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# Optimal control and identification of stochastic systems using differential game theory 

Harry Marvin Burbank<br>New Jersey Institute of Technology

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AT
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Newark, New Jersey<br>1975

APPROVAI OF DISSERIATION
OPTIMAL CONIROL AND IDENTIFICATION OF STOCHASTIC SYSTEMS
USING DIFFERENTIAL GAME THEORY

BY
HARRY MARVIN BURBANK, JR.
FOR

DEPARTMENT OF ELECIRICAL ENGINEERTNG
NEW JERSEY INSTITUTE OF TECHNOLOGY

BY

## FACULIY COMMITTEE

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JUNE, 1975

PREFACE

It was the original intent of this work to provide a theoretical basis to study several problems associated with electrocardiagrams. Preliminary reading and discussions sparked the interest in this area. It became clear that non-stationary statistics were involved, leading to the formulations given in the dissertation. As the theoretical study progressed, the applicability of the work to the area of electroencephalograms was also evident. In each step , with every hurdle encountered, the intricacies involved lent a stronger and stronger intuitive notion that a fruitful path was being followed. Hopefully now, the joining of theory and application can be pursued with an approach that will bear much fruit.

The completion of this dissertation would not have been possible without the patience, understanding and endurance of family and some good friends. To them is given a simple, sincere thank you. Also, grateful thanks are given to Margaret, Sister Clarissa, Bobbie, Karen, Regina, Diana, Rosemary, Karen, Ruth, and Gary who all helped put it together.

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#### Abstract

This dissertation deals with linear systems subjected to stochastic disturbances. The class of stochastic processes considered is the class of second order stochastic processes characterized by having finite continuous covariance. The properties of the covariance provide means to formulate optimization problems without the difficulties present when the covariance is not finite or continuous.

The first aspect studied was several classes of optimal control problems. The effects of the stochastic processes were approximated by the effects of its first two moments. This procedure resulted in allowing optimal system controls to be found whatever the first two moments of the stochastic input were, or "worst case" optinal controls were found. Differential game theory was used to solve the "worst case" problem.

Then, a model reference adaptive control system was employed to permit simultaneous parameter identification and control to be obtained in an on-line environment. The parameter identification was accomplished using gradient or steepest descent techniques. The control inputs were updated as the parameters were changed yielding sub-optimal control of the physical system. In addition, minimum error covariance estimation of linear systeme with second order stochastic disturibances was developed.


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## INTRODUCIION

In this dissertation a class of optimal control and identification problems is studied. The approach developed permits treatment of some physical situations not yet fully explored and allows a different approach to physical situations previously dealt with. The overall problem is to optimally control a plant in the presence of noise disturbances and sumultaneously identify parameters of the plant. The general linear problem is pictured below.


Figure 1.1 General linear problem
In this situation, the measurement $\bar{y}$ corrupted by noise $w$, of the state $x$ is to be driven by the control $u$, which is optimal in some manner, while in the presence of additional noise $v$, at the input to the plant. Simultaneously, the vector $\gamma$, which is the collection of ala elements of $A, B, C$, and $H$ which are not known, is to be determined in some way.

The optimal control solution can be carried out offline. Then the identification can be performed online, while the actual plant is being controlled.

The problem of Figure 1.1 is linear, however both the plant and measurement may actually be nonlinear. In this work, only linear problems are dealt with leaving nonlinear problems for future research. The identification scheme employed is discussed in Chapter 6. The stochastic optimal control problem is detailed in Chapters 2 and 3. Throughout the dissertation only con-tinuous-time problems are studied, since stochastic dis-crete-time problems submit to some approaches which just do not carry over to continuous time. This point will be clarified in Chapter 2. Also the measurement $y$, will be assumed to be equal to the stare $x$, within the dissertation. This corresponds to $w \equiv 0$ and $H(\gamma, t)=I$. Having completed the solution of this problem, the exten-ion to the case of noise corrupted measurement is possible. These modifications slightly change Figure 1.1 to the following.


Figure 1.2 General linear problem
without measurement

The state equation is

$$
\frac{d x}{d t}(t)=A(\gamma, t) x(t)+B(\gamma, t) u(t)+C(\gamma, t) v(t), t \in\left[t_{0}, t_{f}\right] 1.1
$$

where
x is an n vector
$A(f), t)$ is an nxn matrix
$u$ is an $r$ vector
$B(\gamma, t)$ is an nxr matrix
$v$ is an $s$ vector
$C(y, t)$ is an nxs matrix
and
$\dot{\gamma}$ is an $m$ vector.

The state of the system is x and the measurement is $y=x$. The optimal control is $u$ and the additive stochastic disturbance is v. The unknown gains, time constants, etc. which are the elements of the matrices $A, B$ and $C$ form the vector $\varphi$.

The overall problem of figure 1-2 essentially is the combination of two problems: stochastic optimal control and identification. In this work, differential game theory will be employed in the "optimal" part of the solution. Generally speaking background material required for identification problems is presented in the texts by Sage and Melsa (96) and Graupe (31). Discrete-time process identification is given in the text by Mendel (76). Identification
is the subject of the survey articles by Astran and Eykhoff (7) and Balakrishman (14) and the entire special issue of the December, 1974 IEEE Transaction on Automatic Control. Deterministic optimal control background is presented in the texts by Athans and Falb (9), Hsu and Meyer (38), and Leitmann (67). The variational techniques employed in this work are found in Kirk (45), Sage (95) and Citron (22). Some of the proofs depend on a matrix formulation similar to that in Athans (10). Stochastic optimal control for some types of stochastic processes is given in Astrom (8), and Sage and Melsa (97). The separation principle is found in Wonham (109). Various extensions and alternatives to the separation principle exist as in e.g. Athans (12). The original work on deterministic differential game theory is in Isaacs (39). Other texts are Blaquiere ('18.) and Friedman (27). Articles by Ho (33), Kuo (50) and (52), and Kuo and Burbank (55) present solutions to wide classes of two person, zero-sum differential games. Pure stochastic differential games are discussed in Behn and Ho (16) and by Willman in (108). Estimation theory is needed for the study of stochastic optimal control, and especially for the separation theorem, which states that the optimal feedback control solution is obtained only after the best estimate of the state has been found.
. and Sage and Melsa (97). The monograph by Lee (64) formulates estimation type problems concisely. The detailed class of problems will be developed in Chapters 2 and 3. References dealing with these details will be given in those chapters.

For problems of stochastic optimal control, the approaches used to date require estimation of the state and much a priori knowledge of the stochastic disturbances $v$ and w. The linear estimation problem was first presented in Kalman (40). Many extensions have been made such as in Mehra (74) where errors in initial values of the variance are discussed. In Lee (64) the estimation problem is solved using a Bayesian approach. In Kushner (62), Sage and Melsa (97) and Meditch (73) linear anäronlinear estimation, añ stochastic control are discussed. Astrom (8) uses the Ito formulation for the same type of problems.

The class of stochastic processes studied is the class of second order processes. Davis (23) and Boonton (19) deal with prediction and estimation for this class of inputs in the classical manner. A deterministic disturbance in a differential game is given by Krikelis and Rekasius (49). Their problem is a conflict of interest game between a deterministic optimal control and a deterministic disturbance. The differential game approach to be employed in this research considers a stochastic disturbance. It is
assumed that the effects of the stochastic disturbance are approximated by its first two moments. The continuous time vector case is studied. In Yoshikawa (1.10) a scalar two stage discrete stochastic differential game is solved in terms of the complete probability density function. The problem here is based on the assumption that the density function can be approximated by its first two moments, but a wider class of problems is considered than in (110). Differential games with imperfect information are covered by Leondes and Pearson (69), Leondes and Stuart (68) and Kushner and Chamberlain (61). In these references noise corrupted measurements or incomplete state situations are considered.

The complete probability density function for a class of nonlinear Bayesian estimation problems is approximated for the discrete time case only by Alspach (2), and Sorensen and Alspach (105). In another article (l) they consider non Gaussian Bayesian estimation. Sain and Liberty (98) computationally obtain a density function for a quadratic performance index.

The class of differential games studied here is characterized as games of imperfect information in that parameter identification is required. It is assumed though that the stochastic disturbance can be approximated by its
first two moments which is not the most general case. However the problem of simultaneous optimal control and identification included in this research is seen to be an extension of known results even with this assumption. The approach developed here is one in which the plant is optimally controlled in the presence of any stochastic process $v$, considering the effects of the disturbances as follows. The first two moments of $v$ are assumed to provide sufficient information to obtain physically workable solutions to the overall problem. This assumption is crucial to all that follows but it is noted that the effects of any finite number of moments could be included using the approach of this dissertation as a starting point. The manner in which the first two moments propagate through the plant or system then, is the means to describe how the stochastic disturbance affects, in a degrading manner, the optimal operation of the plant. These two moments are then considered to be inputs to the same plant as the optimal control u. There is apparent the conflict between the effects of the optimal control $u$ and the effects of the two moments of the stochastic disturbance v. For each given performance index, a quantitative measure of these effects is obtained.

Using the conflict characterization, a differential game is defined between "man" who chooses to optimally
control the plant versus "nature" who seeks to choose moments of stochastic disturbances which degrade optimal performance . The "man-nature" differential game approach to other kinds of problems appears e.g. in Kuo (51) and (53).

The approach to be employed using differential game theory with imperfect information in the sense that the parameter $\gamma$ is unknown is mentioned by Ho in (34) as a possible area for future research. This approach is formulated in this work for the first time as applied to the problems of figure 1.2: One contribution then is the differential game approach formulation. Actually, the formulation is equivalent to an absolute-worst-case controller design problem as described in Ragade and Sarma (92).

The specific class of problems considered in the research is for stochastic processes with finite variances which are also continuous in the mean square sense with respect to time. To the author's knowledge, this class of stochastic processes with unknown $\gamma$ has not been fully dealt with before in a control problem or in an identification problem. This class of stochastic process is best characterized by the term nonstationary since two independent time variables are required to mathematically describe the moments of the process. Nonstationary inputs are dealt with in Boonton (19), but the resulting integral equations are very difficult to deal with. Baggeroer (13) illustrates the
solution to similar integral equations as does Shinbrot (104). Davis (23) formulates a class of nonstationary prediction problems.

As will be seen in detail in Chapters 2 and 3, the specific problem characterization of this work is one in which the stochastic disturbances are non-stationary with correlation times of the same order of magnitude as the system or plant. This situation appears in many biomedical monitoring and control problems. In particular, as seen in Kawabata (41), EEG waveforms exhibit these properties. It is speculated that the theory developed will permit much more thorough analysis and understanding of any EEG related phenomena, such as time series analysis of the waveforms, modeling of the system generating the waveform, and/or control of the system generating the waveform by drugs. For example, the state of a person's consciousness, sleeping, awake, alert, etc. is influenced by drugs.

The therapeutic use of drugs and the covert use of drugs are both problems requiring deeper understanding. The effect of these drugs on EEG waveforms, and the modeling of the system creating these waveforms would possibly enhance this understanding from a new viewpoint. Several of the previous points are considered in Nunez (82).

Basically this work presents an approach to the solution
of an optimal control problem in which stochastic disturbances are present. Techniques for on-line identification are included. The approach is to consider an absolute-worst-case situation as described mathematically with differential game theory. The particular class of problems dealt with is for nonstationary, continuous-time stochastic disturbances. These disturbances are typically found in many biomedical areas, and especially in EEG related phenomena.*

Chapter 2 provides material needed to formulate the specific problem. This formulation is presented in Chapter 3. Chapters 4 and 5 provide necessary and sufficient conditions for the solution of the "optimal" part of the problem. Chapter 6 provides the solution to the identification portion of the problem. In Chapter 7, examples are worked out to illustrate the theory. The results are summarized in Chapter 8 and the extensions possible are presented in Chapter 9. The derivations of the state equations is given in Appendix A. Appendix B contains derivations of performance index and endpoint condition transformations. The proofs of the necessary condition theorems are in Appendix $C$, and the proofs of the sufficient condition theorems in Appendix D.

## CHAPTER 2

## STOCHASTIC PROCESSES

When a stochastic process is only a part of a larger problem, some assumptions must be made about the stochastic process and only then can the solution to the larger problem begin. This chapter develops the main points required to classify stochastic processes and in particular to delineate the class of stochastic processes chosen for this study.

### 2.1 Discrete and Continuous Stochastic Processes

A stochastic process is, in words, a random phenomena that changes with time, or some other parameter. The real, scalar stochastic process v(.,.), a family or ensemble of functions, depends on the outcome of an event wwics, the sample space and, a parameter $t$, usually assumed to be time where $t \in[. . .,-1,0,1, \ldots] \triangleq I$ for a discrete-time process, and $t(-\infty, \infty) \triangleq E$ for a continuous-time process.

For fixed $t=t_{j} \in I$ (or $\left.E\right), v\left(t_{j}\right.$, ) is a random variable.
For fixed $w=w_{j} \in \Omega_{,} v\left(., w_{j}\right)$ is a (deterministic) function of time.

For fixed $t=t_{j}$ and $w=w_{j}, v\left(t_{j}, w_{j}\right)$ is a number.

For each fixed $t_{j}$, the random variable $v\left(t_{j},.\right)$ is defined on a sample space $\Omega$, where there exists a Borel field $B$ of subsets of $\Omega$, and a probability measure $P$ on $B$. The
probability space ( $\Omega, B, P$ ) is the basis for the measuretheoretic (axiomatic) study of probability, as discussedin Dubes (25). The sample space $\Omega$, can in general, be either discrete, continuous, or mixed. That is, $\Omega=\{j ; j \in I\}$ or $\Omega=E$ or some combination. Similarly, $t_{j} \in E$ or $t_{j} \in I$. The stochastic process then may be a combination of discretetime or continuous time with discrete, continuous, or mixed sample space. Considering only continuous sample spaces $\Omega$, a quantitative description of the stochastic process is desired. Prabhu (90), states that a stochastic process $\mathrm{v}(.,$.$) is statistically determined if the nth order joint$ distribution function $F\left(\xi_{1}, \xi_{2}, \ldots, \xi_{n} ; t_{1}, t_{2}, \ldots, t_{n}\right)=$ $P\left(v\left(t_{1},.\right) \leq \xi_{1} ; v\left(t_{2},.\right) \leq \xi_{2} ; \ldots ; v\left(t_{n} ;\right) \leq \xi_{n}\right)$
2.1.1
is known for all $n$ and $t_{1}, t_{2}, \ldots, t_{n}$ where $F$ satisfies the symmetry and compatibility conditions. For the discrete-time scalar stochastic process $v(t,),. t \in[0,1, \ldots, k], k<\infty$, there are finitely many distribution functions, and a complete statistical description of the process is possible with them. On any open or closed subset of the real line, however, there are infinitely many instants of time, hence infinitely many joint distribution functions and in theory, the continuoustime stochastic process can never be statistically determined. This point is of major significance in estimation theory where density functions have to be defined. It is possible in
theory to completely determine all the distribution functions of a discrete-time process and utilize them, whereas continuous-time problems do not submit to this method of study. For this reason, the present research deals with continuous-time dynamic and stochastic processes, since the discrete situation can be studied later. However, the reverse is not true for many kinds of analysis.

The first order distribution function

$$
F\left(\xi_{1} ; t_{1}\right)=P\left(v\left(t_{1} r .0\right) \leq \xi_{1}\right)
$$

over the ensemble $v\left(t_{1} ; i\right)$, for all $t_{1} \in\left[t_{o}, t_{f}\right]$ exists as does the second order joint distribution function.

$$
F\left(\xi_{1}, \xi_{2} ; t_{1}, t_{2}\right)=P\left(v\left(t_{1}, \lambda\right) \leq \xi_{1} ; v\left(t_{1}, \lambda\right) \leq \xi_{2}\right)
$$

over the ensembles $v\left(t_{1}, .,\right)$ and $v\left(t_{2}, .,\right)$ for all $t_{1}, t_{2} \in\left[t_{0}, t_{f}\right]$. From these two functions much useful information is available. Theorems involving only tnese two functions and their properties appear in Bhat (17) and Hoel (35). Of course, knowledge of these first two distribution functions above does not imply the process v(.,.) is completely statistically determined. It is noted in Dubes (25) and Sage and Melsa 07 ) that in many physical situations knowledge of the first two distribution functions is all that is necessary for satisfactory performance or results. Techniques are available as in Parzen (86) for implementing knowledge of the first $n, n<\infty$, distribution functions as an approximation to full knowledge for all n .

### 2.2 Continuous Stochastic Processes

Next is an illustrative example from Astrom (8) which will be used to classify various types of continuous stochastic processes. This example delineates those properties of continuous stochastic processes most useful in the analysis of a larger problem. A set of "reasonable" assumptions might be the following as a starting point. The real scalar continuous-time stochastic process $v(t,),. t \in\left[t_{o}, t_{f}\right] \triangleq T$ which is a continuous random variable for each fixed t\&T should be

1) second order, i.e. have finite variance,
2) continuous in the mean square sense $\forall t \in T$,
3) a process such that $v(t,$.$) is independent of$

$$
v(\tau, .) \forall t \neq \tau \dot{\in} T,
$$

and
4) zero mean.

It is shown in Astrom (8) that the mean-square value of $v(.,),. \Phi_{v v}(t, t)=E\left\{x^{2}(t)\right\}=0 \quad \forall t \in T$.

If this process is the input (forcing function) of a linear ordinary differential equation, then the difference between the solution with zero input and the solution with this input is zero in the mean square. According to Astrom (8), this is not a "sensible" stochastic state model. Remarks in Papoulis (84) lead to the same conclusion.

In order to describe a "sensible" state model, one or more of assumptions 1) through 4) must be relaxed. It is well known that 4) can be relaxed with no loss of generality in most circumstances. However there does remain the accountability for non-zero means.

If assumption 1) is dropped and in addition stationarity imposed, the process would be basically a white noise process, i.e., the variance is infinite.

Relaxing assumption 2) leads to the time derivative of the stochastic process being undefined (in the mean-square sense). If this process was the input to a linear O.D.E., then the derivatives in the equation would also be undefined. The Ito calculus or the Stratonovich calculus allows analytic treatment of this type of process. Basically, an independent interval type process exists. For normal stationary transition probability, a Weiner process is formed and it is well known that Weiner processes have no defined time derivatives (in the mean-square sense).

Assumption 3) could be dropped and nonstationary processes with mean square sense time derivatives and finite continuous variances are obtained. Such processes are sometimes called second order processes.

### 2.3 Classification

It is desired to grossly classify continuous-time stochastic properties by which of assumptions 1) through 3)
are relaxed. Whether assumption 4) holds or not will not affect the generality of any class.
2.3.1 Class 1.

This is the class of processes with infinite variances. If the process is also stationary it is a white noise process.

### 2.3.2 Class 2.

This is the class of independent increment processes. The best known example is the Weiner process for which the transition probability is Gaussian and stationary. No member of this class of processes is continuous (mean square) in time, hence does not have a defined time derivative.

### 2.3.3 Class 3.

This is the class of second order processes. Such processes are nonstationary with finite variances and continuous (mean square) derivatives.

### 2.3.4 Class 4.

This is the class of stochastic processes with correlation times very much less than the smallest time constants of the system which they enter. Such a process v(.,.) has a variance given by

$$
V_{V v}(t, \tau)=Q(t) \delta(t-\tau)
$$

2.3.4.1

This class of processes is discussed in Bryson and Ho (20) and has been treated in stochastic control and estimation
problems. However, at $t=\tau$, the variance is infinite. As in (20), the Dirac delta function is the limit of a pulse with amolitude $1 / 2 \in$ of duration 2E This pulse can be thought of as having large variance for a: very short time. If $2 \epsilon$ is much less than the smallest time constant of the system, the time correlation of the process dies out in times of order of magnitude relevant to the system. Essentially this implies the nonstationarity property is not of significance and only one time axis is required to describe the statistics of the process. This may not be true in all physical situations. Therefore, this work will deal with class 3 processes and it is noted that approximations other than those with impulses of class 4 processes are included as members of class 3 processes.

### 2.4 Conclusions

This chapter provides the reasoning used to select the class of processes studied in the research. Continuoustime processes are selected since discrete-time analyses do not always allow the extension to the continuous-time case, whereas the reverse is possible. Class 3) stochastic processes are chosen for study since they have not been studied to date in control or identification problems. In fact this class of processes has only been the subject of a few papers most of which formulate but do not solve problems. The other three classes of processes have been studied extensively.

## CHAPTER 3

## PROBLEM FORMULATION

In this chapter the approach to the solution of the stochastic optimal portion of the problem begins. The identification portion of the problem starts in Chapter 6. Throughout Chapters 3, 4, and 5 all elements of $\gamma$ are assumed to be known. Therefore the notational dependence of all variables on the vector $y$ of parameters to be identified, will be eliminated until Chapter 6.

For any optimal problem, four data are needed as discussed in Lee and Markus (65). The dynamic system equations, the performance index, endpoint conditions, and classes of admissible controls must be specified. In addition if the dynamic system has stochastic inputs, some assumptions must be made regarding the class of processes. Also a priori assumptions on the initial values of the moments of the stochastic processes are needed.

From these four data and the assumptions about the stochastic processes, the stochastic optimal problem is formulated. It is sought to derive necessary and sufficient conditions for solutions to be optimal. Also if possible the existence and uniqueness of solutions is to be established. Further, in many problems closed loop (feedback) control laws
are to be found. In this work, the existence of solutions is assumed.

### 3.1 Stochastic Formulation

The four data for the classes of stochastic optimal control problems considered follow. This description covers a wide range of physical situations. An actual physical process is what is being described.

### 3.1.1 Dynamic System

The dynamic system considered is the linear time varying system given by
$\frac{d x}{d t}(t,)=.A(t) x(t,)+.B(t) u(t)+C(t) v(t,$.
3.1.1.1
for $t \in\left[\mathrm{t}_{\mathrm{O}}, \mathrm{t}_{\mathrm{f}}\right] \triangleq \Gamma_{1}$
where
x is an nxl state vector
A is an nxn matrix
u is an rxl optimal control vector
$B$ is an nxr matrix
v is an sxl stochastic disturbance vector
and
$C$ is an nxs matrix.

The stochastic process $v$ is assumed to be a Class 3) type process defined in Chapter 2 , usually called a second order process. No assumptions are made about its distribution (or density) functions. It is assumed that for each fixed
$t \in \Gamma_{1}, v(t,$.$) is a continuous random variable, which implies$ a continuous sample space.

### 3.1.2 Performance Index

The scalar quantitative measure of performance termed the performance index is mathematically given by

$$
J=J(u)=K\left(x\left(t_{f}, .\right), t_{f}\right)+\int_{t_{0}}^{t_{f}} f_{00}(x(t, .), u(t), v(t, .), t) d t
$$

for dynamic optimization problems. The functional J contains two terms, a terminal cost term $K$, and an integral cost term of the function $\mathrm{f}_{00}$. Most performance indices for stochastic optimal problems do not explicitly show dependence on the stochastic input $v$. The effect of this disturbance is assumed to be wholly contained in its effect on the state $x$. It is seen from 3.1.1.1 that $x$ is a stochastic process as well as v. It is a priori assumed that $u$ is a deterministic function of time. From 3.1.2.1, $J$ is a random variable. As such, it is not a "sensible" quantitative measure. In many stochastic problems, the random variable $J$ is made deterministic and is given by

$$
J_{D}=J_{D}(u)=E\{J\},
$$

where $E\{$.$\} denotes expectation. Other means of making J$ a deterministic measure exist in the literature. All of these require expectation, but not in the same way as shown in 3.1.2.2. For exampla, the measure

$$
J_{D 1}=J_{D I}(u)=E\left\{[J-E(J)]^{2}\right\}
$$

$$
3.1 .2 .3
$$

is discussed in articles by Sain and/or Liberty (98), (99), (100), and (101). Other related formulations are given in Murphy (81), and Rekasius (93). In the articles by Pugachev (89), and Andreev (3), (4), (5), and (6), the problem of synthesizing optimal systems for a wide variety of performance criteria is investigated.

Typical deterministic measures are minimum energy, fuel, and time. These are for deterministic problems equivalent to $v(t,)=.0 \forall t \in \Gamma_{1}$.

They are

$$
\begin{align*}
& J_{M E}^{(u)}=\left\langle x\left(t_{f}\right), Q\left(t_{f}\right) x\left(t_{f}\right)\right\rangle+\int_{t_{0}}^{t_{f}}\{\langle x, R x\rangle+\langle u, S u\rangle\} d t \\
& \left.J_{M F}^{(u)}=\int_{t_{0}}^{t_{f}}{ }_{f}^{r} \sum_{j=1} s_{j}\left|u_{j}\right|\right\} d t \\
& J_{M T}^{(u)}=\int_{t_{O}}^{t_{f}} d t=t_{f}-t_{O}
\end{align*}
$$

These three performance indices are most widely used. Even if the noise $v$ is present the same type of index is still used, except that the $E\{J\}$ or a function of $E\{J\}$ is required for a "sensible" measure. A slight generalization for the minimum energy criteria $J_{M E}$ would explicitly include the energy due to the stochastic disturbance $v$, given in a general form by 3.1.2.1. The minimum energy, or quadratic, form of 3.1.2.1 is
$J_{S M E}(u)=E\left\{J_{S}\right\}=E\left\{\left\langle x\left(t_{f},.\right), Q\left(t_{f}\right) x\left(t_{f},.\right)\right\rangle\right.$

$$
+\int_{t_{0}}^{t_{f}}\left\{\left\langle x, R_{1} x>+<u, R_{3} u>+<v, R_{4} v>\right\} d t\right\}
$$

$$
3.1 .2 .7
$$

In this case, minimum energy control of state is desired in the presence of minimum disturbance energy, where $J_{S N E}(u)$ is a more general stochastic minimum energy criteria. Such criteria have indirectly been studied in maximum signal-tonoise ratio problems for stationary Gaussian white noise processes. A modern formulation of this problem is found in Holtzman (37) and Athans and Schweppe (12).

### 3.1.3 Endpoint Conditions

Normally in deterministic optimal problems, only the initial value of the state $x\left(t_{0}\right)$, the initial time $t_{o}$, the final state $x\left(t_{f}\right)$, and the finite final time $t_{f}$ are required as known or as to be determined if not specified. These data are present in the transversality conditions when variational techniques are used.

When stochastic inputs are present, much more a priori information is required such as the mean value and variance of the initial state $x\left(t_{0}\right)$, and the correlation between the stochastic disturbance and the initial state. These data are explicitly required. For many problems, the distribution function (or density function) of the noise $v$ and the state $x$ must also be specified.

### 3.1.4 Classes of Admissible Controls

The last data needed to formulate a deterministic optimal control problem is the set $U$, in which the optimal control $u$ belongs. Whether the set $U$ is open or closed puts bounds or no bounds on the values $u$ can have. Most minimum energy problems require $U$ to be an open set and minimum fuel or time problems require $U$ to be a closed set, i.e., $u$ is bounded above and below. The physical application determines which condition is present in a specific example.

### 3.2 A Transformation

For stochastic optimal control problems, the four data of section 3.1 acquire characteristics which add to the degree of difficulty in completion of the solution as compared to deterministic optimal control problems. Basically the only stochastic optimal problems solved to date are linear ones for which the separation theorem holds. This restriction requires that a Kalman estimate be obtained in addition to the four data. In this work an alternate approach is given. As a first step, a transformation of the four data is made using mainly the expectation operator. This transformation changes the problem from stochastic to deterministic. The form of the data after transformation suggests a differential game approach as will be discussed in Section 3.3.

### 3.2.1 Dynamic System

The stochastic dynamic system governed by the state equation 3.1.1.1, is transformed directly by taking the expectations of both sides resulting in

$$
\frac{d}{d t} \mu_{x}(t)=A(t) \mu_{x}(t)+B(t) u(t)+C(t) \mu_{v}(t)
$$ for all $t \in \Gamma_{1}$

where

$$
E\{x(t, .)\} \triangleq \mu_{x}(t)
$$

and

$$
E\{v(t, .)\} \triangleq \mu_{v}(t)
$$

$$
3.2 .1 .3
$$

The mean $\mu_{x}$, of the state $x$ and the mean $\mu_{v}$ of the stochastic input $v$ are deterministic functions of time. The control $u$ is assumed a priori to be a deterministic function of time. The mean $\mu_{x}$, is the first moment of the state $x$, and similarly for $\mu_{v}$. As mentioned in Chapter 1 , the effects of the stochastic disturbance $v$ will be approximated by its first and second moments. The first moment propagates through the system as in equation 3.2.1.1. The complete description of the propagation of the second moment of $v$ on the state $x$ is given by the two equations,

$$
\begin{align*}
& \frac{\partial V^{\prime}}{\partial \tau} v x(t, \tau)=A(\tau) V^{\prime}{ }_{v x}(t, \tau)+C(\tau) v^{\prime}{ }_{V V}(t, \tau) \\
& \quad \text { for }(t, \tau) \in \Gamma_{1} x \Gamma_{1} \triangleq \Gamma
\end{align*}
$$

and

$$
\begin{aligned}
& \frac{\partial}{\partial t} V_{X x}(t, \tau)=A(t) V_{x x}(t, \tau)+C(t) V_{V x}(t, \tau) \\
& \text { for }(t, \tau) \in \Gamma
\end{aligned}
$$

$$
3.2 .1 .5
$$

where

$$
\begin{aligned}
& \left.E\{[v(t, .)-E\{v(t, .)\}]\}\left[\mathcal{G}(\tau, .)-E\{v(\tau, .)\}^{\prime}\right]^{\sim}\right\} \triangleq v_{v v}(t, \tau) \\
& \quad \text { for }(t, \tau) \in \Gamma,
\end{aligned}
$$

$$
\left.E\{[v(t, .)-E\{v(t, .)\}] \overline{[x}(\tau, .)-E\{x(\tau, .)\}]^{\prime} \sim\right\} 今 v_{v x}(t, \tau)
$$

$$
\text { for }(t, \tau) \in \Gamma
$$

$$
3.2 .1 .7
$$

$$
E\left\{[x(t, .)-E\{x(t, .)\}][x(\tau, .)-E[x(\tau, .)\}]^{2}\right\} \triangleq V_{x x}(t, \tau)
$$

$$
\text { for }(t, \tau) \in \Gamma
$$

$$
3.2 .1 .8
$$

The covariance of $v$,

$$
\mathrm{V}_{\mathrm{vv}}(\mathrm{t}, \tau) \text { is an sxs matrix }
$$

and

$$
\begin{aligned}
& V_{V V}(t, \tau)=\Phi_{V V}(t, \tau)-\mu_{V}(t) \mu_{V}(t) \\
& \quad \text { for }(t, \tau) \in \Gamma
\end{aligned}
$$

$$
3.2 .1 .9
$$

where

$$
\begin{aligned}
& E\left\{v(t, .) v^{3 \rho}(\tau, .)\right\} \triangleq \Phi_{v v}(t, \tau) \\
& \text { for }(t, \tau) \in \Gamma
\end{aligned}
$$

and $\Phi_{V V}(t, \tau)$ is the sxs matrix called the autocorrelation of $v$.

The cross-covariance of $v$ and $x$, $\mathrm{V}_{\mathrm{VX}}(\mathrm{t}, \tau)$ is an sxn matrix
and

$$
\begin{aligned}
& V_{V_{X}}(t, \tau)=\Phi_{V X}(t, \tau)-\mu_{v}(t) \mu_{X}(\tau) \\
& \quad \text { for }(t, \tau) \in \Gamma
\end{aligned}
$$

where

$$
\begin{aligned}
& E\left\{v(t, .) x^{-}(\tau, .)\right\} \triangleq \Phi_{v X}(t, \tau) \\
& \quad \text { for }(t, \tau) \in \Gamma
\end{aligned}
$$

and $\Phi_{V x}(t, \tau)$ is the sxn matrix called the cross correlation of $v$ and $x$.

