Spring 1989

Kinematic synthesis of mechanisms for multiply separated positions

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Order Number 9003134

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New Jersey Institute of Technology, 1989

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KINEMATIC SYNTHESIS OF MECHANISMS FOR MULTIPLY SEPARATED POSITIONS

by
Hyoung Jun Kim

Dissertation submitted to the Faculty of the Graduate School
of the New Jersey Institute of Technology in partial fulfillment
of the requirements for the degree of
Doctor of Engineering Science
1989

APPROVAL SHEET

Title of Thesis: **KINEMATIC SYNTHESIS OF MECHANISMS FOR MULTIPLY SEPARATED POSITIONS**

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Doctor of Engineering Science, 1989

Thesis and Abstract Approved: ______________________ ____________
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ABSTRACT

Title of Thesis: **KINEMATIC SYNTHESIS OF MECHANISMS FOR MULTIPLY SEPARATED POSITIONS**

Hyoung Jun Kim, Doctor of Engineering Science, 1989
Thesis directed by: Dr. Raj S. Sodhi

The rigid body motion is studied in a combination of finitely and infinitesimally separated positions in planar, spherical, and spatial kinematics. A general new method for determining the locations of points and/or lines in a rigid body moving through finitely and infinitesimally separated positions is developed. These points and/or lines would satisfy the constraints of various types of binary links for planar, spherical, and spatial mechanisms.

A unified form of circle-point curve equation is derived for finitely and multiply separated position problems in planar and spherical motions. A graphical method to construct the circle-point and center-point curves and Ball point is also investigated for the PP-PP multiply separated positions problem in planar motion. Instantaneous geometric motion of a rigid body is studied in terms of the instantaneous screw axis for the infinitesimally separated positions in spatial kinematics. Also the finite spatial motion problem is recast in terms of determining the screw parameters directly.

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"Unified Form of Burmester Curve Equations in planar and Spherical Motions," to be submitted to Journal, *Mechanism and Machine Theory.*

"Personal Computer Based CAD System for the Synthesis of Planar Four-Bar Mechanisms," Presented at the 3rd International Conference on Robotics and Factories of the Future, Detroit, August 1988., by H.J. Kim and R. S. Sodhi.

"Synthesis of Planar Four-Bar Mechanisms for Generating a Prescribed Coupler Curve with Five Precision Points," Proceedings of the 10th Applied Mechanisms Conference, New Orleans, Dec. 1987, by H.J. Kim and R. S. Sodhi.

"On the Synthesis of Planar Four-Bar Mechanisms for Path and Position Generations," NJIT Research Report No. RS 8801 1988, by H.J. Kim and R. S. Sodhi.

"Derivation of Burmester Curve Equations Using Displacement Matrix and Matrix Theorem of Linear Equations," NJIT Research Report No. RS 8701, 1987, by H.J. Kim and R. S. Sodhi.

"On the Synthesis of Planar Four-Bar Mechanisms for Generating a prescribed Coupler Curve," *The Journal of Korean Society for Agricultural Machinery*, Vol. 7, N0. 2, 1982, pp. 8-17, by H. J. Kim and K. U. Kim.

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ACKNOWLEDGEMENTS

Throughout this research work the far-sighted approach of my advisor has been a constant source of inspiration to me. I therefore take this opportunity to acknowledge my association with Dr. Raj S. Sodhi during the past four years and our many hours of valuable discussions. I would also like to thank the New Jersey Institute of Technology for its financial assistance in the form of research assistantship. I also want to express my gratitude to my family who have done more for me than I can ever hope to repay.

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CHAPTER 1

INTRODUCTION

1 - 1. Mechanism

The word 'mechanism' has many meanings. Originally a 'contrivance' or 'device' in a machine, an assemblage of working parts designed to produce some required effect, the word is now also attached to a phenomenon which can be explained by applying scientific logic or some other reasoning that we accept as logical. Thus, there is the 'mechanism' of catalysis, or a heat transfer through a boundary layer, and so on. In this study I take Hunt's interpretation of the mechanism. Hunt [1] has interpreted a 'mechanism' essentially classically, as a means of transmitting, controlling, or constraining relative movement. In this sense, Reuleaux's [2] definition of a mechanism as a "combination of rigid resistant bodies so formed and connected that they move upon each other with definite relative motion," can be helpful to understand the mechanism.

A mechanism is said to have N degree-of-freedom if an arbitrary infinitesimal change in its position requires independent infinitesimal increments in N of its coordinates. Thus the traditional mechanisms, if modeled as a series of rigid bodies, are one degree-of-freedom since they depend on only one arbitrary input parameter. The types of the mechanisms can be gears, cams, links, or combination of them. Until recently one degree-of-freedom linkages have been used quite extensively in industry to obtain unusual motions because they are simple and cheap to build and provide good service as compared to cams which are much more difficult to manufacture. However, cams have the advantage of being much easier to design than the linkages.

As is well known, there is now a world-wide interest in open-loop linkages since these form the basic structure of mechanical manipulators, mechanical hands, walking machines, and other so-called robotic devices and smart products. Open-loop linkages are inherently multi-degree-of-freedom mechanisms, and in addition there are many possible multi-degree-of-freedom closed-loop mechanisms. Such mechanisms require independent input devices and sophisticated controllers which themselves demand efficient algorithms, suitably packaged software and computer back-up. Therefore one-degree-of-freedom mechanisms are more desirable than multi-degree-of-freedom mechanisms.

Because most actual mechanisms move with planar motion, the kinematics study has been biased toward two dimensions. On the other hand, theoretical developments in three dimensional kinematics have led to a significant awareness of the potential use of spatial mechanisms. The use of spatial mechanisms, however, remains rather rare in practice, except in specific areas of gear technology, spatial cam-and-follower mechanisms, shaft couplings, swash- and wobble-plate devices, and other related items. The universal joint, a swivelling electric fan, and a pair of bevel gears show the practical use of spherical mechanisms which are one classification of spatial mechanisms. This study deals with the planar, spherical, and general spatial mechanisms.

1 - 2. Synthesis of Mechanisms

The study of kinematics of mechanisms can be considered as two fundamental concepts. Firstly, the analysis process which includes the study of motion characteristics of all the points of the rigid bodies in an existing mechanism. The characteristics include such useful concepts as displacement, velocity, acceleration, etc.. Secondly, the synthesis process in which a physical mechanism is to be designed to produce the desired motion. It is the reverse of the analysis process.

Kinematic synthesis of mechanisms is often stated as having three stages. First, the "type synthesis" regarding the selection of type of device used such as; gears, cams, linkages, or a combination of them. Secondly, the "number synthesis" which determines how many links and joints are to be used, and in what schematic pattern. Thirdly, the "dimensional synthesis" which determines the essential dimensions of the entire mechanism perform the desired motion.

If we choose the linkage type of mechanism as one degree-of-freedom closed loop, the dimensional synthesis of the mechanism can be catagorized by the kind of motion: 1) path synthesis is to move a point on the linkage through a series of prescribed points on a specified path, 2) position synthesis is to move a rigid body through a set of prescribed positions in plane or space, 3) function synthesis is to have a linkage where input and output are related by a function. As the synthesis of mechanisms is briefly catagorized in Figure 1-1, it can be said that the synthesis is the process of designing a mechanism, that is, coming up with suitable dimensions and parameters of the mechanism.

1 - 3. Multiply Separated Positions

If any of the points in a moving body is displaced by a finite distance, the positions before and after the displacement can be called "finitely separated." On the other hand, infinitesimally separated positions (ISP) are the limit of the finitely separated positions (FSP). If the displaced position approaches

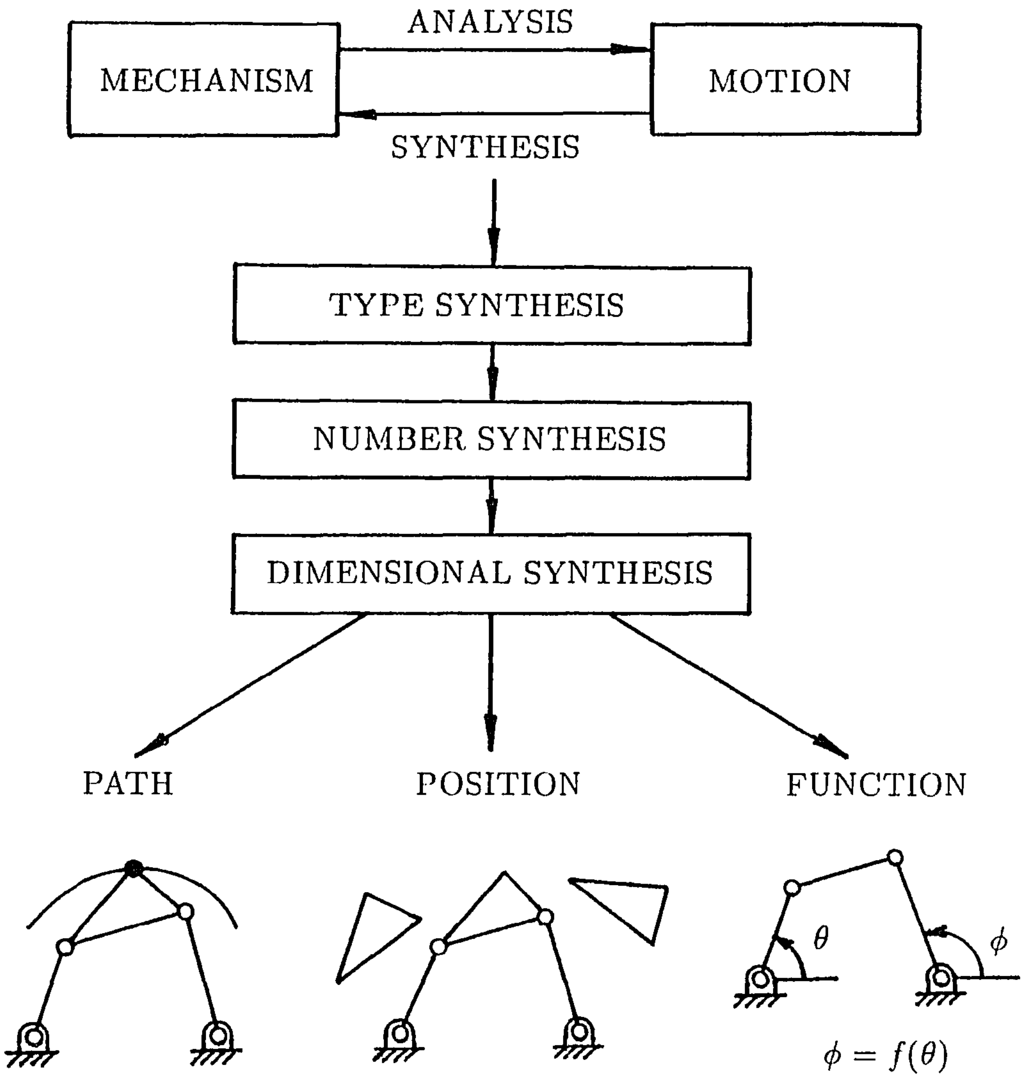

Figure 1-1. Synthesis of mechanisms

the undisplaced one, the positions are called "infinitesimally separated." The infinitesimally separated positions can be expressed in many different ways. These may be instantaneous center, instantaneous screw axis, curvature of the path, velocity state of the moving body, etc.. The concept of multiply separated positions (MSP) involves various combinations of finitely and infinitesimally separated positions as shown in Figure 1-2.

There are three combinations for three multiply separated positions (PPP, PP-P, P-P-P), five combinations for four multiply separated positions (PPPP, PPP-P, PP-PP, PP-P-P, P-P-P-P), and seven combinations for five multiply separated positions (PPPPP, PPPP-P, PPP-PP, PPP-P-P, PP-PP-P, PP-P-P-P, P-P-P-P-P), where the symbol P represents a single position of the moving body, the combination P-P represents two finitely separated positions, and PP two infinitesimally separated positions. This notation was originally introduced by Tesar [3] and provides a shorthand notation to be used in discussing multiply separated positions.

This study develops the analytical and graphical methods for MSP problems. The infinitesimally separated positions are specified by locating the instantaneous center (IC) in planar motion, the instantaneous rotational axis (IRA) in spherical motion, and the instantaneous screw axis (ISA) in spatial motion. The reasons for the interest in specifying the IC, IRA, and ISA are to enhance the rigid body motion at a specified position, to make a point or a line in the coupler dwell at a specified position, and further to include dynamic effects in the synthesis.

1 - 4. Analytical Synthesis Procedure

In dimensional synthesis, there are two ways to come up with a solu-

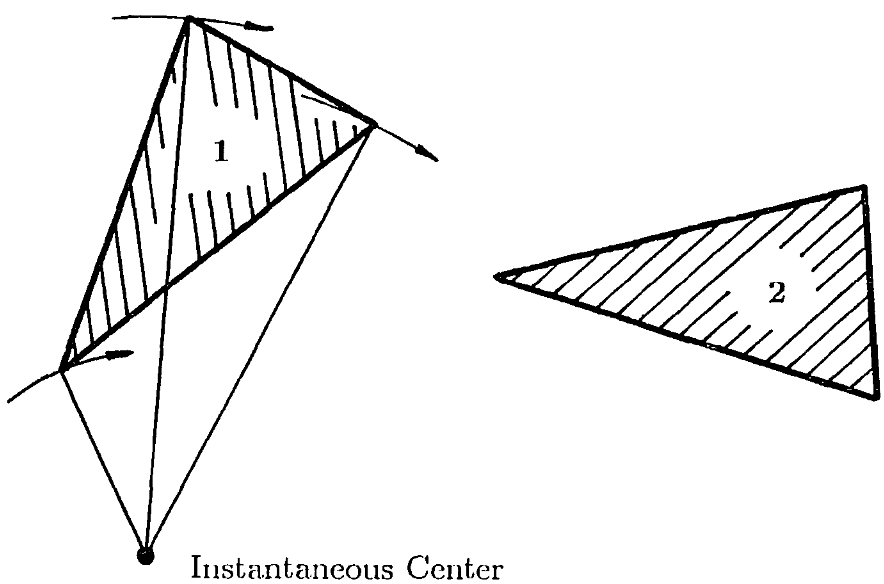

Figure 1-2. Multiply separated positions

tion linkage. One is to find the link lengths and their possible variables. The other one is to locate the special points and/or lines in moving and fixed bodies which is used on this study. The fundamental problem in the kinematic synthesis of mechanical linkages is to locate a point, or set of points, fixed in the moving rigid body that will pass through a series of points in space that satisfy geometrical constraints imposed by specific types of mechanical guiding links. For example, in the synthesis of planar four-bar linkages, the problem becomes one of locating a set of points in the moving plane which, as the plane assumes specified positions, will assume a series of positions that lie on a circular-arc. The circular-arc constraint is a result of assuming that the guidance is to be provided by two rigid links, each with two pivots. One pivot of each link is to be attached to the rigid body and the second to a fixed reference member. In spatial mechanisms there are many more possibilities for the geometric form of constraining links or link-pair combinations. The basic synthesis problem, however, remains the same, thus the general procedure is explained by taking a planar four-bar linkage as an example.

1). Specify the position of the moving body for the finite number of positions and the infinitesimal position by giving instantaneous center at the specified position. The maximum number of possible positions is determined by the type of guiding link being considered.

2). Select (x, y) as the Cartesian coordinates of a point in a moving body σ and (X_i, Y_i) as its ith coordinates in a fixed system Σ. Let the first position of σ coincide with Σ as shown in Figure 1-3.

3). Find the coordinates in Σ of the point in ith moving body by using the

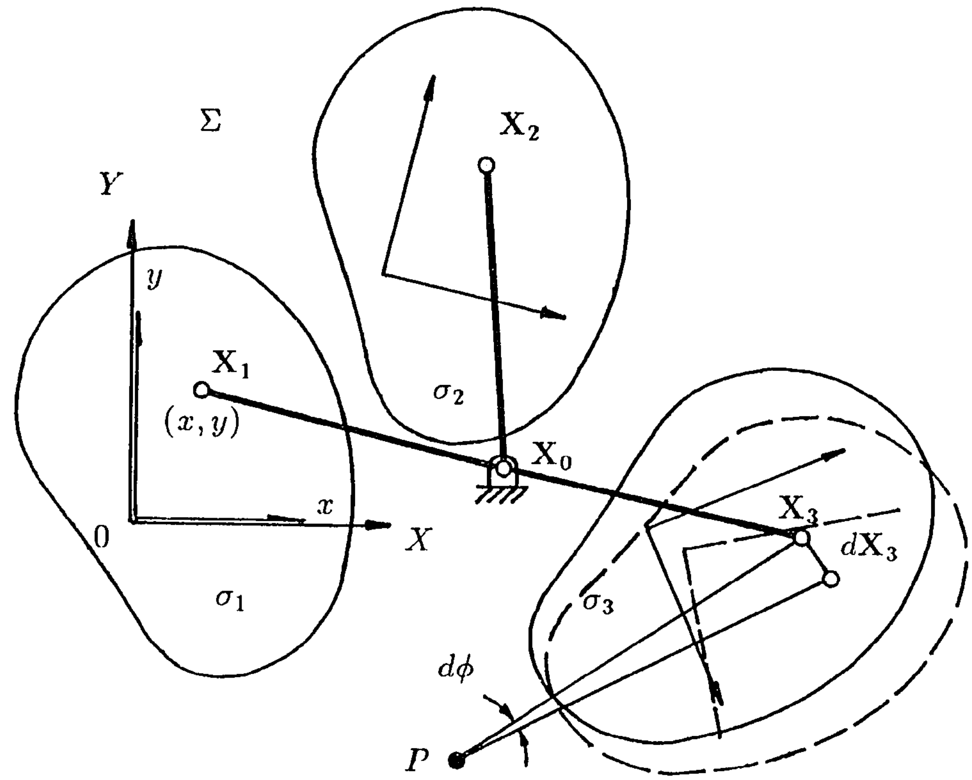

Figure 1-3. Geometric constraint of crank

following displacement matrix equations:

$$\mathbf{X}_i = \mathbf{R}_i\mathbf{x} + \mathbf{T}_i,$$

where $\mathbf{X}_i = \begin{pmatrix} X_i \\ Y_i \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$. The matrices $\mathbf{R}_i$ and $\mathbf{T}_i$ are determined by the specified finitely separated positions.

4). Find the infinitesimal displacements for the infinitesimally separated positions by using the following displacement matrix equations:

$$\frac{d\mathbf{X}_j}{d\phi} = \frac{d\mathbf{R}_j}{d\phi}\mathbf{x} + \frac{d\mathbf{T}_j}{d\phi}.$$

The matrices $\frac{d\mathbf{R}_j}{d\phi}$ and $\frac{d\mathbf{T}_j}{d\phi}$ are determined by the specified instantaneous center or instantaneous screw axis at the specified position.

5). Establish the geometric constraint equations at each design position as imposed by the guiding link to be synthesized. If we take a planar four-bar linkage as an example, each of guiding cranks must satisfy a condition of constant length, as shown in Figure 1-3, where points $\mathbf{X}_i$ and $\mathbf{X}_0$ are representative of a typical guiding link. This leads to following constraint equations for the finitely separated positions:

$$(\mathbf{X}_i - \mathbf{X}_0)^T(\mathbf{X}_i - \mathbf{X}_0) = (\mathbf{X}_1 - \mathbf{X}_0)^T(\mathbf{X}_1 - \mathbf{X}_0), \qquad i = 2, 3, ..., n.$$

6). Establish the generated constraint equations for the instantaneous motion by differentiating the above geometric constraint equations:

$$\left(\frac{d\mathbf{X}_j}{d\phi}\right)^T (\mathbf{X}_j - \mathbf{X}_0) = 0.$$

7). Operate on the unknown coordinates in each design position using the displacement matrix equations to eliminate all parameters except those that define the synthesized linkage in the design position. The constraint equations then become design equations.

8). Solve the resulting set of nonlinear design equations.

1 - 4. Objectives

There is little work done on the multiply separated positions problem in general three-dimensional kinematics, even though Tesar, Dowler, and their associates [3, 7, 21, 24] have extensively studied MSP problem for planar, spherical mechanisms. The purpose of this study is to develop a new mathematical approach to solve the MSP problems for planar, spherical, and spatial mechanisms. There is no available graphical procedure for the PP-PP case of MSP problem in planar kinematics. Thus another purpose of this study is to develop a graphical method for the PP-PP case of the MSP problem in planar mechanisms. Specifically, in this study, the following aspects are considered.

1). A unified form of circle-point curve equation is derived for FSP and MSP problems in planar motion.

2). A unified form of the spherical circle-point curve equation is derived for FSP and MSP problems in spherical motion.

3). A graphical method to construct the circle-point and center-point curves and the Ball point is investigated for the PP-PP case of MSP problem in planar motion.

4). The finite three-dimensional motion problem is recast in terms of determining the screw or rotational parameters.

5). Instantaneous geometric motion of a rigid body is defined in terms of the instantaneous screw axis for the infinitesimally separated positions in spatial motion.

6). The infinitesimal screw or rotational parameters are determined using a new mathematical method for three-dimensional motion.

7). The RRSS spatial mechanism to move through the multiply separated positions is synthesized for illustrative purpose.

CHAPTER 2

Analytical Synthesis of Planar Four-Bar Mechanisms for Multiply Separated Positions

2 - 1. Introduction

The technique of synthesizing a planar four-bar linkage to carry a lamina precisely through several given positions has been known for a long time. Basically, the problem of synthesizing a four-bar mechanism to move its coupler through a number of positions is simply that of locating points in the moving plane which lie on the same circle in all of the design positions. Such a point can be used as the moving pivot of a crank of the four-bar linkage. The center of the circle on which the positions of the point lie is the fixed pivot of the crank.

For three positions any point on the moving plane can be used as a pivot since a circle can always be drawn through three points. For four positions, the points whose positions all lie on the same circle are found to lie on a cubic curve, called the circle-point curve. If the circle is the locus for the moving pivot of a crank, then the fixed pivot lies at the center of that circle. Therefore, for each point on the circle-point curve there exists a point which represents the corresponding fixed pivot. The locus of these fixed pivots is also a cubic curve, called the center-point curve. Thus there is a one-to-one correspondence between points on these two curves. For five positions, the problem is solved by solving the four design position problem twice for two different sets of four out of the five design positions.

It is known that there are, at most, four points which lie on the same circle in all five of the design positions [4]. Thus there is a maximum of six linkages which may be designed for a five finitely separated position problem.

