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Mathematical modeling and simulation of human motion using 3-dimensional, multi-segment coupled pendulum system : derivation of a generalized formula for equations of motion

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ABSTRACT

MATHEMATICAL MODELING AND SIMULATION OF HUMAN MOTION USING A 3-DIMENSIONAL, MULTI-SEGMENT COUPLED PENDULUM SYSTEM: DERIVATION OF A GENERALIZED FORMULA FOR EQUATIONS OF MOTION

by
Loay Ahmed-Wasfe Al-Zube

The use of mathematical models to investigate the dynamics of human movement relies on two approaches: forward dynamics and inverse dynamics. In this investigation a new modeling approach called the Boundary Method is outlined. This method addresses some of the disadvantages of both the forward and the inverse approach. The method yields as output both a set of potential movement solutions to a given motor task and the net muscular impulses required to produce those movements. The input to the boundary method is a finite and adjustable number of critical target body configurations. In each phase of the motion that occurs between two contiguous target configurations the equations of motion are solved in the forward direction as a two point ballistic boundary value problem. In the limit as the number of specified target configurations increases the boundary method approaches a stable algorithm for doing inverse dynamics.

A 3-Dimensional, multi-segment coupled pendulum system, that mathematically models human motion, will be presented along with a derivation of a generalized formula that constructs the equations of motion for this model. The suggested model is developed to utilize the boundary method. The model developed in this thesis will lead to a long rang goal, which is the development of a diagnostic tool for any motion analysis laboratory that will answer the question of finding optimal movement patterns, to prevent injury and improve performance in human subjects.

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APPROVAL PAGE

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الإسراء

بسم الله الرحمن الرحيم

وَقَضَى رَبُّكَ أَلاَّ تَعْبُدُواْ إِلاَّ إِيَّاهُ

وَبِالْوَالِدَيْنِ إِحْسَاناً

إِمَّا يَبْلُغَنَّ عِندَكَ الْكِبَرَ أَحَدُهُمَا أَوْ كِلاَهُمَا

فَلاَ تَقُل لَّهُمَا أُفٍّ وَلاَ تَنْهَرْهُمَا

وَقُل لَّهُمَا قَوْلاً كَرِيماً

(23)

This thesis is dedicated to my Father and my Mother,

Thank you so much for being there for me, Thank you so much for taking care of me, Thank you so much for teaching me, Thank you so much for believing in me, Thank you so much for every thing you did and still doing for me.

Your Son
Loay

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CHAPTER 1

INTRODUCTION

1.1 Objective

Human movement analysis and modeling using mathematics, coupled with theoretical concepts in physiology and mechanics is constantly expanding and becoming more and more important in human performance and rehabilitation studies.

Abnormal stresses that exceed the ultimate strength of any biological tissue can cause material interaction that results in injury; therefore finding optimal motor patterns that can prevent abnormal stresses and also enhance performance is a worthwhile goal to achieve.

The objective of this thesis is to develop a 3-dimensional, multi-segment coupled pendulum system that mathematically models human motion. The model of human motion described in this thesis is in a form that can be directly used by a new method for solving human motion problems called the boundary method. This method was developed in this laboratory by HM Lacker [1] and colleagues and until this thesis was restricted to planar motion applications only. The boundary method is designed to find new movement techniques to solve motor tasks. The model developed in this thesis will lead to a long rang goal, which is the development of a diagnostic tool for any motion analysis laboratory that will answer the question of finding optimal movement patterns, to prevent injury and improve performance in human subjects.

1.2 Background Information

If many attempted repetitions of a motor drill can result in neural network reorganization that affects subsequent motor performance, then an important question arises as to the best choice of drill that will produce the desired coordination dynamics most efficiently and safely.

There are two current methods to investigate the dynamics of movement; they are called forward dynamics and inverse dynamics. The former uses forces to predict the motion and the later uses a system's motion to predict the forces required to produce that motion.

The forward method requires knowledge of the net forces acting on the system as inputs to calculate the system acceleration from the equations of motion (EOM). Integrating the acceleration of the system twice yields the system motion trajectory.

In contrast the inverse method starts with system trajectories, and derives their accelerations by differentiating twice and then uses the equations of motion to calculate the net joint forces. The type of model used in a particular research project depends on the goal of the study and the type of data that has either been collected or is otherwise made available.

Underlying every model mathematically are its equations of motion (EOM), which relate the actual movement (Kinematics) of the body to the applied forces (Kinetics), which causes the movement. How the equations of motion are solved determines if the modeling approach is forward or inverse.

A description of the inverse and the forward models of movement will be presented in the next chapter, with a brief discussion of the advantages, disadvantages, and limitation of each approach, then a new modeling approach is outlined called the boundary method which has been developed in our laboratory.

It is important to emphasize that each modeling approach has its advantages and limitations that may make one or the other more appropriate depending on the goal of the study and the data available to achieve it.

1.3 Equations of Motion

There are basically two approaches to formulating equations of motion of a mechanical system. The energy (Work) – based approach of Lagrange and the force – based approach of Newton and D'Alembert. If the choice of independent dynamic variables are position only, then the equations of motion that are formulated using these two methods will be of second order.

The Lagrange method is usually more direct than either Newton's or D'Alembert's methods for arriving at the correct set of independent motion equations.

In this thesis a generalized formula for the equations of motion (EOM) of a 3-dimensional, multi-segmented coupled pendulum system will be derived using the Energy (Work) - based approach of Lagrange. The EOM derived in this thesis applies to a 3-dimensional pendulum system where both the number of segments, s, and the branching pattern of those segments is arbitrary provided the connected pattern corresponds to an acyclic graph and that at least one segment is anchored to the origin of the coordinate system.

These EOMs are presented in a form that allows the boundary method to be utilized; the method was previously limited to only 2-Dimensions.

The energy (Work) - based approach of Lagrange is based on the difference between the kinetic and potential energy of the system. More precisely the Lagrangian function is defined as:

Lagrangian = Kinetic Energy – Potential Energy

$$L = K - P$$

Using the above definition of the Lagrangian function, the ^{i}th equation of motion corresponding to the ^{i}th degree of freedom takes the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \underline{\dot{\alpha}}_i}\right) - \frac{\partial L}{\partial \underline{\alpha}_i} = 0 \qquad (1.1)$$

where $i = 1, \cdots, 2s$ (s is the number of segments) and $\underline{\alpha}_i$ represents either the segment angle θ or ϕ of the ith segment (Figure 1.1) and $\underline{\dot{\alpha}}$ represents the angular velocity of the corresponding segment angle.

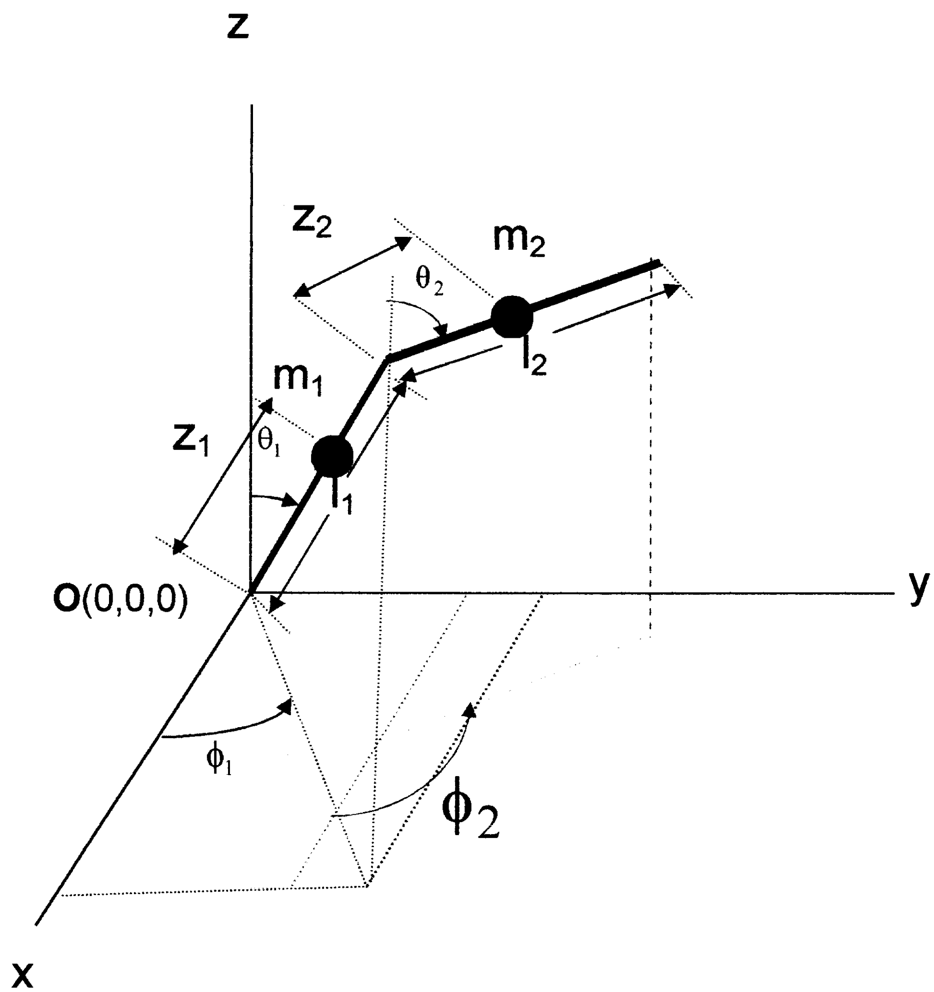

Figure1.1 3-Dimensional, multi-segment (2 segments) coupled pendulum system.

The segments angles (θ_1, ··· ,θ_S, Φ_1, ··· ,Φ_S), will be the generalized coordinate system that will be considered in this thesis (to describe position data), this choice of generalized coordinates implicitly allows for body movements to occur without violating the constraints of constant segment length.

CHAPTER 2

DYNAMICAL MODELING APPROACHES

2.1 Inverse Dynamics

Most mathematical models of human motion satisfy the generalized second order system:

$$\mathbf{F}(X,\dot{X}) = \mathbf{M}(X) * \ddot{X}(t) \tag{2.1}$$

Where;

$\mathbf{F}(\dot{X}(t),X(t))$:Net generalized system force vector
$\mathbf{M}(X(t))$: generalized mass matrix (a function of position only)
$\ddot{X}(t)$ Generalized system acceleration vector

In inverse dynamics the motion X(t) is given and equation (2.1) is used to solve for F(t). This is inverted from the usual way the EOM are solved in mechanical problems where the forces are given and the motion is calculated by solving the system of differential Equation (2.1).

The conventional approach for inverse dynamics starts with formulating the equations of motion using the force-angular momentum method of Newton-Euler (ref. Bressler, B) [2] . A sequential or iterative procedure is then applied. This procedure solves for the net joint reaction force and net joint torque (moment of muscle force) acting at the next joint end of a given segment using the joint torque acting at a similar joint end of the previous adjacent segment (Winter DA 1984a [3]; Winter D 1990 [4]). The motion data required for each separate iteration are only the translational and angular accelerations of that rigid-body segment and not the entire motion of the system.

Another way to solve the inverse problem is to use the following algorithm:

1) Transform, if necessary, the position data into the chosen generalized coordinate system for equation (2.1) to obtain the system motion vector $X(t)$.

2) Twice differentiate $X(t)$ to obtain $\ddot{\boldsymbol{X}}(\boldsymbol{t}) = a(t)$.

3) Substitute $X(t)$ into $\mathbf{M}(X(t))$.

4) Matrix multiply to find: $F(t) = \mathbf{M}(X(t))\, a(t)$.

5) Transform, if necessary, F(t) to the equivalent resultant system force vector whose components are the desired dynamic torque at each joint.

This algorithm illustrates the fact that the motion of the segments can be treated simultaneously to yield net dynamic joint torque.

Inverse dynamics solutions can often be much improved by adding force data, such as ground reaction forces, to the motion data. Kuo (1998) [5] proposed to over complete the algorithm by adding ground reaction force and torque data to the equations of motion to be treated as another generalized external force that has been applied to the system in a given time varying fashion so that the ground constraint is always satisfied.