And the covariance of $x$,
$V_{X X}(t, \tau)$ is an nxn matrix
and

$$
\begin{aligned}
& V_{X X}(t, \tau)=\Phi_{X X}(t, \tau)-\mu_{x}(t) \mu_{x}^{\sim}(\tau) \\
& \quad \text { for }(t, \tau) \in \Gamma
\end{aligned}
$$

$$
3.2 .1 .13
$$

where

$$
\begin{aligned}
& E\left\{x(t, .) x^{e}(\tau, .)\right\} \triangleq \Phi_{x X}(t, \tau) \\
& \text { for }(t, \tau) \in \Gamma
\end{aligned}
$$

$$
3.2 .1 .14
$$

and $\Phi_{x X}(t, \tau)$ is the nxn matrix called the autocorrelation of $x$.

The following moments also appear and are defined below.
The variance of $v$

$$
V_{v v}(t, t) \text { is an sxs matrix }
$$

and

$$
\begin{aligned}
& V_{v v}(t, t)=\Phi_{v v}(t, t)-\mu_{v}(t) \mu_{v}^{i}(t) \\
& \quad \text { for } t \in \Gamma_{I}
\end{aligned}
$$

where

$$
\begin{aligned}
& E\left\{v(t, .) v^{T}(t, \ldots)\right\} \triangleq \Phi v v(t, t) \\
& \text { for } t \in \Gamma_{I}
\end{aligned}
$$

and $\Phi_{v v}(t, t)$ is the sxs matrix called the mean square value of $v$.

The cross variance of $v$ and $x$

$$
V_{V x}(t, t) \text { is an sxn matrix }
$$

and

$$
\begin{aligned}
& V_{v x}(t, t)=\Phi_{v x}(t, t)-\mu_{v}(t) \mu_{x}^{d}(t) \\
& \quad \text { for } t \in \Gamma_{1}
\end{aligned}
$$

where

$$
E\left\{v(t, .) x^{\text {² }}(t, .)\right\} 今 \Phi_{V x}(t, t)
$$

$$
\text { for } t \in \Gamma_{1}
$$

and $\Phi_{V X}(t, t)$ is the sxn matrix called the mean cross value of $v$ and $x$.

The variance of $x$

$$
V_{X X}(t, t) \text { is an nxn matrix }
$$

and

$$
\begin{aligned}
& V_{x x}(t, t)=\Phi_{x x}(t, t)-\mu_{x}(t) \mu \frac{r}{x}(t) \\
& \quad \text { for } t \in \Gamma_{1}
\end{aligned}
$$

where

$$
\begin{aligned}
& E\left\{x(t, .) x^{\prime \prime}(t, .)\right\} \triangleq \Phi_{X X}(t, t) \\
& \text { for } t \in \Gamma_{1}
\end{aligned}
$$

and
$\Phi_{\mathrm{XX}}(t, t)$ is the $n x n$ matrix called the mean square value of $x$.

The detailed derivation of equations 3.2.1.1, 3.2.1.4, and 3.2.1.5 is contained in Appendix A. The derivation uses material from Papoulis (84), Lebedev (63), Pugachev (39), and Sage and Melsa (97). These equations describe how the first and second moments of a Class 3) stochastic process propagate through a linear system. Similar equations could be obtained for any other class of stochastic processes.

### 3.2.2 Performance Index

The transformation of performance indices was first performed in estimation problems. Typically, minimum variance criteria resulted for criteria quadratic in the state $x$. The major portion of criteria after transformation used in the literature depend on criteria which were originally quadratic. For example, if $x$ is a stochastic process, then from 3.1.2.4

$$
\begin{aligned}
& E\left\{J_{\text {ME }}\right\}=E\left\{\left\langle x\left(t_{f}, \cdot\right), Q x\left(t_{f}, .\right)\right\rangle\right. \\
& \left.\quad+\int_{t_{O}}^{t_{f}}\{\langle x, R x\rangle+\langle u, S u\rangle\} d t\right\} \\
& \quad=\operatorname{tr}\left[Q \Phi_{X X}\left(t_{f}, t_{f}\right)\right]+\int_{t_{0}}^{t_{f}}\left\{\operatorname{tr}\left[R \Phi_{X X}(t, t)\right]+\langle u, R u>\} d t\right.
\end{aligned}
$$

$$
\begin{align*}
= & \operatorname{tr}\left[Q V_{X X}\left(t_{f}, t_{f}\right)-Q \mu_{X}\left(t_{f}\right) \mu_{X}^{2}\left(t_{f}\right)\right] \\
& +\int_{t_{0}}^{t_{f}}\left\{\operatorname{tr}\left[R V_{X X}(t, t)-R \mu_{x}(t) \mu_{x}^{2}(t)\right]+\langle u, R u\rangle\right\} d t \\
= & \operatorname{tr}\left[Q V_{X X}\left(t_{f}, t_{f}\right)\right]-\left\langle\mu_{x}\left(t_{f}\right), Q \mu_{X}\left(t_{f}\right)\right\rangle \\
& +\int_{t_{0}}^{t_{f}}\left\{\operatorname{tr}\left[R V_{X X}(t, t)\right]-\left\langle\mu_{x}(t), R \mu_{x}(t)\right\rangle+\langle u, R u\rangle\right\} d t .
\end{align*}
$$

Similarly, it can be shown that

$$
\begin{aligned}
E\left\{J_{S}\right\}= & \operatorname{tr}\left[Q\left(t_{f}\right) V_{X X}\left(t_{f}, t_{f}\right)\right]-\left\langle\mu_{x}\left(t_{f}\right), Q\left(t_{f}\right) \mu_{X}\left(t_{f}\right)\right\rangle \\
+ & \int_{t_{0}}^{t_{f}}\left\{\operatorname{tr}\left[R_{1} V_{X X}(t, t)\right]-\left\langle\mu_{X}(t), R_{1} \mu_{x}(t)\right\rangle+\left\langle u, R_{3} u\right\rangle\right. \\
& \left.\quad+\operatorname{tr}\left[R_{4} V_{V V}(t, t)\right]-\left\langle\mu_{v}(t), R_{4} \mu_{v}(t)\right\rangle\right\} d t
\end{aligned}
$$

The other types of performance indices mentioned in the references of Section 3.1.2 are one of the following:

$$
E\left\{J_{M E}^{2}\right\}, E\left\{\left[J_{M E}-E\left\{J_{M E}\right\}\right]^{2} \mathrm{I},\left[\mathrm{E}\left\{J_{\mathrm{ME}}\right\}\right]^{2}\right.
$$

or for $J_{S}$,

$$
E\left[J_{S}^{2}\right\}, E\left\{\left[J_{S}-E\left\{J_{S}\right\}\right]^{2}\right\},\left[E\left\{J_{S}\right\}\right]^{2}
$$

In some of those references, the equations are worked out, and they contain the first and second moments of $v$ and $x$ as variables, as do equations 3.2.2.1 and 3.2.2.2. Slight generalizations of these types of criteria are the ones chosen for this research. The transformations of these particular indices is contained in Appendix B. Four different criteria are used to
derive necessary conditions in Chapter 4. The physical significance of these four criteria is also detailed in Appendix $B$.

### 3.2.3 Endpoint Conditions

From the equations of the dynamic system and the performance indices, it will be seen that a modification in the form of the endpoint conditions is necessary. The fixed endpoint case is used in Chapter 4 derivations of necessary conditions, as well as fixed finite final time. Other kinds of endpoint conditions, such as free final state and free final time are left for future research.

### 3.2.4 Classes of Admissible Controls

In Section 3.3, the moments of $v$ will be designated as controls. Therefore, these moments must be members of a set which is either open or closed. For Class 3) stochastic processes there are no finite bounds on either the mean or the covariance of $v$ which exist due to the theory of stochastic processes, so it is assumed that they belong to open sets.

### 3.3 Differential Game Approach

In this section the variables of equations 3.2.1.1, 3.2.1.4, and 3.2.1.5 are renamed for ease of notation and for clarity in the formulation of the differential game. Then the equations of the dynamic system, the performance index, the endpoint conditions, and the classes of admissible controls
are specified in full such that necessary and sufficient conditions can be obtained.
3.3.1 Formulation

Equation 3.2.1.1 is changed to

$$
\begin{aligned}
& \frac{d z_{1}}{d t}(t)=A(t) z_{1}(t)+B(t) u_{1}(t)+C(t) u_{2}(t) \\
& \quad \text { for all } t \in \Gamma_{1} .
\end{aligned}
$$

$$
3.3 .1 .1
$$

where
$\mu_{X}(t) \triangleq z_{1}(t)$ an $n x l$ vector,
$u(t) \triangleq u_{1}(t)$ an $r x l$ vector,
and
$\mu_{v}(t) \triangleq u_{2}(t)$ an sxl vector.
Equation 3.2.1.4 becomes
$\frac{\partial}{\partial \tau} z_{2}^{*}(t, \tau)=A(\tau) z_{2}^{2}(t, \tau)+C(\tau) u_{3}^{2}(t, \tau)$
for all $(t, \tau) \in \Gamma$
where

$$
V_{v x}^{\prime}(t, \tau) \triangleq z_{2}^{\frac{T}{2}}(t, \tau) \text { an nxs matrix }
$$

and

$$
V_{v v}^{\sim}(t, \tau) \triangleq u_{3}^{\Gamma}(t, \tau) \text { an sxs matrix. }
$$

Equation 3.2.1.5 becomes

$$
\frac{\partial}{\partial t} z_{3}(t, \tau)=A(t) z_{3}(t, \tau)+C(t) z_{2}(t, \tau)
$$

$$
\text { for all }(t, \tau) \in \Gamma
$$

where

$$
V_{X x}(t, \tau) \triangleq z_{3}(t, \tau) \text { an nxn matrix. }
$$

The state of the system becomes $z_{1}, z_{2}, z_{3}$ with the "man" chosen controls $u_{1}$ and the opposing "nature" controls $u_{2}$, $u_{3}$ 。

A general performance index of the form

$$
\begin{align*}
J=J & \left(u_{1}, u_{2}, u_{3}^{\eta}\right)=\int_{t_{0}}^{t_{0}} \int_{t_{0}}^{t_{f}} f_{2}\left(z_{1}, z_{2}^{\stackrel{r}{t}}, z_{3}, u_{1}, u_{2}, u_{3}^{\prime o}\right) d t d \tau \\
& +\int_{t_{0}}^{t_{f}} f_{1}\left(z_{1}, u_{1}, u_{2}\right) d t
\end{align*}
$$

will be employed. The "man" controls $u_{1}$ seek to minimize $J$ while the "nature" controls $u_{2}, u_{3}^{\sim}$ seek to maximize $J$.

The initial and final times are fixed. The initial state is specified and given by

$$
\begin{aligned}
& z_{1}\left(t_{0}\right)=\pi_{10} \\
& z_{2}^{2}\left(t_{,} t_{0}\right)=\pi_{20}(t) \\
& z_{3}\left(t_{0}, \tau\right)=\pi_{30}(\tau)
\end{aligned}
$$

$$
3.3 .1 .5
$$

The terminal state is also specified and given as

$$
\begin{aligned}
& z_{1}\left(t_{f}\right)=\pi_{11} \\
& z_{2}^{L}\left(t, t_{f}\right)=\pi_{21}(t) \\
& z_{3}\left(t_{f}, \tau\right)=\pi_{31}(t)
\end{aligned}
$$

$$
3.3 .1 .6
$$

Finally it is assumed that $u_{1} \in U_{1}=E^{r}$,i.e. $u_{1}$ takes values in $r$ dimensional Euclidean space which is an open set. Similarly $u_{2} \in U_{2}=E^{s}$ and $u_{3}^{N_{H}} \in U_{3}=E^{s} X E S^{s}$.

### 3.3.2 Differential Game Theory

At this point some remarks about deterministic differential games are necessary as background for the next section. In the description of the four data for a one-sided optimal control problem given in Section 3.1 and 3.2 the dynamic system and performance index change considerably. The discussion of endpoint conditions and classes of admissible controls remains the same except more variables are present.

A two-person zero-sum differential game is a two-sided optimal control problem. The two sides are exemplified as two sets of controls $u_{1}$ and $u_{2}$, which drive dynamic systems with goals that are in conflict with each other. Mathematically, the dynamic system is

$$
\frac{d x}{d t}(t)=A(t) x(t)+B(t) u_{1}(t)+C(t) u_{2}(t), t \in\left[t_{0}, t_{f}\right]
$$

for a class of linear differential games. Equations 3.3.2.1 is sometimes obtained by combining the equations for two systems, one the "man" system given by

$$
\frac{d x_{m}}{d t}(t)=A_{m}(t) x_{m}(t)+B_{m}(t) u_{1}(t), t \in\left[t_{0}, t_{f}\right]
$$

and the other the "nature" system given by

$$
\frac{d x}{d t} n(t)=A_{n}(t) x_{n}(t)+c_{n}(t) u_{2}(t), t \in\left[t_{0}, t_{f}\right]
$$

The performance index would include the effects of both controls, and for system 3.3.2.1 would functionally be

$$
J_{D G}=J_{D G}\left(u_{1}, u_{2}\right)=K\left(x\left(t_{f}\right), t_{f}\right)+\int_{t_{o}}^{t_{f}} f\left(x, u_{1}, u_{2}\right) d t
$$

a minimum energy type criteria is

$$
\begin{align*}
J_{M E} & =J_{M E}\left(u_{1}, u_{2}\right)=\left\langle x\left(t_{f}\right), R_{1} x\left(t_{f}\right)>+\int_{t_{0}}^{t_{f}}\left\{\left\langle x, R_{2} x\right\rangle+\left\langle u_{1}, R_{3} u_{1}\right\rangle\right.\right. \\
& \left.-<u_{2}, R_{4} u_{2}>\right\} d t
\end{align*}
$$

The conflict is present since $u_{1}$ seeks to minimize $J_{M E}$
 taneously establish
$\min _{\max } \Psi_{\mathrm{ME}}\left(\mathrm{u}_{1}, u_{2}\right)$ $u_{1} \in U_{1} \quad u_{2} \in U_{2}$

$$
3.3 .2 .6
$$

where $U_{1}$ and $U_{2}$ are classes of admissible controls. The solution is sought by $u_{1}$ in the knowledge that $u_{2}$ seeks to maximize $J_{M E}$ and vice versa. Therefore the optimal point is

$$
J_{M E}\left(x, u_{1} *, u_{2}\right) \leqq J_{M E}\left(x^{*}, u_{1} *, u_{2}^{*}\right) \leqq J_{M E}\left(x, u_{1}, u_{2}^{*}\right)
$$

which is the definition of a saddle point. From 3.3.2.7 it is seen that $u_{1}$ seeks to minimize $J_{M E}$ knowing $u_{2}$ * is maximizing $J_{M E}$ and vice versa.

These key points are needed to formulate the class of differential game problems in the next section.

### 3.3.3 Statement of the Problem

It is desired to find the pair of controls ( $u_{1} ; u_{2}, u_{3}$ ) which establish a a saddle point for the performance index $J$ of equation 3.3.1.4 in the sense that

$$
\min _{u_{1} \in U_{1}} \max _{u_{2} \in U_{2}} J\left(u_{1}, u_{2}, u_{3}, t_{0}, t_{f}\right)
$$

occurs, while transferring the initial state $z_{1}\left(t_{0}\right), z_{2}^{\prime \prime}\left(t, t_{0}\right)$, $z_{3}\left(t_{0}, \tau\right)$ to fixed terminal state $z_{1}\left(t_{f}\right), z_{2}^{j}\left(t, t_{f}\right) z_{3}\left(t_{f}, \tau\right)$ in fixed finite time $t_{f}-t_{0}$ subject to the constraints given by 3.3.1.1, 3.3.1.2 and 3.3.1.3. Necessary conditions for four different specific problems are given in Chapter 4.

### 3.4 Conclusions

A stochastic optimal control problem for dynamic systems with Class 3) stochastic disturbances was given. The nature of the problem was transformed to deterministic using mainly the expectation operator. The effects of the stochastic disturbance after this transformation are assumed to be contained in the propagation of the first two moments through the dynamic system. The degrading effect of these moments is in conflict with the efforts to optimally control the dynamic system. This conflict situation was formally cast into a differential game. The various data needed to mathematically describe the differential game were specified and finally the statement of the problem in this context was given.

## CHAFTER 4

NECESSARY CONDITIONS

Four different classes of problems are considered in this chapter. For the first three classes, the dynamic system remains the same but the performance index changes. For the fourth class of problems the dynamic system also changes. Then sets of necessary conditions for each problem are given.

## 4.1 pröblem Statements

The four data required to define each class of problems are specified in this section.

### 4.1.1 Problem 1.

It is desired to find the pair of controls ( $u_{1} ; u_{2}, u_{3}$ ) which establish a saddle point for the perfonnance index
$J\left(u_{1}, u_{2}, u_{3}^{\prime}\right)=-\frac{1}{2} \int_{t_{0} t_{0}}^{t_{f}^{f}} \int_{i}^{t}\left[\operatorname{tr}\left[R_{1} z_{3}(t, \tau) z_{3}^{\prime}(t, \tau) R_{1}^{f}\right]+\right.$
$\left.\operatorname{tr}\left[R_{2} u_{3}(t, \tau) u_{3}^{\prime}(t, \tau) R_{2}^{\prime}\right]\right\} d t d \tau+\frac{1}{2} \int_{t_{0}}^{t f}\left\{u_{1}(t), R_{3} u_{1}(t)>-<u_{2}(t), R_{4} u_{2}(t)>\right\} d t$
where $R_{1}$ is assumed non-negative definite and symmetric and $R_{2}, R_{3}$, and $R_{4}$ are assumed to be positive definite and symmetric, in the sense that

$$
\begin{align*}
& \min \max _{\operatorname{meU}_{1}} u_{u_{2} \in U_{2}} J\left(u_{1}, u_{2}, u_{3}^{\prime}\right) \\
& u_{3} \in U_{3}
\end{align*}
$$

occurs, while transferring the specified initial state

$$
\begin{aligned}
& z_{1}\left(t_{0}\right)=\pi_{10} \\
& z_{2}\left(t_{1}, t_{0}\right)=\pi_{20}(t) \\
& z_{3}\left(t_{5}^{\prime}, \tau\right)=\pi_{30}(\tau)
\end{aligned}
$$

to the specified final state

$$
\begin{aligned}
& z_{1}\left(t_{f}\right)=\pi_{11} \\
& z_{2}^{\prime}\left(t, t_{f}\right)=\pi_{21}(t) \\
& z_{3}\left(t_{f}, \tau\right)=\pi_{31}(\tau)
\end{aligned}
$$

in fixed finite time $t_{f}-t_{0}$, where the dynamic system is governed by
$\frac{d z}{d t^{1}}(t)=A(t) z_{1}(t)+B(t) u_{1}(t)+C(t) u_{2}(t), \forall t \in \Gamma_{1}$
4.1.1.5
$\frac{\partial z^{\prime}}{\partial \tau^{2}}(t, \tau)=\Lambda(\tau) z_{2}^{\prime}(t, \tau)+C(\tau) u_{3}^{\prime}(t, \tau), \forall(t, \tau) \in \Gamma$
4.1.1.6
$\frac{\partial z}{\partial t^{3}}(t, \tau)=A(t) z_{3}(t, \tau)+C(t) z_{2}(t, \tau), \forall(t, \tau) \in \Gamma$
and the admissible classes of controls are the open sets

$$
\begin{align*}
& u_{1} \in U_{1}=E^{r} \\
& u_{2} \in U_{2}=E^{S} \\
& u_{3}^{\prime} \in U_{3}=E^{S} \times E^{S}
\end{align*}
$$

### 4.1.2 Problen 2.

Problem 2 involves the functional performance index
$J\left(u_{1}, u_{z}, u_{3}^{\prime}\right)=\int_{t_{0} t_{0}}^{t_{f} t_{f}} f_{0}\left(z_{1}(t), z_{2}(\tau), z_{2}^{\prime}(t, \tau), z_{3}(t, \tau) ; u_{i}(t), u_{1}(\tau), u_{2}(t), u_{z}(\tau)\right.$,
instead of 4.1.1.1, all other data remaining the same.

### 4.1.3 Problem 3.

Problem 3 involves the specific performance index

$$
\begin{aligned}
& J\left(u_{1}, u_{2}, u_{3}^{\prime}\right)=\int_{t_{0}}^{t} f_{0}^{t} f^{f}\left\{f_{\infty}\left(z_{1}^{\prime}(t) ; z_{1}^{\prime \prime}(\tau), z_{2}^{\prime}(t, \tau), z_{y}(t, \tau), u_{1}(t), u_{1}(\tau), u_{z}(t), u_{z}(\tau)\right)\right. \\
& \left.+\sum_{k=1}^{S} \sum_{m=1}^{S} f_{k m}\left(z_{i}^{\prime}, z_{3}\right) u_{s k m}^{e}\right\} d t d \tau
\end{aligned}
$$

instead of 4.1.1.1, all other data remaining the same.

### 4.1.4 Problem 4

Consider a dynamic system described by

$$
\begin{array}{ll}
\frac{d z}{d t^{3}}(t)=A(t) z_{2}(t)+B(t) u_{1}(t)+C(t) u_{2}(t) & \text { 4.1.4.1 } \\
& \\
\frac{d z}{d t^{2}}(t)=F \cdot(t) z_{2}(t)+z_{2}(t) A^{\prime}(t)+u_{3}(t) C^{\prime}(t) & \text { 4.1.4.2 } \\
\frac{d z}{d t^{3}}(t)=A(t) z_{3}(t)+z_{3}(t) A^{\prime}(t)+C(t) z_{2}(t)+z_{2}^{\prime}(t) C^{\prime}(t) & 4.1 .4 .3
\end{array}
$$

with the functional
$J\left(u_{1}, u_{2}, u_{3}\right)=\int_{t_{0}}^{t_{f}} \operatorname{tr}\left\{\frac{a_{1}}{2} R_{1}(t) u_{1}(t) u_{1}{ }^{\prime}(t)\right.$
$\left.+\underset{4}{d_{2}} R_{2}(t)\left[u_{3}(t)+u_{2}(t) u_{2}^{\prime}(t)\right]^{2}\right\} d t$
Given the state equations 4.1.4.1 through 4.1.4.3 and the performance index 4.1.4.4, find the optimal strategy
$u_{i} \in U_{i}, i=1,2,3$ such that
$J\left(u_{1}{ }^{*}, u_{2}, u_{3}\right) \leq J\left(u_{1}{ }^{*}, u_{2}{ }^{*}, u_{3}{ }^{*}\right) \leq J\left(u_{1}, u_{2}{ }^{*}, u_{3}{ }^{*}\right)$
for all $u_{i} \in U_{i}, i=1,2,3$
where

$$
\begin{aligned}
& z_{1}\left(t_{0}\right)=z_{10} \\
& z_{2}\left(t_{0}\right)=z_{20} \\
& z_{3}\left(t_{0}\right)=z_{30}
\end{aligned}
$$

$$
4.1 .4 .6
$$

are specified
and
$d_{1}+d_{z} \neq 1$
$\mathrm{d} \geq 0$
$\mathrm{d}_{2} \geq 0$
4.1.4.7
4.2 Statement of Necessary Conditions

### 4.2.1 Theorem 4.1

In order that the pair of controls ( $u_{1} ; u_{2}, u_{3}^{\prime}$ ) be extremal for Problen 1 of Section 4.1.1, it is necessary that there exists nonzero continuous functions $\lambda_{1}(t), t \in \Gamma_{1}$, and $\lambda_{2}(t, \tau), \lambda_{3}(t ; \tau),(t, \tau) \in \Gamma$ which are solutions of

$$
\begin{array}{ll}
\frac{d \lambda^{*}}{d t^{1}}(t)=-A^{*}(t) \lambda_{1}^{*}(t), \forall t \in \Gamma_{1} & \text { 4.2.1.1 } \\
\frac{\partial \lambda^{*}}{\partial \tau^{2}}(t, \tau)=-A^{\prime}(\tau) \lambda_{2}^{*}(t, \tau)-\lambda_{3}^{*}(t, \tau) C(\tau), \forall(t, \tau) \in \Gamma & 4.2 .1 .2 \\
\frac{\partial \lambda^{*}}{\partial \tau^{3}}(t, \tau)=-A^{-}(t) \lambda_{3}^{*}(t, \tau)-R_{1}^{\prime} R_{1} z_{3}^{*}(t, \tau), \forall(t, \tau) \in \Gamma & 4.2 .1 .3
\end{array}
$$

through which the trajectories governed by

$$
\begin{aligned}
& \frac{d z}{d t}{ }_{1}^{*}(t)=A(t) z_{1}^{*}(t)+B(t) u_{1}^{*}(t)+C(t) u_{2}^{*}(t), \forall t \in \Gamma_{1} \\
& \text { 4.2.1.4 } \\
& \frac{\partial z^{* *}}{\partial \tau^{2}}(t, \tau)=A(\tau) z_{z}^{\circ *}(t, \tau)+C(\tau) u_{3}^{\mu^{*}}(t, \tau), \forall(t, \tau) \in \Gamma \\
& \text { 4.2.1.5 } \\
& \frac{\partial z^{*}}{\partial t^{3}}(t, \tau)=A(t) z_{3}^{*}(t, \tau)+C(t) z_{2}^{*}(t, \tau), \forall(t, \tau) \in \Gamma \\
& \text { 4.2.1.6 }
\end{aligned}
$$

are transferred from
$z_{1}\left(t_{0}\right)=\pi_{10}$
$z_{2}^{\prime}\left(t, t_{0}\right)=\pi_{20}(t)$
$z_{3}\left(t_{0}, \tau\right)=\pi_{30}(\tau)$
4.2.1.7
to
$z_{i}^{\prime}\left(\mathcal{E}_{f}\right)=\pi_{11}$
$z_{i}^{\prime}\left(t, t_{f}\right)=\pi_{21}$
$z_{3}\left(t_{f}, \tau\right)=\pi_{31}(\tau)$
4.2.1.8
in fixed finite time $t_{f}-t_{0}$, by the controls in the admissible classes
$u_{i}^{*} \in U_{1}=E r$
$\mathrm{u}_{2}{ }_{*}^{*} \in \mathrm{U}_{2}=E^{S}$
$u_{y}{ }^{*} \in U_{y}=E^{S}$ X $E^{S}$
which must satisfy

$$
\begin{array}{ll}
u_{1}^{*}(t)=-\mathrm{R}_{3}^{-1} B^{-}(t) \lambda_{1}^{*}(t), \forall v_{1}^{\prime \prime} \in \Gamma_{1} & \text { 4.2.1.10 } \\
u_{2}^{*}(t)=R_{4}^{-1} C^{-}(t) \lambda_{1}^{*}(t), \forall t \in \Gamma_{1} & 4.2 .1 .11 \\
u_{3}^{* *}(t, \tau)=C^{-}(\tau) \lambda_{2}^{*}(t, \tau) R_{2}^{-1}\left(R_{2}^{-1}\right)-\forall(t, \tau) \in \Gamma & \text { 4.2.1.12 } \\
\text { where }()^{*} \text { denotes extrenal. } &
\end{array}
$$

## Remark:

After substitution of equations 4.2.1.10, 4.2.1.11 and
4.2.1.12 into 4.2.1.4 and 4.2.1.5, there are $2 n(1+s+n)$ differential equations with $n(1+s+n)$ initial conditions and $n(1+s+n)$ final conditions, forming a two point boundary value problem (TPBVP) with $2 n(1+s+n)$ differential equations and $2 n(1+s+n)$ endpoint conditions. In theory then, it is possible to solve for the extremal controls and trajectories from the set of necessary conditions of Theorem 4.1.