However, it may happen that none of these linkages is a desirable solution. Therefore the probability of a practicable solution for a five finitely separated position problem is greatly reduced from that of a four finitely separated position problem.

The finite four position problem has been studied very extensively because one can plot the circle-point curve by using Burmester's methods. Analytical derivation of the curve can be found from several papers [4-5]. In the case of four multiply separated position problem, Tesar and his associates [7] provided initiative work.

This chapter of the study provides the analytical method used to find the circle-point curve equations for the cases of PP-P-P, P-PP-P, P-P-PP, and PP-PP.

2 - 2. Finite and Infinitesimal Displacements

Consider a moving system σ in continuous motion relative to a fixed system Σ. We select a point P fixed in σ represented by the constant position vector $\mathbf{x}$, and $\mathbf{X}$ is the position vector of the coincident point of P in Σ. Thus the transformation between the coordinates of a point $P(x, y)$ in the moving system and $P(X, Y)$ in the fixed system can be expressed as follows:

$$\mathbf{X} = \mathbf{R}\mathbf{x} + \mathbf{T}, \tag{2.1}$$

where

$$\mathbf{R} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

The transformation depends on the position parameters a, b, ϕ where ϕ may be considered as the independent parameter of the constraint motion. The functions $a = a(\phi)$ and $b = b(\phi)$ represent the generalized constraints provided between the moving and fixed systems by the mechanism. In addition, the motion is assumed to occur over any time interval so that $\phi \neq f(t)$. In the initial position of the motion, the relative position of the coordinate systems may be chosen arbitrarily. Generally, the systems are assumed to be coincident by requiring that $a_1 = b_1 = \phi_1 = 0$ and

$$\mathbf{X}_1 = \mathbf{x}. \tag{2.2}$$

Thus equation (2.1) becomes for the finitely separated positions

$$\mathbf{X}_i = \mathbf{R}_i\mathbf{x} + \mathbf{T}_i, \qquad i = 2, 3, ..., n \tag{2.3}$$

where

$$\mathbf{R}_i = \begin{pmatrix} \cos\phi_i & -\sin\phi_i \\ \sin\phi_i & \cos\phi_i \end{pmatrix}, \qquad \mathbf{T}_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix},$$

and a_i, b_i, ϕ_i are the parameters governing the relative position of σ_i and Σ. If the moving positions are given by a lamina, that is $\mathbf{C}_i(X_i, Y_i)$, θ_i, the parameters are determined as follows:

$$\phi_i = \theta_i - \theta_1, \tag{2.4}$$

$$\mathbf{T}_i = \mathbf{C}_i - \mathbf{R}_i\mathbf{C}_1. \tag{2.5}$$

If we are interested only in the study of kinematic geometry of the motion of plane σ and we exclude the case of pure translation (i.e., ϕ = constant), we may write the 1st derivative of equation (2.1) with respect to ϕ:

$$\frac{d\mathbf{X}}{d\phi} = \frac{d\mathbf{R}}{d\phi}\mathbf{x} + \frac{d\mathbf{T}}{d\phi}. \tag{2.6}$$

The instantaneous center at the jth position is expressed by $\mathbf{I}_j(I_{xj}, I_{yj})$. And the vector $\mathbf{i}$ represents the corresponding points of $\mathbf{I}_j$ on the moving system σ. Then we may write the following equations from the equations (2.3) and (2.6)

$$\mathbf{I}_j = \mathbf{R}_j \mathbf{i} + \mathbf{T}_j, \tag{2.7}$$

$$\frac{d\mathbf{I}_j}{d\phi} = \frac{d\mathbf{R}_j}{d\phi}\mathbf{i} + \frac{d\mathbf{T}_j}{d\phi}. \tag{2.8}$$

Since the vector $\mathbf{I}_j$ of the instantaneous center does not change in the fixed system at the instant of the jth position, we may have

$$\frac{d\mathbf{I}_j}{d\phi} = \mathbf{0}. \tag{2.9}$$

Substituting equations (2.7) and (2.8) into equation (2.9) yields

$$\frac{d\mathbf{R}_j}{d\phi}\mathbf{R}_j^{-1}(\mathbf{I}_j - \mathbf{T}_j) + \frac{d\mathbf{T}_j}{d\phi} = \mathbf{0}. \tag{2.10}$$

Now the $\frac{d\mathbf{T}_j}{d\phi}$ is determined as follows

$$\frac{d\mathbf{T}_j}{d\phi} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\mathbf{I}_j - \mathbf{T}_j). \tag{2.11}$$

Thus the infinitesimal linear transformation at the jth position can be written as

$$\frac{d\mathbf{X}_j}{d\phi} = \frac{d\mathbf{R}_j}{d\phi}\mathbf{x} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\mathbf{I}_j - \mathbf{T}_j), \tag{2.12}$$

where

$$\frac{d\mathbf{R}_j}{d\phi} = \begin{pmatrix} -\sin\phi_j & -\cos\phi_j \\ \cos\phi_j & -\sin\phi_j \end{pmatrix}, \quad \text{and} \quad \mathbf{I}_j = \begin{pmatrix} I_{xj} \\ I_{yj} \end{pmatrix}.$$

2 - 3. Constraint Equations

The pinned crank is used as a constraint on a moving point which travels on a circular arc in the fixed plane. The equation of this circle can be written, in general, as

$$(X - X^*)^2 + (Y - Y^*)^2 = R^2, \tag{2.13}$$

where (X^*, Y^*) and R are the coordinates of the center-point and the radius of the circle respectively. And the derivative form of equation (2.13) is

$$(X - X^*)\frac{dX}{d\phi} + (Y - Y^*)\frac{dY}{d\phi} = 0. \tag{2.14}$$

Note that the derivatives of R, X^*, Y^* vanish.

2 - 4. Derivation of Circle-Point Equation for the Case of PP-P-P, P-PP-P, or P-P-PP

The rigid body motion is represented by the three finitely separated positions and one infinitesimally separated position. For the three finitely separated positions, substitution of equation (2.3) into equation (2.13) yields

$$\begin{aligned}&(x\cos\phi_i - y\sin\phi_i + a_i - X^*)^2\\&+(x\sin\phi_i + y\cos\phi_i + b_i - Y^*)^2 = R^2, \qquad i = 2, 3,\end{aligned} \tag{2.15}$$

and from equation (2.2), equation (2.13) yields for the first design equation

$$(x - X^*)^2 + (y - Y^*)^2 = R^2. \tag{2.16}$$

Equating the left hand sides of equations (2.15) and (2.16) and rearranging yields

$$(-x\cos\phi_i + y\sin\phi_i + x - a_i)X^* + (-x\sin\phi_i - y\cos\phi_i + y - b_i)Y^*$$

$$+\left(x(a_i\cos\phi_i + b_i\sin\phi_i) + y(b_i\cos\phi_i - a_i\sin\phi_i) + \frac{a_i^2 + b_i^2}{2}\right) = 0, i = 2, 3. \quad (2.17)$$

For the one infinitesimally separated position at the jth finite position, substitution of equations (2.3) and (2.12) into equation (2.14) yields

$$(x\cos\phi_j - y\sin\phi_j + a_j - X^*)(-x\sin\phi_j - y\cos\phi_j + I_{jy} - b_j)$$

$$+ (x\sin\phi_j + y\cos\phi_j + b_j - Y^*)(x\cos\phi_j - y\sin\phi_j - I_{jx} + a_j) = 0. \quad (2.18)$$

Rearranging equation (2.18) yields

$$(x\sin\phi_j + y\cos\phi_j - I_{jy} + b_j)X^* + (-x\cos\phi_j + y\sin\phi_j + I_{jx} - a_j)Y^*$$

$$+\Big((-x\sin\phi_j - y\cos\phi_j - b_j)I_{xj} + (x\cos\phi_j - y\sin\phi_j + a_j)I_{yj}\Big) = 0. \quad (2.19)$$

In order to simplify the equations, the following matrices are introduced, based on equations (2.17) and (2.19)

$$\mathbf{a} = \begin{pmatrix} 1-\cos\phi_2 & \sin\phi_2 & -a_2 \\ 1-\cos\phi_3 & \sin\phi_3 & -a_3 \\ \sin\phi_j & \cos\phi_j & b_j - I_{jy} \end{pmatrix}, \quad (2.20)$$

$$\mathbf{b} = \begin{pmatrix} -\sin\phi_2 & 1-\cos\phi_2 & -b_2 \\ -\sin\phi_3 & 1-\cos\phi_3 & -b_3 \\ -\cos\phi_j & \sin\phi_j & I_{jx} - a_j \end{pmatrix}, \quad (2.21)$$

$$\mathbf{c} = \begin{pmatrix} a_2\cos\phi_2 + b_2\sin\phi_2 & b_2\cos\phi_2 - a_2\sin\phi_2 & (a_2^2 + b_2^2)/2 \\ a_3\cos\phi_3 + b_3\sin\phi_3 & b_3\cos\phi_3 - a_3\sin\phi_3 & (a_3^2 + b_3^2)/2 \\ -I_{xj}\sin\phi_j + I_{yj}\cos\phi_j & -I_{yj}\sin\phi_j - I_{xj}\cos\phi_j & -b_jI_{xj} + a_jI_{yj} \end{pmatrix}$$

(2.22)

Thus equations (2.17) and (2.19) become

$$\mathbf{a}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} X^* + \mathbf{b}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} Y^* + \mathbf{c}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \mathbf{0}. \tag{2.23}$$

The above constraint equarions are linear in the X^* and Y^*. Thus for the solution, we require

$$\left| \mathbf{a}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \quad \mathbf{b}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \quad \mathbf{c}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \right| = 0. \tag{2.24}$$

Equation (2.24) can be written

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} A_{ijk} x^m y^n = 0, \tag{2.25}$$

where

$$A_{ijk} = \begin{vmatrix} a_{1i} & b_{1i} & c_{1i} \\ a_{2j} & b_{2j} & c_{2j} \\ a_{3k} & b_{3k} & c_{3k} \end{vmatrix},$$

$$m = \text{the number of } 1'\text{s among the } i, j, k,$$

$$n = \text{the number of } 2'\text{s among the } i, j, k,$$

which upon expansion and rearrangement yields the circle point equation

$$\begin{aligned} &A_{111}x^3 + A_{222}y^3 + (A_{112} + A_{121} + A_{211})x^2y + (A_{122} + A_{212} + A_{221})xy^2 \\ &+(A_{123} + A_{132} + A_{213} + A_{231} + A_{312} + A_{321})xy + (A_{113} + A_{131} + A_{311})x^2 \\ &+(A_{223} + A_{232} + A_{322})y^2 + (A_{133} + A_{313} + A_{331})x \\ &+(A_{233} + A_{323} + A_{332})y + A_{333} = 0. \end{aligned} \tag{2.26}$$

The center point for the circle point can be determined by using any two equations in the equation set (2.23). For the four finitely separated positions, the circle-point equation has the same form of equations (2.25), (2.26) if we use

$$\mathbf{a} = \begin{pmatrix} 1-\cos\phi_2 & \sin\phi_2 & -a_2 \\ 1-\cos\phi_3 & \sin\phi_3 & -a_3 \\ 1-\cos\phi_4 & \sin\phi_4 & -a_4 \end{pmatrix}, \tag{2.27}$$

$$\mathbf{b} = \begin{pmatrix} -\sin\phi_2 & 1-\cos\phi_2 & -b_2 \\ -\sin\phi_3 & 1-\cos\phi_3 & -b_3 \\ -\sin\phi_4 & 1-\cos\phi_4 & -b_4 \end{pmatrix}, \tag{2.28}$$

$$\mathbf{c} = \begin{pmatrix} a_2\cos\phi_2 + b_2\sin\phi_2 & b_2\cos\phi_2 - a_2\sin\phi_2 & (a_2^2+b_2^2)/2 \\ a_3\cos\phi_3 + b_3\sin\phi_3 & b_3\cos\phi_3 - a_3\sin\phi_3 & (a_3^2+b_3^2)/2 \\ a_4\cos\phi_4 + b_4\sin\phi_4 & b_4\cos\phi_4 - a_4\sin\phi_4 & (a_4^2+b_4^2)/2 \end{pmatrix}. \tag{2.29}$$

2 - 5. Example for the case of P-PP-P

An illustrative example for the Section 2-4 is to synthesize a planar four-bar linkage to pass through the specified three finitely separated positions and have an instantaneous center at a specified position. The data of desired motion is given in Table 2-1. By letting the first position of σ coincide with Σ and using equations (2.4) and (2.5), we have

$$\phi_2 = 20°, \qquad \phi_3 = 65°,$$

$$a_2 = -62.72, \qquad a_3 = 27.71,$$

$$b_2 = -37.20, \qquad b_3 = -12.07.$$

Using equation (2.25) the circle-point equation becomes

Table 2-1. Design data for P-PP-P in planar motion

1st position	$C_1(\ 37,\ \ 6)$, $\Theta_1= 70^\circ$
2nd position	$C_2(-30,\ 46)$, $\Theta_2= 90^\circ$
3rd position	$C_3(-27,\ 24)$, $\Theta_3=135^\circ$
instantaneous center at 2nd position	$I_2(-30,\ 10)$

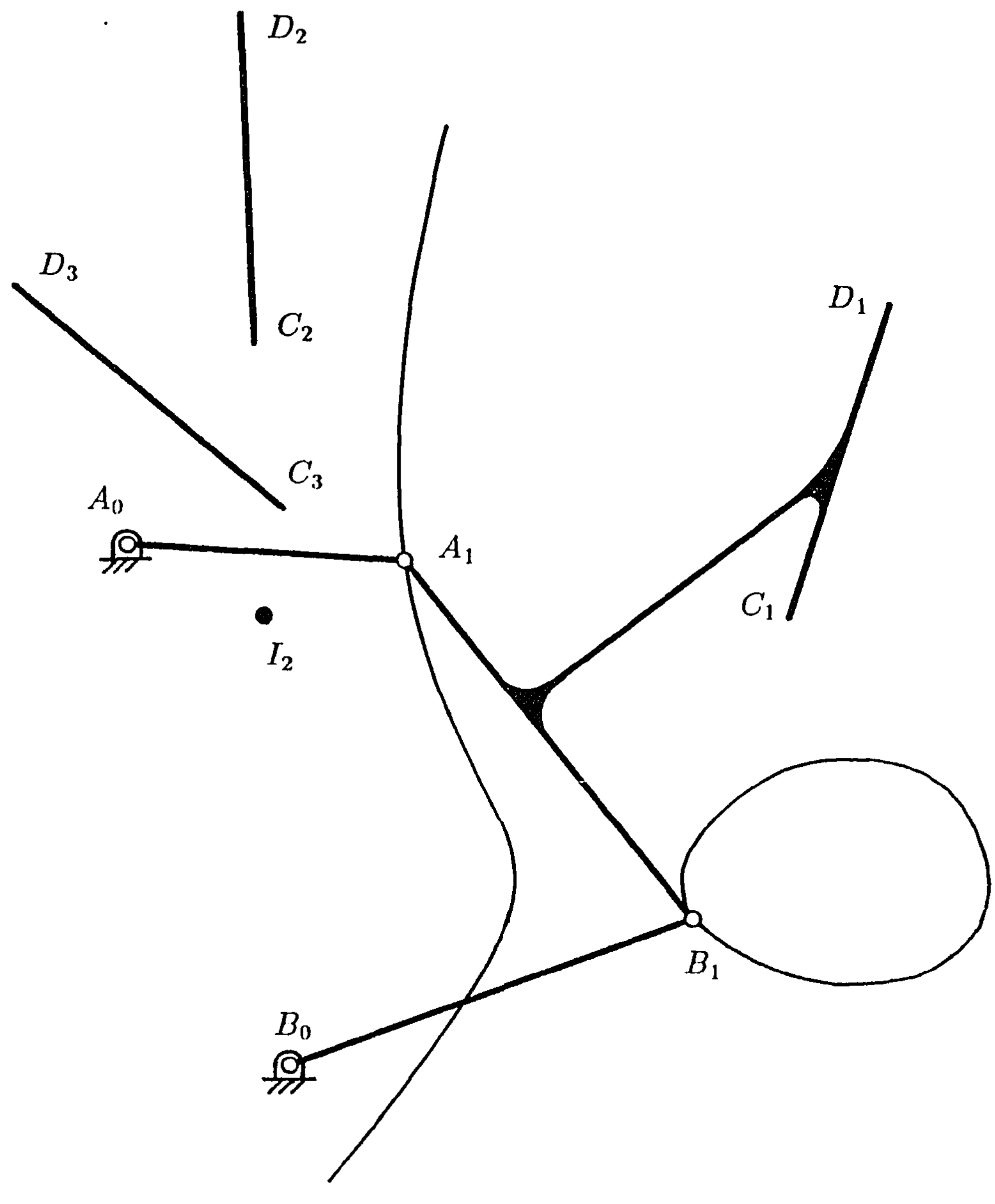

Figure 2-1. The circle-point curve and a solution linkage for the prescribed P-PP-P planar motion

$$(x^2 + y^2)(15.92x - 9.23y) - 2001.57x^2 + 341.79y^2$$

$$+2059.15xy + 69646.54x + 36931.05y + 778796.51 = 0,$$

and the circle-point curve is drawn in Figure 2-1. Choosing two points, one for the driving crank the other for the driven crank, a four-bar linkage is synthesized as shown in Figure 2-1.

2 - 6. Derivation of Circle-Point Equation for the Case of PP-PP

The rigid body motion is represented by the two finitely separated positions and the two infinitesimally separated positions. For the two finitely separated positions, substitution of equation (2.3) into equation (2.13) for the second position yields

$$(x \cos\phi_2 - y \sin\phi_2 + a_2 - X^*)^2 + (x \sin\phi_2 + y \cos\phi_2 + b_2 - Y^*)^2 = R^2, \quad (2.30)$$

and from equation (2.2), equation (2.13) yields for the first design position

$$(x - X^*)^2 + (y - Y^*)^2 = R^2. \quad (2.31)$$

Equating the left hand sides of equations (2.15) and (2.16) and rearranging yields

$$(-x \cos\phi_2 + y \sin\phi_2 + x - a_2)X^* + (-x \sin\phi_2 - y \cos\phi_2 + y - b_2)Y^*$$

$$+ \left(x(a_2 \cos\phi_2 + b_2 \sin\phi_2) + y(b_2 \cos\phi_2 - a_2 \sin\phi_2) + \frac{a_2^2 + b_2^2}{2} \right) = 0. \quad (2.32)$$

For the two infinitesimally separated positions, substitution of equations (2.3) and (2.12) into equation (2.14) yields

$$(x\cos\phi_j - y\sin\phi_j + a_j - X^*)(-x\sin\phi_j - y\cos\phi_j + I_{jy} - b_j)$$

$$+ (x\sin\phi_j + y\cos\phi_j + b_j - Y^*)(x\cos\phi_j - y\sin\phi_j - I_{jx} + a_j) = 0, \quad (2.33)$$

$$j = 1, 2.$$

Rearranging equation (2.24) and considering $a_1 = b_1 = \phi_1 = 0$ yield

$$(y - I_{1y})X^* + (-x + I_{1x})Y^* + (-yI_{x1} + xI_{y1}) = 0 \quad (2.34)$$

and

$$(x\sin\phi_2 + y\cos\phi_2 - I_{2y} + b_2)X^* + (-x\cos\phi_2 + y\sin\phi_2 + I_{2x} - a_2)Y^*$$

$$+\Big((-x\sin\phi_2 - y\cos\phi_2 - b_2)I_{x2} + (x\cos\phi_2 - y\sin\phi_2 + a_2)I_{y2}\Big) = 0. \quad (2.35)$$

In order to simplify the equations, the following matrices are introduced based on equations (2.32), (2.34), and (2.35)

$$\mathbf{a} = \begin{pmatrix} 1 - \cos\phi_2 & \sin\phi_2 & -a_2 \\ 0 & 1 & -I_{1y} \\ \sin\phi_2 & \cos\phi_2 & b_2 - I_{2y} \end{pmatrix}, \quad (2.36)$$

$$\mathbf{b} = \begin{pmatrix} -\sin\phi_2 & 1 - \cos\phi_2 & -b_2 \\ -1 & 0 & I_{1x} \\ -\cos\phi_2 & \sin\phi_2 & I_{2x} - a_2 \end{pmatrix}, \quad (2.37)$$

$$\mathbf{c} = \begin{pmatrix} a_2\cos\phi_2 + b_2\sin\phi_2 & b_2\cos\phi_2 - a_2\sin\phi_2 & (a_2^2 + b_2^2)/2 \\ I_{y1} & -I_{x1} & 0 \\ -I_{x2}\sin\phi_2 + I_{y2}\cos\phi_2 & -I_{y2}\sin\phi_2 - I_{x2}\cos\phi_2 & -b_2I_{x2} + a_2I_{y2} \end{pmatrix}$$

$$(2.38)$$

Thus equations (2.32), (2.34), and (2.35) become

$$\mathbf{a}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} X^* + \mathbf{b}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} Y^* + \mathbf{c}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0. \tag{2.39}$$

The above constraint equations are linear in X^* and Y^*. Thus for the solution, we require

$$\left| \mathbf{a}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \quad \mathbf{b}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \quad \mathbf{c}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \right| = 0. \tag{2.40}$$

Equation (2.40) can be written

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} A_{ijk} x^m y^n = 0, \tag{2.41}$$

where

$$A_{ijk} = \begin{vmatrix} a_{1i} & b_{1i} & c_{1i} \\ a_{2j} & b_{2j} & c_{2j} \\ a_{3k} & b_{3k} & c_{3k} \end{vmatrix},$$

$$m = \text{the number of } 1'\text{s among the } i, j, k,$$

$$n = \text{the number of } 2'\text{s among the } i, j, k,$$

which upon expansion and rearrangement yields the circle point equation

$$\begin{aligned} &A_{111}x^3 + A_{222}y^3 + (A_{112} + A_{121} + A_{211})x^2y + (A_{122} + A_{212} + A_{221})xy^2 \\ &+(A_{123} + A_{132} + A_{213} + A_{231} + A_{312} + A_{321})xy + (A_{113} + A_{131} + A_{311})x^2 \\ &+(A_{223} + A_{232} + A_{322})y^2 + (A_{133} + A_{313} + A_{331})x \\ &+(A_{233} + A_{323} + A_{332})y + A_{333} = 0. \end{aligned} \tag{2.42}$$

The center point for the circle point can be determined using any two equations of the equation set (2.39).

