m|} (Q_k(\delta, |E|))\right)\right) \tag{6.69}$$

Substituting known quantities and rewriting (6.69),

$$\frac{dF}{dE_m} = 2 \left(\sum_{k=1}^{N} \Delta P_k N_{km} + \sum_{k=1}^{N} \Delta Q_k L_{km} \right) \tag{6.70}$$

The H_{km}, J_{km}, N_{km} and L_{km} terms in (6.68) and (6.70) were derived in the section of Newton's Method pages 68 -70. There are six expressions for the Hessian:

$m = j$

$$\frac{d^2F}{d\delta_m\, d\delta_j} = 2\Big(\sum_{k=1}^{N} H_{km}^{\ 2} + \Delta P_k \frac{d^2P_k}{d\delta_j d\delta_m} + \sum_{k=1}^{N} J_{km}^{\ 2} + Q_k \frac{d^2Q_k}{d\delta_j d\delta_m} \Big) \qquad (6.71)$$

$m \neq j$

$$\frac{d^2F}{d\delta_m\, d\delta_j} = 2\Big(\sum_{k=1}^{N} H_{kj}\, H_{km} + \Delta P_k \frac{d^2P_k}{d\delta_j d\delta_m} + \sum_{k=1}^{N} J_{kj}\, J_{km} + \Delta Q_k \frac{d^2Q_k}{d\delta_j d\delta_m} \Big) \qquad (6.72)$$

$m \neq j$

$$\frac{d^2F}{d|E_m| d|E_j|} = 2\Big(\sum_{k=1}^{N} N_{kj}\, N_{km} + \Delta P_k \frac{d^2P_k}{d|E_m| d|E_j|} + \sum_{k=1}^{N} L_{kj}\, L_{km} + \Delta Q_k \frac{d^2Q_k}{d|E_m| d|E_j|} \Big) \qquad (6.73)$$

$m = j$

$$\frac{d^2F}{d|E_m| d|E_j|} = 2\Big(\sum_{k=1}^{N} N_{km}^{\ 2} + \Delta P_k \frac{d^2P_k}{d|E_m| d|E_j|} + \sum_{k=1}^{N} L_{km}^{\ 2} + \Delta Q_k \frac{d^2Q_k}{d|E_m| d|E_j|} \Big) \qquad (6.74)$$

$m = j$

$$\frac{d^2F}{d|E_m|d\delta_j} = 2\left(\sum_{k=1}^{N} H_{km}\,N_{km} + \Delta P_k \frac{d^2P_k}{d|E_m|d\delta_j} + \right.$$

$$\left.\sum_{k=1}^{N} J_{km}\,L_{km} + \Delta Q_k \frac{d^2Q_k}{d|E_m|d\delta_j}\right) \qquad (6.75)$$

$m \neq j$

$$\frac{d^2F}{d|E_m|d\delta_j} = 2\left(\sum_{k=1}^{N} H_{kj}\,N_{km} + \Delta P_k \frac{d^2P_k}{d|E_m|d\delta_j} + \right.$$

$$\left.\sum_{k=1}^{N} J_{kj}\,L_{km} + \Delta Q_k \frac{d^2Q_k}{d|E_m|d\delta_j}\right) \qquad (6.76)$$

In matrix form, the general equation for the solution of the Load Flow Problem by this method is:

$$\begin{bmatrix} \dfrac{dF}{d\delta_m} \\[2ex] \dfrac{dF}{d|E_m|} \end{bmatrix} \begin{bmatrix} \dfrac{d^2F}{d\delta_m d\delta_j} & \dfrac{d^2F}{d|E_m|d\delta_j} \\[2ex] \dfrac{d^2F}{d|E_m|d\delta_j} & \dfrac{d^2F}{d|E_m|d|E_j|} \end{bmatrix} \begin{bmatrix} \Delta\delta_m \\[2ex] \Delta|E_m| \end{bmatrix}$$

(6.77)

Procedure for Solution.