### 4.2.2 Theorem 4.2

In order that the pair of controls ( $u_{1} ; u_{2}, u_{j}^{\prime}$ ) be extremal for Problem 2 of Section 4.1.2, it is necessary that there exists non-zero continuous functions $\lambda_{1}(t), t \in \Gamma_{1}$, and $\lambda_{2}(t, \tau), \lambda_{3}(t, \tau),(t, \tau) \in T$ which are solutions of
through which the trajectories governed by

$$
\begin{array}{ll}
{\frac{d z_{2}}{d t}}^{*}(t)=A(t) z_{1}^{*}(t)+B(t) u_{1}^{*}(t)+C(t) u_{2}^{*}(t), t \in \Gamma_{1} & 4.2 .2 .4 \\
{\frac{\partial z_{2}^{*}}{\partial \tau}}_{*}^{*}(t, \tau)=A(\tau) z_{2}^{* *}(t, \tau)+C(\tau) u_{1}^{* *}(t, \tau),(t, \tau) \in \Gamma & 4.2 .2 .5 \\
{\frac{\partial z_{3}}{\partial t}}_{*}^{*}(t, \tau)=A(t) z_{3}^{*}(t, \tau)+C(t) z_{2}^{*}(t, \tau),(t, \tau) \in \Gamma & 4.2 .2 .6
\end{array}
$$

are transferred from

$$
\begin{aligned}
& z_{1}\left(t_{0}\right)=\pi_{10} \\
& z_{2}^{-}\left(t_{,}, t_{0}\right)=\pi_{20}(t) \\
& z_{y}\left(t_{0}, \tau\right)=\pi_{30}(\tau)
\end{aligned}
$$

4.2.2.7
to

$$
\begin{aligned}
& z_{d}\left(t_{f}\right)=\pi_{11} \\
& z_{i}^{\prime}\left(t, t_{f}\right)=\pi_{21}(t) \\
& z_{3}\left(t_{f}, \tau\right)=\pi_{31}(\tau)
\end{aligned}
$$

$$
4.2 .2 .8
$$

$$
\begin{align*}
& \left.\int_{t_{0}}^{t_{i}}\left\{\frac{(\partial f 0}{\partial z_{1}}(t)\right)+A^{-}(t) \lambda_{1} *(t)+\frac{d \lambda}{d t^{2}} *(t)\right\} d \tau=0, \int_{t_{0}}^{t_{f}}\left\{\left(\frac{\partial f 0}{\partial z_{1}}(\tau) * d t=0\right.\right. \\
& \frac{\partial \lambda_{2}}{\partial \tau}{ }^{*}(t, \tau)=-A^{-}(\tau) \lambda_{2}^{*}(t, \tau)-\lambda_{3}^{*} *(t, \tau) C(\tau)-\left(\frac{\partial f_{0}}{\partial z_{2}}\right)_{*}^{*} \\
& \frac{\partial \lambda^{3}}{\partial E^{3}}{ }^{*}(t, \tau)=-A^{-}(\tau) \lambda_{3} *(t, \tau)-\frac{(\partial f o)}{\partial z_{3}} *
\end{align*}
$$

in fixed finite time $t_{f}{ }^{-t}{ }_{0}$, by the controls in the admissible classes

$$
u_{1} * \in U_{1}=E^{r}
$$

$$
u_{2}{ }^{*} \in U_{2}=E^{s}
$$

$$
u_{3}^{0}{ }_{3} \in U_{3}=E^{s} x E^{s} \quad 4.2 .2 .9
$$

which must satisfy
where ( )* denotes extremal.

$$
\begin{align*}
& \left.\int_{t_{0}}^{f_{\{ }}\left\{\frac{\left(\partial f_{0}\right.}{\partial u_{1}}(t)\right)+B^{\prime}(t) \lambda_{1} *(t)\right\} d \tau=0, \int_{t_{0}}^{t_{i}}\left\{\frac{\left(\partial f_{0}\right.}{\partial u_{1}}(\tau), * d t=0\right. \\
& \left.\int_{t_{0}}^{t}\left\{\frac{\left(\partial f_{0}\right.}{\partial u_{i}(\tau)}\right)+C^{\rho}(t) \lambda_{1} *(t)\right\} d \tau=0, \int_{t_{0}}^{f^{f}}\left\{\frac{\left(\frac{\partial f 0}{\partial u_{2}}(\tau)\right.}{}\right) * d t=0 \\
& \left(\frac{\partial f 0}{\partial u_{3}}\right)_{*}+C^{-}(\tau) \lambda_{2}^{-*}(t, \tau)=0
\end{align*}
$$

4.2.3 Theorem 4.3

In order that the pair of controls ( $u_{i} ; u_{2}, u_{j}^{\prime}$ ) be extremal for Problem 3 of Section 4.2.3, it is necessary that in addition to Theorem 4.2, the following conditions
$\frac{\partial}{\partial \tau} \frac{\partial H}{\partial u_{s}}=0$
4.2.3.1
$\square \frac{\partial}{\partial u_{s}}\left\{\left[\square \frac{\partial}{\partial \tau} \frac{\partial H}{\partial u_{s}}, \square\right\} \square=0\right.$
4.2.3.2
$\square \frac{\partial}{\partial u_{s}^{\prime}}\left\{\square \frac{\partial^{2}}{\partial \tau^{2}}{\left.\frac{\partial H}{\partial u_{s}^{\prime}}, \square\right\} \square \geq 0}^{\partial}\right.$
hold where
$H \equiv f_{\mathrm{OO}}\left(z_{i}(t), z_{i}(\tau), z_{\alpha}^{-}(t, \tau), z_{3}(t, \tau), u_{1}(t), u_{i}(\tau), u_{\alpha}(t), u_{z}(\tau)\right)$
$+\sum_{k=1}^{S} \sum_{m=1}^{S} f_{k m}\left(z_{z}^{-}(t, \tau), z_{3}(t, \tau)\right) u_{j-1}^{-}(t, \tau)$
$+\left\langle\lambda_{1}(t), A(t) z_{1}(t)+B(t) u_{1}(t)+C(t) u_{z}(t)>\right.$
$+\operatorname{tr}\left[\left[A(\tau) z_{i}^{\prime}(t, \tau)+C(\tau) u_{j}^{\prime}(t, \tau)\right]\left[\lambda_{i}^{\prime}(t, \tau)\right]\right]$
$+\operatorname{tr}\left[\left[\Lambda(t) z_{y}(t, \tau)+C(t) z_{z}(t, \tau)\right]\left[\lambda_{s}^{\prime}(t, \tau)\right]\right]$
4.2.3.4
and where for $G$ an $n \times m$ matrix
$\square G \square \triangleq \sum_{i=1}^{n} \sum_{j=1}^{m} g_{i j}$ • 4.2.3.5

Remark:
The performance index for this class of problems is nonlinear in $z_{1}, z_{<}^{\prime}, z_{j}, u_{1}$ and $u_{z}$ but linear in $u_{s}^{\prime}$. Therefore, the optimal
controls $u_{1}$ and $u_{2}$ are found as nonsingular controls but $u_{j}^{\prime}$ is a singular control. The equations 4.2.3.1 and 4.2.3.2 allow the solution for $u_{3}^{*}$ to be carried out. Equation 4.2.3.3 represents a strengthened necessary condition similar to the Legendre Clebsch condition of nonsingular problems. For this class of problems $u_{1}, u_{2}$ and $u_{j}^{\prime}$ are not bounded, i.e. they take values in open sets.
4.2.4. Theorem 4.4

In order that $u_{i}{ }^{*}(t) \in U_{i}, t \in \Gamma_{i}, i=1,2,3$ be the optimal strategies for Problem 4 of Section 4.1.4, it is necessary that there exist a nonzero vector function $\lambda_{1} *(t)$ and nonzero matrix functions $\lambda_{2}{ }^{*}(t)$ and $\lambda_{3}{ }^{*}(t)$ such that:
a) $\lambda_{1} *(t), \lambda_{2} *(t), \lambda_{3} *(t), z_{1} *(t), z_{2} *(t), z_{3} *(t)$ are solutions of $\frac{d z_{1}}{d t} *(t)=\left(\frac{\partial H)}{\partial \lambda_{1}}{ }^{*}=A(t) z_{1}{ }^{*}(t)+B(t) u_{1}{ }^{*}(t)+C(t) u_{2}{ }^{*}(t) \quad\right.$ 4.2.4.1 $\frac{d z_{2}}{d t}(t)={\frac{(\partial H)}{\partial \lambda_{2}}}_{*}=F(t) z_{2} *(t)+z_{2}^{*}(t) A^{\mu}(t)+u_{3}^{*}(t) C^{-}(t) \quad$ 4.2.4.2
$\frac{d z_{3}}{d t}{ }^{*}(t)=\frac{(\partial H)}{\partial \lambda_{3}}{ }^{*}=A(t) z_{3}{ }^{*}(t)+z_{3}{ }^{*}(t) A^{\prime}(t)+C(t) z_{2}{ }^{*}(t)$
$+z_{2}^{*}{ }^{*}(t) C^{\prime}(t)$
4.2.4.3
$\frac{d \lambda_{1}}{d t} *(t)=-\frac{(\partial H)}{\partial z_{1}}{ }^{*}=-A^{-}(t) \lambda_{1} *(t)$
$\frac{d \lambda_{2}}{d t} *(t)=-\frac{(\partial H)}{\partial z_{2}} *=-\left[F^{\prime}(t) \lambda_{2} *(t)+\lambda_{2} *(t) A(t)+C^{\prime}(t) \lambda_{3} *(t)\right.$
$\left.+C^{-}(t) \lambda_{3}^{*} *(t)\right]$
$\frac{d \lambda_{3}}{d t} *(t)=-\frac{(\partial H)}{\partial Z_{3}}=-\left[A^{-}(t) \lambda_{3} *(t)+\lambda_{3} *(t) A(t)\right]$
with the boundary conditions
$z_{1}{ }^{*}\left(t_{0}\right)=z_{10}$
$z_{2}{ }^{*}\left(t_{0}\right)=z_{20}$
$z_{3}{ }^{*}\left(t_{0}\right)=z_{30}$
and
$\lambda_{1} *\left(t_{f}\right)=0$
$\lambda_{2}{ }^{*}\left(t_{f}\right)=0$
$\lambda_{3}{ }^{*}\left(t_{f}\right)=0$
b) The extremal strategy $\left(u_{1}{ }^{*}(t), u_{2}{ }^{*}(t), u_{3}{ }^{*}(t)\right)$ satisfies the following min-max principle

$$
\begin{align*}
& H\left(z_{1}^{*}(t), z_{2}^{*}(t), z_{3}^{*}(t), u_{1}^{*}(t), u_{2}(t), u_{3}(t), \lambda_{1}^{*}(t), \lambda_{2}^{*}(t), \lambda_{3}^{*}(t)\right) \\
& \leqq H\left(z_{1}^{*}(t), z_{2}^{*}(t), z_{3}^{*}(t), u_{1}^{*}(t), u_{2}^{*}(t), u_{3}^{*}(t), \lambda_{1}^{*}(t), \lambda_{2}^{*}(t), \lambda_{3}^{*}(t)\right) \\
& \leq H\left(z_{1}^{*}(t), z_{2}^{*}(t), z_{3}^{*}(t), u_{1}(t), u_{2}^{*}(t), u_{3}^{*}(t), \lambda_{2}^{*}(t), \lambda_{2}^{*}(t), \lambda_{3}^{*}(t)\right)
\end{align*}
$$

for all $u_{i}(t) \in U_{i}, i=1,2,3, t \in \Gamma_{1}$ and c .)
$H\left(z_{1}{ }^{*}(t), z_{2}{ }^{*}(t), z_{3}{ }^{*}(t), u_{1}{ }^{*}(t), u_{2}{ }^{*}(t), u_{3} *(t), \lambda_{2}{ }^{*}(t), \lambda_{2}^{*}(t), \lambda_{3}{ }^{*}(t)\right)$
$=$ constant
4.2.4.10
for all $t \in \Gamma_{1}$.

### 4.3 Conclusions

Four theorems containing sets of necessary conditions for Problens 1 thru 4 were stated. The equations for the propogation of the first two monents through a linear time varing system are derived in Appendix A. The relation of the performance indices of the four problems to physical criteria is explored in Appendix B. Also in Appendix B the physical implications of the endpoint conditions are established. The proofs of Theorems 4.1, 4.2, 4.3 and 4.4 are given in Appendix $C$.

## CHAPTER 5

SUFFICILNT CONDITIONS

Sufficient conditions are developed for Problem 1 using the properties of convexity and concavity. The particular performance index of equation 4.1 .1 .1 is shown to satisfy these properties. A similar theorem is given for the functional form of the performance index of Problem 2. No theorem was developed for Problem 3 since this is a singular problem and in general sufficient conditions are very difficult to develop for singular optimal problems. A set of sufficient conditions for Problem 4 was derived using the same techniques as in Problems 1 and 2 but is not included in this chapter. The uniqueness of the optimal solution of Problem 1 is established in Section 5.3.

### 5.1 Theorem 5.1

For Problem 1 of Section 4.1 .1 it is sufficient that the pair of external controls $\left(u_{1} ; u_{2}, u_{3}^{\prime}\right)$ are optimal in the sense of establishing a saddle point defined by

$$
\begin{aligned}
& J\left(z_{1}, z_{2}^{\prime}, z_{3}, u_{1} *, u_{2}, u_{3}^{*}, \lambda_{1}^{*}, \lambda_{2}^{*}, \lambda 3^{*}\right) \leqq \\
& J\left(z_{1}^{*}, z_{2}^{*}, z_{3}^{*}, u_{1}^{*}, u_{2}^{*}, u_{3}^{*} *, \lambda_{1} *, \lambda_{2} *, \lambda_{3}^{*}\right) \leqq \\
& J\left(z_{1}, z_{2}^{*}, z_{3}, u_{1}, u_{2}^{*}, u_{3}^{*}, \lambda_{1} *, \lambda_{2} *, \lambda_{3}^{*}\right)
\end{aligned}
$$

for $J$ where

$$
\begin{align*}
& J=J\left(z_{1}, z_{2}^{\prime}, z_{3}, u_{1}, u_{2}, u_{3}^{\prime}, \lambda_{1}, \lambda_{2}, \lambda_{3}\right)= \\
& -\frac{t_{2}}{t_{1}} \int_{t_{0} t^{t} f_{0}}^{f}\left\{t r\left[R_{1} z_{3}(t, \tau) z_{3}^{\prime}(t, \tau) R_{1}^{\prime}\right]+\operatorname{tr}\left[R_{2} u_{3}(t, \tau) u_{3}^{\prime}(t, \tau) R_{1}^{\prime}\right]\right. \\
& +\operatorname{tr}\left[\left[A(\tau) z_{2}^{\prime}(t, \tau)+C(\tau) u_{3}^{\prime}(t, \tau)-\frac{\partial z_{2}^{\prime}}{\partial \tau}(t, \tau)\right]\left[\lambda_{2}^{\prime}(t, \tau)\right]\right] \\
& \left.+\operatorname{tr}\left[\left[A(t) z_{3}(t, \tau)+C(t) z_{2}(t, \tau)-\frac{\partial z_{3}}{\partial t_{3}}(t, \tau)\right]\left[\lambda_{3}^{\prime}(t, \tau)\right]\right]\right\} d t d \tau \\
& +\frac{1}{2} \int^{t_{0}}\left\{\left\langleu_{1}(t), R_{3} u_{1}(t)>-<u_{2}(t), R_{4} u_{2}(t)>\right.\right. \\
& \left.+<\lambda_{1}(t), A(t) z_{1}(t)+B(t) u_{1}(t)+C(t) u_{2}(t)-\frac{d}{d t} z_{1}(t)>\right\} d t
\end{align*}
$$

or

$$
\begin{aligned}
J= & J\left(z_{1}, z_{2}^{\prime}, z_{3}, u_{1}, u_{2}, u_{3}^{\prime}, \lambda_{1}, \lambda_{2}, \lambda_{3}\right)= \\
& \int_{t_{0}}^{t_{f}} \int_{f_{0}}^{f_{\Omega_{2}}}\left(z_{2}, z_{3}, u_{3}^{\prime}, \lambda_{2}, \lambda_{3}\right) d t d \tau+\int_{t_{0}}^{f_{f}} \Omega_{1}\left(z_{1}, u_{1}, u_{2}, \lambda_{1}\right) d t
\end{aligned}
$$

$$
5.1 .3
$$

if the necessary conditions of Theorem 4.1 are satisfied. Proof:

The proof follows immediately from Theorem 5.1.l and Theorem 5.1.2.
5.1.1 Theorem 5.1.1

For Problem 1 of Section 4.1 .1 it is sufficient that the pair of extremal controls ( $u_{1} ; u_{2}, u_{3}^{\prime}$ ) are optimal in the sense of establishing a saddle point, as in 5.1.1, for the functional
form of the performance index $J$ in 5.1 .3 if in addition to the necessary conditions of Theorem 4.1 being satisfied, $\Omega_{1}$ is convex WRT $u_{1}$ and concave WRT $u_{2}$
and
$\Omega_{2}$ is concave WRT $u_{3}^{*}$ and $z_{3}$.
5.1.1.2

Proof:
The complete proof is contained in Appendix D.l.

### 5.1.2 Theorem 5.1.2

It is both necessary and sufficient that the function $\Omega_{1}$ is convex in $u_{1}$ and concave in $u_{2}$, and that the function $\Omega_{2}$ is concave in $z_{3}$ and $u_{3}^{\prime}$ if:
$\mathrm{R}_{1}$ is non-negative definite and symmetric
and

$$
R_{2}, R_{3}, R_{4} \text { are positive definite and symmetric }
$$

$$
5.1 .2 .2
$$

Proof:
The complete proof is also contained in Appendix D.1.
5.2 Theorem 5.2

For Problem 2 of Section 4.1 .2 it is sufficient that the pair of extremal controls $\left(u_{1} ; u_{2}, u_{3}\right)$ are optimal in the sense of establishing a saddle point defined by equation 5.1.1 for $J$ where

$$
\begin{align*}
& J= \int_{t_{0}}^{t_{0}} \int_{t_{0}}^{t_{f}} f o\left(z_{1}(t), z_{1}(\tau), z_{2}^{\prime}(t, \tau), z_{3}(t, \tau), u_{1}(t), u_{1}(\tau),\right. \\
& \\
&\left.u_{2}(t), u_{2}(\tau) u_{3}(t, \tau)\right) d t d \tau
\end{align*}
$$

if in addition to the necessary conditions of Theorem
4.2 being satisfied
fo is convex in $u_{1}(t), u_{1}(\tau), z_{1}(t)$ and $z_{1}(\tau)$
and
fo is concave in $u_{2}(t), u_{2}(\tau), z_{1}(t), z_{1}(\tau), u_{3}^{\prime}(t, \tau)$,

$$
z_{2}^{\prime}(t, \tau) \text { and } z_{3}(t, \tau) .
$$

### 5.3 Uniqueness of Problem 1 Solution

The definiteness of the matrices $R_{2}, R_{3}$ and $R_{4}$ provide the means to establish equation 5.1 .1 as strict inequalities. This means the extremal solution obtained from Theorem 4.1 is the only solution which is optimal. If the solution obtained from Theorem 4.1 is the only solution of the resulting TPBVP, then it is by the strict inequalities the unique solution. This point is clarified in Appendix D.I.

Since global convexity and concavity is used, global sufficient conditions result. Then if the TPBVP has only one solution, Theorem 4.1 provides necessary and sufficient conditions for unique global solutions for the specific performance index of equation 5.1.2.

These remarks do not apply for the functional form of the performance index given in equation 5.1.3, i.e., it is not shown to be the unique solution although it is the global optimal solution.

### 5.4 Conclusions

It is established that the necessary conditions of Theorem 4.1 are also sufficient conditions for Problem 1 of Section 4.l.1. This is accomplished by considering the properties of convexity and concavity in Theorem 5.1.1, and then proving Theorem 5.1.2 which gives the requirements on the weighting matrices of the performance index such that the convexity and concavity conditions are satisfied. In view of the results of these two theorems, the optimal solutions are the unique global optimal solutions if they exist.

In addition a sufficient condition theorem for Problem 2 is stated, also through the properties of convexity and concavity.

IDENTIFICATION
The overall problem of simultaneous identification and control is now presented. It is the object of some techniques to identify while a process or plant is being controlled while other methods are used to identify only. Simultaneous identification and control require techniques suitable for "on-line" implementation. Methods for identification only are considered as "off-line" methods. It is the purpose of this chapter to study the "on-line" situation. Further, the simultaneous control is to be optimal or sub-optimal with respect to a given criteria. The control generated from Chapters 4 and 5 is to be implemented in such a way that if identification is complete, i.e. all parameters are known exactly, it would be the actual optimal control. If some parameters are not known exactly, then the control is actually sub-optimal. As the identification progresses in real time, the parameters are more closely known and if the dependence of the control on the parameters is updated, the sub-optimal control becomes closer to optimal. This scheme is carried out using a model reference adaptive control method as described in Section 6.3.

### 6.1 Background

The basic question of identifiability of parameters is discussed by several authors inthe December, 1974 Special Issue of the I.E.E.E. Transactions on Automatic Control. For the problens considered in this work, it will be assumed that the systens are controllable, observable, and in canonical form, hence identifiable.

Also, as previously mentioned, only parameter identification will be covered, since model identification would preclude the use of the simultaneous optimal control which is solved for offline. Further, the parameters are assumed to be constants.

Many different identification methods are currently known. Ioosely:speaking, they can be classified in the following way.

First, frequency domain or spectrum analysis techniques are used in time series analysis. Also, random input methods are used which imply that all modes of the plant will be excited, hence identifiable. These first two methods are usually offline in that identification only is sought.

Secondly, when on-line identification and control is desired, the state space, or time domain representation is most often used. The two main classifications are model reference adaptive control and nonlinear estimation. As discussed in Sage(96) and Graupe (31), even if linear systems are to be identified, non-linear estimation must be employed. The reason for this is that the parameters to be identified are collected in a vector and adjoined to the usual state vector resulting in nonlinear state equations, since the parameters of the actual state matrix multiply the actual state variables. Depending on the a priori statistical information about the original state, measurement and inputs, various estimation schemes, such as Kalman filtering of a linearized representation of the nonlinear system can be used to identify the parameters. Further, the controlling input usually is not available as an optimal control, but must be chosen to aid the identification as mentioned in (96).

### 6.2 Model Reference - Gradient Approach

For two main reasons, the model reference adaptive control method was chosen to be able to simultaneously identify and optimally (or suboptimally) control. First, it is seen from Section 6.1, that on-line controls available for optimization are not included in most other identification methods. Second, no methods are currently available when the stochastic disturbances are Class 3), (i.e. second order) processes. For other kinds of stochastic inputs, i.e. other than Class 3), model reference adaptive control schemes have been studied. Many adaptive schemes are not concerned with "optimal" control, but more with parameter tracking or trajectory following.

In the model reference system used in this work several issues arise which would significantly degrade performance in a real - world application. The stability of the overall system of plant, model, and adaptive loop with feedback control from the model is questionable. Using Lyapunov stability theory Kuo(59) derived conditions such that the overall system is stable. This method to insure stability is applicable directly to the overall system of Section 6.3, and that system can be shown to satisfy the conditions given (59).

The stochastic input disturbance requires that an estimate, hopefully optimal in some sense, of the state is available. In the case where the complete state is available, it has been shown by Kurtaran and Menachem (60) that the actual state is the best estimate in the sense that it is the mininum error variance estimate.

For the case where only a noisy measurement of the state is available, a best estimateof the actual state is needed. This estimate has been derived and is discussed in Section 6.4. For plants wi.th some parameters to be identified and only a noisy measurement of the state available, questions about sensitivity arise which are not covered here.

Finally, the parameter adjustment criteria was chosen as an integral square error criteria, since stability was easily established. The gradient or steepest descent algorithm as discussed in Kirk (45), Wilde and Beightler (107), and Sage (96) was modified and used such that a real time implementation could be readily obtained. It is seen that in a real world application this algorithm requires much less digital hardware than say a Kalman estimator, since no calculus is involved in the parameter adjustment, whereas in Kalman estimation a matrix Ricatti differential equation must be solved by the digital hardware.
6.3 Implementation

This section describes in detail how to implement the optimal control and simultaneously identify parameters in a model reference adaptive scheme. The controls used are those derived in Theorem 4.1 since closed loop strategies are readily obtainable and easily implemented.

### 6.3.1 Optimal Closed Loop Strategies

The necessary conditions of Section 4.2.1 imply the open loop problem solution, i.e. the control laws for $u_{i} *$ and $u_{i} *$ depend on end times and endpoint conditions only
$u_{1} *(t)=u_{1}^{*}\left(t_{0}, z_{1}\left(t_{0}\right), t_{f}, z_{1}\left(t_{f}\right)\right), t \in \Gamma_{1}$
6.3.1.1
and
$u_{z}^{*}(t)=u_{2}^{*}\left(t_{0}, z_{1}\left(t_{0}\right), t_{f}, z_{1}\left(t_{f}\right)\right), t \in \Gamma_{1} \quad$ 6.3.1.2 In many situations it is more desired to form a closed loop control by feedback of the state $z_{1}(t)$, for all $t \in \Gamma_{1} \cdots$ These control laws are desired to be in the form
$u_{1}$ *
and
$u_{z}{ }^{*}(t)=u_{z}{ }^{*}\left(t, z_{1} *(t)\right), t \in \Gamma_{1}$.
6.3.1.3
$(t)), t \in \Gamma_{1}$

These would be nonlinear control laws. It is even more desired to find linear time - varying feedback gains for ease of implementation and stability purposes. These points are discussed in Athans and Falb (9) for deteministic optimal control problems and in Lee (64) and Wonham(109) for stodiastic optimal control problems. For this class of problems, control laws given by
$u_{i} *(t)=W_{1}(t) z_{1}{ }^{*}(t), t \in \Gamma_{1}$
6.3.1.5
and
$u_{<}^{*}(t)=W_{c}(t) z_{1}^{*}(t), t \in \Gamma_{2}$
are to be found.
$u_{1}^{*}(t)=-R_{3}^{-1} B^{\prime}(t) \lambda_{1} *(t), t \in \Gamma_{1}$
6.3.1.7
and
$u_{2}^{*}(t)=R_{4}-1 C^{\rho}(t) \lambda_{1} *(t), t \in \Gamma_{1} \quad$ 6.3.1.8
so letting
$\lambda_{1}{ }^{*}(t)=K(t) z_{1}{ }^{*}(t), t \in \Gamma_{1}$
6.3.1.9
where
$K(t)$ is an $n \times n$ matrix
would result in
$u_{1}^{*}(t)=-R_{3}^{-1} B^{-}(t) K(t) z_{1}^{*}(t), t \in \Gamma_{1}$
6.3.1.10
giving
$W_{1}(t)=-R_{3}^{-1} B^{-}(t) K(t), t \in \Gamma_{1}$
6.3.1.11
and
$u_{z}^{*}(t)=R_{4}^{-1} C^{-}(t) K(t) z_{1}^{*}(t), t \in \Gamma_{1}$ 6.3.1.12
giving
$W_{z}(t)=R_{4}^{-1} C^{\prime}(t) K(t), t \in \Gamma_{1}$
6.3.1.13

Differentiating 6.3.1.9 gives
$\frac{d \lambda}{d t^{1}} *=\frac{d K}{d t}(t) z_{1}^{*}(t)+K(t) \frac{d z}{d t^{2}}{ }^{*}(t), t \in \Gamma_{1}$
6.3.1.14

Substituting from the necessary conditions of Theorem 4.2.1 it can be shown that $K(t)$ must satisfy

$$
\frac{d K}{d t}(t)=-A^{-}(t) K(t)-K(t) A(t)-K(t) D(t) K(t), t \in \Gamma_{i}
$$

where
$D(t)=-B(t) R_{9}^{-}+B^{-}(t)+C(t) R_{4}^{-1} C^{-}(t), t \in r_{.} \quad$ 6.3.1.16
and further $K(t)=K^{\prime}(t)$ if $D(t)$ is symméric as is shown in
Kirk (45) and Ogata ( 83 ).
The initial conditions required for the solution of
equation 6.3.1.15 are obtained from equation 6.3.1.9 at $t=t_{0}$.
This control law gives a closed loop feedback solution
which is solved offline and implemented on-line either by
storing the function $K(t) \forall t \in \Gamma$, or by on line sirmalation. This is a very useful and practical means for generating an optimal control, and even more useful when simultaneous on-line identification is required.

### 6.3.2 Model Reference Identification and Simultaneous Updated

Sub-Optimal Control

The overall method used is explained by considering Figure 6.3.2.1


Figure 6.3.2.1 Model reference adaptive system

As is seen, the only digital block is that one used to generate a new set of values for the parameters being identified. The closed loop feedback controls can be simulated using analog equipment.

The steepest descent algorithm is carried out by the digital block. The rate of convergence, region of convergence, sampling rate, and weights are all very involved problems in themselves. For a particular real-world application, these problems are generally dealt with after several trial runs. No comprehensive theory exists for predetermined solutions. After completion of this step in some manner a workable, near-optimal system is obtained.

For a completely identified system, the closed loop controller $u_{1}$ would be optimal for both the model and the plant with respect to the criteria of Theorem 4.1. If some parameters are not identified exactly, then suboptimal control is applied while identification is being carried out. By adjusting the parameters in the analog simulation of the feedback gain $K$ and model $z_{1}$, the control $u_{1}$ is updated and as the parameters become closer and closer to their true values, the suboptimal controls become closer and closer to the optimal controls.

The actual plant is governed by
$\frac{d x}{d t}(t, \sigma)=A(t) x(t,)+.B(t) u_{2}(t)+C(t) v(t, \ldots), t \in \Gamma_{1} \quad$ 6.3.2.1
where some parameters of $A, B$, and $C$ are unknown.
The model is

$$
\frac{d z}{d t^{2}}(t)=A(\gamma, t) z_{1}(t)+B \cdot(\gamma, t) u_{2}(t)+C(\gamma, t) u_{z}(t), t \in \Gamma_{1} \quad 6.3 .2 .2
$$

where $\gamma$ is a vecitor of the unknown parameters.
In equation 6.3.2.2, the controls $u_{1}(t)$ and $u_{<}(t)$ are simulated in tenns of the time-varying feedback gain matrix $K(t)$ governed by
$\frac{d K}{d t}(t)=-A^{\prime}(\gamma, t) K(t)-K(t) A(\gamma, t)-K(t) D(\gamma, t) K(t), t \in \Gamma_{1} \quad$.3.2.3
where
$D(\gamma, t)=-B(\gamma, t) R_{3}^{R^{-2}} B^{-}(\gamma, t)+C(\gamma, t) R_{4}^{R^{-2}} C(\gamma, t)$.
Define the error between plant and model state as

$$
e(\gamma, t)=z_{1}(t)-x(t) \quad 6 \cdot 3 \cdot 2.5
$$

and the weighted criteria used to generate updated values of the parameters as

$$
J_{I}(\gamma, i)=E\left[\int_{t_{i}}^{t_{i+1}}<e(\gamma, t), w_{i} e(\gamma, t)>d t\right\}
$$

where $W_{i}$ is a constant positive definite weighting matrix, $t_{i}, t_{i}+l \in \Gamma_{1}, i=1, \ldots, N$ where $N$ is the number of sampling intervals.

The method of steepest descent is used to numerically minimize
$J_{I}(\gamma)$ with respect to $\gamma$ such that the system will be identified, i.e. the model parameters are equal to the actual plant parameters when $J_{I}(\gamma)$ becomes sufficiently small, in comparison to some preselected numerical value. The function is minimized by evaluating the slope or gradient at a given initial point, and then moving in the direction of steepest slope to a new point. The process is carried out numerically until $J_{I}(\gamma)$ is less than some preselected value, or until no further decrease in $J_{I}(\gamma)$ can be obtained. The slope is approximately numerically as
$\frac{\partial J}{\partial \gamma} \simeq \frac{\Delta J}{\Delta \gamma}$
6.3.2.7
for each iteration where
$\Delta J_{I}(\gamma)=J_{I}(\gamma, i+1)-J_{I}(\gamma, i)$
6.3.2. 8
and
$\Delta y=\gamma_{\text {ORIGI_IAL }}-Y_{\text {IJEI CIIA:NGE }}$
6.3.2.9
and $\gamma_{\text {ORIGINAL }}$ is the initial guess at the parameters used in solving
the "worst-case" optimal control problem offline.
At each iteration a new value of $\gamma$ is obtained from
$\begin{array}{lr}\gamma_{i}=\gamma_{i+1}-\Delta \gamma & 6.3 .2 .10 \\ \text { and } & \\ \Delta \gamma=\theta \frac{\partial J_{T}}{\partial \gamma}(\gamma, i+1) & 6.3 .2 .11\end{array}$
where $\theta$ is a weighting factor chosen to aid convergence. As each new value of $\gamma$ is obtained, the model generating the state $z_{1}$ and feedback gain $K$ are updated.

The overall system is suboptimally controlled until identification
is complete, then optimally controlled from that point on. This overall system is an adaptive system using a reference model, or a model-reference adaptive control system. A complete digital simulation of the overall system was written and is discussed in Chapter 7.

### 6.4 Optimal Estimation with Second Order Stochastic Disturbances

In this section an estimate is obtained for the dase where only a noisy measurement of the state, not the state itself, is available. The results could be used in conjunction with the identification procedure of the first portion of this chapter, winere the estimate $\hat{\hat{x}}$, replaces the state x in the model reference system. If this was to be done, a sensitivity analysis would be necessary before implementation. Assuming identification was complete, i.e. all parameters were known, the estimate $\hat{x}$ could be used for many purposes, just as the estimate from a Kalman filter is used.

Consider a plant with stochastic disturbance
$\frac{d x}{d t}(t,)=.A(t) x(t,)+.B(t) u(t)+C(t) v\left(t_{,}.\right) \quad$ 6.4.1
and with noisy measurement
$z(t,)=.H(t) x\left(t_{,}\right)+w\left(t_{,}\right)$.
It is desired to find an optimal filter such that $\hat{x}$ is the besi estimate, in the sense of minimum error covariance, where the filter is constrained by
$\frac{d \hat{x}}{d t}(t)=F(t) \hat{x}(t)+G(t) z(t)+D(t) u(t) \quad 6.4 .3$ where $F, G$, and $D$ are to be detemined. The filter is also to be unbiased. For this condition,

$$
F(t)=A(t)-G(t) H(t)
$$

and

$$
D(t)=B(t)
$$

where the means $\mu_{v}(t)$ and $\mu_{v}(t)$ must be either zero or known a pricri. If $v(t)$ and $w(t)$ have essentially the same a priori daca known that is discussed in Appendix B, then the optimal value of $G(t)$ can be found and the estimate $\hat{x}$ generated as given in:

Theorem 6.1
For the conditions of 6.4.1 through 6.4.5 the optimal estimate $\hat{x}$ is given by
$\frac{A \hat{x}}{d t}(t)=A(t) \hat{X}(t)+G(t)[z(t)-H(t) \hat{x}(t)]+B(t) u(t) \quad$ 6.4.6
where $G(t)$ is determined by and must satisfy
$G(t)=V_{W X} \sim(t, \tau)\left[V_{W X} \sim(t, \tau) H(\tau)+V_{W W}(t, \tau)\right]^{-1}+S(t) \quad$ 6.4.7
and where $V_{W X} \tilde{(t, \tau)}$ is found from
$\frac{\partial V_{W} \tilde{X}^{\prime}}{\partial \tau}(t, \tau)=V_{W X} \tilde{X}(t, \tau)[A(\tau)-G(\tau) H(\tau)]^{\wedge}+V_{W V}(t, \tau) C^{\prime}(\tau)-V_{W W}(t, \tau) G^{\prime}(\tau)$
Proof: The proof leading to equations 6.4 .6 through 6.4 .8 is developed in Appendix F. The method is similar to the calculus of variations derivation of the Kalman filter given in Sage (97).

Remark: The actual a priori statistical data, the differential aquacions describing the evolution of the error covariance, and the performance index used are all detailed in Appendix F.