2 - 7. Example for the case of PP-PP

An illustrative example for the Section 2-6 is to synthesize a planar four-bar linkage to pass through the specified two finitely separated positions and have two instantaneous centers at the specified positions. The data of desired motion is given in Table 2-2. By letting the first position of σ coincide with Σ and using equations (2.4) and (2.5), we have

$$\phi_2 = -68^\circ, \qquad a_2 = 9.256, \qquad b_2 = 11.694.$$

Using equation (2.41) the circle-point equation becomes

$$(x^2 + y^2)(14.44x - 4.91y) - 1005.50x^2 + 217.52y^2$$

$$+1283.74xy + 20235.01x - 15354.31y - 124800.81 = 0,$$

and the circle-point curve is drawn in Figure 2-2. Choosing two points, one for the driving crank the other for the driven crank, a four-bar linkage is synthesized as shown in Figure 2-2.

Table 2-2. Design data for PP-PP in planar motion

1st position	$C_1(25,\ 36)$, $\Theta_1 = 0^\circ$
2nd position	$C_2(52,\ 2)$, $\Theta_2 = -68^\circ$
instantaneous center at 1st position	$I_1(40,\ 0)$
instantaneous center at 2nd position	$I_2(-22,\ 22)$

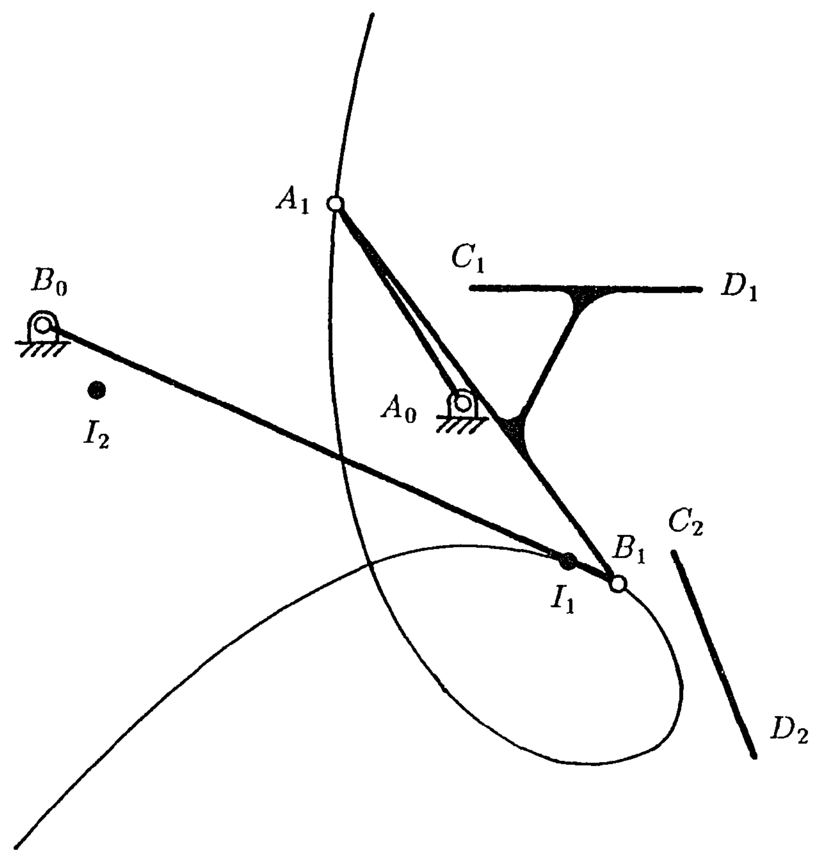

Figure 2-2. The circle-point curve and a solution linkage for the prescribed PP-PP planar motion

CHAPTER 3

Graphical Synthesis of Planar Four-Bar Mechanisms for Multiply Separated Positions

3 - 1. Introduction

Burmester used the concepts of poles, image poles, circle-point and center-point curves to develop geometric methods for synthesizing mechanisms. These ideas were later extended by Alt [8], Beyer [9], and Hain [10].

A pole is the point in the fixed plane about which the moving lamina rotates for a pair of design positions. The image pole is the pole as seen relative to the moving plane. Figure 3-1 shows the poles and the image poles for the three finitely separated positions.

Müller [11] developed numerous synthesis methods for infinitesimally separated position problems. For infinitesimally separated position problems, the desired motion of a rigid body can be given by the instantaneous center. The instantaneous centers are found by locating the intersection of the normal to the path tangent for each end of the coupler. And the instantaneous centers can be handled in the same manner as a pole for finitely separated positions problem.

When the finitely separated position problem and infinitesimally separated position problem are combined, they are called multiply separated position problems. Previous work in this area using an analytical numerical approach is that of Tesar and his associates [3,7]. Graphical solutions to the multiply separated position problems have been presented by Volmer [12], Dijksun [13], Hain [10] and Waldron [14]. Tesar and Carrero [15] have developed graphical solutions to FSP, ISP, and MSP problems.

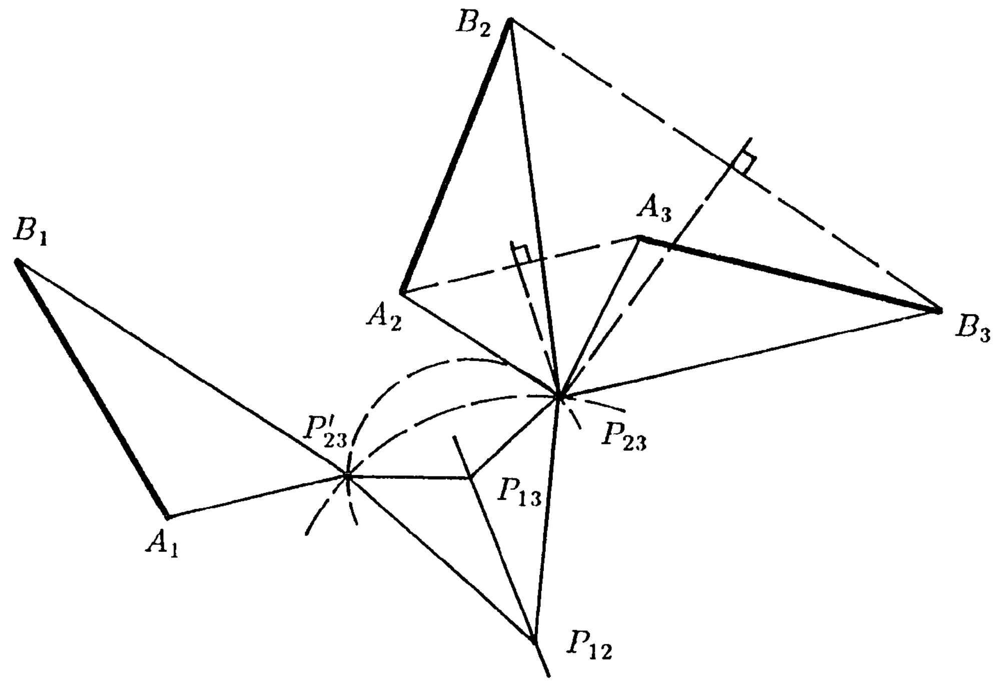

Figure 3-1. Poles and image poles for three finitely separated positions

Since there is no work done the PP-PP case of the MSP problems, the objective of this chapter is to review the conventional methods for the solution of the problem of the P-P-P-P case, apply the method to the PP-P-P and PP-PP MSP cases, and find the simpler method for construction of both the circle-point and center-point curves for the PP-PP case of MSP problems.

3 - 2. Circle-point and Center-Point Curves for P-P-P-P, PP-P-P, and PP-PP

Before the MSP problem is considered, the four FSP problem is reviewed. If we consider the four specified finitely separated positions of a moving body, the four positions of an arbitrary selected point belonging to that body will not, in general, fall on a circle. However, some points whose four positions do fall on circles can be obtained. Such points would be suitable locations for crank-pins, with crank pivots at the circle centers, for guiding the body through the specified positions.

In Figure 3-2, let point A be a point belonging to a moving body and so chosen that its four positions, A_1, A_2, A_3, A_4, fall on a circle with center A_c. In dealing with the four positions, six poles are involved: P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}. In Figure 3-2, if we draw the line l_{12} which is the perpendicular bisector of A_1A_2, the pole P_{12} is on the line l_{12}. Similarly we can locate all the lines which pass through the poles. Then these lines all pass through a point A_c.

By studying the angles between the various lines in Figure 3-2, we see that the angle between the lines l_{14}, l_{24} and the angle between the lines l_{13}, l_{23} are equal or else differ by 180 degree. And for the lines l_{24}, l_{34} and the lines l_{12}, l_{13}, we have same condition as for the lines l_{14}, l_{24} and the lines l_{13}, l_{23}.

Now with these ideas, the center point A_c can be located by using the

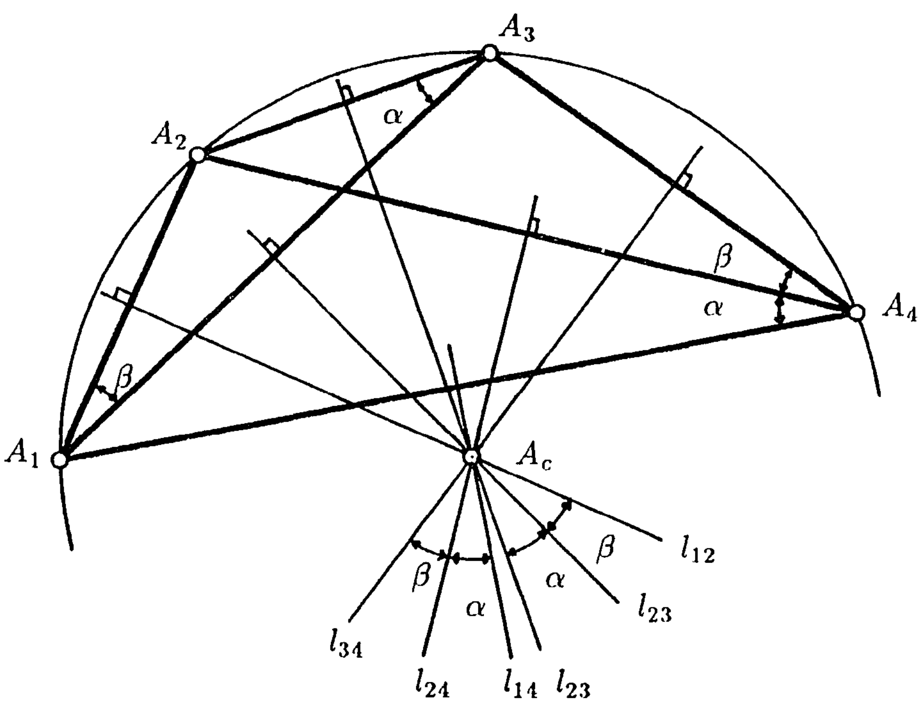

Figure 3-2. Angles between the various pole lines for the four finitely separated positions

poles. As an example, we take the poles P_{12}, P_{23}, P_{14}, P_{34} as shown in Figure 3-3. By choosing the point C which is on the perpendicular bisector of P_{12} P_{23} as chord, the point C' is located to make similar triangle $C'P_{14}P_{34}$ to $CP_{12}P_{23}$. Then the point A_c is at the intersections of two circles which have centers C, C' passing through the points P_{12}, P_{23} and P_{14}, P_{34} respectively, because the angles α, α' and angles β, β' are equal or differ by 180 degree. For the additional points of the center-point curve, we choose the other point C on the perpendicular bisector of the chord.

But with these four poles, the entire curve is not completed. In order to select another set of poles, we use the following terminologies. Two poles whose subscripts do not contain a common numeral are called "opposite poles." For example, P_{12} and P_{34} are opposite poles. Two pairs of opposite poles form an "opposite pole quadrangle". "Adjacent poles" are those whose subscripts do contain a common numeral such as P_{12} and P_{23}. The "sides" of an opposite pole quadrangle are the lines joining adjacent poles. Figure 3-4 shows the several configurations of the opposite pole quadrangles.

In order to complete the center-point curve, we take the other pair of sides of the opposite pole quadrangle or another configuration of opposite pole quadrangle.

If the moving plane is considered as the reference frame, the fixed pivot is observed to move through four different positions relative to an observerer on the moving plane. Then the moving pivot becomes a fixed pivot at the center of the four apparent positions of the fixed pivot as seen by the observer. In Figure 3-5, the point A_0^i is the inverted point of A_0. Since image poles are inverted from the poles, the circle-point curve can be plotted using the image poles by the same way as the center-point curve.

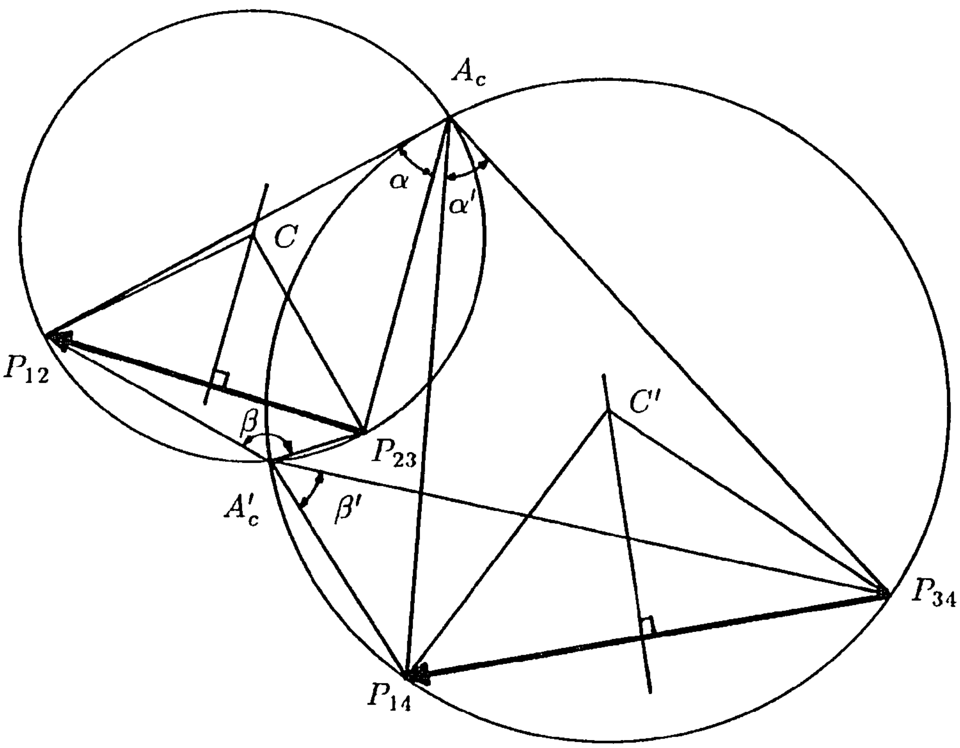

Figure 3-3. Construction for finding points of the circle-point curve

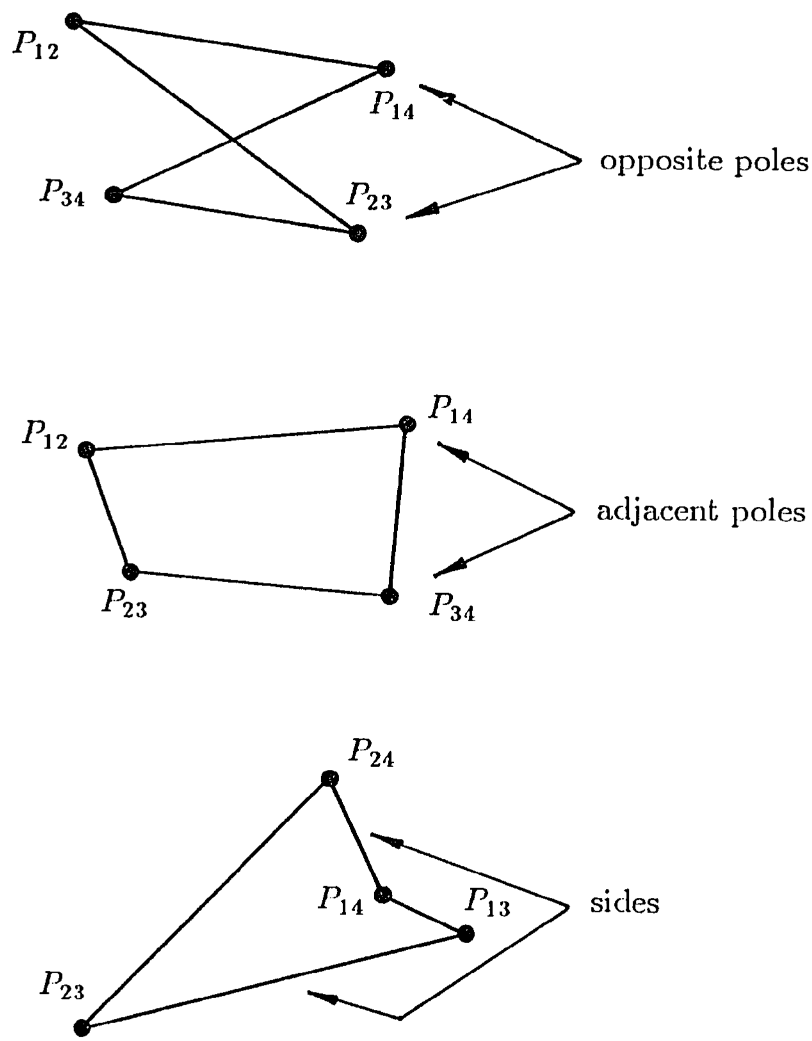

Figure 3-4. Some possible configurations of opposite pole quadrangles

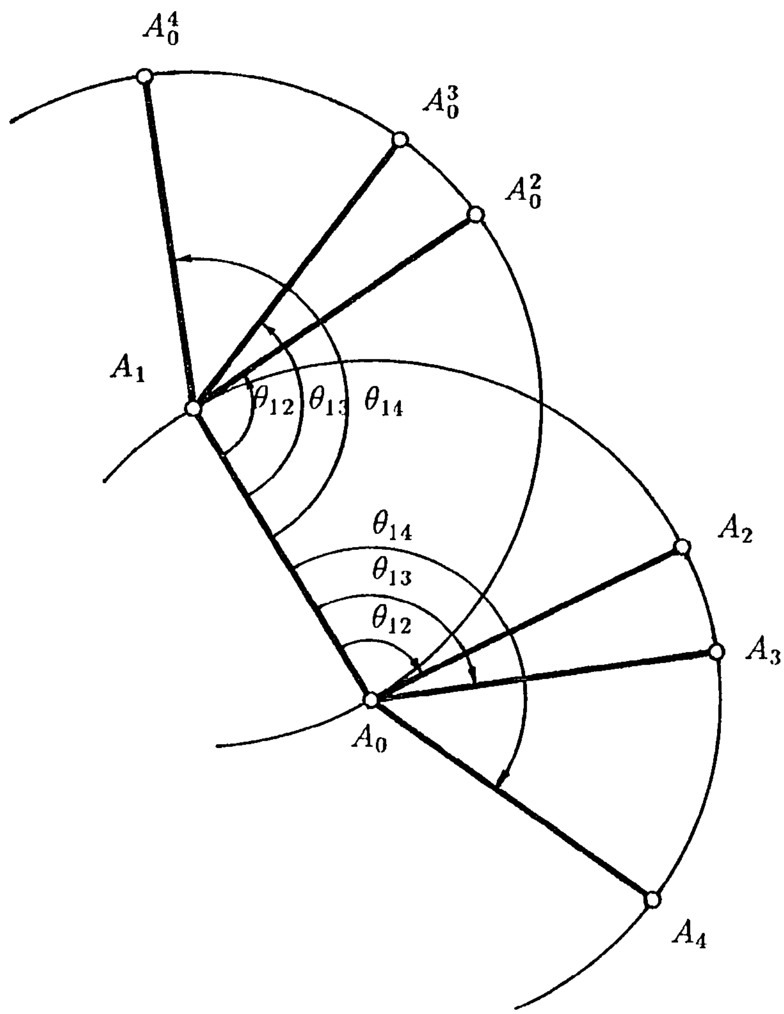

Figure 3-5. Inversion of center point A_0 on the first moving plane

There is one special circle point on the circle-point curve. This is the Ball point **B**. It is the circle point whose four positions lie on a line. That is, the radius of the circle on which its positions lie has become infinite. It is the point which is used as the pivot between coupler and slider if it is desired to use a slider-crank rather than a four-bar linkage. The graphical location of the Ball point is as follows. Any three image poles which have only three distinct subscripts between them are called a 'closed set' of poles. For example, P_{12}, P_{13}, P'_{23} is a closed set since only 1, 2, and 3 appear as subscripts. We choose any two closed sets of image poles and construct two circles on which each of these sets lie. One of the two intersections of the circles will be at an image pole. The other is the Ball point **B**.

For the PP-P-P case of the multiply separated positions, A_1B_1, A_2B_2, A_3B_3, are specified as the three finitely separated positions, and the instantaneous center I_1 is specified as an infinitesimally separated position at the first position of the lamina as shown in Figure 3-6. Thus it is considered that the positions 1 and 4 are the ISP and $I_1 = P_{14}$. And since positions 1 and 4 are same, $P_{24} = P_{21}$, and $P_{34} = P_{31}$, and we have four distinctive poles which make one configuration of opposite pole quadrangle.