1.) Unless a better approximation is available set all unknown voltage magnitudes and angles equal to the values for the system's slack bus.

2.) Using the most recent approximations, for the unknown voltage magnitudes and angles, conpute the real and reactive powers for each load bus and the real power for each generator bus.

3.) Check, if all equations to be minimized by the Hessian Method, (6.66), are within the predetermined degree of accuracy. If all equations are, the problem is solved. If all equations are not, continue.

4.) Compute all quantities necessary for the gradient and Hessian matrices, (6.68), (6.70), (6.71) - (6.76).

5.)Form the Hessian matrix and augment it with the gradient column matrix.

6.) Solve for the voltage magnitude and angle corrections using the Gauss Elimination Method.

7.) Update all voltage magnitudes and angle approximations with the values computed in step 6.

8.) Repeat steps 2 and 3 until all equations are minimized.

Notes on the Hessian Method. In a comparison with Newton's Method for solving the Load Flow Problem, the Hessian Methid is slower, requires more computer memory and produces a solution with the same degree of accuracy.

The Hessian Method is slower because it works on a slightly different problem, the derivatives of the Load Flow equations. Data published by Sasson's group indicated that on the test metworks used to evaluate this method, it required approximately 1 to 2 more iterations than Newton's Method. It requires more computer memory because the Hessian matrix is less sparse than the Jacobian matrix. The methods most outstanding feature is the ease with which it can be converted, compared to Newton's Method, to solve the optimal Load Flow Problem previously mentioned. The Hessian Method is ideal for the power engineer who requires a fast and accurate solution to the system's Optimal Load Flow Problem and can afford to spend more time and energy to solve the standard Load Flow Problem.

The ϵ - Coupling Method.[5]

The e– coupling, (epsilon - coupling), method is the last method that will be discussed in this section. It is based on the e– coupling method used in control theory and was developed for Load Flow Studies by Dr. J. Medanic and Mr. B. Avramovic at the Institute for Automation and Telecommunications in Beograd, Yugoslavia. It operates on the standard set of Load Flow equations and not, as the Hessian Method, on a modified set of equations.

5 J. Medanic and B. Avramovic,"Solution of Load-Flow Problems in Power Systems by e - coupling Method", Proc. of IEE, August 1975, p 801

The method combines the speed and storage requirements of the Decoupled Newton's Method with the high degree of accuracy available with the conventional Newton's Method. The results of some preliminary tests done by Dr. Medanic and Mr. Avaramovic indicate that the e - coupling method is at least equal to or superior to Newton's Method in terms of speed and computer memory required for solution.

In the conventional Newton's Method, the general form of the equation written in matrix notation is:

$$\left[\begin{array}{c} \Delta P_k \\ \hline \Delta Q_k \end{array}\right] = \left[\begin{array}{c|c} H & N \\ \hline J & L \end{array}\right] \left[\begin{array}{c} \Delta \delta_k \\ \hline \Delta |E_k| \end{array}\right] \tag{6.78}$$

All terms in (6.78) are defined in the section on Newton's Method pages 68 - 71 .

The e - coupling method takes advantage of the relative weak coupling between the real power and the voltage magnitude and between the reactive power and the voltage angle in the following manner:

$$A = \begin{bmatrix} H & N \\ J & L \end{bmatrix} \tag{6.79}$$

e - coupling transforms the A matrix of (6.79), which is the standard Jacobian matrix in the conventional Newton's Method, into a function of e, so that, when e = 1 the standard Newton's Method problem will exist and when e = 0 the decoupled problem will result. Under thses conditions the e must be introduced into the A matrix of (6.79) as:

$$A(e) = \begin{bmatrix} H & eN \\ eJ & L \end{bmatrix} \qquad (6.80)$$

Subsituting e = 0 or e = 1 into (6.80) will prove that all specified conditions have been met by this formulation. Equation (6.78) can now be rewritten as a function of e:

$$\begin{bmatrix} \Delta P_k \\ \Delta Q_k \end{bmatrix} = \begin{bmatrix} H & eN \\ eJ & L \end{bmatrix} \begin{bmatrix} \Delta \delta_k \\ \Delta |E_k| \end{bmatrix} \qquad (6.81)$$