2.1.1 Advantages of the Inverse Method

The major advantage of inverse modeling is that it's required input, motion kinematics and ground reaction forces, is readily measurable by sophisticated motion analysis systems and force plates available in many research laboratories.

The predictions obtained from inverse modeling can provide means for quantitatively comparing different movement techniques with respect to the forces, the mechanical energy, and the power required to produce them.

2.1.2 Disadvantages of the Inverse Method

Any noise in the motion data is amplified when it is differentiated twice to obtain the required acceleration input to the dynamic equations. Even when data can be appropriately filtered; however, problems often still exist when the conventional algorithm is applied to dynamical problems. At an unstable equilibrium, small external forces can produce large changes in system configuration even over relatively small time intervals.

For such systems the result will be that the torque output will not reproduce the motion that they were supposed to cause. This is shown when the dynamic torque output of an inverse method is used as input to a forward method that solves the same system of differential equations and the resulting motion fails to match the initial motion data.

Risher, Schutte and Runge (1997) [6] called this phenomenon *inverse dynamics simulation failure*.

A primary disadvantage of inverse models is that they require kinematics as inputs. Because of this, the inverse method can be used only to analyze movements that have *already been performed*; they are incapable of generating system kinematics as output and, therefore, cannot be used to predict new movement patterns. That is, they cannot suggest better alternatives to present observable motions but can only compare existing motions.

2.2 Forward Dynamics

Movement begins with neural drive to the muscles, resulting in muscle force activity at the joints, and ultimately causing displacements of the joint (i.e., the observed motion).

Forward models solve Equation (2.1) for the acceleration (Forward direction):

$$a = \mathbf{M}(X)^{-1}\ \mathbf{F}(X,\dot{X}) \tag{2.2}$$

All forces acting on the system must be input to the model, while accelerations are unknown and constitute model output. These dynamic accelerations are used to solve for position (and velocity) segment trajectories as a function of time.

Forward models simulate physically realizable movements when provided with input data resembling the net joint force produced from neural input to muscles. In addition to net muscle force, forward model input in a Newton-Euler formulation of the equations of motion requires inclusion of all other body and contact forces that contribute to the system's motion such as gravity, ground and joint reaction forces.

In the forward approach the differential equations of motion are usually solved as initial value problems, therefore both the initial system configuration (segment angles) and initial system velocity (segment angular velocities) are required inputs.

A challenging but potentially powerful forward modeling approach is to construct a rather complete model of the human body that closely mimics the real sequence of physiological events beginning with neural drive to the muscles and ultimately ending in joint displacements of the body. An example of this approach is the work of Hatze H and Venter A 1981 [7], Pandy et al.[1990] [8], and Zajac (1993) [9] on models of jumping.

As with inverse solutions all forward solutions require accurate anthropometric segment data.

2.2.1 Advantages of the Forward Method

When the goal of the research is to predict new physically realizable motions forward models are ordinarily utilized.

The hierarchy of forward modeling more closely mimics the real sequence of physiological movement events. In summary, the potential of the forward modeling approach is that new and better movement solutions can be found that will optimize any sufficiently well defined performance goal; however, the technique is very difficult to apply towards this goal.

2.2.2 Disadvantages of the Forward Method

The forward method requires muscle forces as inputs. Unfortunately, noninvasive instrumentation to measure such forces does not exist presently. Therefore, forward models must rely on estimates or even educated guesses of forces that must have acted on the system in order to produce an observed movement.

Many of the motor tasks for which an optimal solution is desired are highly skilled movements that are likely to be very sensitive to small changes in the dynamic forces applied to the system. This inherent instability of the motion requires a very accurate guess of the net muscle moments that must have acted at each joint of the system in order to produce the improved movement and even educated guesses of what these net muscle moments might be are not available.

2.3 Boundary Method

To summarize the previous sections:

(1) The inverse method can effectively use motion data collected in the lab to critically compare two given performance techniques but cannot output new motions (they are input to the method).

(2) The forward method has the potential to find new movements to skilled motor tasks, but to find such new solutions very accurate input data (muscle forces) is required that cannot be known or even reasonably guessed a priori.

Prof. H. M. Lacker [1] and colleagues have proposed a new modeling approach that is based on a generalization of the forward modeling method of Mochon and McMahon (Mochon S and McMahon TA 1980) [10], this modeling approach is called the *Boundary Method.* It was developed to address some of the disadvantages of both inverse and forward models described above.

The hypothesis behind this approach is that if a motor task or a significant portion of a motor task could be accomplished without muscle involvement, other than to establish the correct initial body configuration and initial system velocity, then it might be desirable to do so.

Although muscles are required to initiate movement, once initiated, movement can and generally does continue without further muscular activity because of the momentum initially imparted to the segments, and their interaction both with the gravitational field and with each other. The motion of the system and its parts only stops because of the presence of dissipative forces such as friction that eventually removes the macroscopic mechanical energy initially imparted to the system.

Macroscopic motion which is imparted to a system by the sudden injection of energy into it is sometimes called *ballistic* because it is similar to what happens to a projectile shot out of a cannon. After the initial explosive force generates the projectile's initial velocity it continues to move towards the target by the action of gravity alone. If the projectile is composed of many internal parts or segments, each rigid but connected by movable joints then these segments will also continue to move relative to each other as the system moves towards its target. If the initial velocity of each of its segments is given, the motion of the system and of each of its parts is relatively easy to calculate, since the only external force acting on the system is gravity. Frictional or viscous forces can also be included if they are believed to play an important role.

For a ballistic motion force terms relating to the action of contracting muscles are not present because they are not active except to give the segments their starting velocity at the beginning of the movement. Of course, it would be difficult in a complicated system of joints and segments to know what the correct initial velocity of each of the segments should be in order that the system might arrive from the action of gravity alone at the target configuration in a desired configuration at the specified target time.

Mathematically the correct initial velocity can be determined by solving the ballistic equations of motion as a two point boundary value problem rather than as an initial value problem. The input required to solve an initial value problem is the system's initial position and its initial velocity. In a two-point boundary value problem the initial velocity of the system is not used as input for the motion solution but is replaced by knowledge of the desired target position of the system. The two-points in a two-point boundary value problem are the initial and target points. In a system consisting of many segments these two points really represent the configuration of the system at each of these times. The coordinates of each point, for example, could be represented by the angle that the beginning of each segment makes relative to its horizontal position. Instead of asking to find the motion of a projectile given its initial position and velocity, a two-point boundary value problem solves for what the initial velocity must be in order that the resulting motion solution hits the target configuration in a specified time starting from a given initial configuration.

Ballistic motions, if they exist, are often quite efficient because muscles need act only at the beginning of a movement. Gravity and the spontaneous transfer of momentum between segments accomplish the rest. Such ballistic motions however would often require great skill to perform especially if they occur over a long duration. This is because small errors in the initial velocities of the segments magnify into large errors in the target configuration as the duration of the ballistic movement increases.

What happens if no solution exists that moves the body for a designated time from a given starting configuration to a desired target configuration using muscles only to deliver the initial segment velocities? Unlike many initial value problems, the solution of a boundary value problem is not guaranteed to either exist or even be unique if it does exist. Perhaps if a single ballistic solution does not exist, the motion could be successfully performed if two ballistic phases are allowed. One ballistic phase is used to get to an important intermediate target configuration. This configuration serves as the initial configuration of a second ballistic phase where muscle activity is again permitted to actively change the momentum of some or all of the segments to complete the motor task in the second ballistic phase.

The *boundary method* approach for this two ballistic phase motion will consist of three input configurations (initial, intermediate and target) and the target time will be the sum of the two times to complete each phase. Both ballistic phases are solved as separate and independent two-point boundary value problems. The entire motion will now consist of the two ballistic solutions contiguously pieced together. The complete motion will be continuous since the end configuration of the first ballistic phase solution is the beginning configuration of the second ballistic phase. Since both phases are solved separately and independently the final segment velocities at the end of the first ballistic phase solution will not, in general, match the initial velocities that are given by the solution of the second phase.

The method therefore generates as output the sudden change in segment velocity that is required to be delivered by the muscles at the crucial intermediate configuration in order to complete the task in the designated time. If there are several two-phase ballistic solutions each obtained with a different intermediate configuration, then each would represent a different possible movement strategy that solves the desired motor task. Since the solution of a boundary value problem is not guaranteed to be unique, there may be several single phase ballistic solution strategies as well to choose from. Different motion solution strategies will predict that muscles deliver different amounts of mechanical energy to the system and one of many possible selection criteria could be to search for that strategy that minimizes muscular effort, assuming, of course, that this can be defined in some reasonable way.

Of course, one need not restrict the search for motion strategies that are limited to one or two ballistic phases. If more ballistic phases then two are permitted then more intermediate configurations must be specified as input. This results in more times when muscle activity occurs. In the limit of a large number of phases, the boundary method approaches a type of (stable) inverse dynamics since essentially the input consists of a given motion and the output consists of the large number of impulsive forces that are required to produce it. At the other extreme, a single ballistic phase solution is a type of forward dynamic solution where the forces are known but no muscle force terms are used in the equations of motion since the motion is assumed to be ballistic.

In summary, the Boundary Method breaks any movement task into a finite and adjustable number of separate contiguous phases. Each phase is solved as an independent ballistic two-point boundary value problem using Lagrangian Mechanics. The generated output will always be the continuous motion obtained by piecing together the independent two-point boundary value solutions of the separate but contiguous ballistic phases. In addition, the generated output will also consist of a predicted series of impulsive "hammer blows" that represent the net impulsive muscle forces that are needed to suddenly change the segment velocities at the beginning of each contiguous phase. The Boundary Method yields muscular activity at a finite but adjustable number of discrete times during the motion.

CHAPTER 3

GENERALIZED EQUATIONS OF MOTION OF 3-DIMENSIONAL MULTI-SEGMENTS COUPLED PENDULUM MODEL

3.1 The Relation matrix

Assuming that any moving system consists of a number of connected segments and point masses on these segments Figure (1.1).

The relation matrix **R** shows the relationship of connection between segments of the system, where each row of **R** represents a point mass of the system, and has the information about the path from the origin of the system to the point mass. Each column represents a segment, and has the information about the usage of the segment for every path to point masses (Lacker et al.) [1]. For a system with S segments and P point masses, the relation matrix **R** has the form as:

$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1S} \\ r_{21} & r_{22} & \cdots & r_{2S} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ r_{P1} & r_{P2} & \cdots & r_{PS} \end{pmatrix}$$

The path from the origin to the p-th point mass may consist of many segments. Assuming that the p-th point mass is on the i-th segment of the system. Then, the i-th segment is the last segment of the path, and all other segments of the path are called as forefather segments of the i-th segment. Using the above information each element r_{ps} of **R** can be determined as follows:

$$r_{ps} = \begin{cases} z_P & \text{if } s = i, \\ L_s & \text{if the } s\text{-th segment is a forefather segment of the } i\text{-th segment} \\ 0 & \text{otherwise.} \end{cases} \quad \text{----- (3.1)}$$

Where

z_P : mass center of the p-th particle from the joint with its forefather segment
L_s : length of the s-th segment
$p = 1, 2, 3, \cdots$ P (p : total number of particles)
$s = 1, 2, 3, \cdots$ S (S : total number of segments) .