Figure 3-7 shows the PP-PP case of MSP, consisting of two finitely separated positions, A_1B_1, A_2B_2, and two infinitesimally separated positions, I_1, I_2 which are the instantaneous centers of positions 1 and 2 respectively. It is considered that positions 1 and 4 are the ISP and $I_1 = P_{14}$. Similarly for the position 2 and 3, we have $I_2 = P_{23}$. Since positions 1 and 4 are the same, and also the positions 2 and 3 are the same, $P_{12} = P_{13} = P_{24} = P_{34}$. Thus we have three distinctive poles which can make a pair of opposite sides. Since these sides shares the common pole, the curve can be completed by

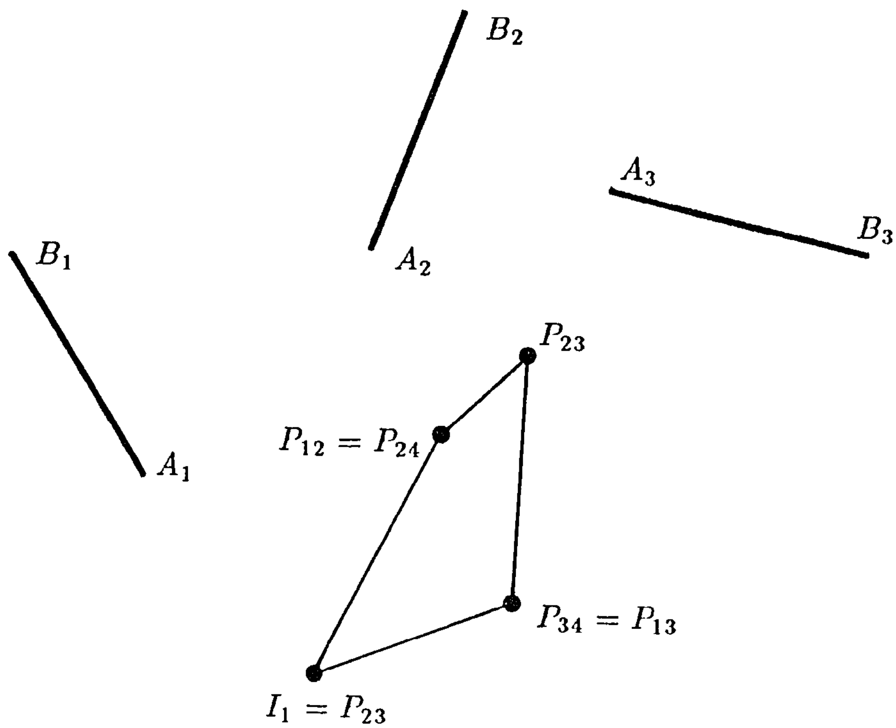

Figure 3-6. An opposite pole quadrangle for PP-P-P

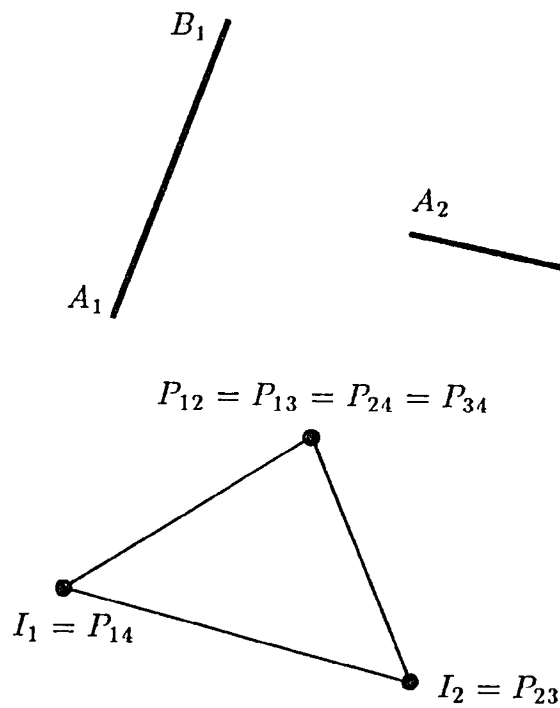

Figure 3-7. An opposite pole quadrangle for PP-PP

only that sides.

3 - 3. Simpler Method for Construction of Both Circle-point and Center-Point Curves for PP-PP

Section 3-2 showed one of the several construction methods [9,38]. This section will show another simpler construction method for the case of PP-PP.

Figure 3-8 shows two finitely separated positions, A_1B_1, A_2B_2, and two infinitesimally separated positions described by I_1, I_2 which are the instantaneous centers at the positions 1 and 2 respectively. The two finitely separated positions 1 and 2 define a pole P_{12}. We define the lamina's angular displacement ϕ_{12} as angular motion between the two positions of the moving plane about the pole. If we draw the line l_1 on the 1st moving plane position which passes through the pole P_{12}, its 2nd position l_2 on the fixed frame can be drawn by rotating the line l_1 by ϕ_{12} about P_{12} as shown in Figure 3-8. If there is a circle point A_1 on the line l_1, the the 2nd position A_2 of the circle point is on the line l_2 with the condition $P_{12}A_1 = P_{12}A_2$. Constant link length of the crank makes the center point A_c lie on the center line l_c which is the bisector line of $\angle A_1P_{12}A_2$. Thus line A_2A_c is the mirror image of the line A_1A_c about l_c. And the instantaneous centers, I_1 and I_2, are located on the link lines, A_cA_1 and A_cA_2, respectively. If we make the mirror image instantaneous center I_2' of I_2 about the center line l_c, the points A_c, A_1, I_1, I_2' are on the same line. Thus the circle point A_1 is the intersection of the lines l_1 and I_1I_2' and the center point A_c is the intersection of the lines l_c and I_1I_2'

Figure 3-9 shows how to locate the Ball point. Since line joining I_1I_2' is parallel to line l_c, the center point is located at infinity. Thus the Ball point

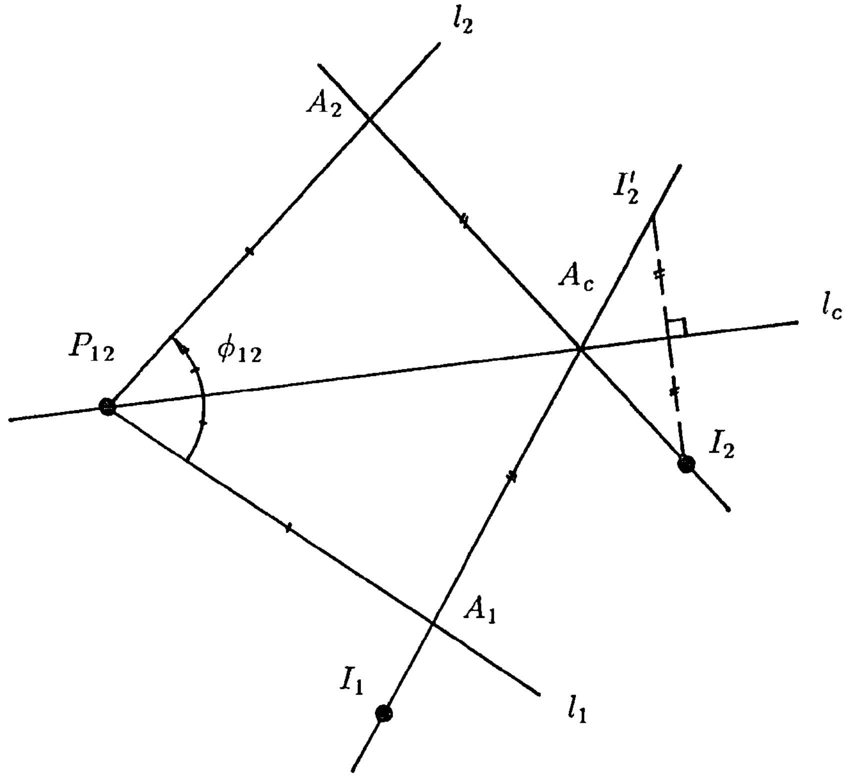

Figure 3-8. A straight line on which A_1, A_c, I_1 and I_2' are located

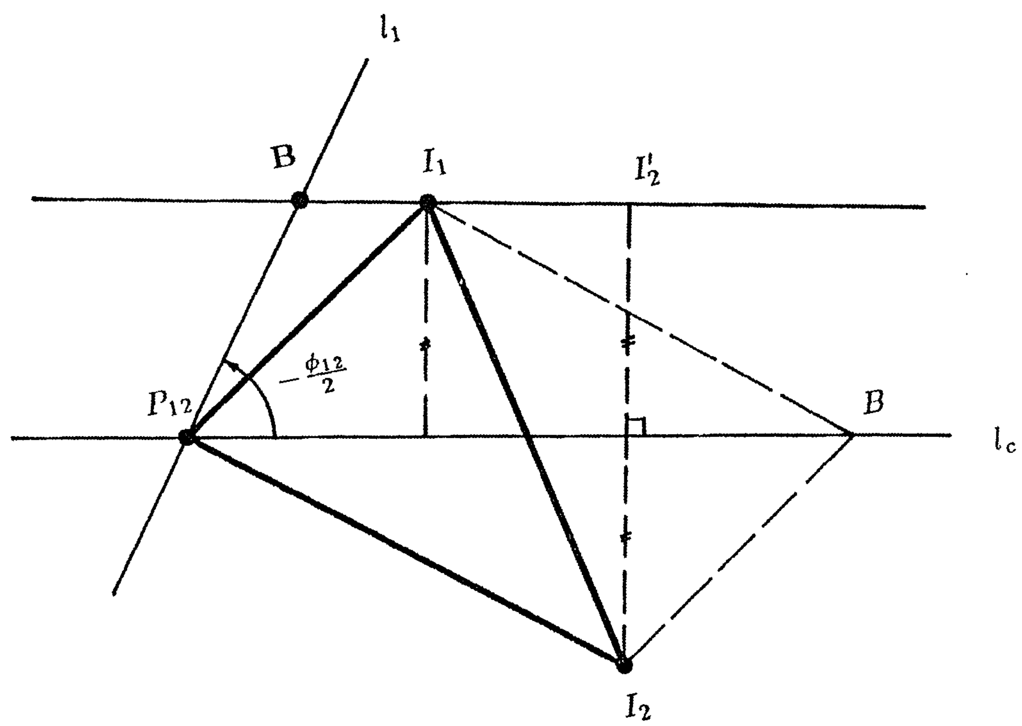

Figure 3-9. Location of the Ball point

is at the intersection of lines l_1 and $I_1 I_2'$. Because the distance between the point I_1 and the line l_c and the distance between the point I_2' and the line l_c are same, the center point line l_c for the Ball point is the line joining $P_{12}B$ where the point B is located by making a parallelogram $I_1 P_{12} I_2 B$.

Figure 3-10 shows a way to plot the points on the center curve and circle curve, given two laminae and two instantaneous centers. The procedure is as follows:

1. Locate a pole P_{12}.
2. Measure the lamina's angular displacement ϕ_{12}.
3. Locate a point B by making a parallelogram $I_1 P_{12} I_2 B$.
4. Draw a line l_c passing through the pole P_{12} and B.
5. Draw a line l_1 by rotating the line l_1 by $-\frac{\phi_{12}}{2}$ about the pole P_{12}.
6. Locate a mirror image instantaneous center I_2' of I_2 about the line l_c.
7. Draw a line $I_1 I_2'$ passing through the points I_1 and I_2'.
8. The lines l_1 and $I_1 I_2'$ intersect at the Ball point **B**.
9. Rotate the line l_c about the pole P_{12} and follow the steps 4, 5, 6, 7.
10. The lines l_1 and $I_1 I_2'$ intersects at the circle point A_1 and the lines l_c and $I_1 I_2'$ intersects at the center point A_c.
11. As many additional points of the curves as desired can be plotted by drawing additional lines l_1.

3 - 4. Design Example

In order to design a four-bar linkage capable of assuming the desired positions, in principle, one has only to select two points from either cubic curve and locate the corresponding points on the remaining cubic curve. But this procedure does not always guarantee that the resulting linkage can

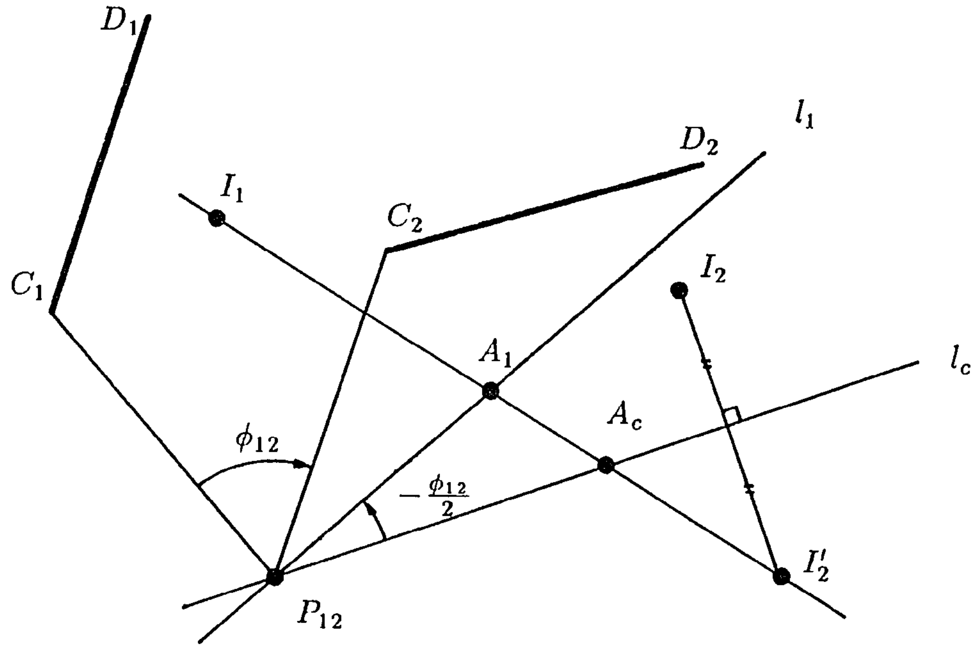

Figure 3-10. Construction of the circle-point and center-point curves

perform the desired motion. As is well known, this technique produces three types of defective solutions. One of these is a solution linkage which will not traverse all the design positions without being disconnected and reassembled in a different configuration. The appearance of this type of defective solution is concerned with the appearance of two distinct branches of the coupler curve in the linkage. For this reason, the elimination of this type of defective solution is referred to as the branch problem.

The second type of defective solution arises if the linkage is required to have a crank capable of complete rotation. It is frequently necessary to drive the linkage by means of a continuously rotating motor. Since Grashof's rules are used to distinguish linkage types which have at least one continuously rotating crank, the problem of identifying such solutions is referred to as the Grashof problem.

The third type of defective solution is also a result of the requirement for a continuously rotating driving crank. When the crank is driven in a uniform direction it may happen that the coupler of the linkage moves through the design positions in the wrong order. The classification of solutions that move through the design positions in wrong order is referred to as the order problem.

No convenient solution of the Grashof problem is yet available, although Beyer [9] and Filemon [16] have done useful work on it graphically. Geometric solutions of the branch and order problem for four finitely separated positions are now available through the work of Filemon [16] and Waldron [13]. Kim and Sodhi [18] solved these three problems by using the computer graphics techniques.

Waldron [19] studied the problem of PP-P-P MSP case by solving the branch and order problems. Since there is no available solution for the problem of PP-PP, an example is presented here to show the designing process using the generated cubic curves. Unless the rotational directions of the lamina about the instantaneous centers are required, the order problem is not significant. But it is noteworthy that, even in this rather trivial two finitely separated positions case, the branch problem can still arise. Figure 3-11 shows a solution of the PP-PP problem in which the two design positions can not be reached from one another without disconnecting the linkage. This is indicated by the change in sign of the angle ψ between the two positions. For the two FSP and two ISP, any point on the circle-point curve can be chosen as the driven crank circle point. The corresponding center point can be located on the center-point curve by using the graphical method developed in Section 4-3. In principle, any other point on the circle-point curve can be chosen as the driving crank circle-point. However, there is a portion of the cubic curve for which driving crank circle-points will give solution linkages which must change branch to move between the two finitely separated positions. This portion is in the shaded region in Figure 3-12. The construction used was first applied by Filemon [16] to the 4FSP case. It has been discussed in references [17,19] and will, therefore, not be further treated here. Figure 3-13 shows an alternative solution to the problem of Figure 3-11 which does not require a change of branch.

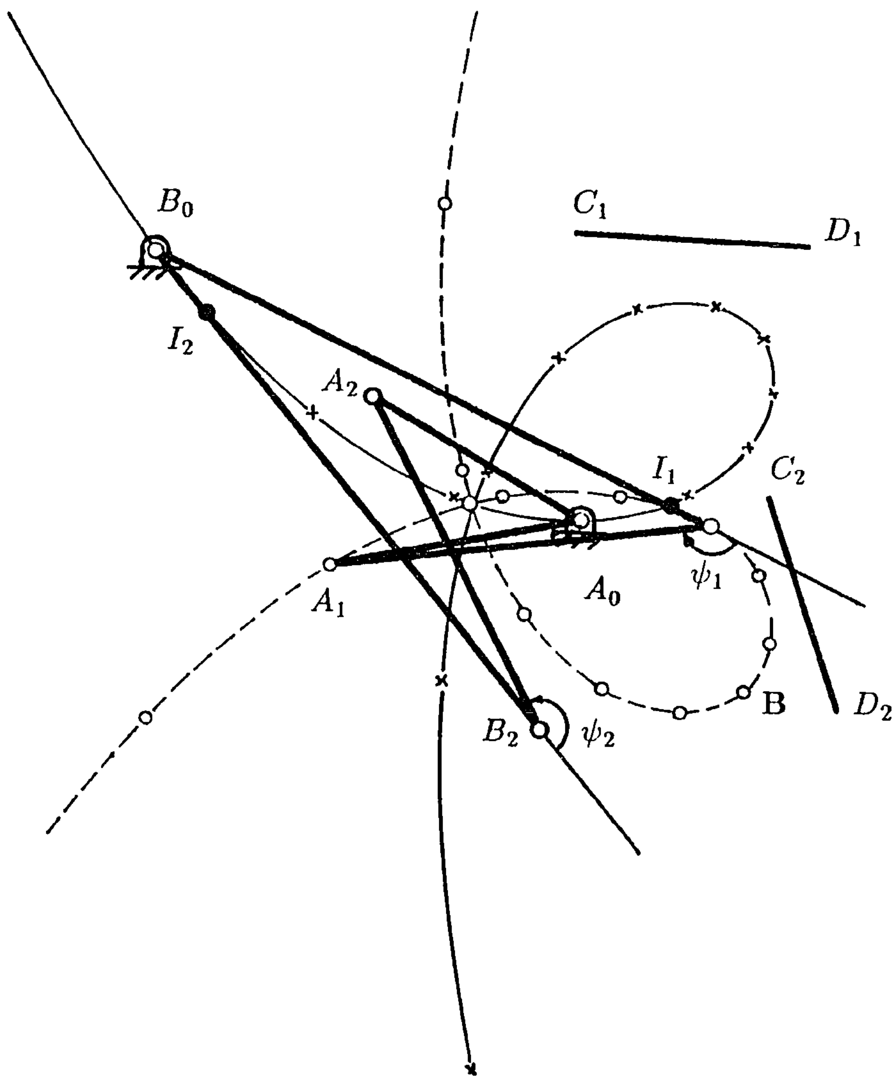

Figure 3-11. Branch problem in PP-PP

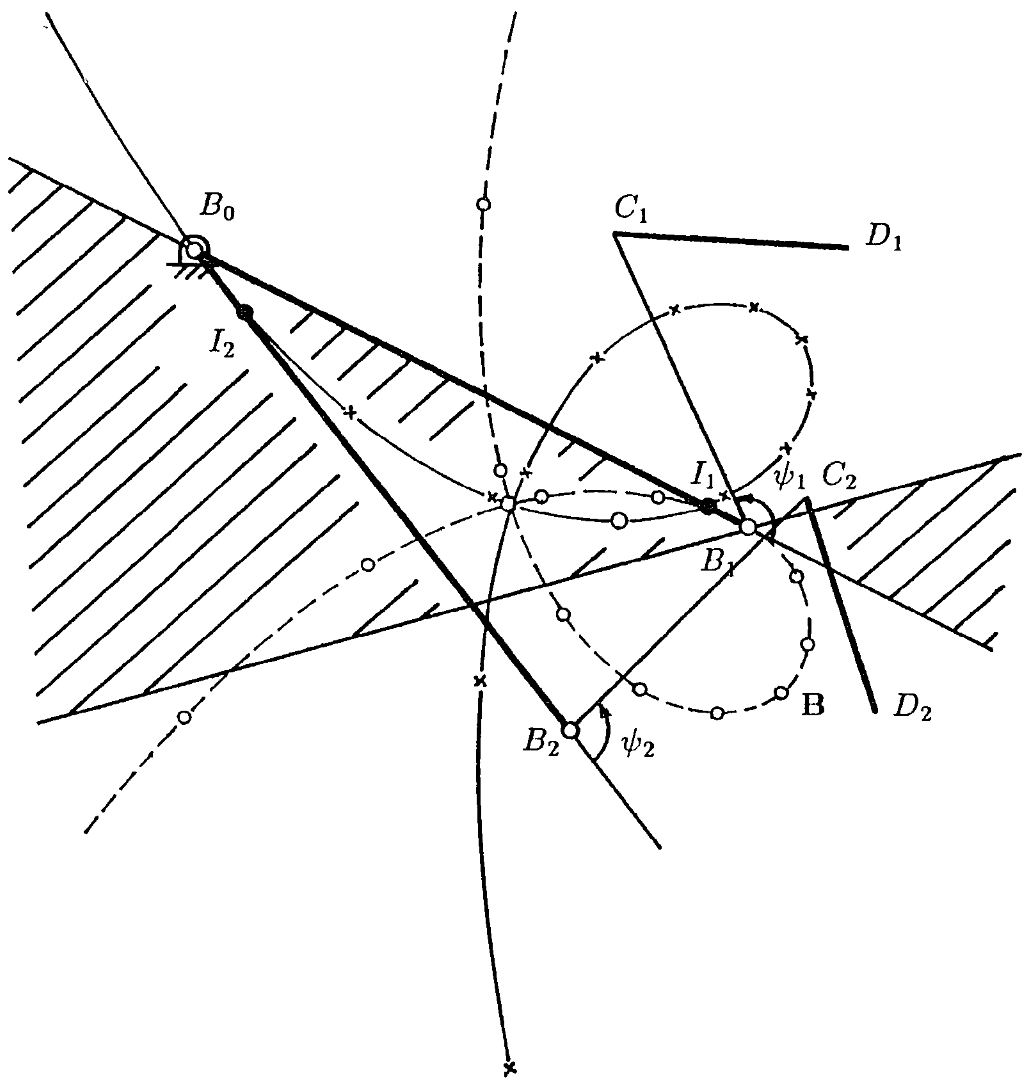

Figure 3-12. Region which has the branch problem

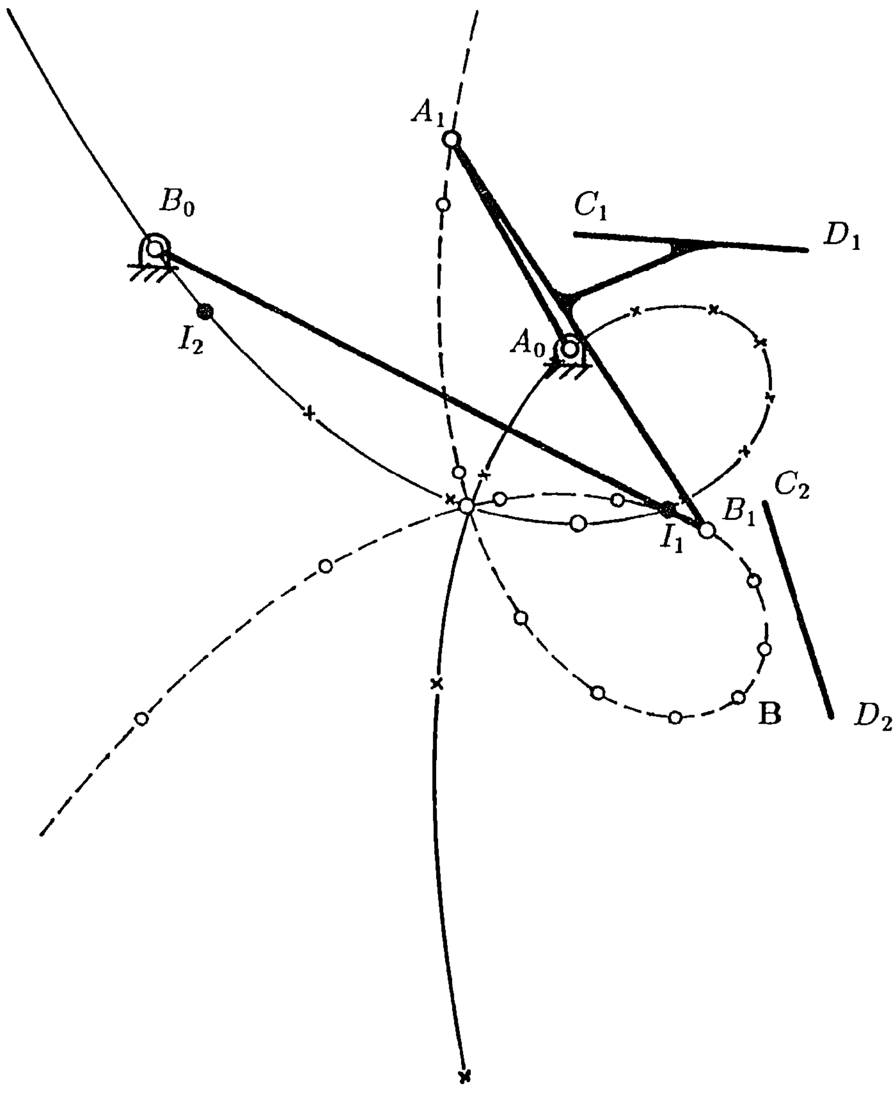

Figure 3-13. A solution linkage for PP-PP without branch problem

CHAPTER 4

Analytical Synthesis of Spherical Mechanisms for Multiply Separated Positions

4 - 1. Introduction

In 1988, Chiang [20] published *Kinematics of Spherical Mechanisms*. This book shows several examples of spherical mechanisms, such as the universal joint, a swivelling electric fan, and a pair of bevel gears, and examines the spherical counterparts of well-known theorems in plane kinematics.

