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ABSTRACT

SOME CONTRIBUTIONS TO MODELING USAGE SENSITIVE WARRANTY SERVICING STRATEGIES AND THEIR ANALYSES

by Rudrani Banerjee

Providing a warranty as a part of a product's sale is a common practice in industry. Parameters of such warranties (e.g., its duration limits, intensity of use) must be carefully specified to ensure their financial viability. A great deal of effort has been accordingly devoted in attempts to reduce the costs of warranties via appropriately designed strategies to service them. many such strategies, that aim to reduce the total expected costs of the warrantor or / and are appealing in other ways such as being more pragmatic to implement - have been suggested in the literature. Design, analysis and optimization of such servicing strategies is thus a topic of great research interest in many fields.

In this dissertation, several warranty servicing strategies in two-dimensional warranty regimes, typically defined by a rectangle in the age-usage plane, have been proposed, analyzed and numerically illustrated. Two different approaches of modeling such usage sensitive warranty strategies are considered in the spirit of Jack, Iskandar and Murthy (2009) and Iskandar (2005). An 'Accelerated Failure Time' (AFT) formulation is employed to model product degradation resulting due to excessive usage rate of consumers.

The focus of this research is on the analysis of warranty costs borne by the manufacturer (or seller or third party warranty providers) subject to various factors such as product's sale price, consumer's usage rate, types and costs of repair actions. By taking into account the impact of the rate of use of an item on its lifetime, a central focus of our research is on warranty cost models that are sensitive to the usage rate. Specifically, except the model in Chapter 4 where the rate at which an item is used is considered to be a random variable; all other warranty servicing policies that we

consider, have usage rate as a fixed parameter, and hence are policies conditional on the rate of use. Such an approach allows us to examine the impact of a consumer's usage rate on the expected warranty costs. For the purpose of designing warranties, exploring such sensitivity analysis may in fact suggest putting an upper limit on the rate of use within the warranty contract, as for example in case of new or leased vehicle warranties.

A Bayesian approach of modeling 2-D Pro-rated warranty (PRW) with preventive maintenance is considered and explored in the spirit of Huang and Fang (2008). A decision regarding the optimal PRW proportion (paid by the manufacturer to repair failed item) and optimal warranty period that maximizes the expected profit of the firm under different usage rates of the consumers is explored in this research. A Bayesian updating process used in this context combines expert opinions with market data to improve the accuracy of the parameter estimates. The expected profit model investigated here captures the impact of juggling decision variables of 2-D pro-rated warranty and investigates the sensitivity of the total expected profit to the extent of mis-specification in prior information.

SOME CONTRIBUTIONS TO MODELING USAGE SENSITIVE WARRANTY SERVICING STRATEGIES AND THEIR ANALYSES

by Rudrani Banerjee

A Dissertation Submitted to the Faculty of New Jersey Institute of Technology and Rutgers, The State University of New Jersey – Newark in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Mathematical Sciences

Department of Mathematical Sciences, NJIT Department of Mathematics and Computer Science, Rutgers-Newark

May 2011

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APPROVAL PAGE

SOME CONTRIBUTIONS TO MODELING USAGE SENSITIVE WARRANTY SERVICING STRATEGIES AND THEIR ANALYSES

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to my beloved Dadawa

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CHAPTER 1

INTRODUCTION

1.1 Motivation of the Research

The interaction between producers and buyers in a market is an interesting phenomenon in real life. Both groups try to optimize their satisfaction in terms of factors like revenue, cost, product behavior etc. It is obvious that there is always a trade-off between the interests of manufacturer and consumer, and setting an optimal ground of trade is very important to maintain equilibrium in the market. The manufacturer tries to optimize his profit which in turn implies reduction in costs (production, marketing, maintenance etc.) and maximization of revenue. But it is also important to maintain the standard of produce in the market, since it affects the goodwill of the firm. On the contrary, the consumer tries to minimize his cost and optimize his satisfaction by comparing identical products in the market on the basis of cost, demand, post sales support facilities etc. Thus this game of setting optimal grounds of trade between the two groups needs investigation from all possible subjects of interest.

One such subject concerning the manufacturer is Product Warranty Analysis. The definition of warranty goes as follows a contractual agreement that requires a manufacturer (producer) or, seller of a product to provide an assurance of satisfactory product performance during a specified length of time (called, the warranty period) after purchase, by replacement or, repair of failed items within the covered period. Clearly, it serves as a promotional tool and an important component of the product's marketing strategy to attract potential buyers. On the other hand, warranties not only serve as an instrument that address the protectional needs of the buyer / consumer (against unwelcome and unforeseen disruption of service due to product failures), but also that of the producer, by controlling costs through appropriate warranty servicing strategies.

Within the broad structure of a warranty framework; many variations on how warranties are formulated are possible and have been considered. One of the overriding concerns in all such situations is the need to adequately model realistic warranty servicing strategies and the corresponding expected costs.

In recent times, contemporary research on warranty modeling has focused on both the intensity of usage of the product as well as its lifetime characteristics as important determinants of the time to product failure. Warranty policies of this type are referred to as two-dimensional (2-D) warranties. Consumer products generally degrade due to aging of the item as well as the rate of usage. Thus consideration of usage level in a warranty becomes inevitable, although it increases the complexity of the problem.

In the literature of two-dimensional warranties, there are two different approaches to the inclusion of usage as a relevant factor in modeling warranties, one of which considers item failures and costs conditional on the rate of usage and thus treats the latter as a parameter specific to the buyer (user). The other approach considers usage rate as a random variable with a distribution that reflects our beliefs about the aggregate profile of use of the product's target customers. In our study we have considered both approaches; the first approach in Chapters 2, 3 and 5; and the second approach in Chapter 4.

1.2 Introduction to Warranty Analysis

Modern manufacturing and sales are characterized by speedily developing technology, exposure to the global market, fierce competition, well-informed and demanding consumers. These factors have posed serious challenges to the manufacturers and policy makers across the globe jockeying for competitive advantage. In purchase decisions, consumers typically compare the characteristics of different products of competing brands. When these comparable products are similar or nearly identical, it becomes very difficult to choose a particular brand solely based on product related characteristics such as model specifications and other features, such as price or, financing offered by the manufacturer. In this case, provision of post-sales support adds to the product's appeal, and is thus a useful marketing tool. Such support is generally provided by the manufacturer in the form of repair/replace warranty, maintenance servicing or, money-back guarantee. When a new product is manufactured, each generation of the product often becomes more complex than previous ones. If the consumers are not sure about the product reliability; an attractive warranty servicing scheme signals higher product quality and provides greater assurance to customers in the sense that the manufacturer will provide some remedial action (repair/replace/money-back) to compensate for the failure of the item during a preassigned time period.

Servicing post-sales support in the form of warranty or maintenance involves additional costs to the manufacturer and has significance impact on the profit of the firm. These costs, in fact, are unpredictable future costs, which typically range from 2% to as much as 15% of net sales McGuire [57]; depending on various factors such as product reliability, usage level, warranty terms and coverage. Product reliability is influenced by the decisions made during the design and manufacturing of the product. Product warranties thus play an increasingly important role in consumer and commercial transactions.

The use of warranties is extensive and they serve many purposes. These include protection for manufacturer and buyer, signalling of product quality, an important element of marketing strategy, assuring buyers against items which do not perform as promised and play an important role in the dispute resolution between buyer and manufacturer. These in turn pose serious challenges to legislators in terms of formulating sensible warranty policy legislation that will protect the societal (buyers' and manufacturers') interests. Therefore, analyzing the different aspects of warranty has received greater attention of the researchers from many different disciplines and the corresponding research literature is substantial and vast.

As in any real life problems of practical significance, the problem of warranties must deal with relevant issues arising from its social, behavioral, economic, political / legal dimensions, their mutual interactions, as well as its analytical (mathematical and statistical) aspects. Without minimizing the importance of the social / institutional factors, the scope of this thesis and our main concern here is necessarily with addressing the issues of modeling and quantitative analysis of warranties that account for the built-in risk factors such as uncertainty in product lifetime, consumers use patterns, and severity of operating conditions of the product relative to a baseline environment.

During the warranty period, the producer guarantees the product's correct operation and the repair/replacement or compensation for damage resulting from failure or poor performance by paying all or part of the resulting losses. This is the consumer's recourse. The post-sale functioning of the product (i.e., the capability of the product to perform its assigned job) is influenced by a number of uncertain factors such as the age, usage level, preventive maintenance actions etc. Thus the impact of warranty periods and policies must be modeled and analyzed in a probabilistic way. They must necessarily take into account the difference in operating conditions, their variation in reliability over time and quality characteristics, together with appropriate economic factors such as the product's cost, cost of repair, compensation for losses from idle time, marketing costs, etc. All these factors, along with the type of product (see Section 1.2.1) allow for a great deal of flexibility in designing different models in warranty analysis.

To choose an appropriate warranty period is, thus a basic concern of the producer. This choice must balance the trade-offs between increasing adverse (cost) impact on producers of servicing efforts with increasing warranty period and corresponding benefits of increased market share. An extremely short warranty period may affect the sales by repelling the consumers, while a too long one will lead to losses from compensation of customer claims. Even with a given value of warranty period, warranty servicing strategies (product maintenance plans, repair-replace options and costs) as well as uncontrolled risk factors such as consumers' usage profile of the product can significantly influence the total servicing costs, and must be adequately modeled and analyzed.

1.2.1 Some Standard Terminology in Warranty Literature

In this research, the strategies proposed and discussed typically consists of different types of rectification action at item failures. The rectification actions are either a replacement or different types of repairs. To avoid ambiguities, we list below some standard nomenclatures that are used in the relevant literature.

- Replacement / Perfect Repair action under warranty, refers to replacing the failed product by an identically similar products from manufacturer's stock of new products. Clearly this type of rectification action is the best as it restores the product to its initial stage. The corresponding rate of degradation of the product resulting from a replacement is same as a new product.
- Minimal repair refers to repairs which restore the aging condition of an equipment (which fail at time t, say) to its corresponding condition just prior to failure (i.e., at time t-). In the engineering community, such repairs are often called "bad as old", while replacements are called "good as new" repairs. Some literature on minimal repairs type rectification can be found in Singpurwalla and Balaban [92], Ascher and Feingold [2], Block et al. [13], Phelps [85], Jack and Schouten [42] and many others.
- Imperfect repair on the other hand, refers generically to repairs which restore a failed equipment to a condition intermediate between that achieved by a minimal repair and a replacement. The notion of an imperfect repair is not unique from a technical viewpoint, and has been specified in different ways by

different authors. Some of them are 'degree of repair' (Kijima [50], Yun and Kang [105], Varnosafaderani and Chukova [98]), 'randomized repair' (Brown and Proschan [15]), 'age-dependent minimal repair' (Block *et. al.* [13]) and others.

- Free Replacement Warranty (FRW) [9] refers to warranty policies in which the entire cost of servicing (which can be a replacement by a new item or sort type of repair) of warranted items at failures, are borne by the manufacturer or seller (i.e., free to consumer). This type of warranty policies are thus 'consumer friendly'.
- Pro-rated Warranty (PRW) [9] refers to warranty policies in which the cost of servicing of failed items under warranty is shared between the manufacturer and the consumer. Such warranty policies charge a fixed percentage of warranty costs to the consumers and the remaining percentage is incurred by the warranty provider. Pro-rated warranties are therefore relatively more 'manufacturer friendly'.

1.2.2 Classification of Products

The demand of products in the market vary among groups of consumers. It is very important for the firm to identify their target consumer population since it provides an insight about the quantity and frequency of product demand, and in turn helps to maintain the availability of products in the market. The target population varies according to the variety of items sold. A classification of products into groups describes the market of buyers and a clear image of product demand, as follows.

1. Consumer durables: (e.g., household appliances, cars) these are bought by individual households.

- 2. Industrial and commercial products: (e.g., equipment used in a hospital to provide medical care, aircrafts used by airline operators) these are bought in lots by businesses for the production of services or products.
- 3. *Government acquisitions:* (e.g., new fleet of tanks or jet fighters) these are often technologically advanced equipments required (in lots or singles) for purposes of security related issues.

1.2.3 Role of Warranties

Trade is a crucial part of society. Hence, issues related to trade including warranties have different roles in the market (and broadly in the society). The utility of warranties are different for the buyers and the producers and can be summarized as

- 1. From a **Buyer**'s point of view, the main role of a warranty is protectional, it provides a means of rectification if the item, when properly used, fails to perform as intended or as specified by the seller. A second role is informational. Many buyers infer or, perceive that a product with a relatively longer warranty period to be more reliable and long lasting than one with a shorter warranty period.
- 2. From a **Producer**'s view-point, warranties play a protectional as well as a promotional role. Warranty terms may, and often do, specify the use and conditions of use for which the product is intended and provide for limited coverage or no coverage at all in the event of misuse of the product. Thus they protect a manufacturer/service provider/reseller against unjustified claims of warranty coverage. The promotional aspect of warranties is exemplified by their role as an instrument of advertisement. Since buyers often infer a more reliable product when a long warranty is offered, this has been used as an effective advertising tool. This is often particularly important when marketing

new and innovative products, which may be viewed with a degree of uncertainty by many potential consumers.

1.2.4 Types of Warranty Policies

The schematic in Figure 1.1 depicts the basic classification of warranties as described by Blischke and Murthy in 'Warranty Cost Analysis' (WCA) [9], a standard reference work on the subject.



Figure 1.1 Taxonomy for warranty policies from Warranty Cost Analysis [9].

1.2.5 Review of Warranty Models

To review the literature in warranty modeling and analysis, we have referred to several textbooks ([9], [10], [11]), articles and review articles ([70], [8], [67], [68], [18], [42], [34]). Information regarding the development of this research topic, included in this chapter are presented following these articles and classical textbooks. One of the earliest formal work on warranty modeling known to us is by Barlow and Hunter |4|. Blischke [7] authored the first review paper on warranties, focusing on mathematical models for warranty cost analysis. The three-part review paper (Product Warranty Management - I, II, III; Blischke and Murthy [8], Murthy and Blischke [67],[68]) proposed a taxonomy for new product warranties and discussed various issues. Since then the literature on warranties (for both new and used products) has grown considerably with two review papers, three books and many journals and conference papers. The review papers by Chukova et al. [18], Thomas and Rao [97], discussed some warranty management issues and suggested topics for future research. Murthy and Djamaludin [70] reviewed the literature that has appeared over the period 1990 to 2002. It builds on the review papers by Murthy and Blischke ([67],[68]) and Thomas and Rao [97]. Their review looks at different aspects of warranties for new products. The main thrust is on issues that are of high relevance to manufacturer from a product life cycle perspective.

Details of different policies, which can be grouped into three categories (types A, B and C) can be found in Blischke and Murthy ([9], Chapter 2). The type A policies (single item sale, not involving product development) can be divided into onedimensional (1-D) and two-dimensional (2-D) policies. In two-dimensional policies, the warranty is characterized by a region in the two-dimensional plane representing age and usage. Blischke and Murthy [8],[9] defined four different shapes for such warranty regions. Singpurwalla and Wilson [94] suggested many other shapes based on sellers' and consumers' preferences. Under a non-renewing warranty, the terms of the warranty do not change during the warranty period. As a result, if an item fails during warranty, it is rectified by the seller and returned to the buyer without any changes to the original warranty terms. Under a renewing warranty, the warranty terms can change, for example, after failure, the item is returned with a new warranty either identical to, or different from, the original warranty terms. Each of these can be further subdivided into two groups : simple policies and combination policies. Two simple policies that have been investigated by different authors are free replacement warranty (FRW) and pro-rata warranty (PRW). Combination warranties involve different FRW or PRW terms over different periods of warranty.

The manufacturer of a product incurs additional costs resulting from servicing of claims under warranty. Warranty claims occur due to item failures. An item is said to have failed when it is unable to perform its function in a satisfactory manner. Item functioning is influenced by several factors. These include engineering decisions during product designing and manufacturing, customer's usage intensity, operating environment, maintenance effort expended by users etc., each of which have an impact on the inherent reliability of the product. Blischke and Murthy [9] have defined several costs of interest to manufacturers and buyers. They include:

- Warranty cost per unit sale.
- Warranty cost over the lifetime of an item (Life cycle cost LCC-I): This is buyer oriented and includes elements such as purchase cost, maintenance and repair costs following expiration of the warranty coverage, operating costs and disposal costs.
- Warranty costs over the product life cycle (Life cycle cost LCC-II): This is dependent on the interval over which buyers purchase the product. This life cycle begins with the launch of the product in the marketplace and ends with its withdrawal.

• Cost per unit time: This is useful for managing warranty servicing resources such as parts inventories, labor and costs over time with dynamic sales.

The costs are clearly different for buyers and manufacturers. They are random variables, since claims under warranty and the cost to service claims are uncertain. The warranty cost per unit sale is important in the context of pricing the product. The sale price must exceed the total of the manufacturing and warranty costs, otherwise the manufacturer incurs a loss. On an average, the warranty cost per item decreases as product reliability increases. The life cycle cost of a product is of relevance to both buyer and manufacturer in the context of complex and expensive products.

1.2.6 One-dimensional Warranty

The first type of warranty policies to exist in the market are One-dimensional (1-D). Such a warranty is characterized by the aging profile of the product i.e., the degradation of the item is modeled as a function of age only. Due to simplicity of modeling, 1-D warranty has gathered lot of attention from many researchers. Analysis of a warranty policy can be done from different view-points. Some of the references are given in this section.

1.2.6.1 Modeling Failures. Failures over the warranty period can be modeled either at the component level or at the product (or item) level. In component level modeling, the item is characterized in terms of its components and failure of each component modeled separately. The modeling of first failure needs to be treated different from that for subsequent failures. The latter depends on whether the component is repairable or not, the type of repair action used in the case of repairable item and, the type of item (used or new) used in the case of replacement of a failed item. The time to first failure is typically modeled by the lifetime distribution of a new product, while the type of formulation needed for modeling subsequent failures, as stated earlier depends on the nature of rectification (repair or replace) action.

When every failure results in a replacement by a new item and the replacement times are negligible, then the point process of failures is a renewal process. If all failures are minimally repaired and the repair times are negligible then the failure process is a suitable point process formulation with specified intensity function (Blischke and Murthy ([9], Chapter 2) or Murthy ([10], Chapter 3). When the rectification can involve either minimal repair or replacement by new components, then the formulation needed is complicated and is given by the G-Renewal process (see, Kijima and Sumita [51]).

In the system (product) level modeling, the item state is modeled as a binary variable (working or failed) and failures over time is a non-homogeneous Poisson process (NHPP) with mean value function $\Lambda(t)$, that equals the cumulative hazard function of a new equipment's lifetime distribution.

1.2.6.2 Cost analysis. The cost of each rectification is comprised of several cost elements (handling, material, labor, facilities, etc). Often it is modeled by a single variable which is the aggregate of the different costs. In general, the aggregate cost is a random variable and needs to be modeled by a probability distribution function. Most of the cost analyzes are based on the following simplifying assumptions:

- i) All buyers are alike in their usage.
- ii) All items are statistically similar and independent.
- iii) All failures, result in immediate claims.
- iv) All claims are valid (no fraudulent claims).
- v) The manufacturer has the logistic support (spares and facilities) needed to carry out the rectification actions without any delays.

vi) The time to rectify a failed item (either through repair or replacement) is sufficiently small relative to the mean time between failures, so that repairs can be assumed to be effectively instantaneous.

Further, the life cycle of the product LC is modeled as a deterministic variable, although this assumption can be relaxed to treat LC as a random variable with a specified life-distribution. All the model parameters (including costs and of the various distributions involved) are known. Bulk of the literature deals with expected warranty costs and few deal with higher moments or characterization through a distribution function. Blischke ([10], Chapter 8) discusses the statistical techniques for warranty cost analysis. A brief review of 1-D warranty cost analysis is included here as follows.

Free replacement warranties: Blischke and Murthy ([9], Chapter 4) and Blischke ([10], Chapter 10) considered and analyzed the expected warranty costs for both repairable and non-repairable products. Kaminsky and Krivstov [49] dealt with the case where failures are modeled by a G-Renewal process. Sahin and Polatoglu [88], Polatoglu and Sahin [86] and Sahin and Polatoglu [89] derived the probability distribution for warranty cost and some related variables.

Pro-rata warranties: Blischke and Murthy ([9], Chapter 5) and Patankar and Mitra ([10], Chapter 11) investigated the expected cost of the pro-rated warranties. Menzefricke ([59],[60]) dealt with both the mean and variance of total pro-rated warranty cost. Sahin and Polatoglu [88], Polatoglu and Sahin [86] and Sahin and Polatoglu [89] also derived the probability distribution for pro-rated warranty cost and some related variables.

Combination warranties: The expected warranty cost analysis for a variety of combination policies can be found in Blischke and Murthy ([9], Chapter 6) and Blischke ([10], Chapter 12). Bohoris and Young [14] deal with the warranty cost analysis of a hybrid warranty. Simulation approach: The warranty cost analysis can be analytically challenging and often requires solving complicated renewal functions. An alternate approach is to obtain estimates of the costs through simulation. Hill *et al.* [30] and Murthy *et al.* [71] have considered such problems.

Extended warranties: A warranty that is an integral part of product's sale is called the *base warranty*. It is offered by the manufacturer at no additional cost and is factored into the sale price. Extended warranty provides additional coverage over the base warranty and is obtained by the buyer by paying a premium. Extended warranties are optional, not tied to the sale and can be either offered by the manufacturer or a third party (for example, several credit card companies offer extended warranties for products bought using their credit cards etc.). The expected cost incurred by the provider of extended warranties can be computed using models similar to base warranty costs. The cost of extended warranty is related to product reliability and usage intensity. The reasons for purchase of extended warranties have been analyzed extensively in the marketing literature. Padmanabhan ([10], Chapter 18) discussed the alternate theories and the design of extended warranty policies. Padmanabhan and Rao [78] examined the extended warranty with heterogeneous customers with different attitudes to risk, captured through a utility function. Patankar and Mitra [81] considered the case where items are sold with pro-rata warranty where the customer is given the option of renewing the initial warranty by paying a premium, provided the product does not fail during the initial warranty period. Mitra and Patankar [61] dealt with a model where the product is sold with a rebate policy, and the buyer has the option to extend the warranty should the product not fail during the initial warranty period. Yeh and Peggo [103] looked at extended warranty policies with different options for consumers. Rinsaka and Sandoh [87] dealt with problems related to extension of the base warranty period.

Service contracts: A service contract is similar to an extended warranty. Bulk of the literature on service contracts is mainly qualitative. Murthy and Asgharizadeh [66] and Murthy [65] deal with the modeling and analysis of service contracts using a game-theoretic approach.

Most of the literature on warranty servicing through the mid- 1990s is summarized in Blischke and Murthy ([9],[10]). Another review article authored by Murthy and Djamaludin [70] captured the recent advances in different warranty studies. Models where repaired items are assumed to have independent and identically distri-

buted lifetimes different from that of a new item include those of Biedenweg [6] and Nguyen and Murthy [76],[77]. Biedenweg [6] showed that the optimal strategy is to replace with a new item at any failure occurring up to a certain time measured from the initial purchase and then repair all other failures that occur during the remainder of the warranty period. This technique of splitting the warranty period into distinct intervals for replacement and repair is also used by Nguyen and Murthy [76],[77].

In Nguyen and Murthy [76], where the warranty period is partitioned into two disjoint intervals, any item failures occurring during the second interval of the warranty period are rectified using a stock of used items. Nguyen and Murthy [77] extended Biedenweg's [6] model by adding a third interval where failed items are either replaced or repaired and a new warranty is given at each failure. The first warranty servicing model involving minimal repair and assuming constant repair and replacement costs is that of Nguyen [77]. As in Biedenweg [6], the warranty period is split into a replacement interval followed by a repair interval. Under this strategy a failed item is always replaced by a new one in the first interval, irrespective of its age at failure. Thus, if the failure occurs close to the beginning of the warranty then the item will be replaced at a higher cost than that of a repair and yet there will be very little reduction in its effective age. This is the major limitation of this model and makes the strategy clearly sub-optimal.

In a later paper, and with the same assumptions as Nguyen [77], Jack and Schouten [42] investigated the structure of the manufacturer's optimal servicing strategy over a warranty period [0, W], using a dynamic programming model. It is shown that the repair-replacement decision on failure should be made by comparing the item's current age with a time-dependent control limit function $\tilde{h}(x)$ (some typical plots of $\tilde{h}(x)$ versus x can be seen in [42]). The item is replaced on failure at time x if and only if its age is greater than $\tilde{h}(x)$.

The repair-replacement decisions under such a policy is characterized by the three intervals I - III. In intervals I and III, the optimal strategy is to always repair the failed item. The shape of $\tilde{h}(x)$ in interval II then determines the number of replacements that will occur. In this interval, if $\tilde{h}(x)$ lies above the indicated line L_1 , then at most one replacement will be carried out. In general, the shape of $\tilde{h}(x)$ depends on the relationship between the item's mean time to first failure (MTTF) and the length of the warranty period (W), and also the cost of replacement relative to repair. For example, if MTTF >> W, then $\tilde{h}(x)$ has a straight line form throughout the three intervals, and the optimal policy is 'always repair' during [0, W]. Alternatively, if MTTF << W, then the lengths of intervals I and III become very small, and the optimal policy correspondingly approaches 'always replace' during [0, W]. However, a manufacturer will not offer a long warranty when the MTTF is small, and so this latter case is unlikely to occur. A more realistic scenario is that the MTTF will be comparable to W, and then only a small number of replacements will be carried out.

The optimal strategy of Jack and Schouten [42] yields the smallest expected warranty servicing cost, but the computation of the control limit policy involve considerable computational effort. The strategy also requires continuous monitoring of the item's age by the manufacturer, which is not very practical since such monitoring may involve additional costs which in turn would compromise the optimality of Jack-Schouten policy. A pragmatic variation of the above strategy is the new strategy proposed by Jack *et al.* [40], which again involves splitting the warranty period [0, W]into three distinct intervals for carrying out repairs and replacements. A maximum of one replacement is allowed in the middle interval and there is no need to monitor the item's age. A fair amount of substantial work, including our work with extensions to two-dimensional warranties reported in this thesis, derive their genesis from the above pragmatic framework of Jack *et al.* [40].

1.2.7 Two-dimensional Warranty

The two-dimensional (2-D) warranties have received a lot less attention relative to the 1-D case due to analytical complexities. This type of warranty modeling takes into account the effects of aging and usage level on the degradation profile of the product. Such a warranty is represented by a 2-D region, where the horizontal and vertical axes respectively represents age (x) and total usage level (u). The warranty expires at the first instance when either the age or usage level exceeds their respectively pre-assigned threshold values. An example of such a warranty is the so called '5 year-50,000 mile' warranty for new automobiles which would provide warranty coverage for a new car until it is 5 years old, or has been driven 50,000 miles, whichever occurs first.

1.2.7.1 Modeling failures. Two different approaches have been used to model item failures. The first is to use a two-dimensional distribution function to describe the joint distribution of age and usage. In this case, failures are modeled by a two-dimensional point process formulation (see, Iskandar [35], Murthy *et al.* [72] and Hunter ([10], Chapter 7]). The second approach involves modeling usage as a function of time so that failures are effectively modeled by a one-dimensional point process formulation. Iskandar [35] suggested a linear model for usage of the form U(x) = Yx where Y is the random usage rate, required to model the varying usage across the consumer population. Moskowitz and Chun [62] also used a one-dimensional approach to capture the effects of usage and age on the failure process. They modeled the failures by a Poisson process with intensity function as a linear function of age and usage. Singpurwalla and Wilson [93] followed a different approach – conditional on the total usage, the time to failure is modeled by a univariate distribution function.
The total usage as a function of age is modeled by another univariate distribution. Combining these two, they derived a two-dimensional distribution for failure involving both age and usage. Singpurwalla [91] dealt with modeling the survival under multiple time scales in dynamic environments with the usage rate changing dynamically. Gertsbakh and Kordonsky [28] and Ahn *et al.* [1] reduced the usage and time to a single scale. The former used a linear relationship and the latter a linear relationship after log transformation.

1.2.7.2 Cost analysis. A two-dimensional warranty is characterized by a region in a two-dimensional plane. Different shapes for the region characterize different policies and many different shapes have been proposed (see Blischke and Murthy ([9], Chapter 8) and Singpurwalla and Wilson [94]).

Free replacement warranties: The expected warranty costs for a variety of policies can be found in Moskowitz and Chun ([62] and [10], Chapter 13), Singpurwalla and Wilson [94], Blischke and Murthy ([9], Chapter 8), Murthy et al. [72] and Chun and Tang [22]. Kim and Rao [53] dealt with the cost analysis based on a bivariate exponential distribution.