3.2 Angular Coordinate System

Assuming that the generalized position vector is:

$$\vec{X} = (X1, \cdots, X_{2s})^T = (\vec{\theta}, \vec{\Theta})^T \tag{3.2}$$

Where,

$\vec{\theta} = (\theta_1, \cdots, \theta_s)^T$
$\vec{\Phi} = (\Phi 1, \cdots, \Phi s)^T$

Note that:

$$X_i = \begin{cases} \theta_i & i = 1, \cdots, s \\ \Phi_{i-s} & i = s+1, \cdots, 2s \end{cases}$$

The generalized velocity is:

$$\vec{V} = (V_1, \cdots, V_{2s})^T = (\dot{\vec{\theta}}, \dot{\vec{\Phi}})^T = \dot{\vec{X}} \tag{3.3}$$

After determining the relation matrix **R** , x- and y- and z-coordinates of a point mass can be written as follows :

$$\mathbf{x} = [\, x_1 \;\; x_2 \cdots x_P \,]^T = \mathbf{R} \cdot [\, (\text{Cos}\Phi_1 * \text{Sin}\theta_1) \;\; (\text{Cos}\Phi_2 * \text{Sin}\theta_2) \;\cdots\; (\text{Cos}\Phi_S * \text{Sin}\theta_S)]^T \tag{3.4}$$

$$\mathbf{y} = [\, y_1 \;\; y_2 \cdots y_P \,]^T = \mathbf{R} \cdot [\, (\text{Sin}\Phi_1 * \text{Sin}\theta_1) \;\; (\text{Sin}\Phi_2 * \text{Sin}\theta_2) \;\cdots\; (\text{Sin}\Phi_S * \text{Sin}\theta_S)]^T \tag{3.5}$$

$$\mathbf{z} = [\, z_1 \;\; z_2 \cdots z_P \,]^T = \mathbf{R} \cdot [\text{Cos}\theta_1 \;\; \text{Cos}\theta_2 \;\cdots\; \text{Cos}\theta_S\,]^T \tag{3.6}$$

where

θ_i : Angle of the $_i$-th segment with respect to the positive Z-direction .
Φ_i: Angle of the $_i$-th segment with respect to the positive Y-direction.
As in Figure.---

From equation (3.4), (3.5), and (3.6) , x- and y- and z-component of the velocity of a point mass can be determined as follows :

$$\dot{\mathbf{x}} = [\, \dot{x}1 \;\; \dot{x}2 \cdots \dot{x}p \,]^T = \mathbf{R} \cdot \underline{\text{CSX}} \cdot \vec{V} \tag{3.7}$$

$$\dot{\mathbf{y}} = [\, \dot{y}1 \;\; \dot{y}2 \cdots \dot{y}p \,]^T = \mathbf{R} \cdot \underline{\text{CSY}} \cdot \vec{V} \tag{3.8}$$

$$\dot{\mathbf{z}} = [\, \dot{z}1 \;\; \dot{z}2 \cdots \dot{z}p \,]^T = -\mathbf{R} \cdot \underline{\text{CSZ}} \cdot \vec{V} \tag{3.9}$$

Where

$$\underline{\text{CSX}} = \left(\begin{array}{cccc:cccc} \text{Cos}\Phi_1 * \text{Cos}\theta_1 & 0 & \cdots & 0 & -\text{Sin}\Phi_1 * \text{Sin}\theta_1 & 0 & \cdots & 0 \\ 0 & \text{Cos}\Phi_2 * \text{Cos}\theta_2 & \cdots & 0 & 0 & -\text{Sin}\Phi_2 * \text{Sin}\theta_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & & \text{Cos}\Phi_S * \text{Cos}\theta_S & 0 & 0 & & -\text{Sin}\Phi_S * \text{Sin}\theta_S \end{array}\right)$$

$$\underline{CSY} = \left(\begin{array}{cccc|cccc} \sin\Phi_1 * \cos\theta_1 & 0 & \cdots & 0 & \cos\Phi_1 * \sin\theta_1 & 0 & \cdots & 0 \\ 0 & \sin\Phi_2 * \cos\theta_2 & \cdots & 0 & 0 & \cos\Phi_2 * \sin\theta_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & & \sin\Phi_S * \cos\theta_S & 0 & 0 & & \cos\Phi_S * \sin\theta_S \end{array}\right)$$

$$\underline{CSZ} = \left(\begin{array}{cccc|ccc} \sin\theta_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sin\theta_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & & \sin\theta_S & 0 & \cdots & 0 \end{array}\right)$$

3.3 Potential Energy

Adding up potential energies of all the point masses of a system gives the total potential energy of this system.

$$P(X) = \sum_{i=1}^{P} g\, m_i\, z_i$$

$$P = \sum_{i=1}^{P} g\, m_i\, L_i \cos\theta_i$$

$$= g\ \mathbf{z}^T\ \mathbf{M_D}\ [1\ \ 1\ \ 1\ \cdots\ 1]^T_{1xP}$$

$$= g\ [\mathrm{Cos}\theta_1\ \ \mathrm{Cos}\theta_2\ \cdots\ \mathrm{Cos}\theta_S] \cdot \mathbf{R}^T \cdot\ \mathbf{M_D} \cdot \mathbf{I}^T{}_{1xP} \qquad (3.10)$$

$$= [\mathrm{Cos}\theta_1\ \ \mathrm{Cos}\theta_2\ \cdots\ \mathrm{Cos}\theta_S] \cdot [\ \mathrm{P}]_{Sx1}$$

$$\therefore\ \mathrm{P} = \sum_{i=1}^{P} \mathrm{P_i}\ \mathrm{Cos}\theta_i$$

Where

$$\mathbf{M_D} = \begin{bmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & \dots & 0 \\ . & . & & . \\ . & . & & . \\ 0 & 0 & & m_P \end{bmatrix}$$

$$\mathbf{I} = [\ 1\ 1 \cdots\ 1]_{\ 1xP}$$

3.4 Kinetic Energy and the Generalized Mass Matrix

Adding up kinetic energies of all the point masses of a system gives the total kinetic energy of this system. When the mass and the velocity of the p-th point mass are m_P and v_P , respectively, the kinetic energy of the point mass is given as $e_p = (1/2)\ m_P\ v^2_P$. Substituting $v^2_P = (\ \dot{x}^2_P + \dot{y}^2_P + \dot{z}^2_P)$, the total kinetic energy **K** of a system with P points masses can be written as follows :

$$K = \frac{1}{2} * \sum_{P=1} m_P * (x^{\cdot 2}_P + y^{\cdot 2}_P + z^{\cdot 2}_P) \text{ , where } P \text{ is the \# of mass points}$$

$$= \frac{1}{2} * (\mathbf{x}^{\cdot T} \cdot \mathbf{M_D} \cdot \mathbf{x}^{\cdot} + \mathbf{y}^{\cdot T} \cdot \mathbf{M_D} \cdot \mathbf{y}^{\cdot} + \mathbf{z}^{\cdot T} \cdot \mathbf{M_D} \cdot \mathbf{z}^{\cdot}) \qquad (3.11)$$

Substituting equations (3.7), (3.8) and (3.9) into equation (3.11) we get :

$$K = \frac{1}{2} * (\vec{V}^T \cdot \underline{CSX}^T \cdot \mathbf{R}^T \cdot \mathbf{M_D} \cdot \mathbf{R} \cdot \underline{CSX} \cdot \vec{V} + \vec{V}^T \cdot \underline{CSY}^T \cdot \mathbf{R}^T \cdot \mathbf{M_D} \cdot \mathbf{R} \cdot \underline{CSY} \cdot \vec{V} + \vec{V}^T \cdot \underline{CSZ}^T \cdot \mathbf{R}^T \cdot \mathbf{M_D} \cdot \mathbf{R} \cdot \underline{CSZ} \cdot \vec{V})$$

Defining,

$$\mathbf{C} = \mathbf{R}^T \cdot \mathbf{M_D} \cdot \mathbf{R} = [c_{ij}]_{SxS} \qquad (3.12)$$

i,j = 1, $\cdots$, s (number of segments)

$$\sum_{p=1}^{p} c_{ij} = \sum m_P \, r_{Pi} \, r_{Pj} \text{ , (} P \text{ is the \# of mass points)}$$

$$K = \frac{1}{2} * (\vec{V}^T \cdot \underline{CSX}^T \cdot \mathbf{C} \cdot \underline{CSX} \cdot \vec{V} + \vec{V}^T \cdot \underline{CSY}^T \cdot \mathbf{C} \cdot \underline{CSY} \cdot \vec{V} + \vec{V}^T \cdot \underline{CSZ}^T \cdot \mathbf{C} \cdot \underline{CSZ} \cdot \vec{V})$$

$$= \frac{1}{2} * \vec{V}^T \cdot (\underline{CSX}^T \cdot \mathbf{C} \cdot \underline{CSX} + \underline{CSY}^T \cdot \mathbf{C} \cdot \underline{CSY} + \underline{CSZ}^T \cdot \mathbf{C} \cdot \underline{CSZ}) \cdot \vec{V}$$

Then, the total kinetic energy can be written in a compact form as:

$$K(X,V) = \frac{1}{2} * \vec{V}^T \cdot \mathbf{M(X)} \cdot \vec{V} \qquad (3.13)$$

Where:

$$\mathbf{M}(X) = \underline{CSX}^T \cdot \mathbf{C} \cdot \underline{CSX} + \underline{CSY}^T \cdot \mathbf{C} \cdot \underline{CSY} + \underline{CSZ}^T \cdot \mathbf{C} \cdot \underline{CSZ} \qquad (3.14)$$

$$\mathbf{M}(X) = \begin{pmatrix} M11 & M12 \\ M21 & M22 \end{pmatrix}$$

Where each sub-Matrix $\boldsymbol{M}_{IJ}$ ($I = 1,2; J = 1,2$) is sxs and defined as:

$$M_{11_{ij}} = c_{ij} \; [\mathrm{Sin}\theta i \; \mathrm{Sin}\theta j + \mathrm{Cos}\theta i \; \mathrm{Cos}\theta j * \mathrm{Cos}(\Phi i - \Phi j)]$$

$$M_{12_{ij}} = c_{ij} \; [\mathrm{Cos}\theta i \; \mathrm{Sin}\theta j * \mathrm{Sin}(\Phi i - \Phi j)]$$

$$M_{21_{ij}} = c_{ij} \; [-\mathrm{Sin}\theta i \; \mathrm{Cos}\theta j * \mathrm{Sin}(\Phi i - \Phi j)]$$

$$M_{22_{ij}} = c_{ij} \; [\mathrm{Sin}\theta i \; \mathrm{Sin}\theta j * \mathrm{Cos}(\Phi i - \Phi j)]$$

With:
$i,j = 1, \cdots, s$ (number of segments)

Note that $M_{11_{ij}}$ and $M_{22_{ij}}$ are both symmetric that is:
$M_{11_{ij}} = M_{11_{ji}}$
$M_{22_{ij}} = M_{22_{ji}}$
Also,
$M_{12_{ij}} = M_{21_{ji}}$, These prosperities make **M(*X*)** symmetric

$$\mathbf{M}(X) = \mathbf{M}^{\mathbf{T}}(X) \qquad (3.15)$$

That is:

$M_{IJ_{ij}} = M_{JI_{ji}}$

3.5 Lagrangian Function

For a system of S independent variables, the Lagrangian equations take the form (Wells 1967,60):

$$L = L(\vec{X}, \vec{V}) = \text{Kinetic Energy - Potential Energy}$$

$$L(\vec{X},\vec{V}) = \{ 0.5 * \vec{V}^{T} \cdot \mathbf{M(X)} \cdot \vec{V} \} - \{ g \ [\cos\theta 1 \ \ \cos\theta 2 \ \cdots \ \cos\theta_{S}] \cdot \mathbf{R}^{T} \cdot \mathbf{M_D} \cdot \mathbf{I}^{T}_{1xP} \} \tag{3.16}$$

3.6 Equations of Motion from Lagrangian Function

Haveing 2 degrees of freedom for each segment of the system, implies 2s equations of motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{V}_i} \right) - \frac{\partial L}{\partial \vec{X}_i} = 0 \tag{3.17}$$

where i = 1 , ⋯ , 2s (s is the number of segments)

Or splitting the system in to two parts, one for the θ's and the other for the Φ's, we get:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i} \right) - \frac{\partial L}{\partial \theta_i} = 0 \qquad \text{, where } i = 1, \cdots, s$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Phi}_i} \right) - \frac{\partial L}{\partial \Phi_i} = 0 \qquad \text{, where } i = 1, \cdots, s$$

Since the only part of the Lagrangian dependent on $\vec{V}$ is the Kinetic Energy, the above two equations can be written in the form:

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = 0 \quad \text{, where } i = 1, \cdots, s$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\Phi}_i}\right) - \frac{\partial L}{\partial \Phi_i} = 0 \quad \text{, where } i = 1, \cdots, s$$

CHAPTER 4

DERIVATION OF A GENERAL FORMULA FOR OBTAINING THE EQUATIONS OF MOTION OF A 3-DIMENSIONAL, MULTI-SEGMENTS COUPLED PENDULUM SYSTEM