The spherical four-bar linkage, being analogous to the four-bar linkage in plane mechanisms, is the basic form of all spherical mechanisms. A typical spherical four-bar linkage A_0ABB_0 is shown in Figure 4-1.

Spherical motion is defined as the motion of a rigid body which moves about a fixed point O, all points in the body thus being constrained to lie on a system of spheres all concentric about O. Spherical motion is sometimes examined by the movement of one spherical shell which slides on a fixed reference shell, the two shells being concentric and of the same radius. Thus the position of the rigid body is determined by specifying the two points C and D on the shell, provided that O, C, and D are not collinear.

As a counterpart of an instantaneous center in plane kinematics, an instantaneous rotational axis (IRA) is used for an infinitesimal motion of the rigid body. Similarly a pole axis is equivalent to a pole. Since the body moves about a fixed point O, it is sufficient to specify only the coordinates of the point I on the shell in order to locate the IRA as shown in Figure 4-1.

Generally the concepts of plane kinematics for the rigid body guidance problem hold for spherical kinematics. The three FSP problems were in-

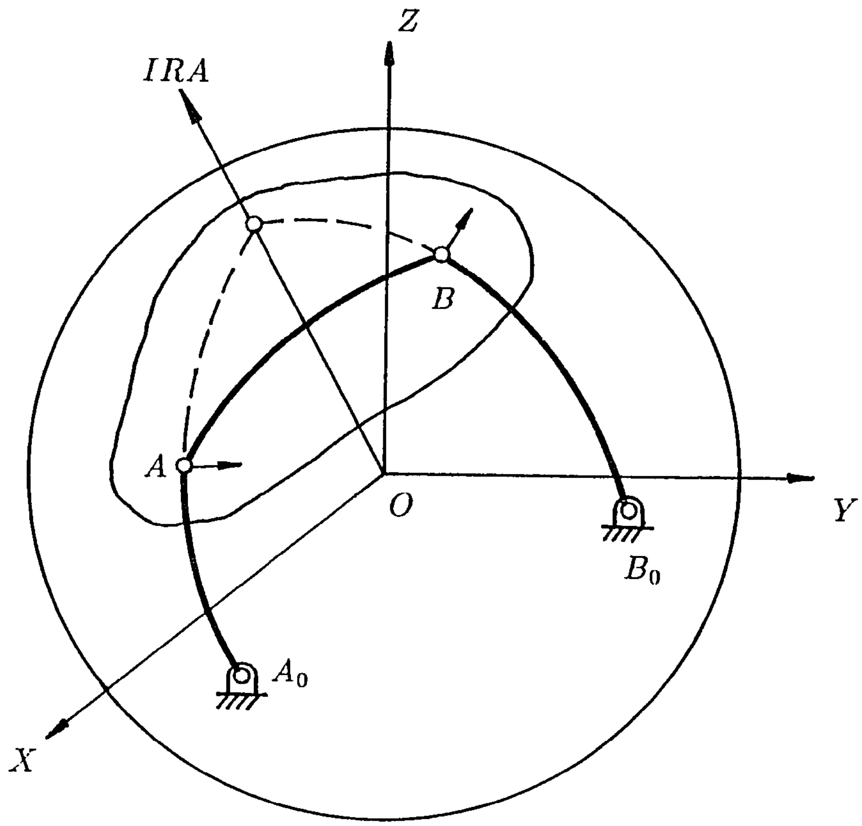

Figure 4-1. Spherical four-bar linkage

vestigated by Hain [9] in a quite similar way in plane kinematics. General equations for the three MSP problem have been derived by Dowler *et al.* [21]. For the four FSP problem, it is possible to establish a geometric rule, similar to that in plane kinematics, for the locus of the circle-point, which we call the spherical circle-point curve. However, as it is quite impracticable to carry out graphical construction on the spherical surface, algebraic methods are used to find the locus of the spherical circle-point [22,23]. All five cases of three MSP and all seven cases of four MSP are treated by Dowler *et al.* [24]. However, their method was not applied to general spatial kinematics.

This chapter deals with the four MSP problems in the spherical four-bar linkages. This method is different from Dowler's and can be applied to general spatial kinematics which will be seen in Chapter 5.

4 - 2. Finite Rotational Displacements

A displacement of the moving sphere can be regarded as a rotation about an axis through O. The linear transformation of the finite rotational displacement can be expressed by selecting (x, y, z) as the Cartesian coordinates of a point in a moving system σ and (X, Y, Z) as its coordinates in a fixed system Σ. Let σ_i denote the ith position of σ, and (X_i, Y_i, Z_i) the coordinates in Σ of the ith position of the point (x, y, z) in σ. Knowing the position of σ_i relative to Σ, we can express (X_i, Y_i, Z_i) in terms of (x, y, z):

$$\mathbf{X}_i = \mathbf{R}_i \mathbf{x}, \tag{4.1}$$

where the matrix $\mathbf{R}_i$ is function of the parameters governing the relative position of σ_i and Σ. If we let the 1st position of σ coincide with Σ, the displacement to σ_i may be described as a rotational displacement which is equivalent to a rotation ϕ_i about an axis which is parallel to the unit vector $\mathbf{u}_i(u_i, v_i, w_i)$ and passes through a point O. In this case we have

$$\mathbf{R}_i = \begin{pmatrix} u_i^2\text{vers}\phi_i + \cos\phi_i & u_iv_i\text{vers}\phi_i - w_i\sin\phi_i & u_iw_i\text{vers}\phi_i + v_i\sin\phi_i \\ u_iv_i\text{vers}\phi_i + w_i\sin\phi_i & v_i^2\text{vers}\phi_i + \cos\phi_i & v_iw_i\text{vers}\phi_i - u_i\sin\phi_i \\ w_iu_i\text{vers}\phi_i - v_i\sin\phi_i & v_iw_i\text{vers}\phi_i + u_i\sin\phi_i & w_i^2\text{vers}\phi_i + \cos\phi_i \end{pmatrix} \tag{4.2}$$

where $\text{vers}\phi_i = 1 - \cos\phi_i$.

The matrix $\mathbf{R}_i$ is orthogonal. This can be shown by the inversion of the rotational matrix which can be formed in terms of the reverse displacement, achieved by replacing the angle ϕ_i by its negative $-\phi_i$.

4 - 3. Determination of Rotational Parameters

The rotational parameters for two given finitely separated positions can be found by several methods [22,37]. Here a direct method to find the rotational parameters is introduced.

It is sometimes sufficient to examine the relative movement of one spherical shell sliding on a fixed reference shell, the two shells being concentric and of the same unit radius. In order to locate the shell on the sphere, it is enough to specify two points on the sphere. If we are given two positions, position 1 and position j, in terms of two points say C, D on a sphere then the rotational parameters can be determined as follows. As shown in Figure 4-2, we consider the displacement from position C_1D_1 to C_iD_i as one where by C_1D_1 rotates about $\$_i$ to position C_iD_i. The $\$_i$, is defined as rotational axis and is perpendicular to the two vectors, $\vec{C_1C_i}$ and $\vec{D_1D_i}$. The following vector equations are obtained:

$$\vec{C_1C_i} \cdot \mathbf{u}_i = 0, \tag{4.3}$$

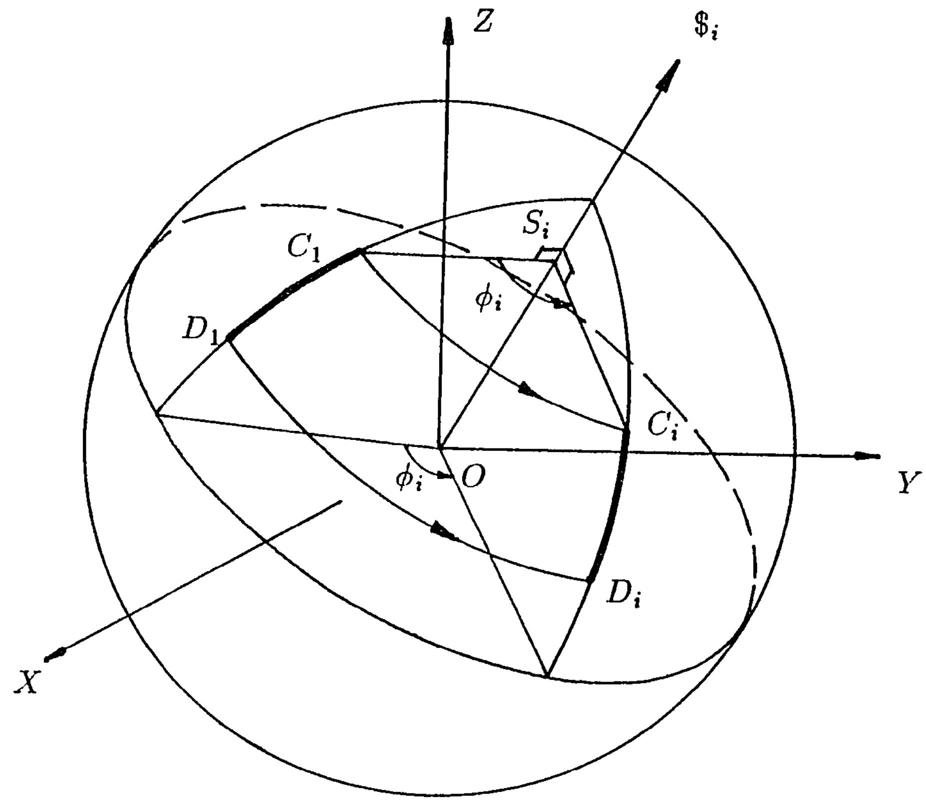

Figure 4-2. Finite spherical motion

$$\vec{D_1 D_i} \cdot \mathbf{u}_i = 0, \tag{4.4}$$

$$\mathbf{u}_i \cdot \mathbf{u}_i = 1. \tag{4.5}$$

Solving equations (4.3), (4.4), (4.5) for the scalar components (u_i, v_i, w_i) we get

$$u_i = \pm \left(\frac{(bc' - cb')^2}{(ab' - ba')^2 + (bc' - cb')^2 + (ca' - ac')^2} \right)^{\frac{1}{2}}, \tag{4.6}$$

$$v_i = \frac{ca' - ac'}{bc' - cb'} u_i, \tag{4.7}$$

$$w_i = \frac{ab' - ba'}{bc' - cb'} u_i, \tag{4.8}$$

where

$$a = C_{xi} - C_{x1}, \qquad a' = D_{xi} - D_{x1},$$

$$b = C_{yi} - C_{y1}, \qquad b' = D_{yi} - D_{y1},$$

$$c = C_{zi} - C_{z1}, \qquad b' = D_{zi} - D_{z1}.$$

In order to find the rotation angle about the rotational axis, the point S, which makes the plane $C_1 S C_i$ perpendicular to the axis, is located on the axis. Then

$$\mathbf{S} = (\mathbf{C}_1 \cdot \mathbf{u}_i)\mathbf{u}_i = (\mathbf{C}_i \cdot \mathbf{u}_i)\mathbf{u}_i. \tag{4.9}$$

From the two vectors, $\vec{SC_1}$ and $\vec{SC_i}$, the rotation angle ϕ_i becomes

$$\phi_i = \cos^{-1} \frac{\vec{SC_1} \cdot \vec{SC_i}}{(\vec{SC_1} \cdot \vec{SC_1})^{1/2} (\vec{SC_i} \cdot \vec{SC_i})^{1/2}}. \tag{4.10}$$

Since the sphere has the unit radius, substituting equation (4.9) into (4.10) yields

$$\phi_i = \cos^{-1} \frac{\mathbf{C}_1 \cdot \mathbf{C}_i - (\mathbf{C}_1 \cdot \mathbf{u}_i)^2}{1 - (\mathbf{C}_1 \cdot \mathbf{u}_i)^2}. \tag{4.11}$$

The direction of the rotational angle ϕ_i is determined by using the cross vector property

$$\text{sign} = \mathbf{u}_i \cdot (S\vec{C}_1 \times S\vec{C}_i) = \mathbf{u}_i \cdot (\mathbf{C}_1 \times \mathbf{C}_i). \tag{4.12}$$

If sign is negative value, ϕ_i is negative.

4 - 4. Dermination of Infinitesimal Rotational Parameters

Infinitesimal displacements of a rigid body may be described by a series of successive infinitesimal rotational displacements. Consider a moving system σ in continuous motion relative to a fixed system Σ. We select a point P fixed in σ which is represented by the constant position vector $\mathbf{x}$, and $\mathbf{X}$ is the position vector of a coincident point P in Σ. Thus we may express equation (4.1) as follow:

$$\mathbf{X} = \mathbf{R}\mathbf{x}. \tag{4.13}$$

If we are interested only in the study of kinematic geometry of space, we may write the first derivative of equation (4.13) with respect to ϕ:

$$\frac{d\mathbf{X}}{d\phi} = \frac{d\mathbf{R}}{d\phi}\mathbf{x}. \tag{4.14}$$

The instantaneous rotational axis (IRA) at the jth position is located by describing a unit vector $\mathbf{V}_j$ passing through the origin in the fixed system Σ. The corresponding vector in the moving system σ is expressed by $\mathbf{v}$. Then we may write following equations from equations (4.13) and (4.14)

$$\mathbf{V}_j = \mathbf{R}_j\mathbf{v}. \tag{4.15}$$

$$\frac{d\mathbf{V}_j}{d\phi} = \frac{d\mathbf{R}_j}{d\phi}\mathbf{v}. \tag{4.16}$$

Since the vector $\mathbf{V}_j$ of IRA does not change in fixed system at the instant of the given position, we may have

$$\frac{d\mathbf{V}_j}{d\phi} = \mathbf{0}. \tag{4.17}$$

Substituting equations (4.15) and (4.16) into equation (4.17) yields

$$\frac{d\mathbf{R}_j}{d\phi}\mathbf{R}_j^T\mathbf{V}_j = \mathbf{0}. \tag{4.18}$$

In order to determine the $\frac{d\mathbf{u}_j}{d\phi}$ for the given position and unit vector $\mathbf{V}_j$ of IRA, the matrix $\frac{d\mathbf{R}_j}{d\phi}$ is written as follows:

$$\frac{d\mathbf{R}_j}{d\phi} = \mathbf{H}_1 + \begin{pmatrix} \mathbf{H}_2\frac{d\mathbf{u}_j}{d\phi} & \mathbf{H}_3\frac{d\mathbf{u}_j}{d\phi} & \mathbf{H}_4\frac{d\mathbf{u}_j}{d\phi} \end{pmatrix}, \tag{4.19}$$

where

$$\mathbf{H}_1 = \begin{pmatrix} {u_j}^2\sin\phi_j - \sin\phi_j & u_jv_j\sin\phi_j - w_j\cos\phi_j & u_jw_j\sin\phi_j + v_j\cos\phi_j \\ u_jv_j\sin\phi_j + w_j\cos\phi_j & {v_j}^2\sin\phi_j - \sin\phi_j & v_jw_j\sin\phi_j - u_j\cos\phi_j \\ u_jw_j\sin\phi_j - v_j\cos\phi_j & v_jw_j\sin\phi_j + u_j\cos\phi_j & {w_j}^2\sin\phi_j - \sin\phi_j \end{pmatrix} \tag{4.20}$$

$$\mathbf{H}_2 = \begin{pmatrix} 2u_j(1-\cos\phi_j) & 0 & 0 \\ v_j(1-\cos\phi_j) & u_j(1-\cos\phi_j) & \sin\phi_j \\ w_j(1-\cos\phi_j) & -\sin\phi_j & u_j(1-\cos\phi_j) \end{pmatrix}, \tag{4.21}$$

$$\mathbf{H}_3 = \begin{pmatrix} v_j(1-\cos\phi_j) & u_j(1-\cos\phi_j) & -\sin\phi_j \\ 0 & 2v_j(1-\cos\phi_j) & 0 \\ \sin\phi_j & w_j(1-\cos\phi_j) & v_j(1-\cos\phi_j) \end{pmatrix}, \tag{4.22}$$

$$\mathbf{H}_4 = \begin{pmatrix} w_j(1-\cos\phi_j) & \sin\phi_j & u_j(1-\cos\phi_j) \\ -\sin\phi_j & w_j(1-\cos\phi_j) & v_j(1-\cos\phi_j) \\ 0 & 0 & 2w_j(1-\cos\phi_j) \end{pmatrix}. \tag{4.23}$$

Substituting equation (4.19) into equation (4.18) we have

$$\mathbf{H}_1\mathbf{R}_j^T\mathbf{V}_j = -\left(\mathbf{H}_2\frac{d\mathbf{u}_j}{d\phi} \quad \mathbf{H}_3\frac{d\mathbf{u}_j}{d\phi} \quad \mathbf{H}_4\frac{d\mathbf{u}_j}{d\phi} \right)\mathbf{R}_j^T\mathbf{V}_j. \tag{4.24}$$

Solving equation (4.24) for $\frac{d\mathbf{u}_j}{d\phi}$ using the partition matrix property in Appendix II, we have

$$\frac{d\mathbf{u}_j}{d\phi} = -\left((\mathbf{R}_j^T\mathbf{V}_j)_x\mathbf{H}_2 + (\mathbf{R}_j^T\mathbf{V}_j)_y\mathbf{H}_3 + (\mathbf{R}_j^T\mathbf{V}_j)_z\mathbf{H}_4\right)^{-1}\mathbf{H}_1\mathbf{R}_j^T\mathbf{V}_j. \tag{4.25}$$

Then $\frac{d\mathbf{R}_j}{d\phi}$ can be determined by equation (4.19).

4 - 5. **Constraint Equations for R-R Link on a Sphere**

Two spherical links, as shown in Figure 4-1, connected between each of a pair of spherical circle points and corresponding spherical center points which lie on axes passing through the origin and the center of the circular arcs, would guide the moving body through a series of prescribed positions. The constraint equations are based on the constant link length of guiding links. The coordinates of the spherical center point are $\mathbf{X}_0 = (X_0, Y_0, Z_0)$. Then we have

$$(\mathbf{X}_i - \mathbf{X}_0)^T(\mathbf{X}_i - \mathbf{X}_0) = R^2 \quad i = 1, 2, ..., n. \tag{4.26}$$

The coordinates of the points $\mathbf{X}_i = (X_i, Y_i, Z_i)$ are determined in terms of 1st position of moving body from the finite rotational displacement equation (4.1). And the derivative form of equation (4.26) is

$$(\mathbf{X}_j - \mathbf{X}_0)^T \frac{d\mathbf{X}_j}{d\phi} = 0. \tag{4.27}$$

Note that the derivatives of $\mathbf{X}_0, R$ are zeros. The infinitesimal displacement $\frac{d\mathbf{X}_j}{d\phi}$ is dertermined from the infinitesimal rotational displacement equation (4.14).