Pro-rata warranties: The expected warranty cost analysis for a variety of policies can be found in Iskandar [35], Blischke and Murthy ([9], Chapter 8), Wilson and Murthy ([10], Chapter 14) and Chun and Tang [22]. Patankar and Mitra [81] and Eliashberg *et al.* [25] studied some warranty reserve problem.

Combination warranties: The expected warranty cost analysis for combination policies can be found in Iskandar *et al.* [37] and Wilson and Murthy ([9], Chapter 14).

Fleet warranties: These are also referred to as cumulative warranties. Berke and Zaino [5] and Zaino and Berke [107] and Blischke and Murthy [11] dealt with the warranty cost analysis for a variety of such policies. Yeh and Chen [104] considered economic order quantities for items bought in lots with a cumulative free-replacement warranty.

1.3 Scope of the Dissertation and Research Contribution

1.3.1 Scope

In this dissertation, we have considered different servicing strategies under the 2-D warranty regime, analyzed their cost behavior from a manufacturer's point of view and justified their use under appropriate circumstances. It can be observed throughout this thesis that our focus on warranty strategies is not only concerned with reducing their costs, but also on the realism and relative ease of implementing them. For example, using a stochastic choice between a minimal repair or, replacement may be more realistic with two available skill levels of repairmen than achieving any arbitrary 'degree of repair' in practise. It is worth mentioning at this point that the warranty strategies analyzed here are optimal under specific assumptions, but might not be optimal if these conditions are not fulfilled.

By taking into account the impact of the rate of use of an item on its lifetime, a central focus of our research is on warranty cost models that are sensitive to the usage rate. Specifically, except the model in Chapter 4 where the rate at which an item is used is considered to be a random variable; all other warranty servicing policies that we consider, have usage rate as a fixed parameter, and hence are policies conditional on the rate of use. Such an approach allows us to examine how the expected warranty cost changes as a function of the usage rate. For the purpose of designing warranties, exploring such sensitivity analysis may in fact suggest putting an upper limit on the rate of use within the warranty contract, as is sometimes the case. For example, agreements on leased automobiles typically include a maximum allowable mileage that can be driven per year during the lease.

A second overriding feature of our models and analyzes, that follow as a consequence of both the age and usage rate being important factors influencing item failures and hence their warranty costs, is the corresponding necessity of 2-D framework to define use-specific warranties. While different possible shapes of such 2-D warranty regions and possible justifications thereof have been explored by Singpurwalla and Wilson [94]; in our work here, we confine ourselves to the pragmatic rectangular shaped warranty region as is standard in most literature (see Section 2.2.1.3).

1.3.2 Outline of the Dissertation

An outline of our research described in the subsequent chapters of this dissertation, is as follows.

Our work in Chapters 2 through 4 investigate 2-D warranty cost models under several different frameworks, and their corresponding analyzes as is listed below. The cost model and corresponding optimization in Chapter 2 extends,

- (1) the work of Jack *et al.* [40] on 2-D warranties by incorporating a 'degree of repair' option and
- (2) the work of Yun *et al.* [106] to the 2-D setup (see Sections 2.2.2 and 2.3).

In Chapter 3, a servicing strategy that allows a probabilistic choice between two possible rectification actions (minimal repair or, replacement) for atmost *one* failure during warranty is proposed. It is shown that, this enables us,

- to achieve a substantial reduction in total expected warranty cost compared to 'minimal repairs only' strategy; and
- (2) provides a corresponding generalized specification of possibly age-dependent probability p(t) of choosing a replacement at age t. This strategy reduces to that of Jack *et al.* [40] when $p \equiv 1$.

In Chapter 4, a different approach to 2-D warranty models (Iskandar [36], Chukova and Johnston [19], Yun and Kang [105], Chukova *et al.* [20], Varnosafaderani and Chukova [98]), where the usage rate is subject to uncertainty is considered. Here we

propose a strategy that allows randomization with a constant probability (p, 1 - p)of (imperfect repair, minimal repair) choice for atmost *two* failures during warranty. From the results it can be seen that our proposed strategy,

- again leads to reduction in total expected costs, relative to the 'minimal repairs only' strategy, and
- (2) is an extension of the Varnosafaderani-Chukova [98] cost model, which becomes a special case of our model when p = 1.

Finally, in Chapter 5, a Bayesian approach of modeling 2-D pro-rated warranty (PRW) strategy with preventive maintenance (PM) action is considered and explored in the spirit of the one-dimensional research of Huang and Fang [34]. Under the Accelerated Failure Time (AFT) formulation of item degradation process, an approach of determining optimal PRW cost proportion to be borne by the manufacturer is proposed and investigated. The expected profit model obtained in this context, captures the impact of juggling decision variables of 2-D PRW and investigates the sensitivity of the total expected profit to the extent of mis-specification in prior information. A Bayesian updating process is also employed to improve the quality of managerial decision. It can be seen that inclusion of consumer's usage rate in the integrated model has significant effects on the profit, warranty cost and pro-ration proportion.

In contrast to the cost models in which usage rate is given and thus essentially plays the role of a parameter, other frameworks that treat equipment usage as stochastic are possible as suggested in a review article Singpurwalla and Wilson [94]. Such models would account for the dependence and trade-offs between usage and failure time via suitable joint distributions and associated counting process of failures, to provide alternative framework for exploring 2-D warranties; but are not within the scope of this thesis.

1.3.3 Research Contributions

1.3.3.1 Overall Contribution. The broad dimension of our research contribution is to highlight the importance of recognizing the impacts of use intensity on warranty costs. This is achieved by including the rate of use as a factor in our models *either* as a warranty parameter (in conditional cost models) *or* as a random variable (with a distribution that specifies the use profile among all consumers of the product). Our work thus includes two-dimensional extensions of one-dimensional warranties, as well as new servicing strategies that are proposed and analyzed.

1.3.3.2 Specific Contribution. The notion of 2-D warranty is not new in the market of industrial products. As we have referred in later chapters, such warranties are available for automobiles, heavy machineries, defense equipments and many other products. Our specific contribution in this context can be itemized as follows:

- the servicing strategy with an imperfect repair option (Chapter 2) extends the work of Jack, Iskandar and Murthy [40] by introducing at most one imperfect repair in the middle interval, defined by the 'degree of repair' in the spirit of Yun, Murthy and Jack [106].
- to increase the realism and relative ease of implementation of the strategy in Chapter 2, we have considered a randomized repair strategy (Chapter 3), in which a stochastic choice between a minimal repair or, replacement is practised. In this context we have explored the two cases where the probability of randomization is *either* a constant *or* dependent on age at item failure.
- an alternative approach of 2-D modeling with imperfect repairs, extending the research of Varnosafaderani and Chukova [98] is also proposed (Chapter 4), that endorses the notion of imperfect repairs [50].
- finally, an integrated model of production, sales, warranty and maintenance is proposed in the 2-D regime, that captures the various aspects of product

manufacturing subject to a specific customer usage rate and the interaction between profit, warranty and costs.

CHAPTER 2

ANALYSIS OF A 2-D WARRANTY SERVICING STRATEGY WITH AN IMPERFECT REPAIR OPTION

2.1 Background and Motivation

If a warranted product fails under warranty, the manufacturer rectifies it with a repair or replacement. A replacement which costs the same as a new item can increase the total warranty cost. If the rectification action is a minimal repair, the item is restored to the state as it was just before failure and the corresponding cost is comparatively much smaller. However, there is always a trade-off between rectification cost and product reliability. An expensive repair will typically increase the reliability of the item, reducing the total number of failures over the warranty term. Conversely, less expensive restoration options, while attractive to the warranty provider in the near term, may end up being more costly over the life of the warranty; since cheaper repairs will not arrest future degradation as effectively as more expensive restorations. Thus, from the point of view of equipment reliability for the unexpired time to end of warranty, replacements appear to be the best strategy of rectification. But a replacement only strategy will cost the manufacturer too much. One way of controlling the warranty cost without entirely sacrificing the reliability issue is to practise some repair action that is better than minimal repair but worse than a replacement. Such repairs are often termed as an *imperfect repairs*. Under an imperfect repair, the item is restored up to a specified degree (denoted by $\delta \in [0, 1]$), such that its reliability profile after such restoration becomes *better* than an old item, but worse than that of a new one, in a well-defined sense. The cost of imperfect repair is also bracketed between the respective costs of replacement and minimal repair.

In this chapter, we consider a 2-D warranty strategy where: if the item fails for the first time in some interval of the warranty period, it is imperfectly repaired and all failures that occur at times preceding and following the imperfect repair are minimally repaired. Such a warranty policy not only reduces the expected cost compared to perfect repair strategy, but also improves the product reliability at the end of the warranty interval, relative to the 'minimal repairs only' policy. The rest of this chapter is as organized in the following way. Section 2.2 describes the set-up of the 2-D warranty problem with failures and rectifications. Section 2.3 comprises the proposed servicing strategy, model formulation, analysis, optimization and numerical illustration. Finally, some concluding remarks are included in Section 2.4.

2.2 Usage Rate Based Servicing Strategies

In the case of two-dimensional warranties, there are effects of both age and usage on the product degradation and failure needs to be modeled accordingly. The usage can be the output (e.g., number of pages printed/scanned for a printer/scanner), distance traveled (e.g., kilometers covered for an automobile) and the number of times or hours the product has been used (e.g., number of times or hours used for a vacuum cleaner).

The modeling approach assumes that the usage rate Y varies from customer to customer but is constant for a given customer. Therefore Y is a random variable that can be modeled using a density function g(y). Conditional on Y = y, the total usage u at age x is given by

$$u = yx, \qquad 0 \le u < \infty \tag{2.1}$$

Given usage rate y, the conditional hazard (failure rate) function $h_y(x) \geq 0$ is assumed to be non-decreasing in item's age x and usage rate y. Failures over time are modeled by a *counting process*. If failed items are replaced (by new ones), then this counting process is a renewal process associated with the conditional distribution $F_y(x)$, which can be derived from $h_y(x)$. If failed items are repaired then the counting process is characterized by a conditional intensity function $\lambda_y(x)$, which is a nondecreasing function of x and y. If all repairs are minimal [11] and repair times are negligible, then $\lambda_y(x) = h_y(x)$.

2.2.1 Modeling Failures

We consider a repairable item sold with a two dimensional non-renewing free replacement warranty of period W and maximum usage level U, that requires the manufacturer to either repair or replace the item when it fails. Failure occurs if warranty exceeds time W or total usage exceeds U. We make the following additional assumptions:

- 1. All item failures are detected immediately and result in immediate claims by the consumer.
- 2. All claims are valid and must be rectified by the manufacturer immediately through repairs.
- 3. Repair and replacement times are small relative to the mean time between item failures and therefore can be ignored.
- 4. For the duration of the warranty, no separate preventive maintenance except those (if any), that are built in with the warranty, is carried either by the manufacturer or by the consumer.
- 5. The product's hazard rate function is monotone non-decreasing in its age and usage rate.

A product can be considered as a system containing several interconnected components. When the components are statistically independent, the reliability of the product is a function of the individual component reliabilities. During the design stage, decisions are made about component reliabilities in order to ensure that the product has the desired reliability at some *nominal* usage rate y_0 . When the usage rate differs from this nominal value used in the design, the reliabilities of some of the components can be affected and this in turn affects the total product reliability. As the usage rate increases, the rate of degradation increases and this, in turn, accelerates the time to failure. As a result, the product reliability decreases (increases) as the usage rate increases (decreases).

2.2.1.1 Modeling First Failure. The effect of usage rate on degradation can be modeled by "Accelerated Failure Time model" (AFT) ([11],[40]). If T_0 (T_y , respectively) denotes the time to first failure under usage rate $y_0(y)$, then the standard AFT model postulates,

$$\frac{T_y}{T_0} = \left(\frac{y_0}{y}\right)^{\gamma},\tag{2.2}$$

where $\gamma \in [1, \infty)$ is the so called acceleration parameter.

Note, for usage rates y more (less) than the nominal usage rate y_0 , the resulting actual time T_y to failure is a fraction (multiple) of the nominal failure time T_0 . Let

$$\mathcal{F} = \{F(x;\alpha) : \alpha \in A\}$$
(2.3)

be a scale parameter family indexed by a scale parameter $\alpha \in A \subset (0, \infty)$ for some index set A. If the cumulative distribution function (CDF) of T_0 is $F(x; \alpha_0) \in \mathcal{F}$; then, by 2.2, the CDF of AFT T_y is

$$F(x;\alpha(y)) = F_0\left(\left(\frac{y}{y_0}\right)^{\gamma}x;\alpha_0\right)$$
(2.4)

i.e., CDF of T_y is the same as that for T_0 but with scale parameter given by

$$\alpha(y) = \left(\frac{y_0}{y}\right)^{\gamma} \alpha_0 \quad \text{where} \quad \gamma \ge 1.$$
(2.5)

The hazard rate and the cumulative hazard function associated with the CDF $F(x, \alpha(y))$ are given by

$$h(x;\alpha(y)) = \frac{f(x;\alpha(y))}{\overline{F}(x;\alpha(y))}$$
(2.6)

and

$$H(x;\alpha(y)) = \int_0^x h(u,\alpha(y))du$$
(2.7)

where $f(x; \alpha(y))$ and $\overline{F}(x; \alpha(y))$, respectively are the associated density function (PDF) and survival function.

2.2.1.2 Modeling Subsequent Failures. The times of subsequent failures are influenced by the type of action taken to rectify a failed item. For a non-repairable product, the only option is to replace the failed item by a new one. In the case of a repairable product, the subsequent failures depend on the type of repair carried out. If it is a minimal repair, reliability of the product after repair is same as that just before failure. If it is an imperfect repair [24], reliability after repair is better than minimal repair but is inferior to that of a new item.

Here we confine our attention to minimal repair and assume that repair times are negligible (relative to the mean time between failures) and so can be ignored. It is well known that failures over time under such a minimal repairs only policy occur according to a non-homogeneous Poisson process (NHPP) with intensity function having the same form as the hazard rate for time to first failure. Thus, if the product has usage rate y, the failure intensity function is

$$\lambda_y(x) = h(x; \alpha(y)) \tag{2.8}$$

where $h(x; \alpha(y))$ is the hazard rate given by Equation (2.6).

2.2.1.3 Warranty Policy and Coverage. The product is sold with a twodimensional warranty with warranty region the rectangle $[0, W) \times [0, U)$, where W is the time limit and U the usage limit. The warranty expires at the first instance when the age of the item reaches W or its usage reaches U, whichever occurs first. Clearly if the usage rate y is at most U/W then the warranty expires at age W and an estimate of the total usage is yW. When y is greater than U/W, the warranty expires at age U/y when the usage limit U is reached. With usage rate y, if W_y denotes the calendar time when warranty expires, then

$$W_y = \min(W, \frac{U}{y}) = \begin{cases} W, & y \le U/W; \\ U/y, & y > U/W. \end{cases}$$
(2.9)

2.2.2 2-D Servicing Strategies of Jack *et al.* (2009) and Yun *et al.* (2008) Jack *et al.* [40] have considered a 2-D warranty servicing strategy using minimal repairs, except for the first failure to be 'rectified' (i.e., 'repaired') by a replacement. Such a strategy can be described via three disjoint intervals $[0, K_y)$, $[K_y, L_y)$ and $[L_y, W_y)$ with $0 < K_y < L_y < W_y$, along the age (time) scale where failures in the initial interval $[0, K_y)$ when the item is relatively new undergo only minimal repair; the first failure (if any) in the middle interval $[K_y, L_y)$ rectified by a replacement and all subsequent failures therein, as well in the interval $[L_y, W_y)$ when the item is relatively old getting only quick fixes via minimal repairs. Such a strategy minimizes what is known to be near-optimal among 1-D warranty policies (viz., Jack *et al.* [41], Jiang *et al.* [46]).

In the 1-D replacement / repair warranty (FRW, F-free, to the consumer) policies, Yun *et al.* [106] have investigated the impact of allowing 'imperfect repair' (IR) as a mode of rectifying the first failure in the middle interval $[K_y, L_y)$ to restore the unit to a working condition. They describe the degree of 'imperfect repair' via a parameter $\delta \in [0, 1]$ with $\delta = 0$ (1, respectively) being equivalent to minimal repair (replacement), and assume that it is possible to restore a failed equipment with any chosen degree (δ) of repair.

2.3 Proposed Servicing Strategy for 2-D Warranties with Imperfect Repairs

For 2-D warranties, alternatives to 'minimal repair' in the middle interval $[K_y, L_y)$ in Jack *et al.* [40] approach is restricted to replacements (i.e., 'perfect repairs') only. We propose and investigate a new 2-D servicing strategy. Our current work described here, is an attempt to extend the model and analysis of 2-D warranties by allowing imperfect repairs. For a given usage rate y the value of the parameters K_y and L_y are selected to minimize the expected warranty servicing cost. If K_y^* and L_y^* denote the optimal values then, as y varies the set of points (K_y^*, L_y^*) define a closed curve as indicated in Figure 2.1. Let Γ denote the region enclosed by this curve.



Figure 2.1 The ideal Γ region of Jack, Iskandar, Murthy (2009), in the 2-D warranty space where the first failure is replaced by a new item.

Our new servicing strategy:

For items sold with 2-D warranties, the first failure in the region Γ is rectified with an imperfect repair and all other failures are repaired minimally.

The region Γ depends on the type of model used for item failures and on the cost of minimal repair and imperfect repair. Let C_m denote the cost of a minimal repair and $C_i(\delta_y(x), x)$ (> C_m) denote the cost of an imperfect repair conditioned on y. Here, given usage rate y, the chosen degree of repair $\delta_y(x) \in [0, 1]$ denotes the conditional proportional reduction factor in the hazard rate after failure at age x. We will consider two different strategies:

- i) if $\delta_y(x)$ is a function of both age(x) and usage(y).
- ii) if $\delta_y(x) (= \delta_y)$ is a function of usage rate(y) only.

2.3.1 Model Formulation

For a failed unit restored by minimal repairs, the hazard rate function of post-repair lifetime continues uninterrupted, as if there was no failure. If repair times are small relative to the mean time between failures (so that minimal repairs can be treated as being instantaneous) then item failures over time follow a non-homogeneous Poisson process (NHPP) with intensity function $\lambda_y(x) = h(x; \alpha(y))$. The intensity function is also referred to as the rate of occurrence of failures (ROCOF).

In contrast, an imperfect repair improves the items operating characteristics in the sense that the hazard rate of item's lifetime after such a repair is typically smaller than before failure. This can be modeled as follows. For a given usage rate y, if the failure occurs at age x_i the conditional hazard rate just before failure is $h(x_i -; \alpha(y))$ and after repair, is

$$h(x_i+;\alpha(y)) = h(x_i-;\alpha(y)) - \delta_y(x_i)(h(x_i-;\alpha(y)) - h(0;\alpha(y)))$$

as suggested by Yun *et al.* [106], where $\delta_y(x_i)$ can take values in the interval [0, 1]. The reduction in the hazard rate is a linear function of $\delta_y(x_i)$. $\delta_y(x_i)$ is a decision variable with a higher value indicating a greater improvement in the reliability after repair.

Imperfect repairs are also assumed to be instantaneous. The concept of imperfect repair is appropriate for a complex system containing a very large number of components. The system hazard rate can be expressed in terms of the component hazard rates. (For example, in the case of a series configuration, the system hazard is the sum of the component hazard rates.) The hazard rates are usually assumed to be increasing functions of time (reflecting the degradation effect of age). System failure occurs due to failure of one or more components, depending on the system's failure logic. Under minimal repairs, only failed components are so repaired and there is no effect on the system hazard rate. Under imperfect repair, the failed components and also some of the non-failed components are correspondingly repaired in order to achieve the desired reduction in the system hazard rate. This implies that the cost of an imperfect repair is greater than that of a minimal repair and this cost increases as the degree of hazard rate reduction increases.

2.3.2 Model Analysis

In this section the conditional expected warranty servicing cost $J(K_y, L_y, \Delta_y(K_y, L_y))$ for a given usage rate y is derived as a function of parameters K_y , L_y (subject to the constraints $0 \le K_y \le L_y \le W_y$) and the set of imperfect repair functions $\Delta_y(K_y, L_y)$ $\equiv \{\delta_y(x) : K_y \le x \le L_y\}.$

2.3.2.1 Conditional Expected Warranty Cost. For a given usage rate y, let T_1 denote the time of the first failure under usage rate y after age K_y . The conditional density function (PDF) and survival function for T_1 are respectively, given by

$$f_1(t; \alpha(y)) = \frac{f(t; \alpha(y))}{\overline{F}(K_y; \alpha(y))}, \text{ and}$$
$$\overline{F}_1(t; \alpha(y)) = 1 - F_1(t; \alpha(y)) = \frac{\overline{F}(t; \alpha(y))}{\overline{F}(K_y; \alpha(y))}, t \ge K_y,$$

Over $[0, K_y)$ – all failures are minimally repaired with average cost C_m , so the failures occur according to a NHPP with conditional intensity function $\lambda_y(x) = h(x; \alpha(y))$ and the conditional expected warranty servicing cost for this interval is given by

$$C_m \int_0^{K_y} h(x; \alpha(y)) dx \tag{2.10}$$

Over $[K_y, W_y)$ – We need to consider the following two cases:

(A)
$$K_y \leq T_1 = x \leq L_y$$
 and (B) $T_1 = x > L_y$

Case (A) $K_y \leq T_1 = x \leq L_y$

The conditional expected cost, conditional on $K_y \leq T_1 = t_1 \leq L_y$, is obtained as follows. The first failure in $[K_y, L_y]$ occurs at age t_1 and is imperfectly repaired with cost $C_i(\delta(t_1), t_1)$. All failures over the remaining interval $(t_1, W_y]$ are minimally repaired. As a result, the failures over this interval occur according to an NHPP with conditional intensity function:

$$\lambda_y(x) = h(x; \alpha(y)) - \delta_y(t_1)(h(t_1; \alpha(y)) - h(0; \alpha(y))), \quad t_1 \le x \le W_y.$$

The expected cost of servicing failures over $(t_1, W_y]$ is given by

$$C_m \int_{t_1}^{W_y} [h(x; \alpha(y)) - \delta_y(t_1) \{ h(t_1; \alpha(y)) - h(0; \alpha(y)) \}] dx.$$
(2.11)

As a result, the conditional expected warranty cost for usage rate y and $K_y \leq T_1 = t_1 \leq L_y$ is given by

$$J(K_y, L_y, \delta_y(K_y, L_y) | K_y \le x \le L_y) = C_m \int_0^{K_y} h(x; \alpha(y)) dx + C_m \int_{t_1}^{W_y} \left[h(x; \alpha(y)) - \delta_y(t_1) \{ h(t_1; \alpha(y)) - h(0; \alpha(y)) \} \right] dx + C_i(\delta(t_1), t_1).$$
(2.12)

Case (B) $T_1 = x > L_y$

The conditional expected cost, conditional on $T_1 = t_1 > L_y$, is obtained as follows. Note that there is no failure in $[K_y, L_y]$ and failures over the remaining interval $(L_y, W_y]$ occur according to an NHPP with intensity function

$$\lambda_y(x) = h(x; \alpha(y)), \quad L_y \le x \le W_y$$

As a result, the conditional expected warranty cost is given by

$$J(K_y, L_y, \delta_y(K_y, L_y)|x > L_y) = C_m \int_0^{K_y} h(x; \alpha(y)) dx + C_m \int_{L_y}^{W_y} h(x; \alpha(y)) dx.$$
(2.13)

For a given usage rate y the expected warranty cost is obtained by unconditioning on T_1 i.e.,

$$EJ(K_y, L_y, \delta_y(K_y, L_y)) = E(E\{J(\cdot)|T_1\})$$

= $P(T_1 > L_y)E\{J(\cdot)|T_1 \equiv x > L_y\} + \int_{K_y}^{L_y} E\{J(\cdot)|T_1 \equiv x\}dF_1(x)$

equivalently,

$$EJ(K_{y}, L_{y}, \delta_{y}(K_{y}, L_{y})) = EJ(K_{y}, L_{y}, \delta_{y}(x) | T_{1} \equiv x > L_{y}) \overline{F}_{1}(L_{y}; \alpha(y))$$

+
$$\int_{K_{y}}^{L_{y}} EJ(K_{y}, L_{y}, \delta_{y}(x)) | K_{y} \leq T_{1} \equiv x \leq L_{y}) f_{1}(x; \alpha(y)) dx.$$
(2.14)

Using the cumulative hazard function $H(t; \alpha(y)) = \int_0^t h(u; \alpha(y)) du$ and combining terms containing $\delta_y(K_y, L_y)$, Equation (2.14) can be further rewritten as

$$J(K_y, L_y, \delta_y(K_y, L_y)) = \Psi(K_y, L_y) + \Phi(\Delta(K_y, L_y), K_y, L_y),$$
(2.15)

where

$$\Psi(K_{y}, L_{y}) = C_{m}(H(K_{y}; \alpha(y)) - [H(W_{y}; \alpha(y)) - H(L_{y}; \alpha(y))] \frac{\overline{F}(L_{y}; \alpha(y))}{\overline{F}(K_{y}; \alpha(y))} + \int_{K_{y}}^{L_{y}} [H(W_{y}; \alpha(y)) - H(x; \alpha(y))] \frac{f(x; \alpha(y))}{\overline{F}(K_{y}; \alpha(y))} dx), \Phi(\Delta_{y}(K_{y}, L_{y}), K_{y}, L_{y}) = \int_{K_{y}}^{L_{y}} [C_{i}(\delta_{y}(x), x) - C_{m}\delta_{y}(x)\{h(x; \alpha(y)) - h(0; \alpha(y))\}(W_{y} - x)] \frac{f(x; \alpha(y))}{\overline{F}(K_{y}; \alpha(y))} dx.$$
(2.16)

2.3.3 Optimization of Strategy 1

The optimization problem is given by

$$\min_{K_y, L_y, \Delta_y(K_y, L_y)} J(K_y, L_y, \Delta_y(K_y, L_y))$$

=
$$\min_{K_y, L_y, \Delta_y(K_y, L_y)} \{ \Psi(K_y, L_y) + \Phi(\Delta(K_y, L_y), K_y, L_y) \}.$$

Note that this involves selecting optimally the two parameters K_y and L_y for a given y (subject to the constraints $0 \le K_y \le L_y \le W_y$) and the function $\Delta_y(K_y, L_y) \equiv \{\delta_y(x): K_y \le x \le L_y\}$ (subject to the constraints $0 \le \delta_y(x) \le 1$).

Let K_y^* and L_y^* denote the optimal solution. We obtain this using a two-stage approach. In stage 1, for a fixed K_y and L_y , we obtain the optimal $\Delta_y^*(K_y, L_y)$ that minimizes $J(K_y, L_y, \Delta_y(K_y, L_y))$. Then, in stage 2, we obtain the optimal (K_y^*, L_y^*) by minimizing $J(K_y, L_y, \Delta_y^*(K_y, L_y))$.

Stage 1

To determine $\Delta_y^*(K_y, L_y)$ we need to focus on $\Phi(\Delta_y(K_y, L_y), K_y, L_y)$ given by Equation 2.16 and this can be rewritten as

$$\Phi(\Delta_y(K_y, L_y), K_y, L_y) = \int_{K_y}^{L_y} [C_i(\delta_y(x), x) - \delta_y(x)\xi_y(x)] \frac{f(x; \alpha(y))}{\overline{F}(K_y; \alpha(y))} dx$$

where

$$\xi_y(x) = C_m\{h(x; \alpha(y)) - h(0; \alpha(y))\}(W_y - x), \quad K_y \le x \le L_y.$$

Assume the baseline survival distribution F_0 of the product's lifetime is such that $\xi_y(x)$ is concave in the item's age x. This postulate is satisfied by many parametric lifetime models that are increasingly degrading with age. In particular, the following is a sufficient condition for such concavity.

Proposition 2.3.1 If $h(x; \alpha_0)$ is increasing and concave (i.e., baseline survival time T_0 with d.f. $F(\cdot; \alpha_0)$ is IFR with concave hazard rate), implies

$$g(x) = \{h(x; \alpha_0) - h(0; \alpha_0)\}(W - x)$$

is concave in $\{0 \le x \le W\}$.

Proof: Assuming $h(x; \alpha_0)$ is twice differentiable, it can be seen that g(x) is concave in $\{0 \le x \le W\}$. If $h''(x; \alpha_0)$ does not exist, then the proof follows from the general definition of concavity. Since

$$F(x;\alpha(y)) = F((\frac{y}{y_0})^{\gamma}x;\alpha_0) \Leftrightarrow h(x;\alpha(y)) = (\frac{y}{y_0})^{\gamma}h((\frac{y}{y_0})^{\gamma}x;\alpha_0), \text{thus}$$
$$\xi_y(x) = C_r(\frac{y}{y_0})^{\gamma}\{h((\frac{y}{y_0})^{\gamma}x;\alpha_0) - h(0;\alpha_0)\}(W_y - x)$$

is also concave in $x \in [0, W_y]$.