4.1 General Formation of the Equations of motion

Equation (3.17) can be written as:

$$\frac{d}{dt}\left(\frac{\partial K}{\partial V_i}\right) - \frac{\partial K}{\partial X_i} = -\frac{\partial P}{\partial X_i} \qquad (4.1)$$

The first term in the above equation will be evaluated by; first, doing the partial derivative of kinetic energy with respect to the velocity, and secondly deriving the result with respect to time:

$$\left(\frac{\partial K}{\partial V_i}\right) = \frac{1}{2} * \sum_{j=1}^{2s} \sum_{k=1}^{2s} M_{jk}(X) \frac{\partial}{\partial V_i}(V_j V_k)$$

$$= \frac{1}{2} * \left\{ \sum_{\substack{j=1 \\ (j \neq i)}}^{2s} M_{ji}(X) \frac{\partial}{\partial V_i}(V_j V_i) + \sum_{\substack{k=1 \\ (k \neq i)}}^{2s} M_{ik}(X) \frac{\partial}{\partial V_i}(V_i V_k) + M_{ii}(X) \frac{\partial}{\partial V_i} V^2_i \right\}$$

$$= \frac{1}{2} * \sum_{\substack{j=1 \\ (j \neq i)}}^{2s} M_{ji}(X) V_j + \frac{1}{2} * \sum_{\substack{k=1 \\ (k \neq i)}}^{2s} M_{ik}(X) V_k + M_{ii} V_i$$

$$= \frac{1}{2} * \sum_{j=1}^{2s} M^T_{ij}(X) V_j + \frac{1}{2} * \sum_{j=1}^{2s} M_{ij}(X) V_j \qquad (4.2)$$

From the above derivation Equation (4.2), and using the fact that $M=M^T$ Equation (3.15), we can write the partial derivative of Kinetic energy with respect to the velocity in a Matrix-Vector multiplication as shown below:

$$\left(\frac{\partial K}{\partial V_i} \right) = (\nabla_V K)_i = \sum_{j=1}^{2s} \mathrm{M_{ij}(X)\,V_j} = (\,\mathbf{M}(\boldsymbol{X})\,\vec{V}(t)\,)_\mathrm{i} \tag{4.3}$$

Where, ($\mathbf{M}(\boldsymbol{X})\,\vec{V}$) is the generalized momentum of the system.

Now, deriving the previous partial derivative of the kinetic energy, Equation (4.3), with respect to time, using the product rule:

$$\frac{d}{dt}\left(\frac{\partial K}{\partial V_i} \right) = \frac{d}{dt}(\mathrm{M}(\boldsymbol{X})\,\vec{V}(t))_\mathrm{i} = \sum_{j=1}^{2s} \frac{d}{dt}(\,\mathrm{M_{ij}}(X(t))\,V_j(t)\,)$$

$$= \sum_{j=1}^{2s} (\,\mathrm{M_{ij}}(X(t))\,\dot{V}_j(t)\,) + \sum_{j=1}^{2s} \frac{d}{dt}(\,\mathrm{M_{ij}}(X(t)))\,V_j(t) \tag{4.4}$$

Using the chain rule, for the second term of Equation (4.4):

$$\frac{d}{dt}(\,\mathrm{M_{ij}}(X(t))) = \sum_{k=1}^{2s} \frac{\partial M_{ij}}{\partial X_k}\frac{dX_k}{dt} = \sum_{k=1}^{2s} \frac{\partial M_{ij}}{\partial X_k} V_k$$

Substituting in Equation (4.4):

$$\frac{d}{dt}\left(\frac{\partial K}{\partial V_i} \right) = \sum_{j=1}^{2s} (\,\mathrm{M_{ij}}(X(t))\,\dot{V}_j(t)\,) + \sum_{j=1}^{2s}\sum_{k=1}^{2s} \frac{\partial M_{ij}}{\partial X_k} V_k\,V_j \tag{4.5}$$

Using the above results Equation (4.5), the i^{th} equation of motion for a conservative system can be written in the following format:

$$(\mathbf{M(X)}\,\dot{\vec{V}}(t))_i + \sum_{j=1}^{2s}\sum_{k=1}^{2s} \frac{\partial M_{ij}}{\partial X_k} V_k\, V_j \;-\; \frac{\partial K}{\partial X_i} = -\frac{\partial P}{\partial X_i} \qquad (4.6)$$

The partial derivative of the kinetic energy with respect to position, the second term in Equation (4.1), can be written in the following way:

$$\frac{\partial K}{\partial X_i} = \frac{1}{2} V^T \left(\frac{\partial}{\partial X_i} M_{jk}(X)\right) V = \frac{1}{2} * \sum_{j=1}^{2s}\sum_{k=1}^{2s} \frac{\partial}{\partial X_i} [\,M_{jk}(X)] \qquad (4.7)$$

Adding Equation (4.7) to Equation (4.6), this will give us the general i^{th} equation of motion for a conservative system:

$$(\mathbf{M(X)}\,\dot{\vec{V}}(t))_i + \frac{1}{2} \sum_{j=1}^{2s}\sum_{k=1}^{2s} V_j\, V_k \left(\frac{\partial}{\partial X_k}[\,2\, M_{ij}(X)\,] - \frac{\partial}{\partial X_i}[\,M_{jk}(X)]\right) = -\frac{\partial P}{\partial X_i} \qquad (4.8)$$

Defining $S_i\,(X,V)$as:

$$S_i\,(X,V) = \frac{1}{2} \sum_{j=1}^{2s}\sum_{k=1}^{2s} V_j\, V_k \quad \frac{\partial}{\partial X_k}[\,2\, M_{ij}(X)\,] - \frac{\partial}{\partial X_i}[\,M_{jk}(X)] \qquad (4.9)$$

where $i = 1, \cdots, 2s$

Substituting S_i (X,V) Equation (4.9) in to Equation (4.8), the i^{th} equation of motion for a conservative system, in a Matrix-Vector format will become:

$$M(X)\quad \dot{V} \qquad = -\,\nabla \vec{P}(X) - \vec{S}\,(X,V) \qquad (4.10)$$

Where, $\nabla\vec{P}(X) = \begin{pmatrix} \nabla_\theta P \\ ---- \\ \nabla_\Phi P \end{pmatrix}$ with; $\nabla_\theta P = \dfrac{\partial P}{\partial \theta s}$ and, $\nabla_\Phi P = \dfrac{\partial P}{\partial \Phi s}$

$$(\nabla_{\theta} P)_i = \frac{\partial P}{\partial \theta_i}$$

$$(\nabla_{\theta} P)_i = - g \; [\text{Sin}\theta_1 \;\; \text{Sin}\theta_2 \cdots \text{Sin}_S] \cdot \mathbf{R}^T \cdot \mathbf{M_D} \cdot \mathbf{I}^T_{1xP} \qquad (4.11)$$

$$(\nabla_{\Phi} P)_i = \frac{\partial P}{\partial \Phi_i} = \text{Zero} \qquad , \; i = 1 , \cdots , s$$

No Φ dependent in the Potential Energy Equation (3.10).

Equation (4.10) can be written as well in a Sub-Matrix form in the following way:

$$\left(\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array} \right) \left[\begin{array}{c} \ddot{\theta} \\ \hline \ddot{\Phi} \end{array} \right] = - \left(\begin{array}{c} \nabla_{\theta} P \\ \hline \nabla_{\Phi} P \end{array} \right) - \left(\begin{array}{c} S_{\theta} \\ \hline S_{\Phi} \end{array} \right) \qquad (4.12)$$

Note that:

$$\begin{cases} S_{\theta_i} = S_i , & i = 1 , \cdots , s \\ S_{\Phi_{i-s}} = S_i , & i = s+1 , \cdots , 2s \end{cases}$$

Splitting Equation (4.12) in to two parts we get:

$$M_{11} \ddot{\theta} + M_{12} \ddot{\Phi} = - \nabla_{\theta} P - S_{\theta}$$

$$M_{21} \ddot{\theta} + M_{22} \ddot{\Phi} = -\nabla_{\Phi} P - S_{\Phi}$$

Where;

$$S_{\theta_i} = \frac{1}{2}\{ S_{\theta 11_i} + S_{\theta 12_i} + S_{\theta 21_i} + S_{\theta 22_i} \} = \frac{1}{2}\sum_{J=1}^{2}\sum_{I=1}^{2} S_{\theta IJ_i} \tag{4.13}$$

Using the Sub-Matrices of **M(X)**, the θ part of Equation (4.9) becomes:

$$(S_{\theta IJ})_i = \sum_{j=1}^{s}\sum_{k=1}^{s} (V_I)_j \left(\frac{\partial}{\partial (X_J)_k}[2\,(M_{IJ})_{ij}] - \frac{\partial}{\partial \theta_i}[(M_{IJ})_{jk}] \right) (V_J)_k \tag{4.14}$$

And,

$$S_{\Phi_i} = \frac{1}{2}\{ S_{\Phi 11_i} + S_{\Phi 12_i} + S_{\Phi 21_i} + S_{\Phi 22_i} \} = \frac{1}{2}\sum_{J=1}^{2}\sum_{I=1}^{2} S_{\Phi IJ_i} \tag{4.15}$$

Using the Sub-Matrices of **M(X)**, the Φ part of Equation (4.9) becomes:

$$(S_{\Phi IJ})_i = \sum_{j=1}^{s}\sum_{k=1}^{s} (V_I)_j \left(\frac{\partial}{\partial (X_J)_k}[2\,(M_{IJ})_{ij}] - \frac{\partial}{\partial \Phi_i}[(M_{IJ})_{jk}] \right) (V_J)_k \tag{4.16}$$

Where;

$$V_I = \begin{cases} \dot{\theta} & I=1 \\ \dot{\Phi} & I=2 \end{cases}$$

$$V_J = \begin{cases} \dot{\theta} & J=1 \\ \dot{\Phi} & J=2 \end{cases}$$

$$X_J = \begin{cases} \theta & J=1 \\ \Phi & J=2 \end{cases}$$

4.2 Evaluating $S_\theta(i)$

Writing Equation (4.14) explicitly by substituting the corresponding values of $(X_J)_k$, $(V_I)_j$, and $(V_J)_k$:

$$S_{\theta 11}(i) = \sum_{k=1}^{s} \dot{\theta}_k \sum_{j=1}^{s} \dot{\theta}_j \left[\frac{\partial}{\partial \theta_k} (2M)_{11}(i,j) - \frac{\partial}{\partial \theta_i} M_{11}(j,k) \right]$$

$$S_{\theta 12}(i) = \sum_{k=1}^{s} \dot{\theta}_k \sum_{j=1}^{s} \dot{\Phi}_j \left[\frac{\partial}{\partial \theta_k} (2M)_{12}(i,j) - \frac{\partial}{\partial \theta_i} M_{12}(j,k) \right]$$

$$S_{\theta 21}(i) = \sum_{k=1}^{s} \dot{\Phi}_k \sum_{j=1}^{s} \dot{\theta}_j \left[\frac{\partial}{\partial \Phi_k} (2M)_{21}(i,j) - \frac{\partial}{\partial \theta_i} M_{21}(j,k) \right]$$

$$S_{\theta 22}(i) = \sum_{k=1}^{s} \dot{\Phi}_k \sum_{j=1}^{s} \dot{\Phi}_j \left[\frac{\partial}{\partial \Phi_k} (2M)_{22}(i,j) - \frac{\partial}{\partial \theta_i} M_{22}(j,k) \right]$$

Evaluating the double sums and partial derivatives of $S_{\theta 11}(i)$, $S_{\theta 12}(i)$, $S_{\theta 21}(i)$ and $S_{\theta 22}(i)$ required subjecting each S_θ to four possible choices as shown in the following section :

$$S_{\theta 11}(i) = \sum_{k=1}^{s} \dot{\theta}_k \sum_{j=1}^{s} \dot{\theta}_j \left[\frac{\partial}{\partial \theta_k} (2M)_{11}(i,j) - \frac{\partial}{\partial \theta_i} M_{11}(j,k) \right]$$

$$\Rightarrow \frac{\partial}{\partial \theta_k} (2M)_{11}(i,j) :$$

$$\begin{cases} 0 & i \neq k \quad j \neq k \\ 2\, \dot{\theta}_i \sum_{j=1}^{s} \dot{\theta}_j \frac{\partial}{\partial \theta_i} (M)_{11}(i,j) & i = k \quad j \neq k \\ 2 \sum_{k=1}^{s} \dot{\theta}^2_k \frac{\partial}{\partial \theta_k} (M)_{11}(i,k) & i \neq k \quad j = k \\ 2\, \dot{\theta}^2_k \frac{\partial}{\partial \theta_k} (M)_{11}(k,k) \rightarrow \text{Zero} & i = k \quad j = k \end{cases}$$