4 - 6. Derivation of Spherical Circle-Point Equation for the Case of PP-P-P, P-PP-P, or P-P-PP

The rigid body motion is represented by the three finitely separated positions and the one infinitesimally separated position. For the three finitely separated positions, substitution of equation (4.1) into equation (4.26) yields

$$(\mathbf{R}_i\mathbf{x} - \mathbf{X}_0)^T(\mathbf{R}_i\mathbf{x} - \mathbf{X}_0) = R^2, \quad i = 2, 3. \tag{4.28}$$

For the first design position, equation (4.26) becomes

$$(\mathbf{x} - \mathbf{X}_0)^T(\mathbf{x} - \mathbf{X}_0) = R^2. \tag{4.29}$$

Since the matrix $\mathbf{R}_i$ is orthogonal, equating the left hand sides of equations (4.28) and (4.29) and rearranging yields

$$\mathbf{x}^T(\mathbf{R}_i - \mathbf{I})^T\mathbf{X}_0 = 0. \quad i = 2, 3 \tag{4.30}$$

For the one infinitesimally separated position at the jth finite position, substitution of equations (4.1) and (4.14) into equation (4.27) yields

$$(\mathbf{R}_j\mathbf{x} - \mathbf{X}_0)^T \frac{d\mathbf{R}_j}{d\phi}\mathbf{x} = 0. \tag{4.31}$$

Rearranging equation (4.31) with the orthogonal matrix property shown in Appendix III yields

$$\mathbf{x}^T \left(\frac{d\mathbf{R}_j}{d\phi} \right)^T \mathbf{X}_0 = 0. \tag{4.32}$$

In order to simplfy the equations, the following matrices are introduced based on equations (4.30) and (4.32)

$$\mathbf{a} = \begin{pmatrix} (\mathbf{R}_2 - \mathbf{I})^1 \\ (\mathbf{R}_3 - \mathbf{I})^1 \\ (\frac{d\mathbf{R}_j}{d\phi})^1 \end{pmatrix}, \tag{4.33}$$

$$\mathbf{b} = \begin{pmatrix} (\mathbf{R}_2 - \mathbf{I})^2 \\ (\mathbf{R}_3 - \mathbf{I})^2 \\ (\frac{d\mathbf{R}_j}{d\phi})^2 \end{pmatrix}, \tag{4.34}$$

$$\mathbf{c} = \begin{pmatrix} (\mathbf{R}_2 - \mathbf{I})^3 \\ (\mathbf{R}_3 - \mathbf{I})^3 \\ (\frac{d\mathbf{R}_j}{d\phi})^3 \end{pmatrix}, \tag{4.35}$$

where the superscipts 1, 2, 3 of the matrices are the first, second, third rows of the matrices respectively. Thus equations (4.30) and (4.32) become

$$\mathbf{a} \begin{pmatrix} x \\ y \\ z \end{pmatrix} X_0 + \mathbf{b} \begin{pmatrix} x \\ y \\ z \end{pmatrix} Y_0 + \mathbf{c} \begin{pmatrix} x \\ y \\ z \end{pmatrix} Z_0 = \mathbf{0}. \tag{4.36}$$

The nontrivial solution of equation (4.36) requires

$$\left| \mathbf{a} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \mathbf{b} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \mathbf{c} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right| = 0, \tag{4.37}$$

Equation (4.37) can be written in a tripple sum polynomial form as:

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} A_{ijk}x^l y^m z^n = 0, \tag{4.38}$$

where

$$A_{ijk} = \begin{vmatrix} a_{1i} & b_{1i} & c_{1i} \\ a_{2j} & b_{2j} & c_{2j} \\ a_{3k} & b_{3k} & c_{3k} \end{vmatrix}, \tag{4.39}$$

$$l = \text{the number of } 1'\text{s among the } i, j, k,$$

$$m = \text{the number of } 2'\text{s among the } i, j, k,$$

$$n = \text{the number of } 3'\text{s among the } i, j, k.$$

which upon expansion and rearrangement yields the circle-point equation

$$\begin{aligned} & A_{111}x^3 + A_{222}y^3 + A_{333}z^3 \\ & +(A_{112} + A_{121} + A_{211})x^2y + (A_{113} + A_{131} + A_{311})x^2z \\ & +(A_{122} + A_{212} + A_{221})xy^2 + (A_{133} + A_{313} + A_{331})xz^2 \\ & +(A_{223} + A_{232} + A_{322})y^2z + (A_{233} + A_{323} + A_{332})yz^2 \\ & +(A_{123} + A_{132} + A_{213} + A_{231} + A_{312} + A_{321})xyz = 0. \end{aligned} \tag{4.40}$$

Equation (4.39) is a spherical cubic cone which, together with equation $x^2 + y^2 + z^2 = 1$ of the unit sphere, represents the spherical circle-point curve. The spherical center point for the spherical circle point can be determined using first two equations of equation (4.36).

For the four finitely separated positions, the cubic cone equation has the same form of equation (4.38) (4.40) if we use

$$\mathbf{a} = \begin{pmatrix} (\mathbf{R}_2 - \mathbf{I})^1 \\ (\mathbf{R}_3 - \mathbf{I})^1 \\ (\mathbf{R}_4 - \mathbf{I})^1 \end{pmatrix}, \tag{4.41}$$

$$\mathbf{b} = \begin{pmatrix} (\mathbf{R}_2 - \mathbf{I})^2 \\ (\mathbf{R}_3 - \mathbf{I})^2 \\ (\mathbf{R}_4 - \mathbf{I})^2 \end{pmatrix}, \tag{4.42}$$

$$\mathbf{c} = \begin{pmatrix} (\mathbf{R}_2 - \mathbf{I})^3 \\ (\mathbf{R}_3 - \mathbf{I})^3 \\ (\mathbf{R}_4 - \mathbf{I})^3 \end{pmatrix}, \tag{4.43}$$

where the superscipts 1, 2, 3 of the matrices are the first, second, third rows of the matrices respectively.

4 - 7. Example for the case of P-PP-P

An illustrative example for the Section 4-6 is to synthesize a spherical four-bar linkage to pass through the specified three finitely separated positions and have an instantaneous rotational axis at a specified position. The data of the desired motion is given in Table 4-1. By letting the first position of σ coincide with Σ and using the method developed in the previous sections, we have

$$\mathbf{R}_2 = \begin{pmatrix} 0.92534 & -0.29529 & 0.23781 \\ 0.31693 & 0.94669 & -0.05770 \\ -0.20809 & 0.12876 & 0.96960 \end{pmatrix}$$

$$\mathbf{R}_3 = \begin{pmatrix} 0.51414 & -0.73249 & 0.44623 \\ 0.40360 & 0.66567 & 0.62768 \\ -0.75681 & -0.14261 & 0.63788 \end{pmatrix}$$

$$\frac{d\mathbf{R}_2}{d\phi} = \begin{pmatrix} -0.38714 & -0.54479 & 0.82995 \\ 0.63102 & -0.20015 & 0.18219 \\ -0.76047 & 0.22218 & -0.19271 \end{pmatrix}$$

Using equation (4.40) the circle-point equation becomes

$$-0.005698x^3 + 0.012970y^3 + 0.012669z^3 - 0.023788x^2y - 0.034149x^2z$$

$$-0.028458xy^2 - 0.006383xz^2 - 0.024042y^2z + 0.005746yz^2 + 0.010665xyz = 0.$$

With $x^2+y^2+z^2 = 1$, the circle-point curve is drawn in Figure 4-3. Choosing two points, one for the driving crank the other for the driven crank, a spherical four-bar linkage is synthesized as shown in Figure 4-3 with

$$\mathbf{A}_1 = (-0.61608, -0.02373, 0.78733), \quad \mathbf{A}_0 = (-0.30000, 0.10000, 0.94868),$$

$$\mathbf{B}_1 = (0.43939, 0.16852, 0.88235), \quad \mathbf{B}_0 = (0.10000, 0.70000, 0.70711).$$

4 - 8. Derivation of Spherical Circle-Point Equation for the Case of PP-PP

The rigid body motion is represented by the two finitely separated positions and the two infinitesimally separated position. For the two finitely separated positions, substitution of equation (4.1) into equation (4.26) yields

$$(\mathbf{R}_2\mathbf{x} - \mathbf{X}_0)^T(\mathbf{R}_2\mathbf{x} - \mathbf{X}_0) = R^2. \tag{4.44}$$

For the first design position, equation (4.26) becomes

$$(\mathbf{x} - \mathbf{X}_0)^T(\mathbf{x} - \mathbf{X}_0) = R^2. \tag{4.45}$$

Since the matrix $\mathbf{R}_i$ is orthogonal, equating the left hand sides of equations (4.28) and (4.29) and rearranging yields

Table 4-1. Design data for P-PP-P in spherical motion

1st position	$C_{1x} = -0.59588$, $C_{1y} = -0.55331$, $C_{1z} = 0.58204$ $D_{1x} = -0.25878$, $D_{1y} = -0.19157$, $D_{1z} = 0.94675$
2nd position	$C_{2x} = -0.24959$, $C_{2y} = -0.74626$, $C_{2z} = 0.61710$ $D_{2x} = 0.04226$, $D_{2y} = -0.31801$, $D_{2z} = 0.94715$
3nd position	$C_{3x} = 0.35866$, $C_{3y} = -0.24349$, $C_{3z} = 0.90115$ $D_{3x} = 0.42975$, $D_{3y} = 0.36229$, $D_{3z} = 0.82708$
IRA at 2nd position	$V_{2x} = -0.01836$, $V_{2y} = 0.76483$, $V_{2z} = 0.64398$

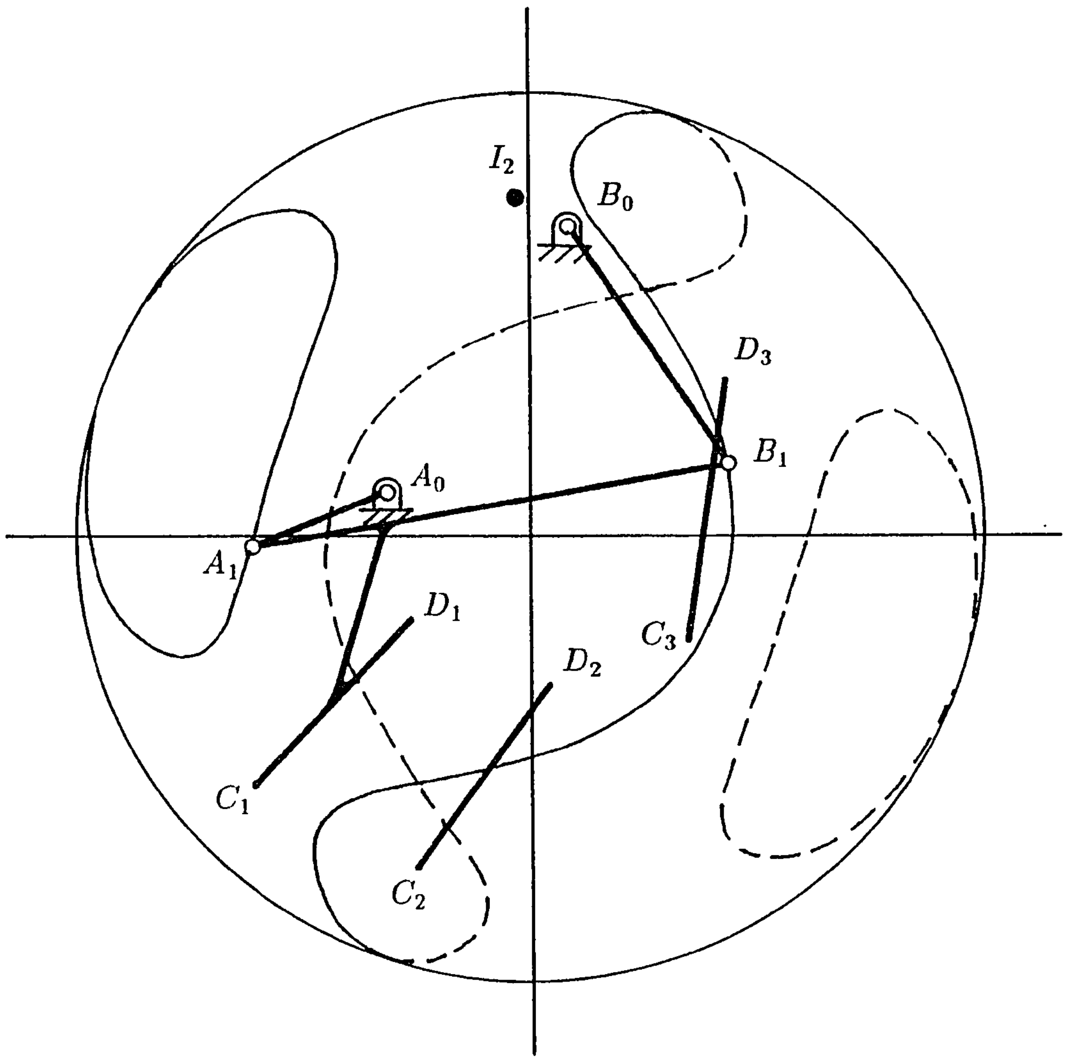

Figure 4-3. The circle-point curve and a solution linkage for the prescribed P-PP-P spherical motion

$$\mathbf{x}^T(\mathbf{R}_2 - \mathbf{I})^T\mathbf{X}_0 = 0. \tag{4.46}$$

For the two infinitesimally separated positions, substitution of equations (4.1) and (4.14) into equation (4.27) yields

$$(\mathbf{R}_j\mathbf{x} - \mathbf{X}_0)^T \frac{d\mathbf{R}_j}{d\phi}\mathbf{x} = 0, \quad j = 1, 2. \tag{4.47}$$

Rearranging equation (4.47) with the orthogonal matrix property shown in Appendix III yields

$$\mathbf{x}^T \left(\frac{d\mathbf{R}_j}{d\phi}\right)^T \mathbf{X}_0 = 0, \quad j = 1, 2 \tag{4.48}$$

In order to simplfy the equations, the following matrices are introduced based on equations (4.46) and (4.48)

$$\mathbf{a} = \begin{pmatrix} (\mathbf{R}_2 - \mathbf{I})^1 \\ \left(\frac{d\mathbf{R}_1}{d\phi}\right)^1 \\ \left(\frac{d\mathbf{R}_2}{d\phi}\right)^1 \end{pmatrix}, \tag{4.49}$$

$$\mathbf{b} = \begin{pmatrix} (\mathbf{R}_2 - \mathbf{I})^2 \\ \left(\frac{d\mathbf{R}_1}{d\phi}\right)^2 \\ \left(\frac{d\mathbf{R}_2}{d\phi}\right)^2 \end{pmatrix}, \tag{4.50}$$

$$\mathbf{c} = \begin{pmatrix} (\mathbf{R}_2 - \mathbf{I})^3 \\ \left(\frac{d\mathbf{R}_1}{d\phi}\right)^3 \\ \left(\frac{d\mathbf{R}_2}{d\phi}\right)^3 \end{pmatrix}, \tag{4.51}$$

where the superscipts 1, 2, 3 of the matrices are the first, second, third rows of the matrices respectively. Thus equations (4.46) and (4.48) become

$$\mathbf{a}\begin{pmatrix} x \\ y \\ z \end{pmatrix} X_0 + \mathbf{b}\begin{pmatrix} x \\ y \\ z \end{pmatrix} Y_0 + \mathbf{c}\begin{pmatrix} x \\ y \\ z \end{pmatrix} Z_0 = \mathbf{0}. \tag{4.52}$$

The nontrivial solution of equation (4.35) requires

$$\left| \mathbf{a}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \mathbf{b}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \mathbf{c}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \right| = 0, \tag{4.53}$$

Equation (4.53) can be written

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} A_{ijk} x^l y^m z^n = 0, \tag{4.54}$$

where

$$A_{ijk} = \begin{vmatrix} a_{1i} & b_{1i} & c_{1i} \\ a_{2j} & b_{2j} & c_{2j} \\ a_{3k} & b_{3k} & c_{3k} \end{vmatrix}, \tag{4.55}$$

$$l = \text{the number of } 1'\text{s among the } i, j, k,$$

$$m = \text{the number of } 2'\text{s among the } i, j, k,$$

$$n = \text{the number of } 3'\text{s among the } i, j, k.$$

which upon expansion and rearrangement yields the circle-point equation

$$A_{111}x^3 + A_{222}y^3 + A_{333}z^3$$

$$+(A_{112} + A_{121} + A_{211})x^2y + (A_{113} + A_{131} + A_{311})x^2z$$

$$+(A_{122} + A_{212} + A_{221})xy^2 + (A_{133} + A_{313} + A_{331})xz^2$$

$$+(A_{223}+A_{232}+A_{322})y^2z+(A_{233}+A_{323}+A_{332})yz^2$$

$$+(A_{123}+A_{132}+A_{213}+A_{231}+A_{312}+A_{321})xyz=0. \qquad (4.56)$$

Equation (4.56) is a spherical cubic cone which, together with equation $x^2+y^2+z^2=1$ of the unit sphere, represents the spherical circle-point curve. The spherical center point for the spherical circle-point can be determined using first two equations of equation (4.52).

4 - 9. **Example for the case of PP-PP**

An illustrative example for the Section 4-8 is to synthesize a spherical four-bar linkages to pass through the specified two finitely separated positions and have two instantaneous rotational axes at the specified positions. The data of desired motion is given in Table 4-2. By letting the first position of σ coincide with Σ and using the method developed in the previous sections, we have

$$\mathbf{R}_2=\begin{pmatrix} 0.79816 & -0.55624 & 0.23136 \\ 0.32617 & 0.72188 & 0.61032 \\ -0.50650 & -0.41168 & 0.75761 \end{pmatrix}$$

$$\frac{d\mathbf{R}_2}{d\phi}=\begin{pmatrix} 0.00000 & -0.64398 & 0.76483 \\ 0.64398 & 0.00000 & 0.01836 \\ -0.76483 & -0.01836 & 0.00000 \end{pmatrix}$$

$$\frac{d\mathbf{R}_2}{d\phi}=\begin{pmatrix} 0.34782 & 0.75856 & 0.62381 \\ -0.53031 & 0.82428 & -0.69154 \\ 0.20661 & 0.42045 & 0.36660 \end{pmatrix}$$

Using equation (4.37) the circle-point equation becomes

$$0.122x^3+0.765y^3+0.260z^3+0.477x^2y-0.881x^2z$$

$$+0.309xy^2 - 0.661xz^2 - 0.197y^2z + 0.023yz^2 - 0.772xyz = 0.$$

With $x^2+y^2+z^2 = 1$, the circle-point curve is drawn in Figure 4-4. Choosing two points, one for the driving crank the other for the driven crank, a spherical four-bar linkage is synthesized as shown in Figure 4-4 with

$$\mathbf{A}_1 = (-0.37584, -0.26315, 0.88853), \quad \mathbf{A}_0 = (-0.30000, 0.10000, 0.94868),$$

$$\mathbf{B}_1 = (0.56665, 0.24788, 0.78579), \quad \mathbf{B}_0 = (0.10000, 0.70000, 0.70711).$$

Table 4-2. Design data for PP-PP in spherical motion

1st position	C_{1x} =-0.24959, C_{1y} =-0.74626, C_{1z} = 0.61710 D_{1x} = 0.04226, D_{1y} =-0.31801, D_{1z} = 0.94715
2nd position	C_{2x} = 0.35866, C_{2y} =-0.24349, C_{2z} = 0.90115 D_{2x} = 0.42975, D_{2y} = 0.36229, D_{2z} = 0.82708
IRA at 1st position	V_{1x} =-0.01836, V_{1y} = 0.76483, V_{1z} = 0.64398
IRA at 2nd position	V_{2x} =-0.49569, V_{2y} = 0.01321, V_{2z} = 0.86840

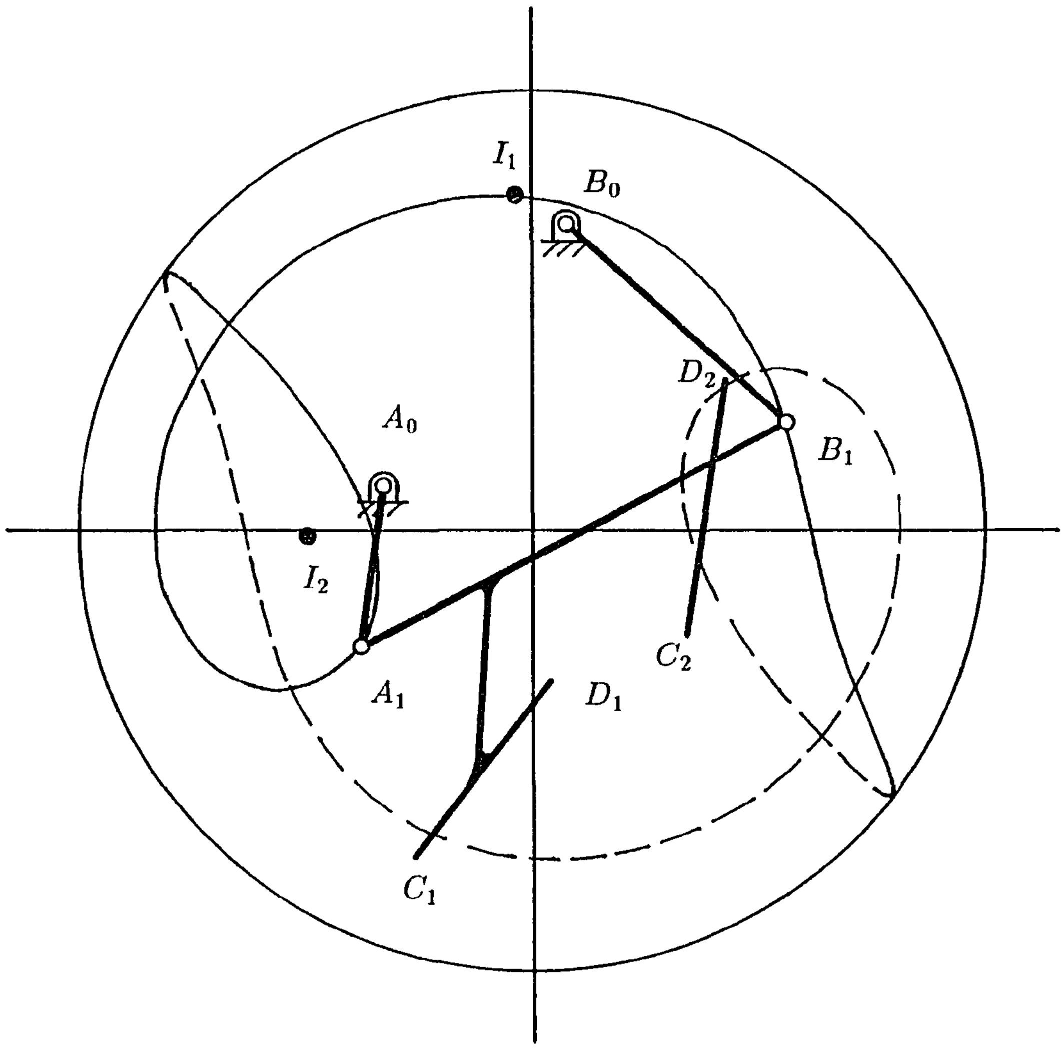

Figure 4-4. The circle-point curve and a solution linkage for the prescribed PP-PP spherical motion

CHAPTER 5

Analytical Synthesis of Spatial Mechanisms for Multiply Separated Positions

5 - 1. Introduction

Position synthesis of spatial mechanisms involves the determination of mechanism dimensions which will move a rigid body through a series of prescribed positions. These positions could be finitely separated, infinitesimally separated or a combination of both finitely and infinitesimally separated. Instead of determining the mechanism dimensions directly, the dimensional synthesis process is performed by locating the points or lines in a rigid body moving through finitely and infinitesimally separated positions.