We need to determine the optimal form for $\delta_y(x)$ for every point x along the time axis. The optimal $\delta_y(x)$ must result in $[C_i(\delta_y(x), x) - \delta_y(x)\xi_y(x)]$ being a minimum for each $x \in [K_y, L_y]$. As result, $\delta_y^*(x)$ can be obtained by examining:

$$v(z_y, x) = [C_i(z_y, x) - \xi_y(x)z_y]$$

for each $x \in [K_y, L_y]$. For a fixed x, $C_i(z_y, x)$ is an increasing function of z_y as shown in Figure 2.2. $\xi_y(x)z_y$, the second term in $v(z_y, x)$, is linear in z_y and so is a straight line when plotted as a function of z_y , as shown in Figure 2.2.

We need to consider the following two cases.

Case (1): The line $\xi_y(x)z_y$ lies below the curve $C_i(z_y, x)$. This corresponds to (a) in Figure 2.2. In this case, $\delta_y^*(x) = 0$. This is because the cost of any imperfect repair with $\delta_y^*(x) > 0$ is not worth the reduction in the expected warranty servicing cost when compared with only minimal repair $\delta_y^*(x) = 0$.

Case (2): The straight line $\xi_y(x)z_y$ and the curve $C_i(z_y, x)$ intersect. This corresponds to (b) in Figure 2.2 and in this case we have $\delta_y^*(x) > 0$. Since $0 \le \delta_y^*(x) \le 1$ then either $\delta_y^*(x) = 1$ (the boundary solution) or $0 < \delta_y^*(x) < 1$ (an interior point solution). In the latter case, the optimal value is obtained from the usual first order condition. This yields $\delta_y^*(x) = z_y^*$ for a given y with z_y^* given by

$$\frac{\delta C_i(z_y, x)}{\delta z_y} = \xi_y(x). \tag{2.17}$$

Let the straight line κz_y be a tangent to the curve $C_i(z_y, x)$ at $z_y = \tilde{z}$. This is shown by (c) in Figure 2.2 κ and \tilde{z} are obtained by solving the simultaneous equations given



Figure 2.2 Plots of $C_i(z_y, x)$ and $\xi_y(x)z_y$ vs. z_y .

below:

$$C_i(\tilde{z}, x) = \kappa \tilde{z}$$
 and $\frac{\delta C_i(z_y, x)}{\delta z_y}|_{z_y = \tilde{z}} = \kappa.$ (2.18)

where $\xi_y(x)$ is a concave function as shown in Figure 2.3 with $\xi_y(0) = 0$ and $\xi_y(W_y) = 0$. Define

$$\xi_{y(max)} = \max_{0 \le x \le W_y} \xi_y(x).$$
(2.19)

Proposition 2.3.2 If $\xi_{y(max)} < \kappa$ then $\delta_y^*(x) = 0$ for all x. If $\xi_{y(max)} > \kappa$ then $\delta_y^*(x) > 0$ for $0 \le \tau_{1y} \le x \le \tau_{2y} \le W_y$ where τ_{1y} and τ_{2y} are the solutions of the equation $\xi_y(x) = \kappa$. For x outside the interval $[\tau_{1y}, \tau_{2y}), \ \delta_y^*(x) = 0$.

Note: This implies that $\delta_y^*(x)$ has a shape as shown in Figure 2.4, and note that $\delta_y^*(x)$ does not depend on K_y and L_y .



Figure 2.3 Plot of $\xi_y(x)$ against age(x).



Figure 2.4 Plot of $\delta_y^*(x)$ vs. age (x) for Strategy 1.

Stage 2

Let $\Delta_y^*(K_y, L_y) \equiv \{\delta_y^*(x) : 0 \le x \le W_y\}$ which is obtained from Stage 1. K_y^* and L_y^* , the optimal values for K_y and L_y , are obtained by solving the following minimization problem

$$\min_{K_y, L_y} J(K_y, L_y, \Delta_y^*(K_y, L_y)) = \min_{K_y, L_y} \{ \Psi(K_y, L_y) + \Phi(\Delta_y^*(K_y, L_y), K_y, L_y) \}$$

subject to the constraint $0 \le K_y \le L_y \le W_y$. These can be obtained from the usual first-order conditions:

$$\frac{\partial}{\partial K_y} J(K_y, L_y, \Delta_y(K_y, L_y)) = 0 \quad \text{and} \quad \frac{\partial}{\partial L_y} J(K_y, L_y, \Delta_y(K_y, L_y)) = 0 \quad (2.20)$$

if they lie inside the interval $[0, W_y]$. It is not possible to derive any analytical results from these conditions and the optimal values need to be obtained using a computational approach.

2.3.4 Optimization of Strategy 2

The optimization problem is given by

$$\min_{K_y, L_y, \delta_y} J(K_y, L_y, \delta_y) = \min_{K_y, L_y, \delta_y} \{ \Psi(K_y, L_y) + \Phi(\delta_y, K_y, L_y) \}.$$

where $\Psi(K_y, L_y)$ is same as Equation 2.16 and

$$\Phi(\delta_y, K_y, L_y) = \int_{K_y}^{L_y} [C_i(\delta_y) - C_m \delta_y \{h(x; \alpha(y)) - h(0; \alpha(y))\} \\ (W_y - x)] \frac{f(x; \alpha(y))}{\overline{F}(K_y; \alpha(y))} dx.$$
(2.21)

Here the cost of imperfect repair $C_i(\delta_y)$ will not depend on the age at failure. This problem involves selecting optimally three parameters δ_y ($0 \le \delta_y \le 1$), K_y and L_y ($0 \le K_y \le L_y \le W_y$) for a given y.

We use the two-stage approach. In stage 1, given y, we fix K_y and L_y and obtain the optimal $\delta_y^*(K_y, L_y)$ that minimizes $J(K_y, L_y, \delta_y)$. Then in stage 2, we obtain the optimal (K_y^*, L_y^*) by minimizing $J(K_y, L_y, \delta_y^*)$.

Stage 1: $\delta_y^*(K_y, L_y)$ is obtained by solving the following optimization problem:

$$\min_{\delta_y | K_y, L_y} \phi(\delta_y, K_y, L_y) = \min_{\delta_y | K_y, L_y} \{ \phi_1(K_y, L_y) C_i(\delta_y) + \phi_2(K_y, L_y) \delta_y \}.$$
(2.22)

where
$$\phi_1(K_y, L_y) = \int_{K_y}^{L_y} f(x; \alpha(y)) dx,$$

and
$$\phi_2(K_y, L_y) = C_m \int_{K_y}^{L_y} \{h(x; \alpha(y)) - h(0; \alpha(y))\} (W_y - x) f(x; \alpha(y)) dx$$

 δ_y^* can either be an interior point or one of the end-points of the interval [0,1]. If δ_y^* is an interior point then it is obtained from the first order condition:

$$\frac{\partial}{\partial\delta}\phi(\delta_y, K_y, L_y) = 0, \quad \text{or} \quad \phi_1(K_y, L_y)\frac{\partial}{\partial\delta}C_i(\delta_y) = \phi_2(K_y, L_y). \quad (2.23)$$

Here the optimal δ_y^* will be a function of K_y and L_y .

Stage 2: K_y^* and L_y^* is obtained from the following optimization problem:

$$\min_{K_y, L_y} J(K_y, L_y, \delta_y^*) = \min_{K_y, L_y} \{ \Psi(K_y, L_y) + \Phi(\delta_y^*, K_y, L_y) \}$$
(2.24)

subject to the constraint $0 \le K_y \le L_y \le W_y$. We need to use computational approach to obtain these optimal values. The optimal reduction when an imperfect repair is carried out is given by $\delta_y^*(K_y^*, L_y^*)$.

In the final stage, the minimal expected warranty cost $J^* \equiv J^*(K_y^*, L_y^*, \delta_y^*)$ is obtained.

2.3.5 Special Case: Weibull Failure Distribution

The distribution function for the time to first failure under the nominal usage rate y_0 denoted by T_0 is a Weibull distribution with scale parameter $\alpha_0 > 0$ and shape parameter $\beta > 1$, so

$$F(x; \alpha_0) = 1 - \exp\left(-\frac{x}{\alpha_0}\right)^{\beta}$$
 and $\overline{F}(x; \alpha_0) = \exp\left(-\frac{x}{\alpha_0}\right)^{\beta}$.

Therefore, using the AFT formulation (Equation 2.2) the following functions can be derived for T_y , the time to first failure under the usage rate y

$$\begin{aligned} \text{CDF:} \quad F(x;\alpha(y)) &= 1 - \exp\left(-\frac{x}{\alpha(y)}\right)^{\beta} = 1 - \exp\left(-\left(\frac{y}{y_{0}}\right)^{\gamma}\frac{x}{\alpha_{0}}\right)^{\beta}. \\ \text{Survival function:} \quad \overline{F}(x;\alpha(y)) &= \exp\left(-\frac{x}{\alpha(y)}\right)^{\beta} = \exp\left(-\left(\frac{y}{y_{0}}\right)^{\gamma}\frac{x}{\alpha_{0}}\right)^{\beta} \\ \text{Hazard function:} \quad h(x;\alpha(y)) &= \beta\left(\frac{y}{y_{0}}\right)^{\gamma\beta}\left(\frac{x^{\beta-1}}{\alpha_{0}^{\beta}}\right), \quad (\beta > 1). \\ \text{Cumulative hazard function:} \quad H(x;\alpha(y)) &= \left(\frac{y}{y_{0}}\right)^{\gamma\beta}\left(\frac{x^{\beta}}{\alpha_{0}^{\beta}}\right). \end{aligned}$$

2.3.5.1 Strategy 1 Computations. Let C_r denote the cost of repair that achieves 100% reduction in the system hazard rate (equivalent to replacement), C_m (< C_r) denote the cost of minimal repair. Then the cost of imperfect repair for usage rate yis given by the expression

$$C_i(z_y, x) = C_m + (C_r - C_m)z_y^q, \quad q > 1,$$

where z_y is the proportional reduction factor in the failure rate under fixed usage rate y. Thus $z_y \in [0, 1]$ is a decision variable, with a greater value indicating a greater improvement in the reliability of the item after repair.

From Equation 2.17, we have

$$\xi_y(x) = C_m \{ h(x; \alpha(y)) - h(0; \alpha(y)) \} (W_y - x) = C_m \beta \left(\frac{y}{y_0}\right)^{\gamma \beta} \left(\frac{x^{\beta - 1}}{\alpha_0^{\beta}}\right) (W_y - x).$$

$$\frac{\partial}{\partial x}\xi_y(x) = 0$$
 gives the maximum at age $x = \frac{\beta - 1}{\beta}W_y$,

since $\xi_y(x)$ is concave in x for each y. The maximum value of $\xi_y(x)$ is

$$\xi_{y(max)} = \max_{0 \le x \le W_y} \xi_y(x) = \frac{C_m(\frac{y}{y_0})^{\gamma\beta}}{\alpha_0^{\beta}} \left(\frac{\beta - 1}{\beta}\right)^{\beta - 1} W_y^{\beta}.$$

Clearly, $\xi_{y(max)} > 0$ for all y.

From Equation 2.18 we have:

$$\widetilde{z} = \left(\frac{C_m}{(C_r - C_m)(q-1)}\right)^{\frac{1}{q}} \quad \text{and} \quad \kappa = (C_r - C_m)q\left(\frac{C_m}{(C_r - C_m)(q-1)}\right)^{\frac{q-1}{q}}.$$

For each y, τ_{1y} and τ_{2y} are the solutions of the equation:

$$C_m \beta \left(\frac{y}{y_0}\right)^{\gamma \beta} \left(\frac{x^{\beta - 1}}{\alpha_0^{\beta}}\right) (W_y - x) - (C_r - C_m) q \left(\frac{C_m}{(C_r - C_m)(q - 1)}\right)^{\frac{q - 1}{q}} = 0.$$

The optimum $\delta_y^*(x)$ for strategy 1 is

$$\delta_y^*(x) = \left\{ \left(\frac{C_m \beta}{(C_r - C_m)q} \right) \left(\frac{y}{y_0} \right)^{\gamma\beta} \left(\frac{x^{\beta-1}}{\alpha_0^{\beta}} \right) (W_y - x) \right\}^{\frac{1}{q-1}},$$

for $0 < \tau_{1y} < x < \tau_{2y} < W_y.$

Here $\delta_y^*(x)$ does not depend on the values of K_y and L_y . As mentioned previously, we need to calculate the values of K_y^* and L_y^* using computational methods.

2.3.5.2 Strategy 2 Computations. Given y, the cost function

$$C_i(\delta) = C_m + (C_r - C_m)\delta^q$$
, therefore $\frac{\partial}{\partial\delta}C_i(\delta) = (C_r - C_m)q\delta^{q-1}$.

From the first order condition, i.e.,

$$\frac{\partial}{\partial \delta}\phi(\delta, K_y, L_y) = 0 \Leftrightarrow \text{i.e.}, \quad \frac{\partial}{\partial \delta} \Big(\frac{\phi_1(K_y, L_y)C_i(\delta) - \phi_2(K_y, L_y)\delta}{\overline{F}(K_y, \alpha(y))} \Big) = 0.$$

Therefore for Strategy 2,

$$\delta^*(K_y, L_y) = \left(\frac{\phi_2(K_y, L_y)}{(C_r - C_m)p\phi_1(K_y, L_y)}\right)^{\frac{1}{q-1}},$$

where $\phi_1(K_y, L_y) = \int_{K_y}^{L_y} f(x; \alpha(y))dx$
and $\phi_2(K_y, L_y) = C_m \int_{K_y}^{L_y} [\{h(x; \alpha(y)) - h(0; \alpha(y))\}(W_y - x)]f(x; \alpha(y))dt.$

Note: Unlike strategy 1, here the optimum reduction proportion depends on K_y and L_y .

2.3.6 Numerical Example: Strategy 1 and 2 for $C_r = 2$ and $\beta = 2$

We normalize costs so that the cost of minimal repair, Cr = 1 and consider a range of values for the cost of replacement (perfect repair), i.e., C_r varying from 2 to 10. We assume the nominal values warranty period, W = 2, total usage limit, U = 2, Weibull scale (baseline) parameter, $\alpha_0 = 1$, Weibull shape parameter, $\beta = (2,3)$, nominal usage rate, $y_0 = 1$, the AFT model parameter, $\gamma = 2$ and imperfect cost function parameter, q = 4.

The numerical results are obtained from high-performance workstations (2.3 GHz Intel Core 2 Quad Q8200 processors) in the *Department of Mathematical Sciences* / *Center of Applied Mathematics and Statistics* computing lab and the average runtime for each pair (C_r,β) is approximately 850 minutes. The optimal values of parameters and the corresponding minimal cost is demonstrated in Table 2.1. Table 2.2, presents a comparison of costs of the proposed strategies to those of Jack *et al.* [40], where the figures in brackets are percentage cost savings.

For computational purposes, we have developed MATLAB programs that are appropriate for the different warranty models considered in this dissertation. The corresponding MATLAB codes have been used to carry out all illustrative numerical computations in this and subsequent chapters. A sample MATLAB program is included in the Appendix A.

		(age-dependent degree of repair)			(constant degree of repair)			
y	Wy	K_y^*	L_y^*	$J(K_y^*, L_y^*, \delta_y^*)$	K_y^*	L_y^*	δ_y^*	$J(K_y^*, L_y^*, \delta_y^*)$
0.1	2.00	1.9004	1.9000	0.0004	1.2500	1.3100	0.0360	0.0004
0.3	2.00	1.9000	1.9000	0.0324	0.2990	1.8900	0.1440	0.0324
0.5	2.00	1.8000	1.8000	0.2500	0.2080	1.9200	0.2840	0.2460
0.7	2.00	1.4301	1.9860	0.9602	0.2380	1.9200	0.4520	0.8870
0.9	2.00	0.8048	1.977	2.1880	0.5400	1.8800	0.6560	2.1361
1.0	2.00	0.636	1.768	3.1950	0.6470	1.8470	0.7670	3.0130
1.2	1.66	0.590	1.4502	4.0250	0.6057	1.5300	0.8762	4.0210
1.4	1.42	0.5329	1.1929	5.1022	0.5690	1.3900	0.9780	5.1010
1.6	1.25	0.5029	0.9951	6.1899	0.4790	1.0120	1.0000	6.1290
1.8	1.11	0.473	0.9080	7.6135	0.4790	1.0100	1.0000	7.3985
2.0	1.00	0.447	0.7719	8.7756	0.4430	0.8740	1.0000	8.5730
2.5	0.80	0.3777	0.5542	13.2400	0.3690	0.6490	1.0000	11.4020
3.0	0.67	0.3199	0.4130	18.7374	0.3150	0.4910	1.0000	13.8050
3.5	0.57	0.3045	0.3420	21.6691	0.2750	0.4190	1.0000	15.4690
4.0	0.50	0.2717	0.3206	33.2484	0.2430	0.3420	1.0000	16.0890
4.5	0.44	0.2437	0.2650	35.8719	0.2190	0.2609	1.0000	15.3930
5.0	0.40	0.2068	0.2467	60.1243	0.1980	0.2423	1.0000	13.0792

Strategy 1

Strategy 2

y	Strategy 1	Strategy 2	Jack <i>et.al</i> (2009)
0.9000	2.1880 (11.11)	2.1361 (13.22)	2.4614
1.0000	3.1950 (1.12)	3.0130 (6.75)	3.2312
1.2000	4.0250 (1.78)	4.0210 (1.88)	4.0980
1.4000	5.1022 (0.02)	5.1010 (0.04)	5.1032
1.6000	6.1899 (1.42)	6.1290 (2.39)	6.2790
1.8000	7.6135 (0.03)	7.3985 (2.85)	7.6157
2.0000	8.7756 (3.77)	8.5730 (5.99)	9.1197
2.5000	13.2400 (4.05)	11.4020 (17.37)	13.7987
3.0000	18.7374 (5.32)	13.8050 (30.24)	19.7906
3.5000	21.6691 (21.69)	15.4690 (44.10)	27.6721
4.0000	33.2484 (11.21)	16.0890 (57.03)	37.4444
4.5000	35.8719 (25.67)	15.3930 (68.10)	48.2577
5.0000	60.1243 (02.51)	13.0792 (78.79)	61.6739

Table 2.2 Comparison of Costs with respect to Jack et al. (2009)

Note: Bracketed figures in boldface, are percentage savings in average cost relative to Jack *et al.* [40] strategy of atmost one replacement.

Relative Cost Savings Percentage =
$$100 \times \left(\frac{\text{Cost of Strategy } i}{\text{Cost of Jack } et al. Strategy} - 1\right); \quad i = 1, 2.$$



Figure 2.5 (Γ region for strategies 1 and 2 when $C_r = 2$ and $\beta = 2$.

The two axes are total usage (u) level and age (x) respectively. The maximum usage limit (U) is 2 (× 10000 miles) and the warranty period (W) is 2 years. It can be seen that the region Γ obtained from numerical computation is similar to Figure 2.1.



Figure 2.6 The plot of δ_y versus y for different values of C_r and β .

As C_r increases the value of δ_y^* decreases given y. Intuitively this makes sense because if the cost of replacement (C_r) increases, the cost of imperfect repair $C_i(\delta_y, x)$ which is a function of C_m , C_r and δ_y increases and can be controlled by reducing the value of δ_y .



Figure 2.7 Plots of (δ_y^*, C_r, y) when $\beta = 2$.

As usage rate y increases, the number of failures increase (due to AFT model) resulting in higher expected cost. Thus, to reduce the item's hazard rate (or, number of failures) the manufacturer provides higher degree of imperfect repairs δ_y^* . Also, if replacement cost (C_r) increases, the expected cost increase and can be reduced by choosing smaller degree of repairs δ_y^* , justifying the decrease in δ_y^* for higher C_r .

2.3.7 Qualitative Interpretation of Results

- 1. Strategy 2 is more cost-effective compared to Strategy 1, since maintaining a setup that can execute any degree of repair $\delta_y(x)$, $K_y \leq x \leq L_y$ is more expensive compared to the fixed δ_y case.
- 2. But Strategy 1 is more consumer friendly in the sense that the degree of repair being dependant on age has a greater appeal to the customer and signals higher reliability of the item after repair.
- 3. Finally, for Strategy 2, it can be seen that when y is large enough (≥ 1.6), the optimal δ_y^* is 1 (equivalent to replacement), since any repair of degree less than 1 will not result in the minimization of the total warranty cost.
- 4. The Gamma regions obtained in Figure 2.5 show some interesting feature of the cost model,
 - if y, is relatively high, depending on the behavior of $\delta_y(x)$ or, δ and the costs C_r , C_m , the length of the middle interval $[K_y^*, L_y^*]$ vary for every y.
 - if age at failure in [K_y*, L_y*] is comparatively less, then the length of the middle interval is relatively longer, since at this stage an imperfect repair is worth the cost given the early age of the item.
 - for older ages in warranty the length of the interval $[K_y^*, L_y^*]$ decreases, since there is no point of imperfect repair and a minimal repair will be an optimal strategy.

Thus, these features contribute to the shape of the Gamma region as shown in Figure 2.5.

2.4 Concluding Remarks

Our proposed servicing strategy extends the work of Jack *et al.* [40] by introducing at most one imperfect repair in the middle interval. Since a replacement is costlier than a repair; the manufacturer/warranty provider has a natural incentive to do repairs rather than a replacement. Under standard degradation assumption such as increasing failure rate (IFR), the post repair reliability of the costliest (replacement) option is the highest and that of the cheapest (minimal repair) option is the lowest. However, practising an imperfect repair in the middle interval will reduce the expected warranty cost relative to replacements without completely trading-off the reliability of the item.

While minimal repairs only will be clearly optimal from a purely cost minimization perspective, and a replacement only policy the costliest one, choosing an intermediate degree of repair optimally in a framework with a built-in provision for such choices for the first failure if any within an intermediate age-bracket allows for a reasonable trade-off between costs and post-repair reliability.

CHAPTER 3

ANALYSIS OF A 2-D WARRANTY SERVICING STRATEGY WITH A BROWN-PROSCHAN TYPE REPAIR OPTION

3.1 Background and Motivation

In Chapter 2, we considered a servicing strategy with several minimal repairs and one imperfect repair and analyzed the expected warranty cost. The concept of an imperfect repair is tempting to the manufacturer since it reduces cost considerably. But maintaining a servicing set up (e.g. a team of servicemen with different skill levels) which enables any percentage of restoration to the item say 30% or 60% or 90%, is not practically feasible in most cases. Especially for small scale producers, the servicing cost might exceed the total revenue resulting in losses to maintain such varying facility. Therefore to reduce the hurdle of servicing and make the warranty servicing easily conductible, we consider two easily amendable degree of restoration i.e., 0% (minimal repair) and 100% (replacement), and probabilistically assign them if the item fails during the warranty.

A new 2-D warranty servicing strategy in the above set up is considered. We demonstrate the modeling, analysis and optimization of total expected costs accompanied by a numerical illustration with Weibull failure model. After a brief recap of the 2-D warranty setup and the accelerated failure time (AFT) formulation that reflects the role of usage rate, Section 3.2 comprises the proposed servicing strategy. In Sections 3.3, 3.4 and 3.5, we separately investigate the cases where probability of a replacement is constant, or dependent on age; with corresponding model analysis, optimization, numerical illustration and conclusions. The idea of randomizing the choice of repair options between replacements and minimal repairs were originally suggested by Brown and Proschan [15], and by Block, Borges and Savits [13], who

explored the resulting failure processes generated by such repairs, but did not investigate them in warranty contexts.

3.1.1 The 2-D Warranty Model

We assumes that for a given customer, the usage rate Y is constant. Conditional on Y = y, the total usage u of an unit at age x is thus

$$u = yx, \qquad 0 \le u < \infty$$

3.1.2 Modeling Failures

The distribution of failure time conditional on a customer's usage rate y is the appropriate distribution to model an unit's failures, with corresponding conditional hazard rate h(x; y) at age x.

3.1.2.1 Modeling First Failure. We use an 'Accelerated Failure Time (AFT) model' ([74],[11]) to describe the impact of a given usage rate y on the unit's time to failure. If y_0 (y, respectively) represent the *nominal* (typical, resp.) usage rate with corresponding time to failure T_0 (T_y , resp.); then the standard AFT model postulates,

$$\frac{T_y}{T_0} = \left(\frac{y_0}{y}\right)^{\gamma},$$

where $\gamma \geq 1$ is the acceleration parameter. If $F(\cdot; \alpha_0)$ with a scale parameter α_0 denote the baseline CDF of T_0 , then the accelerated failure time T_y has CDF $F(\cdot; \alpha(y))$ with scale parameter given by

$$\alpha(y) = \left(\frac{y_0}{y}\right)^{\gamma} \alpha_0,$$

and conditional hazard rate $h(\cdot; \alpha(y))$. Note $\alpha(y_0) = \alpha_0$.

3.1.2.2 Modeling Subsequent Failures. For a repairable product, the subsequent failures depend on the type of rectification action carried out. Under minimal

We further assume

- 1. All item failures are detected immediately and result in immediate claims by the consumer.
- 2. All claims are valid and must be rectified by the manufacturer immediately through repairs.
- 3. Repair and replacement times are small relative to the mean time between item failures and therefore can be ignored.
- 4. For the duration of the warranty, no separate preventive maintenance except those (if any), that are built in with the warranty, is carried either by the manufacturer or by the consumer.
- 5. The product's hazard rate function $h(\cdot; \alpha(y))$ is monotone non-decreasing in its age and usage rate.

3.1.2.3 Warranty Policy and Coverage. Consider a repairable item sold with a 2-D non-renewing free replacement warranty of period W and maximum usage limit U. Thus the 2-D warranty region is the rectangle $[0, W) \times [0, U)$. Given y, the usage sensitive warranty expires when the item currently in use reaches an age $W_y = min(W, \frac{U}{y})$.
3.2 Proposed 2-D Warranty Servicing Strategy with a Brown-Proschan Repair Option

Jack *et al.* [40] have considered a 2-D warranty servicing strategy using minimal repairs and at most one replacement. Such a strategy is described via three disjoint intervals $[0, K_y), [K_y, L_y)$ and $[L_y, W_y)$ with $0 \le K_y \le L_y \le W_y$, along the age (time) scale where failures in $[0, K_y)$ undergo only minimal repairs; the first failure in $[K_y, L_y)$ if any, rectified by a replacement and all subsequent failures therein, and in $[L_y, W_y)$ are repaired minimally. For a given usage rate y, the optimal values of the parameters K_y^* and L_y^* minimize the expected warranty servicing cost. As y varies, the set of points (K_y^*, L_y^*) defines a closed region, analogous to Jack *et al.* [40].

Our work described here, is an attempt to extend the model and analysis of 2-D warranties by allowing a warranty servicing action, henceforth referred to as *Brown-Proschan repair*, which randomizes the choice of restoration between a replacement or, minimal repair with a probability p and (1-p) respectively, that was first introduced by Brown and Proschan [15], although not in the cost of warranty servicing context. The servicing strategy we consider and analyze, can be described as follows:

With the warranty period partitioned into three intervals as described at the beginning of this Section; the first failure (if any) in the middle interval $[K_y; L_y)$ undergoes a 'Brown-Proschan repair'; all other failures undergo minimal repair.

It is clear that our strategy reduces to the strategy of Jack *et al.* [40] when p = 1 and that of minimal repairs when p = 0. Pragmatically however, there may be practical reasons to choose a *Brown-Proschan repair* with 0 ; e.g., consider a repair crew, each with one of two skill levels (minimal, as-good-as-new) in proportions <math>(1 - p, p) respectively, to whom repair jobs are assigned randomly. Such randomized assignments will result in a total expected servicing cost bracketed between the corresponding costs of the strategy of minimal repairs only and that of Jack *et al.* [40].

Intuition suggests that higher the chance of choosing a replacement, smaller should be the total expected cost of servicing the warranty (see e.g., Tables 3.2 and 3.3), under reasonable degradation profiles of the unit's failure time and replacement versus minimal repair cost ratio.

3.3 Repair Strategy with Constant Probability of Replacement

Let us consider that the probability p of Brown-Proschan repair is fixed irrespective of the time to first failure after age K_y . Clearly this assumption simplifies the analysis of the strategy since the manufacturer need to determine of a single probability p as oppose to a spectrum of degrees of repairs (δ) as in the previous chapter. This inherent simplicity which can be easily understood and be appealing to a warranty provider, is an argument in its favor as a realistic model apart from pragmatic justification of utilizing repair crews with different skill levels, as mentioned in the previous Section.