In Appendis $F$ it is shown that $S(t)$ is a weighting matrix. The error $x(t)$ is defined as
$\tilde{x}(t,)=.x(t,)-.\hat{x}(t)$
The results given above are just briefly quoted to show the preliminary work done in extending the "worst case" control cheorems of Chapter 4 and 5, and the identification method of the previous sections of Chapter 6.

### 6.5 Conclusions

In this chapter, the model reference adaptive control system used to simultaneously identify parameters and suboptimally control a physical plant is given. The real-time or on-line implementation is possible and in fact provided the reasoning on which the choice of this method was made. The parameters are identified using
steepest descent or gradient methods suitable for on-line use. The preliminary estimation results of the last section are included to show the extension made to the case where only a noisy measurement of the actual system state is available.

## CHAPTER 7

APPLICATITONS
Consider a physical plant, e.g. a cardiac cell, which can be mathematically descrined by linear ordinary differential equations with time varying coefficients. Assuming the parameters, ise. gains and time constants of the equations are known then it is desired to optimally control the plant with respect to some measure of performance. If, in addition, the plant is subjected to stochastic disturbances, the optimal control of the plant becomes much more difficult. Assuming same of the plant parameters are not known further complicates the implementation of some optimal scheme.

This chapter illustrates through example, a technique for the simultaneous optimal control and parameter identification of a physical plant in the presence of stochastic inputs.

The theory required for "worst case" optimal control in a stochastic environment was presented in Chapters 4 and 5. All the computations can be done off-line before the operation of the plant begins. As the plant operation progresses the unknown parameters are updated. These updated parameters are fed to the model in such a way that the "optimal" controls are generated from these new values of the parameters after each update. Until the identification is complete, though, actual optimal control is not possible, though qualitatively "good" suboptimal control is actually obtained over the time interval of interest as can be seen by comparing actual and non-identified trajectories.
7.1 An Illustrative Example

An illustrative example showing in detail the optimal computations and then a simulation of simultaneous sub-optimal control and identification is given. The simulation was completely performed on a digital computer using the Fortran language. Ideally, a hybrid computer simulation would perhaps be more suitable as a means to illustrate all the various aspects of the theory, but large enough facilities were not available. The Fortran program listings are contained in Appendix E. The particular example is specified and set up in Section 7.1.1 . The offline optimal computations are detailed in Section 7.1.2 . The results of the simulation of implementing both the identification and control are given in Section 7.1.3.

A particular example was simulated on the digital computer. The actual, real-world physical plant is described in this section. The general equation for the plant is $\frac{d x}{d t}(t,)=.A(\gamma, t) x(t,)+.B(\gamma, t) u(t)+C(\gamma, t) v(t,),. t \in \Gamma_{1}$ 7.1.1.1

Choosing $n=r=s=1$ gives $x, u$, and $v$ as scalars, hence
$\frac{d x}{d t}(t,)=.a(\gamma) x(t,)+.b(\gamma) u(t)+c(\gamma) v(t,.) . t \in \Gamma_{1} \quad$ 7.1.1.2
Assuming $c$ is known, $a(\gamma)=a$ and $b(\gamma)=b$ yields $\gamma=\left[\begin{array}{l}a \\ b\end{array}\right]$
and omitting the dependence of a and b on $\gamma$,
$\frac{d x}{d t}(t, \ldots)=a x(t,)+.b u(t)+c v(t,). t \in \Gamma \ldots$
where a and b are to be identified. Equation 7.1.1.4
in analog computer form is


Figure 7.1.1.1 Analog computer diagram of example which physically is a lag with time constant $1 /|a|$, and with the sum of deterministic control $u$ : with gain $b$ and stochastic disturbance $v$ with gain $c$ as input.

It is desired to identify the parameters $a$ and $b$ while simultaneously optimally or at least sub optimally controlling the plant in the presence of the stochastic disturbance $v$, which is assumed to be a Class 3), i.e. second order, stochastic process.

In order to solve the optimal control problem offline, initial guesses at the values of the parameters are required. In the particular example simulated these were

$$
\begin{aligned}
& a=-3.0 \\
& b=2.0
\end{aligned}
$$

Also, the other data required was selected as

$$
\begin{align*}
c & =1.0 \\
t_{0} & =0.0 \\
t_{f} & =1.0
\end{align*}
$$

More data is required for the optimal solution and is specified in the next section.

### 7.1.2 Optimal Solution

For purposes of obtaining the optimal problem solution, the physical plant is described by

$$
\begin{aligned}
& \frac{d x}{d t}(t, .)=-3.0 x(t, .)+2.0 u(t)+1.0 v(t, .), t \in[0.0 ; 1.0] \\
& \text { as specified in the previous section. }
\end{aligned}
$$

Using the theory of Appendix A, the first two moments
of the state $x(t,$.$) become$

$$
\begin{aligned}
& \frac{d z_{1}}{d t}(t)=-3.0 z_{1}(t)+2 u_{1}(t)+u_{2}(t), t \in[0.0,1.0] \\
& \frac{\partial z_{2}}{\partial \tau}(t, \tau)=-3.0 z_{2}(t, \tau)+u_{3}(t, \tau),(t, \tau) \in[0.0,1.0] \times[0.0,1.0] \\
& \frac{\partial z_{3}}{\partial t}(t, \tau)=-3.0 z_{3}(t, \tau)+z_{2}(t, \tau),(t, \tau) \in[0.0,1.0] \times[0.0,1.0] 7.1 .2 .4
\end{aligned}
$$

The variables $z_{1}, z_{2}, z_{3}, u_{1}, u_{2}$ and $u_{3}$ are all scalars.
The weighting matrices were chosen as

$$
\begin{aligned}
& R_{1}=0.0 \\
& R_{2}=\sqrt{2} \\
& R_{3}=1.0 \\
& R_{4}=1.0
\end{aligned}
$$

$$
7.1 .2 .5
$$

giving
$J\left(u_{1}, u_{2}, u_{3}\right)=1 / 2 \int_{t_{0}}^{t^{f}}\left\{u_{1}^{2}-u_{2}^{2}\right\} d t-1 / 2 \int_{t_{0} t_{0}}^{f^{t} f^{f}}\left\{2 u_{3}^{2}\right\} d t d \tau \quad$ 7.1.2.6
The initial states used were

$$
z_{1}\left(t_{0}\right)=z_{1}(0)=5.0
$$

corresponding to $\pi_{1}=5.0$ where this is the measurable prior mean.
It was desired to drive the state (in terms of $\mathrm{z}_{1}$ ) to

$$
z_{1}\left(t_{f}\right)=z_{1}(1)=0.1
$$

The initial cmoss covariance chosen was

$$
z_{2}\left(t, t_{0}\right)=1.0 e^{-t}
$$

7.1.2.9
and the initial covariance

$$
z_{3}\left(t_{0}, \tau\right)=10.0 e^{-\tau}
$$

which correspond to a cross variance

$$
z_{2}\left(t_{0}, t_{0}\right)=1.0
$$

and a variance

$$
z_{3}\left(t_{0}, t_{0}\right)=10.0
$$

which are measurable as is the time constant $T=1.0$ used in 7.1.2.9 and 7.1.2.10 for the exponential correlation distribution discussed in Appendix B. The final time endpoints are discussed later. For the above data, application of equations 4.2.1.1 through 4.2.2.12 yield

$$
\begin{aligned}
& \frac{d z^{*}}{d t}(t)=-3.0 z_{1}^{*}(t)+2.0 u_{1} *(t)+u_{2}^{*}(t) \\
& {\frac{\partial z_{2}}{\partial \tau}}^{*}(t, \tau)=-3.0 z_{2}^{*}(t, \tau)+u_{3} *(t, \tau) \\
& \frac{\partial z}{\partial t} 3^{*}(t, \tau)=-3.0 z_{3} *(t, \tau)+z_{2} *(t, \tau) \\
& \frac{d \lambda_{1}}{d L^{2}}{ }^{*}(t)=3.0 \lambda_{1} *(t) \\
& \frac{\partial \lambda_{2}}{\partial \tau} 2^{*}(t, \tau)=3.0 \lambda_{2}^{*}(t, \tau)-\lambda_{3}^{*}(t, \tau) \\
& \frac{\partial \lambda_{3}}{\partial{ }_{3}}{ }^{*}(t, \tau)=3.0 \lambda_{3} *(t, \tau) \\
& u_{1} *(t)=-2.0 \lambda_{1} *(t) \\
& u_{2}{ }^{*}(t)=\lambda_{1}{ }^{*}(t) \\
& u_{3}{ }^{*}(t, \tau)=0.5 \lambda_{2} *(t, \tau) \\
& \text { 7.1.2.13 }
\end{aligned}
$$

Putting 7.1.2.13 into 7.1.2.11 yields the following $2 n(1+s+n)=6$ differential equations with the four endpoint conditions of 7.1.2.7 through 7.1.2.10,
$\frac{d z_{I}}{d t} *(t)=-3.0 z_{1} *(t)-3.0 \lambda_{1} *(t)$
$\frac{\partial z}{\partial \tau}_{2}{ }^{*}(t, \tau)=-3.0 z_{2}^{*}(t, \tau)+0.5 \lambda_{2}^{*}(t, \tau)$
$\frac{\partial z_{3}}{\partial t}{ }^{*}(t, \tau)=-3.0 z_{3}{ }^{*}(t, \tau)+z_{2}^{*}(t, \tau)$
$\frac{d \lambda_{1}}{d t}{ }^{*}(t)=3.0 \lambda_{1}{ }^{*}(t)$
$\frac{\partial \lambda_{2}}{\partial \tau}(t, \tau)=3.0 \lambda_{2}^{*}(t, \tau)-\lambda_{3} *(t, \tau)$
$\frac{\partial \lambda^{2}}{\partial t}{ }^{*}(t, \tau)=3.0 \lambda_{3} *(t, \tau)$
7.1.2.14

Two more final time endpoint conditions are required.
These conditions can be established by assuming the following Eorm for the initial conditions of $\lambda_{2}$ and $\lambda_{3}$

$$
\begin{align*}
& \lambda_{2}\left(t_{,} t_{0}\right)=L e^{-t} \\
& \lambda_{3}\left(t_{0}, \tau\right)=K e^{-\tau}
\end{align*}
$$

With this form the differential equations can be solved in terms of the constants $L$ and $K$. 'These constants can be determined algebraically by choosing $z_{2}\left(t, t_{f}\right)$ and $z_{3}\left(t_{f}, \tau\right)$ as functions of $t$ and $\tau$ respectively, realizing that $z_{2}$ and $z_{3}$ will have solutions due to their respective transition matrices and initial time endpoint conditions as well as those of $\lambda_{2}$ and $\lambda_{3}$. From a physical viewpoint, it would be desired to drive $z_{2}\left(t_{f}, t_{f}\right)$ and $z_{3}\left(t_{f}, t_{f}\right)$ to values much smaller than the initial cross variance and variance. Specifying the final cross variance and variance as
$Z_{2}\left(t_{f}, t_{f}\right)=0.683$
$23\left(\tau_{f}, t_{f}\right)=0.303$
requires
$z_{2}\left(t, t_{f}\right)=0.061 e^{-t}+0.033 e^{3 t}$
$z_{3}\left(t_{f}, \tau\right)=0.490 e^{-\tau}+0.170 e^{-3 \tau}+0.589 e^{3 \tau}$
where the constants $K$ and $L$ are algebraically determined from
7.1.2.16 and 7.1.2.17 as
$\mathrm{L}=0.00694$
$K=-0.333$
The solutions for the differential equations may be obtained by three methods. From Appendix A, the solutions are directly found if the transition matrix is known. Assuming the form of equation 7.1.2.15 the two-dimensional Laplace transform technique in Kuo (54) can be applied. Digital computer simulation of the in( $1+s+n$ ) differential equations could also be used. Algebraic solutions for the constants in 7.1.2.15 is used for the first two methods above. Shooting techniques would allow these constants to be evaluated on a digital computer. The complete analytic solution is

$$
\begin{aligned}
& z_{1} *(t)= 5.0 e-t-0.015 \sinh 3 t \\
& z_{2}^{*}(t, \tau)= 1.00 e^{-t} e^{-3 \tau}+0.00347 e^{3 t} e^{-3 \tau}+0.000578 e^{-t} e^{3 \tau} \\
&+0.00173 e^{3 \tau^{\prime}} e^{3 \tau}-0.00520 e^{3 t} e^{-\tau} \\
& z_{3}^{*}(t, \tau)=10.0 e^{-3 t} e^{-\tau}-0.500 e^{-3 t} e^{-3 \tau}-0.000578 e^{-3 t} e^{3 \tau} \\
&+0.499 e^{-t} e^{-3 \tau}+0.000578 e^{3 t} e^{-3 \tau}+0.000289 e^{-t} e^{3 \tau} \\
&+0.000209 e^{3 t} e^{3 \tau}-0.000368 e^{3 t} e^{-\tau}
\end{aligned}
$$

$$
\begin{aligned}
& \lambda_{1} *(t)=0.0150 e^{3 t} \\
& \lambda_{2} *(t, \tau)=0.00694 e^{-t} e^{3 \tau}+0.0208 e^{3 t} e^{3 \tau}-0.0208 e^{3 t} e^{-\tau} \\
& \lambda_{3}^{*}(t, \tau)=0.0833 e^{3 t} e^{-\tau} \\
& u_{1}^{*}(t)=-0.0300 e^{3 t} \\
& u_{2}^{*}(t)=0.0150 e^{3 t} \\
& u_{3}^{*}(t, \tau)=0.00347 e^{-t} e^{3 \tau}+0.0104 e^{3 t} e^{3 \tau}-0.0104 e^{3 t} e^{-\tau}
\end{aligned}
$$

The value of the performance index $J$ is

$$
J\left(u_{1}^{*}, u_{2}{ }^{*}, u_{3}^{*}\right)=-0.452
$$

Fron Section 6.3.1, the analytic solution for the time varying gain is

$$
K(t)=0.09 /\left(29.955 e^{-6 t}+0.045\right) \quad 7.1 .2 .20
$$

where for this problem
$d K(t)=6 K(t)+3 K^{2}(t)$ ..... 7.1.2.21
dt
and

$$
K\left(t_{0}\right)=0.003
$$

$$
7.1 .2 .22
$$

All these solutions were obtained and plotted using various digital techniques and are presented graphically in the following figures.


Figure 7.1.2.1 Plot of $z_{1}$


Figure 7.1.2.2 Plot of $\lambda_{1}$


Figure 7.1.2.3 Plot of $u_{1}$


Figure 7.1.2.4 Plot of $\mathrm{u}_{2}$


Figure 7.1.2.5 Plot of K


Figure 7.1.2.6 Plot of $z_{2}$


Figure 7.1.2.7 Plot of $z_{3}$


Figure 7.1.2.8 plot of $\lambda_{2}$


Figure 7.1.2.9 Plot of $\lambda_{3}$


Figure 7.1.2.10 Plot of $u_{3}$

### 7.1.3 Simultaneous Identification Solution

The complete model reference system of Figure 6.3.2.1 was simulated and run on a digital computer. The results of the previous section were implemented as shown in Figure 6.3.2.1 with allowance to change the parameters a and b . n continuous system was approximated by dividing the interval [0.0.1.0] into 400 sub-intervals for integration purposes. Every ten sub-intervals, the error was numerically sampled, and a resulting change in parameters $a$ and b calculated. This corresponds to there being 40 sampling intervals in one second, therefore convergence must be fast enough to cone to completion before 40 clanges occur. Similarly, overshoot of the minimization of the functional must be prohibited. These resulted in selection of a heuristic scaling of the factor $\theta$ such that smaller percentage changes occured as the percent change in the error function decreased. The stopping criteria was selected as $0.05 \%$ of the value of the error during the first sampling period. This corresponds roughly to a gradient of less than 0.083 . that is the magnitude of the gradient of the function at this stopping point is very small. The problem was run assuming initial guesses of

$$
a=3.0
$$

$$
b=2.0 \quad \text { 7.1.3.1 }
$$

with actual parameters of

$$
\begin{align*}
& a_{\mathrm{PLIP}}=-4.0 \\
& \mathrm{~b}_{\mathrm{PLT}}=3.0
\end{align*}
$$

For the above particular set of actual plant parameters, the
identification procedure resulted in

$$
a=-3.89707
$$

$\mathrm{b}=3.11534$
after 0.6 seconds corresponding to 24 identification sub-intervals having elapsed. For purposes of convergence, the weight $W$ of the integral square error measure was assigned as

$$
W=500
$$

7.1.3.4
after several trial runs established the range of $W$ such that overshoot did not occur, yet convergence did progress rapidly enough such that the stopping criteria was met in less that 40 sampling intervals. The results of the overall simulation are presented graphically in the following figures obtained with digital plotting routines.

First the convergence of the parameters to the true values is pictured.


Figure 7.1.3.1 Identification of a


Figure 7.1.3.2 Identification of $b$

Using the actual time varying values of a and b in Figure 7.1.3.1, the state of the model is


Figure 7.1.3.3.
Trajectory of $z_{1}$ fron model reference simulation
whereas the state of the model from the optimal solution run with the same values as the true parameters and with the sane initial conditions as would be used offline is


Figure 7.1.3.4 Trajectory of $z_{1}$ from optimal solution Similarly the adaptive scheme provides a suboptimal control $u_{1}$ as


Figure 7.1.3.5 Trajectory of $u_{1}$ from model reference simulation
whereas the optimal control $u_{1}$ from the same run as Figure 7.1.3.4 is


Figure 7.1.3.6 Trajectory of $u_{1}$ from optimal solution Comparing the last four figures it is seen that "qualitatively good" suboptimal control is obtained in terns of the closeness of trajectories of the state $z_{1}$ and control $u_{1}$ from the model reference scheme and the optimal run with true values substituted. Therefore in this sense the model reference scheme is close to optimal or suboptimal with regard to the criteria of Chapters 4 and 5 .

An example of the stochastic process v is show below


Figure 7.1.3.7 Plot of noise $v$ used in model reference simulation

The typical effects of the noise $v$ on a steepest descent path are illustrated next.


Figure 7.1.3.8
Regions of parameter space
Region 1 shows the area close to the minimum where the parameters are close. In this region the noise effects are most pronounced. Region 2 is where the noise effects are observed but not quantitatively significant with respect to the gradient search. In this region the parameters are far enough unequal such that the gradient procedure continues with no randomness. In Region 3, the parameters are so far apart that the error due to this is very much greater than the error due to the noise. The outer bound of Region 3 is the limit of the region of convergence. These regions were not numerically established but could have been by just executing many runs with various data. Generally speaking, the circle for Regions 1 and 2 could be determined in terms of the norm of the variance of the noise $v$ as is mentioned in Bryson and Ho (20).

In the actual results obtained, different noise sequences were generated for runs with the same unknown parameters. The average value of parameters as identified from the runs was

$$
\begin{aligned}
& a_{A V G}=-3.89266 \\
& b_{A V G}=3.112383
\end{aligned}
$$

In the next two figures are samples of the identification of $a$ and $b$ for several different noise sequences


Figure 7.1.3.9 Identification of a for other noise inputs


Figure 7.1.3.10 Identification of $b$ for other noise inputs

The results of runs with different actual parameter values are given in Figures 7.1.3.11 and 7.1.3.12.


Figure 7.1.3.11 Identification of a for other values


Figure 7.1.3.12 Identification of $b$ for other values

As mentioned in Chapter 6, the difficult problens of predetermining optimal. sampling rates, rates of convergence, regions of convergence and weights have not been solved explicitly, but rather through establishing workable values by several trial runs. The flowcharts and program listings of the adaptive simulation are given in Appendix E.

### 7.2 Conclusions

The illustrative example presented in this chapter was completly solved with respect to all aspects required for simultaneous on-line identification and control. The same techniques could be applied in principle to any problem such that Theorems 4.1, 4.2, 5.1, and 5.2 are applicable. That is, the case for vector state, control, and stochastic disturiance can be solved.

## CHAPTER 8

CONCLUSIONS

The study of second order stochastic processes as input disturbances to linear time-varying systems was dealt with in a number of ways. This class of stochastic processes had been virtually unstudied previously. .The main advantages of assuming these disturbances are physically present, are that the processes are continuous in the mean-square sense and the first two moments are continuous and finite. As such, these moments and the actual processes are time integrable without any of the difficulties and limitations encountered in white noise or colored noise stochastic processes whose covariances can only be written with Dirac delta functions. Further, the continuous time case can be treated independently of the discrete time case in many areas where this is not true for white noise or colored noise processes.

The first method of study was to approach a stochastic optimal control problem in a way such that optimal control was possible regardless of disturbance, by approximately the effects of the disturbance by the first two moments and casting the resulting moment equations into the form of a differential game. This approach was taken as a means to finding the "worst-case" optimal controls in the sense that optimal control was found for any set of first two moments of the stochastic disturbance. The initial studies of this approach led to both singular and non-singular performance indices. Also, the determination of optimal performance weighting constants was performed, corresponding to a greater degree of freedom in optimal system synthesis than is normally allowed.

The next area studied was to establish a method suitable for implementation in real-time for an actual
physical plant, that simultaneously identified system parameters and optimally or at least sub-optimally controlled the system. The particular technique used was a model reference adaptive system. The main advantage of this system is that it could be established in a recursive manner with an algorithm that is relatively easy to implement in terms of hardware and software. A complete illustrative example was given as simulated on a digital computer. Both identification using steepest descent and "worst case" suboptimal control were obtained. The overall model reference system can be shown to satisfy Lyapunov stability criteria, and the simulation verified that this stability did exist. The particular problems associated with gradient minimization, such as rate of convergence and region of convergence, were solved by several trial runs of the simulation rather than by explicit analytical techniques.

Finally, a best in the sense of minimum error variance, linear estimate was obtained for linear time-varying systems
with a stochastic input which was a second order process, as well as a measurement including another second order stochastic process possibly correlated with the input process. This estimate is actually an unbiased minimum variance estimate and was derived using variational techniques similar to those used in deriving the "worst case" optimal controls.

The three aspects of control, identification, and estimation were studied for linear stochastic systems. The stochastic processes utilized and studied throughout were second order processes, characterized mainly by having finite continuous covariances which was a very useful analytic property.

## CHAPTER 9

## AREAS FOR FUTURE RESEARCH

The classes of "worst case" optimal control problems, i.e., the differential games could be extended to cover a slightly wider class in several ways. The addition of a terminal cost term to the performance index and the inclusion of a measurement of the state are examples of the possible extensions.

The study of nonlinear systems or linearized nonlinear systems is a major area for further investigation. In several ways, the differential game-moment treatment is more amenable to linearized nonlinear systems than present methods due to the presence already of both means and covariances.

The model reference adaptive control system has inherent in it several interesting side issues. Sensitivity, stability and overall adaptive optimality are possible points to consider further. The special problem of a true optimal control having a component for control only and a component for identification only may possibly be looked at in a setting very similar to the "worst case" situation already studied.

Similarly, the use of the "worst case" covariances and cross-covariances could be compared to lack of a priori statistical knowledge in the minimum error variance estimation problem.

Finally, a good solid area seemingly ripe to explore using the techniques for optimal control, identification and estimation for second order processes is in the study of various EEG phenomena. It is the fervent desire of this author to try out the different theories on several facets of these particular practical problems.

APPENDIX A.

DERIVATION CF SYSIEM EQUATIONS

The equations governing the propagation of the first two moments of the Class 3) stochastic process $v$ through the dynamic system $\frac{d}{d t} x(t, \cdot)=A(t) x(t, \cdot)+B(t) u(t)+C(t) v(t, \cdot), t \in\left[t_{0}, t_{f}\right] \triangleq \Gamma_{1} A-1$ are derived. Here
$x$ is nxl
A is nxn
$B$ is nxr
u is rxl
C is rxs
$v$ is sxl .
As is well known, the solution of A-1 is

$$
x(t, \cdot)=\Phi\left(t, t_{0}\right) x\left(t_{0}, \cdot\right)+\int_{t_{0}}^{t} \Phi(t, \tau)[B(\tau) u(\tau)+C(\tau) v(\tau, \cdot)] d \tau, t \in \Gamma_{1} .
$$

where
$\Phi\left(t, t_{0}\right)$ is the transition matrix found from
$\frac{d \Phi}{d t}\left(t, t_{0}\right)=A(t) \Phi\left(t, t_{0}\right), t \in \Gamma_{1} \quad A-3$
with initial condition
$\Phi\left(t_{0}, t_{0}\right)=I$. A-4

Taking the expectation of A-1 with
$E\{x(t, \cdot)\} \triangleq \mu_{x}(t)$ and $E\{v(t, \cdot)\} \triangleq \mu_{v}(t), E\left\{\frac{d x}{d t}(t, \cdot)\right\}=\frac{d}{d t} E\{x(t, \cdot)\}=$
$\frac{d}{d t} \mu_{x}(t)=E\left\{A(t) x(t, \cdot)+B(t) u(t)+C(t) \mu_{v}(t)\right\}=$
$A(t) E\{x(t, \cdot)\}+B(t) u(t)+C(t) E\{v(t, \cdot)\}$
or finally
$\frac{d}{d t} \mu_{x}(t)=A(t) \mu_{x}(t)+B(t) u(t)+C(t) \mu_{v}(t), t \in \Gamma_{1}$
A-5

The solution of equation A-5 is given by taking the expectation of $\mathrm{A}-2$ and is

$$
\begin{aligned}
& E\{x(t, \cdot)\}=\mu_{x}(t)= \\
& E\left\{\Phi\left(t, t_{0}\right) x\left(t_{0}, \cdot\right)+\int_{t_{0}}^{t} \Phi(t, \tau)[B(\tau) u(\tau)+C(\tau) v(\tau, r)] d \tau\right\}=
\end{aligned}
$$

$\Phi\left(t, t_{0}\right) E\left\{x\left(t_{0}, \cdot\right)\right\}+\int_{t_{0}}^{t} \Phi(t, \tau)[B(\tau) u(\tau)+C(\tau) E\{v(\tau, \cdot)\}] d \tau, t \in \Gamma_{1}$,
or finally with $E\left\{x\left(t_{0}\right)\right\} \triangleq \mu_{x}\left(t_{0}\right)$, the prior mean, $\mu_{x}(t)=$
$\Phi\left(t, t_{0}\right) \mu_{x}\left(t_{0}\right)+\int_{t_{0}}^{t} \Phi(t, \tau)\left[B(\tau) u(\tau)+C(\tau) \mu_{v}(\tau)\right] d \tau, t \in \Gamma_{1}$.
A-6

In applications either A-6 or A-5 would be used to determine
$\mu_{x}(t), t \in \Gamma_{1}$ depending on the specific case as mentioned in Sage and Melsa (97).

The cross-covariance $V_{v x}(t, \tau)$, an nxs matrix, is found by differentiating the expectation
$\frac{\partial}{\partial \tau} E\left\{\left[v(t, \cdot)-\mu_{v}(t)\right]\left[x(\tau, \cdot)-\mu_{x}(\tau)\right]^{\prime}\right\}=\frac{\partial}{\partial \tau} V_{v x}(t, \tau)$
and using A-1 and A-5 evaluated at $t=\tau$ for all $\tau \in \Gamma_{1}$, which after transposing become
$\frac{d}{d t} X^{\prime}(\tau, \cdot)=X^{\prime}(\tau, \cdot) A^{\prime}(\tau)+u^{\prime}(\tau) B^{\prime}(\tau)+v^{\prime}(\tau, \cdot) C^{-}(\tau), \tau \in \Gamma_{1}$ A-7
and
$\frac{d}{d t} \mu_{x}^{\prime}(\tau)=\mu_{x}^{\prime}(\tau) A^{\prime}(\tau)+u^{\prime}(\tau) B^{\prime}(\tau)+\mu_{v}^{\prime}(\tau) C^{\prime}(\tau), \tau \in \Gamma_{1}$
A-8

Then

$$
\begin{aligned}
& \frac{\partial E}{\partial \tau}\left\{\left[v(t, \cdot)-\mu_{v}(t)\right]\left[x(\tau, \cdot)-\mu_{x}(\tau)\right]^{\prime}\right\}=\frac{\partial}{\partial \tau} v_{v x}(t, \tau) \\
& E\left\{\left[v(t, \cdot)-\mu_{v}(t)\right]\left[\frac{d}{d \tau} x^{\prime}(\tau, \cdot)-\frac{d}{d \tau} \mu_{x}^{\prime}(\tau)\right]\right\}= \\
& E\left\{[ v ( t , \cdot ) - \mu _ { v } ( t ) ] \left[x^{\prime}(\tau, \cdot) A^{\prime}(\tau)+u^{\prime}(\tau) B^{\prime}(\tau)+v^{\prime}(\tau, \cdot) C^{\prime}(\tau)\right.\right. \\
& \left.\left.\quad-\mu_{x}^{\prime}(\tau) A^{\prime}(\tau)-u^{\prime}(\tau) B^{\prime}(\tau)-\mu_{v}^{\prime}(\tau) C^{\prime}(\tau)\right]\right\}=
\end{aligned}
$$

$E\left\{\left[v(t, \cdot)-\mu_{v}(t)\right] \quad\left[x^{-}(\tau, \cdot)-\mu_{x}^{\prime}(\tau)\right] A^{\prime}(\tau)\right.$

$$
\left.+\left[v^{-}(\tau, \cdot)-\mu_{v}^{\prime}(\tau)\right] C^{-}(\tau)\right\}=
$$

$V_{V X}(t, \tau) A^{-}(\tau)+V_{V}(t, \tau) C^{-}(\tau),(t, \tau) \in \Gamma_{1} x \Gamma_{1} \triangleq \Gamma$
or finally
$\frac{\partial}{\partial \tau} V_{V X}\left(t_{1}\right)=V_{V X}(t, \tau) A^{\prime}(\tau)+V_{V V}(t, \tau) C^{\prime}(\tau),(t, \tau) \in \Gamma$.
A-9

The solution of A-9 can be determined by using A-2 and A-6 and forming
$E\left\{\left[v(t, \cdot)-\mu_{v}(t)\right]\left[x(\tau, \cdot)-\mu_{x}(\tau)\right]\right\}=v_{v x}(t, \tau)=$ $E\left\{\left[v(t, \cdot)-\mu_{v}(t)\right]\left[x^{\prime}\left(t_{0}, \cdot\right) \Phi^{\prime}\left(\tau, t_{0}\right)+\int_{t_{0}}^{t}\left[v^{\prime}(s, \cdot) C^{\prime}(s)\right.\right.\right.$ $\left.+u^{\prime}(s) B^{\prime}(s)\right] \times \Phi^{\prime}(\tau, s) d s-\mu_{x}^{\prime}\left(t_{0}\right) \Phi^{\prime}\left(\tau, t_{0}\right)-\int_{t_{0}}^{t}\left[\mu_{v}^{\prime}(s) C^{\prime}(s)\right.$ $\left.\left.+u^{\prime}(s) B^{-}(s)\right] \times \Phi^{-}(\tau, s) d s\right\}=$
$E\left\{\left[v\left(t_{i}\right)-\mu_{v}(t)\right] \quad\left[x^{\prime}\left(t_{0} ;\right)-\mu_{x}^{\prime}\left(t_{0}\right)\right] \Phi^{-}\left(\tau, t_{0}\right)+\right.$
$t$
$\left.\left.\int_{t_{0}}\left[v^{-}(s, \cdot)-\mu_{v}^{\prime}(s)\right] C^{\wedge}(s) \Phi^{-}(\tau, s) d s\right]\right\}=$
$v_{v x}\left(t, t_{0}\right) \Phi^{\prime}\left(\tau, t_{0}\right)+\int_{t_{0}}^{t} v_{v v}(t, s) C^{\sim}(s) \Phi^{-}(\tau, s) d s,(t, \tau) \in \Gamma$
or finally
$v_{v x}(t, \tau)=V_{v x}\left(t, t_{0}\right) \Phi^{\prime}\left(\tau, t_{0}\right)+\int_{t_{0}}^{t} V_{v v}(t, s) C^{\prime}(s) \Phi^{\prime}(\tau, s) d s,(t, \tau) \in \Gamma . A-10$
Taking the transpose of A-9 and A-10 gives
$\frac{\partial}{\partial \tau} V_{V X}^{\prime}(t, \tau)=A(\tau) V_{V X}^{\prime}(t, \tau)+C(\tau) V_{V V}^{\prime}(t, \tau),(t, \tau) \in \Gamma$
and
$v_{v x}^{\prime}(t, \tau)=\Phi\left(\tau, t_{0}\right) V_{v x}^{e}\left(t, t_{0}\right)+\int_{t_{0}}^{t} \Phi(\tau, s) C(s) V_{v v}(t, s) d s,(t, \tau) \in \Gamma$
The other possible cross-covariance is
$V_{v x}(\tau, t) \triangleq E\left\{\left[v(\tau, \cdot)-\mu_{v}(\tau)\right]\left[x(t, \cdot)-\mu_{x}(t)\right] \cdot\right\}$
and from the definitions
$V_{V_{X}}(\tau, t)=V_{V_{X}}^{\sim}(t, \tau)$
hence can be determined by interchanging $t$ and $\tau$ upon solution of either A-9 or A-10.