With e = 0 , the equations for the general decoupled problem are:

$$\Delta P_{Dk} = H \Delta \delta_{Dk} \qquad (6.82)$$

$$\Delta Q_{Dk} = L \Delta |E_{Dk}| \qquad (6.83)$$

The subscript Dk represent the decoupled problem at the k^{th} bus. The general solution to (6.82) and (6.83) is:

$$\Delta \delta_{Dk} = H^{-1} \Delta P_{Dk} \qquad (6.84)$$

$$\Delta |E_{Dk}| = L^{-1} \Delta Q_{Dk} \qquad (6.85)$$

Actually (6.82) and (6.83) are solved by the Gauss Elimination Method and no matrix inversion takes place. With e = 1, the problem is the standard Newton's Method Load Flow Problem.

$$\Delta P_k = H \, \Delta\delta_k + N \, \Delta|E_k| \tag{6.86}$$

$$\Delta Q_k = J \, \Delta\delta_k + L \, \Delta|E_k| \tag{6.87}$$

The general solution to (6.81) is:

$$\begin{bmatrix} \Delta\delta_k \\ \Delta|E_k| \end{bmatrix} = \begin{bmatrix} H & eN \\ eJ & L \end{bmatrix}^{-1} \begin{bmatrix} \Delta P_k \\ \Delta Q_k \end{bmatrix} \tag{6.88}$$

Or

$$\begin{bmatrix} \Delta\delta_k \\ \Delta|E_k| \end{bmatrix} = A(e)^{-1} \begin{bmatrix} \Delta P_k \\ \Delta Q_k \end{bmatrix} \tag{6.89}$$

As in Newton's Method the Jacobian matrix would be augmented by the column of residuals ΔP_k and ΔQ_k and the problem solved by Gauss Elimination. It is at this point that the e - coupling method employs a different scheme. The matrix $A(e)^{-1}$ is expanded about e = 0 in a truncated Taylor Series. For notational ease define:

$$M(e) = A(e)^{-1} \tag{6.90}$$

and the truncated Taylor series of M(e) as,

M_a (e) - the subscript a identifies it as as approximation to the total series.

The series M_a (e) about e = 0 is:

$$M_a(e) = M(0) + e\,\frac{dM}{de}(0) \qquad (6.91)$$

The terms of the series in (6.91) are defined as:

$$M(0) = A(0)^{-1} = \begin{bmatrix} H^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} \qquad (6.92)$$

and

$$\frac{dM}{de}(0) = \frac{d}{de} A(e)^{-1}\Big|_{e=0} \qquad (6.93)$$

$$\frac{dM}{de}(0) = -\,A(e)^{-1}\,\frac{dA(e)}{de}\,A(e)^{-1}\Big|_{e=0} \qquad (6.94)$$

$$\frac{dM}{de}(0) = -\,A(0)^{-1}\,\frac{dA}{de}(0)\,A(0)^{-1} \qquad (6.95)$$

$$\frac{dM}{de}(0) = -\begin{bmatrix} H^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix}\begin{bmatrix} 0 & N \\ J & 0 \end{bmatrix}\begin{bmatrix} H^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} \qquad (6.96)$$

$$\frac{dM}{de}(0) = \begin{bmatrix} 0 & -H^{-1}\,N\,L^{-1} \\ -\,L^{-1}\,J\,H^{-1} & 0 \end{bmatrix} \qquad (6.97)$$

Substituting (6.97) and (6.92) into (6.91), the truncated series of the inverse Jacobian matrix in terms of the original Jacobian submatrices becomes:

$$M_a(e) = \begin{bmatrix} H^{-1} & -e\,H^{-1}\,N\,L^{-1} \\ -e\,L^{-1}\,J\,H^{-1} & L^{-1} \end{bmatrix} \qquad (6.98)$$