3.3.1 Model Formulation

Our objective here is to model the expected warranty servicing cost denoted by $J(K_y, L_y, p)$ for a given usage rate y, and find the optimal values of the parameters that minimize the cost. Let C_m denote the cost of a minimal repair and C_r (> C_m) denote the cost of a replacement. For a given usage rate y, let T_1 denote the time of the first failure under usage rate y after age K_y . The conditional CDF of T_1 is given by

$$F_1(t;\alpha(y)) = \frac{F(t;\alpha(y)) - F(K_y;\alpha(y))}{\overline{F}(K_y;\alpha(y))}.$$
(3.1)

All failures over $[0, K_y)$ are minimally repaired, so the failures occur according to an non-homogeneous poisson (NHPP) process with conditional intensity function $h(x; \alpha(y))$ and the conditional expected warranty servicing cost for this interval is given by

$$C_m \int_0^{K_y} h(x; \alpha(y)) dx = C_m H(K_y; \alpha(y)),$$

where $H(x; \alpha(y))$ is the cumulative hazard function at age x. For failures occurring after age K_y we need to consider two cases:

(1)
$$K_y \leq T_1 = x \leq L_y$$
 and (2) $T_1 = x > L_y$

The conditional expected cost, conditional on $K_y \leq T_1 \leq L_y$, is obtained as follows. The first failure in $[K_y, L_y)$ occurs at age $T_1 \equiv t$ and is either replaced with probability p or, minimally repaired with probability (1 - p). All failures over the remaining interval $(t, W_y]$ are minimally repaired. The expected cost function of servicing failures over $[t, W_y]$ is given by

$$p[C_r + C_m \int_t^{W_y} h(x - t; \alpha(y)) dx] + (1 - p)C_m [1 + \int_t^{W_y} h(x; \alpha(y)) dx]$$

= $p[C_r + C_m H(W_y - t; \alpha(y))] + (1 - p)C_m [1 + H(W_y; \alpha(y)) - H(t; \alpha(y))]$

As a result, the expected warranty cost over the intervals $[K_y, L_y)$ and $[L_y, W_y]$ for usage rate y conditioned on $K_y \leq T_1 \leq L_y$ is given by

$$J(K_y, L_y, p | K_y \le T_1 \le L_y) = \int_{K_y}^{L_y} \left[p\{C_r + C_m H(W_y - t; \alpha(y))\} + (1 - p)C_m \{1 + H(W_y; \alpha(y)) - H(t; \alpha(y))\} \right] \frac{f(t; \alpha(y))}{\bar{F}(K_y; \alpha(y))} dt.$$

The expected cost, conditioned on $T_1 > L_y$, is obtained as follows. Note that, in this case, there is no failure in $[K_y, L_y]$ and failures over the remaining interval $(L_y, W_y]$ occur according to an NHPP with intensity function $h(t; \alpha(y))$, for $L_y \leq T_1 \leq W_y$. Therefore, the conditional expected warranty cost given by $T_1 > L_y$ is,

$$J(K_y, L_y, p|T_1 > L_y) = C_m[H(W_y; \alpha(y)) - H(L_y; \alpha(y))]$$

By removing the conditioning on T_1 using Equation (3.1), the total expected warranty servicing cost for a given usage rate y, is therefore

$$J(K_{y}, L_{y}, p) = C_{m}H(K_{y}; \alpha(y)) + \int_{K_{y}}^{L_{y}} \left[p\{C_{r} + C_{m}H(W_{y} - t; \alpha(y))\} + (1 - p)C_{m}[1 + H(W_{y}; \alpha(y)) - H(t; \alpha(y))] \right] \frac{f(t; \alpha(y))}{\bar{F}(K_{y}; \alpha(y))} dt + C_{m}[H(W_{y}; \alpha(y)) - H(L_{y}; \alpha(y))] \frac{\bar{F}(L_{y}; \alpha(y))}{\bar{F}(K_{y}; \alpha(y))}.$$
(3.2)

3.3.2 Model Analysis and Optimization

We assume the probability $p \in [0, 1]$ is known. Hence, the optimization problem $\min_{K_y, L_y} J(K_y, L_y)$ involves selecting the optimal K_y^* and L_y^* for a given y (subject to the constraints $0 \leq K_y \leq L_y \leq W_y$). We obtain this using a two-stage approach. In stage 1, for a fixed K_y , obtain the optimal $L_y^*(K_y)$ that minimizes $J(K_y, L_y)$. Then, in stage 2, we obtain the optimal K_y^* by minimizing $J(K_y, L_y^*(K_y))$. Thus for a fixed K_y , the optimal $L_y^*(K_y)$ can be obtained from the first order condition $\frac{\partial}{\partial L_y}J(K_y, L_y) = 0$; i.e.,

$$pC_m\xi_y(L_y)\frac{f(L_y;\alpha(y))}{\bar{F}(K_y;\alpha(y))} = 0,$$
(3.3)

where

$$\xi_{y}(t) = \frac{C_{r}}{C_{m}} - 1 + H(W_{y} - t; \alpha(y)) - H(W_{y}; \alpha(y)) + H(t; \alpha(y))$$

= $\frac{C_{r}}{C_{m}} - 1 - g(t)$, and (3.4)

$$g(t) := H(W_y; \alpha(y)) - H(t; \alpha(y)) - H(W_y - t; \alpha(y)).$$
(3.5)

Since $C_m \in (0, \infty)$, $p \in (0, 1]$ and $\frac{f(L_y; \alpha(y))}{\bar{F}(K_y; \alpha(y))} > 0$ (as K_y and L_y are in the support of $f(\cdot; \alpha(y))$); Equation (3.3) reduces to $\xi_y(L_y) = 0$.

Note the optimal $L_y^*(K_y) \equiv L_y^*$ does *not* depend on the probability *p*.

Finally the optimum K_y^* is obtained by solving

$$\frac{\partial}{\partial K_y}J(K_y, L_y^*(K_y)) = 0.$$

Unlike L_y^* , it can be seen that the optimal $K_y^*(p) \equiv K_y^*$ does depend on p. We have used computational approach to find the optimal values of K_y^* and L_y^* .

3.3.3 Special Case: Weibull Failure Distribution

Let the time to first failure under the nominal usage rate y_0 denoted by T_0 follow a Weibull distribution with scale parameter $\alpha_0 > 0$ and shape parameter $\beta > 1$, i.e., $\overline{F}(x; \alpha_0) = 1 - F(x; \alpha_0) = \exp(-\frac{x}{\alpha_0})^{\beta}$. Using the corresponding AFT model, the survival function, hazard function and cumulative hazard function for T_y , the time to first failure under the usage rate y can be derived as follows,

$$\overline{F}(x;\alpha(y)) = 1 - F(x;\alpha(y)) = \exp\left(-\left(\frac{y}{y_0}\right)^{\gamma} \frac{x}{\alpha_0}\right)^{\beta},$$
$$h(x;\alpha(y)) = \beta\left(\frac{y}{y_0}\right)^{\gamma\beta}\left(\frac{x^{\beta-1}}{\alpha_0^{\beta}}\right), \text{ and}$$
$$H(x;\alpha(y)) = \left(\frac{y}{y_0}\right)^{\gamma\beta}\left(\frac{x^{\beta}}{\alpha_0^{\beta}}\right).$$

Finally we derive the special forms of Equations (3.2)-(3.4), and compute the optimal values K_y^*, L_y^* with the corresponding minimal cost $J(K_y^*, L_y^*, p)$, given in Tables 3.1 and 3.2, respectively.

3.4 Age Dependent Probability of Replacement (Block-Borges-Savits) Strategy: For Some Generic Function p(t)

If we are to exercise a choice between a replacement vs. a minimal repair probabilistically, Block, Borges and Savits [13], henceforth referred to as BBS, suggest that the probability of such choices should depend on the item's age at failure. Let p(t) (and (1 - p(t)), respectively) denote the probability of choosing a replacement (minimal repair, respectively) of an item that fails at age $t (\geq K_y)$. Consistent with their suggestion for choosing an age-dependent repairs, we will refer to the corresponding warranty servicing policy of choosing a replacement or minimal repair in the middle interval $[K_y, L_y)$ as the BBS Strategy. Note that the technical condition

$$\int_0^\infty p(t)\Lambda(dt) = +\infty,$$

where Λ denote the hazard measure induced by a new item's life-distribution, which ensures the finiteness with probability one of the sojourn-time between consecutive replacements with a random number of minimal repairs in between when the BBS repair option is repeatedly exercised at each failure, is inapplicable in out context, which exercises such a choice only once and then switches to minimal repairs at all subsequent failures for the duration of the warranty. The choice of the randomizing function p(t), is thus in principle, quite arbitrary subject only to the constraint that it is a mapping of the half-line $[0, \infty)$ into [0, 1].

In our view, realistic models of the probability p(t) should be chosen in a way that reflects the extent of the failed item's degradation (at age t of failure) and our attitude towards the potential usefulness of a replacement versus minimal repair. For example, a monotone increasing choice of p(t) may be appropriate for items whose replacements are increasingly important with increasing age at failure from a mission critical or, safety perspective, when the items degradation profile has an increasing hazard (failure) rate function. Conversely, functions p(t) which are monotone decreasing may be relevant, if *either*, the item has a decreasing failure rate (DFR), *or*, we are in a situation where replacements are much more costly than minimal repairs and the working unit's degradation status is unimportant.

One can also imagine other scenarios where a bell-shaped unimodal choice of p(t) would be a reasonable model to pursue. The odds of a minimal repair versus replacement is the ratio $\{1 - p(t)\}/p(t)$, which can be constructed for practical purposes to favor minimal repair (replacements, respectively) if p(t) < (>, respectively) p_0 , where p_0 is an externally specified threshold. A bell-shaped unimodal p(t) together with a specified threshold p_0 would partition the warranty duration into three intervals:

an 'early', 'intermediate' and 'late' phases when the item's age is considered 'young', 'mature' and 'old' respectively.

Consider an equipment degrading with increasing failure rate (IFR). If it fails in an 'early' phase, the hazard rate value at the time of failure is relatively low and likely to be close to the value of the hazard rate when the item was new (age zero). In such cases the potential gains of a replacement due to a stochastically larger lifetime to the next failure compared to that under a minimal repair may not be worthwhile depending on the ratio of replacement versus minimal repair costs. A bell-shaped unimodal p(t) which is increasing in the 'early' phase would correspondingly indicate odds in favor of a minimal repair, and thus endorse the previous argument for a preference of minimal repair over replacement due to small values of p(t) in the early phase.

Similarly, for failures that occur during the 'late' phase, an increasing hazard rate (IFR) assumption implies that although its value at the point of failure is already high, a replacement is possibly not cost effective since we are nearing the end of the warranty duration. The resulting preference of minimal repairs during the 'late' phase is likewise endorsed by reasonable choices of bell-shaped unimodal p(t) which would be decreasing during the 'late' phase.

Finally, if a working item has reached the 'intermediate' phase in age, although the item has progressively deteriorated to a certain extent, the trade-offs between time to next failure and costs of minimal repair versus replacement is less clear. A bell-shaped unimodal p(t) such that $\max_t p(t)$ occurs at a point t in the 'intermediate' phase would schemate capture such situations.

In this Section we consider some specific functions p(t) in $[K_y, L_y]$ and investigate the behavior of costs using a computational approach. It can be noted that the forms of function p(t) considered here depicts the degradation profile of the product. The three probability functions considered for illustrative purposes, are the following:



Figure 3.1 Plots of the probability functions for $C_r = 2$, $C_m = 1$, $K_y = 0$, $L_y = 2$ and a = b = 2.

- (i) If $p_1(t) = 1 e^{-t}$, $t \in [K_y, L_y]$ i.e., the probability function p(t) is increasing in time to first failure t after age K_y . Such a function is relevant for items with increasing failure rates (IFR) distributions, which are prone to failures at later ages. In this case as the operating time increase, the rate of failures and the probability of replacement is simultaneously increased. Also when t is relatively small the products hazard rate is low and expected number of failures is less compare to later ages, and there is no point in replacement at an early age. Whereas at later ages if a replacement is performed, it will not only reduce the hazard rate but also the expected number of failures over the warranty term resulting in minimal cost.
- (ii) If $p_2(t) = \frac{1}{(C_r C_m) + t}$, $t \in [K_y, L_y]$ i.e., the probability function p(t) is decreasing in time to first failure t after age K_y . Such a function is relevant for items with decreasing failure rate (DFR) distributions, which are prone to failures at early ages. Here as the operating time increase, the rate of failures and the probability of replacement simultaneously decrease. Also when t is relatively less the products hazard rate is high and expected number of failures is more

compare to later ages. Thus an early replacement will reduce the total number of failures and minimize the expected cost over the warranty period.

There are obviously many choices for p(t) decreasing, e.g., $p(t) = [(L_y - t)/(L_y - K_y)]^a$, $K_y \leq t \leq L_y$, a > 0, decreases from 1 to 0 as the item's age increases from K_y to L_y . The shape of p(t) over $[K_y, L_y]$ is determined by the parameter $a \in [0, \infty)$, viz. p(t) is concave for 0 < a < 1, linear for a = 1, strictly convex if a > 1, and constant (equivalent to Brown-Proschan repair, Section 3.2-3.3) if a = 0.

(iii) If $p_3(t) = (t - K_y)^{a-1}(L_y - t)^{b-1}$, $t \in [K_y, L_y]$, a > 1, b > 1, i.e., the probability function p(t) is a unimodal function of time to first failure t after age K_y . In this case the item's hazard rate initially increase and then decrease, as a result the probability of replacement is low when the item is relatively young or old. A replacement near the peak of the hazard rate (determined by the parameters a and b) will reduce the expected number of failures resulting in minimal cost. For computational purposes we have considered a = 2 and b = 2.

Table 3.3 in Section 3.6 shows the results of computation with a Weibull failure distribution model.

3.5 Age Dependent Probability of Replacement (Block-Borges-Savits) Strategy: For General Function p(t)

Finally, let us suppose that the probability p(t) of replacement in $[K_y, L_y]$ is an unknown function of time to first failure $t \in [K_y, L_y]$. Our objective in this case to optimally derive the function p(t), $t \in [K_y, L_y]$ along with K_y^* and L_y^* such that the expected warranty cost is minimum.

3.5.1 Model Formulation

The model formulation is similar to the constant probability case Section 3.3, except p is replaced by p(t). The expected warranty servicing cost for a given usage rate y, is

$$J(K_{y}, L_{y}, p(t)) = C_{m}H(K_{y}; \alpha(y)) + \int_{K_{y}}^{L_{y}} \left[p(t)\{C_{r} + C_{m}H(W_{y} - t; \alpha(y))\} + (1 - p(t))C_{m}[1 + H(W_{y}; \alpha(y)) - H(t; \alpha(y))] \frac{f(t; \alpha(y))}{\overline{F}(K_{y}; \alpha(y))} dt + C_{m}[H(W_{y}; \alpha(y)) - H(L_{y}; \alpha(y))] \frac{\overline{F}(L_{y}; \alpha(y))}{\overline{F}(K_{y}; \alpha(y))},$$

$$= \frac{C_{m}}{\overline{F}(K_{y}; \alpha(y))} \left[(H(K_{y}; \alpha(y)) + H(W_{y}; \alpha(y)) + 1)\overline{F}(K_{y}; \alpha(y)) - \int_{K_{y}}^{L_{y}} H(t; \alpha(y))f(t; \alpha(y))dt - (H(L_{y}; \alpha(y)) + 1)\overline{F}(L_{y}; \alpha(y)) + \int_{K_{y}}^{L_{y}} p(t) \left\{ \frac{C_{r}}{C_{m}} - 1 - g(t) \right\} f(t; \alpha(y))dt \right],$$

$$= \Psi(K_{y}, L_{y}) + \Phi(K_{y}, L_{y}, p(t)), \qquad (3.6)$$

with

$$\Psi(K_{y}, L_{y}) := \frac{C_{m}}{\bar{F}(K_{y}; \alpha(y))} \Big[(H(K_{y}; \alpha(y)) + H(W_{y}; \alpha(y)) + 1) \bar{F}(K_{y}; \alpha(y)) \\ - \int_{K_{y}}^{L_{y}} H(t; \alpha(y)) f(t; \alpha(y)) dt - (H(L_{y}; \alpha(y)) + 1) \bar{F}(L_{y}; \alpha(y)) \Big],$$
(3.7)

$$\Phi(K_y, L_y, p(t)) := \frac{C_m}{\bar{F}(K_y; \alpha(y))} \int_{K_y}^{L_y} p(t) \Big\{ \frac{C_r}{C_m} - 1 - g(t) \Big\} f(t; \alpha(y)) dt,$$
(3.8)

where

$$g(t) := H(W_y; \alpha(y)) - H(t; \alpha(y)) - H(W_y - t; \alpha(y)).$$

(labeled earlier as Equation (3.5)).

3.5.2 Model Analysis and Optimization

The optimization problem $\min_{K_y, L_y, p(t)} J(K_y, L_y, p(t))$ involves two stages. At the first stage is for fixed y, K_y and L_y , select the function $p^*(t)$ that minimizes $J(K_y, L_y, p^*(t))$. At the second stage, for fixed y the optimal K_y^* and L_y^* are obtained (subject to the constraints $0 \le K_y \le L_y \le W_y$) such that $J(K_y^*, L_y^*, p^*(t))$ is minimum.

Stage 1:

Clearly, the contribution of the probability p(t) of practising a replacement (perfect repair) in $(K_y, L_y]$ on the expected cost is captured by $\Phi(K_y, L_y, p(t))$ in Equation (3.8). Thus, the optimal choice $p^*(t)$ of the age-dependent replacement probability p(t) can be obtained by studying $\Phi(K_y, L_y, p(t))$.

Proposition 3.5.1 Suppose x_1 and $x_2(>x_1)$ are the roots of the equation $\frac{C_r}{C_m} - 1 - g(t) = 0$, where g(t) is given in Equation (3.5).

CASE A

If $1 + g\left(\frac{W_y}{2}\right) > \frac{C_r}{C_m}$, then the expected warranty servicing cost is minimized by the following choices of $p^*(t)$, depending on the positions of the roots x_1, x_2 relative to K_y and L_y .

$$(a) If 0 \le K_y < L_y \le x_1, \text{ then } p^*(t) = 0 \text{ for all } t \in (K_y, L_y].$$

$$(b) If K_y \le x_1 < L_y \le x_2, \text{ then } p^*(t) = \begin{cases} 0, t \in (K_y, x_1], \\ 1, t \in (x_1, L_y]. \end{cases}$$

$$(c) If K_y \le x_1 < x_2 \le L_y, \text{ then } p^*(t) = \begin{cases} 0, t \in (K_y, x_1] \cup [x_2, L_y] \\ 1, t \in (x_1, x_2). \end{cases}$$

$$(d) If x_1 \le K_y < L_y \le x_2, \text{ then } p^*(t) = \begin{cases} 1, t \in (K_y, L_y] \\ 0, o.w. \end{cases}$$

$$(e) If x_1 \le K_y < x_2 \le L_y, \text{ then } p^*(t) = \begin{cases} 1, t \in (K_y, x_2] \\ 0, t \in (x_2, L_y]. \end{cases}$$

$$(contd. after Figure 3.2)$$



Figure 3.2 Graphs of the functions $\rho := C_r/C_m$, 1 + g(t) versus age t.

CASE A: (a)
$$K_y < L_y \le x_1$$
, (b) $K_y \le x_1 < L_y \le x_2$, (c) $K_y \le x_1 < x_2 \le L_y$, (d)
 $x_1 \le K_y < L_y \le x_2$ (e) $x_1 \le K_y < x_2 \le L_y$, (f) $x_2 \le K_y < L_y \le W_y$,
CASE B: (g) $1 + g\left(\frac{W_y}{2}\right) < \frac{C_r}{C_m}$, (h) $1 + g\left(\frac{W_y}{2}\right) = \frac{C_r}{C_m}$.

 $(\underline{CASE A} \ contd.)$

(f) If
$$x_2 \le K_y < L_y \le W_y$$
, then $p^*(t) = 0$ for all $t \in (K_y, L_y]$

The corresponding optimal values of K_y^* and L_y^* are obtained as follows: $K_y^* \in \left[0, \frac{W_y}{2}\right)$ is the smaller root of the equation

$$\bar{F}(K_y;\alpha(y)) - \bar{F}(L_y;\alpha(y)) - \int_{K_y}^{L_y} h(W_y - t;\alpha(y))\bar{F}(t;\alpha(t))dt = 0;$$

and $L_y^* \in \left(\frac{W_y}{2}, W_y\right]$ is the larger root of the equation

$$\frac{C_r}{C_m} - 1 - g(L_y) = 0.$$

CASE B

If $1+g\left(\frac{W_y}{2}\right) \leq \frac{C_r}{C_m}$, then $p^*(t) = 0$ for all $t \in (K_y, L_y]$ and the corresponding $K_y^* = L_y^*$, i.e., the optimal strategy is to carryout minimal repair throughout the warranty period. Remark: If the roots of the equation $\frac{C_r}{C_m} - 1 - g(t) = 0$ are equal, the strategy reduces to the 'minimal repairs only' strategy.

<u>Proof of CASE A</u>: Equation (3.6) above shows that, the expected cost is the sum of two components Φ and Ψ . It is easy to note that, in the integral equation

$$\Phi(K_y, L_y, p(t)) = C \int_{K_y}^{L_y} p(t) \Big\{ \frac{C_r}{C_m} - 1 - g(t) \Big\} f(t; \alpha(y)) dt$$

constant $C := \frac{C_m}{\bar{F}(K_y; \alpha(y))} > 0, \ p(t) \ge 0, \ f(t; \alpha(y)) \ge 0$ for all t, the expression g(t)in (3.5) is concave (assuming $h'(t; \alpha(y))$ exists) and symmetric about $t = W_y/2$, with maximum value

$$\max_{t} \quad g(t) = g\left(\frac{W_y}{2}\right) = H(W_y; \alpha(y)) - 2H\left(\frac{W_y}{2}; \alpha(y)\right).$$

Therefore, sign of the function $\frac{C_r}{C_m} - 1 - g(t)$ depends on the cost ratio $\frac{C_r}{C_m}$ and age t. Further simplification of Equation (3.7) gives

$$\Psi(K_y, L_y) = C_m H(W_y; \alpha(y)) > 0$$

(same as the expected cost of minimal repair only strategy), since, integrating by parts we get,

$$\int_{K_y}^{L_y} H(t;\alpha(y))f(t;\alpha(y))dt = H(K_y;\alpha(y))\bar{F}(K_y;\alpha(y)) - H(L_y;\alpha(y))\bar{F}(L_y;\alpha(y)) + F(L_y;\alpha(y)) - F(K_y;\alpha(y)).$$

And $\Phi(K_y, L_y, p(t)) < \Psi(K_y, L_y)$ for all p(t), K_y and L_y . Thus a smaller $\Phi(K_y, L_y, p(t))$ will reduce the expected cost. So our objective is to choose optimally the function $p^*(t) \in [0, 1]$ such that the resultant $\Phi(K_y, L_y, p^*(t)) (< 0)$ is as small as possible. The following are the different possibilities of choosing $p^*(t)$.

Case(1a): If $0 \le K_y < L_y \le x_1$ as shown in Figure 3.2(a), then $\frac{C_r}{C_m} - 1 - g(t) > 0$ for all $x \in (K_y, L_y]$, therefore the value of the integral $\Phi(K_y, L_y, p(t))$ is minimum iff $p^*(t) = 0$ for all $t \in (K_y, L_y]$.

Case(1b): If $K_y \leq x_1 < L_y \leq x_2$ as shown in Figure 3.2(b), then

$$\frac{C_r}{C_m} - 1 - g(t) > 0, \quad t \in (K_y, x_1]$$
$$\leq 0, \quad t \in (x_1, L_y]$$

so, the corresponding probability that minimizes the total expected cost $J(K_y, L_y, p(t))$ is

$$p^*(t) = \begin{cases} 0, & x \in (K_y, x_1], \\ 1, & x \in (x_1, L_y]. \end{cases}$$

Case(1c): If $K_y \leq x_1 < x_2 \leq L_y$ as shown in Figure 3.2(c), then

$$\frac{C_r}{C_m} - 1 - g(t) \ge 0, \quad t \in (K_y, x_1] \cup [x_2, L_y]$$
$$< 0, \quad t \in (x_1, x_2)$$

thus, the corresponding probability that minimizes the total expected cost $J(K_y, L_y, p(t))$ is

$$p^*(t) = \begin{cases} 0, & t \in (K_y, x_1] \cup [x_2, L_y] \\ 1, & t \in (x_1, x_2). \end{cases}$$

Case(1d): If $x_1 \leq K_y < L_y \leq x_2$ as shown in Figure 3.2(d), then

$$\frac{C_r}{C_m} - 1 - g(t) < 0, \quad t \in (K_y, L_y]$$

hence, the corresponding probability that minimizes the total expected cost $J(K_y, L_y, p(t))$ is

$$p^{*}(t) = \begin{cases} 1, & t \in (K_{y}, L_{y}] \\ 0, & o.w. \end{cases}$$

Case(1e): If $x_1 \leq K_y < x_2 \leq L_y$ as shown in Figure 3.2(e), then

$$\frac{C_r}{C_m} - 1 - g(t) < 0, \quad t \in (K_y, x_2]$$
$$\geq 0, \quad t \in (x_2, L_y]$$

therefore, the corresponding probability that minimizes the total expected cost $J(K_y, L_y, p(t))$ is

$$p^*(t) = \begin{cases} 1, & t \in (K_y, x_2] \\ 0, & t \in (x_2, L_y]. \end{cases}$$

Case(1f): If $x_2 \leq K_y < L_y \leq W_y$ as shown in Figure 3.2(f), then $\frac{C_r}{C_m} - 1 - g(t) > 0$, $x \in (K_y, L_y]$ therefore, the corresponding probability that minimizes the total expected cost $J(K_y, L_y, p(t))$ is $p^*(t) = 0$ for all $t \in (K_y, L_y]$.

Stage 2:

At this stage for a fixed K_y , the optimal $L_y^*(K_y)$ can be obtained from the first order

condition $\frac{\partial}{\partial L_y} J(K_y, L_y) = 0$; i.e.,

$$p(L_y)C_m\xi_y(L_y)\frac{f(L_y;\alpha(y))}{\bar{F}(K_y;\alpha(y))} = 0,$$
(3.9)

where $\xi_y(t) = \frac{C_r}{C_m} - 1 - g(t)$.

Since $C_m \in (0, \infty)$, $p(L_y) \in (0, 1]$ and $\frac{f(L_y;\alpha(y))}{F(K_y;\alpha(y))} > 0$ (as K_y and L_y are in the support of $f(\cdot; \alpha(y))$); Equation (3.9) reduces to $\xi_y(L_y) = 0$. Thus the optimal L_y^* depends on the behavior of $\xi_y(L_y)$ for $K_y \leq L_y \leq W_y$ as shown in the following subcases.

2(i) If
$$1 + g\left(\frac{W_y}{2}\right) \ge \frac{C_r}{C_m}$$
, then $\xi_y(L_y) \ge 0$ and $\frac{\partial}{\partial L_y} J(K_y, L_y) \ge 0$, $\forall L_y \in [K_y, W_y]$, so $L_y^*(K_y) = K_y, \forall K_y \in [0, W_y].$

- 2(ii) If $1 + g\left(\frac{W_y}{2}\right) < \frac{C_r}{C_m}$, then $\xi_y(L_y) = 0$ has two roots in the interval $[0, W_y]$, one at $L_1 \in [0, W_y/2)$ and the other at $L_2 \in (W_y/2, W_y]$:
 - (a) If $K_y \in [0, L_2]$, then $L_y \in [K_y, W_y]$ and $L_y^*(K_y) = L_2$, due to convexity of $\xi_y(L_y)$.

(b) If
$$K_y \in [L_2, W_y]$$
, then $L_y \in [K_y, W_y]$ and $L_y^*(K_y) = K_y$.

Note that the optimal $L_y^*(K_y) \equiv L_y^*$ does not depend on p(t).