Utilizing the fact that: $M(i,j)_{11} = M_{11}^T(i,j)$

$$\Rightarrow - \frac{\partial}{\partial \theta_i} (M)_{11}(j,k) :$$

$$\begin{cases} 0 & j \neq i \quad k \neq i \\ -\, \dot{\theta}_i \sum_{k=1}^{s} \dot{\theta}_k \frac{\partial}{\partial \theta_i} (M)_{11}(i,k) & j = i \quad k \neq i \\ -\, \dot{\theta}_i \sum_{j=1}^{s} \dot{\theta}_j \frac{\partial}{\partial \theta_i} (M)_{11}(j,i) & j \neq i \quad k = i \\ -\, \dot{\theta}^2_i \frac{\partial}{\partial \theta_i} (M)_{11}(i,i) \rightarrow \text{Zero} & j = i \quad k = i \end{cases}$$

$$S_{\theta 12}(i) = \sum_{k=1}^{s} \dot{\theta}_k \sum_{j=1}^{s} \dot{\Phi}_j \left[\frac{\partial}{\partial \theta_k}(2\,M)_{12}(i,j) - \frac{\partial}{\partial \theta_i} M_{12}(j,k) \right]$$

$$\Rightarrow \frac{\partial}{\partial \theta_k}(2\,M)_{12}(i,j) \ :$$

$$\begin{cases} 0 & i \neq k \quad j \neq k \\ \dot{\theta}_i \sum_{j=1}^{s} \dot{\Phi}_j \frac{\partial}{\partial \theta_i}(2\,M)_{12}(i,j) & i = k \quad j \neq k \\ \sum_{k=1}^{s} \dot{\Phi}_k \dot{\theta}_k \frac{\partial}{\partial \theta_k}(2\,M)_{12}(i,k) & i \neq k \quad j = k \\ \dot{\Phi}_k \dot{\theta}_k \frac{\partial}{\partial \theta_k}(2\,M)_{12}(k,k) \rightarrow \text{Zero} & i = k \quad j = k \end{cases}$$

$$\Rightarrow - \frac{\partial}{\partial \theta_i}(M)_{12}(j,k) \ : \ \{\text{Utilizing the fact that: } M(i,j)_{12}=M_{21}(j,i)\}$$

$$\begin{cases} 0 & j \neq i \quad k \neq i \\ -\dot{\Phi}_i \sum_{k=1}^{s} \dot{\theta}_k \frac{\partial}{\partial \theta_i}(M)_{12}(i,k) & j = i \quad k \neq i \\ -\dot{\theta}_i \sum_{j=1}^{s} \dot{\Phi}_j \frac{\partial}{\partial \theta_i}(M)_{12}(j,i) \equiv -\dot{\theta}_i \sum_{j=1}^{s} \dot{\Phi}_j \frac{\partial}{\partial \theta_i}(M)_{21}(i,j) & j \neq i \quad k = i \\ -\dot{\Phi}_i \dot{\theta}_i \frac{\partial}{\partial \theta_i}(M)_{12}(i,i) \rightarrow \text{Zero} & j = i \quad k = i \end{cases}$$

$$S_{\theta 21}(i) = \sum_{k=1}^{s} \dot{\Phi}_k \sum_{j=1}^{s} \dot{\theta}_j \left[\frac{\partial}{\partial \Phi_k} (2M)_{21}(i,j) - \frac{\partial}{\partial \theta_i} M_{21}(j,k) \right]$$

$$\Rightarrow \frac{\partial}{\partial \Phi_k} (2M)_{21}(i,j) :$$

$$\begin{cases} 0 & i \neq k \quad j \neq k \\ \dot{\Phi}_i \sum_{j=1}^{s} \dot{\theta}_j \frac{\partial}{\partial \Phi_i} (2M)_{21}(i,j) & i = k \quad j \neq k \\ \sum_{k=1}^{s} \dot{\Phi}_k \dot{\theta}_k \frac{\partial}{\partial \Phi_k} (2M)_{21}(i,k) & i \neq k \quad j = k \\ \dot{\Phi}_k \dot{\theta}_k \underbrace{\frac{\partial}{\partial \Phi_k} (2M)_{21}(k,k)}_{\text{Zero}} & i = k \quad j = k \end{cases}$$

$$\Rightarrow - \frac{\partial}{\partial \theta_i} (M)_{12}(j,k) \quad : \{\text{Utilizing the fact that: } M(i,j)_{12} = M_{21}(j,i)\}$$

$$\begin{cases} 0 & j \neq i \quad k \neq i \\ - \dot{\theta}_i \sum_{k=1}^{s} \dot{\Phi}_k \frac{\partial}{\partial \theta_i} (M)_{21}(i,k) & j = i \quad k \neq i \\ - \dot{\Phi}_i \sum_{j=1}^{s} \dot{\theta}_j \frac{\partial}{\partial \theta_i} (M)_{21}(j,i) \equiv - \dot{\Phi}_i \sum_{j=1}^{s} \dot{\theta}_j \frac{\partial}{\partial \theta_i} (M)_{12}(i,j) & j \neq i \quad k = i \\ - \dot{\Phi}_i \dot{\theta}_i \underbrace{\frac{\partial}{\partial \theta_i} (M)_{21}(i,i)}_{\text{Zero}} & j = i \quad k = i \end{cases}$$

$$S_{\theta 22}(i) = \sum_{k=1}^{s} \dot{\Phi}_k \sum_{j=1}^{s} \dot{\Phi}_j \left[\frac{\partial}{\partial \Phi_k} (2M)_{22}(i,j) - \frac{\partial}{\partial \theta_i} M_{22}(j,k) \right]$$

$$\Rightarrow \frac{\partial}{\partial \Phi_k} (2M)_{22}(i,j) :$$

$$\begin{cases} 0 & i \neq k \quad j \neq k \\ 2\, \dot{\Phi}_i \sum_{j=1}^{s} \dot{\Phi}_j \frac{\partial}{\partial \Phi_i} (M)_{22}(i,j) & i = k \quad j \neq k \\ 2 \sum_{k=1}^{s} \dot{\Phi}^2_k \frac{\partial}{\partial \Phi_k} (M)_{22}(i,k) & i \neq k \quad j = k \\ 2\, \dot{\Phi}^2_k \frac{\partial}{\partial \Phi_k} (M)_{22}(k,k) \rightarrow \text{Zero} & i = k \quad j = k \end{cases}$$

$$\Rightarrow - \frac{\partial}{\partial \theta_i} (M)_{22}(j,k) \quad : \{ \text{Utilizing the fact that: } M(i,j)_{22} = M_{22}^T(i,j) \}$$

$$\begin{cases} 0 & j \neq i \quad k \neq i \\ - \dot{\Phi}_i \sum_{k=1}^{s} \dot{\Phi}_k \frac{\partial}{\partial \theta_i} (M)_{22}(i,k) & j = i \quad k \neq i \\ - \dot{\Phi}_i \sum_{j=1}^{s} \dot{\Phi}_j \frac{\partial}{\partial \theta_i} (M)_{22}(j,i) & j \neq i \quad k = i \\ - \dot{\Phi}^2_i \frac{\partial}{\partial \theta_i} (M)_{22}(i,i) & j = i \quad k = i \end{cases}$$

$$S_{\theta 11}(i) = 2\sum_{\substack{k=1 \\ k\neq i}}^{s} \frac{\partial M11(i,k)}{\theta_k} \dot{\theta}^2_k \qquad (4.17)$$

Utilizing the fact that;

$$\sum_k A_{ik}X_k = (A\vec{X})_i \qquad (4.18)$$

Equation (4.17) can be written as a matrix vector multiplication:

$$\vec{S}_{\theta 11} = 2 \left(\frac{\partial M11}{\theta_k} \vec{\dot{\theta}^2} \right)_i \qquad (4.19)$$

Where,

$$\left(\vec{\dot{\theta}^2} \right)_i = (\dot{\theta}^2_1, \dot{\theta}^2_2, \cdots, \dot{\theta}^2_s)^T = (\dot{\theta}_i)^2 \qquad , i = 1, \cdots, s$$

$$\left(\frac{\partial M11}{\theta_k} \right)_{ik} = \frac{\partial M11_{ik}(\theta_i, \theta_k, \Phi_i, \Phi_k)}{\theta_k} \qquad , i \,\&\, k = 1, \cdots, s$$

$$S_{\theta 12}(i) = 2\sum_{\substack{k=1 \\ k\neq i}}^{s} \frac{\partial M12(i,k)}{\theta_k} \dot{\theta}_k \dot{\Phi}_k + \dot{\theta}_i \sum_{\substack{j=1 \\ j\neq i}}^{s} \dot{\Phi}_j \, 2 \frac{\partial M12(i,j)}{\theta_i}$$

$$- \dot{\Phi}_i \sum_{\substack{k=1 \\ k\neq i}}^{s} \dot{\theta}_k \frac{\partial M12(i,k)}{\theta_i} - \dot{\theta}_i \sum_{\substack{j=1 \\ j\neq i}}^{s} \dot{\Phi}_j \frac{\partial M21(i,j)}{\theta_i} \qquad (4.20)$$

Defining:

$$\left[\overrightarrow{\dot{\theta}\,\dot{\Phi}}\right]_i = (\dot{\theta}_1\dot{\Phi}_1, \dot{\theta}_2\dot{\Phi}_2, \cdots, \dot{\theta}_s\dot{\Phi}_s)^T \qquad , i = 1, \cdots, s \tag{4.21}$$

$$\left[\overrightarrow{\dot{\theta}}\right]_i = (\dot{\theta}_1, \dot{\theta}_2, \cdots, \dot{\theta}_s)^T \qquad , i = 1, \cdots, s \tag{4.22}$$

$$\left[\overrightarrow{\dot{\Phi}}\right]_i = (\dot{\Phi}_1, \dot{\Phi}_2, \cdots, \dot{\Phi}_s)^T \qquad , i = 1, \cdots, s \tag{4.23}$$

$$\left[\overrightarrow{Q1}\right]_i = \left[2\,\frac{\partial M12}{\theta_i}\right]_{ij} \left[\overrightarrow{\dot{\Phi}}\right]_i \qquad , i\&j = 1, \cdots, s \tag{4.24}$$

$$\left[\overrightarrow{Q2}\right]_i = \left[\frac{\partial M12}{\theta_i}\right]_{ik} \left[\overrightarrow{\dot{\theta}}\right]_i \qquad , i\&k = 1, \cdots, s \tag{4.25}$$

$$\left[\overrightarrow{Q3}\right]_i = \left[\frac{\partial M21}{\theta_i}\right]_{ij} \left[\overrightarrow{\dot{\Phi}}\right]_i \qquad , i\&j = 1, \cdots, s \tag{4.26}$$

$$U1_i = \dot{\theta}_i\ Q1_i \tag{4.27}$$

$$U2_i = \dot{\Phi}_i\ Q2_i \tag{4.28}$$

$$U3_i = \dot{\theta}_i\ Q3_i \tag{4.29}$$

Using the definitions of the Equations (4.21) through (4.29), Equation (4.20) is written in a matrix-vector form as:

$$\vec{S}_{\theta 12} = \left(2\,\frac{\partial M12}{\theta_k}\,\vec{\dot{\theta}\dot{\Phi}}\right)_i + \vec{U1} - \vec{U2} - \vec{U3} \tag{4.30}$$

$$S_{\theta 21}(i) = 2\sum_{\substack{k=1\\k\neq i}}^{s} \frac{\partial M21(i,k)}{\Phi_k}\,\dot{\theta}_k\,\dot{\Phi}_k + \dot{\Phi}_i \sum_{\substack{j=1\\j\neq i}}^{s} \dot{\theta}_j\,2\,\frac{\partial M21(i,j)}{\Phi_i}$$

$$- \dot{\theta}_i \sum_{\substack{k=1\\k\neq i}}^{s} \dot{\Phi}_k\,\frac{\partial M21(i,k)}{\theta_i} - \dot{\Phi}_i \sum_{\substack{j=1\\j\neq i}}^{s} \dot{\theta}_j\,\frac{\partial M12(i,j)}{\theta_i} \tag{4.31}$$