Synthesis of spatial mechanisms under a series of finite displacements has been extensively studied [25-29]. The problems of finite displacements are not concerned with the manner in which the motion takes place and the only consideration for the synthesized mechanism is reaching the position before and the position after the motion. For such a mechanism, transition of the rigid body between the two positions can not be predicted. On the other hand, specifying two design positions infinitesimally separated from one another is equivalent to specifying a position of the rigid body and the velocity state of that body as it moves through that position. The spatial instantaneous motion of a rigid body can be prescribed by specifying the instantaneous screw axis (ISA) at a position. Synthesis for infinitesimally separated positions has been studied by several researchers [30-34].

Little research has been done on the synthesis of spatial mechanisms for multiply separated positions [35-36]. In this paper, the objective is to perform the synthesis of spatial mechanisms for combined finitely and infinitesimally separated positions using screw axes.

In this Chapter, the instantaneous geometric motion of a rigid body is studied in terms of the instantaneous screw axis for the infinitesimally separated positions. Thus the multiply separated positions are defined by specifying the several finitely separated positions and their Instantaneous Screw Axes. Equations are developed to locate special points or lines in the rigid body moving through the specified motion. These point or lines would satisfy the constraints of various type of links for spatial mechanisms. Spatial RRSS mechanism is synthesized for illustrative purpose.

5 - 2. Finite Screw Displacements

It is well known that a displacement in spatial motion, regardless of how a motion actually occurs, may always be regarded as a rotation about a screw axis and a translation along the axis. In order to obtain the explicit expressions for the linear transformation of the finite screw displacement, we select (x, y, z) as the Cartesian coordinates of a point in a moving system σ and (X, Y, Z) as its coordinates in a fixed system Σ. Let σ_i denote the jth position of σ, and (X_i, Y_i, Z_i) the coordinates in Σ of the jth position of the point (x, y, z) in σ. Knowing the position of σ_i relative to Σ, we can express (X_i, Y_i, Z_i) in terms of (x, y, z):

$$\mathbf{X}_i = \mathbf{R}_i \mathbf{x} + \mathbf{T}_i, \tag{5.1}$$

where the matrices $\mathbf{R}_i$ and $\mathbf{T}_i$ are functions of the parameters governing the relative position of σ_i and Σ. If we let the first position of σ coincide with Σ, the displacement to σ_i may be described as a screw displacement which is equivalent to a translation d_i along, and a rotation ϕ_i about an axis which is parallel to the unit vector $\mathbf{u}_i(u_i, v_i, w_i)$ and passes through a point $\mathbf{S}_i(S_{xj}, S_{yj}, S_{zj})$ in Σ. In this case we have

$$\mathbf{R}_i = \begin{pmatrix} u_i^2 \mathrm{vers}\phi_i + \cos\phi_i & u_i v_i \mathrm{vers}\phi_i - w_i \sin\phi_i & u_i w_i \mathrm{vers}\phi_i + v_i \sin\phi_i \\ u_i v_i \mathrm{vers}\phi_i + w_i \sin\phi_i & v_i^2 \mathrm{vers}\phi_i + \cos\phi_i & v_i w_i \mathrm{vers}\phi_i - u_i \sin\phi_i \\ w_i u_i \mathrm{vers}\phi_i - v_i \sin\phi_i & v_i w_i \mathrm{vers}\phi_i + u_i \sin\phi_i & w_i^2 \mathrm{vers}\phi_i + \cos\phi_i \end{pmatrix} \tag{5.2}$$

and

$$\mathbf{T}_i = \begin{pmatrix} S_{xj} \\ S_{yj} \\ S_{zj} \end{pmatrix} + d_i \begin{pmatrix} u_i \\ v_i \\ w_i \end{pmatrix} - \mathbf{R}_i \begin{pmatrix} S_{xj} \\ S_{yj} \\ S_{zj} \end{pmatrix} \tag{5.3}$$

where $\mathrm{vers}\phi_i = 1 - \cos\phi_i$.

5 - 3. Determination of Screw Parameters

The problem at this stage is to find the screw parameters for two given finitely separated positions. Bottema and Roth [37] have used Rodrigues equation to find screw parameters. Suh [22] has used homogeneous transformation using four non-coplanar points. We introduce here a direct method to find the screw parameters.

If we are given two positions, position 1 and position j, in terms of three non-collinear prescribed points say, C, D, E, then the screw parameters can be determined as follows. As shown in Figure 5-1, we consider the displacement from position $C_1D_1E_1$ to $C_iD_iE_i$ as one where by $C_1D_1E_1$ first translates parallel to the screw axis $\$_i$ to position $C^tD^tE^t$ and then rotates about $\$_i$ to position $C_iD_iE_i$. Thus the displacement between $C^tD^tE^t$ and $C_iD_iE_i$ is a pure rotation about $\$_i$, therfore we have

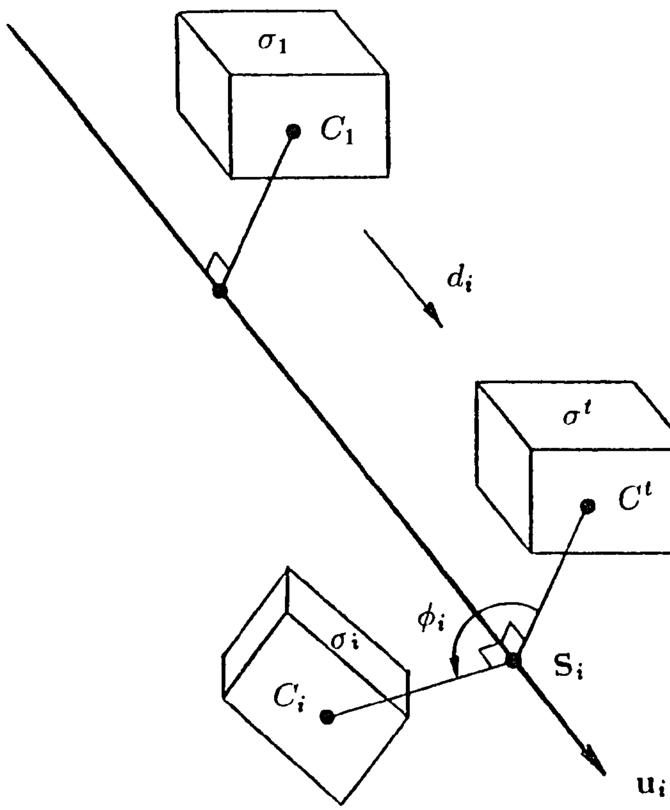

Figure 5-1. Finite screw motion

$$(\mathbf{C}_i - \mathbf{C}^t) \cdot \mathbf{u}_i = 0, \tag{5.4}$$

$$(\mathbf{D}_i - \mathbf{D}^t) \cdot \mathbf{u}_i = 0, \tag{5.5}$$

$$(\mathbf{E}_i - \mathbf{E}^t) \cdot \mathbf{u}_i = 0. \tag{5.6}$$

We take d_i as the screw translation along the screw axis, which carries $C_1D_1E_1$ to position $C^tD^tE^t$, then we can express

$$\mathbf{C}^t = \mathbf{C}_1 + d_i\mathbf{u}_i, \tag{5.7}$$

$$\mathbf{D}^t = \mathbf{D}_1 + d_i\mathbf{u}_i, \tag{5.8}$$

$$\mathbf{E}^t = \mathbf{E}_1 + d_i\mathbf{u}_i. \tag{5.9}$$

Combining equations (5.4)-(5.6) and (5.7)-(5.9) we get

$$\big((\mathbf{C}_i - \mathbf{C}_1) - (\mathbf{D}_i - \mathbf{D}_1)\big) \cdot \mathbf{u}_i = 0, \tag{5.10}$$

$$\big((\mathbf{C}_i - \mathbf{C}_1) - (\mathbf{E}_i - \mathbf{E}_1)\big) \cdot \mathbf{u}_i = 0. \tag{5.11}$$

Solving equations (5.10), (5.11) with $\mathbf{u}_i \cdot \mathbf{u}_i = 1$ for the scalar components (u_i, v_i, w_i) we get

$$u_i = \pm\left(\frac{(bc' - cb')^2}{(ab' - ba')^2 + (bc' - cb')^2 + (ca' - ac')^2}\right)^{\frac{1}{2}}, \tag{5.12}$$

$$v_i = \frac{ca' - ac'}{bc' - cb'}u_i, \tag{5.13}$$

$$w_i = \frac{ab' - ba'}{bc' - cb'}u_i, \tag{5.14}$$

where

$$a = (C_{xj} - C_{x1}) - (D_{xj} - D_{x1}), \quad a' = (C_{xj} - C_{x1}) - (E_{xj} - E_{x1}),$$

$$b = (C_{yj} - C_{y1}) - (D_{yj} - D_{y1}), \quad b' = (C_{yj} - C_{y1}) - (E_{yj} - E_{y1}),$$

$$c = (C_{zj} - C_{z1}) - (D_{zj} - D_{z1}), \quad c' = (C_{zj} - C_{z1}) - (E_{zj} - E_{z1}).$$

In order to determine the screw translation d_i, we consider the equations (5.4) and (5.7)

$$(\mathbf{C}_i - \mathbf{C}_1 + d_i\mathbf{u}_i) \cdot \mathbf{u}_i = 0, \tag{5.15}$$

Solving equation (5.15) for d_i

$$d_i = (\mathbf{C}_i - \mathbf{C}_1) \cdot \mathbf{u}_i. \tag{5.16}$$

We choose the point S_i on the screw axis which satisfies the following equation:

$$(\mathbf{C}_i - \mathbf{S}_i) \cdot \mathbf{u}_i = 0. \tag{5.17}$$

Since C_i is obtained from C^t by only rotation around the screw axis, we have

$$(\mathbf{C}_i - \mathbf{S}_i) \cdot (\mathbf{C}_i - \mathbf{S}_i) = (\mathbf{C}_1 + d_i\mathbf{u}_i - \mathbf{S}_i) \cdot (\mathbf{C}_i + d_i\mathbf{u}_i - \mathbf{S}_i). \tag{5.18}$$

And for D_1 and D_i we get

$$(\mathbf{D}_i - \mathbf{S}_i) \cdot (\mathbf{D}_i - \mathbf{S}_i) = (\mathbf{D}_1 + d_i\mathbf{u}_i - \mathbf{S}_i) \cdot (\mathbf{D}_i + d_i\mathbf{u}_i - \mathbf{S}_i). \tag{5.19}$$

Solving equations (5.17), (5.18), (5.19) for the components (S_{xj}, S_{yj}, S_{zj}) of S_i we get

$$\begin{pmatrix} S_{xj} \\ S_{yj} \\ S_{zj} \end{pmatrix} = \begin{pmatrix} 2(\mathbf{C}_1 + d_i\mathbf{u}_i - \mathbf{C}_i)^T \\ 2(\mathbf{D}_1 + d_i\mathbf{u}_i - \mathbf{D}_i)^T \\ \mathbf{u}_i^T \end{pmatrix}^{-1} \begin{pmatrix} (\mathbf{C}_1 + d_i\mathbf{u}_i) \cdot (\mathbf{C}_1 + d_i\mathbf{u}_i) - \mathbf{C}_i \cdot \mathbf{C}_i \\ (\mathbf{D}_1 + d_i\mathbf{u}_i) \cdot (\mathbf{D}_1 + d_i\mathbf{u}_i) - \mathbf{D}_i \cdot \mathbf{D}_i \\ \mathbf{C}_i \cdot \mathbf{u}_i \end{pmatrix}. \tag{5.20}$$

Rotation angle ϕ_i may be obtaind by two vectors $\vec{S_iC_i}$ and $\vec{S_iC^t}$:

$$\phi_i = \cos^{-1}\left(\frac{(\mathbf{C}_i - \mathbf{S}_i)\cdot(\mathbf{C}_1 + d_i\mathbf{u}_i - \mathbf{S}_i)}{(\mathbf{C}_i - \mathbf{S}_i)\cdot(\mathbf{C}_i - \mathbf{S}_i)}\right), \quad 0 \le \phi_i \le \pi. \tag{5.21}$$

Direction of ϕ_i is defined by cross product property:

$$\text{sign} = \mathbf{u}_i \cdot [(\mathbf{C}_1 + d_i\mathbf{u}_i - \mathbf{S}_i) \times (\mathbf{C}_i - \mathbf{S}_i)]. \tag{5.22}$$

If sign is negative, ϕ_i is negative.

5 - 4. **Spatial Instantaneous Motion**

When two positions of the lamina become infinitesimally apart in planar motion, the pole P_{12} becomes the instantaneous center P. By prescribing the instantaneous center at the specified position, the motion can be directed at least in the manner as illustrated in Figure 1-2.

Similarily, the istantaneous screw axis helps instantaneous motion in three dimensions. But there is a translation motion as well as a rotation motion. Thus we have to consider the instantaneous translation motion. The instantaneous pitch plays that role and is defined by

$$h = \frac{\dot{s}}{\dot{\phi}} = \frac{ds}{d\phi} \tag{5.23}$$

As shown in the above equation, the instantaneous pitch is defined from the time dependent properties, and it turns out a geometric propertiy. But h in the above equation cannot give visual geometric motion. In order to give more specific geometric motion, the instantaneous pitch angle θ as shown in Figure 5-2 is introduced as follows.

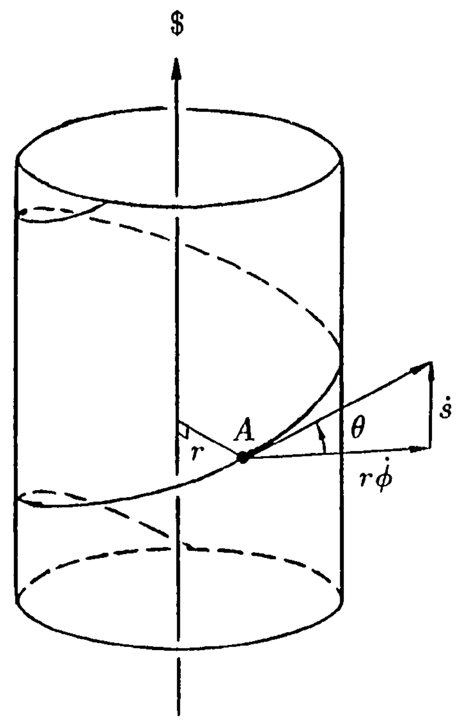

Figure 5-2. Instantaneous pitch angle

$$\theta = \tan^{-1}(\frac{s}{h}) \tag{5.24}$$

So the instantaneous spatial motion can be given by the location of the instantaneous screw axis as well as the instantaneous pitch or the instantaneous pitch angle.

5 - 5. Infinitesimal Screw Displacements

Infinitesimal displacements of a rigid body may be described by a series of successive infinitesimal screw displacements. Consider a moving system σ in continuous motion relative to a fixed system Σ. We select a point P fixed in σ which is represented by the constant position vector $\mathbf{x}$, and $\mathbf{X}$ is the position vector of the coincident point of P in σ. Thus we may express equation (5.1) as follow:

$$\mathbf{X} = \mathbf{R}\mathbf{x} + \mathbf{T}. \tag{5.25}$$

If we are interested only in the study of kinematic geometry of space and we exclude the case of pure translation (i.e., ϕ = constant), we may write the 1st derivative of equation (5.25) with respect to ϕ:

$$\frac{d\mathbf{X}}{d\phi} = \frac{d\mathbf{R}}{d\phi}\mathbf{x} + \frac{d\mathbf{T}}{d\phi}. \tag{5.26}$$

5 - 6. Determination of Infinitesimal Screw Parameters

The ISA at the jth position is located by describing a position $\mathbf{I}_j$ (I_{xj}, I_{yj}, I_{zj}) and its unit vector $\mathbf{V}_j$. If we take two points $\mathbf{I}_j$ and $\mathbf{I}_j + \mathbf{V}_j$ in the fixed system Σ, we may have the vectors $\mathbf{i}$ and $\mathbf{i} + \mathbf{v}$ for the coresponding points on the moving system σ. Then we may write following equations from equation (5.25)

$$\mathbf{I}_j = \mathbf{R}_j \mathbf{i} + \mathbf{T}_j, \tag{5.27}$$

$$\mathbf{I}_j + \mathbf{V}_j = \mathbf{R}_j(\mathbf{i} + \mathbf{v}) + \mathbf{B}_j, \tag{5.28}$$

and from equation (5.26)

$$\frac{d\mathbf{I}_j}{d\phi} = \frac{d\mathbf{R}_j}{d\phi}\mathbf{i} + \frac{d\mathbf{T}_j}{d\phi}. \tag{5.29}$$

$$\frac{d(\mathbf{I}_j + \mathbf{V}_j)}{d\phi} = \frac{d\mathbf{R}_j}{d\phi}(\mathbf{i} + \mathbf{v}) + \frac{d\mathbf{T}_j}{d\phi}. \tag{5.30}$$

Equations (5.27) and (5.28) give

$$\mathbf{V}_j = \mathbf{R}_j \mathbf{v}. \tag{5.31}$$

Also Equations (5.29) and (5.30) give

$$\frac{d\mathbf{V}_j}{d\phi} = \frac{d\mathbf{R}_j}{d\phi}\mathbf{v}. \tag{5.32}$$

Since the vector $\mathbf{V}_j$ of ISA does not change in fixed system at the instant of the given position, we may have

$$\frac{d\mathbf{V}_j}{d\phi} = \mathbf{0}. \tag{5.33}$$

Substituting equations (5.31) and (5.33) into equation (5.32) yields

$$\frac{d\mathbf{R}_j}{d\phi}\mathbf{R}_j^T \mathbf{V}_j = \mathbf{0}. \tag{5.34}$$

In order to determine the $\frac{d\mathbf{u}_j}{d\phi}$ for the given position and unit vector $\mathbf{V}_j$ of ISA, the matrix $\frac{d\mathbf{R}_j}{d\phi}$ is written as follows:

$$\frac{d\mathbf{R}_j}{d\phi} = \mathbf{H}_1 + \left(\mathbf{H}_2 \frac{d\mathbf{u}_j}{d\phi} \quad \mathbf{H}_3 \frac{d\mathbf{u}_j}{d\phi} \quad \mathbf{H}_4 \frac{d\mathbf{u}_j}{d\phi} \right), \tag{5.35}$$

where

$$\mathbf{H}_1 = \begin{pmatrix} {u_j}^2 \sin\phi_j - \sin\phi_j & u_j v_j \sin\phi_j - w_j \cos\phi_j & u_j w_j \sin\phi_j + v_j \cos\phi_j \\ u_j v_j \sin\phi_j + w_j \cos\phi_j & {v_j}^2 \sin\phi_j - \sin\phi_j & v_j w_j \sin\phi_j - u_j \cos\phi_j \\ u_j w_j \sin\phi_j - v_j \cos\phi_j & v_j w_j \sin\phi_j + u_j \cos\phi_j & {w_j}^2 \sin\phi_j - \sin\phi_j \end{pmatrix} \tag{5.36}$$

$$\mathbf{H}_2 = \begin{pmatrix} 2u_j(1-\cos\phi_j) & 0 & 0 \\ v_j(1-\cos\phi_j) & u_j(1-\cos\phi_j) & \sin\phi_j \\ w_j(1-\cos\phi_j) & -\sin\phi_j & u_j(1-\cos\phi_j) \end{pmatrix}, \tag{5.37}$$

$$\mathbf{H}_3 = \begin{pmatrix} v_j(1-\cos\phi_j) & u_j(1-\cos\phi_j) & -\sin\phi_j \\ 0 & 2v_j(1-\cos\phi_j) & 0 \\ \sin\phi_j & w_j(1-\cos\phi_j) & v_j(1-\cos\phi_j) \end{pmatrix}, \tag{5.38}$$

$$\mathbf{H}_4 = \begin{pmatrix} w_j(1-\cos\phi_j) & \sin\phi_j & u_j(1-\cos\phi_j) \\ -\sin\phi_j & w_j(1-\cos\phi_j) & v_j(1-\cos\phi_j) \\ 0 & 0 & 2w_j(1-\cos\phi_j) \end{pmatrix}. \tag{5.39}$$

Substituting equation (5.35) into equation (5.34) we have

$$\mathbf{H}_1 \mathbf{R}_j^T \mathbf{V}_j = -\left(\mathbf{H}_2 \frac{d\mathbf{u}_j}{d\phi} \quad \mathbf{H}_3 \frac{d\mathbf{u}_j}{d\phi} \quad \mathbf{H}_4 \frac{d\mathbf{u}_j}{d\phi} \right) \mathbf{R}_j^{-1} \mathbf{V}_j. \tag{5.40}$$

Solving equation (5.40) for $\frac{d\mathbf{u}_j}{d\phi}$ using the partition matrix property in Appendix II, we have

$$\frac{d\mathbf{u}_j}{d\phi} = -\Big((\mathbf{R}_j^T \mathbf{V}_j)_x \mathbf{H}_2 + (\mathbf{R}_j^T \mathbf{V}_j)_y \mathbf{H}_3 + (\mathbf{R}_j^T \mathbf{V}_j)_z \mathbf{H}_4 \Big)^{-1} \mathbf{H}_1 \mathbf{R}_j^T \mathbf{V}_j. \tag{5.41}$$

Now $\frac{d\mathbf{R}_j}{d\phi}$ can be determined by equation (5.35). Using equations (5.27) and (5.29) by defining the instantaneous pitch h_j of the ISA, the $\frac{d\mathbf{T}_j}{d\phi}$ is determined as follows:

$$\frac{d\mathbf{T}_j}{d\phi} = h_j\mathbf{V}_j - \frac{d\mathbf{R}_j}{d\phi}\mathbf{R}_j^T(\mathbf{I}_j - \mathbf{T}_j). \tag{5.42}$$

5 - 7. Constraint Equations for S-S and R-R Binary Links

Refering to [22,36], since there are many possibilities for the geometric form of constructing links or link-pair combinations, an almost infinite variety of spatial mechanisms can be formed to guide a rigid body through a series of specified positions in space. The number of possible positions that can be prescribed depends on the constraints provided by the guiding link used. In this section, I will show the constraint equations for a couple of the binary links, R-R and S-S, to proceed to the next section which will demonstrate the foregoing method to synthesize a RRSS mechanism.