For the solution of A-9 or A-10, the function of $t$ at the boundary $V_{v x}\left(t, t_{0}\right)$ must be known. This function represents the a priori knowledge of the randomess of the state at initial time correlated with the stochastic imput.

The equations of the auto-covariance or covariance of x are found by differentiating the expectation

$$
\frac{\partial E}{\partial t}\left\{\left[x(t, \cdot)-\mu_{x}(t)\right]\left[x(\tau, \cdot)-\mu_{x}(\tau)\right]\right\} \stackrel{\Delta}{\triangle} \frac{\partial}{\partial t} V_{x x}(t, \tau)
$$

using A-1 and A-6 gives
$\left.\frac{\partial E}{\partial t} E\left[x(t, \cdot)-\mu_{x}(t)\right]\left[x(\tau, 0)-\mu_{x}(\tau)\right]\right\}=\frac{\partial}{\partial t} V_{x x}(t, \tau)=$
$E\left\{\frac{d}{d t}\left[x(t, 0)-\mu_{x}(t)\right]\left[x(\tau, 0)-\mu_{x}(\tau)\right]^{-}\right\}=$
$E\left\{\left[A(t) x(t, \cdot)+B(t) u(t)+C(t) v\left(t, \cdot{ }^{\cdot}\right)-A(t) \mu_{x}(t)-B(t) u(t)\right.\right.$

$$
\left.\left.-c(t) \mu_{v}(t)\right]\left[x(\tau, \cdot)-\mu_{x}(\tau)\right]^{\rho}\right\}=
$$

$E\left\{A(t)\left[x(t, \cdot)-\mu_{x}(t)\right]+C(t)\left[v(t, \cdot)-\mu_{v}(t)\right] \quad\left[x(\tau, \cdot)-\mu_{x}(\tau)\right]^{\wedge}\right\}$ $(t, \tau) \in \Gamma$
or finally

$$
\frac{\partial V_{X x}}{\partial t}(t, \tau)=A(t) V_{X x}(t, \tau)+C(t) V_{v x}(t, \tau),(t, \tau) \in \Gamma .
$$

The solution of A-14 can be found by using A-2 in

$$
\begin{aligned}
& E\left\{\left[x(t, \cdot)-\mu_{x}(t)\right]\left[x(\tau, \cdot)-\mu_{x}(\tau)\right]\right\}= \\
& E\left\{\left[\Phi\left(t, t_{0}\right) x\left(t_{0}, \cdot\right)+\int_{t_{0}}^{t} \Phi(t, s)[B(s) u(s)+C(s) v(s, \cdot)] d s\right.\right. \\
& \quad-\Phi\left(t, t_{0}\right) \mu_{x}\left(t_{0}\right)-\int_{t_{0}}^{t} \Phi(t, s)\left[B(s) u(s)+C(s) \mu_{v}(s)\right] d s x \\
& \left.\quad\left[x(\tau, \cdot)-\mu_{x}(\tau)\right]\right\}=
\end{aligned}
$$

$E\left\{\Phi\left(t, t_{0}\right)\left[x\left(t_{0}, \cdot\right)-\mu_{x}\left(t_{0}\right)\right]+\int_{t_{0}}^{t} \Phi(t, s) C(s)\left[v(s, \cdot)-\mu_{v}(s)\right] d s x\right.$ $\left.\left[x(\tau, \cdot)-\mu_{x}(\tau)\right]\right\}=$
$\bar{\Phi}\left(t, t_{0}\right) v_{x x}\left(t_{0}, t\right)+\int_{t_{0}}^{t} \Phi(t, s) C(s) V_{v x}(s, \tau) d s,(t, \tau) \in \Gamma$
or finally
$V_{x x}(t, \tau)=\Phi\left(t, t_{0}\right) V_{x x}\left(t_{0}, \tau\right)+\int_{t_{0}}^{t} \Phi(t, s) C(s) V_{V x}(s, \tau) d \tau,(t, \tau) \in \Gamma . A-15$
Here the prior correlation of $x$ at $t_{0}$ with itself for all $\tau$ must be known to camplete the solution, i.e. $V_{x x}\left(t_{0}, \tau\right)$ is known.

Equations A-5, A-9, and A-14 are the differential equation representation of how the moments of $v$ propagate through a system given by A-1. Equations A-6, A-10, and A-15 are the solutions of A-5, A-9, and A-14 with the specified initial conditions. For either representation these equations are for $v$ a Class 3 stochastic process. They are the state-space form of the $n^{\text {th }}$ order differential equations in Papoulis (84) or the operator equations of Lebedev (63). In Sage and Melsa (97), Bryson and Ho (20), and Astrom (8), the same type of derivation is given for other classes of stochastic processes.

It is shown by various theorems in Bhat (17), Hoel (35), and Prabhu (90) that the interchange of operators required for the derivations in this appendix is valid for Class 3 stochatic processes. Further all the specified derivatives and iritegrals are shown to exist in these same three references.

It also can be shown that differentiating A-6, A-10, and A-15 gives A-5, A-9, and A-14 respectively; since in the derivations the
solution set was not obtained by directly integrating the differential equations. More will be said about the initial conditions in Appendix B .

The complete set in differential form is collected below.
The mean of x is
$\frac{d}{d t} \mu_{x}(t)=A(t) \mu_{x}(t)+B(t) u(t)+C(t) \mu_{v}(t), t \in \Gamma_{1} \quad A-16$
where
$\mu_{\mathrm{x}}$ is nxl
$A$ is nxn
$B$ is nxr
u is rxl
C is nxs
$\mu_{v}$ is sxl.
The cross variance of $x$ and $v$ is

$$
\frac{\partial}{\partial \tau} V_{v x}^{\prime}(t, \tau)=A(\tau) V_{V x}^{\prime}(t, \tau)+C(\tau) V_{V V}^{-}(t, \tau),(t, \tau) \in \Gamma
$$

where

$$
\begin{aligned}
& v_{v x} \text { is sxn } \\
& v_{v x}^{\prime} \text { is nxs } \\
& v_{v y} \text { is } s \times s \\
& v_{v v}^{\prime} \text { is } 5 \times s .
\end{aligned}
$$

The auto-covariance of x is
$\frac{\partial}{\partial t} V_{x x}(t, \tau)=A(t) V_{x x}(t, \tau)+C(t) V_{v x}(t, \tau),(t, \tau) \in \Gamma$
where $\quad V_{X X}$ is $n \times n$.

It is now desired to derive the equivalent equations for Class 4) stochastic inputs, which are characterized by having covariance. $V_{W}(t, \tau)=\Psi(t) \delta(t-\tau)$
where $\delta(t-\tau)$ is the scalar symetric Dirac delta function defined by

A-20
For the system of A-l with $v(t,$.$) having covariance given$ above, the mean of $v$ propagates the same as for Class 3) processes and is given by equations A-5 and A-6.

The cross covariance is
$V_{x v}(t, \tau)=E\left\{\left[x(t,)-.\mu_{x}(t)\right]\left[v(\tau, .)-\mu_{v}(\tau)\right]^{\prime \prime}\right\}$
and can be evaluated by post multiplying equation A-6 minus
A-2 by $\left[v(\tau, .)-\mu_{v}(\tau)\right]^{-}$and taking the expectation,
$x(t,)-.\mu_{x}(t)=\Phi\left(t, t_{0}\right)\left[x\left(t_{0},.\right)-\mu_{x}\left(t_{0}\right)\right]+\int_{t_{0}}^{t} \Phi\left(t_{,} s\right) C(s)\left[v(s,)-.\mu_{v}(s)\right] d s$
giving

$$
\begin{aligned}
v_{x v}(t, \tau) & =\Phi\left(t, t_{0}\right) E\left\{\left[x\left(t_{0}, .\right)-\mu_{x}\left(t_{0}\right)\right]\left[v(\tau, .)-\mu_{v}(\tau)\right]^{\prime}\right\} \\
& +\int_{t_{0}} \Phi(t, s) C(s) E\left\{\left[v(s, .)-\mu_{v}(s)\right]\left[v(\tau, .)-\mu_{v}(\tau)\right]^{\sim}\right\} d s A-22
\end{aligned}
$$

Normally it is assumed that $x\left(t_{0},.\right)$ is uncorrelated with $v(\tau,),. \tau \in \Gamma_{1}$, then
$v_{x v}(t, \tau)=\int_{t_{0}}^{t} \Phi(t, s) C(s) V_{v v}(s, \tau) d s,(t, \tau) \in \Gamma$.
and knowing the form of $V_{v v}(s, t)$ from $A-19$ gives
$V_{X V}(t, \tau)=\int_{t_{0}}^{t} \Phi(t, s) C(s) \delta(s-\tau) d s$
From A-20, this becomes
$V_{\mathrm{xv}}(\mathrm{t}, \tau)= \begin{cases}0 & t_{0}<t<\tau \\ C(\tau) \Psi(\tau) / 2 & t_{0}<t=\tau \\ \Phi(t, \tau) C(\tau) \Psi(\tau) & t_{0}<\tau<t\end{cases}$ A-23
and it is clear that a discontinuity occurs at $t=\tau$.
Post multiplying A-21 by $\left[x(\tau, .)-\mu_{x}(\tau)\right]^{\prime} \quad$ and
taking the expectation gives

$$
\begin{aligned}
& V_{X V}(t, \tau)=E\left\{\left[x(t, .,)-\mu_{x}(t)\right]\left[x(\tau, .)-\mu_{x}(\tau)\right]^{\prime}\right\}=\Phi\left(t, t_{0}\right) V_{x X}\left(t_{0}, t_{0}\right) \Phi^{\prime}\left(\tau, t_{0}\right) \\
& +\Phi\left(t, t_{0}\right) E\left\{\left[x\left(t_{0}, .\right)-\mu_{x}\left(t_{0}\right)\right]\left[\int_{t_{0}}^{\tau} \Phi(\tau, s) C(s)\left[v(s, .)-\mu_{v}(s)\right] d s\right]^{\rho}\right\} \\
& +E\left\{\left[\int_{t_{0}}^{t} \Phi(t, s) C(s)\left[v(s, .)-\mu_{v}(s)\right]\right]\left[\Phi\left(\tau, t_{0}\right)\left[x\left(t_{0}, .\right)-\mu_{x}\left(t_{0}\right)\right]\right]^{n} d s\right\} \\
& +E\left\{\left[\int_{t_{0}}^{t} \Phi(t, s) C(s)\left[v(s, .)-\mu_{v}(s)\right] d s\right]\left[\int_{t_{0}}^{\tau} \Phi(\tau, \sigma) C(\sigma)\left[v(\sigma, .)-\mu_{v}(\sigma)\right] d \sigma\right]\right\}
\end{aligned}
$$

Again it is assumed that $x\left(t_{0}, 0\right)$ and $v(\tau, .$,$) are uncorrelated$ resulting in the second and third terms going to zero.

Rearranging,

$$
\begin{aligned}
& V_{X x}(t, \tau)=\Phi\left(t, t_{0}\right) v_{X x}\left(t_{0}, t_{0}\right) \Phi^{\prime}\left(\tau, t_{0}\right) \\
& +\int_{t_{0}}^{t} \int_{0}^{\tau} \Phi(t, s) C(s) E\left\{\left[v(s, .)-\mu_{v}(s)\right]\left[v(\sigma, .)-\mu_{v}(\sigma)\right]^{\prime}\right\} C^{\prime}(\sigma) \Phi^{\prime}(t, \sigma) d \sigma d s
\end{aligned}
$$

$$
\begin{aligned}
& V_{x x}(t, \tau)=\Phi\left(t_{1} t_{0}\right) V_{x x}\left(t_{0}, t_{0}\right) \Phi^{\prime}\left(\tau, t_{0}\right) \\
& +\int_{t_{0}}^{t} \int_{0}^{\tau} \Phi(t, s) C(s) \Psi(s) \delta(s-\sigma) C^{\prime}(\sigma) \Phi^{\prime}(\tau, \sigma) d \sigma d s
\end{aligned}
$$

In this last expression the order in which the double integration is carried out must be selected carefully. If $t>\tau$, then first integrate with respect to $s$ in order to obtain a range where the delta function exists. If $\tau>t$, then first integrate with respect to $\sigma$ for the same reason. Rewriting A-28 with this reasoning gives $V_{x X}(t, \tau)=\Phi\left(t, t_{0}\right) V_{x X}\left(t_{0}, t_{0}\right) \Phi^{-}\left(\tau, t_{0}\right)$

$$
+\int_{t_{0}}^{\min [t, \tau]} \Phi(t, \eta) C(\eta) \Psi(\eta) C^{\prime}(\eta) \Phi^{\wedge}(\tau, \eta) d \eta
$$

Restricting attention to only the case where $t=\tau$ gives

$$
\begin{aligned}
& V_{X X}(t, \tau)=\Phi\left(t, t_{0}\right) V_{x x}\left(t_{0}, t_{0}\right) \Phi^{-}\left(\tau, t_{0}\right) \\
&+\int_{t_{0}}^{t} \Phi(t, \eta) C(n) \Psi(n) C^{\prime}(n) \Phi^{\prime}(t, \eta) d n
\end{aligned}
$$

$$
A-30
$$

Now, A-30 is the solution of an ordinary differential equation which can be obtained by differentiating and using Leibnitz's rule for differentiation under an integral which is
 A-31

$$
\frac{d V^{d x}}{d t}(t, t)=\frac{d \Phi}{d t}\left(t, t_{0}\right) V_{x x}\left(t_{0}, t_{0}\right) \Phi^{-}\left(t, t_{0}\right)+\Phi\left(t, t_{0}\right) V_{x x}\left(t_{0}, t_{0}\right) \frac{d \Phi\left(t, t_{0}\right)}{d t}
$$


$+\Phi(t, t) C(t) \Psi(t) C^{\prime}(t) \Phi^{\prime}(t, t) \frac{d t-0}{d t}$
But $\Phi(t, t)=I$ and $\frac{d \Phi}{d t}(t, \tau)=A(t) \Phi(t, \tau), t \in \Gamma_{I}$
which gives

$$
\begin{aligned}
& \frac{d V}{d t^{*}}(t, t)=A(t)\left[\Phi\left(t, t_{0}\right) V_{X X}\left(t_{0}, t_{0}\right) \Phi^{\prime}\left(t, t_{0}\right)\right]+\left[\Phi\left(t, t_{0}\right) V_{X X}\left(t_{0}, t_{0}\right) \Phi^{\sim}\left(t, t_{0}\right] A^{-}(t)\right. \\
& +A(t)\left[\int_{t_{0}}^{t} \Phi(t, n) C(n) \Psi(n) C^{\prime}(\eta) \Phi^{\prime}(t, \eta) d n\right]+\left[\int_{t_{0}}^{t} \Phi(t, \eta) C(\eta) \Psi(n) C^{-}(\eta) \Phi^{-}(t, n) d \eta\right] A^{\prime}(t) \\
& +C(t) \Psi(t) C^{-}(t) \quad A-33
\end{aligned}
$$

$$
\text { and from } A-30
$$

$$
\frac{d V_{x x}}{d t}(t, t)=A(t) V_{x x}(t, t)+V_{x x}(t, t) A^{\prime}(t)+C(t) \Psi(t) C^{-}(t), t \in \Gamma_{1} A-34
$$

Equation A-34 is used throughout much of the literature dealing with estimation theory. 'The major difference between this equation and equations A-17 and A-18 which hold for Class 3) stochastic processes is seen in that equation A-34 holds only in the plane with $t=\tau$, but equations A-17 and A-18 hold throughout the square $(t, \tau) \in\left[t_{0}, t_{f}\right] \times\left[t_{0}, r t_{f}\right]$. Further, as remarked on page $106, \mathrm{~V}_{\mathrm{XV}}(\mathrm{t}, \tau) \mid \mathrm{t}=\tau$ is a point of discontinuity in the square, and equation A-34 includes the effects of $V_{X V}(t, t)$ hence includes the effects of this discontinuity, in fact $A-34$ only is true along a line where $\mathrm{V}_{\mathrm{XV}}(\mathrm{t}, \tau)$ is everywhere discontinuous.

Therefore, it is seen that much more information is implied by equations A-17 and A-18 for Class 3) processes than by equation $A-34$ for Class 4) processes.

An alternate derivation of equation $A-34$ can be performed by differentiating the definition of the covariance of $x$, as
$\frac{d}{d t} E\left\{\left[x(t,)-.\mu_{x}(t)\right]\left[x(t,)-.\mu_{x}(t)\right]\right\}=\frac{d V}{d t} x(t, t)=$
$E\left\{\left[\frac{d x}{d t}(t,)-.\frac{d \mu}{d t}(t)\right][x(t,)-.x(t)\}\right\}+E\left\{\left[x(t,)-.\mu_{x}(t)\right]\left[\frac{d x}{d t}(t, .)-\mu_{x}(t)\right]^{-}\right\}=$
$E\left\{\left[A(t)\left[x(t,)-.\mu_{x}(t)\right]+C(t)\left[v(t,)-.\mu_{v}(t)\right]\right]\left[x(t,)-.\mu_{x}(t)\right]-\right\}$
$+E\left\{\left[x(t,)-.\mu_{x}(t)\right]\left[\left[v(t, .)-\mu_{v}(t)\right]^{-} C^{\prime}(t)+\left[x(t, .)-\mu_{x}(t)\right]^{-} A^{\prime}(t)\right]\right\}=$
$A(t) V_{X X}(t, t)+C(t) V_{V x}(t, t)+V_{X v}(t, t) C^{\prime}(t)+V_{X x}(t, t) A^{\prime}(t)$
A-35
This equation shows the explicit dependence of $V_{X X}(t, t)$ on
$v_{X V}(t, t)$
From A-25,
$V_{X v}(t, t)=C(t) \Psi(t) / 2$
and
$V_{V X}(t, t)=V_{X V}^{-}(t, t)=\psi^{-}(t) C^{-}(t) / 2$
therefore, substituting in A-35 gives

and since $\Psi(t)$ is symmetric, $A-36$ is the same as $A-34$.
Evaluating A-30 at $t=\tau$ and premultiplying by $\Phi(t, \tau)$ gives
$V_{X X}(\tau, \tau)=\Phi(t, \tau) \Phi\left(\tau, t_{0}\right) V_{X X}\left(t_{0}, t_{0}\right) \Phi^{\prime}\left(\tau, t_{0}\right)+$
$\int_{t_{0}}^{\tau} \Phi(t, \tau) \Phi(\tau, \eta) C(\eta) \Psi(\eta) C^{\prime}(\eta) \Phi^{-}(\tau, \eta) d \eta$ A-37
but since $\Phi(t, \tau) \Phi(\tau, \beta)=\Phi(t, \beta)$ for $t>\tau$ if it is restricted to $t>\tau$, A-37 is equal to equation $A-29$. Hence for $t>\tau$,
$V_{x x}(t, \tau)=\Phi(t, \tau) V_{x x}(\tau, \tau) \quad A-38$
and similarly for $\tau>t$,
$V_{x x}(t, \tau)=V_{X x}(t, t) \Phi^{\prime}(\tau, t) \quad n-39$
These last two equations enable more information to be obtained since they permit the covariance of $x$ to be found over the whole square.

However, there is no convenient differential equation form for A-38 and A-39, and the transition matrix is required. The major differences then in the equations for Class 3) and Class 4) stochastic processes are:

1) All the first two moments of state and input are continuous everywhere in the square for Class 3) processes but not for Class 4),
2) The evaluation of all the first two moments of state and input for Class 3) processes does not require knowledge of the transition matrix whereas this knowledge is required for Class 4) processes,
3) For Class 4) processes, two separate evaluations are required for the cases $t>\tau$ and $\tau>t$, after obtaining all data at $t=\tau$, but this separation of the square is not at all needed for Class 3) processes, and
4) The equations developed for Class 4) processes only hold for "white" or "not time correlated" inputs whereas the equations
for the Class 3) processes allow any type of correlation, including memory. "Non-white" or "time correlated" models can be obtained for Class 4) inputs with "white" noise driving a prefilter becoming the input to the plant.

The "colored noise" representation is
$\frac{d x}{d t}(t,)=.A(t) x(t,)+.B(t) u(t)+C(t) v(t,) \quad A-$.
where
$v\left(t_{0} .,\right)=\Lambda(t) \gamma\left(t_{f},.\right)+\beta\left(t_{p}.\right)$
A-41
and
$\frac{d \gamma}{d t}(t,)=.E(t) \gamma(t,)+.T(t) \alpha(t,) \quad A-$.
with
$V_{\alpha \beta}(t, \tau)=0 \quad(t, \tau) \in \Gamma$
$V_{\beta \alpha}(t, \tau)=0 \quad(t, \tau) \in \Gamma$
$V_{\beta \beta}(t, \tau)=\Psi_{\beta}(t) \delta(t-\tau)$
$V_{\alpha \alpha}(t, \tau)=\Psi_{\alpha}(t) \delta(t-\tau)$
Since $\alpha(t,$.$) is "white noise", the covariance of \gamma$ is found using A-29, as

$$
V_{\gamma \gamma}(t, \tau)=\Phi_{\gamma}\left(t_{,} t_{O}\right) V_{\gamma \gamma}\left(t_{0}, t_{0}\right) \Phi^{-}\left(\tau, t_{0}\right)
$$


where
$\frac{d \Phi}{d t} \Phi_{Y}\left(t, t_{0}\right)=E(t) \Phi_{\gamma}\left(t, t_{0}\right)$
and
$\Phi_{\gamma}\left(t_{0}, t_{0}\right)=I$
From A-43, the explicit dependence of $V_{\gamma \gamma}$ on $t$ and $\tau$ is exhibited by the $\tau$ argument in the transition matrix $\Phi_{\gamma}$. Then the state covariance $V_{X x}$ will depend on $\tau$ through this means. The complete solution is obtained from the augmented model

$$
\begin{array}{r}
\frac{d}{d t}\left[\begin{array}{l}
x(t, .) \\
\gamma(t, .)
\end{array}\right]=\left[\begin{array}{cc}
A(t) & C(t) \Lambda(t) \\
0 & E(t)
\end{array}\right]\left[\begin{array}{l}
x(t, .) \\
\gamma(t, .)
\end{array}\right]+\left[\begin{array}{ll}
B(t) & 0 \\
0 & 0
\end{array}\right] u(t) \\
+\left[\begin{array}{cc}
C(t) & 0 \\
0 & T(t)
\end{array}\right]\left[\begin{array}{l}
\beta(t, .) \\
\alpha(t, .)
\end{array}\right]
\end{array}
$$