Finally the optimum K_y^* is obtained as follows. If from the previous step in Stage 2, $L_y^*(K_y) = K_y$, $\forall K_y \in [0, W_y]$, then the optimal strategy is to carry out minimal repair throughout the warranty period and the expected warranty cost will be $C_m H(W_y; \alpha(y))$, i.e.,

$$J(K_y, L_y^*, p^*(t)) = \begin{cases} J(K_y, L_2, p^*(t)), & \text{if } 0 \le K_y \le L_2 \\ J(K_y, K_y, p^*(t)) & \text{if } L_2 < K_y \le W_y. \end{cases}$$
(3.10)

From Equation (3.6),

$$\begin{aligned} \frac{\partial}{\partial K_y} J(K_y, L_y^*(K_y), p^*(t)) &= C_m \frac{h(K_y; \alpha(y))}{\bar{F}(K_y; \alpha(y))} \Big[\int_{K_y}^{L_y} p^*(t) \xi_y(t) f(t; \alpha(y)) dt \\ &- p^*(K_y) \xi_y(K_y) f(K_y; \alpha(y)) \Big] \\ &= C_m \frac{h(K_y; \alpha(y))}{\bar{F}(K_y; \alpha(y))} \Big[\int_{K_y}^{L_y} \xi_y(t) f(t; \alpha(y)) dt \\ &- \xi_y(K_y) f(K_y; \alpha(y)) \Big], \quad \text{for} \quad p^*(t) = 1, \\ &= C_m \frac{h(K_y; \alpha(y))}{\bar{F}(K_y; \alpha(y))} \zeta(K_y), \end{aligned}$$

where

$$\begin{aligned} \zeta(K_y) &= \int_{K_y}^{L_y} H(W_y - t; \alpha(y)) f(t; \alpha(y)) dt - \bar{F}(L_y; \alpha(y)) + \bar{F}(K_y; \alpha(y)) \\ &+ H(W_y - L_y; \alpha(y)) - H(W_y - K_y; \alpha(y)) \\ &= \int_{K_y}^{L_y} \left[h(t; \alpha(y)) - h(W_y - t; \alpha(y)) \right] \bar{F}(t; \alpha(y)) dt. \end{aligned}$$

The value of K_y^* depends on the behavior of $\zeta(K_y)$ for $K_y \in [0, L_2]$. $\zeta(0) < 0, \zeta(L_2) = 0$ and $\zeta'(K_y) = h(K_y; \alpha(y)) - h(W_y - K_y; \alpha(y)) \overline{F}(K_y; \alpha(y))$, i.e., $\zeta(K_y)$ is increasing on $[0, W_y/2)$, decreasing on $(W_y/2, L_2]$ and attains a maximum at $K_y = W_y/2$. The equation $\zeta(K_y) = 0$ has two roots in the interval $[0, L_2]$ one at $K_1 \in [0, W_y/2)$ and other at $K_2 = L_2$. Thus the optimal value of K_y that minimizes cost is $K_y^* = K_1$.

Unlike L_y^* , it can be seen that the optimal K_y^* does depend on $p^*(t)$. We have used a computational approach to find optimal values K_y^* and L_y^* .

<u>Proof of CASE B</u>: In this case, $\Phi(K_y, L_y, p(t)) \ge 0$ and is clearly minimized if and only if $p^*(t) = 0$. Thus the optimal strategy reduces to the 'minimal repairs only' strategy and the corresponding $K_y^* = L_y^*$.

3.6 Numerical Illustration

Let the cost of minimal repair $C_m = 1$, cost of replacement (perfect repair) $C_r = 2$, warranty period W = 2 (years), total usage limit $U = 2(\times 10^4 \text{ km per year})$, Weibull baseline parameters $\alpha_0 = 1$, $\beta = 2$, nominal usage rate $y_0 = 1$ and the AFT model parameter $\gamma = 2$. In Table 3.1 and 3.2, we assign values 0 (minimal repairs), 0.2, 0.4, 0.6, 0.8 and 1 (replacement) to the constant probability of replacement p and compute the corresponding minimal costs $J(K_y^*, L_y^*)$ and optimal (K_y^*, L_y^*) for different y's. The figures in brackets are the percentage cost savings relative to the strategy of always minimal repair (i.e., p = 0). In Table 3.3, the behavior of expected warranty costs is demonstrated numerically by plugging in the different probability functions given in Section 3.4 in the expected cost model Equation (3.2).

The MATLAB program used to compute the optimal parameters (K_y^*, L_y^*) and the corresponding minimal costs $J(K_y^*, L_y^*)$ is included in Appendix A.

3.6.1 Qualitative Interpretation of Results

The results of numerical computation in Tables 3.1, 3.2 and 3.3 demonstrate the following features of the expected cost model. From Table 3.1, it can be seen that

- (i) For fixed usage rate (y), the total expected warranty servicing cost decreases as the probability of choosing a replacement increases.
- (ii) For a fixed probability p, the total expected warranty servicing cost is increasing in usage rate y as one intuitively expect.
- (iii) The corresponding percentage savings in costs relative to minimal repair only strategy (p = 0), typically has a maximum at some intermediate usage rate y, similar to the profile of cost savings of Jack *et al.* [40] (p = 1).
- (iv) For any function p, the expected warranty cost is bracketed between the costs of Jack *et al.* [40] (minimum) and minimal repair only strategy (maximum).

From Table 3.2, it can be seen that

- (i) The usage sensitive warranty period W_y is decreasing in usage rate y.
- (ii) The optimal K_y^* decreases to zero as usage rate y increase.
- (iii) The corresponding optimal L_y^* decreases to W_y as usage rate y increase.
- (iv) Unlike L_y^* , the optimal K_y^* is dependent on the probability of replacement p.

Finally, from Table 3.3, it can be seen that

- (i) Unlike L_y^* , the optimal K_y^* is dependent on the probability of replacement $p_i(t)$, i = 1, 2, 3.
- (ii) The expected warranty cost varies for different forms of $p_i(t)$, i = 1, 2, 3.
- (iii) The optimal $K_y(p_i(t))$ is decreasing in usage rate y.
- (iv) The expected warranty servicing cost is always bracketed between the costs of p(t) = 0 and p(t) = 1 strategy.

y	W_y	$K_y^*(p=0.2)$	$K_y^*(p = 0.4)$	$K_y^*(p=0.6)$	$K_y^*(p=0.8)$	$K_y^*(p=1)$	L_y^*
0.85	2.00	0.79	0.81	0.81	0.81	0.82	1.21
0.9	2.00	0.66	0.66	0.66	0.66	0.66	1.49
1.0	2.00	0.67	0.67	0.66	0.66	0.66	1.71
1.2	1.67	0.61	0.61	0.61	0.61	0.60	1.51
1.4	1.43	0.57	0.57	0.56	0.56	0.56	1.33
1.6	1.25	0.54	0.53	0.52	0.52	0.51	1.19
1.8	1.11	0.49	0.49	0.48	0.48	0.48	1.06
2.0	1.00	0.45	0.45	0.44	0.44	0.44	0.97
2.5	0.8	0.34	0.34	0.34	0.34	0.34	0.78
3.0	0.67	0.26	0.26	0.26	0.26	0.26	0.66
3.5	0.57	0.20	0.20	0.20	0.20	0.20	0.57
4.0	0.5	0.16	0.16	0.16	0.16	0.16	0.50
4.5	0.44	0.13	0.13	0.13	0.13	0.13	0.44
5.0	0.4	0.11	0.11	0.11	0.11	0.11	0.40
5.5	0.36	0.09	0.09	0.09	0.09	0.09	0.36
6.0	0.33	0.08	0.08	0.08	0.08	0.08	0.33

Table 3.1 Optimal Warranty Parameters (K_y^*, L_y^*) for Constant Replacement Probability p

Note, unlike K_y^* , optimal choices of L_y^* does <u>not</u> depend on the choice of probability of replacement p.

y	$J^*(p=0)$	$J^*(p=0.2)$	$J^*(p = 0.4)$	$J^*(p = 0.6)$	$J^{*}(p = 0.8)$	$J^*(p=1)$
0.85	2.09	2.09(0.1)	2.08(0.2)	2.08(0.3)	2.08(0.4)	2.08(0.5)
0.9	2.62	2.59(1.3)	2.56(2.5)	2.53(3.8)	2.49(5.0)	2.46(6.2)
1.0	4.00	3.84(3.9)	3.69(7.8)	3.54(11.6)	3.38(15.4)	3.23(19.2)
1.2	5.76	5.42(5.9)	5.09(11.6)	4.76(17.3)	4.43(23.1)	4.10(28.9)
1.4	7.84	7.29(7.1)	6.74(14.0)	6.21(20.8)	5.66(27.8)	5.10(34.9)
1.6	10.24	9.44(7.9)	8.65(15.5)	7.88(23.0)	7.08(30.8)	6.28(38.7)
1.8	12.96	11.87(8.4)	10.82(16.5)	9.76(24.7)	8.71(32.8)	7.62(41.2)
2.0	16.00	14.6(8.8)	13.25(17.2)	11.90(25.7)	10.55(34.1)	9.12(43.0)
2.5	25.00	22.67(9.3)	20.47(18.1)	18.27(26.9)	16.07(35.7)	13.80(44.8)
3.0	36.00	32.58(9.5)	29.43(18.2)	26.28(27.0)	23.14(35.7)	19.79(45.0)
3.5	49.00	44.38(9.4)	40.25(17.9)	36.11(26.3)	31.98(34.7)	27.67(43.5)
4.0	64.00	58.04(9.3)	52.89(17.4)	47.74(25.4)	42.64(33.4)	37.44(41.5)
4.5	81.00	73.6(9.1)	67.43(16.7)	61.27(24.4)	55.19(31.9)	48.26(40.4)
5.0	100.00	90.92(9.1)	83.69(16.3)	76.46(23.5)	69.36(30.6)	61.67(38.3)
5.5	121.00	110.34(8.8)	102.22(15.5)	94.09(22.2)	85.96(29.0)	77.04(36.3)
6.0	144.00	131.21(8.9)	121.93(15.3)	112.66(21.8)	103.38(28.2)	94.35(34.5)

Table 3.2 Expected Warranty Servicing Costs for Constant Replacement Probability \boldsymbol{p}

The figures in brackets are the percentage cost savings with respect to 'minimal repairs only' (i.e., p = 0) strategy. The expected costs for an intermediate value of $p \in (0, 1)$ is bracketed between Jack *et al.* [40] and 'minimal repairs only' strategy costs.

y	$K_y^*(p_1(t))$	$J^*(p_1(t))$	$K_y^*(p_2(t))$	$J^*(p_2(t))$	$K_y^*(p_3(t))$	$J^*(p_3(t))$	L_y^*
0.85	0.86	2.038	0.86	2.038	0.86	2.038	1.21
0.9	0.63	2.5414	0.71	2.5201	0.67	2.5335	1.49
1.0	0.56	3.6067	0.79	3.5031	0.73	3.7632	1.71
1.2	0.49	4.8325	0.75	4.7823	0.68	4.8545	1.51
1.4	0.45	6.2075	0.70	6.3622	0.64	6.3854	1.33
1.6	0.41	7.7468	0.64	8.2655	0.59	8.1334	1.19
1.8	0.38	9.4571	0.58	10.501	0.52	10.2865	1.06
2.0	0.35	11.3431	0.51	13.0964	0.47	12.3467	0.97
2.5	0.28	16.8636	0.36	21.2833	0.33	18.8745	0.78
3.0	0.22	23.6978	0.26	31.8494	0.24	28.5324	0.66
3.5	0.18	31.9168	0.20	44.4862	0.20	37.9877	0.57
4.0	0.14	42.1298	0.16	59.1082	0.16	49.7319	0.50
4.5	0.12	53.5022	0.13	75.7416	0.13	66.9176	0.44
5.0	0.10	67.0670	0.11	94.1313	0.11	75.6549	0.40
5.5	0.8	83.4194	0.9	114.7269	0.9	98.8531	0.36
6.0	0.7	100.4835	0.8	136.6894	0.8	121.7698	0.33

Table 3.3 Optimal Warranty Parameters (K_y^*, L_y^*) and Expected Servicing Costs J^* for Some Choices of Age-dependent Replacement Probability Function p(t)

Note, optimal choices of L_y^* does <u>not</u> depend on the choice of probability function p(t). The functions $p_i(t)$, i = 1, 2, 3 are those cited in Section 3.4 with $C_m = 1, C_r = 2, W = 2, U = 2$, Weibull ($\alpha_0 = 1, \beta = 2$) baseline and accelerated failure time parameter $\gamma = 2$.

3.7 Concluding Remarks

Our proposed servicing strategy extends the work of Jack *et al.* [40] by introducing a randomized choice between replacement and minimal repair in the middle interval. Since a replacement is costlier than a minimal repair $(C_r > C_m)$; the manufacturer or warranty provider has a natural incentive to do minimal repairs rather than a replacement. However, allowing a randomized choice between minimal repairs and replacement will have an impact on the reliability of the item in use at the end of warranty and, under reasonable assumptions on the aging profile of the item's life distribution, will typically be increasing in the probability of replacement, and hence higher than the corresponding reliability with minimal repairs only (p = 0). The corresponding analysis of resulting final reliability at the end of warranty is the subject of a future work.

Also as remarked in Section 3.3; under plausible assumptions such as an increasing failure rate (IFR) property of the unit's failure time and the relative cost ratio of the replacement vs minimal repair we may intuitively expect the total average cost of warranty to be decreasing in p, since a replacement in the middle interval is likely to result in less degradation compared to minimal repairs only and correspondingly to less number of expected failures in the remaining time to end of warranty. Exploring such conditions would also be a topic of future research.

Finally in Section 3.4-3.5, the impact of an age-dependent probability of replacement on the expected warranty servicing costs is investigated. The results are similar to the constant probability of repair p case, in the sense that the expected cost is bracketed between the respective costs of Jack *et al.* [40] and 'minimal repairs only'. Qualitatively, we have discussed the necessity of considering an age-dependent probability of replacement and its impact on products with the different types of degradation profiles (e.g., IFR, DFR, etc.).

CHAPTER 4

ANALYSIS OF A 2-D WARRANTY SERVICING STRATEGY WITH TWO RANDOMIZED REPAIR OPTIONS

4.1 Background and Motivation

Reduction of the total expected warranty cost is a serious issue in the warranty theory. Thus, contemporary research focuses on designing warranty servicing strategies and analytically justify their use under appropriate circumstances. If an item fails under warranty, it is rectified by a replacement or some form of repair. Clearly, the cost induced due to replacement by an identically similar item is maximum that a warranty provider can incur. On the contrary a minimal repair costs least, which restores the failed item to the state right before failure. An imperfect repair instead restores the failed item up to a specified degree, with 100% and 0% restorations implying a replacement and minimal repair, respectively. As a result, the cost and resulting item reliability of an imperfect repair is bracketed between those of minimal repairs and replacements. Here, as in other chapters of the thesis, our focus is on two-dimensional (2-D) warranty policies, that explicitly account for the influence of lifetime characteristics as well as the usage intensity of the item and allows for 'degree of repair' options based on the co-ordinates of failure instances in the (age, usage) plane, where such failures occur among pre-defined subspaces that constitute a finite partition of the 2-D warranty region. For modeling purposes describing the results of our research in this chapter, the 2-D region is partitioned into four subregions $(\Omega_i; i = 1, 2, 3, 4)$, originally considered by Varnosafaderani and Chukova [98], which we modify and generalize to include a randomized choice of 'degree of repair', is optimized for total costs together with our model parameters that are subject to choice.

Iskandar *et al.* [36] defined an approach of modeling 2-D warranty by partitioning the 2-D rectangle into three disjoint *two-dimensional subregions* Ω_1, Ω_2 and Ω_3 such that $\Omega_1 \cup \Omega_2 \cup \Omega_3 = \Omega$ with regions Ω_1 and $\Omega_1 \cup \Omega_2$ having similar shapes, where all repairs are minimal, except the first repair in the middle subregion Ω_2 which is perfect (replacement). Yun *et al.* [105] modified the strategy so that the first repair in Ω_2 is imperfect and all others are minimal. Chukova *et al.* [21] extended [36] where regions Ω_1 and $\Omega_1 \cup \Omega_2$ were not necessarily of similar shapes. Chukova *et al.* [20] further extended the strategy to *n* disjoint subregions. Their strategy is to repair all failures occurring in Ω_1 and Ω_n minimally; but, the first failures (if any) in subregions $\Omega_i, i = 2, 3, ..., n - 1$ are rectified by a replacement which can be followed by several minimal repairs in that subregion. This strategy was modified by Varnosafaderani *et al.* [98] by extending [105] to four (and *n*, in general) subregions where the first failures (if any) in each of middle subregions Ω_2 and Ω_3 (when n = 4) are imperfectly repaired, all others therein, and in Ω_1 and Ω_4 are minimally repaired. Our work in the four subregion context of [98] is an attempt to randomize (with a fixed probability *p*) the choice of minimal and imperfect repairs for the first failures (if any) in the middle subregions i.e., Ω_2 and Ω_3 , and analyze the corresponding expected cost functions.

4.1.1 Imperfect Repair Strategy of Varnosafaderani and Chukova (2010) We follow the symbolic notations of Varnosafaderani and Chukova [98] to model the expected warranty servicing cost with randomization of 'degree of repair'. The 2-D warranty region is denoted by the rectangle $\Omega = [0, K) \times [0, L)$, where K and L are the time and usage limits. The warranty expires when either the age exceeds K or the usage exceeds L [9]. We consider The *restricted* strategy in which Ω is partitioned into four disjoint subregions, $\Omega_i = ([0, K_i] \times [0, L_i]) \setminus ([0, K_{i-1}] \times [0, L_{i-1}]), i = 1, 2, 3, 4,$ $K_0 = L_0 = 0, K_n \equiv K, L_n \equiv L$ such that $\Omega_1 \cup \Omega_2 \cup ... \cup \Omega_4 = \Omega$ and

$$\frac{L_1}{K_1} = \frac{L_2}{K_2} = \frac{L_3}{K_3} = r_1(>0) \quad \text{and} \quad \frac{L_4}{K_4} = \frac{L}{K} = r_2(>0).$$
(4.1)

The total warranty cost is a function of 4 decision variables (K_1, K_2, K_3, r_1) that uniquely define the subregions and the strategy. The imperfect repair strategy (say, S_4^{δ}) proposed and investigated by Varnosafaderani and Chukova [98] is as follows. Failures in Ω_1 and Ω_4 are minimally repaired with cost C_m , the first failures (if any) in each of the intermediate subregions Ω_2 and Ω_3 , is imperfectly repaired with cost C_i (which is proportional to the degree of repair denoted by $\delta \in (0, 1)$), and any further failure in these subregions is repaired minimally.

Let A(t) and U(t) be the virtual (operating) age and virtual (operating) usage of the product at the calendar time t, and the random variable R (with distribution $G(\cdot)$) be the usage rate of a typical customer. Then, the items total usage can be modeled as U(t) = RA(t) ([36], [98]). Conditional on R = r, the process is a one-dimensional counting process $\{\tilde{N}(t|r); t \ge 0\}$, with intensity function $\tilde{\lambda}(t|r) = \theta_0 + \theta_1 r + (\theta_2 + \theta_3 r)t^2$ [36]. Consider the time at sale of the item to be zero. Therefore the *virtual age* of the item at time t, prior to the first imperfect repair is $A_0(t) = t$. After each imperfect repair, it gets adjusted and becomes

$$A_{i}(t) = A_{i-1}(t) - \delta A_{i-1}(u_{i}),$$

where u_i is the time of the i^{th} imperfect repair, i = 1, 2 and $\delta \in [0, 1]$ is the degree of imperfect repair which is considered to be fixed throughout the warranty period. Correspondingly, every imperfect repair reduces (changes) the hazard rate (intensity) of the item (process) with respect to the virtual age. Thus, conditional on the times (u_1, u_2) of the imperfect repairs, the intensity function of the process is given by

$$\tilde{\lambda}(t|r) = \begin{cases} \lambda(t|r), & 0 \le t \le u_1 \\ \lambda(A_1(t)|r), & u_1 \le t \le u_2 \\ \lambda(A_2(t)|r), & u_2 \le t \le \infty. \end{cases}$$

$$(4.2)$$

Since all repairs between the imperfect repairs are minimal, the processes before, between and after the imperfect repairs, can be viewed as nonhomogeneous Poisson processes with intensity functions given in expression (4.2).

4.2 Proposed Randomized Repair Strategy

In the S_4^{δ} setup, we investigate a variation of the Varnosafaderani-Chukova strategy, which can be defined as follows.

Any failure in Ω_1 is minimally repaired. In each of the subregions Ω_i , i = 2, 3, the first failure (if any) is either imperfectly repaired to a degree $\delta \in [0, 1]$ with probability $p \in [0, 1]$, or minimally repaired (equivalent to $\delta = 0$) with probability (1 - p) and all subsequent failures are minimally repaired. Any failure in Ω_4 is minimally repaired.

Randomizing the choice between a given degree $\delta \in [0, 1]$ of repair and minimal repair ($\delta = 0$) with a constant probability p can be pragmatically justified on the same grounds as in the Brown-Proschan strategy (Chapter 3, Section 3.2). Clearly p = 1reduces to the strategy of Varnosafaderani-Chukova [98] and p = 0 is the 'minimal repairs only' strategy. Thus given the warranty limits K, L and the probability p, there are 5 decision variables ($K_1, K_2, K_3, r_1, \delta$), that uniquely determines the new warranty strategy of randomized repairs.

4.2.1 Probability Distribution of the Times of Imperfect Repair

The distribution function of the time to first failure $T_{1|r}$, conditioned on R = r, is $F_{T_1|r}(t) = 1 - e^{-\Lambda(t|r)}$, and its density function is $f_{T_1|r}(t) = \lambda(t|r)e^{-\Lambda(t|r)}$. Let the random variables $T_{K_1|r}$ and $T_{K_2|r}$ denote the time of the first failure (or, imperfect repair) in intermediate subregions Ω_2 and Ω_3 respectively. For $t \ge K_1$, the distribution and density functions of $T_{K_1|r}$ are

$$F_{T_{K_1|r}}(t) = 1 - e^{-[\Lambda(t|r) - \Lambda(K_1|r)]},$$

$$f_{T_{K_1|r}}(t) = \lambda(t|r)e^{-[\Lambda(t|r) - \Lambda(K_1|r)]}$$

and zero otherwise. The distribution of $T_{K_2|r}$ depends on $T_{K_1|r}$ as follows. Suppose there are no failures in Ω_2 then the CDF of $T_{K_2|r}$ is same as that of $T_{K_1|r}$. But if there is atleast one failure in Ω_2 , it is then imperfectly repaired with probability p and minimally repaired with probability (1 - p). Thus for $t \ge K_2$, the distribution and density functions of $T_{K_2|r}$ are

$$\begin{split} F_{T_{K_2|r}}(t) &= \int_{K_2}^t f_{T_{K_1|r}}(u_1) du_1 + \int_{K_1}^{K_2} \left[p \Big(1 - e^{-[\Lambda(A_1(t)|r) - \Lambda(A_1(K_2)|r)]} \Big) \\ &+ (1-p) \Big(1 - e^{-[\Lambda(t|r) - \Lambda(K_2|r)]} \Big) \right] f_{T_{K_1|r}}(u_1) du_1, \\ f_{T_{K_2|r}}(t) &= f_{T_{K_1|r}}(t) + \int_{K_1}^{K_2} \left[p\lambda(A_1(t)|r) e^{-[\Lambda(A_1(t)|r) - \Lambda(A_1(K_2)|r)]} \\ &+ (1-p)\lambda(t|r) e^{-[\Lambda(t|r) - \Lambda(K_2|r)]} \right] f_{T_{K_1|r}}(u_1) du_1 \end{split}$$

and zero otherwise.

4.2.2 Modeling Expected Warranty Servicing Cost

Our objective is to derive the expected cost equation EC^{Ω} , under any usage rate r. The two possible situations as shown in Figures 4.1 and 4.2, are

> Case(A): $r_1 \leq r_2$, corresponding cost denoted by EC_A^{Ω} , Case(B): $r_2 \leq r_1$, corresponding cost denoted by EC_B^{Ω} .



Figure 4.1 Different sub-cases of warranty strategy for $r_1 \leq r_2$.



Figure 4.2 Different sub-cases of warranty strategy for $r_2 \leq r_1$.

We need to consider three subcases for each of cases A and B as shown in Figures 4.1 and 4.2,

$$(A-1) \quad r \le r_1 \qquad (B-1) \quad r \le r_2$$
$$(A-2) \quad r_1 \le r \le r_2 \quad (B-2) \quad r_2 \le r \le r_3$$
$$(A-3) \quad r_2 \le r \qquad (B-3) \quad r_1 \le r$$

The expected cost for sub-cases (A-1), (A-2) and (A-3) are denoted by $EC^{\Omega}_{A_1}, EC^{\Omega}_{A_2}$ and $EC^{\Omega}_{A_3}$, combining which we get the expected cost for Case(A) i.e.,

$$EC_A^{\Omega} = \int_0^{r_1} EC_{A_1}^{\Omega} dG(r) + \int_{r_1}^{r_2} EC_{A_2}^{\Omega} dG(r) + \int_{r_2}^{\infty} EC_{A_3}^{\Omega} dG(r).$$
(4.3)

Similarly, the cost of *Case* (B) is computed by taking the sum of average costs of four subregion.

4.2.2.1 Analysis of Case(A): $r_1 \leq r_2$. Now, we consider the three subcases individually to model the total expected warranty servicing cost.

$$\operatorname{Case}(\underline{\mathbf{A-1}}) r \leq r_1$$

The cost of each subregion Ω_i , i = 1, 2, 3, 4, denoted by $EC_{A_1}^{\Omega_i}$ and the expected cost conditioned on $r \leq r_1$, i.e., $EC_{A_1}^{\Omega} = \sum_{i=1}^4 EC_{A_1}^{\Omega_i}$ is evaluated.

- 1. The expected cost over the subregion Ω_1 is $EC_{A_1}^{\Omega_1} = C_m \Lambda(K_1|r)$, since all repairs are minimal with average cost C_m .
- 2. If there are no failures in Ω_2 , the cost in this subregion is denoted by $EC_{A_1}^{\Omega_2} = 0$. Suppose there is atleast one failure (repair) in Ω_2 , then the first failure at $u_1 \in [K_1, K_2]$ is imperfectly repaired with probability p and the resultant intensity function becomes $\lambda(A_1(\cdot)|r)$; or, it is minimally repaired with probability (1-p) and the intensity function $\lambda(\cdot|r)$ remains unchanged. Thus for $K_1 < T_{K_1|r} \equiv u_1 \leq K_2$, the expected servicing cost over Ω_2 is,

$$EC_{A_1}^{\Omega_2} = \int_{K_1}^{K_2} \left[p[C_i + C_m \{ \Lambda(A_1(K_2)|r) - \Lambda(A_1(u_1)|r) \}] + (1-p)[C_m + C_m \{ \Lambda(K_2|r) - \Lambda(u_1|r) \}] \right] f_{T_{K_1|r}}(u_1) du_1$$

- 3. Over Ω_3 , $EC_{A_1}^{\Omega_3} = 0$, if there are no failures. Suppose there is atleast one failure (repair) in Ω_3 , then the expected cost is modeled considering the following list of events:
 - the first failure in Ω₂ is imperfectly repaired with probability p and the intensity function alters to λ(A₁(·)|r); the first failure in Ω₃ is also imperfectly repaired with probability p and the intensity function becomes λ(A₂(·)|r).
 - the first failure in Ω₂ is imperfectly repaired with probability p and the intensity function alters to λ(A₁(·)|r); the first failure in Ω₃ is minimally repaired with probability (1−p) and the intensity function remains unchanged i.e., λ(A₁(·)|r).
 - the first failure in Ω₂ is minimally repaired with probability (1−p) and the intensity function remains unchanged i.e., λ(·|r); the first failure in Ω₃ is imperfectly repaired with probability p and the intensity function changes to λ(A₁(·)|r).

- the first failure in each of the subregions Ω_2 and Ω_3 is minimally repaired with probability (1 - p) respectively and the intensity function remains unchanged i.e., $\lambda(\cdot|r)$.
- there are no failures in Ω_2 , the first failure in Ω_3 is either imperfectly repaired with probability p with intensity function changing to $\lambda(A_1(\cdot)|r)$, or, minimally repaired with probability (1 - p) and the intensity function remains unchanged i.e., $\lambda(\cdot|r)$

The total expected cost over Ω_3 is, therefore,

$$\begin{split} EC_{A_1}^{\Omega_3} &= p^2 \int_{K_2}^{K_3} \int_{K_1}^{K_2} \left[C_i + C_m \{ \Lambda(A_2(K_3)|r) - \Lambda(A_2(u_2)|r) \} \right] \\ &\lambda(A_1(u_2)|r) e^{-[\Lambda(A_1(u_2)|r) - \Lambda(A_1(K_2)|r)]} f_{T_{K_1|r}}(u_1) du_1 du_2 \\ &+ p(1-p) \int_{K_2}^{K_3} \int_{K_1}^{K_2} \left[C_m + C_m \{ \Lambda(A_1(K_3)|r) - \Lambda(A_1(u_2)|r) \} \right] \\ &\lambda(A_1(u_2)|r) e^{-[\Lambda(A_1(u_2)|r) - \Lambda(A_1(K_2)|r)]} f_{T_{K_1|r}}(u_1) du_1 du_2 \\ &+ (1-p) p \int_{K_2}^{K_3} \int_{K_1}^{K_2} \left[C_i + C_m \{ \Lambda(A_1(K_3)|r) - \Lambda(A_1(u_1)|r) \} \right] \\ &\lambda(u_2|r) e^{-[\Lambda(u_2|r) - \Lambda(K_2|r)]} f_{T_{K_1|r}}(u_1) du_1 du_2 \\ &+ (1-p)^2 \int_{K_2}^{K_3} \int_{K_1}^{K_2} \left[C_m + C_m \{ \Lambda(K_3|r) - \Lambda(u_1|r) \} \right] \\ &\lambda(u_2|r) e^{-[\Lambda(u_2|r) - \Lambda(K_2|r)]} f_{T_{K_1|r}}(u_1) du_1 du_2 \\ &+ \int_{K_2}^{K_3} \left[p[C_i + C_m \{ \Lambda(A_1(K_3)|r) - \Lambda(A_1(u_1)|r) \} \right] \\ &+ (1-p)[C_m + C_m \{ \Lambda(K_3|r) - \Lambda(u_1|r) \} \right] f_{T_{K_1|r}}(u_1) du_1. \end{split}$$

- 4. Over Ω_4 , $EC_{A_1}^{\Omega_4} = 0$, if there are no failures. Suppose there is atleast one failure (repair) in Ω_4 , then the expected cost is modeled considering the following list of events:
 - no failure occurs in both Ω_2 and Ω_3 , with probability $e^{-[\Lambda(K_3|r) \Lambda(K_1|r)]}$.