Defining:

$$\left(\vec{Q4}\right)_i = \left(2\,\frac{\partial M21}{\Phi_i}\right)_{ij} \left(\vec{\dot{\theta}}\right)_i \quad , i\&j = 1, \cdots, s \tag{4.32}$$

$$\left(\vec{Q5}\right)_i = \left(\frac{\partial M21}{\theta_i}\right)_{ik} \left(\vec{\dot{\Phi}}\right)_i \quad , i\&k = 1, \cdots, s \tag{4.33}$$

$$\left(\vec{Q6}\right)_i = \left(\frac{\partial M12}{\theta_i}\right)_{ij} \left(\vec{\dot{\theta}}\right)_i \quad , i\&j = 1, \cdots, s \tag{4.34}$$

$$U4_i = \dot{\Phi}_i\ Q4_i \tag{4.35}$$

$$U5_i = \dot{\theta}_i\ Q5_i \tag{4.36}$$

$$U6_i = \dot{\Phi}_i\ Q6_i \tag{4.37}$$

Using the definitions of the Equations (4.32) through (4.37), Equation (4.31) is written in a matrix-vector form as:

$$\vec{S}_{\theta 21} = \left[2 \frac{\partial M21}{\Phi_k} \dot{\theta} \dot{\vec{\Phi}} \right]_i + \vec{U4} - \vec{U5} - \vec{U6} \tag{4.38}$$

$$S_{\theta 22}(i) = 2 \sum_{\substack{k=1 \\ k \neq i}}^{s} \frac{\partial M22(i,k)}{\Phi_k} \dot{\Phi}^2_k + \dot{\Phi}_i \sum_{\substack{j=1 \\ j \neq i}}^{s} \dot{\Phi}_j \, 2 \frac{\partial M22(i,j)}{\Phi_i}$$

$$- \dot{\Phi}_i \sum_{\substack{j=1 \\ j \neq i}}^{s} \dot{\Phi}_j \, 2 \frac{\partial M22(i,j)}{\theta_i} - \dot{\Phi}^2_i \frac{\partial}{\theta_i} \left(Sin^2(\theta i) \right) \tag{4.39}$$

Defining:

$$\left[\vec{Q7} \right]_i = \left[2 \frac{\partial M22}{\Phi_i} \right]_{ij} \left[\dot{\vec{\Phi}} \right]_i \quad , i\&j = 1, \cdots, s \tag{4.40}$$

$$\left[\vec{Q8} \right]_i = \left[2 \frac{\partial M22}{\theta_i} \right]_{ij} \left[\dot{\vec{\Phi}} \right]_i \quad , i\&j = 1, \cdots, s \tag{4.41}$$

$$U7_i = \dot{\Phi}_i \, Q7_i \tag{4.42}$$

$$U8_i = \dot{\Phi}_i \, Q8_i \tag{4.43}$$

Using the definitions of the Equations (4.40) through (4.43), Equation (4.39) is written in a matrix-vector form as:

$$\overrightarrow{S_{\theta 22}} = \left[2\,\frac{\partial M22}{\Phi_k}\,\overrightarrow{\dot{\Phi}^2}\right]_i + \overrightarrow{U7} - \overrightarrow{U8} - \dot{\Phi}^2_i\,\frac{\partial}{\theta_i}\,(\,Sin^2(\theta i\,)) \qquad (4.44)$$

Summing up Equations (4.19), (4.30), (4.38), and (4.44), then evaluating the partial derivatives (Appendix A) and combining the similar terms, then multiplying it with a half will result in Equation (4.13), the value of S_{θ_i}.

Defining the following Matrices, which resulted from doing partial derivatives and combining the similar terms in appendix A.

$R1(i,j)_{sxs} = Sin(\theta_i)\,Cos\,(\theta_j) - Sin(\theta_j)\,Cos\,(\theta_i)\,Cos(\Phi_i - \Phi_j)$

$R2(i,j)_{sxs} = Sin(\theta_i)\,Sin(\theta_j)\;\;Sin(\Phi_i - \Phi_j)$

$R3(i,j)_{sxs} = Cos(\theta_j)\,Sin(\theta_i + \Phi_i - \Phi_j)$

$R4(i,j)_{sxs} = Cos(\theta_i + \theta_j)\;\;Sin(\Phi_i - \Phi_j)$

$R5(i,j)_{sxs} = Sin(\theta_i)\;\;Cos(\theta_i + \Phi_i - \Phi_j)$

$R6(i,j)_{sxs} = Sin(\theta_j)\,Cos(\theta_i - \Phi_i + \Phi_j)$

Using the above definitions Equation (4.13) can be written as:

$$S_{\theta_i} = \frac{1}{2}\,\{\;S_{\theta 11_i} + S_{\theta 12_i} + S_{\theta 21_i} + S_{\theta 22_i}\;\}$$

$$= \mathbf{R1}\,\overrightarrow{\dot{\theta}^2} + \mathbf{R2}\,\overrightarrow{\dot{\Phi}^2} + \mathbf{R3}\,\overrightarrow{\dot{\theta}\dot{\Phi}} + \dot{\theta}_i\left[\mathbf{R4}\,\overrightarrow{\dot{\Phi}}\right] - \dot{\Phi}_i\left[\mathbf{R5}\,\overrightarrow{\dot{\theta}}\right]$$

$$- \dot{\Phi}_i\left[\mathbf{R6}\,\overrightarrow{\dot{\Phi}}\right] - \Phi^2_i\;\;Cos\,(\theta_i)\,Sin(\theta_i) \qquad (4.45)$$

4.3 Evaluating $S_{\Phi}(i)$

Writing Equation (4.16) explicitly by substituting the corresponding values of $(X_J)_k$, $(V_I)_j$, and $(V_J)_k$:

$$S_{\Phi 11}(i) = \sum_{k=1}^{s} \dot{\theta}_k \sum_{j=1}^{s} \dot{\theta}_j \left[\frac{\partial}{\partial \theta_k} (2M)_{11}(i,j) - \frac{\partial}{\partial \Phi_i} M_{11}(j,k) \right]$$

$$S_{\Phi 12}(i) = \sum_{k=1}^{s} \dot{\theta}_k \sum_{j=1}^{s} \dot{\Phi}_j \left[\frac{\partial}{\partial \theta_k} (2M)_{12}(i,j) - \frac{\partial}{\partial \Phi_i} M_{12}(j,k) \right]$$

$$S_{\Phi 21}(i) = \sum_{k=1}^{s} \dot{\Phi}_k \sum_{j=1}^{s} \dot{\theta}_j \left[\frac{\partial}{\partial \Phi_k} (2M)_{21}(i,j) - \frac{\partial}{\partial \Phi_i} M_{21}(j,k) \right]$$

$$S_{\Phi 22}(i) = \sum_{k=1}^{s} \dot{\Phi}_k \sum_{j=1}^{s} \dot{\Phi}_j \left[\frac{\partial}{\partial \Phi_k} (2M)_{22}(i,j) - \frac{\partial}{\partial \Phi_i} M_{22}(j,k) \right]$$

Evaluating the double sums and partial derivatives of $S_{\Phi 11}(i)$, $S_{\Phi 12}(i)$, $S_{\Phi 21}(i)$ and $S_{\Phi 22}(i)$ required subjecting each S_{Φ} to four possible choices as shown in the following section :

$$S_{\Phi 11}(i) = \sum_{k=1}^{s} \dot{\theta}_k \sum_{j=1}^{s} \dot{\theta}_j \left[\frac{\partial}{\partial \theta_k} (2M)_{11}(i,j) - \frac{\partial}{\partial \Phi_i} M_{11}(j,k) \right]$$

$$\Rightarrow \frac{\partial}{\partial \theta_k} (2M)_{11}(i,j) :$$

$$\begin{cases} 0 & i \neq k \quad j \neq k \\ 2\, \dot{\theta}_i \sum_{j=1}^{s} \dot{\theta}_j \frac{\partial}{\partial \theta_i} (M)_{11}(i,j) & i = k \quad j \neq k \\ 2 \sum_{k=1}^{s} \dot{\theta}^2_k \frac{\partial}{\partial \theta_k} (M)_{11}(i,k) & i \neq k \quad j = k \\ 2\, \dot{\theta}^2_k \frac{\partial}{\partial \theta_k} (M)_{11}(k,k) \rightarrow \text{Zero} & i = k \quad j = k \end{cases}$$

$$\Rightarrow - \frac{\partial}{\partial \Phi_i} (M)_{11}(j,k) \quad :\{ \text{Utilizing the fact that: } M(i,j)_{11} = M_{11}^{T}(i,j) \}$$

$$\begin{cases} 0 & j \neq i \quad k \neq i \\ - \dot{\theta}_i \sum_{k=1}^{s} \dot{\theta}_k \frac{\partial}{\partial \Phi_i} (M)_{11}(i,k) \quad \text{(Same)} & j = i \quad k \neq i \\ - \dot{\theta}_i \sum_{j=1}^{s} \dot{\theta}_j \frac{\partial}{\partial \Phi_i} (M)_{11}(j,i) \quad \text{(Same)} & j \neq i \quad k = i \\ - \dot{\theta}^2_i \frac{\partial}{\partial \Phi_i} (M)_{11}(i,i) \rightarrow \text{Zero} & j = i \quad k = i \end{cases}$$

$$S_{\Phi 12}(i) = \sum_{k=1}^{s} \dot{\theta}_k \sum_{j=1}^{s} \dot{\Phi}_j \left[\frac{\partial}{\partial \theta_k}(2M)_{12}(i,j) - \frac{\partial}{\partial \Phi_i} M_{12}(j,k) \right]$$

$$\Rightarrow \frac{\partial}{\partial \theta_k}(2M)_{12}(i,j) :$$

$$\begin{cases} 0 & i \neq k \quad j \neq k \\ \dot{\theta}_i \sum_{j=1}^{s} \dot{\Phi}_j \frac{\partial}{\partial \theta_i}(2M)_{12}(i,j) & i = k \quad j \neq k \\ \sum_{k=1}^{s} \dot{\Phi}_k \dot{\theta}_k \frac{\partial}{\partial \theta_k}(2M)_{12}(i,k) & i \neq k \quad j = k \\ \underbrace{\dot{\Phi}_k \dot{\theta}_k \frac{\partial}{\partial \theta_k}(2M)_{12}(k,k)}_{\text{Zero}} & i = k \quad j = k \end{cases}$$

$$\Rightarrow -\frac{\partial}{\partial \Phi_i}(M)_{12}(j,k) \quad :\{ \text{Utilizing the fact that: } M(i,j)_{12} = M_{21}(j,i) \}$$

$$\begin{cases} 0 & j \neq i \quad k \neq i \\ -\dot{\Phi}_i \sum_{k=1}^{s} \dot{\theta}_k \frac{\partial}{\partial \Phi_i}(M)_{12}(i,k) & j = i \quad k \neq i \\ -\dot{\theta}_i \sum_{j=1}^{s} \dot{\Phi}_j \frac{\partial}{\partial \Phi_i}(M)_{12}(j,i) \equiv -\dot{\theta}_i \sum_{j=1}^{s} \dot{\Phi}_j \frac{\partial}{\partial \Phi_i}(M)_{21}(i,j) & j \neq i \quad k = i \\ \underbrace{-\dot{\Phi}_i \dot{\theta}_i \frac{\partial}{\partial \Phi_i}(M)_{12}(i,i)}_{\text{Zero}} & j = i \quad k = i \end{cases}$$

$$S_{\Phi 21}(i) = \sum_{J=1}^{s} \dot{\Phi}_k \sum_{k=1}^{s} \dot{\theta}_j \left[\frac{\partial}{\partial \Phi_k} (2M)_{21}(i,j) - \frac{\partial}{\partial \Phi_i} M_{21}(j,k) \right]$$