rewritten with obvious definitions as

```
\(\frac{d x^{*}}{d t}(t,)=.A^{*}(t) x^{*}(t,)+.B^{*}(t) u(t)+C^{*}(t) v^{*}(t, .\).
with \(v^{*}\) a "white noise" process. The covariance of \(x^{*}\) is
found using A-29 as

where
\(\frac{d \Phi^{*}}{d t}\left(t, t_{0}\right)=A^{*}(t) \Phi^{*}\left(t, t_{0}\right)\)
and
\(\Phi^{*}\left(t_{0}, t_{0}\right)=I\)
and
\(V_{V^{\star} V^{*}}(t, \tau)=\left[\begin{array}{cc}\Psi_{B}(t) & 0 \\ 0 & \Psi_{\alpha}(t)\end{array}\right] \quad \delta(t-\tau)=\Psi_{V^{*}}(t) \delta(t-\tau)\)

The solution of A-46 enables the retrieval of the covariance of the original state \(x\) driven by "colored noise" \(v\) to be obtained.

The "colored noise" \(v\) with covariance depending on \(t\) and \(\tau\) is obtained through the artificial use of the prefilter of equation \(A-42\). It is obvious then that \(\Lambda(t), \Xi(t)\) and \(T(t)\) must be linear time varying or linear constant matrices. Nonlinear memory type elements could not be present. Therefore, only processes with Markov properties can be treated. This restriction is not present in the development of the second moment equations for Class 3) processes.

\section*{APPENDIX B}

\section*{TPRANSFORMATIONS CF PERFORNANCE INDEX AND ENDPOINT CONDITIONS}

\section*{B. 1 Generalized Performance Index}

The typical "minimm energy" performance index including all energies present in a physical plant was obtained in Section 3.1:2 as the expectation of
\(J_{S}=\left\langle x\left(t_{f}, \cdot\right), Q\left(t_{f}\right) x\left(t_{f}, \cdot\right)\right\rangle\)
\(+\int_{t_{0}^{f}}^{t_{0}}\left\{\left\langle x(t, \cdot), R_{2} x(t, \cdot)\right\rangle+\left\langle u(t), R_{2} u(t)\right\rangle+\left\langle v(t, \cdot), R_{3} v(t, \cdot)\right\rangle\right\} d t \quad\) B. 1-1

Redefine B.1-1 as
\[
J_{s}=K\left(t_{f}, x\left(t_{f}\right)\right)+J_{x s}+J_{u s}+J_{v s}
\]
where the various terms are obviously defined. For \(x\) a stochastic process, the criteria most often used in the literature is
\(J_{m, x} \triangleq E\left\{J_{x S}\right\}=E\left\{\int_{t_{0}}^{t_{f}}<x(t, \cdot), R_{1} x(t, \cdot)>d t\right.\)
which is a measure of the mean value of state energy. However, other means to make \(J_{x s}\) a deterministic number appear. Sain and Liberty (98) use the minimm variance value of state energy
\[
J_{m v, x} \Delta E\left\{\left[J_{x s}-\left\{J_{x s}\right\}\right]^{2}\right\} \quad \text { B.1-4 }
\]

Pugachev (91) and Andreev (5) use the other measures
\(J_{m s, x} \stackrel{\Delta}{=} E\left\{J_{x s}{ }^{2}\right\}\)
B.1-5
a minimum mean square value of state energy and
\(J_{s, x} \triangleq\left[E\left\{J_{x s}\right\}\right]^{2} \quad\) B.1-6
a minimum mean-squared value of state energy. Murphy (81)
introduces arbitrary weighting of these measures. Rekasius (93) and Sherman(103) define further possible modifications. The original work of Kalman and Bucy (40) in estimation theory, and its extensions such as in Mehra (74), Sage and Melsa (97), Bryson and Ho (20), Astrom (8) and Kushner (61), all use the mean value of cost.

A more general performance index would be the weighed sum of these four, defined as
\[
\operatorname{IP}_{x} \triangleq a_{1 x} J_{m j x}+a_{2 x} J_{m v, x}+a_{3 x} J_{m s, x}+a_{4 x} J_{s, x}
\]
where \(a_{i x}, i=1, \cdots, 4\) are constants.
Similarly, define
\[
I P_{v} \triangleq a_{l v} J_{m, v}+a_{2 v} J_{m v, v}+a_{3 v} J_{m s, v}+a_{4 v} J_{s, v}
\]
and
\[
\begin{aligned}
& \mathrm{IP}_{u} \triangleq a_{l u} J_{m, u}+a_{2 u m v, u} J_{n}+a_{3 u m s, u} J_{m}+a_{4 u} J_{s, u} \\
& \text { and finally the overall measure }
\end{aligned}
\]
\[
I P \triangleq I P_{x}+I P_{v}+I P_{u} \quad \text { B. } 1-10
\]

Several interesting sidelights arise when all of the terms are collected as in B.1-10. Consideration of this measure may result in solutions to the propositions of Guillemin (29) about finding Nature's exror criteria.

The optimal selection of the \(a_{i x}, a_{i v}\) and \(a_{i u}, i=1, \cdots, 4\) is discussed with respect to optimal system synthesis by Andreyev: (6) and others. A particular version of this type of problem is included as Problem 4).

The separation of state, control and disturbance energies precludes the occurence of cross-terms between these variables. There would be many cross terms present if the four measures of \(J_{s}\) in B.1-1 were taken and summed. These cross terms are not considered here as is done in most of the literature.

\section*{B. 2 Performance Indices of Problems 1 and 2}

The choice of
\[
\begin{align*}
& a_{3 x}=-\frac{1}{2} \\
& a_{3 v}=-\frac{1}{2} \\
& a_{4 u}=+\frac{1}{2} \\
& a_{4 v}=+\frac{1}{2}
\end{align*}
\]
and the rest of the a's zero in B.1-10 yields a performance index related to the Problem 1 performance index, though not exactly equal. The relation is established through the inequality

which can be shown to hold using some basic properties of covariances as in Papoulis (84), some theorems on positive definite matrices as in Hohn (36) and Graybill (32), and some inequalities as in Mitrinovic (77), (78), Beckenbach and Bellman (15) and Marcus and Minc (72).

In addition to the terms of the problem 1 performance index, a term linear in \(t r R_{2} u_{3}^{\prime}(t, t)\) will arise. Using the theory in Kleindorfer and Kleindorfer (46) and Athans (11), the cost of this term can be shown to be included in the RHS of B.2-2.

Alternately, the performance index of problem 1 can be treated as a function of other indices as in Petrov (87) and Andreyev (6) yielding a similar relation.

A detailed analysis of B.1-10 would show that all terms would be functions of the first and second moments of state and disturbance. Including all of these moments under a double integral yields the functional performance index of Problem 2.

\section*{B. 3 Transformation of Problem 3 Performance Index}

The performance index of Problem 3, equation 4.1.3.1 is
\[
\begin{aligned}
& \left.+\sum_{k=1}^{S} \sum_{m=1}^{S} f_{k m}\left(z_{i}^{\prime}, z_{3}\right) u_{j k m}^{\prime}\right\} d t d \tau \\
& \text { B. 3-1 }
\end{aligned}
\]
where B.3-1 represents a functional linear in \(u_{3}^{\prime}\) and nonlinear in \(z_{1}, z_{2}^{\prime}, z_{3}, u_{1}\) and \(u_{2}\). The problem then becomes singular with respect to \(u_{3^{\prime}}^{\prime}\) but remains nonsingular with respect to \(u_{1}\) and \(u_{2}\). Since the state equations are also linear in \(u_{3}\), the Hamiltonian, \(H\) is linear in \(u_{3}^{\prime}\) and the necessary condition of equation 4.2.2.12 does not enable \(u_{3}^{\prime *}\) to be found. Therefore the higher order necessary conditions of Theorem 4.3 had to be developed.

A typical performance index in the form of equation 4.1.3.1 arises with the choice of
\[
\begin{aligned}
& a_{4 x}=-\frac{1}{2} \\
& a_{4 u}=+\frac{1}{2} \\
& a_{3 v}=-\frac{1}{2}
\end{aligned}
\]
in the generalized measure of equation B.1-10. With the aid of inequality B. 2-2, a specific measure singular, hence linear, in \(u_{3}^{\prime}\) and non-singular in \(u_{1}\) and \(u_{2}\), but nonlinear in \(z_{3}\) is \(J\left(u_{1}, u_{2}, u_{3}^{\prime}\right)=\int_{t_{0}}^{t_{f}} \int_{f} f\left(-1 / 2 \operatorname{tr}\left[R_{1} z_{3}(t, \tau) z_{3}^{\prime}(t, \tau) R_{1}^{\prime}\right]+1 / 2<u_{1}(t), R_{2} u_{1}(t)>\right.\)
\[
\left.-1 / 2 \operatorname{tr}\left[R_{3} u_{3}^{\prime}(t, \tau)\right]-1 / 2<u_{2}(t), R_{3} u_{2}(\tau)>\right\} d t d \tau \quad \text { B.3-2 }
\]

Other terms in the form of \(f_{l k}\left(z_{2}, z_{3}\right) u_{3_{1 k}}\) would arise naturally by considering the cross terms which were neglected in developing equation B.1-10.

Since B. \(3-1\) is linear in \(u_{3}\), Problem 3 is a singular control problem and much more difficult to solve than problems with quadratic criteria. This fact led to the formulation of Problem 2) and Problem 1).

\section*{B. 4 Endpoint Conditions}

The choice of performance indices for problems 1), 2) and 3) constrains the problems to be fixed endpoint problems. If a terminal cost term were present, free endpoint conditions would arise, but this case is not covered in this research.

Therefore it becomes necessary to specify the state \(z_{1}, z_{2}^{\prime}, z_{3}\) at \(t_{0}\) and at fixed finite final time \(t_{f}\) as
\[
\begin{array}{ll}
z_{1}\left(t_{0}\right)=E\left\{x\left(t_{0}, \cdot\right)\right\}=\mu_{x}\left(t_{0}\right) & \text { B. 4-1 } \\
z_{1}\left(t_{f}\right)=E\left\{x\left(t_{f}, \cdot\right)\right\}=\mu_{x}\left(t_{f}\right) & \text { B.4-2 } \\
z_{2}^{\prime}\left(t_{,}, t_{0}\right)=E\left\{\left[v\left(t_{,}\right)-\mu_{v}(t)\right]\left[x\left(t_{0}, \cdot\right)-\mu_{x}\left(t_{0}\right)\right]^{\prime}\right\} & \text { B.4-3 } \\
z_{2}^{\prime}\left(t_{,}, t_{f}\right)=E\left\{\left[v\left(t_{,}\right)-\mu_{v}(t)\right]\left[x\left(t_{f}, \cdot\right)-\mu_{x}\left(t_{f}\right)\right]^{\prime}\right\} & \text { B.4-4 } \\
z_{3}\left(t_{0}, \tau\right)=E\left\{\left[x\left(t_{0}, \cdot\right)-\mu_{x}\left(t_{0}\right)\right]\left[x(\tau, \cdot)-\mu_{x}(\tau)\right]^{\rho}\right\} & \text { B.4-5 } \\
z_{3}\left(t_{f}, \tau\right)=E\left\{\left[x\left(t_{f}, \cdot\right)-\mu_{x}\left(t_{f}\right)\right]\left[x(\tau, \cdot)-\mu_{x}(\tau)\right]^{\prime}\right\} & \text { B.4-6 }
\end{array}
\]

Graphically B.4-1 through B. 4-6 are given in the following figures. In these figures the functions plotted are scalars, but a similar figure would apply for each element of the vector-matrix state. Assume
\[
z_{3}(t, \tau)=\Phi(t) e^{-\frac{|t-\tau|}{T}}
\]
and
\[
\Phi(t)=z_{3}(t, t)
\]
B.4-8


Figure B.4.1 Endpoint conditions
then
\[
z_{3}\left(t_{0}, \tau\right)=\Phi\left(t_{0}\right) e^{-\left|t_{0}-\tau\right|} \frac{T}{T}
\]
and for \(\tau \geq t_{0}\)
\[
\begin{align*}
z_{3}\left(t_{0}, \tau\right) & =\Phi\left(t_{0}\right) e^{\frac{-\left(\tau-t_{0}\right)}{T}} \\
& =\Phi\left(t_{0}\right) e^{\frac{t_{0}}{T}} e^{\frac{\tau}{T}} \\
& =\operatorname{Ke} \frac{-\tau}{T}
\end{align*}
\]
and this is the form assumed in example 1 for \(T=1\). Similar remarks hold for \(z_{2}^{\prime}\left(t, t_{0}\right)\). Then since \(z_{2}^{\prime}\left(t_{0}, t_{0}\right)\) and \(z_{3}\left(t_{0}, t_{0}\right)\) can be measured as is described in Kalman (40), and \(T\) also can be measured for a physical process, with the form in B.4-9, the initial point conditions are known.

The conditions at \(t_{f}\) result from the initial condition response and response due to the transition matrices of \(A(t)\) and \(-A^{\wedge}(t)\). It would be desired to drive \(z_{3}\left(t_{f}, t_{f}\right)\) and \(z_{2}\left(t_{f}, t_{f}\right)\) to a smaller value than at \(t_{0}\), and this can be accomplished through the above analysis and selection of initial conditions on \(\lambda_{2}\) and \(\lambda_{3}\) which satisfy the TPBVP. The mean value \(z_{1}\left(t_{0}\right)\) can be measured hence is assumed given.

\section*{APPENDIX C}

NECESSARY CONDITIONS
It was proposed that the geometric approach of Kuo in ( 50 ) and (52), and Leitmann(67) be employed in deriving necessary conditions. The choice of Class 3) stochastic disturbances, and the resulting form of equations for the dymanic system and performance index after transformation precluded the use of the geometric approach. It was chosen to employ variational techniques instead for all four classes of problems. Standard forms of variacional approaches are found in Kirk(45). Citron (22), Bryson and Ho (20) and Athans and Falb (9). Since some of the state and control variables are matrices rather than vectors, the results of standard forms of variational approaches could not be directly applied. Instead, matrix variations had to be defined and the entire proof had to be carried out. A formal extension of Pontriagin's minimum principle, found in Pontriagin et al (88) , to the matrix variable case was given by Athans in (10), but since the form of the equations included dependence on two independent variables, \(t\) and \(\tau\), the result of that derivation could not be directly applied either. The dynamic optimisation of criteria and constraints with more than one independent variable is briefly described for the vector case in Gottfried and Weisman (30) : . Introductory distributed parameter optimal control theory is covered in Sage(95), but this the results of this theory cannot be directly applied either.

\section*{C. 1 Proof of Theorern 4.1:}

For Problem 1 of Section 4.1.1 defined by equations 4.1.1.2
through 4.1.1.8 with the performance index of 4.1.1.1, define
\(\lambda(t)\) an \(n \times 1\) vector
\(\lambda_{2}(t, \tau)\) an nxs matrix
\(\lambda_{\rho}(t, \tau)\) an \(n \times n\) matrix
and form the augmented performance index
\[
\text { noting that } \mathrm{Ja}=\mathrm{J} \text { of 4.1.1.1. . }
\]

Define \(\Omega_{z}\) as the scalar function
\[
\text { Define } \Omega_{2} \text { as the scalar function }
\]
\[
\Omega_{2}\left(z_{z}^{\prime}, z_{3}, \frac{\partial z_{2}^{\prime}}{\partial \tau}, \frac{\partial z_{3}^{\prime}}{\partial \tau}, u_{j}^{\prime}, \lambda_{2}, \lambda_{3}\right)=-\frac{1}{2} \operatorname{tr} R_{1} z_{3} z_{3}^{\prime} R_{1}^{\prime}
\]
\[
-\frac{1}{2} \operatorname{tr} \mathrm{R}_{2} u_{y} \mathrm{u}_{3}^{\prime} \mathrm{R}_{2}+\operatorname{tr}\left[\left[\mathrm{Az}_{2}^{\prime}+\mathrm{Cu}_{3}^{\prime}-\frac{\partial z_{2}}{\partial \bar{\tau}}\right]\left[\lambda_{2}^{\prime}\right]\right]
\]
\[
\begin{aligned}
& J a=J\left(z_{1}, z_{<}^{\prime}, z_{j}, u_{1}, u_{e}, u_{j}^{\prime}, \lambda_{1}, \lambda_{2}, \lambda_{3}\right)= \\
& -\frac{1}{2} \int_{t_{0} t_{0}}^{t_{f} t_{j} f_{i n}}\left[R_{s} z_{s}(t, \tau) z_{s}^{-}(t, \tau) R^{\prime}\right]+\operatorname{tr}\left[R_{z} u_{s}(t, \tau) u_{s}^{\prime}(t, \tau) R_{z}^{\prime}\right] \\
& +\operatorname{tr}\left[\left[A(\tau) z_{2}^{\prime}(t, \tau)+C(\tau) u_{3}^{\prime}(t, \tau)-\frac{\partial z_{2}^{\prime}}{\partial \tau}(t, \tau)\right]\left[\lambda_{2}^{\prime}(t, \tau)\right]\right] \\
& \left.+\operatorname{tr}\left[\left[A(t) z_{j}^{\prime}(t, \tau)+C(t) z_{e}^{\prime}(t, \tau)-\frac{\partial z_{3}}{\partial t}(t, \tau)\right]\left[\lambda_{j}^{\prime}(t, \tau)\right]\right]\right\} d t d \tau \\
& +\frac{1}{2} \int_{t_{0}}^{t_{f}}\left\{<u_{1}(t), R_{3} u_{1}(t)>-\left\langle u_{2}(t), R_{4} u_{2}(t)>\right.\right. \\
& +\left\langle\lambda_{1}(t), A(t) z_{1}(t)+B(t) u_{1}(t)+C(t) u_{2}(t)-\frac{d z_{1}}{d t}(t)>\right\} d t
\end{aligned}
\]
\(+\operatorname{tr}\left[\left[A z_{3}+C z_{2}-\frac{\partial z_{3}}{\partial t}\right]\left[\lambda_{3}^{\prime}\right]\right]\)
and define \(\Omega_{1}\) as the scalar function
\(\left.\left.\Omega_{1}\left(z_{1}, \frac{d z_{1}}{d t}, u_{1}, u_{2}, \lambda_{1}\right)=\frac{1}{2}<u_{1}, R_{3} u_{1}\right\rangle-\frac{1}{2}<u_{2}, R_{4} u_{2}\right\rangle\)
\(+\left\langle\lambda_{1}, \mathrm{Az}_{1}+\mathrm{Bu}_{1}+\mathrm{Cu}_{2}-\frac{\mathrm{dz}}{\mathrm{dt}}\right\rangle\)
Then
\(J a=J=\int_{t_{0}}^{t_{f}} \int_{0}^{t_{f}} \Omega_{2}\left(z_{2}^{\prime} \cdot z_{3}, \frac{\partial z_{2}^{\prime}}{\partial \tau}, \frac{\partial z_{3}}{\partial t_{3}}, u_{3}^{\prime}, \lambda_{2}, \lambda_{3}\right) d t d \tau\)
\(+\int_{t_{0}}^{t_{f}} \Omega_{1}\left(z_{1}, \frac{d z_{1}}{d t}, u_{1}, u_{2}, \lambda\right) d t\)

Define the variations
\[
\begin{aligned}
& \delta z_{1}^{\prime}(t)=z_{1}^{\prime}(t)-z_{1}^{*}(t) \\
& \delta z_{2}^{\prime}(t, \tau)=z_{2}^{\prime}(t, \tau)-z_{2}^{\prime *}(t, \tau) \\
& \delta z_{3}(t, \tau)-z_{3}(t, \tau)-z_{3}^{*}(t, \tau) \\
& \delta u_{1}^{\prime}(t)=u_{1}^{\prime}(t)-u_{1}^{*}(t) \\
& \delta u_{2}(t)=u_{2}(t)-u_{i}^{*}(t) \\
& \delta u_{3}^{\prime}(t, \tau)=u_{3}^{\prime}(t, \tau)-u_{3}^{\prime} *(t, \tau) \\
& \delta \lambda_{1}(t)=\lambda_{1}(t)-\lambda_{1}^{*}(t)
\end{aligned}
\]
\[
\delta \lambda_{2}(t, \tau)=\lambda_{2}(t, \tau)-\lambda_{2}^{*}(t, \tau)
\]
\[
\delta \lambda_{s}(t, \tau)=\lambda_{s}(t, \tau) \sim \lambda_{s} *(t, \tau)
\]
and
where ( )* denotes () evaluated at extremal conditions.

Then
\(\delta J=\int_{t_{0}}^{t_{f} t_{f}} \int_{z} \delta \Omega_{z} d t d \tau+\int_{t_{0}}^{t_{f}} \delta \Omega_{1} d t\)
The variation of a scalar function \(g(A)\) of an nom matrix \(A\) is
\[
\delta g(A)=\operatorname{tr}\left[\delta A \frac{\partial g}{\partial A}^{(A))}\right)_{*}^{\wedge}
\]
where
\[
\frac{\partial g}{\partial A}(A) \quad \text { C.1~9 }
\]
is an \(n \times m\) gradient matrix and
\(\delta A=A-A^{*}\)
Using C. 1-8,

\(+\operatorname{tr}\left[\frac{\delta z_{2}}{\partial \tau}\left(\frac{\partial \Omega_{2}}{\partial\left(\frac{\partial Z_{2}}{\partial \tau}\right) *}\right)^{*}\right]+\operatorname{tr}\left[\delta \frac{\partial z_{3}}{\partial t}\left(\frac{\partial \Omega_{2}}{\left.\partial)^{\left(\frac{\partial Z_{3}}{3}\right.}\right) *}\right]^{\partial t}\right]\)
\(+\operatorname{tr}\left[\delta u_{3}^{-}\left(\frac{\partial \Omega_{2}}{\partial u_{j}^{\prime}}\right)_{*}^{\prime}\right]+\operatorname{tr}\left[\delta \lambda_{2}\left(\frac{\partial \Omega_{2}}{\partial \lambda_{2}}\right)_{*}^{\prime}\right]+\operatorname{tr}\left[\delta \lambda_{3}\left(\frac{\partial \Omega_{2}}{\partial \lambda_{3}}\right)_{*}^{*}\right]\)
\(\delta \Omega_{1}=\left\langle\delta z_{1}, \frac{\left(\partial \Omega_{1}\right)_{*}^{\partial z_{1}}}{*}\right\rangle+\left\langle\delta \frac{d z_{1}}{d t},\left(\frac{\partial \Omega_{1}}{\left.\frac{\partial\left(\frac{d z_{1}}{d t}\right.}{d t}\right)^{2}}\right\rangle\right.\)
\(+\left\langle\delta u_{1}, \frac{\left(\partial \Omega_{1}\right)_{*}}{\partial u_{1}}\right\rangle+\left\langle\delta u_{i}, \frac{\left(\partial \Omega_{1}\right)^{\partial u_{2}} *}{*}\right\rangle+\left\langle\delta \lambda_{1},\left(\frac{\left.\left.\left.\partial \Omega_{1}\right)^{\partial \lambda_{1}}\right)_{*}\right\rangle}{}\right.\right.\) C. 1 - 12

It is seen from C.1-2 and C.1-3 that
\(\left.\frac{\partial \Omega}{\partial\left(\frac{d z 1}{d t}\right)}\right|_{*}=-\lambda_{1}^{*}\)
\[
\begin{array}{lc}
\left.\frac{\partial \Omega_{2}}{\partial\left(\frac{\partial z_{2}}{-}\right)}\right|_{*}=-\lambda_{2}^{*} & \text { C.1-14 } \\
& \\
\left.\frac{\partial \Omega}{\partial\left(\frac{\partial z_{3}}{\partial t}\right)}\right|_{*}=-\lambda_{3}^{*} & \text { C.1-15 }
\end{array}
\]

These last three equations are substituted in C.1-11 and C.1-12 and then those equations put in C.1-7 gives.
\[
\left.+\operatorname{tr}\left[\begin{array}{lll}
\delta \lambda_{3} & \frac{\partial j L_{2}}{\partial \lambda_{3}} & \text { * }
\end{array}\right]\right\} d t d \tau
\]
\[
+\int_{t_{0}^{f}}^{t}\left\{<\delta z_{1},\left(\frac{\partial \Omega_{1}}{\partial z_{1}}\right)_{*}>-<\delta d z_{1}, \lambda_{1}^{*}>+<\delta u_{1}, \frac{\left(\frac{\partial \Omega_{1}}{\partial u_{1}} *\right.}{*}\right.
\]

Using integration by parts it can be shown that
\[
-\int_{t_{0}}^{t_{0}\left\langle\delta_{d z_{2}}^{d t}\right.}, \lambda_{1}^{\left.\lambda^{*}\right\rangle} d t=\int_{t_{0}^{f}}^{t_{1}}\left\langle\delta_{z_{1}}, \frac{d^{\lambda^{*}}}{d t} d t-\left\langle\delta z_{1}\left(t_{f}\right), \lambda_{1}^{*}\left(t_{f}\right)\right.\right.
\]
\[
\begin{aligned}
& +\operatorname{tr}\left[\frac{\delta \partial z_{3}^{\prime}}{\partial \tau}\left(-\lambda_{2}^{\prime}\right){ }_{*}\right]+\operatorname{tr}\left[\frac{\partial z_{3}}{\partial t}\left(-\lambda_{3}^{\prime}\right){ }_{*}\right]
\end{aligned}
\]
\(+\left\langle\delta z_{1}\left(t_{0}\right), \lambda_{1} *\left(t_{0}\right)\right\rangle\)
and.
\[
\begin{align*}
& -\int_{t_{0}}^{t_{f}} \operatorname{tr}\left[\delta z_{2}^{\prime}\left(t, t_{f}\right) \lambda_{z}^{\sim} *\left(t, t_{f}\right)\right] d t+\int_{t_{0}}^{t_{f}} \operatorname{tr}\left[\delta z_{2}^{\prime}\left(t, t_{0}\right) \lambda_{2}^{\sim} *\left(t, t_{0}\right)\right] d t
\end{align*}
\]
and
\[
\begin{aligned}
& \int_{t_{0} t_{0}}^{t_{f} t_{f}} \operatorname{tr}\left[\delta \frac{\partial z_{3}}{\partial t}\left[-\lambda_{3}^{-} *\right]\right] d t d \tau=\int_{t_{0}}^{t_{f} t_{f}} \int_{f} \operatorname{tr}\left[\delta z_{3} \frac{\partial \lambda_{3}}{\partial t}{ }^{*}\right] d t d \tau \\
& -\int_{t_{0}^{*}}^{t_{f}} \operatorname{tr}\left[\delta z_{3}\left(t_{f}, \tau\right) \lambda_{3}^{-*}\left(t_{f}, \tau\right)\right] d \tau+\int_{t_{0}}^{t_{f}} \operatorname{tr}\left[\delta z_{3}\left(t_{0}, \tau\right) \lambda_{3}^{-*}\left(t_{0}, \tau\right)\right] d \tau \quad \text { c.1-19 }
\end{aligned}
\]

These last three equations are substituted into C.I-16. Terms multiplied by the same arbitrary variation are collected and the fundamental theorem of tire calculus of variations is applied, \(i\), e. if \(\delta J=0\), then the coefficients multiplying each arbitrary variation must be zero. This results in
\[
\begin{aligned}
& \frac{\left(\partial \Omega_{\alpha}\right)}{\partial Z_{2}^{\prime} *}+\frac{\partial \lambda_{2}^{2}}{\partial \tau}=0 \\
& \frac{\left(\partial \Omega_{2}\right)^{-}}{\partial z_{3}}+{\frac{\partial \lambda_{3}}{\partial t^{*}}}^{*}=0 \\
& \left(\frac{\left.\partial \Omega_{1}\right)^{\partial}}{\partial z_{1}}+\frac{d \lambda}{d t} t^{*}=0\right. \\
& \frac{\left(\partial \Omega_{j}\right)^{\prime}}{\partial u_{3}^{\prime}}=0
\end{aligned}
\]
\[
\begin{align*}
& \frac{\left(\frac{\partial \Omega}{\partial u_{1}}\right)_{*}}{}=0 \\
& \frac{\left(\frac{\partial \Omega}{\partial u_{2}^{1}}\right)_{*}}{}=0 \\
& \frac{(\partial \Omega)^{\prime}}{\partial \lambda_{2}^{2}}{ }_{*}^{\prime}=0 \\
& \left.\frac{(\partial 2}{\partial \lambda_{3}^{2}}\right)_{*}^{\prime}=0 \\
& \frac{(\partial \Omega)^{2}}{\partial \lambda_{1}}=0 \\
& \text { and } \\
& \delta z_{1}\left(t_{0}\right)=\delta z_{1}\left(t_{f}\right)=0 \\
& \delta z_{z}^{\prime}\left(t, t_{0}\right)=\delta z_{z}^{\prime}\left(t, t_{f}\right)=0 \\
& \delta z_{j}\left(t_{0}, \tau\right)=\delta z_{3}\left(t_{f}, \tau\right)=0
\end{align*}
\]

Equation C.1-2l implies the specified endpoints of equations
4.1.1.3 and 4.1.1.4.

Equations C.1-20 become, for the functions \(\Omega_{2}\) and \(\Omega_{1}\) of C.1-2 and C.1-3,
\[
\frac{d \lambda^{*}}{d t^{2}}(t)=-A^{-}(t) \lambda_{1}^{*}(t), \forall t \in \Gamma_{1}
\]
\(\frac{\partial \lambda^{2}}{\partial \tau^{2}}(t, \tau)=-A^{-}(\tau) \lambda_{2}^{*}(t, \tau)-\lambda_{s}^{*} *(t, \tau) \subset(\tau), \forall(t, \tau) \in \Gamma\)
\(\frac{\partial \lambda^{j}}{\partial t^{j}}(t, \tau)=-A^{-}(t) \lambda_{3}^{*}(t, \tau)-R_{1}^{p} R_{1} z_{3}^{*}(t, \tau), \forall(t, \tau) \in \Gamma\)
\(\frac{d z}{d t^{1}} *(t)=A(t) Z_{1}^{k}(t)+B(t) u_{1}^{*}(t)+C(t) u_{2}^{*}(t), \forall t \in \Gamma_{1}\)
\[
\begin{array}{ll}
\frac{\partial z^{*}}{\partial t^{*}}(t, \tau)=A(\tau) z_{2}^{\sim *}(t, \tau)+C(\tau) u_{0}^{*}(t, \tau), \forall(t, \tau) \in \Gamma & C .1-26 \\
\frac{\partial z}{\partial t^{*}} *(t, \tau)=A(t) z_{3}^{*}(t, \tau)+C(t) z_{2}^{*}(t, \tau), \forall(t, \tau) \in \Gamma & C .1-27 \\
u_{1}^{*}(t)=-R_{3}^{-1} B^{-}(t) \lambda_{1}^{*}(t), \forall t \in \Gamma & \text { C.1-28 } \\
& \\
u_{2}^{*} *(t)=R_{4}^{-1} C^{-}(t) \lambda_{1}^{*}(t), \forall t \in \Gamma & C .1-29 \\
u_{3}^{*} *(t, \tau)=C^{-}(\tau) \lambda_{2}^{*}(t, \tau)\left(R_{2}^{-1}\right)\left(R_{2}^{-1}\right), \forall(t, \tau) \in \Gamma & C .1-30
\end{array}
\]
which are the necessary conditions stated in Theorem 4.1. It is seen that the inverses of \(R_{<}, R_{s}\) and \(R_{4}\) are required, but not for \(R_{1}\), hence
\(R_{1}\) is assumed to be non-negative definite
and
\(R_{4}, R_{s}, R_{4}\) are assumed to be positive definite.

Renark: Substitution of equations C.1-28, C.1-29 and C.1-30 into C.1-25 and C.1-26 result in \(2 \mathrm{n}(\mathbf{1}+\mathrm{s}+\mathrm{m})\) differential equations to be solved. There are \(n(1+s+n)\) endpoint conditions specified by 4.1.1.3 and \(n(1+s+n)\) more by 4.1.1.4. So the complete solution of the problem requires the solution of a \(2 n(l+s+n)\) dimension two point boundary value problem (TPBVP) since half of the endpoint conditions are specified at \(t_{0}\) and the other half at \(t_{f}\).

\section*{C. 2 Proof of Theorem 4.2}

For Problem 2 of Section 4.1.2 defined by equations 4.1.1.2 through 4.1.1.8 with the performance index of 4.1.2.1 define
\[
\lambda_{1}(t) \text { an } n \times 1 \text { vector }
\]
\(\lambda_{2}(t, \tau)\) an \(n \times s\) matrix
\[
\lambda_{y}(t, \tau) \text { an } n \times n \text { matrix }
\]
and fonn the augmented performance index
where
\[
\begin{aligned}
& \Omega=\Omega\left(z_{1}(t), z_{1}(\tau), z_{2}^{\prime}(t, \tau), z_{3}(t, \tau), \frac{d z_{1}}{d t}(t), \frac{\partial z_{j}^{\prime}}{\partial \tau}(t, \tau), \frac{\partial z_{j}}{\partial t}(t, \tau),\right. \\
& \\
& \left.u_{1}(t), u_{1}(\tau), u_{2}(t), u_{2}(\tau), u_{3}^{\prime}(t, \tau), \lambda_{1}(t), \lambda_{2}(t, \tau), \lambda_{3}(t, \tau)\right)
\end{aligned}
\]
and \(\Omega\) is defined in \(\mathrm{C} .2-1\), and it is noted that \(J a=J\) of equation 4.1.2.1.

In addition to the variations in C.1-5, define
\[
\begin{aligned}
& \delta \mathbf{z}_{1}(\tau)=\left.\delta \mathbf{z}_{1}(t)\right|_{t=\tau} \\
& \delta u_{1}(\tau)=\left.\delta u_{1}(t)\right|_{t=\tau}
\end{aligned}
\]
\[
\delta u_{c}(\tau)=\left.\delta u_{c}(t)\right|_{t=\tau}
\]
\[
\begin{align*}
& J a=\int_{t_{0} f t_{0}}^{t} f_{\left\{f o\left(z_{1}(t), z_{1}(\tau), z_{2}^{\prime}(t, \tau), z_{3}(t, \tau), u_{1}(t), u_{1}(\tau), u_{2}(t), u_{2}(\tau), u_{3}^{\prime}(t, \tau)\right)\right.} \\
& +\operatorname{tr}\left[\left[A(\tau) z_{2}^{\prime}(t, \tau)+C(\tau) u_{3}^{\prime}(t, \tau)-\frac{\partial z_{2}^{\prime}}{\partial \tau}(t, \tau)\right]\left[\lambda_{2}^{\prime}(t, \tau)\right]\right] \\
& +\operatorname{tr}\left[\left[A(t) z_{3}(t, \tau) C(t) z_{2}(t, \tau)-\frac{\partial z_{3}}{\partial t}(t, \tau)\right]\left[\lambda_{3}^{-}(t, \tau)\right]\right] \\
& +\left\langle A(t) z_{1}(t)+B(t) u_{1}(t)+C(t) u_{2}(t)-\frac{d z_{1}}{d t}(t), \lambda_{1}(t)>\right\} d t d \tau \\
& \text { or } \\
& J a=\int_{t_{0} t_{0}^{t}}^{t} \int_{0}^{t} \Omega d t d \tau
\end{align*}
\]

Then,
\[
\delta J a=\delta J=\int_{t_{0}}^{t^{f} t_{0}} f_{\delta} \delta \Omega d t d \tau
\]
where
\[
\begin{aligned}
& \left.\delta \Omega=\operatorname{tr} \delta z_{2}^{-}\left(\frac{\partial \Omega}{\partial z_{2}^{\prime}}\right)^{\prime}+\operatorname{tr} \delta z_{j}\left(\frac{\partial \Omega}{\partial z_{3}}\right)^{\prime}+\operatorname{tr} \delta \frac{\partial z_{j}^{-}}{\partial \tau} \frac{\left(\frac{\partial \Omega}{\partial\left(\partial z_{z}^{-}\right.}\right)^{\partial \tau}}{}\right)^{\prime} \\