- atleast one failure (repair) occur in Ω_2 which is either imperfectly repaired with probability p, and the corresponding intensity function changes to $\lambda(A_1(\cdot)|r)$, or, minimally repaired with probability (1-p), intensity function remains unchanged i.e., $\lambda(\cdot|r)$; but no failure occur in Ω_3 with probability $e^{-[\Lambda(A_1(K_3)|r)-\Lambda(A_1(K_2)|r)]}$.
- atleast one failure (repair) occur in Ω₃ which is either imperfectly repaired with probability p, and the corresponding intensity function changes to λ(A₁(·)|r), or, minimally repaired with probability (1−p), intensity function remains unchanged i.e., λ(·|r); but no failure occur in Ω₂.
- the first failure in Ω₂ is imperfectly repaired with probability p and the intensity function alters to λ(A₁(·)|r); the first failure in Ω₃ is also imperfectly repaired with probability p and the intensity function becomes λ(A₂(·)|r).
- the first failure in Ω₂ is imperfectly repaired with probability p and the intensity function alters to λ(A₁(·)|r); the first failure in Ω₃ is minimally repaired with probability (1−p) and the intensity function remains unchanged i.e., λ(A₁(·)|r).
- the first failure in Ω₂ is minimally repaired with probability (1−p) and the intensity function remains unchanged i.e., λ(·|r); the first failure in Ω₃ is imperfectly repaired with probability p and the intensity function changes to λ(A₁(·)|r).
- the first failure in each of the subregions Ω_2 and Ω_3 is minimally repaired with probability (1 - p) respectively and the intensity function remains unchanged i.e., $\lambda(\cdot|r)$.

The total expected cost over Ω_4 is, therefore,

$$\begin{split} EC_{A_1}^{\Omega_4} &= C_m[\Lambda(K|r) - \Lambda(K_3|r)]e^{-[\Lambda(K_3|r) - \Lambda(K_1|r)]} \\ &+ \int_{K_1}^{K_2} \left[pC_m\{\Lambda(A_1(K)|r) - \Lambda(A_1(K_3)|r)\} \right] \\ &+ (1-p)C_m\{\Lambda(K|r) - \Lambda(K_3|r)\} \right] e^{-[\Lambda(A_1(K_3)|r) - \Lambda(A_1(K_2)|r)]} \\ &\quad f_{T_{K_1|r}}(u_1)du_1 + \int_{K_2}^{K_3} \left[pC_m\{\Lambda(A_1(K)|r) - \Lambda(A_1(K_3)|r) + (1-p)C_m\{\Lambda(K|r) - \Lambda(K_3|r)\} \right] f_{T_{K_1|r}}(u_1)du_1 \\ &+ p^2 \int_{K_2}^{K_3} \int_{K_1}^{K_2} \left[C_m\{\Lambda(A_2(K)|r) - \Lambda(A_2(K_3)|r)\} \right] \lambda(A_1(u_2)|r) \\ &\quad e^{-[\Lambda(A_1(u_2)|r) - \Lambda(A_1(K_2)|r)]} f_{T_{K_1|r}}(u_1)du_1du_2 + p(1-p) \\ &\quad \int_{K_2}^{K_3} \int_{K_1}^{K_2} \left[C_m\{\Lambda(A_1(K)|r) - \Lambda(A_1(K_3)|r)\} \right] \lambda(A_1(u_2)|r) \\ &\quad e^{-[\Lambda(A_1(u_2)|r) - \Lambda(A_1(K_2)|r)]} f_{T_{K_1|r}}(u_1)du_1du_2 \\ &+ (1-p)p \int_{K_1}^{K_2} \left[C_m\{\Lambda(A_1(K)|r) - \Lambda(A_1(K_3)|r)\} \right] \lambda(u_2|r) \\ &\quad e^{-[\Lambda(u_2|r) - \Lambda(K_2|r)]} f_{T_{K_1|r}}(u_1)du_1du_2 \\ &+ (1-p)^2 C_m[\Lambda(K|r) - \Lambda(K_3|r)] \int_{K_2}^{K_3} \int_{K_1}^{K_2} \lambda(u_2|r) \\ &\quad e^{-[\Lambda(u_2|r) - \Lambda(K_2|r)]} f_{T_{K_1|r}}(u_1)du_1du_2. \end{split}$$

The expected cost for *Case* (A-1) is obtained by summing the costs over four subregions and is expressed as

$$EC_{A_1}^{\Omega} = \varphi(K_1, K_2, K_3, K).$$

To obtain the cost of sub-cases (A-2) and (A-3) we need to adjust the arguments of the function $\varphi(\cdot, \cdot, \cdot, \cdot)$ as shown here.

$Case(\underline{A-2}) r_1 \leq r \leq r_2$

It can be noted from Figure 4.1 Case (A-2), that the warranty of each rectangular

sub-regions $\Omega_i, i = 1, 2, 3$ expires at the following time points τ_i respectively, where

$$\tau_1 = \frac{L_1}{r}, \quad \tau_2 = \frac{L_2}{r}, \quad \tau_3 = \frac{L_3}{r}.$$
 (4.4)

Thus the required expected warranty cost here is

$$EC_{A_2}^{\Omega} = \varphi(\tau_1, \tau_2, \tau_3, K).$$

 $\operatorname{Case}(\underline{\mathbf{A-3}}) \ \boldsymbol{r_2} \leq \boldsymbol{r}$

This sub-case is similar to sub-case (A-2), the only difference is the total warranty period (K) expires at age $\tau = \frac{L}{r}$, and the expected cost is

$$EC_{A_3}^{\Omega} = \varphi(\tau_1, \tau_2, \tau_3, \tau).$$

Finally, unconditioning on R as shown in Equation (4.3) we obtain the total expected warranty servicing cost for *Case* (A).

4.2.2.2 Analysis of Case(B): $r_2 \leq r_1$. In this case, the computation of the expected cost is similar to *Case A*. One can obtain a generic function $\varphi(\cdot, \cdot, \cdot, \cdot)$, the arguments uniquely defining the subregions. Following Figure 4.2, the expected cost for each of three different sub-cases corresponding to Case(B) can be obtained and is listed here.

 $Case(\underline{B-1}) \ r \leq r_2$

The expected cost is same as Case A i.e.,

$$EC_{B_1}^{\Omega} = \varphi(K_1, K_2, K_3, K).$$
(4.5)

 $\operatorname{Case}(\underline{\mathbf{B-2}}) r_2 \leq r \leq r_1$

The expected cost is given by

$$EC_{B_2}^{\Omega} = \varphi(K_1, K_2, K_3, \tau) \quad \text{where} \quad \tau = \frac{L}{r}.$$
(4.6)

$Case(\underline{\mathbf{B-3}}) \ r_1 \leq r$

Again, in this case the expected cost is same as Case A i.e.,

$$EC_{B_3}^{\Omega} = \varphi(\tau_1, \tau_2, \tau_3, \tau), \qquad (4.7)$$

where τ_1 , τ_2 , τ_3 and τ are given in Equations (4.4) and (4.6) respectively. Finally, the expected servicing cost for *Case B* is obtained by unconditioning on *R*, i.e.,

$$EC_B^{\Omega} = \int_0^{r_2} EC_{B_1}^{\Omega} dG(r) + \int_{r_2}^{r_1} EC_{B_2}^{\Omega} dG(r) + \int_{r_1}^{\infty} EC_{B_3}^{\Omega} dG(r)$$

4.2.3 Numerical Illustration

We consider a FRW example given by [98], where K = 2 (years), L = 2 (20000 km), $r_2 = 1, \theta_0 = 0.1, \theta_1 = 0.2, \theta_2 = 0.7, \theta_3 = 0.7$. The usage rate R is uniformly distributed over $[r_l, r_u]$, and three usage level considered are 'light' (i.e., $[r_l = 0.1, r_u = 0.9]$), 'medium' (i.e., $[r_l = 0.7, r_u = 1.3]$) and 'heavy' (i.e., $[r_l = 1.1, r_u = 2.9]$). Let the cost of replacement $C_r = 1$, the cost ratio of minimal repair to replacement $\varsigma = \frac{C_m}{C_r} = C_m$ varies from 0.1 to 0.9 with increments of 0.1 and the cost ratio of imperfect repair to replacement $\delta = \frac{C_i}{C_r} = C_i$ takes values in $(\varsigma, 1)$ with increments of 0.1. For each ς the optimal $K_i, i = 1, 2, 3$ are also obtained by grid search method over [0.1,2.0) with steps of size 0.1 and r_1 is sought over [0.2, 3) with increments of 0.2. Tables 4.1, 4.2 and 4.3 show the minimal costs for three intermediate probabilities of imperfect repairs i.e., p = [0.25, 0.5, 0.75] along with those of Varnosafaderani and Chukova [98] (same as p = 1) and 'minimal repairs only' strategies.

Table 4.1 Minimal Costs of Warranty for Light Usage	Bate
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$p = 1^*$	$r_1^*, \delta^*) \hspace{0.2cm} \left \hspace{0.2cm} EC^{\Omega}(K_1^*, K_2^*, K_3^*, r_1^*, \delta^*) \right $	0.2, 0.2) 0.3218 (0.1, 0.2, 0.3, 0.2, 0.2)	$1.0, 0.3) \left \begin{array}{c} 0.5893 \\ 0.5893 \\ \end{array} \right (0.7, 1.0, 1.7, 1.0, 0.3) \\ \end{array}$	$1.0,0.4) \left \begin{array}{c} 0.7682 \\ \end{array} (0.5,1.1,1.8,1.0,0.4) \right \\ \end{array} \right $	$1.0, 0.5) \left \begin{array}{c} 0.9137 \ (0.4, 1.1, 1.9, 1.0, 0.5) \end{array} \right $	$1.0,0.6) \left 1.0377 \ (0.3,1.1,1.9,1.0,0.6) \right $	$1.0, 0.7) \left \begin{array}{c} 1.1464 \\ 1.1464 \\ (0.3, 1.1, 1.9, 1.0, 0.7) \end{array} \right $	$1.0,0.8) \left \begin{array}{c} 1.2452 \\ (0.3,1.1,1.9,1.0,0.8) \end{array} \right $	$1.0,0.9) \left \begin{array}{c} 1.3383 \\ 1.3383 \\ \end{array} (0.3,1.0,1.9,1.0,0.9) \\ \end{array} \right.$	I
p = 0.75	$EC^{\Omega}(K_1^*, K_2^*, K_3^*, $	$0.3212\ (0.1, 0.2, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3$	$0.5989\ (0.7, 1.0, 1.7, 1$	$0.7905\ (0.6, 1.1, 1.8, 1)$	$0.9473 \ (0.5, 1.2, 1.9, 1)$	$1.2009 \ (0.4, 1.1, 1.9, 1.9, 1.9, 1.9, 1.9, 1.9, 1.9$	$1.2999 \ (0.3, 1.1, 1.9, 1.9, 1.9, 1.9, 1.9, 1.9, 1.9$	$1.4117\ (0.3, 1.1, 1.9, 1.9, 1.9, 1.9, 1.9, 1.9, 1.9$	$1.5355\ (0.3, 1.0, 1.9, 1$	Ι
p = 0.5	$EC^{\Omega}(K_{1}^{*},K_{2}^{*},K_{3}^{*},r_{1}^{*},\delta^{*})$	$0.3208\ (0.1, 0.2, 0.3, 0.2, 0.2)$	$0.6129\ (0.8, 1.0, 1.7, 1.0, 0.3)$	$0.8333\ (0.5, 1.0, 1.8, 1.0, 0.4)$	$0.9765\ (0.5, 1.2, 1.9, 1.0, 0.5)$	$1.3280\ (0.4, 1.2, 1.9, 1.0, 0.6)$	$1.5468\ (0.3,1.1,1.9,1.0,0.7)$	$1.7373\ (0.3,1.1,1.9,1.0,0.8)$	$1.8118\ (0.3, 1.0, 1.9, 1.0, 0.9)$	Η
p = 0.25	$EC^{\Omega}(K_1^*,K_2^*,K_3^*,r_1^*,\delta^*)$	$0.3200\ (0.1, 0.2, 0.3, 0.2, 0.2)$	$0.6311 \ (0.8, 1.0, 1.7, 1.0, 0.3)$	$0.8998\ (0.6,1.1,1.8,1.0,0.4)$	$1.1231 \ (0.4, 1.0, 1.8, 1.0, 0.5)$	$1.4545\ (0.4, 1.2, 1.9, 1.0, 0.6)$	$1.7995\ (0.3, 1.1, 1.9, 1.0, 0.7)$	$1.9989\ (0.3, 1.1, 1.9, 1.0, 0.8)$	$2.2551 \ (0.3, 1.0, 1.9, 1.0, 0.9)$	I
	C^{MR}	0.32	0.64	0.96	1.28	1.60	1.92	2.24	2.56	2.88
	V	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9

* Cost of Varnosafaderani-Chukova (p = 1) [98].

 $C^{MR} := \text{Cost of 'minimal repairs only'}$ (p = 0) strategy.
Table 4.2 Minimal Costs of Warranty for Medium Usage Rate

$p = 1^*$	$EC^{\Omega}(K_{1}^{*},K_{2}^{*},K_{3}^{*},r_{1}^{*},\delta^{*})$	$0.3649\ (0.5, 0.6, 0.7, 0.2, 0.2)$	$0.6574 \ (0.6, 1.1, 1.7, 1.0, 0.3)$	$0.8553 \ (0.5, 1.2, 1.8, 1.0, 0.4)$	$1.0165\ (0.4, 1.1, 1.9, 1.0, 0.5)$	$1.1541 \ (0.4, 1.1, 1.9, 1.0, 0.6)$	$1.2747\ (0.3, 1.1, 1.9, 1.0, 0.7)$	$1.3843\ (0.3, 1.1, 1.9, 1.0, 0.8)$	$1.4874\ (0.3, 1.0, 1.9, 1.0, 0.9)$	I
p = 0.75	$EC^{\Omega}(K_1^*,K_2^*,K_3^*,r_1^*,\delta^*)$	$0.3644 \ (0.5, 0.6, 0.7, 0.2, 0.2)$	$0.6725\ (0.6, 1.2, 1.8, 1.0, 0.3)$	$0.8792\ (0.4, 1.3, 1.8, 1.0, 0.4)$	$1.1224 \ (0.4, 1.1, 1.9, 1.0, 0.5)$	$1.2931 \ (0.4, 1.0, 1.8, 1.0, 0.6)$	$1.4443 \ (0.3, 1.1, 1.9, 1.0, 0.7)$	$1.6277 \ (0.3, 1.1, 1.9, 1.0, 0.8)$	$1.8843 \ (0.3, 1.0, 1.9, 1.0, 0.9)$	I
p = 0.5	$EC^{\Omega}(K_1^*,K_2^*,K_3^*,r_1^*,\delta^*)$	$0.3640\ (0.5, 0.6, 0.7, 0.2, 0.2)$	$0.6907 \ (0.6, 1.2, 1.7, 1.0, 0.3)$	$0.8833 \ (0.4, 1.3, 1.8, 1.0, 0.4) \ \\$	$1.2733 \ (0.4, 1.0, 1.9, 1.0, 0.5)$	$1.4415 \ (0.5, 1.0, 1.9, 1.0, 0.6) \\$	$1.7002 \ (0.3, 1.1, 1.9, 1.0, 0.7)$	$1.9230 \ (0.3, 1.1, 1.9, 1.0, 0.8)$	$2.2350 \ (0.3, 1.0, 1.9, 1.0, 0.9)$	I
p = 0.25	$EC^{\Omega}(K_1^*,K_2^*,K_3^*,r_1^*,\delta^*)$	$0.3638\ (0.5, 0.6, 0.7, 0.2, 0.2)$	$0.7105\ (0.6, 1.0, 1.7, 1.0, 0.3)$	$0.8999 \ (0.5, 1.3, 1.8, 1.0, 0.4)$	$1.3432\ (0.4, 1.0, 1.8, 1.0, 0.5)$	$1.6999 \ (0.5, 1.0, 1.9, 1.0, 0.6)$	$1.9980 \ (0.3, 1.1, 1.9, 1.0, 0.7)$	2.2334(0.3,1.1,1.9,1.0,0.8)	$2.5541 \ (0.3, 1.0, 1.9, 1.0, 0.9)$	I
	C^{MR}	0.3637	0.7274	1.0911	1.4549	1.8186	2.1823	2.5460	2.9097	3.2735
	\sim	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9

* Cost of Varnosafaderani-Chukova (p = 1) [98].

 $C^{MR} := \text{Cost of 'minimal repairs only'} (p = 0)$ strategy.

Table 4.3 Minimal Costs of Warranty for Heavy Usage Rate

$p = 1^*$	$EC^{\Omega}(K_{1}^{*},K_{2}^{*},K_{3}^{*},r_{1}^{*},\delta^{*})$	$0.1470\ (0.4, 0.5, 0.6, 0.2, 0.2)$	$0.2928\ (1.7, 1.8, 1.9, 0.2, 0.3)$	$0.4252\ (1.0, 1.1, 1.9, 0.8, 0.4)$	$0.5332\ (0.6, 1.0, 1.7, 1.0, 0.5)$	$0.6306\;(0.4,1.0,1.8,1.0,0.6)$	$0.7223\ (0.4, 1.0, 1.8, 1.0, 0.7)$	$0.8095\ (0.3, 0.9, 1.9, 1.0, 0.8)$	$0.8947\ (0.3, 0.9, 1.9, 1.0, 0.9)$	I
p = 0.75	$EC^{\Omega}(K_{1}^{*},K_{2}^{*},K_{3}^{*},r_{1}^{*},\delta^{*})$	$0.1470\ (0.4, 0.5, 0.6, 0.2, 0.2)$	$0.2825\ (1.5, 1.7, 1.9, 0.2, 0.3)$	$0.4275\ (1.0,1.1,1.8,0.8,0.4)$	$0.5532\ (0.6, 1.0, 1.8, 1.0, 0.5)$	$0.6514 \ (0.4, 1.0, 1.8, 1.0, 0.6)$	$0.7638\ (0.4, 1.0, 1.8, 1.0, 0.7)$	$0.8355\ (0.3, 0.9, 1.9, 1.0, 0.8)$	$0.9315\ (0.3, 0.9, 1.9, 1.0, 0.9)$	I
p = 0.5	$EC^{\Omega}(K_1^*,K_2^*,K_3^*,r_1^*,\delta^*)$	$0.1460\ (0.4, 0.5, 0.6, 0.2, 0.2)$	$0.2823\ (1.5, 1.8, 1.9, 0.2, 0.3)$	$0.4301\ (1.0, 1.4, 1.9, 0.8, 0.4)$	$0.5782\ (0.6, 1.0, 1.7, 1.0, 0.5)$	$0.6791 \ (0.4, 1.0, 1.8, 1.0, 0.6)$	$0.7943 \ (0.4, 1.0, 1.8, 1.0, 0.7)$	$0.8639\ (0.3, 0.9, 1.9, 1.0, 0.8)$	$0.9723\ (0.3, 0.9, 1.9, 1.0, 0.9)$	I
p = 0.25	$EC^{\Omega}(K_1^*,K_2^*,K_3^*,r_1^*,\delta^*)$	$0.1460\ (0.4, 0.5, 0.6, 0.2, 0.2)$	$0.2820\ (1.6, 1.8, 1.9, 0.2, 0.3)$	$0.4345\ (1.3, 1.4, 1.9, 0.8, 0.4)$	$0.5815\ (0.6, 1.0, 1.7, 1.0, 0.5)$	$0.6999 \ (0.4, 1.0, 1.8, 1.0, 0.6)$	$0.8389\ (0.4, 1.0, 1.8, 1.0, 0.7)$	$0.8873\ (0.3, 0.9, 1.9, 1.0, 0.8)$	$0.9994\ (0.3, 0.9, 1.9, 1.0, 0.9)$	I
	C^{MR}	0.1460	0.2919	0.4379	0.5839	0.7299	0.8758	1.0218	1.1678	1.3138
	S	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9

* Cost of Varnosafaderani-Chukova (p = 1) [98].

 $C^{MR} := \text{Cost of 'minimal repairs only'} (p = 0)$ strategy.

4.2.4 Qualitative Interpretation

Tables 4.1, 4.2 and 4.3, respectively corresponds to 'low', 'medium' and 'high' usage rates. From the tables it can be observed that:

- i) The minimal cost corresponding to an intermediate $p \in (0, 1)$ is bracketed between the 'minimal repairs only' and p = 1 (same as [98]) costs as expected by intuition.
- ii) If probability p is relatively high, the strategy is more cost-effective, thus allowing further reduction of cost compared to 'minimal repairs only' strategy.
- iii) The strategy is optimal for the lowest value of $\delta^* \in (\varsigma, 1)$ irrespective of the values of probability p.
- vi) As the cost of each minimal repair (equivalently, $\varsigma = C_m/C_r$) increases, the optimal ratio $r_1^* = L_i/K_i$, i = 1, 2, 3 converges to 1. This means, if the cost of minimal repairs are relatively high, the expected servicing cost is minimum for square shaped subregions Ω_i , i = 1, 2, 3, i.e., providing a longer warranty limit $(K_i, i = 1, 2, 3)$ or higher total usage limit $(L_i, i = 1, 2, 3)$ is not worth the cost.
- v) If the cost of each minimal repair is too high (here for e.g., $\varsigma = 0.9$) close to replacement, then the best strategy is to replace the item with a new one (i.e., p = 1 and $\delta = 1$). Hence, no computational results are obtained for the proposed strategy and Varnosafaderani-Chukova strategy for $\varsigma = 0.9$ (for search grid-size equal to 0.1).

4.3 Concluding Remarks

Thus in this chapter, a new servicing strategy has been proposed, extending the model of Varnosafaderani and Chukova [98]. Our model offers a randomized choice between imperfect and minimal repairs at the first failures (if any) in the two intermediate subregions. This type of randomization is warranty-friendly, because it provides the warrantor with an alternative option at these two specific failure times thus increasing the flexibility of the strategy. Further, if the chosen probability of imperfect repair is reasonably high, the warrantor can substantially reduce the total servicing cost as shown in Tables 4.1, 4.2 and 4.3.

CHAPTER 5

A DECISION PROBLEM FOR 2-D PRO-RATED WARRANTY STRATEGY WITH MAINTENANCE

5.1 Background and Motivation

In production industries, manufacturers aim at maximizing their total expected profit by controlling costs. These costs are incurred due to several factors viz. costs of manufacturing, marketing, servicing, maintenance etc. Hence, the total profit cannot be optimized without considering these related costs and their interactions. A successful marketing strategy should account for these different aspects of pricing, production, and warranty. Many researchers have considered such integrated decision problems to analyze the effect of individual factors. In this study, we consider a decision problem and analyze the behavior of total expected profit under usage sensitive warranty strategy, popularly known as two-dimensional (2-D) warranties.

Many integrated decision related problems present in literature are based on historical data only. But, it has been pointed out in some recent articles that, decision models of cost optimization based solely on historical data and managerial experience are either too optimistic – indicating an 'ideal' situation, where the true values of governing parameters that influence such costs are considered to be known; or, are insufficient to estimate the true rate of deterioration of a product. A data driven Bayesian updating process based on real life observations can be useful here as a more pragmatic decision model applicable to the real market.

In a typical Bayesian updating process, the model describing deterioration process of a product is updated by collecting failure data of related products from the market and incorporating them in the model. Thus, in this approach a combination of expert opinion from production managers (historical data) and real-life data from market is used to model item failures, which increase the realism of deterioration process model and the consequent optimization of cost estimates.

In an integrated approach to cost optimization, an important aspect of marketing strategy is the provision of post-sales servicing in form of some warranty policy. It not only safeguards the rights and interests of consumers, but also acts as a tool for promoting sales by signaling enhanced product reliability. In particular, a good warranty policy of service and maintenance assists in leveraging the image of a high-quality product, and thus, becomes a powerful weapon in a competitive market. The following flow diagram (Figure 5.1) obtained from Huang [32], shows the interaction between production, pricing, sales and warranty.



Figure 5.1 Interaction of production and marketing factors.

The two main types of warranties considered here, are Free Replacement Warranty (FRW) and Pro-rated Warranty (PRW). The policy of a PRW, which charges the consumer a preset proportion of the cost for each repair during the term of warranty, is a popular warranty policy for relatively high-priced products such as plant facilities and large scale machines. On the contrary, a FRW policy does not charge consumers anything to rectify a failed product during the term of warranty. PRW seems more appealing to manufacturers because the warranty cost is partially paid by customers, and the money saved may be used to extend the warranty term to attract more customers and gain their loyalty. However, the greater the proportion of costs borne by the manufacturers, the higher the operational costs, which may eventually make the product unprofitable. Therefore, the tradeoff between a higher warranty cost and a greater market share is of special importance for managers aiming to control costs or, maximize profits.

5.2 Review of Pro-rated Warranty Policy

In general, PRW policy would be applicable to industry, since plant facilities and large-scale machinery usually need a long-term maintenances service program for which the duration of warranty is longer, but a certain portion of the warranty cost is shared by customers. Murthy and Djamaludin [70] mentioned that FRW is sometimes thought of as an offensive strategy, while PRW is a defensive strategy that distributes the risk of bearing warranty cost between sellers and buyers. PRW can be justifiable in many industrial applications [86]. Murthy and Blischke [69] stated that FRW is most often used for items that are not repairable, while PRW is most often used for items that are repairable.

Most of the studies regarding the PRW issue have mainly focused on cost analysis with considerations of reliability estimation. Menke [58] evaluated the warranty cost for a nonrepairable product under PRW with the assumption of exponential failure process. Blischke and Scheuer [12] extended Menkes research to consider other failure processes. Thomas [96] proposed an approach to determine the optimal warranty term for non-repairable products, by which the equivalent situation of PRW and FRW can also be investigated. Nguyen and Murthy [75] assessed the expected warranty costs for renewable and nonrenewable warranty policies, respectively, under the assumption of a monotone product failure rate. Frees and Nam [27] revised the assumption made by Nguyen and Murthy [75] and used the approach of straight-line approximation to estimate the warranty cost from both long-term and short term perspectives with the policy of PRW. Balcer and Sahin [3] proposed a stochastic failure model to derive the expected warranty cost in which the repair cost is paid by the manufacturer in accordance with the proportion of damage under PRW. Blischke and Murthy [9] derived the expected warranty cost for renewable and nonrenewable PRW from the perspectives of sellers and buyers, respectively, with the assumption of having a Weibull product failure model. Zuo *et al.* [109] considered the optimal decision about replacement versus repair for minimizing the warranty cost with the considerations of the degree of decay and the surplus of the warranty term for repairable products with multi-phase decay.

Wang *et al.* [100] stated that warranty cost and product reliability are positively associated (correlated), and thus, the expected warranty cost can be reduced by improving the product reliability. Ja et al. [38] proposed an approach to determine the optimal warranty term under the condition that manufacturers pay for the expense of minimal repair. Chattopadhyay and Murthy [16] evaluated the expected warranty cost in terms of a component failure mechanism. Jain and Maheshwari [43] proposed a warranty model for a renewed PRW policy in which the failure rate of the product, the cost of preventive maintenance (PM), and the cost of replacement are assumed to be constant, and the proposed model is able to determine the optimal number of preventive maintenance (PM) activities within the warranty period. Zhou [108] developed a mathematical model to investigate a policy that jointly considers product pricing and warranty length for a repairable high-tech product over its effective lifetime. Huang et al. [31] developed a model to determine the optimal combination of product reliability, price, and warranty that can achieve the maximum profit for a repairable product. Wu et al. [101] developed a cost model to determine the optimal burn-in time and warranty term for nonrepairable products under the policies of FRW and PRW.