Replacing the inverse of the Jacobian, $A(e)^{-1}$ in (6.89) by the expression for the truncated series M_a (e) in (6.98) produces:

$$\begin{bmatrix} \Delta\delta_k \\ \Delta|E_k| \end{bmatrix} = \begin{bmatrix} H^{-1} & -e\,H^{-1}\,N\,L^{-1} \\ -e\,L^{-1}\,J\,H^{-1} & L^{-1} \end{bmatrix} \begin{bmatrix} \Delta P_k \\ \Delta Q_k \end{bmatrix} \qquad (6.99)$$

With e = 1 in (6.99), the equations for the standard Load Flow Problem are:

$$\Delta\delta_k = H^{-1}\,\Delta P_k - H^{-1}\,N\,L^{-1}\,\Delta Q_k \qquad (6.100)$$

$$\Delta|E_k| = -L^{-1}\,J\,H^{-1}\,\Delta P_k + L^{-1}\,\Delta Q_k \qquad (6.101)$$

Using the solution equations for the decoupled problem presented in (6.84) and (6.85), the last two equations can be written as:

$$\Delta\delta_k = \Delta\delta_{Dk} - H^{-1}\,N\,\Delta|E_{Dk}| \qquad (6.102)$$

$$\Delta|E_k| = -L^{-1}\,J\,\Delta\delta_{Dk} + \Delta|E_{Dk}| \qquad (6.103)$$

Equations (6.102) and (6.102) are the general form of the equations used by the e - coupling method to solve the

Load Flow Problem. Next a procedure for the use of these equations will be outlined.

Procedure for Solution.

1.) Unless a better approximation is available set all unknown voltage magnitudes and angles equal to the values for the system's slack bus.

2.) Using the most recent approximations for the unknown quantities of voltage magnitude and angle, calculate the real and reactive powers for each load bus and the real power for each generator bus.

3.) Compute all residuals ΔP and ΔQ. Check if the magnitudes of each is within the preset degree of accuracy, if not, continue to step 4. If all residuals are within the prescribed limits, stop, the problem is solved.

4.) Form the H and L Jacobian submatrices and using the technique of triangular factorization construct a table of factors for each.

5.) Using the table of factors for the H and L matrices in (6.81) and the residuals of step 3 solve for the decoupled solutions with equations (6.84) and (6.85).

6.) Compute the following quantities:

$$\alpha = N \ \Delta |E_{Dk}| \tag{6.104}$$

$$\beta = J \ \Delta \delta_{Dk} \tag{6.105}$$

Use the decoupled solutions of step 5. The N and J matrices are the Jacobian submatrices and they need not be permanently stored, since they are use only to calculate α and β.

7.) Find:

$$\Delta\delta_k = \Delta\delta_{Dk} - H^{-1}\alpha \qquad (6.106)$$

$$\Delta|E_k| = \Delta|E_{Dk}| - L^{-1}\beta \qquad (6.107)$$

Use the decoupled solutions of step 5, the α and β quantities of step 6 and the H and L table of factors in step 7.

8.) Update all voltage magnitude and angle approximations with the corrections found in step 7.

9.) Repeat steps 2 and 3.

<u>Modified Procedure for Solution.</u> Steps 1 to 5 are the same as in the previously outlined procedure.

6.) Compute the following quantities:

$$\alpha = N\,\Delta|E_{Dk}| \qquad (6.104)$$

$$\Delta\delta_k = \Delta\delta_{Dk} - H^{-1}\alpha \qquad (6.106)$$

7.) Modify equation (6.105) by replacing $\Delta\delta_{Dk}$, the solution by the decoupled problem by $\Delta\delta_k$ of equation (6.106). Compute:

$$\beta = J\,\Delta\delta_k \qquad (6.108)$$

$$\Delta|E_k| = \Delta|E_{Dk}| - L^{-1}\beta \qquad (6.107)$$

Steps 8 and 9 are also those outlines previously.

This modification may result in a faster convergence to a more accurate solution of the Load Flow Problem by the e - coupling method.