& \left.+\operatorname{tr} \delta \frac{\partial z}{\partial t}, \frac{(\partial \Omega}{\partial\left(\frac{\partial z}{\partial t}\right)^{\prime}}\right)^{\prime}+\operatorname{tr}^{\delta} u_{3}^{\sim}\left(\frac{\partial \Omega}{\partial u_{3}^{\prime}}\right)^{\prime}+\operatorname{tr} \delta \lambda_{2}\left(\frac{\partial \Omega}{\partial \lambda_{2}}\right)^{\prime} \\
& +\operatorname{tr} \delta \lambda_{3}\left(\frac{\partial \Omega}{\partial \lambda_{3}}\right)^{\prime}+\left\langle\delta z_{1}(t), \frac{\partial \Omega}{\partial z_{1}}(t)\right\rangle+\left\langle\delta z_{1}(\tau), \frac{\partial \Omega}{\partial z_{1}}(\tau)\right\rangle \\
& +\left\langle\delta \frac{d z_{1}}{d t}(t), \frac{\partial \Omega}{\partial\left(\frac{d z_{1}}{d t}\right.}(t)\right)>+\left\langle\delta u_{1}(t), \frac{\partial \Omega}{\partial u_{1}}(t)\right\rangle+\left\langle\delta u_{1}(\tau), \frac{\partial \Omega}{\partial u_{1}}(\tau)\right\rangle
\end{aligned}
\]

Since fo does not depend on \(\frac{d z}{d t}(t), \frac{\partial z_{j}^{\prime}}{\partial \tau}(t, \tau)\) or \(\frac{\partial z_{j}}{\partial t}(t, \tau)\)
\[
\begin{array}{ll}
\frac{\left(\frac{\partial \Omega}{\partial\left(\frac{\partial z^{\prime}}{\partial \tau}\right)}\right)^{\prime}=-\lambda_{2}^{\prime}(t, \tau)}{} & \text { C.2-6 } \\
\left(\frac{\partial \Omega}{\partial\left(\frac{\partial z_{3}}{\partial t}\right)}\right)^{\prime}=-\lambda_{3}^{\prime}(t, \tau) & \text { C.2-7 } \\
\frac{\partial \Omega}{\partial\left(\frac{d z_{3}}{d t^{\prime}}\right)}=-\lambda_{4}(t) & \text { C.2-8 }
\end{array}
\]

Substituting C.2-6, C.2-7, and C.2-8 into C.2-5, and then that equation into C.2-4 results in 3 terms to be integrated by parts similar to C.1-17, C.1-18, and C.1-19. After this step, terms are collected, \(\delta J\) is set to zero, and the fundamental theorem of the calculus of variations applied, which results in the following necessary conditions

\(\frac{\partial \lambda_{2}}{\partial \tau}(t, \tau)=-A^{\wedge}(\tau) \lambda_{2}(t, \tau)-\lambda_{s}^{-}(t, \tau) C(\tau)-\frac{\partial f_{0}}{\partial z_{2}^{\prime}}\)
\(\frac{\partial \lambda_{3}}{\partial t}(t, \tau)=-A^{\prime}(t) \lambda_{3}(t, \tau)-\frac{\partial f_{0}}{\partial z_{3}}\)
\(\frac{d z_{1}}{d t}(t)=A(t) z_{1}(t)+B(t) u_{1}(t)+C(t) u_{2}(t)\)
\(\frac{\partial z_{2}^{-}}{\partial \tau}(t, \tau)=A(\tau) z_{2}^{\sim}(t, \tau)+C(\tau) u_{3}^{\beta}(t, \tau)\)
\(\frac{\partial z_{3}}{\partial t}(t, \tau)=A(t) z_{3}(t, \tau)+C(t) z_{2}(t, \tau)\)
\(\left.\int_{t_{0}}^{t_{f}} \frac{\partial f_{0}}{\partial u_{1}}(t)+B^{\prime}(t) \lambda_{1}(t)\right\} d \tau=0, \quad \int_{t_{0}}^{t_{0}}\left\{\frac{\partial f_{0}}{\partial u_{1}(\tau)}\right\} d t=0\)
\(\left.\left.\left.\int_{t_{0}}^{t_{\left\{\frac{\partial f o}{}\right.}^{\partial u_{i 2}(t)}}+C^{\prime}(t) \lambda_{1}(t)\right\} d \tau=0, \int_{t_{0}}^{t^{f^{\prime}}\left\{\frac{\partial f_{0}}{\partial u_{\cdot 2}}(\tau)\right.}\right\}\right\} d t=0\)
\(\frac{\partial f o}{\partial u_{3}^{\prime}}+C^{\prime}(\tau) \lambda_{2}^{-}(t, \tau)=0\)
with the endpoint conditions of C.1-21.

Remark: The equations C.2-9, C.2-15, and C.2-16 can be shown to not present constraint difficulties for the following examples. Only the case for C. 2-15 is given.

Consider four cases.
Case 1: fo \(\left(u_{1}(t), u_{1}(\tau)\right)<u_{1}(t), R_{3} u_{1}(t)>, R_{3}=R_{3}^{-}\) \(\frac{\partial \mathrm{ffo}_{0}}{\partial u_{1}}(\tau)=0\)
\[
\frac{\partial \text { fo }}{\partial u_{1}}(t)=2 R_{3} u_{1}(t)
\]
and C.2-15 becomes
\[
u_{1}(t)=\frac{-1}{2} R_{1}^{-1} B^{-}(t) \lambda_{1}(t)
\]

Case 2: \(f \circ\left(u_{1}(t), u_{i}(\tau)\right)=<u_{1}(t), R_{y} u_{L}(t)>W(t) W(\tau), R_{y}=R_{y}^{\prime}\) where \(W(\).\() is a scalar, time-varying weighting function\)

Then
\[
\begin{aligned}
& \frac{\partial \mathrm{fo}}{\partial u_{1}}(\tau)=0 \\
& \frac{\partial \mathrm{fo}}{\partial u_{1}}(t)=2 \mathrm{R}_{3} W(t) W(\tau) u_{1}(t)
\end{aligned}
\]
and C.2-15 becomes
\[
u_{1}(t)=\frac{-2\left(t_{f}-t_{0}\right)}{W(t) \int_{t_{0}}^{t^{t}} W(\tau) d \tau} \quad R_{3}^{-1} B^{\prime}(t) \lambda_{1}(t)
\]

Case 3: fo \(\left(u_{1}(t), u_{1}(\tau)\right)=<u_{1}(t), R_{3} u_{1}(\tau)>, R_{3}=R_{3}^{e}\)
Then
\[
\begin{aligned}
& \frac{\partial f_{0}}{\partial u_{1}}(\tau)=R_{3}^{j} u_{1}(t) \\
& \frac{\partial f_{0}}{\partial u_{1}}(t)=R_{3} u_{1}(\tau)
\end{aligned}
\]
and C.2-15 becames
\[
\int_{t_{0}}^{t_{u}} f_{u(\tau) d \tau}=\frac{-\left(t_{f}-t_{0}\right)}{2} R_{3}^{+} ; B^{-}(t) \lambda_{1}(t)
\]

Case 4: fo \(\left(u_{1}(t), u_{i}(\tau)\right)=\left\langle u_{1}(t), R_{3}(t, \tau) u_{1}(\tau)\right\rangle\)
\[
\text { with } R_{y}(t, \tau)=R_{3}^{\prime}(\tau, t)
\]
then C.2-15 becomes
\[
\int_{t_{0}^{t}}^{f_{R_{j}}}(t, \tau) u(\tau) d \tau=\frac{-\left(t_{f}-t_{0}\right) B^{-}(t) \lambda_{i}(t)}{2}
\]

\section*{C. 3 Proof of. Theorem 4.3}

The proof is an extension of derivations in Bryson and Ho (20) and Gabasov and Kirillova (28). In (20) the actual derivation is carried out for a one sided problem with a scalar control in one independent variable, \(t\). The result is
\[
\begin{array}{ll}
\frac{d}{d t} \frac{\partial H}{\partial u}=0 & \text { C.3-1 } \\
\frac{\partial}{\partial u} \frac{d}{d t} \frac{\partial H}{\partial u}=0 & \text { C.3-2 } \\
\frac{\partial}{\partial u}\left\{\frac{d^{2}}{d t^{2}} \frac{\partial H}{\partial u}\right\} \leq 0 & \text { C.3-3 }
\end{array}
\]
where H is the Hamiltonian for the one-sided problem and minimization of cost with respect to the scalar \(u\) is carried out. The forn of C.3-3 is similar to the form of the Legendre-Clebsch condition
\[
\begin{array}{ll}
H_{u u} \geq 0 & \text { C. 3-4 }
\end{array}
\]

For singular problems, the equality holds in C.3-4 therefore no information is obtained from this condition. For nonsingular problems C.3-4 implies the sign of the second variation of the augmented performance index thereby indicating whether minimization or maximization is achieved.

When equality holds in C.3-4, the test for maximization or minimization is provided by C.3-3.

The results for a vector control \(u\) are quoted in (28) and were derived in Kelley (43) and Kelley, Kopp and Moyer (42). The equations of Theorem 4.3 are a generalization of the above for a matrix control \(u_{3}^{\prime}\) in two independent variables, \(t\) and \(\tau\) where maximization with respect to \(u_{3}\) is desired.

The proof is extremely lengthy but similar in form to the derivation in Bryson and Ho (20). The second variation is found, and then the optimal state. equations and costate equations are perturbed resulting in the comparison of neighboring trajectories, i.e. variations in the state and costate are examined. It is seen that equations 4.2.3.1 through 4.2.3.3 are equivalent to C.3-1 through C.3-3 if \(u_{3}\) is restricted to be a scalar function of one independent variable, \(t\) realizing that maximization rather than minimization with respect to \(u \hat{3}\) is sought.

\section*{C. 4 Proof of Theorem 4.4}

Define the Hamiltonian
\[
H\left(z_{1}, z_{2}, z_{3}, u_{1}, u_{2}, u_{3}, \lambda_{1}, \lambda_{2}, \lambda_{3}\right)=
\]
\[
\frac{d_{1}}{4} u_{1}^{\prime}(t) R_{1}(t) u_{1}(t)+\frac{d_{2}}{4} \operatorname{tr}\left\{R_{2}(t)\left[u_{3}(t)+u_{2}(t) u_{2}^{\prime}(t)\right]^{2}\right\}
\]
\[
+\lambda_{1}^{-}(t)\left[A(t) z_{1}(t)+B(t) u_{1}(t)+C(t) u_{2}(t)\right]
\]
\[
+\operatorname{tr}\left\{\left[F(t) z_{2}(t)+z_{2}(t) A^{-}(t)+u_{3}(t) C^{-}(t)\right] \lambda_{2}^{-}(t)\right\}
\]
\[
\left.\left.+\operatorname{tr}\left\{\left[A(t) z_{3}(t)+z_{3}(t) A^{-}(t)+C(t) z_{2}(t)+z_{2}^{-}(t) C^{-}\right) t\right)\right] \lambda_{3}^{-}(t)\right\} \quad C .4-1
\]
where the vector \(\lambda_{1}(t)\) and the matrixes \(\lambda_{2}(t), \lambda_{3}(t)\) are Lagrange multipliers. With the aid of the matrix maximum principle of Athans (10), the following result is obtained. In order that \(u_{i}{ }^{*}(t) \in U_{i}, t \in r_{1}\), \(i=1,2,3\) be the optimal strategies for Problem 4, it is necessary that there exist a nonzero vector function \(\lambda_{1}{ }^{*}(t)\) and nonzero matrix functions \(\lambda_{2}{ }^{*}(t)\) and \(\lambda_{3}{ }^{*}(t)\) such that conditions a), b) and c) of Theorem 4.4 hold, as given in Section 4.2.4.

Froof:
Define the variations
\[
\begin{align*}
& z_{i}(t)=z_{i}^{*}(t)+\delta z_{i}(t) \\
& u_{i}(t)=u_{i}^{*}(t)+\delta u_{i}(t)
\end{align*}
\]
and using the standard variational approach, it can be shown that
\[
\begin{aligned}
& \delta J a=\int_{t_{0}}^{f_{f}}\left[\left(\frac{\partial H)}{\partial z_{1}} *+\frac{d \lambda_{1}}{d t} *\right]-\delta z_{1}(t)+\operatorname{tr}\left[\delta z_{2}^{-}\left(\frac{(\partial H)}{\partial z_{2}} *+\frac{d \lambda_{2}}{d t}\right)\right]\right. \\
& +\operatorname{tr}\left[\delta z_{3}^{\prime}\left(\frac{(\partial \mathrm{F})}{\partial z_{3}} *+\frac{d \lambda_{3}}{d t}\right)\right]-\lambda_{1}^{-*}\left(t_{f}\right) \delta z_{1}\left(t_{f}\right)
\end{aligned}
\]
\[
-\operatorname{tr}\left[\lambda_{2} *\left(t_{f}\right) \delta z_{2}\left(t_{f}\right)\right]-\operatorname{tr}\left[\lambda_{3} *\left(t_{f}\right) \delta z_{3}\left(t_{f}\right)\right]
\]
where ( \(\cdot\) * denotes that the function (•) is evaluated at
\(z_{i}=z_{i}{ }^{*}, u_{i}=u_{i}{ }^{*}\) and \(\lambda_{i}=\lambda_{i}{ }^{*}, i=1,2,3\). If the state equations in 4.2.4.1 through 4.2.4.3 are satisfied, and \(\lambda_{i}{ }^{*}(t), i=1,2,3\) are selected so that the coefficients of \(\delta z_{i}(t), i=1,2,3\) in the integral are identically zero, and the boundary conditions of 4.2.4.7 and
4.2.4.8 are satisfied, then
\(\delta J_{a}=\int_{t_{0}}^{t_{f}^{f}} \frac{(\partial H)}{\partial u_{1}}{ }_{i}^{\prime} \delta u_{1}+\frac{(\partial H)}{\partial u_{2}}{ }^{\delta} \delta u_{2}+\operatorname{tr}\left[\delta u_{3}^{\prime}\left(\frac{\partial H)}{\partial u_{3}}{ }_{3}\right]\right\} d t\)
To the first order approximation
\[
\frac{(\partial H)}{\partial u}{ }^{*} \delta u_{1}(t)=H\left(z_{1}{ }^{*}, z_{2}^{*}, z_{3}{ }^{*}, u_{1}{ }^{*}+\delta u_{1}, u_{2}^{*}, u_{3}^{*}, \lambda_{1}{ }^{*}, \lambda_{2}^{*}, \lambda_{3}^{*}\right)-H^{*} \quad \text { C.4-5 }
\]

Therefore, with \(u_{1}=u_{1} *+\delta u_{1}, u_{2}=u_{2} *, u_{3}=u_{3} *\)
\(\delta J_{a}={\underset{t}{t_{0}}}_{t^{f}}^{\left(\frac{\partial H}{\partial u_{1}}\right.}{ }^{-\delta} \delta u_{1} d t=\)
\[
\int^{t}\left\{H_{2}-H^{\star}\right\} d t
\]
C. 4-6

Define a sufficiently small neighborhood of \(u_{i}^{*}\) as \(\delta U_{i}=\left\{\delta U_{i}:\left\|\delta u_{i}\right\|<\beta_{i}\right.\) and \(\left.u_{i}^{*}+\delta u_{i} U_{i}\right\} \quad\) C.4-7
for \(i=1,2,3\) where the \(\beta_{i}\) are positive constants. For \(u_{1} *\) to be a minimizing strategy it is necessary that
\[
\delta J_{a}\left(u_{1}^{*}, \delta u_{1}\right)=\int_{t_{0}}^{t}\left\{H_{1}-H^{*}\right\} d t \geqq 0
\]
for all \(\delta u_{1} \epsilon \delta U_{1}\). It can be shown as in Kirk (45) that in order for C.4-8 to be satisfied for all \(\delta u_{1} \in \delta U_{1}\), it is necessary that \(\mathrm{H}_{2} \geqq \mathrm{H}^{*}\)

In a similar manner, with \(u_{1}=u_{1}{ }^{*}, u_{2}=u_{2}{ }^{*}+\delta u_{2}, u_{3}=u_{3} *\) \(+\delta u_{3}\) where \(u_{2}\) and \(u_{3}\) are maximizing strategies, it can be shown that
\(\mathrm{H}_{2} \leqq \mathrm{H}^{*}\) C. 4-10
and combining C.4-9 and C.4-10 yields conditions b) and c) of Theorem 4.4.

\section*{APPENDIX D}

\section*{SUFFICIENT CONDITIONS}

In one-sided deterministic optimal control problems sufficient conditions determine whether the extremal solution obtained by the necessary conditions is a minimizing or maximizing solution. The technique often used is to see if the second variation of the augmented performance index is positive definite implying a minimizing extremal solution or negative definite implying a maximizing solution. In this approach, the restrictions on the variations lead to local sufficient conditions. This means the extremal solution is compared only to neighboring values of this solution. Global solutions can be obtained by comparing the value of the solution to all possible values, not only those due to neighboring solutions. The theory of convex functions is most often used to establish global sufficient conditions.

In two-sided deterministic optimal control problems, which are two player zero-sum differential games, the second variation technique is not as useful as the notions of convexity and concavity. The latter techniques were given for nonlinear programing problems in Saaty and Bram (94) and Wilde and Beightler (107). The approach of convex and concave functions was used to establish global sufficient conditions for a wide class of differential games in Kuo and Burbank (55) and Kuo (52).

The definition of a saddle point for the functional performance index of equation 5.1.3 is
\[
\begin{align*}
& J\left(z_{1}{ }^{*}, z_{<}^{*}, z_{s}{ }^{*}, u_{1}{ }^{*}, u_{c}{ }^{*}, u_{s}^{\prime *}, \lambda_{1}{ }^{*}, \lambda_{2}{ }^{*}, \lambda_{s}{ }^{*}\right) \mathrm{IJ}^{*} \leq \\
& J\left(z_{1}, z_{<}^{\prime}, z_{s}, u_{1}, u_{L}{ }^{*}, u_{s}^{*}{ }^{*}, \lambda_{L}{ }^{*}, \lambda_{c}{ }^{*}, \lambda_{s}{ }^{*}\right) \Delta_{\Delta}
\end{align*}
\]
where now ( )* denotes optimal, not extremal. This equation can be written as the two separate inequalities
\[
J_{<} J^{\star}<0 \quad \text { D.1-2 }
\]
and
\[
J_{1}-J^{*}>0
\]

Recalling that it is sought to establish
\(\min \max J\left(u_{1} ; u_{c}, u_{3}^{\prime}\right)\)
\(u_{i} \in U_{i} u_{<} \in U_{L}\)
\(u_{s}^{\prime} \in U_{s}\)
the inequalities can be described. \(J_{2}\) is \(J\) evaluated at \(u_{1}=u_{2} *\), the optimal condition, and at \(u_{c}, u_{y}^{\prime}\) some non optimal conditions. The resulting trajectories for \(z_{A}, z_{c}^{\prime}\), and \(z\), will not be optimal since \(u_{c}, u_{j}^{\prime}\) are not. Since \(u_{c}\) and \(u_{3}^{\prime}\) seek to maximize \(J\), when they are not optimal the value of \(J\) should be less than the value \(J^{*}\) which is the maximm attainable at all optimal conditions.

Similarly, \(J_{1}\) is at optimal \(u_{2}{ }^{*}, u_{s}^{* *}\) and non-optimal \(u_{1}\). Since \(u_{1}\) minimizes, the value of \(J\) at non-optimal \(u_{\perp}\) should be greater than the minimum attainable at all optimal conditions, \(J^{*}\). If the inequalities D.1-2 can be established for all non-optimal \(u_{c}, u_{s}^{\prime}\) and D.1-3 for all non-optimal \(u_{1}\), then global sufficient conditions have been found.
D. 1 proof of Theorems 5.1.1 and 5.1.2

Proof of Theorem 5.1.1:
First half:
It is sought to establish D.1-2 for the augmented performance
index of equation 5.1.2.From D.1-1,
\[
\begin{align*}
& \left.\Omega_{2}\left(z_{2}^{\prime *}, z_{3}{ }^{*}, \frac{\partial z_{2}^{*}}{\partial \tau}, \frac{\partial z_{3}}{\partial t}{ }^{*}, u_{3}^{*} *, \lambda_{2}{ }^{*}, \lambda_{3}{ }^{*}\right)\right\} d t d \tau \\
& +\int_{t_{0}}^{t_{f}}\left\{\Omega_{1}\left(z_{1}, \frac{d z_{1}}{d t}, u_{1}{ }^{*}, u_{2}, \lambda_{1} *\right)-\Omega_{1}\left(z_{1} *, \frac{d z}{d t}{ }^{*}, u_{1}{ }^{*}, u_{2}{ }^{*}, \lambda_{1} *\right)\right\} d t
\end{align*}
\]
where \(\Omega_{2}\) and \(\Omega_{1}\) are defined in equations C.1-2 and C.1-3. Using the properties of concavity it can be shown that
\[
\begin{aligned}
& \text { D.1-6 }
\end{aligned}
\]

But, from C.1-20 and C.1-30
\[
\begin{aligned}
& \frac{\left(\partial \Omega_{2}\right)^{\prime}}{\partial z_{2}^{2}}=\lambda_{2}^{-*}(t, \tau) A(\tau)+C^{-}(t) \lambda_{3}^{*}(t, \tau) \\
& \frac{\left(\partial \Omega_{2}\right.}{\partial\left(\frac{\partial z_{2}^{*}}{\partial \tau}\right)^{*}}=-\lambda_{2}^{\beta *}(t, \tau) \\
& \left.\frac{\left(\partial \Omega \Omega_{2}\right.}{\partial\left(\frac{\partial z_{3}}{\partial t}\right.}\right)_{*}^{n}=-\lambda_{3}^{*}(t, \tau) \\
& \left.\frac{\left(\partial \Omega_{2}\right)}{\partial z_{3}}\right)_{*}=-z_{3}^{\prime} *(t, \tau) R_{1}^{\rho} R_{1}+\lambda_{3}^{-} *(t, \tau) A(t)
\end{aligned}
\]
and
\[
\frac{\left(\partial \Omega_{2}\right)^{\prime}}{\partial u_{3}^{\prime}}=\left[C^{-}(\tau) \lambda_{2}^{-*}(t, \tau)-u_{3}^{-} *(t, \tau) R_{2}^{-} R_{2}\right]^{\rho}=0
\]

After substituting, D.1-6 becomes
\[
\begin{aligned}
& \Omega_{2}\left(z_{2}^{\prime}, z_{3}, \frac{\partial z_{2}^{\prime}}{\partial \tau}, \frac{\partial z_{3}}{\partial t}, u_{3}^{\prime}, \lambda_{2}^{*}, \lambda_{3}^{*}\right)-\Omega_{2}\left(z_{2}^{\prime *}, z_{3}^{*}, \frac{\partial z^{\circ}}{\partial \tau^{2}}, \frac{\partial z_{3}^{*}}{\partial t}, u_{3}^{-} *, \lambda_{2}^{*}, \lambda_{3}^{*}\right) \leq \\
& +\operatorname{tr}\left[\delta z_{2}^{\prime}\left[\lambda_{2}^{-*}(t, \tau) A(\tau)+C^{\prime}(\tau) \lambda_{3}^{*}\left(t_{p} \tau\right)\right]\right]-\operatorname{tr}\left[\delta\left(\frac{\partial z_{2}^{\prime}}{\partial \tau}\right) \lambda_{2}^{\prime} *(t, \tau)\right]
\end{aligned}
\]
\[
-\operatorname{tr}\left[\delta\left(\frac{\partial z_{3}}{\partial t}\right) \lambda_{3}^{*}(t, \tau)\right]+\operatorname{tr}\left[\delta z_{3}\left[-z_{3}^{*} *(t, \tau) R_{1} R_{1}+\lambda_{3}^{*} *(t, \tau) A(t)\right]\right]
\]

Using the endpoint conditions
\[
\delta z_{\dot{2}}^{\prime}\left(t, \dot{\tau}_{o}\right)=\delta z_{2}^{-}\left(t, t_{f}\right)=\delta z_{s}\left(t_{0}, \tau\right)=\delta z_{s}\left(t_{f}, \tau\right)=0
\]
and integration by parts it can be shown that
and

Substituting D.1-7 into D.1-5, and then substituting D.1-8 and D.1-9 gives
in view of C.1-23 and C.1-24.
Using the properties of concavity, and integration by parts
\[
\int_{t_{0}}^{t}\left\{\Omega_{1}\left(z_{1}, \frac{d z_{1}}{d t}, u_{1} *, u_{2}, \lambda_{1}^{*}\right)-\Omega_{1}\left(z_{1} *, \frac{d z_{1}}{d t}{ }^{*}, u_{1} *, u_{2}^{*}, \lambda_{1} *\right)\right\} d t \leq
\]
\[
\left.\int_{t_{0}}^{t_{f}}\left\{<\delta z_{1}, \frac{\left(\frac{\partial \Omega_{1}}{\partial z_{1}}\right)_{*}>+<\delta \frac{d z_{1}}{d t}}{}, \frac{\left(\frac{\partial \Omega}{\partial\left(d_{1}\right.}\right)_{1}^{d t}}{d t}\right)_{*}>+<\delta u_{2}{\frac{\partial \Omega \Omega_{2}}{\partial u_{2}}}_{*}>\right\} d t=
\]
\[
\left.\int_{t_{0}}^{t}\left\langle\delta z_{1}, A^{-}(t) \lambda_{1} *(t)+{\frac{d \lambda^{2}}{d t}}_{1} *\right\rangle+\left\langle\delta u_{L_{2}},-R_{4} u_{2} *(t)+C^{-}(t) \lambda_{2} *(t)\right\rangle\right\} d t=0
\]
in view of C.1-22, C.1-25 and C.1-29. From D.1-10 and D.1-11, D.1-2 is established.
\[
\begin{align*}
& \left.-\Omega_{2}\left(z_{2}^{*}{ }^{*}, z_{3}{ }^{*}, \frac{\partial z_{2}^{*}}{\partial \tau}{ }^{*}, \frac{\partial z}{\partial t}{ }^{*}, u_{3}^{0 *}, \lambda_{2}{ }^{*}, \lambda_{3}{ }^{*}\right)\right\} d t d \tau \leq 0
\end{align*}
\]

Second half:
It is sought to establish D.1-3 for the augmented performance
index of 5.1.2. From D.1-1,
\[
\begin{align*}
& \left.\Omega_{L_{2}}\left(z_{2}^{*}{ }^{*}, z_{3}^{*}, \frac{\partial z_{i}^{*} *}{\partial \tau}, \frac{\partial z_{3}}{\partial t}{ }^{*}, u_{3}^{*}{ }^{*}, \lambda_{2}{ }^{*}, \lambda_{3}{ }^{*}\right)\right\} d t d \tau \\
& +\int_{t_{0}}^{t_{f}}\left\{\Omega_{1}\left(z_{1}, \frac{d z_{1}}{d t}, u_{1}, u_{2} *, \lambda_{1} *\right)-\Omega_{1}\left(z_{1} *, \frac{d z_{1}}{d t} *, u_{1} *, u_{2} *, \lambda_{1} *\right)\right\} d t
\end{align*}
\]
and from C.1-26 and C.1-27, if \(u_{3}^{\prime}\) is \(u_{3}^{\prime *}\), then
\[
z_{\alpha}^{\prime}, z_{3}, \frac{\partial z_{2}^{\prime}}{\partial \tau}, \text { and } \frac{\partial z_{3}}{\partial t} \text { are evaluated at optimal conditions. }
\]

Therefore D.1-12 goes to
\[
J_{1}-J^{*}=\int_{t_{0}}^{t} f\left\{\Omega_{1}\left(z_{1}, \frac{d z_{1}}{d t}, u_{1}, u_{2} *, \lambda_{1}^{*}\right)-\Omega_{1}\left(z_{1} *, \frac{d z_{1}}{d t} *, u_{1} *, u_{2} *, \lambda_{1} *\right)\right\} d t
\]

Using the properties of convexity it can be shown that
\[
\begin{align*}
& \Omega_{1}\left(z_{1}, \frac{d z_{1}}{d t}, u_{1}, u_{2} *_{,} \lambda_{1}^{*}\right)-\Omega_{1}\left(z_{1} *, \frac{d z_{1}}{d t} *, u_{1}^{*}, u_{2}^{*}, \lambda_{1}^{*}\right) \geq \\
& \left\langle\delta u_{1}, \frac{\partial \Omega_{1}}{\partial u_{1}}\right)_{*}>+<\delta z_{1},\left(\frac{\partial \Omega_{1}}{\partial z_{1}}\right)_{*}>+\left\langle\delta\left(\frac{d z_{1}}{d t}\right),\left(\frac{\partial \Omega_{1}}{\partial\left(\frac{d z_{1}}{d t}\right)}\right) *\right.
\end{align*}
\]

From C.1-20 and C.1-28
\[
\left.\frac{\left(\partial \Omega_{1}\right.}{\partial u}\right)_{*}=0
\]
then substituting D.1-15 into D.1-14 and the result into
D.1-13 gives

Using the endpoint condition
\[
\delta z_{1}\left(t_{0}\right)=\delta z_{1}\left(t_{f}\right)=0
\]
and integration by parts, it can be shown that
\[
J_{1}-J^{*} \underset{J_{t_{0}}^{t}}{t_{0}}\left\langle\delta z_{1}, \frac{\left(\frac{\partial \Omega_{1}}{\partial z_{1}}\right)}{}{ }^{+\frac{d \lambda_{1}}{d t}} *\right\rangle d t=0
\]
in view of C.1-22 and C.1-25, thereby establishing D.1-3.
This completes the proof of Theorem 5.1.1.

Proof of Theorem 5.1.2:
The statement of Theorem 5.1.1 as given by equations 5.1.1.1 and 5.1.1.2 follows directly from the proof of Theorem 5.1.1 in this Appendix.

In Saaty and Bram (94) and Parthasarathy and Raghavan (85) it is shown that the necessary and sufficient conditions such that 5.1.1.1 holds are that
\(R_{3}, R_{4}\) are non-negative definite and symmetric.
Since \(R_{3}\) and \(R_{4}\) are assumed to be positive definite and symmetric for Theorem 4.1, they are non-negative definite, hence 5.1.1.1 holds true.

After a lengthy rearrangement of the \(s \times s\) matrix \(u_{3}^{-}\)into an \(s^{2} \times 1\) vector of the colums of \(u_{3}^{\prime}\) and similar rearrangements of \(z_{3}, R_{1}\) and \(R_{2}\), it is seen that the
tr \(R_{2} u_{3} u_{3}^{\prime} R_{2}^{\prime}\) and \(t x R_{1} z_{3} z_{3}^{\prime} R_{1}^{R}\) functions are
equivalent to inner products of the columns of \(u_{3}^{\prime}\) and \(z_{3}\). Using theorems from Hohn (36) and Graybill (32), the weighting matrices of these inner products can be proven to be positive definite or positive semidefinite if \(R_{1}\) and \(R_{\alpha}\) are positive definite or positive semidefinite respectively. Then the same approach is used to establish 5.1.1.2 as was done for 5.1.1.1, the result being
\(R_{1}, R_{2}\) are non-negative definite and symmetric.

Since \(R_{1}\) is assumed positive semidefinite and \(R_{2}\) positive definite, and both symmetric, from Theorem 4.1, equation 5.1.1.2 holds true, and the proof of Theorem 5.1.2 is complete.

Remark:
The positive definiteness of \(R_{i}, R_{3}\) and \(R_{4}\) establish D.1-2 and D.l-3 as strict inequalities, hence only one optimal solution can be obtained, if there is only one solution to the \(2 n(1+s+n)\) dimensional T P B V P.

Remark:
The sufficient conditions are global sufficient conditions since the convexity and concavity requirements are global. Therefore, if only one solution of the TPBVP exists, it is the unique global optimal solution.

\section*{D. 2 Proof of Theorem 5.2}

The details of this proof are similar to the proof of Theorem 5.1.2 except that terms in the variations \(\delta z_{1}(\tau), \delta u_{1}(\tau)\) and \(\delta u_{2}(\tau)\) are included. As in the proof of Theorem 5.1.2 which includes the necessary conditions of Theorem 4.1, this proof requires all the necessary conditions of Theorem 4.2. The convexity and concavity requirements are that
\[
f_{0} \text { is convex in } u_{1}(t), u_{1}(\tau), z_{1}(t) \text { and } z_{1}(\tau)
\]
and
\[
\begin{gathered}
f_{0} \text { is concave in } u_{2}(t), u_{2}(\tau), z_{1}(t), z_{1}(\tau), u_{3}^{\prime}(t, \tau), z_{2}^{\prime}(t, \tau) \\
\text { and } z_{3}(t, \tau)
\end{gathered}
\]

APPENDIX E

\section*{COMPUTER PROGRAMS}

There were two main problens solved. The "worst case" optimal control problem, i.e. the differential game of Chapters 4 and 5, and the parameter identificcition problem of Chapter 6. The four prograns OPITILL, ANLPLT, EIITIPT and VALUE were written to complete the solution of the differential game. The program PROB1P was written to simulate the overall model reference adaptive system described in Chapter 6. This program included both identification and suboptimal control. A description and a listing of each program is contained in this appendix. Flowcharts for the programs OPTINL and PROBiP are included. No flow wharts are given for the other three programs since they only contain straight forvard calculations.

The assistance of Dr. F. Russel in creating a set of files which greailly eased the whole prograrming effort is acknowledged and was deeply appreciated. This set of files provided means to edit ard exccute any of the above five !rograns from a remote terminal in the interactive mode and thereby saved much time and physical labor.

The complete analytic solution to the optimal problen was obtained by hand using two methods. First, the solution form of the equations presented in Appendix A was used. Also, the two dimensional Laplace transform tecinnique of Kuo (54) was extended and applied. Both methods yielded identical results. The complete solution was differentiated by hand insuring that the results actually did satisfy the differential equations and
endpoint conditions of Theorem 4.1 . The convexity and concavity requirements of Theorem 5.1 were satisfied, hence the solution obtained was the unique, optimal, global solution.

The program OPTIML simulated the differential equations of all the variables which were functions of \(t\) only. Numerical integration of the ordinary differential equations was carried out with double precision arithnetic. The results agreed exactly with those obtained by hand. A program to numerically evaluate the functions of two independent variables was not written directly, but could be in terms of either of the methods used to obtain the solutions by hand. Shooting techniques could be employed to solve the TPBVP of these variables rather than the actual algebraic hand calculations which were made.

Both open loop and closed versions of OPTIML were run, the only difference being the form of the differential equations. In

Figure E. 2 is the program listing and Figure E.3.1 through E.3.4 contain the actual plots obtained by computer.


Figure E. 1 Flowchart for optimal problem
```151
```

H. BURBANK E.E.DEPT.