It is not uncommon that the manufacturer does not have sufficient historical data to estimate the deterioration of a newly developed product. In such cases, the results obtained via frequentist models may not be reliable, and Bayesian analysis that additionally take expert opinions into account could be a reasonable approach to improve decision making. Kwon [54] proposed a Bayesian life test sampling plan for products with Weibull lifetime distributions by minimizing the expected average cost, which involves three cost components: testing cost, acceptance cost, and rejection cost. Percy et al. [83] solved the problem of scheduling periodic PM by a Bayesian approach, where the prior knowledge about manufacturing processes of similar systems is included. Papazoglou [79] utilized a Bayesian decision analysis to deal with the problem of reliability certification on the basis that the existing prior assessment of uncertainties and the further information that can be obtained through testing of the components. Perlstein *et al.* [84] developed a Bayesian method to determine the optimal burn-in duration for a batch of products whose life distributions were assumed from a mixture of two different exponential populations. Percy [82] discussed several suitable forms of prior distribution for common models and developed a concept of predictive elicitation to specify the hyper parameter subjective prior distributions. Juang and Anderson [47] utilized a Bayesian approach to determine an optimal adaptive PM policy with minimal repairs. By incorporating minimal repair, major repair, planned replacement, unplanned replacement, and periodic maintenance in the model, the mathematical formulas of the expected cost per unit time are obtained. Huang and Fang [34] considered a more complex decision problem under the policy of PRW, a Bayesian decision model for determining the optimal warranty proportion is proposed in which a periodic PM program is performed during the warranty term to slowdown the deterioration, and a nonhomogeneous Poisson process (NHPP) is employed to describe the successive failure times of the deteriorating product. Accordingly, both the repair cost of each breakdown and the potential sales increase due to a specific warranty proportion are also considered.

Our proposed work, described in what follows, is an extension of the strategy of Huang and Fang [34] from one to two-dimensional warranties. We study the impact of accelerated usage rate on the warranty proportion, sales and profit.

5.2.1 Decision of Optimal 2-D Warranty

We now turn to a description of recent approaches to 2-D warranty servicing strategies which are sensitive to the rate of usage, that will lead us to a proposed decision problem of choosing the optimal warranty proportion parameter. The servicing setup considered here is that of Pro-rated warranties (PRW) where the manufacturer and the consumer share the cost of rectification on item failures.

For completeness, we briefly recall (from Chapter 2, Section 2.2.1), the following description of 2-D warranty policies sensitive to the rate of usage. Consider a repairable item sold with a 2-D non-renewing warranty of period W and maximum usage limit U. Then the 2-D warranty region is the rectangle $[0, W) \times [0, U)$.

We assume that the usage rate Y varies from customer to customer but is constant for a given customer. Therefore, Y is a random variable that can be modeled using a density function g(y). Conditional on Y = y, the total usage u at age x is given by

$$u = yx, \qquad 0 \le u < \infty$$

Fixed y, the usage sensitive warranty expires when the item currently in use reaches an age

$$W_y = \min(W, \frac{U}{y}).$$

Note that under the policy of PRW setting a suitable percentage of warranty costs as the seller's liability, with the balance carried by the buyers, is one of choosing a tradeoff since a low warranty proportion will save warranty costs, but on the contrary, a higher proportion will attract consumers and eventually increase sales. Further, the number of breakdowns during the warranty period becomes crucial in choosing the optimal proportion for PRW; because if the warranty period is too long, the manufacturer will have to repair too many failures and incur losses. There is a close association between the customer usage rate and item degradation resulting failures followed by repairs. Since most products degrade due to both age and usage intensity, usage rate is an important determinant of the rate of item failures and cannot be ignored. Thus, an ideal 2-D PRW policy should be such that the manufacturer shares an optimal $100\rho\%$ of the repair cost at each item failure up to age W_y . Determining this optimal pro-rated proportion ρ with respect to warranty duration W_y is the main focus of this chapter.

5.2.2 Modeling Failure Distributions

Failures over time are modeled by a *counting process*. If failed items are repaired then the counting process is characterized by a conditional intensity function $\lambda_y(x)$, which is a non-decreasing function of age (x) and usage rate (y). Suppose (i) all repairs are minimal (i.e., items after repair are 'as bad as old') and (ii) repair times are negligible compared to the mean time between failures. Then the successive clock-times of breakdowns follow non homogeneous poisson process (NHPP) with the hazard rate of the item under use as the intensity function of NHPP, as is well known. Corresponding to an accelerated failure time (AFT) model with an Weibull baseline hazard distribution (as in Chapter 2), Huang and Bier [34] consider the *power law* intensity function:

$$\lambda_y(x) = \alpha(y)\beta x^{(\beta-1)},\tag{5.1}$$

where the shape parameter denoting the rate of deterioration of the product is β (> 1, indicating IFR Weibull distribution) and the scale parameter is

$$\alpha(y) = \left(\frac{y}{y_0}\right)^{\gamma} \alpha, \tag{5.2}$$

which captures the effect of usage intensity on the failure rate of the item, α is the baseline scale, y_0 is the nominal usage rate, y is a typical usage rate and $\gamma(>1)$ is the acceleration parameter. Therefore, the expected number of failures over the usage

sensitive warranty period $[0, W_y]$ is

$$N_y = \int_0^{W_y} \left(\frac{y}{y_0}\right)^{\gamma} \alpha \beta x^{(\beta-1)} dx = \left(\frac{y}{y_0}\right)^{\gamma} \alpha W_y^{\beta},$$

which is a function of α and β . If α and β are unknown, one can model them using a suitable joint prior distribution. In this case, the expected number of failures N_y is also random, since it captures the uncertainty of α and β .

5.2.3 Pro-rated Warranty Proportion

Suppose the cost of rectification of the item at failure is shared by both the manufacturer and the consumer. The proportion of repair cost that the manufacturer has to pay when the item fails is denoted by $\rho \in [0, 1]$. Clearly if $\rho = 1$ this strategy reduces to Free Replacement Warranty (FRW), where the manufacturer pays full repair cost denoted by $C_m N_y$, where C_m is the cost of a minimal repair.

There are several reasons for which reasons for which Pro-rated warranties (PRW) can be appealing. Some of these are as follows.

- If the repair costs (of expensive items) are too high; they drastically affect the manufacturer's profit margin. Distributing such costs between the manufacturer and the user thus allows the manufacturer to reasonably price a PRW and still be cost effective in the long run.
- Customers covered by PRW, have a financial stake in the process. They being aware of the fact that they share a portion of the repair costs, tend to use the item with more care, reducing excessive usage related stress on the product and the corresponding faster degradation.
- Manufacturers get the choice of undertaking appropriate maintenance under PRW, which eventually reduces the number of failures over the warranty duration.
- Manufacturers can extend the warranty term to a longer period over the useful lifetime of the item, thus promoting brand loyalty.

It is easy to believe that a higher value of the proportion ρ of warranty costs that reflect the manufacturer's liability, will signal the product as highly reliable and is correspondingly likely to attract more buyers, thus accelerating product sales.

5.2.4 The Total Sales Function

Let C_0 denote the planned sale price of a new item, then an estimate of total sales (demand) $S(\rho)$ according to Glickman and Berger [29] and Huang and Fang [34] is modeled as:

$$S \equiv S(\rho) = w_1 C_0^{-v_1} (w_2 + W_y)^{v_2} exp\{-v_3(1-\rho)\},$$
(5.3)

where $w_1 > 0, w_2 > 0, v_1 > 1, 0 < v_i < 1, i = 2, 3.$

Here w_1 is an amplitude factor, w_2 is a constant to allow for nonzero baseline demand without warranty, v_1, v_2, v_3 are the elasticities of the sale price (C_0) , the warranty length (W_y) with usage rate y, and the proportion of warranty cost shared by the manufacturer (ρ) , respectively.

Here 'elasticity' is defined in the usual sense as understood in the context of *Microeconomics*, viz., as the ratio of the relative change in sales volume (equal to marginal demand) to the relative change of a variable of interest; e.g., the warranty length elasticity of sales is

$$\left(\frac{dS}{S}\right) / \left(\frac{dw_y}{w_y}\right) = \left(\frac{w_y}{S}\right) \left(\frac{dS}{dw_y}\right) = \frac{d(\ln S)}{d(\ln w_y)} = v_2 \in (0,1)$$

where $w_y := w_2 + W_y$. Similarly,

$$\frac{d(\ln S)}{d(\ln C_0)} = -v_1 < -1$$
$$\frac{d(\ln S)}{d(\ln \rho)} = v_3 \in (0, 1)$$

are the price-elasticity and manufacturer's warranty proportion-elasticity respectively. The assumed restrictions on the constants v_i , i = 1, 2, 3 reflect the reasonableness of the above Glickman-Berger model which shows that the sales (demand) S, as well as the marginal demand $(d \ln S)$ decreases as the warranty length (W_y) or/and PRW proportion (ρ) increases. The model also implies, as noted by Huang and Fang [34] that a longer warranty length (W_y) , larger PRW proportion (ρ) , as well as lower price C_0 will encourage a larger sales volume S.

5.2.5 Preventive Maintenance Action

Under a PRW policy, a manufacturer often includes some preventive maintenance (PM) action to reduce the hazard rate. Such maintenance actions are generally periodic and similar to an 'imperfect repair', i.e., each PM improves the operating condition of the product. In this 2-D model, suppose a periodic PM action is executed after every *h* units of time. Research related to PM is vast, for e.g. Park *et al.* [80] considered a periodic PM policy along with minimal repairs after breakdowns, and derived the optimal period and number of PM actions, Seo and Bai [90] depicted a periodic PM policy for two cases in which the time of PM can be ignored or not, and others for e.g., Jack and Dagpunar [39], Jung *et al.* [48], Kim *et al.* [52], Wang and Sheu [99], and Wu and Li [102]. Since for our purposes of modeling and analysis, only the extent of reduction in effective age as a consequence of preventive maintenance (PM) activity are of interest, we are not concerned with the specific nature of PM activity undertaken.

Let η denote the proportion of reduction in the effective age of the item as a result of every preventive maintenance. Let x denote the actual age of the item, then x_1^+ is the effective age of the item after first PM action and can be expressed as follows $x_1^+ = (1 - \eta)h$. According to Martorell *et al.* [56], the effective ages of the product immediately before and after the k^{th} PM action can be, respectively, derived as

$$x_k^- = x_{k-1}^+ + h = [k - (k-1)\eta]h,$$

$$x_k^+ = x_k^- - \eta h = [k - (k-1)\eta]h - \eta h = k(1-\eta)h$$



Figure 5.2 Periodic maintenance policy.

Thus under periodic PM, the expected number of failures on $[0, W_y]$ is

$$N_{y}^{pm} \equiv N_{y}^{pm}(W_{y}, h, \eta, |\alpha(y), \beta)$$

$$= \Sigma_{k=0}^{\lfloor W_{y}/h \rfloor - 1} \int_{x_{k}^{+}}^{x_{k+1}^{-}} \lambda_{y}(x) dx + \int_{x_{\lfloor W_{y}/h \rfloor}}^{x_{\lfloor W_{y}/h \rfloor}^{+} + (W_{y} - \lfloor W_{y}/h \rfloor)} \lambda_{y}(x) dx$$

$$= \left(\frac{y}{y_{0}}\right)^{\gamma} \alpha \left[\Sigma_{k=0}^{\lfloor W_{y}/h \rfloor - 1} \left[\left(\left((1 - \eta)k + 1\right)h\right)^{\beta} - \left((1 - \eta)kh\right)^{\beta} \right] + \left[(W_{y} - \eta \lfloor W_{y}/h \rfloor h)^{\beta} - \left((1 - \eta) \lfloor W_{y}/h \rfloor h\right)^{\beta} \right] \right], \qquad (5.4)$$

where $\lfloor x \rfloor$ denotes the 'floor of x' (i.e., the largest integer less or equal to x). So $\lfloor W_y/h \rfloor$ is the number of PM actions performed within the warranty period $[0, W_y]$. The first term in the summands of Equation (5.4) denotes the expected number of failures before the last PM action, while the second term denotes the expected number of failures after the last PM until the end of warranty.

Due to the mechanical aging of the system, the maintenance cost will get higher and higher for sequential PM activities over the warranty term. If ρ is the periodically increasing rate of PM cost, then the total cost of PM is

$$C_{pm} := C_{pm}(C_b, \varrho, h, W_y) = \sum_{k=1}^{\lfloor W_y/h \rfloor} C_b(1 + \varrho kh),$$

where C_b denotes the base cost of maintenance (see Jayabalan *et al.* [45]). In addition, the minimal repair cost that must be paid by the manufacturer is $\rho C_m N_y^{pm}$. By adopting a PRW policy, it is practically recognized that a significant proportion of PRW can increase both the revenues and the additional expenses, and the tradeoff for choosing the optimal proportion would be a crucial decision problem for the manufacturer.

5.2.6 The Profit Function

Let C_p denote the production cost per unit. Considering the revenue and the relative costs, the anticipated profit can be expressed as:

$$\pi_y = (C_0 - C_p)S(\rho) - \{C_{pm} + \rho C_m N_y^{pm}\}S(\rho),$$
(5.5)

where C_0 is planned sale price, C_{pm} is cost of PM actions and C_m is cost of minimal repairs. It can be noted that π_y being a function of N_y^{pm} is itself uncertain and thus needs to be estimated using Bayesian methods.

5.3 A Bayesian Decision Model for 2-D Warranty

Product failures are modeled by a NHPP process with power law intensity function sensitive to usage rate y, as shown in Equation (5.1). To obtain the optimal proportion of warranty cost, shared by the manufacturer a Bayesian prior analysis is performed. This analysis is completely based on prior knowledge of item failure distribution in terms of the the unknown parameters α and β , which are jointly modeled by a suitable prior distribution. To set the ground of prior analysis we define:

• State space: $\Theta = \{\theta = (\alpha, \beta) | \alpha > 0, \beta > 0\}.$

- Set of actions: $\mathcal{A} = \{\rho | 0 < \rho < 1\}$. Here ρ denotes the fraction of repair costs shared by the manufacturer.
- Set of profit functions: $\Pi = \{\pi(\theta, \rho) | \pi : \Theta \times [0, 1] \to (-\infty, \infty)\}$. The profits we gain, if a proportion ρ is chosen as the warrantor's share of repair cost liability, under θ , is given by $\pi(\theta, \rho)$.
- Sample space S: The additional information available to be collected (e.g., successive breakdown times for similar products). The cost of collecting this additional information should also be considered in the decision process.

In this context, the production managers typically specify two sets of prior information, which are,

- i) the expected values and variances of the unknown parameters α and β, i.e.,
 μ_α, σ_α and μ_β, σ_β, respectively, for modeling the product degradation profile
 with prior knowledge, and
- ii) a model describing the total volume of sales of the product.

The results of prior analysis based on these information are presented to the managerial group for consideration and possible modification of production decisions. If these results, based solely on expert opinion in the prior stage, are not convincing, (which is often the case, because such results are too optimistic and represent an 'ideal' situation deviating from the real market scenario), a posterior analysis is performed combining the market data and the prior knowledge.

A 'preposterior analysis' serves as a bridge between the prior and posterior analysis. It determines the necessity of a posterior analysis in terms of costs incurred due to collection of market data versus the information extracted from those data. The crux of the preposterior analysis, is to determine the optimal sample size n^* of failures and the corresponding data to be collected from the market. Clearly, the problem reduces to balancing the trade-off between this sample size n^* and the corresponding cost of information collection, which is schematically illustrated in Section 5.3.2, Figure 5.4.

5.3.1 Prior and Posterior Analysis

The estimated usage sensitive profit function involves the unknown parameters α and β through N_y^{pm} . If our beliefs about (α, β) can be described by some specific joint prior distribution, then from Equation (5.5) the *prior* expected profit is given by the expression

$$E(\pi_y) = (C_0 - C_p)S(\rho) - \{C_{pm} + \rho C_m E(N_y^{pm})\}S(\rho),\$$

where $S(\rho)$ is the sales volume, defined by Equation (5.3), considered as a function of ρ and $E(N_y^{pm})$ denotes the *prior* expected number of failures under PM, for a specific usage rate y, i.e.,

$$E(N_{y}^{pm}) = \alpha(y) \int_{A} \int_{B} \left[\Sigma_{k=0}^{\lfloor W_{y}/h \rfloor - 1} \left(\{ ((1-\eta)k+1)h \}^{\beta} - \{(1-\eta)kh \}^{\beta} \right) \\ + \{ W_{y} - \eta \lfloor W_{y}/h \rfloor h \}^{\beta} - \{ (1-\eta) \lfloor W_{y}/h \rfloor h \}^{\beta} \right] f(\alpha(y), \beta) d\beta d\alpha(y) \\ = \left(\frac{y}{y_{0}} \right)^{2\gamma} \alpha \int_{A} \int_{B} \left[\Sigma_{k=0}^{\lfloor W_{y}/h \rfloor - 1} \left(\{ ((1-\eta)k+1)h \}^{\beta} - \{ (1-\eta)kh \}^{\beta} \right) \\ + \{ (W_{y} - \eta \lfloor W_{y}/h \rfloor h)^{\beta} - ((1-\eta) \lfloor W_{y}/h \rfloor h)^{\beta} \} \right] f\left(\left(\frac{y}{y_{0}} \right)^{\gamma} \alpha, \beta \right) d\beta d\alpha.$$
(5.6)

where $f\left(\left(\frac{y}{y_0}\right)^{\gamma}\alpha,\beta\right)$ denotes the prior probability distribution of (α,β) . A and B respectively, are the supports of the prior distribution of (α,β) .

The expected profit function $E(\pi_y)$ is clearly continuous and differentiable with respect to $\rho \in (0, 1)$. The following proposition states the necessary and sufficient condition for concavity of $E(\pi_y)$.

Proposition 5.3.1 The function $E(\pi_y)$, being continuous and differentiable with respect to ρ , is concave in ρ if and only if the condition

$$1/v_3 < (C_0 - C_p - C_{pm})/C_m E(N_y^{pm}) < 1 + 1/v_3,$$
(5.7)

holds, where v_3 denotes elasticity of the proportion ρ of warranty cost shared by the manufacturer given in Equation (5.3); then there exists a unique root $\rho \in (0, 1)$ that maximizes $E(\pi_y)$.

The proof of this proposition follows by arguments analogous to *Proposition* 3 of Huang and Fang [34], together with necessary modifications to include the effect of usage rate y in $E(\pi_y)$.

The corresponding optimal warranty proportion $\rho_0 \equiv \rho_0(y) := \arg \max_{x} E(\pi_y)$ is

$$\rho_0 = \frac{v_3(C_0 - C_p - C_{pm}) - C_m E(N_y^{pm})}{v_3 C_m E(N_y^{pm})}$$
(5.8)

Therefore, maximum expected profit is $E(\pi_y(\theta, \rho_0))$. However, according to the marketing convention as practised in the industry for such purposes, the unit of warranty proportions considered are usually in 5% increments (e.g., $\rho = 35\%$ or 50%, etc.). Let $A = 100\rho_0$ be the optimal PRW percentage. Correspondingly, we define

$$A_1 := \lfloor A/5 \rfloor 5$$
 and $A_2 := \lceil A/5 \rceil 5$

Here $\lfloor x \rfloor$ and $\lceil x \rceil$ respectively, denotes the 'floor' and 'ceiling' of x. Now, setting

$$\rho_1 := A_1/100 \quad \text{and} \quad \rho_2 := A_2/100,$$
(5.9)

the decision rule for selecting the optimal PRW proportion ρ^* would therefore be

$$\rho^* = \{\rho_j | \max_{j=1,2} E[\pi(\theta, \rho_j)]\}$$
(5.10)

Since parameters α and β are assumed to be unknown, they can be modeled by a suitable joint prior distribution $f\left(\left(\frac{y}{y_0}\right)^{\gamma}\alpha,\beta\right)$. By doing so, one can model the randomness of $E(N_y^{pm})$ and estimate the optimal proportions ρ_0 and ρ^* (as shown in Equations (5.8) and (5.10) respectively) along with the expected profit function.

On the other other of these prior estimates do not meet the expectation of the managerial experts, a further investigation is performed to update the results. This is

typically done by a posterior analysis, where real life data are integrated in the model, which is denoted by $f'\left(\left(\frac{y}{y_0}\right)^{\gamma}\alpha,\beta\right)$. This modified model denotes the posterior joint distribution of α and β , and contain observed failure times of similar products from the market. The posterior analysis is performed by substituting the prior expectation $E(N_y^{pm})$ by the posterior expectation $E'(N_y^{pm})$.

Therefore, the warranty costs shared by the manufacturer under warranty and relation between ρ and W_y for the prior and posterior analysis respectively, are given by

Prior Analysis:

$$C_{w} = \rho C_{m} E(N_{y}^{pm}) + C_{pm},$$

$$\rho = C_{m}^{-1} \{ C_{w} - (\Sigma_{k=1}^{\lfloor W_{y}/h \rfloor} C_{b}(1+\varrho kh)) \} \left[\left(\frac{y}{y_{0}} \right)^{2\gamma} \alpha \int_{A} \int_{B} \left[\Sigma_{k=0}^{\lfloor W_{y}/h \rfloor - 1} [\{ ((1-\eta)k+1)h \}^{\beta} - \{(1-\eta)kh \}^{\beta}] + \{ (W_{y} - \eta \lfloor W_{y}/h \rfloor h)^{\beta} - ((1-\eta) \lfloor W_{y}/h \rfloor h)^{\beta} \} \right] f \left(\left(\frac{y}{y_{0}} \right)^{\gamma} \alpha, \beta \right) d\beta d\alpha \right]^{-1}$$

and

Posterior Analysis:

$$C'_{w} = \rho C_{m} E'(N_{y}^{pm}) + C_{pm},$$

$$\rho' = C_{m}^{-1} \{ C_{w} - (\Sigma_{k=1}^{\lfloor W_{y}/h \rfloor} C_{b}(1+\varrho kh)) \} \left[\left(\frac{y}{y_{0}} \right)^{2\gamma} \alpha \int_{A} \int_{B} \left[\Sigma_{k=0}^{\lfloor W_{y}/h \rfloor - 1} [\{ ((1-\eta)k+1)h \}^{\beta} - \{(1-\eta)kh \}^{\beta}] + \{ (W_{y} - \eta \lfloor W_{y}/h \rfloor h)^{\beta} - ((1-\eta) \lfloor W_{y}/h \rfloor h)^{\beta} \} \right] f' \left(\left(\frac{y}{y_{0}} \right)^{\gamma} \alpha, \beta \right) d\beta d\alpha \right]^{-1}$$

A value C_w of the expected warranty cost (prior-based) can be achieved by varying combinations of ρ and W_y . The corresponding plot of ρ versus W_y for fixed C_W is referred to as a iso-warranty cost curve as in Figure 5.3. From Iso-warranty Cost figure, it can be seen that, the warranty cost increases (decreases, respectively) if the curve shifts to the right (left, respectively).



Figure 5.3 2-D iso-warranty cost curve.

5.3.2 Preposterior Analysis

Suppose the result of the prior analysis is not very persuasive, for example, the estimated profit is much higher (lower) than expected. In such cases, gathering additional information might be desirable. But, before collecting this additional data, one need to investigate if the possible outcome of collecting data is worth the cost of collection. Thus the vital step between prior and posterior analysis termed as the 'preposterior analysis' is performed in which a suitable cost-effective sampling plan is proposed, subject to some constraints as discussed here.

In this context, we need to define the expected value of sample information (EVSI) (sometimes called expected value of imperfect information, EVII) which (see [23]) is

$$EVSI(S^{(i)}) = E_S \left\{ \max_{j=1,2} \{ E[\pi_y(\theta, \rho_j) | S^{(i)}] \} \right\} - \max_{j=1,2} \{ E[\pi_y(\theta, \rho_j)] \}$$

where $S^{(i)}$ denotes the i^{th} sampling plan under consideration. Also, the expected net gain of sample information (ENGS) is defined as

$$ENGS(S^{(i)}) = EVSI(S^{(i)}) - C_I(S^{(i)}),$$

where $C_I(S^{(i)})$ denotes the cost of collecting information (CI) for the i^{th} sampling plan. We use a simple decision rule, that is if $ENGS \leq 0$, then collection of additional information is not cost- worthy; but, if ENGS > 0, then one can collect more information for a posterior analysis. Note that the i^{th} sampling plan should be optimally adopted in order to satisfy the condition

$$ENGS(S^{(i^*)}) = \max_{i} \{ EVSI(S^{(i)}) - C_I(S^{(i)}) \}.$$

Therefore, if we assume that additional information can be collected from the successive failure times of similar products, then the critical task is to determine how many breakdowns ought to be gathered. The corresponding ENGS would be

$$ENGS(S^{(n)}) = \int_{X_1} \int_{X_2} \dots \int_{X_n} \int_{\mathcal{A}} \int_{\mathcal{B}} \left(\frac{y}{y_0}\right)^{\gamma} E'[\pi(\theta, \rho^{*'}(x_1, x_2, ..., x_n))] f\left(\left(\frac{y}{y_0}\right)^{\gamma} \alpha, \beta\right)$$
$$\left(\left(\frac{y}{y_0}\right)^{\gamma} \alpha\right)^n \beta^n [\Pi_{i=1}^n x_i] exp\left\{-\left(\frac{y}{y_0}\right)^{\gamma} \alpha x_n^{\beta}\right\} d\beta d\alpha dx_1 dx_2 ... dx_n$$
$$- E[\pi(\theta, \rho^{*'})] - C_I(n)$$
(5.11)

where *n* denotes the sample size, x_i denotes the *i*th breakdown time, $E'[\cdot]$ is the posterior expectation and $\rho^{*'}(\cdot)$ is the optimal pro-rated proportion based on the market data $x_1, x_2, ..., x_n$. The optimal decisions can be derived from Equation (5.11), which will vary for different samples $x_1, x_2, ..., x_n$.

Clearly, more information (data) would improve the quality of decision regarding maximization of profit, but this effect of increasing profit would gradually decrease as the cost of information collection increases. As a result, the *ENGS* will decrease eventually with increasing sample size as shown in Figure 5.4 obtained from [32], and the decision makers need to determine the optimal sample size n^* where $ENGS(S^{(n^*)})$ is maximum.



Figure 5.4 Preposterior decision problem of selecting optimal sample size n^* .

5.4 Analysis with Natural Conjugate

Bayesian decision analysis is typically not easy to perform since the derivation of posterior distributions might involve the use of numerical integration. Especially, in our case of two random variables (i.e., α and β) in the state space, the analysis would be much more complicated to deal with. Huang and Bier [33] proposed a natural conjugate prior distribution for the power law deteriorating model for repairable systems that is of the form

$$f\left(\left(\frac{y}{y_0}\right)^{\gamma}\alpha,\beta\right) = K\left(\left(\frac{y}{y_0}\right)^{\gamma}\alpha\right)^{(g-1)}\beta^{(g-1)}(e^{-d}z^g)^{(\beta-1)}exp\left\{-\left(\frac{y}{y_0}\right)^{\gamma}\alpha cz^{\beta}\right\}$$
(5.12)

Here K is a normalizing factor and g, d, z, c are four suitably chosen constants. Compared to other approaches, this natural conjugate prior distribution has certain features that enables straight-forward and successful analysis instead of the usually complicated computation. Some of these properties are listed here:

1) the marginal distribution β is Gamma with parameters g and d, expectation and coefficient of variation (CV) are

$$E(\beta) = \frac{g}{d},$$

$$CV(\beta) = \frac{\sigma_{\beta}}{\mu_{\beta}} = g^{-1/2}.$$

2) The conditional distribution of α given β is Gamma with parameters g and cz^{β} , expectation and coefficient of variation (CV) are

$$E(\alpha) = \frac{g}{c} \left(\frac{d}{d+z_1}\right)^g, \text{ and}$$
$$CV(\alpha) = \frac{\sigma_{\alpha}}{\mu_{\alpha}} = \left[\frac{\left(1 + \frac{z_1^2}{d^2 + 2dz_1}\right)(g+1)}{g-1}\right]^{1/2},$$

where $z_1 = \ln(z)$.

Therefore, in our case, for a fixed usage rate y, $E(\alpha(y)) = (y/y_0)^{\gamma} E(\alpha)$ and $CV(\alpha(y))$ = $CV(\alpha)$. The four parameters g, d, c and z can be chosen to obtained the desired prior moments of α and β obtained from historical data (expert opinion). The prior analysis can be performed straightforwardly by calculating $E(N_y^{pm})$ in Equation (5.6) with respect to the four parameters (i.e., g, d, c and z), applying the decision rules shown in Equations (5.8) and (5.10).