$$\Longrightarrow \frac{\partial}{\partial \Phi_k} (2M)_{21}(i,j) \; :$$

$$\begin{cases} 0 & i \neq k \quad j \neq k \\ \dot{\Phi}_i \sum_{j=1}^{s} \dot{\theta}_j \frac{\partial}{\partial \Phi_i} (2M)_{21}(i,j) & i = k \quad j \neq k \\ \sum_{k=1}^{s} \dot{\Phi}_k \dot{\theta}_k \frac{\partial}{\partial \Phi_k} (2M)_{21}(i,k) & i \neq k \quad j = k \\ \dot{\Phi}_k \dot{\theta}_k \frac{\partial}{\partial \Phi_k} (2M)_{21}(k,k) \rightarrow \text{Zero} & i = k \quad j = k \end{cases}$$

$$\Longrightarrow - \frac{\partial}{\partial \Phi_i} (M)_{12}(j,k) \quad :\{ \text{Utilizing the fact that: } M(i,j)_{12} = M_{21}(j,i) \}$$

$$\begin{cases} 0 & j \neq i \quad k \neq i \\ - \dot{\theta}_i \sum_{k=1}^{s} \dot{\Phi}_k \frac{\partial}{\partial \Phi_i} (M)_{21}(i,k) & j = i \quad k \neq i \\ - \dot{\Phi}_i \sum_{j=1}^{s} \dot{\theta}_j \frac{\partial}{\partial \Phi_i} (M)_{21}(j,i) \equiv - \dot{\Phi}_i \sum_{j=1}^{s} \dot{\theta}_j \frac{\partial}{\partial \Phi_i} (M)_{12}(i,j) & j \neq i \quad k = i \\ - \dot{\Phi}_i \dot{\theta}_i \frac{\partial}{\partial \Phi_i} (M)_{21}(i,i) \rightarrow \text{Zero} & j = i \quad k = i \end{cases}$$

$$S_{\Phi 22}(i) = \sum_{J=1}^{s} \dot{\Phi}_k \sum_{k=1}^{s} \dot{\Phi}_j \left[\frac{\partial}{\partial \Phi_k}(2M)_{22}(i,j) - \frac{\partial}{\partial \Phi_i} M_{22}(j,k) \right]$$

$\Rightarrow$ $\frac{\partial}{\partial \Phi_k}(M)_{22}(i,j)$:

$$\begin{cases} 0 & i \neq k \quad j \neq k \\ 2\, \dot{\Phi}_i \sum_{j=1}^{s} \dot{\Phi}_j \frac{\partial}{\partial \Phi_i}(M)_{22}(i,j) & i = k \quad j \neq k \\ 2 \sum_{k=1}^{s} \dot{\Phi}^2_k \frac{\partial}{\partial \Phi_k}(M)_{22}(i,k) & i \neq k \quad j = k \\ 2\, \dot{\Phi}^2_k \frac{\partial}{\partial \Phi_k}(M)_{22}(k,k) \rightarrow \text{Zero} & i = k \quad j = k \end{cases}$$

$\Rightarrow$ $-\frac{\partial}{\partial \Phi_i}(M)_{22}(j,k)$: { Utilizing the fact that: $M(i,j)_{22}=M_{22}^{T}(i,j)$ }

$$\begin{cases} 0 & j \neq i \quad k \neq i \\ -\dot{\Phi}_i \sum_{k=1}^{s} \dot{\Phi}_k \frac{\partial}{\partial \Phi_i}(M)_{22}(i,k) & j = i \quad k \neq i \\ -\dot{\Phi}_i \sum_{j=1}^{s} \dot{\Phi}_j \frac{\partial}{\partial \Phi_i}(M)_{22}(j,i) & j \neq i \quad k = i \\ -\dot{\Phi}^2_i \frac{\partial}{\partial \Phi_i}(M)_{22}(i,i) \rightarrow \text{Zero} & j = i \quad k = i \end{cases}$$

$$S_{\Phi 11}(i) = 2\sum_{\substack{k=1\\k\neq i}}^{s} \frac{\partial M11(i,k)}{\theta_k}\dot{\theta}^2_k + 2\dot{\theta}_i \sum_{\substack{j=1\\j\neq i}}^{s} \dot{\theta}_j \frac{\partial M11(i,j)}{\theta_i}$$

$$- 2\dot{\theta}_i \sum_{\substack{j=1\\j\neq i}}^{s} \dot{\theta}_j \frac{\partial M11(i,j)}{\Phi_i} \quad (4.46)$$

Defining:

$$\left[\overrightarrow{Q9}\right]_i = \left[2\frac{\partial M11}{\theta_i}\right]_{ij}\left[\overrightarrow{\dot{\theta}}\right]_i \quad , i\&j = 1, \cdots, s \quad (4.47)$$

$$\left[\overrightarrow{Q10}\right]_i = \left[2\frac{\partial M11}{\Phi_i}\right]_{ij}\left[\overrightarrow{\dot{\theta}}\right]_i \quad , i\&j = 1, \cdots, s \quad (4.48)$$

$$U9_i = \dot{\theta}_i \; Q9_i \quad (4.49)$$

$$U10_i = \dot{\theta}_i \; Q10_i \quad (4.50)$$

Using the definitions of the Equations (4.47) through (4.50), Equation (4.46) is written in a matrix-vector form as:

$$\overrightarrow{S_{\Phi 11}} = 2\left[\frac{\partial M11}{\theta_k}\overrightarrow{\dot{\theta}^2}\right]_i + \overrightarrow{U9} - \overrightarrow{U10} \quad (4.51)$$

$$S_{\Phi 12}(i) = 2\sum_{\substack{k=1\\k\neq i}}^{s} \frac{\partial M12(i,k)}{\theta_k}\dot{\theta}_k \dot{\Phi}_k + \dot{\theta}_i \sum_{\substack{j=1\\j\neq i}}^{s} \dot{\Phi}_j \, 2\frac{\partial M12(i,j)}{\theta_i}$$

$$- \dot{\Phi}_i \sum_{\substack{k=1\\k\neq i}}^{s} \dot{\theta}_k \frac{\partial M12(i,k)}{\Phi_i} - \dot{\theta}_i \sum_{\substack{j=1\\j\neq i}}^{s} \dot{\Phi}_j \frac{\partial M21(i,j)}{\Phi_i} \quad (4.52)$$

Defining:

$$\left[\vec{Q11}\right]_i = \left[\frac{\partial M12}{\Phi_i}\right]_{ik}\left[\vec{\dot{\theta}}\right]_i \qquad , i\&k = 1 , \cdots , s \tag{4.53}$$

$$\left[\vec{Q12}\right]_i = \left[\frac{\partial M21}{\Phi_i}\right]_{ij}\left[\vec{\dot{\Phi}}\right]_i \qquad , i\&j = 1 , \cdots , s \tag{4.54}$$

$$U11_i = \dot{\Phi}_i \, Q11_i \tag{4.55}$$

$$U12_i = \dot{\theta}_i \, Q12_i \tag{4.56}$$

Using the definitions of the Equations (4.23), (4.27), and (4.53) through (4.56), Equation (4.52) is written in a matrix-vector form as:

$$\vec{S_{\Phi 12}} = \left(2 \frac{\partial M12}{\theta_k} \vec{\dot{\theta}\dot{\Phi}}\right)_i + \vec{U1} - \vec{U11} - \vec{U12} \tag{4.57}$$

$$S_{\Phi 21}(i) = 2 \sum_{\substack{k=1 \\ k\neq i}}^{s} \frac{\partial M21(i,k)}{\Phi_k} \dot{\theta}_k \dot{\Phi}_k + \dot{\Phi}_i \sum_{\substack{j=1 \\ j\neq i}}^{s} \dot{\theta}_j \, 2 \frac{\partial M21(i,j)}{\Phi_i}$$

$$- \dot{\theta}_i \sum_{\substack{k=1 \\ k\neq i}}^{s} \dot{\Phi}_k \frac{\partial M21(i,k)}{\Phi_i} - \dot{\Phi}_i \sum_{\substack{j=1 \\ j\neq i}}^{s} \dot{\theta}_j \frac{\partial M12(i,j)}{\Phi_i} \tag{4.58}$$

Defining:

$$\left[\vec{Q13}\right]_i = \left[\frac{\partial M21}{\Phi_i}\right]_{ik}\left[\vec{\dot{\Phi}}\right]_i \qquad , i\&k = 1, \cdots, s \tag{4.59}$$

$$\left[\vec{Q14}\right]_i = \left[\frac{\partial M12}{\Phi_i}\right]_{ij}\left[\vec{\dot{\theta}}\right]_i \qquad , i\&j = 1, \cdots, s \tag{4.60}$$

$$U13_i = \dot{\theta}_i \; Q13_i \tag{4.61}$$

$$U14_i = \dot{\Phi}_i \; Q14_i \tag{4.62}$$

Using the definitions of the Equations (4.32), (4.35), and (4.59) through (4.62), Equation (4.58) is written in a matrix-vector form as:

$$\vec{S_{\Phi 21}} = \left[2\,\frac{\partial M21}{\Phi_k}\,\vec{\dot{\theta}\dot{\Phi}}\right]_i + \vec{U4} - \vec{U13} - \vec{U14} \tag{4.63}$$

$$S_{\Phi 22}(i) = 2\sum_{\substack{k=1 \\ k\neq i}}^{s} \frac{\partial M22(i,k)}{\Phi_k}\,\dot{\Phi}^2_k \tag{4.64}$$

$$\vec{S_{\Phi 22}} = \left[2\,\frac{\partial M22}{\Phi_k}\,\vec{\dot{\Phi}^2}\right]_i \tag{4.65}$$

Summing up Equations (4.51), (4.57), (4.63), and (4.65) after evaluating the partial derivatives (Appendix A) and combining the similar terms. Then multiplying it with a half will result in Equation (4.15), the value of S_{Φ_i}.

Defining the following Matrices which resulted from doing partial derivatives and combining the similar terms in appendix A:

$R7(i,j)_{sxs} = Sin(\theta_i)\ Sin(\theta_j)\ Sin(\Phi_i - \Phi_j)$

$R8(i,j)_{sxs} = Cos(\Phi_i - \Phi_j)\ Sin(\theta_i + \theta_j)$

$R9(i,j)_{sxs} = Sin(\theta_j)\ Cos(\theta_i) + Cos(\theta_j)\ Sin\ (\Phi_i - \Phi_j - \theta_i)$

$$S_{\Phi_i} = \frac{1}{2}\{ S_{\Phi 11_i} + S_{\Phi 12_i} + S_{\Phi 21_i} + S_{\Phi 22_i} \}$$

$$= \mathbf{R1}\ \overrightarrow{\dot{\theta}^2} + \mathbf{R2}\ \overrightarrow{\dot{\Phi}^2} + \mathbf{R3}\ \overrightarrow{\dot{\theta}\dot{\Phi}} - \dot{\theta}_i \left[\mathbf{R7}\ \overrightarrow{\dot{\Phi}}\right] - \dot{\Phi}_i \left[\mathbf{R8}\ \overrightarrow{\dot{\theta}}\right] + \dot{\theta}_i \left[\mathbf{R9}\ \overrightarrow{\dot{\theta}}\right] \tag{4.66}$$

CHAPTER 5

CONCLUSION

The equations of motion (EOM) for a very general Branched 3D Coupled Pendulum System were derived in this thesis using the Lagrange function.

The form of EOM, Equation (4.12), takes a matrix vector form that generalized Newton's second law :

$$\boldsymbol{M(X)\,A = F}$$

$$\left(\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right) \left(\begin{array}{c} \ddot{\theta} \\ \hline \ddot{\Phi} \end{array}\right) = - \left(\begin{array}{c} \nabla_{\theta} P \\ \hline \nabla_{\Phi} P \end{array}\right) - \left(\begin{array}{c} S_{\theta} \\ \hline S_{\Phi} \end{array}\right)$$

The generalized mass matrix $M(X)$, Equation (3.14), is symmetric and positive definite.