```PROBLEM 1 WITH PLOTS
```

DOUBLE PRECISION YSTART,XSTART,XEND,H,EP,SP,Y,OY,X,TSTART,TEND

```DOUBLE PRECISION A,B,C,R3,R4
```

EXTERNAL FUNCI,DSE,OUT1
DIMENSION Y(2), DY(2),YSTART(2)
DIMENSION ..... Z(2),F(51),TIME(51),U1(51),U2(51)
REAL XX,TIME,FOUl,U2
COMMON NOFNS,A,B,C,R3,R4
NOFNS $=0$
REWIND ..... 40
Z1(0)=5.
C Z1(1)=0.1
PIl=5.
P12=0.1
YSTART(1)=PI|
YSTART(2)=(-A*(PI2-PIl*(DEXP(A*TEND))))/(DD*DSINH(-A*TEND))
WRITE(6,10) (YSTART(I),I=1,2)
10 FORMAT(5X,IIC,S',2E15.6)
$S P=0.02$
CALL ODESP(SP,FUNCI,N,YSTART,TSTART,TEND,H,EP,OSE,OUT1,899) GO TO 98

```
99 CALL OERROR
98 WRITE 6,5\()\) NOFNS
5 FORMAT(1H, 5X,36HTOTAL NO OF FUNCTION EVALUATIONS IS ..... -I6)
ENDFILE 40
DO \(100 \mathrm{~K}=1,2\)
```

            REWIND 40
            DO 200 L=1,51
            READ(40,30) XX,(Z(I),I=1,2)
        30 FORMAT(3E15.6)
            F(L)=Z(K)
    200 TIME(L) =XX
    CALL PLTVAR(TIME,F,51)
    GO TO (1000,1001),K
    1000 WRITE (6,301)
    301 FORMAT (20X,DF(T)=Z1(T)')
            GO TO 100
    1001 WRITE (6,302)
302 FORMAT(20X,'F(T)=LAMBDA1 (T)')
100 CONTINUE
DO 735 J=1,51
Ul(J)=SNGL(-8/R3)\#F(J)
735 U2(J)=SNGL(C/R4)*F(J)
CALL PLTVAR(TIME,Ul,51)
WRITE(6,305)
305 FORMAT (20X,'F(T)=U1(T)!)
CALL PLTVAR(TIME,U2,51)
WRITE (6,306)
306 FORMAT(20X,'F(T)=U2(T)')
STOP
ENO

```

SUBROUTINE FUNCI(Y,T,DY)
DOUBLE PRECISION A,B,CDY,T,DY,R3,R4,DD
DIMENSION Y(2) DY(2)
COMMON NOFNS,A,B,C,R3,R4
\(D 0=-B^{*}(1 . / R 3) * B+C *(1 . / R 4) * C\)
\(D Y(1)=A * Y(1)+D D * Y(2)\)
\(D Y(2)=-A \# Y(2)\)
NOFNS \(=\) NOFNS +1
RETURN
END

SUBROUTINE OUTI (Y,OY,N,X,SPTYPE,\#)
LOGICAL SPTYPE
REAL \(Z\) PXX
DOUBLE PRECISION \(Y\),OY, \(X\)
DIMENSION \(Y(2), D Y(2), Z(2)\)
IF (.NOT.SPTYPE) GO TO 100
\(0020 \mathrm{~K}=1 \mathrm{~N}\)
\(20 Z(K)=\operatorname{SNGL}(Y(K))\)
\(X X=S N G L(X)\)
WRITE(40,10) \(X X,(Z(K): K=1,2)\)
10 FORMAT(3E15.6)
100 RETURN
END
SUBROUTINE PLTVAR(T,F,NP)
```155
```

REAL\#4 LINE( 51) ,F(1),T(1)
DATA BLANK,DOT,STAR,OASH,ZERO/IH ,IH.,IH\#,IHm,IHO/
DO $101 \mathrm{~J}=1,51$
101 LINE (J) =DOT
DO $127 \mathrm{~L}=1.10$
127 LINE ( 5 * $\mathrm{L}+1$ )=DASH
LINE (26) =ZERO
LINE (1)=DASH
PRINT ..... 800
800 FORMAT(1H1)
PRINT 102, LINE
102 FORMAT(1X, 51A1,5X,'TIME',7X,'F(T)I)
DO $103 \mathrm{~J}=1,51$
103 LINE (J) =BLANK
LINE (26)=00T
PM=ABS(F1))
DO $104 \mathrm{M}=2, \mathrm{NP}$
IF(PM •GE. ABS(F(M))) GO TO 104
105 $P M=A B S(F(M))$
104 CONTINUE
IF (PM .EQ. 0.0 ) ..... $P M=1.0$
DO $107 \mathrm{M}=1, \mathrm{NP}$
$J=25.0 \# F(M) / P M+26.5$
LINE (J) =STAR
PRINT 106, LINE,T(M) $\mathrm{F}^{(M)}$
106 FORMAT(1X, 51A19F10.40E15 ..... 6)
$\operatorname{LINE}(J)=$ BLANK
LINE (26) =DOT
107 CONTINUE
RETURN
END

$F(T)=Z 1(T)$

| TIME | Fiti |
| :---: | :---: |
| 0.0000 | 0.500000 E |
| 0.0000 | 0.470793 E |
| 0.0400 | $0.443281 E$ |
| 0.0600 | 0.417366 E |
| 0.0800 | $0.392954 E$ |
| 0.1010 | $0.369956 E$ |
| 0.1200 | $0.348291 E$ |
| 0.1400 | 0.327880 E |
| 0.1600 | 0.308650 E |
| 0.1800 | 0.290532 E |
| 0.2000 | 0.273459 E |
| 0.2200 | 0.257372 E |
| 0.2400 | $0.242211 E$ |
| 0.2600 | $0.227922 E$ |
| 0.2800 | 0.214454 E |
| 0.3000 | 0.201759 E |
| $0,32 \mathrm{no}$ | 0.189790 E |
| 0.3400 | 0.178504 E |
| 0.3600 | 0.167861 E |
| 0.3 añ | $0.157823 E$ |
| 0.4000 | $0.148353 E$ |
| 0.4900 | 0.1394178 |
| 0.4400 | $0.130984 E$ |
| 0.4600 | $0.123022 E$ |
| 0.4800 | 0.1155036 |
| 0.5000 | 0.108399 E |
| 0.5200 | $0.101687 E$ |
| 0.5400 | $0.953403 E$ |
| 0.5600 | $0.893371 E$ |
| 0.5800 | 0.836556 E |
| 0.6000 | $0.782753 E$ |
| 0.6900 | $0.731770 E$ |
| 0.6400 | $0.683421 E$ |
| 0.6600 | 0.637534 E |
| 0.6800 | $0.593942 E$ |
| 0.7000 | 0.552489 E |
| 0.7900 | $0.513026 E$ |
| 0.7400 | 0.475410 E |
| 0.7600 | $0.439507 E$ |
| 0,7800 | $0.405186 E$ |
| 0.8000 | $0.372324 E$ |
| 0.8 ? 00 | 0.340802 E |
| 0.8400 | 0.310508 E |
| 0.8000 | $0.281333 E$ |
| 0.8900 | 0.253170 E |
| 0.9000 | 0.225919 E |
| 0.9200 | 0,199481E |
| 0.9400 | 0.173762 E |
| 0.9600 | 0.148669 E |
| 0.9800 | $0.124111 E$ |
| 1.0000 | 0.100000E |

Figure E. 3.1 Plot of $z_{1}$


Figure E.3.2 Plot of $\lambda_{1}$

$F(T)=U 1(T)$

Figure E.3.3 Plot of $u_{1}$

$F(T)=U \dot{2}(T)$

Figure E.3.4 plot of $u_{2}$

The programs ANLPLIT and PlITPT were written to obtain computer plots of all the states and controls. Program VALUE evaluated the performance index, J for optimal states and controls. They are listed in Figures E.4, E.5, and E.6. The results of these prograns are given in Chapter 7.

```
    PROGRAM ANLPLT
    DIMENSION Y(5),YY(51),TT(51)
    REAL#4 LMBDI
    T=0.
    TF=1.
    DELTAT=(TF-T)/50.
    PIl=5.
    PI2=0.1
    A=-3.
    B=2.
    C=1.
    R3=1.
    R4=1.
    DD=-8*(1./R3)*B+C* (1./R4)*C
    ETA=(-A)*(PI2-PII*EXP(-3.))/(DD*SINH(3.))
    REWIND 30
100 Z1=PIl#EXP(-3.#T) +(DO/(-A))#ETA*(SINH(-A*T))
    Y(1)=Z1
    LMBD1=ETA*EXP(-A*T)
    Y(2)=LMBD1
    Ul=(-B/R3)*LMBDL
    Y(3)=Ul
    U2=(C/R4)*LMBD1
    Y(4)=U2
    G=(-A*ETA*2.)/(EYA*DD-(A*PI1#2. +ETA*DD)*EXP (-6.*T))
    Y(5)=G
    WRITE (30,10) T,(Y(I),I=1,5)
10 FORMAT(6E15.6)
    T=T+DELTAT
    IF(T.LT.TF) GO TO 100
    ENDFILE }3
    DO 5 K=1,5
    REWIND }3
    DO }6\textrm{J}=1,5
    READ(30,10) T,(Y(I),I=1,5)
    YY(J)=Y(K)
6TT(J)=T
5 \mp@code { C A L L ~ P L T V A R ( T T , Y Y , 5 1 ) }
STOP
END
```

Figure E. 4 Listing of program ANLPLI

C H．BURBANK E．E．DEPT．
REAL LMBD2，LMBD3
DIMENSION Y（5），YY（51），TT（51）
$E K=-1 . / 12$ ．
$E L=1.1144$ 。
EME1．
$E N=10$ ．
WRITE（6，1）EK，EL，EM，EN

$A A=E M-E L / 12$.
$A B=-E K / 24$ ．
$A C=E L / 12$ ．
$A D=-E K / 48$ ．
$A E=E K / 16$ ．
WRITE（6，2）AA，AB，AC，AD，AE
 1E15．6）
$A F=E N-E K / 96$ ．
$A G=-0.54(E M-(E L / 12)-.(E K / 144)$.
AH＝－1．＊（（EL／24．）－（EK／288．））
$A I=0.5$（ EM －（EL／12．））
WRITE $(6,3)$ AF，AG，AH，AI

AJ＝－EK／144．
AK＝EL／24．
$A L=-E K / 288$.
$A M=E K / 96$ ．
WRITE $(6,4)$ AJ，AK，AL，AM

AN＝EL
$A O=-E K / 4$ 。
$A P=E K / 4$ ．
WRITE $(6,7)$ AN，AO，AP
7 FORMAT（ $5 \mathrm{X}, \mathrm{I} A \mathrm{~A}=1, E 15.6,1 \mathrm{AO}=1, \mathrm{E} 15.6, \mathrm{~A} \mathrm{~A}=1, E 15.6$ ）
$A Q=E K$
WRITE $(6,8) \quad A Q$
8 FORMAT（5X，$A Q=1, E 15.6)$

C2＝－0．5＊（EM－（EL／12．）－（EK／144．））＊EXP（－3．）$+0.5 *(E M-(E L / 12)) *$
1 EXP（－1．）－（EK／144．）＊EXP（3．）

1 EXP（3．）
$01=(E M-(E L / 12)) \# E X P.(-3)+.(E L / 12) \.# \operatorname{EXP}(3$.
$02=-1$ ．＊$\left(E K / 24_{0}\right) * E X P(-3)-$.1 。\＃$(E K / 48) \# E X P.(3)+.(E K / 16) * E X P.(-1 。)$
WRITE（6．10）D1，02
10 FORMAT（5X，DD．S1，2E15．6）
WRITE（6，11）C1，C2，C3
11 FORMAT（5X，＇C，S1，3E15．6）
WRITE $(6,13)$

```
        13 FORMAT(5X,OZ2(T,1)=01#EXP(-T)+02#EXP (3#T)|)
            WRITE (6,14)
    14 FORMAT (5X,1Z3(1,TAU)=C1*EXP(-TAU) +C2*EXP(-3*TAU) +C3*EXP(3#TAU) 1)
        TAU=0.
    90 T=0.
    REWIND 27
    TF=1.
    DELTAT=(TF-T)/50.
    100 EP1T=EXP(T)
        EM1T=EXP(-T)
        EP3T=EXP(3.#T)
        EM3T=EXP(-3.#T)
        EPITAU=EXP(TAU)
        EMITAU=EXP(-TAU)
        EP3TAU=EXP(3.#TAU)
        EM3TAU=EXP(-3.*TAU)
        Z2=AA*EM1T*EM3TAU +AB*EP3T*EM3TAU +AC*EM1T*EP3TAU +AD*EP3T*EP3TAU
    1 +AE#EP3T*EM1TAU
        Z3=AF#EM3T*EM1TAU +AG*EM3T*EM3TAU*AH*EM3T*EP3TAU*AI*EM1T*EM3TAUS
    1 +AJ#EP3T#EM3TAU +AK*EM1T#EP3TAU +AL#EP3T#EP3TAU +AM#EP3T#EM1TAU
        LMBD2=AN*EM1T#EP3TAU*AO#EP3T*EP3TAU +AP#EP3T#EM1TAU
    LMBD3=AQ*EP3T*EMITAU
    U3=LMBD2/2.
    Y(1)=22
    Y(2)=23
    Y(3)=LM802
    Y(4)=LMBD3
    Y(5)=U3
    WRITE(27,12) T,(Y(I),I=1,5)
    12 FORMAT (6E15.6)
    T=T+DELTAT
    IF(T.LT.TF) GO TO 100
    ENOFILE }2
    OO 5 K=1,5
    REWIND 27
    DO 6 J=1,51
    READ(27,12) T,(Y(1),I=1,5)
    YY(J)=Y(K)
    6 TT(J)=T
    CALL PLTVAR(TT,YY,5i)
    GO TO(1001,1002,1003,1004,1005),K
1001 WRITE (6,301)
    301 FORMAT(20X,IF(T)=Z2(T,TAU)')
    WRITE(6,500) TAU
    500 FORMAT (20X, 'TAU=',F10.4)
    GO TO 5
1002 WRITE (6,302)
    302 FORMAT (20X, 'F(T)=23(T,TAU)')
    WRITE(6,500) TAU
    GO TO 5
```

```
1003 WRITE (6,303)
    303 FORMAT(20X,'F(T)=LAMBDAZ(T,TAU)')
        WRITE (6,500) TAU
        GO TO 5
    1004 WRITE(6,304)
    3 0 4 \text { FORMAT (20X,OF(T)=LAMBDA3(T,TAU) ')}
        WRITE (6,500) TAU
        GO TO 5
    1005 WRITE(6,305)
    305 FORMAT (20X, PF(T)=U3(T,TAU):)
        WRITE(6,500) TAU
        5 CONTINUE
        ENDFILE 27
        TAU=TAU*0.2
        IF(TAU.LT.TF) GO TO 90
        STOP
        END
```

Figure E. 5 Listing of program PlTTPT

```
    PROGRAM VALUE
    H. BURBANK E.E.DEPT.
    F(A)={EXP (A)-1.)/A
    A=~3.
    B=2.
    C=1.
    R3=1.
    R4=1.
    DO= - ** (1./R3)*B+C#(1./R4)*C
    PIl=5.
    PI2=0.1
    EK=-1./12.
    EL=1./144.
    EM=1.
    EN=10.
    RT =(-A)*(PI2-PII#EXP(-3.))/(DD*SINH(3.))
    WRITE(6,2) RT
    2 FORMAT (5X, 1RT=1,E15.6)
    V=0.5*RT#RT#F(6.)
    VV=-0.25
    VA=VV#EL#EL#F(-2.)#F(6.)
    VB=VV*(EK*EK/16.)#F(6.)*F(6.)
    VC=VV#(EK#EK/16.) #F(6.)#F(-2.)
    VD=VV*(-EL#EK/4,0)#F(2.0)#F(6.)#2.
    VE=VV#(EL*EK/4.)*F(2.)*F(2.)*2.
    VF=VV#(-EK#EK/16.)*F(6.)*F(2.)*2.
    WRITE(6,3)V
    WRITE (6,3)VA
    WRITE (6,3)VB
    WRITE (6,3)VC
    WRITE (6,3)VD
    WRITE (6,3)VE
    WRITE (6,3)VF
    FORMAT (5X,E15.6)
    VALUE =V +VA+VB+VC+VD+VE+VF
    WRITE(6,1) VALUE
1 FORMAT (5X,'VALUE=1, E15.6)
    STOP
    END
```

The combined identification and suboptimal control was verified by the simulation of the model reference adaptive system of Chapter 6 by the program P R O B 1 P. The flow chart is contained in Figure E.7. The listing of PROB1P is given in Figure E. 8 . The actual results are contained in Chapter 7. Several examples were run for different sets of undenown parameters.


Figure E. 7 Flowchart for identification problem

## PROGRAM PROBIP

C H．BURBANK E．E．DEPT．
PROBLEM 1 WITH PLOTS
DOUBLE PRECISION YSTART，XSTART，XEND，H，EP，SP，Y，OY，X，TSTART，TEND DOUBLE PRECISION A，B，C，R3，R4，APLT，BPLT，CPLT，V，DD DOUBLE PRECISION PTS，DELTAT，PII，PI2，ASTART，日START，TOTA TOTB DOUBLE PRECISION TINT，TSTRNU，TENDNU，PTSINT，TCHG，PJNORM DOUBLE PRECISION STATE，DIFF，SUM，MEAS，MEASOL，XX，MEASI DOUBLE PRECISION DELA，DELB，DELJ，SCALEK，PJWRTA，PJWRTB，PCTCHG DIMENSION STATE（3），DIFF（30）
EXTERNAL FUNCI，DSE，OUT1
DIMENSION Y（3），OY（3），YSTART（3）
DIMENSION 2（3），F（501），TIME（501），U1（501），U2（501）．，TIMEL（501）
DIMENSION FF（501）
DIMENSION VNOISE（4）
REAL TIME，F，UI，U2
COMMON NOFINS，A，B，C，R3，R4，APLT，BPLT，CPLT，V $V=0$ 。
$A P L T=-4$ ．
BPLT $=3.0$
CPLT＝1．0
MEAS $=0$ 。
IFLAG＝0
NOFNS $=0$
NTIMES $=0$
REWIND 50
REWIND 51
REWIND 52
C NPTS IS NO，S DELTA T，S PLUS 1 FOR 0 ．
C LPTS．IS NO．OF POINTS PRINTED CUT
NPTS＝440
LPTS $=440$
NMULT＝（NPTS－1）／（LPTS－1）
WRITE 6,21 ）NMULT
21 FORMAT（ $5 x, 6$ HNMULT $=016$ ）
PTS $=400$ ．
PII＝5．
PI2＝0．1
C
$N=3$
C
TSTART＝0．
C
TEND＝1．
C
DELTAT＝（TEND－TSTART）／PTS
H＝DELTAT
$S P=D E L T A T$
C
C

TINT=40.
NTINT=40
TSTRNU=TSTART
TCHG = (TEND-TSTART)/TINT
TENDNU=TCHG
PTSINT=PTS/TINT
NPTSIT=(NPTS)/NTINT
WRITE $(6,22)$ PTSINT,NPTSIT
22 FORMAT (5X,IPTS PER INT',E15.6,5X,I4)
C

C
$E P=0.002$
$A=-3$.
$B=2$ 。
ASTART=A
BSTART=B
$C=1.0$
R3=1.
R4=1.
$D D=-B \#(1, / R 3) * B+C *(1, / R 4) * C$

C
C
C
C
C $\quad 21(0)=P I 1$
C $\quad 21(1)=P 12$
C $\quad \operatorname{XPLANT}(0)=P I 1$
YSTART(1) $=$ PII
$\operatorname{YSTART}(2)=(-A *(P I 2-P I 1 *(D E X P(A * T E N D))) /(D D * D S I N H(-A * T E N D))$
YSTART (3) $=$ PII
WRITE $(6,10)$ (YSTART(I), $I=1,3)$
10 FORMAT(5X,IIC,S',3E15.6)
C
C
10000 CALL DDESP(SP,FUNCl,N,YSTART,TSTRNU,TENDNU,H,EP,DSE,OUT1,\&99)
GO TO 98
99 CALL DERROR
GO TO 9999
98 TSTRNU=TSTRNU+TCHG •
TENDNU=TENONU+TCHG
IF (TENDNU.GT.TEND) GO TO 9900
ENDFILE 51
REWIND 51
DO $400 \mathrm{KL=19NPTSIT}$
$400 \operatorname{READ}(51,31) \mathrm{XX},(\mathrm{YSTART}(I), I=1,3 ;$
31 FORMAT(4025.16)
ENDFILE 51
REWIND 51
WRITE $(6,23)$ TSTRNU,TENDNU
23 FORMAT(5X, ONEXT TIME INTERVAL IS FROM',E15.6,
1 (TO',E15.6, 'WITHI)
WRITE $(6,10)$ (YSTART(I),I=1,3) YSTART (1) $=$ YSTART (3)

IF (IFLAG.GT.0) GO TO 10000
MEASOL=MEAS
DO $1300 \mathrm{~J}=1$, NPTSIT
$\operatorname{REAO}(51,31) \quad X X,(S T A T E(I), I=1,3)$
$\operatorname{DIFF}(J)=(S T A T E(1)-S T A T E(3)) * * 2$
1300 WRITE 6,40 )DIFF(J)
40 FORMAT $(5 \times, 025.16)$
ENDFILE 51
REWIND 51
C

NPTSIM=NPTSIT-1
DO $1301 \mathrm{~J}=2$.NPTSIM
1301 SUM=SUM+DIFF(J)
MEAS=SUM*DELTAT*20.*25.
WRITE (6,41)SUM
41 FORMAT(5X, ISUM=1,D25.16)
WRITE 6,1302 )MEAS
IF (NTIMES.EQ.O) MEAS1=MEAS
PCTCHG $=($ MEAS $/ M E A S 1) * 100$ 。
WRITE (6,1312)PCTCHG
PCTCHG=ABS (PCTCHG)
FORMAT (5X,'PERCENT CHANGE IN MEAS=1,E15.6)
1312
FORMAT ( $5 \mathrm{X}, \mathrm{OMEAS}=1, E 15,6$ )
NTIMES=NTIMES+1
IF (NTIMES.GT.1) GO TO 1303
MEAS1=MEAS
DELA=0.1*A
$D E L B=0.1 * B$
TOTA=DELA
TOTB=DELB
$A=A+D E L A$
$B=B+D E L B$
WRITE (6,1304)A,B
1304 FORMAT (5X,1A=1,E15.6.1B=1,E15.6)
GO TO 10000
1303 IF (PCTCHG.GT.20.) GO TO 1305
IF (PCTCHG.GT.15.) GO TO 1306
IF (PCTCHG.GT.10.) GO TO 1307
IF (PCTCHG.GT.5.) GO TO 1308 ..... 171
IF (PCTCHG.GT.2.) GO TO 1313

```IF (PCTCHG.GT.1.) GO TO 1314
    IF(PCTCHG.GT.0.1) GO TO 1315
    IF(PCTCHG.GT.0.05) GO TO 1316
    IFLAG=1
        GO TO 10000
    SCALEK=1.0
        GO TO 1309
1306 SCALEK=0.9
        GO TO 1309
1307 SCALEK=0.8
    GO TO 1309
1308 SCALEK=0.6
    GO TO 1309
    1313 SCALEK=0.5
        GO TO 1309
1314 SCALEK=0.4
        GO TO 1309
1315 SCALEK=0.2
        GO TO 1309
1316 SCALEK=0.1
1309 WRITE(6,1310)SCALEK
    1310 FORMAT (5X,'SCALEK=1,E15.6)
        TOTA=ASTART-A
        TOTB=BSTART-B
        DELJ=MEAS-MEAS\
        PJWRTA=DELJ/TOTA
        PJWRTB=DELJ/TOTB
        PJNORM=DSQRT((PJWRTA*#2) +(PJWRTB##2))
        WRITE (6.1319)PJNORM
    1319 FORMAT (5X,1PJNORM=1,E15.6)
        WRITE(6,1311)DELJ,PJWRTA,PJWRTB
1311 FORMAT(5X,IDELJ=',E15.6,'PJWRTA=1,E15.6.1PJWRTB=1,EE15.6)
    DELA=SCALEK*PJWRTA*0.8
    DELB=SCALEK#PJWRTB#1.3
    A=A +DELA
    8=8+DELB
    WRITE(6,1304)A,B
    GO TO 10000
C
C
9900 WRITE(6,5) NOFNS
    5 FORMAT(1H,5X,36HTOTAL NO OF FUNCTION EVALUATIONS IS ,I6)
    ENDFILE }5
    DO 100 K=1.2
    REWIND 50
    DO 200 L=1,NPTS
        READ (50,30) XX,(Z(I),I=1,3)
        30 FORMAT(4E15.6)
```

```
        F(L)=Z(K)
    200 TIME(L) =XX
        DO 205 JJ=1:LPTS
        LL=1+(JJ~I)*NMULT
        FF(JJ)=F(LL)
            TIMEL (JJ) = TIME (LL)
        CALL PLTVAR(TIMEL.FF,LPTS)
        GO TO (1000,1001),K
    1000 WRITE(6,301)
    301 FORMAT(20X, IF(T)=Z1(T)!)
        GO TO 100
    1001 WRITE (6,302)
    302 FORMAT (20X, FF(T) =L_AMBOA1(T) ()
    100 CONTINUE
        DO }735\textrm{J}=1/LPT
        Ul(J)=SNGL (-8/R3)#FF(J)
    735 U2(J)=SNGL (C/R4)*FF(J)
        CALL PLTVAR(TIMEL,UI LPPTS)
        WRITE (6,305)
    305 FORMAT (20X, IF(T)=Ul (T):)
        CALL PLTVAR(TIMEL,UZ,LPTS)
        WRITE (6.306)
    306 FORMAT(20X,'F(T)=U2(T)')
    9998 ENDFILE 52
        REWIND 52
        DO 500 K=1.NPTS
        READ(52,530)WW
    500 U2(K)=WW
        CALL PLTVAR(TIMEL,UZ,NPTS)
        WRITE (6,531)
    531 FORMAT(20X, FF(T)=V(T)1)
    530 FORMAT (E15.6)
9 9 9 9 ~ S T O P
    END
```

```
SUBROUTINE FUNCI (Y,T,DY)
DOUBLE PRECISION A,B,C,Y,T,DY,R3,R4,DD,APLT,BPLT,CPLT,V
DIMENSION Y(3), DY (3), VNOISE (4)
COMMON NOFNS,A&B,C,R3,R4,APLT,BPLT,CPLT,V
CALL RANGEN(VNOISE,4)
V=VNOISE(4)
DD= - B*(1./R3)*B+C*(1./R4)#C
DY(1)=A*Y(1)+DD*Y(2)
DY(2)=\varpiA*Y(2)
DY(3) =APLT*Y(3) & BPLT*(-1./R3) #B*Y(2) +CPLT*V
NOFNS=NOFNS +1
RETURN
END
```

SUBROUTINE RANGEN(VNOISE,N)
DIMENSION VNOISE(20)
CALL RANORM (VNOISE,4)
DO $1 K=1, N$
1 VNOISE $(K)=$ VNOISE(K)/6. RETURN
END

```
    SUBROUTINE OUTI(Y,DY,N,X,SPTYPE,*)
        DOUBLE PRECISION A,B,C,R3,R4,APLT,BPLT,CPLT,V
        COMMON NOFNS,A,B,C,R3,R4,APLT,BPLT,CPLT,V
        LOGICAL SPTYPE
        REAL Z;XX
        DOUBLE PRECISION Y;DY,X
        DIMENSION Y(3),DY(3),Z(3)
        IF(.NOT.SPTYPE) GO TO 100
    DO 20 K=1,N
    20 Z(K)=SNGL(Y(K))
    XX=SNGL(X)
    WRITE(50,10) XX,(Z(K),K=1,3)
    WRITE(51,11)X,(Y(K),K=1,3)
11 FORMAT(4D25.16)
    10 FORMAT(4E15.6)
    WRITE (52,12)V
    12 FORMAT (E15.6)
100 RETURN
    END
```

Figure E. 8 Listing of program PROBIP

## MINIMMM COVARIANCE ESTIMATIOX

For a plant

$$
\begin{array}{ll}
\frac{d x}{d t}(t, .)=A(t) x(t, .)+B(t) u(t)+C(t) v(t, .) & F .1 \\
\text { with measurement } & \\
z(t, .)=H(t) x(t, .)+w(t, .) & F .2
\end{array}
$$

it is desired to obtain an optimal filter constrained by
$\frac{d \hat{x}}{d t}(t)=F(t) \hat{x}(t)+G(t) z(t)+D(t) u(t)$
dt
where $F, G$, and $D$ are to be found. Define the error
$\tilde{x}(t,)=.x(t,)-.\hat{x}(t)$
then
$\frac{d \tilde{x}}{d t}(t,)=.[A(t)-G(t) H(t)-F(t)] x(t, 1)+F(t) \tilde{x}\left(t_{0},\right)$

$$
+C(t) v\left(t_{0} .\right)-G(t) w(t, .)+[B(t)-D(t)] u(t) \quad F .5
$$

If the filter is to be unbiased, then
$E\{\tilde{x}(t,)\}=.E\left\{\frac{d \tilde{x}}{d t}(t,).\right\}=0 \forall t \in \Gamma_{1}$

$$
\text { F. } 6
$$

Equation F. 6 holds if

$$
D(t)=B(t) \quad \mathrm{F} .7
$$

and

$$
F(t)=A(t)-G(t) H(t)
$$

$$
\text { F. } 8
$$

and if $\mu_{V}(t)=\mu_{W}(t)=0 \forall t \in \Gamma_{1}$ or if not zero they are known $\forall t$ and can be subtracted out. With these conditions substituted, F. 5 becomes
$\frac{d \hat{x}}{}(t)=A(t) \hat{x}(t)+G(t)[z(t)-H(t) \hat{x}(t)]+B(t) u(t) \quad F .9$ dt
from which an optimal untriased estimate can be generated once
$G(t)$ is found. The matrix $G(t)$ is chosen to minimize the error
covariances $V_{X X} V_{W} V^{\prime}$, and $V_{V z}$ at final times, i.e. at $t_{f}$, according to the weighted performance measure


$$
\left.+S(t) \frac{\partial V}{\partial \tau} \tilde{W}(t, \tau)+Q(t) \frac{\partial V}{\partial \tau} V_{X}(t, \tau)\right\} d t d \tau
$$

F. 10

The physical meaning of $J$ is determined from the following diagram.


Figure F. 1
Boundary cost of error covariance
Since
$V \sim \sim\left(t_{f}, t_{f}\right)-V \sim\left(t_{0}, t_{o}\right)=\int_{t_{0}}^{t_{f} t_{f}^{f}} \int_{f}^{f} \frac{\partial^{2} V X X}{\partial t \partial \tau}(t, \tau) d t d \tau \quad$ F. 11
and
$\int_{t_{0}}^{t_{f}} R(t) V \sim_{X X}\left(t, t_{f}\right) d t=\int_{t_{0}}^{t_{f}^{f}} \int_{f}^{t_{0}} R(t) \frac{\partial V}{\partial \tau} \sim(t, \tau) d t d \tau$ F. 12
the terms in F. 10 are seen to be terminal cost tems. Similar relations hold for all four terms in $J$.

The a priori data required is
$V_{V V}(t, \tau), V_{W W}(t, \tau)$ and $V_{W V}(t, \tau) \forall(t, \tau) \in \Gamma \quad$ F. 13
and
$V_{X}\left(t, t_{0}\right), V_{W X}^{\sim}\left(t, t_{0}\right) V_{v x}\left(t, t_{0}\right)$, and $V_{X}\left(t_{0}, t_{0}\right) \forall t \in \Gamma_{I} \quad F, 14$ It can be shown by analysis similar to that in Appendix A, that the error covariance must satisfy $\frac{\partial V_{\sim}}{\partial t} \sim(t, \tau)=[A(t)-G(t) H(t)] V_{X} X(t, \tau)+C(t) V_{V X}^{\sim}(t, \tau)-G(t) V_{W x} \tilde{x}(t, \tau)$ F. 15

Similarly,
$\frac{\partial V}{\partial \tau} \sim \underset{X}{\sim}(t, \tau)=V_{V X}^{\sim}(t, \tau)[A(\tau)-G(\tau) H(\tau)]^{N}+V_{V V}(t, \tau) C^{\prime}(\tau)-V_{V W}(t, \tau) G^{\rho}(\tau)$
and

Substituting F. 15 through F. 17 into F. 10 and choosing the weights

$$
\begin{aligned}
& R(t)=G(t) H(t) \\
& Q(t)=C(t)
\end{aligned}
$$

$$
\text { F. } 18
$$

with $S(t)$ arbitrary, an unconstrained dymamic optimization problem results. Applying variational techniques similar to those used in Appendices C and D, Theorem 6.1 can be proven.

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