The Sphere - Sphere (S-S) Binary Link The sphere-sphere link is shown in Figure 5-3. The S-S link must satisfy the constant length condition only. Assuming a fixed pivot $\mathbf{X}_0$ and corresponding moving pivot $\mathbf{X}$, this leads to the S-S link finite displacement constraint equation.

$$(\mathbf{X}_i - \mathbf{X}_0)^T(\mathbf{X}_i - \mathbf{X}_0) = (\mathbf{X}_1 - \mathbf{X}_0)^T(\mathbf{X}_1 - \mathbf{X}_0) \;\; j = 2, 3, ..., n. \tag{5.43}$$

The S-S infinitesimal displcement constraint equation is found by differentiating the above equation to give

$$(\dot{\mathbf{X}}_i)^T(\mathbf{X}_j - \mathbf{X}_0) = 0. \tag{5.44}$$

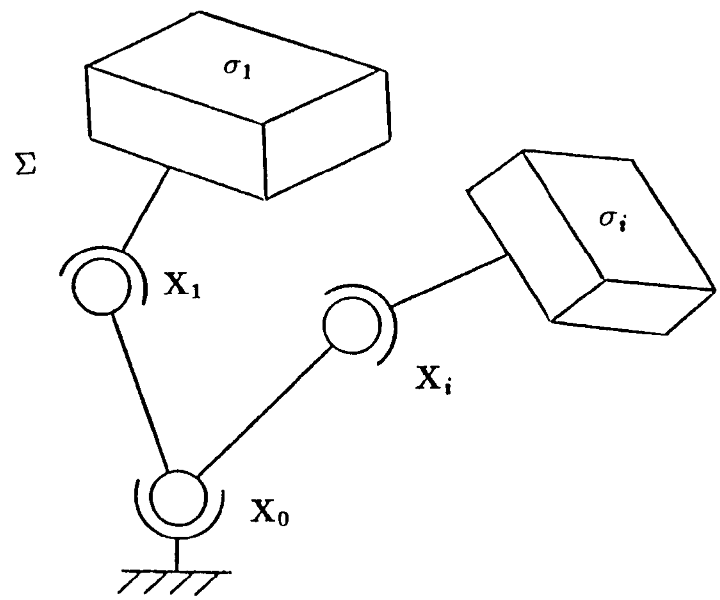

Figure 5-3. S-S Binary link

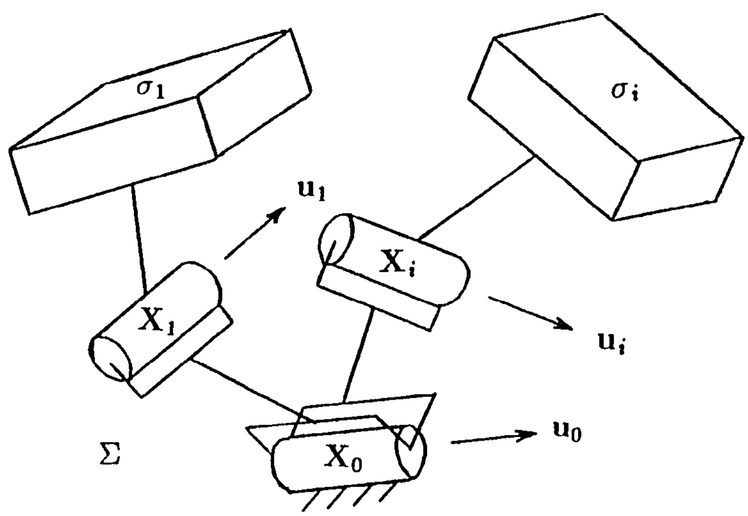

Figure 5-4. R-R Binary link

The Revolute - Revolute (R-R) Binary Link The R-R link shown in Figure 5-4 must satisfy all the constraint equations for the S-S link plus three additional requirements. The spherical joint becomes a revolute joint when it is restricted to rotation in a plane that is perpendicular to the axis $\mathbf{u}_0$ of the revolute joint. One point on the revolute axis must be specified. We will arbitrarily specify a point $\mathbf{X}_0$ that lies at the intersection of the plane of rotation of the spherical joint and the revolute axis $\mathbf{u}_0$. The angle of twist between the fixed axis $\mathbf{u}_0$ and the moving axis $\mathbf{u}$ must remain constant during a displacement, since both axes are fixed in the R-R link. In addition we require a second plane equation that takes note of the fact that point $\mathbf{X}_0$ must have a relative rotation about the moving axis $\mathbf{u}_i$. This constrains $\mathbf{X}_0$ to lie in a plane perpendicular to $\mathbf{u}_i$ at all times. Hence, the R-R link finite displacement constraint equations can be written as follows.

$$(\mathbf{u}_0)^T(\mathbf{X}_i - \mathbf{X}_0) = 0, \quad j = 1, 2, 3, \tag{5.45}$$

$$(\mathbf{u}_i)^T(\mathbf{X}_i - \mathbf{X}_0) = 0, \quad j = 1, 2, 3, \tag{5.46}$$

$$(\mathbf{u}_0)^T(\mathbf{u}_0) = 1, \tag{5.47}$$

$$(\mathbf{u}_1)^T(\mathbf{u}_1) = 1, \tag{5.48}$$

$$(\mathbf{X}_i - \mathbf{X}_0)^T(\mathbf{X}_i - \mathbf{X}_0) = (\mathbf{X}_1 - \mathbf{X}_0)^T(\mathbf{X}_1 - \mathbf{X}_0), \quad j = 2, 3 \tag{5.49}$$

$$((\mathbf{X}_i + \mathbf{u}_i) - (\mathbf{X}_0 + \mathbf{u}_0))^T((\mathbf{X}_i + \mathbf{u}_i) - (\mathbf{X}_0 + \mathbf{u}_0))$$

$$= ((\mathbf{X}_1 + \mathbf{u}_1) - (\mathbf{X}_0 + \mathbf{u}_0))^T((\mathbf{X}_1 + \mathbf{u}_1) - (\mathbf{X}_0 + \mathbf{u}_0)). \tag{5.50}$$

The R-R infinitesimal displcement constraint equation is found by differentiating the above equation to give

$$(\mathbf{u}_0)^T(\dot{\mathbf{X}}_j) = 0, \tag{5.51}$$

$$(\dot{\mathbf{u}}_j)^T(\mathbf{X}_j - \mathbf{X}_0) + (\mathbf{u}_j)^T(\dot{\mathbf{X}}_j) = 0, \tag{5.52}$$

$$(\dot{\mathbf{X}}_j)^T(\mathbf{X}_j - \mathbf{X}_0) = 0, \tag{5.53}$$

$$(\dot{\mathbf{X}}_j + \dot{\mathbf{u}}_j)^T((\mathbf{X}_j + \mathbf{u}_j) - (\mathbf{X}_0 + \mathbf{u}_0)) = 0. \tag{5.54}$$

5 - 8. **Numerical Example**

There are many possible spatial mechanisms. By choosing RRSS mechanism, we illustrate the analytical procedure in which the foregoing results can be applied to the synthesis process.

Refering to Suh [22], an R-R link, for three prescribed positions has no free choice of the parameters and an S-S link has seven positions with no free choice of parameter as the maximum limit in synthesis. Therefore, an RRSS mechanism is possible for a maximum of three positions with no free choice of parameter in synthesis. When the maximum number of positions of three is imposed, then one may have a unique solution for the R-R link but, to find the S-S link, we can choose four parameters arbitrarily from the six S-S parameters.

In the synthesis example, the RRSS mechanism as shown in Figure 5-5 is to pass through the three specified multiply separated positions. These consist of two finitely separated positions and an ISA at one specified position. The data of desired motion is given in Table 5-1.

Using the method developed in this study, we get following linear transformations.

$$\mathbf{X}_1 = \mathbf{x}, \tag{5.55}$$

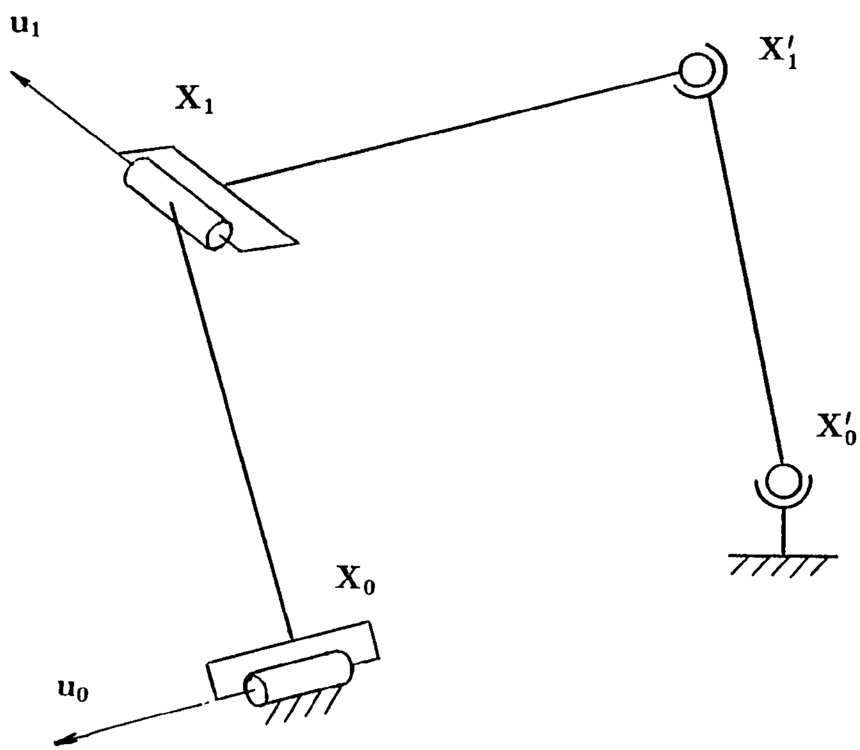

Figure 5-5. RRSS Mechanism

Table 5-1. Design data for P-PP in spatial motion

1st position	$C_{1x} = 0.9303$, $C_{1y} = 0.7365$, $C_{1z} = 0.8576$ $D_{1x} = 0.0229$, $D_{1y} = -0.3289$, $D_{1z} = 0.6529$ $E_{1x} = -0.0224$, $E_{1y} = 0.4179$, $E_{1z} = 1.8532$
2nd position	$C_{2x} = 1.0000$, $C_{2y} = 1.0000$, $C_{2z} = 1.0000$ $D_{2x} = 0.0000$, $D_{2y} = 0.0000$, $D_{2z} = 1.0000$ $E_{2x} = 0.0000$, $E_{2y} = 1.0000$, $E_{2z} = 2.0000$
ISA at 2nd position	$V_{2x} = 0.7992$, $V_{2y} = 0.3786$, $V_{2z} = 0.4669$ $I_{2x} = 1.0532$, $I_{2y} = 1.1467$, $I_{2z} = 0.0000$ $h_2 = -0.9596$

$$\mathbf{X}_2 = \begin{pmatrix} 0.9990 & -0.0368 & 0.0262 \\ 0.0385 & 0.9969 & -0.0681 \\ -0.0236 & 0.0690 & 0.9973 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -0.0145 \\ -0.0618 \\ -0.0899 \end{pmatrix}, \tag{5.56}$$

$$\frac{d\mathbf{X}_2}{d\phi} = \begin{pmatrix} -0.0221 & -0.4197 & 0.2547 \\ 0.4569 & -0.0765 & -0.8618 \\ -0.1917 & 0.8812 & -0.0655 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -0.1189 \\ -0.6284 \\ -1.2183 \end{pmatrix}. \tag{5.57}$$

From equation (5.43) and (5.44), S-S link constraint equations form a set of two synthesis equations with six variables.

$$(\mathbf{X}_2 - \mathbf{X}_0)^T(\mathbf{X}_2 - \mathbf{X}_0) = (\mathbf{X}_1 - \mathbf{X}_0)^T(\mathbf{X}_1 - \mathbf{X}_0), \tag{5.58}$$

$$(\dot{\mathbf{X}}_2)^T(\mathbf{X}_2 - \mathbf{X}_0) = 0, \tag{5.59}$$

where equation (5.59) is the derivation form of equation (5.58) with respect to ϕ. X_{0x}, X_{0y}, X_{0z} and X_{1z} are specified and designated by an astrisk in the results. With initial guesses,

$$\mathbf{X}_0{}' = (0.4000^*, 0.3000^*, 0.0000^*),$$

$$\mathbf{X}_1{}' = (0.1000^*, 0.0000, 0.0000),$$

program BROWN converged in six iterations to the solution

$$\mathbf{X}_0{}' = (0.4000^*, 0.3000^*, 0.0000^*),$$

$$\mathbf{X}_1{}' = (0.1000^*, -0.1155, 0.4446).$$

From equations (5.45)-(5.54), R-R link constraint equations form a set of 12 synthesis equations in 12 variables with no free variables.

$$(\mathbf{u}_0)^T(\mathbf{X}_1 - \mathbf{X}_0) = 0, \tag{5.60}$$

$$(\mathbf{u}_0)^T(\mathbf{X}_2 - \mathbf{X}_0) = 0, \tag{5.61}$$

$$(\mathbf{u}_1)^T(\mathbf{X}_1 - \mathbf{X}_0) = 0, \tag{5.62}$$

$$(\mathbf{u}_2)^T(\mathbf{X}_2 - \mathbf{X}_0) = 0, \tag{5.63}$$

$$(\mathbf{u}_0)^T(\mathbf{u}_0) = 1, \tag{5.64}$$

$$(\mathbf{u}_1)^T(\mathbf{u}_1) = 1, \tag{5.65}$$

$$(\mathbf{X}_2 - \mathbf{X}_0)^T(\mathbf{X}_2 - \mathbf{X}_0) = (\mathbf{X}_1 - \mathbf{X}_0)^T(\mathbf{X}_1 - \mathbf{X}_0), \tag{5.66}$$

$$\begin{aligned} &((\mathbf{X}_2 + \mathbf{u}_2) - (\mathbf{X}_0 + \mathbf{u}_0))^T((\mathbf{X}_2 + \mathbf{u}_2) - (\mathbf{X}_0 + \mathbf{u}_0)) \\ &= ((\mathbf{X}_1 + \mathbf{u}_1) - (\mathbf{X}_0 + \mathbf{u}_0))^T((\mathbf{X}_1 + \mathbf{u}_1) - (\mathbf{X}_0 + \mathbf{u}_0)), \end{aligned} \tag{5.67}$$

$$(\mathbf{u}_0)^T(\dot{\mathbf{X}}_2) = 0, \tag{5.68}$$

$$(\dot{\mathbf{u}}_2)^T(\mathbf{X}_2 - \mathbf{X}_0) + (\mathbf{u}_2)^T(\dot{\mathbf{X}}_2) = 0, \tag{5.69}$$

$$(\dot{\mathbf{X}}_2)^T(\mathbf{X}_2 - \mathbf{X}_0) = 0, \tag{5.70}$$

$$(\dot{\mathbf{X}}_2 + \dot{\mathbf{u}}_2)^T((\mathbf{X}_2 + \mathbf{u}_2) - (\mathbf{X}_0 + \mathbf{u}_0)) = 0, \tag{5.71}$$

where equations (5.68)-(5.71) are the derivation forms of equations (5.60)-(5.67) with respect to ϕ. With initial guesses,

$$\mathbf{X}_0 = (1.0000, 1.0000, 0.0000),$$

$$\mathbf{X}_1 = (0.9000, 0.4500, 0.7500),$$

$$\mathbf{u}_0 = (1.0000, -0.2000, 0.0000),$$

$$\mathbf{u}_1 = (0.6500, -0.6000, -0.4000),$$

program BROWN converged in seventeen iterations to the solution

$$\mathbf{X}_0 = (1.0020, 1.0770, 0.0327),$$

$$\mathbf{X}_1 = (0.8825, 0.4526, 0.7430),$$

$$\mathbf{u}_0 = (0.9852, -0.1706, 0.0159),$$

$$\mathbf{u}_1 = (0.6488, -0.6224, -0.4378).$$

CHAPTER 6

CONCLUSIONS

In this dissertation, a new mathematical approach has been developed for multiply separated positions synthesis problem. This method has been consistently applied to planar, spherical, and spatial mechanisms. A graphical method for PP-PP case of the multiply separated positions problem has also developed for the first time.

In Chapter 2, using the developed analytical method, the unified form of the circle-point curve equations has been derived not only for the multiply separated position problem but also for the finitely separated position problem.

In Chapter 3, one of conventional graphical methods is reviewed for application to the multiply separated position problem. A new graphical method for the PP-PP case of multiply separated position problem is developed to plot the Ball point and the circle-point and center-point curves simultaneously. Since this graphical method uses only straight lines, the construction time for the curves has been much reduced in comparison with the conventional methods.

In Chapter 4, using the developed analytical method the unified form of the cubic cone equations has been derived for not only the multiply separated position problem but also the finitely separated position problem for the spherical four-bar mechanisms.

In Chapter 5, the geometric instantaneous motion of a rigid body has been defined by specifying both the instantaneous screw axis and the instantaneous pitch or the instantaneous pitch angle. Using the developed analytical method, a RRSS mechanism to move through the multiply separated positions is synthesized for the illustrative purpose for the first time.

It is believed that the new mathematical approach developed in this dissertation will be a significant contribution to the art of planar, spherical, and spatial mechanism design. It is also expected that this work will generate further interest in the instantaneous kinematics of mechanisms.

APPENDICES

Appendix I

```
{-----------------------------------------------------}
{                                                     }
{ This program computes the constants A_ijk  for the }
{ unified form of Burmester curve equations to solve }
{ FSP, MSP problems of planar and spherical motions. }
{                                                     }
{                                                     }
{     Hyoung Jun Kim            April 22, 1989.       }
{                                                     }
{-----------------------------------------------------}


{                                                     }
{     Determinant is the pre-declared function        }
{     to calculate 3X3 determinants.                  }
{                                                     }

Type
   matrix33 =array[1..3,1..3] of real;
   matrix333=array[1..3,1..3,1..3] of real;
Var
   a,b,c : matrix33;
   aaa   : matrix333;
Procedure Constant(a,b,c:matrix33  var aaa:matrix333);
   var
      temp       : matrix33;
      i,j,k,l,t : integer;
   begin
      for i:=1 to 3 do
      for j:=1 to 3 do
      for k:=1 to 3 do
         begin
            for l:=1 to 3 do
               begin
                  if l=1  then  t:=i;
                  if l=2  then  t:=j;
                  if l=3  then  t:=k;
                  temp[l,1]:= a[l,t];
                  temp[l,2]:= b[l,t];
                  temp[l,3]:= c[l,t];
               end;
            aaa[i,j,k]:=Determinant(temp);
         end;
   end;
```

Appendix II

If $\mathbf{A}_1, \mathbf{A}_2, ..., \mathbf{A}_n$ are $n \times n$ matrices, $\mathbf{x}$ is $n \times 1$ matrix, and $u_1, u_2, ..., u_n$ are the elements of the $n \times 1$ matrix $\mathbf{u}$, then we have the following partition matrix property

$$(\, \mathbf{A}_1\mathbf{x} \quad \mathbf{A}_2\mathbf{x} \quad \ldots \quad \mathbf{A}_n\mathbf{x}\,)\, \mathbf{u} = \left(\sum_{j=1}^{n} u_j \mathbf{A}_j \right) \mathbf{x}$$

Proof

Let $a_{ij,k}$ be the element of a matrix $\mathbf{A}_k$ and x_j be the jth row element of the matrix $\mathbf{x}$. If we take the ith row of above equation, the left side of the equation, using the following summation rule

$$\sum_{k=1}^{n} \left(\sum_{j=1}^{n} a_{ij,k} x_j \right) u_k = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} u_k a_{ij,k} \right) x_j,$$

equals to the right side of the equation.

Appendix III

An orthogonal matrix $\mathbf{A}$ and any vector $\mathbf{x}$ have the following property

$$\mathbf{x}^T\left(\frac{d\mathbf{A}}{d\phi}\right)^T\mathbf{A}\mathbf{x} = 0.$$

Proof

The basic property of the orthogonal matrix

$$\mathbf{A}^T\mathbf{A} = \mathbf{I}.$$

Differentiation of above equation is

$$\left(\frac{d\mathbf{A}}{d\phi}\right)^T\mathbf{A} + \mathbf{A}^T\frac{d\mathbf{A}}{d\phi} = 0.$$

Since

$$\mathbf{x}^T\left(\frac{d\mathbf{A}}{d\phi}\right)^T\mathbf{A}\mathbf{x} = \left(\mathbf{x}^T\left(\frac{d\mathbf{A}}{d\phi}\right)^T\mathbf{A}\mathbf{x}\right)^T = \mathbf{x}^T\mathbf{A}^T\frac{d\mathbf{A}}{d\phi}\mathbf{x},$$

we have

$$2\mathbf{x}^T\left(\frac{d\mathbf{A}}{d\phi}\right)^T\mathbf{A}\mathbf{x} = \mathbf{x}^T\left(\frac{d\mathbf{A}}{d\phi}\right)^T\mathbf{A}\mathbf{x} + \mathbf{x}^T\mathbf{A}^T\left(\frac{d\mathbf{A}}{d\phi}\right)\mathbf{x}$$

$$= \mathbf{x}^T\left(\left(\frac{d\mathbf{A}}{d\phi}\right)^T\mathbf{A} + \mathbf{A}^T\frac{d\mathbf{A}}{d\phi}\right)\mathbf{x} = 0.$$

Therefore

$$\mathbf{x}^T\left(\frac{d\mathbf{A}}{d\phi}\right)^T\mathbf{A}\mathbf{x} = 0.$$

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