The derived form of EOM is required by the Boundary Method to solve the equations of motion to the suggested model numerically, the solutions from the Boundary Method will help to answer the question of finding the optimal movement patterns that will prevent injury and improve performance in human subjects.

The next step will be to simulate a specific motor task using the Boundary Method, by forming and solving EOM for the desired system using the formula derived in this thesis.

APPENDIX A

PARTIAL DERIVATIVES

In this appendix the partial derivatives of the sub-matrices of the generalized mass matrix M(X) required in Chapter (4), will be evaluated. Substituting the results of these partial derivatives will result in an explicit value for Equations (4.45) and (4.66).

The sub-matrices of the generalized mass matrix M(X) are:

$$M_{11_{ij}} = c_{ij} \; [\text{Sin}\theta i \; \text{Sin}\theta j + \text{Cos}\theta i \; \text{Cos}\theta j * \text{Cos}(\Phi i - \Phi j)]$$

$$M_{12_{ij}} = c_{ij} \; [\text{Cos}\theta i \; \text{Sin}\theta j * \text{Sin}(\Phi i - \Phi j)]$$

$$M_{21_{ij}} = c_{ij} \; [-\text{Sin}\theta i \; \text{Cos}\theta j * \text{Sin}(\Phi i - \Phi j)]$$

$$M_{22_{ij}} = c_{ij} \; [\text{Sin}\theta i \; \text{Sin}\theta j * \text{Cos}(\Phi i - \Phi j)]$$

With:
i,j = 1, ⋯ , s (number of segments)

The required partial derivatives are:

$$\frac{\partial M11(i,k)}{\theta_k} = \text{Sin}(\theta_i)\,\text{Cos}\,(\theta_k) - \text{Sin}(\theta_k)\,\text{Cos}\,(\theta_i)\,\text{Cos}(\Phi_i - \Phi_k)$$

$$\frac{\partial M11(i,j)}{\theta_i} = \text{Cos}(\theta_i)\,\text{Sin}(\theta_j) - \text{Sin}(\theta_i)\,\text{Cos}\,(\theta_j)\,\text{Cos}(\Phi_i - \Phi_j)$$

$$\frac{\partial M11(i,j)}{\Phi_i} = -\,\text{Cos}(\theta_i)\,\text{Cos}\,(\theta_j)\,\text{Sin}(\Phi_i - \Phi_j)$$

$$\frac{\partial M12(i,k)}{\theta_k} = \text{Cos}(\theta_i)\,\text{Cos}(\theta_k)\,\text{Sin}(\Phi_i - \Phi_k)$$

$$\frac{\partial M12(i,j)}{\theta_i} = -\ \mathrm{Sin}(\theta_i)\ \mathrm{Sin}(\theta_j)\ \ \mathrm{Sin}(\Phi_i - \Phi_j)$$

$$\frac{\partial M12(i,k)}{\Phi_i} = \ \mathrm{Cos}(\theta_i)\ \mathrm{Sin}(\theta_k)\ \ \mathrm{Cos}(\Phi_i - \Phi_k)$$

$$\frac{\partial M21(i,j)}{\theta_i} = -\ \mathrm{Cos}(\theta_i)\ \mathrm{Cos}(\theta_j)\ \ \mathrm{Sin}(\Phi_i - \Phi_j)$$

$$\frac{\partial M21(i,k)}{\Phi_k} = \mathrm{Sin}(\theta_i)\ \mathrm{Cos}(\theta_k)\ \ \ \mathrm{Cos}\ (\Phi_i - \Phi_k)$$

$$\frac{\partial M21(i,j)}{\Phi_i} = -\mathrm{Sin}\theta i\ \mathrm{Cos}\theta j * \mathrm{Cos}(\Phi i - \Phi j)$$

$$\frac{\partial M22(i,k)}{\Phi_k} = \mathrm{Sin}\theta_i\ \ \mathrm{Sin}\theta_k\ \ \mathrm{Sin}\ (\Phi_i - \Phi_k)$$

$$\frac{\partial M22(i,j)}{\Phi_i} = -\ \mathrm{Sin}\theta i\ \ \mathrm{Sin}\theta j\ \ \mathrm{Sin}(\Phi i - \Phi j)$$

$$\frac{\partial M22(i,j)}{\theta_i} = \ \mathrm{Cos}\theta i\ \ \mathrm{Sin}\theta j\ \ \mathrm{Cos}(\Phi i - \Phi j)$$

APPENDIX B

VERIFICATION

In this appendix a derivation of the Equations of motion (EOM) for single pendulum model in 3D, Figure (B.1), is presented using Lagrangain function. The results of this derivation will be compared to the result obtained using the formula presented in this thesis as a verification that the formula can predict EOM correctly.

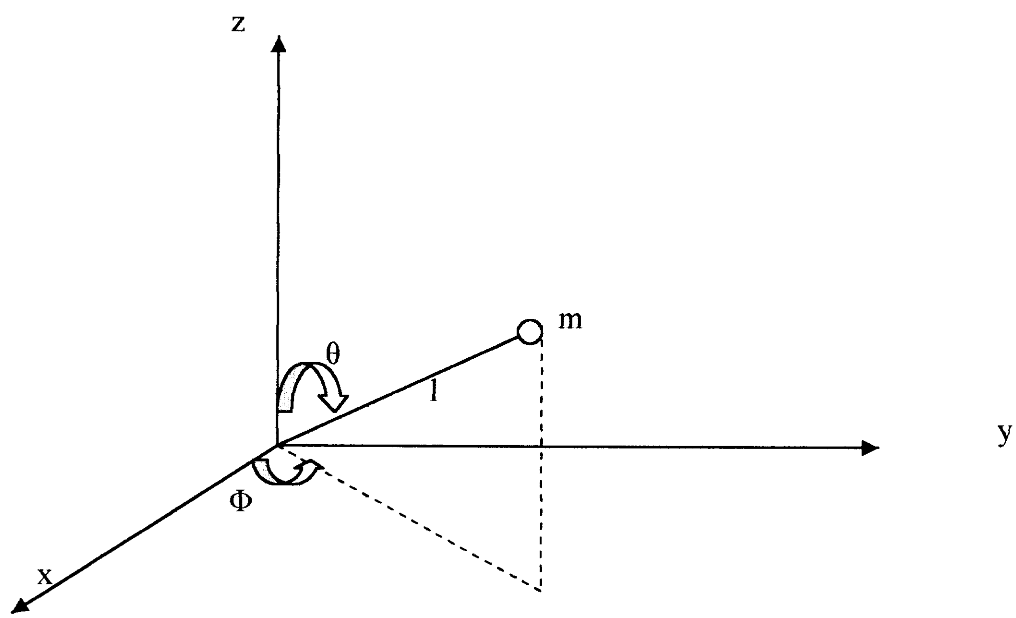

Figure B.1 3-Dimensional, Single-segment pendulum system.

Transforming from the Cartesian coordinate system to the angular coordinate system and via versa is given as:

$$x = l \, Sin(\theta) \, Cos(\Phi) \tag{B.1}$$

$$y = l \, Sin(\theta) \, Sin(\Phi) \tag{B.2}$$

$$z = l \, Cos(\theta) \tag{B.3}$$

The system Potential Energy is given as:

$$PE = m\,g\,l\,Cos(\theta) \tag{B.4}$$

The system Kinetic Energy is given as:

$$KE = \frac{1}{2}\,m\,\{\ \dot{x}^2 + \dot{y}^2 + \dot{z}^2\ \} \tag{B.5}$$

Deriving Equations (B.1), (B.2), and (B.3) with respect to time and then square them results in:

$$\dot{x}^2 = (\,-\,l\,Sin(\theta)\,Sin(\Phi)\,\dot{\Phi} + l\,Cos(\Phi)\,Cos(\theta)\,\dot{\theta}\,)^2$$

$$= l^2\,Sin^2(\theta)\,Sin^2(\Phi)\,\dot{\Phi}^2 + l^2\,Cos^2(\Phi)\,Cos^2(\theta)\,\dot{\theta}^2$$

$$-\,2\,l^2\,Sin(\theta)\,Sin(\Phi)\,Cos(\theta)\,Cos(\Phi)\,\dot{\theta}\,\dot{\Phi} \tag{B.6}$$

$$\dot{y}^2 = (\ l\,Sin(\theta)\,Cos(\Phi)\,\dot{\Phi} + l\,Sin(\Phi)\,Cos(\theta)\,\dot{\theta}\,)^2$$

$$= l^2\,Sin^2(\theta)\,Cos^2(\Phi)\,\dot{\Phi}^2 + l^2\,Sin^2(\Phi)\,Cos^2(\theta)\,\dot{\theta}^2$$

$$+\,2\,l^2\,Sin(\theta)\,Sin(\Phi)\,Cos(\theta)\,Cos(\Phi)\,\dot{\theta}\,\dot{\Phi} \tag{B.7}$$

$$\dot{z}^2 = (\,-\,l\,Sin(\theta)\,\dot{\theta}\,)^2$$

$$= l^2\,Sin^2(\theta)\,\dot{\theta}^2 \tag{B.8}$$

Adding up Equations (B.6), (B.7), and (B.8) gives:

⇨

$$(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = l^2 (\dot{\Phi}^2 \, Sin^2(\theta) + \dot{\theta}^2) \qquad (B.9)$$

Substituting Equation (B.9) in to the Kinetic Energy Equation (B.5):

$$KE = 0.5 \ m \ (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = 0.5 \ m \ l^2 (\dot{\Phi}^2 \, Sin^2(\theta) + \dot{\theta}^2) \qquad (B.10)$$

Deriving equations of motion, using the Energy (Work)-basd approach of Lagrange (L), along with the Potential Energy, Equation (B.4), and the Kinetic Energies, Equation (B.5), results in.

$$L = KE - PE$$

$$L = 0.5 \ m \ l^2 (\dot{\Phi}^2 \, Sin^2(\theta) + \dot{\theta}^2) - m \, g \, l \, Cos(\theta) \qquad (B.11)$$

Equations of motion using L are written in the form :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial V_i} \right) - \frac{\partial L}{\partial X_i} = 0 \quad \text{, where } i=_{1,2} \quad \begin{cases} X_{1,2} = \theta , \Phi \\ V_{1,2} = \dot{\theta} , \dot{\Phi} \end{cases} \qquad (B.12)$$

Solving for Equation of Motion for θ degree of freedom:

$$\partial L / \partial \dot{\theta} = m \, l^2 \, \dot{\theta}$$

$$\frac{d}{dt}(m \, l^2 \, \dot{\theta}) = m \, l^2 \, \ddot{\theta}$$

$$\partial L / \partial \theta = m \ l^2 \ \dot{\Phi}^2 \, Sin(\theta) \, Cos(\theta) + m \, g \, l \ Sin(\theta)$$

$$m\, l^2\, \ddot{\theta} - m\, l^2\, \dot{\Phi}^2\, \mathrm{Sin}(\theta)\, \mathrm{Cos}(\theta) = - m\, g\, l\, \mathrm{Sin}(\theta) \qquad \text{(B.13)}$$

Solving for Equation of Motion for Φ degree of freedom:

$$\partial L / \partial \dot{\Phi} = m\, l^2\, \dot{\Phi}\, \mathrm{Sin}^2(\theta)$$

$$\frac{d}{dt}\left(m\, l^2\, \dot{\Phi}\, \mathrm{Sin}^2(\theta) \right) = m\, l^2 \left(\ddot{\Phi}\, \mathrm{Sin}^2(\theta) + 2\, \dot{\Phi}\, \dot{\theta}\, \mathrm{Sin}(\theta)\, \mathrm{Cos}(\theta) \right)$$

$$\partial L / \partial \Phi = 0$$

$$m\, l^2\, \ddot{\Phi}\, \mathrm{Sin}^2(\theta) = - 2\, m\, l^2\, \dot{\Phi}\, \dot{\theta}\, \mathrm{Sin}(\theta)\, \mathrm{Cos}(\theta)\,) \qquad \text{(B.14)}$$

Comparing Equations (B.13) and (B.14), with the results obtained using the formula derived in this thesis shows that both results were identical. Where in order to have a single 3D pendulum system, from the model suggested in the thesis, both Φi and Φj have to be equal.

When this was done in Equations (4.45) and (4.66) the results were the same as derived in this Appendix.

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