Notes on the e - coupling method. The e - coupling method is relatively new and appears to be a very interesting and successful combination of the conventional and decoupled Newton's Method. While Dr. Medanic and Mr. Avramovic are still conducting more extensive tests on various model power systems, they estimate, based on preliminary results, that e - coupling will require 15 % more time than the Decoupled Newton's Method and be 20 % faster than Newton's Method. Besides the saving in solution time afforded by e - coupling, it also requires less compute memory than Newton's Method and produces a very accurate solution. The storage requirements are equivalent to those of the decoupled method because only the H and L matrices plus a few additional terms need be permanently stored. Another attractive feature of e - coupling is that since it is based on Newton's Method, it can take advantage of sparse matrix ordering and computer storage and the technique of triangular factorization in much the same way to improve its performance.

Chapter VII

CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

The use of the Gauss Elimination Method for the solution of large systems of linear equations has proven to be superior to the method of matrix inversion. The Gauss Elimination Method provides significant savings in the number of calculations necessary for solution and hence produces a reduction in the overall solution time. The decomposition of the system's characteristic matrix by elementary matrices is closely related to the Gauss Elimination Method. The formation of the Table of Factors is very important when many system responses are required for various system excitations. The Table of Factors contaims all the information required to perform repeated solutions without the need to repeat the entire Gauss Elimination Method on the characteristic matrix of the network. The savings in calculation and time can prove to be significant, especially in larger networks. Additional benefits canbe realized if the characteristic matrix is symmetrical. In this case it has been shown that nearly 50 % of the Table of Factors is unnecessary and need not be stored for further solutions.

The use of the admittance matrix in the formulation of Load Flow Methods is very important, since it incorporates the many advantages of it's characteristic sparsity.

The schemes discussed here allow for efficient use of this sparsity property to avoid excessive and unnecessary calculations and storage requirements. The computer storage technique discussed and demonstrated here for large sparse matrices is simple and straightforward. It is easily amendable to any changes or improvements required to adapt it to the computational equipment available. Although the techniques presented here produce significant savings in calculation time and memory space, it it suggested that these two areas be investigated further to find even more efficient methods to take better advantage of matrix sparsity.

The application of Newton's Method to the solution of the Load Flow Problem has met with much success in recent years. The flexibility of this method is no doubt one of it's most attractive advantages. The Decoupled Newton's Method reduces the amount of calculations required by Newton's Method, by nearly 50 %, by using the approximations that real power flow is only a function of voltage angle and reactive power flow is only a function of voltage magnitude. While these are in fact very good assumptions within a certain range of values for voltage angle and magnitude, test results indicate that the equation for reactive power is very unstable in areas some distance from the solution. An effort should be made to investigate this instability and possibly develop a set of factors which have an effect on this instability. The stability of the replacement equation, the imaginary bus current mismatch equation, may provide a good starting point. The assumptions use in the Decoupled Newton's Method can at best produce only an approximate solution. While this may seem to be a drawback of the method, the solutions produced generally are acceptable to those Load Flow Studies not requiring a great deal of accuracy. The Hessian Load Flow Method applies Newton's Method to finding the minimum of the derivative of the square of the Load Flow Equations. Although it has been proven to be as good as or slightly inferior to Newton's Method for the basic Load Flow Problem, it is superior when adapted to the Optimum Load Flow Problem. While very effective in solution of the optimal problem, the Hessian Method is not suggested for the general Load Flow Problem. The e - coupling method is a technique using Newton's Method and the Decoupled Newton's Method together with some ideas from control theory. It combines the speed of the decoupled method with the accuracy of Newton's Method. Although it has not been extensively used, it appears to be even better than Newton's Method yet it still retains many of the advantages associated with Newton's Method. Further investigation into the e - coupling method needs to be done. Application of this method to the Optimal Load Flow Problem should prove to be an interesting area for future work.

The fact that the three new methods presented in this paper are related to Newton's Method should not discourage research into other methods totally unrelated to Newton's Method for the solution of the Load Flow Problem.

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