Now, suppose the results of prior analysis is not convincing and posterior analysis is required. Then the optimum sample size n is carefully obtained using the Monte Carlo curve fitting method (by Muller *et al.* [64]) in the preposterior analysis. If n breakdown times (from the other similar products) are collected as $(x_1, x_2, ..., x_n)$, then the posterior distribution of α and β can be obtained by the property of the natural conjugate family as

$$f'\left(\left(\frac{y}{y_0}\right)^{\gamma}\alpha,\beta\right) \propto L\left(D^{(n)} \left| \left(\frac{y}{y_0}\right)^{\gamma}\alpha,\beta\right) f\left(\left(\frac{y}{y_0}\right)^{\gamma}\alpha,\beta\right) \\ = K'\left(\left(\frac{y}{y_0}\right)^{\gamma}\alpha\right)^{(g+n-1)}\beta^{(g+n-1)}(e^{-d}z^g \prod_{i=1}^n x_i)^{(\beta-1)}exp\left\{-\left(\frac{y}{y_0}\right)^{\gamma}\alpha(cz^{\beta}+x_n^{\beta})\right\}$$
(5.13)

where $L\left(D^{(n)}\left|\left(\frac{y}{y_0}\right)^{\gamma}\alpha,\beta\right) = \left(\left(\frac{y}{y_0}\right)^{\gamma}\alpha\right)^n\beta^n(\prod_{i=1}^n x_i)^{(\beta-1)}exp\{-(\frac{y}{y_0})^{\gamma}\alpha x_n^{\beta}\}$ is the likelihood function, and K' is a normalizing factor to ensure the distribution sums up to unity. The posterior expected number of failures is, therefore,

$$\begin{split} E'(N_y^{pm}) &= \left(\frac{y}{y_0}\right)^{2\gamma} \Big[\sum_{k=0}^{W_y/h]-1} \Big[\int_{\mathcal{A}} \int_{\mathcal{B}} \alpha (h(k(1-\eta)+1))^{\beta} f'\Big(\Big(\frac{y}{y_0}\Big)^{\gamma} \alpha, \beta\Big) d\beta d\alpha \\ &- \int_{\mathcal{A}} \int_{\mathcal{B}} \alpha (hk(1-\eta))^{\beta} f'\Big(\Big(\frac{y}{y_0}\Big)^{\gamma} \alpha, \beta\Big) d\beta d\alpha \Big] \\ &+ \int_{\mathcal{A}} \int_{\mathcal{B}} \alpha (W_y - \eta \lfloor W_y/h \rfloor h)^{\beta} f'\Big(\Big(\frac{y}{y_0}\Big)^{\gamma} \alpha, \beta\Big) d\beta d\alpha \\ &- \int_{\mathcal{A}} \int_{\mathcal{B}} \alpha ((1-\eta) \lfloor W_y/h \rfloor h)^{\beta} f'\Big(\Big(\frac{y}{y_0}\Big)^{\gamma} \alpha, \beta\Big) d\beta d\alpha \Big]. \end{split}$$

The complicated expressions of expected number of failures for both the prior and posterior analyzes are obtained numerically via Monte Carlo (MC) Integration.

5.4.1 Numerical Illustration of Prior and Posterior Analysis

For comparison and illustrative purposes, we have used the same cost and other parameters considered by Huang and Fang [34]; except for the usage rate, the inclusion of which is new.

Suppose we have a heavy industrial equipment which is covered under a nonrenewing 2-D PRW policy with warranty period W = 5.5 (years), usage limit U = 5.5(×10⁵ loads of production), parameters of demand $w_1 = 280,000,000$, $w_2 = 0.8$, v_1 = 2.5, $v_2 = 0.83$, $v_3 = 0.25$, planned price per unit $C_0 = 128,000$ (dollars), unit production cost $C_p = 60,000$ (dollars), prior moments $\mu_{\alpha}=0.46$, $\sigma_{\alpha}=0.21$, $\mu_{\beta}=2.60$, $\sigma_{\beta}=0.64$, maintenance interval h = 4 (months) or 1/3 (year), base cost $C_b = 500$ (dollars), age reduction factor $\eta = 0.7$, annual rate of increase in maintenance cost $\varrho = 0.05$, minimal repair cost $C_m = 2000$ (dollars).

Based on the discussion in the previous sections, the first step is to calculate the estimates of warranty proportion ρ^* and expected profit using the prior knowledge in the prior analysis. The corresponding results are shown in Table 5.1. The change in optimal PRW proportion ρ^* with respect to warranty duration W_y as obtained from the prior analysis is shown in Table 5.3. It can be noted that the behavior of ρ^* w.r.t. W_y is similar to that shown in Figure 5.3.

Now suppose we want to investigate if posterior analysis is worth the cost for the given problem. To do so we need to calculate ENGS as schematically illustrated in Figure 5.4. Clearly, it is difficult to evaluate the complicated integral in (5.11) defining ENGS, except via computational approaches. It can be noticed that in our scenario the posterior density is a joint density of α , β and failure times $X_1, X_2, ..., X_n$. The *posterior* expected number of failures have been obtained by *Gibbs sampling* to draw random samples of α , β , $X_1, X_2, ..., X_n$.

Integrating Equation (5.13), with respect to α (or, β , respectively) yields the following conditional marginal distributions, which are used to generate random samples of α (or, β).

Proposition 5.4.1 The posterior conditional distribution of α given β is Gamma with parameters g + n and $\left(\frac{y}{y_0}\right)^{\gamma} (cz^{\beta} + x_n^{\beta})$, where n is the optimal sample size, i.e.,

$$f(\alpha|\beta) = \frac{\alpha^{g+n-1}e^{-(\frac{y}{y_0})^{\gamma}\alpha(cz^{\beta}+x_n^{\beta})}[(\frac{y}{y_0})^{\gamma}(cz^{\beta}+x_n^{\beta})]^{g+n}}{\Gamma(g+n)}, \quad \alpha > 0.$$
(5.14)

Proposition 5.4.2 The posterior conditional density function of β given α is

$$f(\beta|\alpha) = \frac{\beta^{g+n-1} (e^{-d} z^g \prod_{i=1}^n x_i)^{\beta-1} e^{-(\frac{y}{y_0})^{\gamma} \alpha(c z^{\beta} + x_n^{\beta})}}{\int_0^\infty \beta^{g+n-1} (e^{-d} z^g \prod_{i=1}^n x_i)^{\beta-1} e^{-(\frac{y}{y_0})^{\gamma} \alpha(c z^{\beta} + x_n^{\beta})} d\beta}, \quad \beta > 0.$$
(5.15)

Here g, d, z, c are constant prior parameters given in Equation (5.12). To obtain samples of β from the complicated Equation (5.15), we have used the Metropolis-Hastings (M-H) algorithm (see for e.g., Chib and Greenberg [17]) which can be summarized as follows: Suppose g(x) be the target density function such that $g(x) \propto$ $h(x)\psi(x)$, where h(x) is the density that can be simulated by some known method and $\psi(x)$ is uniformly bounded. Define $v(x,y) = \min\{\frac{\psi(y)}{\psi(x)}, 1\}$ as the candidategenerating-density. Then,

- repeat for j = 1 to N.
- generate y from $h(x^{(j)})$ and u from Uniform(0, 1).

- if $u \le v(x^{(j)}, y)$, set $x^{(j)} = y$.
- else set $x^{(j+1)} = x^{(j)}$.
- return $\{x^{(1)}, x^{(2)}, x^{(3)}, ..., x^{(N)}\}.$

The first 100 values are burned out to avoid dependency on the initial choice of random variables. In our case the target distribution is $f(\beta|\alpha)$ given in Equation (5.15), we have considered $h(\beta) = de^{-d\beta}$, $\beta > 0$, (i.e., exponential density function with mean 1/d), and

$$\psi(\beta) = \beta^{g+n-1} (z^g \prod_{i=1}^n x_i)^{\beta-1} exp\{d - (\frac{y}{y_0})^\gamma \alpha (cz^\beta + x_n^\beta)\}.$$
 (5.16)

Thus following the steps of M-H algorithm, the expression of ENGS is evaluated and the optimal sample sizes n^* 's are determined according to Figure 5.4 as

$$n^* = \begin{cases} 3, & \text{if } y \le 0.7 \\ 5, & \text{if } 0.7 < y < 1.4 \\ 6, & \text{if } y \ge 1.4 \end{cases}$$

It can be noted that as the usage rate y increases, the optimal sample size increase. Intuitively, it makes sense, since high usage rate will result in more failures creating an unstable situation in the product market and this instability in product reliability profile can be efficient captured with greater sample sizes. Let the usage sensitive failure times (in years) of a similar product are

$$n^* = 3, \quad \{1.12, 1.57, 1.79\}$$
$$n^* = 5, \quad \{1.30, 1.82, 2.15, 2.63, 2.96\}$$
$$n^* = 6, \quad \{1.18, 1.61, 1.95, 2.35, 2.88, 3.07\}$$

then, the corresponding posterior analysis results are given in Table 5.2.

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	Total Expected Profit	380175246	354347246	333684846	325936446	303462183	284754995	265500316	253158984	242677533	219838176	201496880	185737790	172324907	163032900	154319550
	Total Warranty Cost	58900754	84728754	105391154	113139554	81893817	61297005	50427684	38969016	30070467	17345824	11343120	9422210	9303093	7919100	7996450
	Total Production Cost	387420000	387420000	387420000	387420000	340020000	305340000	278760000	257760000	240660000	209280000	187800000	172200000	160260000	150840000	143220000
	Total Revenue	826496000	826496000	826496000	826496000	725376000	651392000	594688000	549888000	513408000	446464000	400640000	367360000	341888000	321792000	305536000
	Total Demand	6457	6457	6457	6457	5667	5089	4646	4296	4011	3488	3130	2870	2671	2514	2387
	φ	1.0	1.0	0.9	0.7	0.6	0.5	0.45	0.3	0.2	0.1	0.05	0.05	0.05	< 0.05	< 0.05
	W_y	5.5	5.5	5.5	5.5	4.5833	3.9286	3.4375	3.0556	2.7500	2.2000	1.8333	1.5714	1.3750	1.2222	1.1000
	y	0.3	0.5	0.7	1.0^{*}	1.2	1.4	1.6	1.8	2.0	2.5	3.0	3.5	4.0	4.5	5.0

 Table 5.1
 Prior Analysis Results for Different Usage Rates

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* Case of Huang and Fang [34].

у	W_y	φ	Total Demand	Total Revenue	Total Production Cost	Total Warranty Cost	Total Expected Profit
0.3	5.5000	1.00	6960	890880000	417600000	63489120	409790880
0.5	5.5000	1.00	6960	890880000	417600000	91329120	381950880
0.7	5.5000	1.00	6960	890880000	417600000	119169120	354110880
1.0^{*}	5.5000	0.90	6788	868864000	407280000	135230536	326353464
1.2	4.5833	0.80	5957	762496000	357420000	100381407	304694593
1.4	3.9286	0.70	5282	676096000	316920000	76298490	282877510
1.6	3.4375	0.60	4704	602112000	282240000	59524416	260347584
1.8	3.0556	0.35	4296	549888000	257760000	41976216	250151784
2.0	2.7500	0.10	3961	507008000	237660000	23358017	245989983
2.5	2.2000	< 0.05	3236	414208000	194160000	13180228	206867772
3.0	1.8333	< 0.05	2904	371712000	174240000	10524096	186947904
3.5	1.5714	< 0.05	2662	340736000	159720000	8739346	172276654
4.0	1.3750	< 0.05	2478	317184000	148680000	8630874	159873126
4.5	1.2222	< 0.05	2333	298624000	139980000	7348950	151295050
5.0	1.1000	< 0.05	2215	283520000	132900000	7420250	143199750

 Table 5.2 Posterior Analysis Results for Different Usage Rates

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Table	5.3	Relationship	between	Warranty	Duration	(W_y)	and	Optimal	PRW
Propor	tion (o^*), for Consta	ant Usage	Rates (y)					

		$ ho^+$	
W_y	y = 0.5	$y = 1^*$	y = 1.5
2	1	1	0.9
3	1	0.9	0.8
4	1	0.8	0.6
5	1	0.7	0.5
6	1	0.6	0.3
7	0.9	0.5	0.1
8	0.8	0.4	0.05
9	0.7	0.3	< 0.05
10	0.6	0.2	< 0.05

* Case of Huang and Fang [34].

For a fixed usage rate y, the optimal PRW proportion ρ^* is decreasing in the warranty period W_y . As the latter increases, the number of repairs during warranty increase and the expected warranty cost can be controlled by curtailing the proportion of repair cost paid by the warrantor at each failure. However, for a fixed warranty period W_y , ρ^* is decreasing in y, since higher y results in more failures (or, repairs) and the warranty cost is controlled by reduction of ρ^* .



Figure 5.5 Effects of different usage rates on the prior and posterior estimates of warranty proportion under the PRW scheme.

It can be observed that for relatively low and moderate usage rates y, the estimate of posterior warranty proportion based on failure data obtained from the market, is higher than the prior estimate. This implies that when the item is being used within a nominal usage range, the optimal warranty proportion ρ is relatively high. This will not only attract customers in the market but will result in higher profit levels to the firm. On the contrary, when y is too high (typically more than double the nominal usage level i.e., y = 1), the posterior warranty proportion is less than the prior, this is a reasonable outcome, since when usage (and consequently the failure rate) is high, a greater proportion ρ will only increase the warranty cost, drastically affecting the profit margin of the firm.



Figure 5.6 Effects of different usage rates on the prior and posterior estimates of warranty cost under the PRW scheme.

Similarly, it can be observed that for relatively low and moderate usage rates y, the estimate of posterior warranty cost based on failure data obtained from the market, is higher than the prior estimate. This is because the warranty cost is proportional to the the warranty proportion ρ , which is high for low and moderate y's. Also it can be seen that the warranty cost reduces for high usage rate, since as y increases, ρ and warranty term W_y decreases, in turn reducing the total warranty cost. Although some variation among costs are observable from the plots of prior and posterior warranty costs, those are approximately equal for greater values of y, indicating that a prior analysis is sufficient to draw conclusions if y is too high.



Figure 5.7 Effects of different usage rates on the prior and posterior estimates of expected profit under the PRW scheme.

Interestingly, the expected profit is decreasing in y. For relatively low and high usage rates the posterior estimate of profit is more than the prior estimate, probably because for a nominal (or lower) usage rate, the number of item failures is low resulting in lower repair costs and more profit. As y increases the converse effect is seen for expected profit value, due to excessive number of product failures over warranty.

5.4.2 Sensitivity Analysis

From previous discussion, it is obvious that the prior analysis results are dependent on the pre-specified values of the moments of α and β . It can be clearly seen that, the effect of misjudgment on the values of these parametric moments, provided by the managers (experts) influence the estimates of warranty proportion ρ and expected profit for every usage rate y. Hence, it is interesting for the managers to study the cost behavior for mis-specification of μ_{α} , σ_{α} , μ_{β} , and σ_{β} respectively. Thus, a sensitivity analysis with respect to percentage changes in each of μ_{α} , σ_{α} , μ_{β} , and σ_{β} is performed to estimate the variations in profit and warranty proportion.

We have considered usage rates y = 0.5, 1, 1.5, 2, 3 and for each y, the variations with respect to percentage changes in the value of μ_{α} , σ_{α} , μ_{β} , and σ_{β} are computed, as shown in Tables 5.4 – 5.7. The first column shows the percentage change in the parameters followed by the estimates of warranty proportions and expected value of profit for each y. Figures 5.8 – 5.11 show the results of sensitivity analysis with respect to μ_{α} , σ_{α} , μ_{β} , and σ_{β} for some specific usage rates.

It is understandable that under-estimating μ_{α} (μ_{β}) would cause the underestimation of the warranty cost, leading to an improper decision such as mistakenly extending the warranty term and/or increasing the warranty proportion. Similarly, misjudging σ_{α} (σ_{β}) may result in more risky decisions being taken. Therefore, the managers should be very cautious with the judgments.

Some discussions on the effects of mis-specified prior moments on ρ and expected profit are included below Figures 5.8 - 5.11.

Table 5.4 Impact of Change in μ_{α} on ρ and Total Profit

= 3	Profit	245225024	229267156	226479168	220294404	213954432	211343472	201496880	191420928	191420928	191420928	191420928	191420928	191420928
<i>y</i>	φ	1	0.85	0.6	0.45	0.3	0.15	0.05	< 0.05	< 0.05	< 0.05	< 0.05	< 0.05	< 0.05
l = 2	Profit	281977269	272531269	263085269	253003026	245901056	245252763	242677533	239216627	233598901	233598901	233598901	233598901	233598901
5	φ	1	1	1	0.8	0.6	0.4	0.2	0.05	< 0.05	< 0.05	< 0.05	< 0.05	< 0.05
= 1.5	Profit	336477605	325615605	314753605	303891605	284835510	277687615	273284220	265210960	264173835	258352350	258352350	258352350	258352350
y	φ	1	1	1	1	0.9	0.7	0.5	0.3	0.1	< 0.05	< 0.05	< 0.05	< 0.05
= 1*	Profit	380175246	367261246	354347246	341433246	328519246	315605246	325936446	319135220	317253244	318183264	319118760	319118760	319118760
y	φ	1	1	1	1	1	1	0.7	0.4	0.2	0.05	< 0.05	< 0.05	< 0.05
5.	Profit	380175246	380175246	380175246	380175246	380175246	367261246	354347246	339431670	319943052	307755306	297687168	293683464	290033028
y	φ	1	1	1	1	1	1	1	0.45	0.25	0.15	0.1	0.05	< 0.05
	Change in μ_{α}	-0.3	-0.25	-0.2	-0.15	-0.1	-0.05	0	0.05	0.1	0.15	0.2	0.25	0.3

* Case of Huang and Fang [34].

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Table

		y = .5		$y = 1^*$		y = 1.5		y = 2		J = 3
τ_{α}	θ	Profit	φ	Profit	θ	Profit	θ	Profit	φ	Profit
	0.95	273347250	0.6	280077966	0.7	224856945	0.6	181244195	0.5	157368128
	0.80	277435796	0.4	299967220	0.5	243180940	0.4	214369444	0.25	183294560
	0.70	283611900	0.35	306910248	0.35	253696380	0.25	227769567	0.1	197003664
	0.65	292094928	0.55	304594182	0.15	260522145	0.05	227966677	0	199396800
	0.85	315443600	0.6	310303566	0.3	254610610	0.1	233074006	0	199396800
	0.95	337097250	0.65	317434306	0.4	259219630	0.15	240687395	0.05	200870880
	1.00	354347246	0.7	325936446	0.5	272388940	0.20	242677533	0.0500	201496880
	1.00	367261246	0.75	335909364	0.6	275636670	0.35	250805145	0.05	201809880
	1.00	380175246	0.9	350790264	0.8	281105550	0.5	259144173	0.2	203626400
	1.00	380175246	Т	368030880	0.9	296334060	0.7	261400830	0.45	207688320
	1.00	380175246	1	381950880	Ţ	312172945	0.85	275182908	0.6	216451824
	1.00	380175246	1	395870880	Η	323330945	-	294227597	0.8	219141920
	1.00	380175246		409790880	-	334488945		294227597	0.95	228642336
		-							_	

* Case of Huang and Fang [34].


Figure 5.8 Sensitivity analysis of ρ with respect to μ_{α} and σ_{α} for different usage rates y.

Here five different usage rates are considered as indicated in each block. The x-axis and y-axis corresponds to the percentages of deviation from the true values of the parameters μ_{β} and σ_{β} and the estimate of optimal ρ , respectively. The solid line shows the behavior of ρ for percentage deviation of μ_{α} from the true value 0.46. If the specified value of μ_{α} is less (more) than the true value, estimated optimal ρ is over-estimated (under-estimated, respectively). On the contrary if σ_{α} is less than the true value 0.21, ρ is initially under-estimated, but eventually over-estimated. If the specified σ_{α} is more than the true value, ρ is over-estimated.



Figure 5.9 Sensitivity analysis of Expected Profit with respect to μ_{α} and σ_{α} for different usage rates y.

Here, the solid line shows the behavior of expected profit for percentage deviation of μ_{α} from the true value 0.46. If the specified value of μ_{α} is less (more) than the true value, estimated profit is over-estimated (under-estimated, respectively). On the contrary if the specified value of σ_{α} is less (more) than the true value 0.21, estimated optimal ρ is initially under-estimated, but eventually over-estimated. If the specified σ_{α} is more than the true value, the estimated profit is under-estimated (over-estimated, respectively).

Table 5.6 Impact of Change in μ_{β} on ρ and Total Profit

= 3	Profit	225889312	218015312	206458200	202537324	200754304	199536480	201496880	196520256	196520256	196520256	196520256	196520256	196520256
y	φ	1	1	0.95	0.65	0.4	0.2	0.05	< 0.05	< 0.05	< 0.05	< 0.05	< 0.05	< 0.05
l = 2	Profit	270036669	260990669	251944669	242898669	234995126	238740789	242677533	246339501	246339501	246339501	246339501	246339501	246339501
3	θ	1	1	1	1	0.9	0.55	0.2	< 0.05	< 0.05	< 0.05	< 0.05	< 0.05	< 0.05
= 1.5	Profit	333873565	323147565	312421565	301695565	290969565	271569235	245043380	213673140	198156420	189588420	181020420	172452420	163884420
\overline{y}	θ	1	1	1	1	1	0.8	0.5	0.2	< 0.05	< 0.05	< 0.05	< 0.05	< 0.05
= 1*	Profit	380351880	367431880	354511880	341591880	328671880	328671880	325936446	319135220	318732064	319118760	319118760	319118760	319118760
Ŋ	θ	1	1	1	1	1	1	2.0	0.4	0.05	< 0.05	< 0.05	< 0.05	< 0.05
. 5.	Profit	380175246	380175246	380175246	380175246	380175246	367261246	354347246	337784664	301655662	262670372	241046676	230762676	220478676
Ŋ	φ	1		1	1	1	1	1	0.9	0.6	0.1	< 0.05	< 0.05	< 0.05
	Change in μ_{β}	-0.3	-0.25	-0.2	-0.15	-0.1	-0.05	0	0.05	0.1	0.15	0.2	0.25	0.3

* Case of Huang and Fang [34].

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$3 \text{ on } \rho$
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Table 5.7

		y = .5		$y = 1^*$		y = 1.5		y = 2	3	l = 3
ge in σ_{β}	φ	Profit	φ	Profit	φ	Profit	θ	Profit	φ	Profit
-0.3	1.0	328519246	0.9	265261464	0.8	253901375	0.6	210688276	0.2	204174912
-0.25	1.0	315605246	0.8	283852360	0.7	264197315	0.4	227017477	0.2	205559712
-0.2	1.0	315605246	0.8	294444360	0.7	271691515	0.3	234995977	0.1	206125344
-0.15	0.9	319768230	0.7	325936446	0.6	274897815	0.3	237491377	0.1	206784144
-0.1	0.9	319768230	0.7	325936446	0.6	281057415	0.3	239986777	0.1	207442944
-0.05	1.0	341433246	0.7	325936446	0.5	284754995	0.2	241073133	0.05	201183880
0	1.0	354347246	0.7	325936446	0.5	284754995	0.2	242677533	0.05	201496880
0.05	1.0	354347246	0.7	325936446	0.5	284754995	0.2	244281933	0.05	201809880
0.1	0.9	357478230	0.65	325724406	0.5	284754995	0.2	245886333	0.05	202122880
0.15	0.9	357478230	0.65	325724406	0.4	278299910	0.2	247490733	0.05	202435880
0.2	0.8	358684776	0.6	340529166	0.4	282141510	0.1	247963626	< 0.05	196585632
0.25	0.8	358684776	0.6	348085566	0.4	285983110	0.1	248752026	< 0.05	196585632
0.3	0.8	358684776	0.6	355641966	0.4	289824710	0.1	249540426	< 0.05	196585632

* Case of Huang and Fang [34].



Figure 5.10 Sensitivity analysis of ρ with respect to μ_{β} and σ_{β} for different usage rates y.

Here the solid (dotted) line shows the behavior of optimal warranty proportion ρ for percentage deviation of μ_{β} (σ_{β}) from the true value 2.6 (0.64). It can be seen that if the specified value of μ_{β} is less (more) than the true value, estimated optimal ρ is over-estimated (under-estimated, respectively). On the contrary if the specified value of σ_{β} is less than the true value, estimated optimal ρ is under-estimated for y = 0.5 and over-estimated for rest of the y's. If the specified σ_{β} is more than the true value, the estimated optimal ρ is either the same or is under-estimated for all y's. It is worth noting that the the effect of mis-specification of μ_{β} has more adverse effect on ρ , compares to that of σ_{β} .



Figure 5.11 Sensitivity analysis of Expected Profit with respect to μ_{β} and σ_{β} for different usage rates y.

Here the solid (dotted) line shows the behavior of optimal warranty proportion profit for percentage deviation of μ_{β} (σ_{β}) from the true value 2.6 (0.64). It can be seen that if the specified value of μ_{β} is less (more) than the true value, estimated profit is over-estimated (under-estimated, respectively). On the contrary if the specified value of σ_{β} is less (more) than the true value, estimated profit is under-estimated (over-estimated, respectively).

5.5 Concluding Remarks

In this chapter, we have investigated a profit optimization decision problem for 2-D warranties, by integrating several component models affecting the profit such as production, sales, warranty and maintenance. A decision regarding the optimal prorated warranty (PRW) proportion (paid by the manufacturer to repair failed item) and optimal warranty period that maximizes the expected profit of the firm under different usage rates of the consumers is explored here in a Bayesian framework. The first phase or, *prior analysis* is based on expert opinion and historical data (prior information). The second phase or, *posterior analysis* is based on prior information and market data. In most real life scenarios, managers dealing with such situations believe that the estimates obtained by posterior analysis that blends expert knowledge, prior believes and market data, are generally more accurate.

Our objective in studying this problem in a 2-D warranty context, is primarily to demonstrate the effect of varying usage rates on the final decision regarding the PRW proportion and warranty duration – which can be clearly seen from our results. Thus for items degrading due to both age and usage rate (almost every product in the market), the fact that an integrated decision problem modeling is incomplete without the consideration of usage rates has been demonstrated in this chapter. Finally a sensitivity analysis shows the effect (given below Figures 5.8 - 5.11) of mis-specification (of prior moments) on the firm's profit.

CHAPTER 6

CONCLUSION

6.1 Research Summary

In this dissertation, our focus is on the design and cost analysis of warranty that incorporate usage level of items as important factor that impacts on their failure profile and corresponding costs of service assurance by replacement or repair. We have considered usage sensitive warranty servicing strategies in several different setups, analyzed their theoretical and practical consequences and presented a comparative study of their behavior in terms of expected costs to those in the literature. It may be noted that the usage sensitiveness aspect of these models has given a different direction to this research in terms of applicability in real-life. We have broadly demonstrated considerations of usage sensitivity in warranty models impacts the cost behavior and under appropriate conditions can be a more efficient and realistic approach to designing such policies.

6.2 Some Possible Research Problems for the Future

6.2.1 Using Copulas to Model Warranties

Copulas are mathematical constructs that can fully capture the dependence structure among components of random vectors, and hence offer great flexibility in modeling joint distributions [44]. Formally a n-dimensional copula is any joint distribution on $[0,1]^n$, $n \ge 2$ with uniform marginal distributions [73]. If the products lifetime (X)and its usage (U) have a joint distribution H(x, u) and marginal distributions F(x)and G(u) respectively; their mutual relationship is captured by

$$H(x,y) := P(X \le x, U \le u) = C(F(x), G(u))$$
(6.1)

where C is a suitable copula on the unit square. Thus given H, F, and G there exists a copula C satisfying Equation (6.1). Conversely, given the marginals F, G and a copula C determines the joint distribution H of (X, U).

For 2-D warranty modeling and analysis, the joint distribution of (X, Y), is usually not known. Generally the marginal or conditional (on Y) distribution of item's failure time (X) is assumed to be Weibull (or Gamma) for computational purposes. Exploring the different types of joint - marginal distribution combinations of X and Y have received some importance in literature [55]. Problems demonstrating the use of copula functions in this context is an area that needs further investigation. In future work, we would like to use copula functions as the basic toolkit for identifying the distributional patterns associated with the 2-D warranty modeling.

6.2.2 Bayesian Warranty Policies

In the case of new products in the market there may not be enough historical data to completely specify a model, for the product's lifetime that can be adequately justified statistically. Absence of such knowledge can reflect the ignorance of *either* (i) the true value(s) of underlying parameter(s) of a parametric lifetime distribution model, *or* (ii) the distribution of the lifetime itself – except for some structural nonparametric assumptions about the product's degradation profile.

A Bayesian approach which updates the product's lifetime profile as failure and service cost data accumulate is appropriate here. While there is some literature on the subject (see e.g., [26], [63], [95]); further research focused on a Bayesian approach to warranties can profitably receive more attention.

In the setup of parametric life-distribution models with unknown parameter value(s) together with a prior distribution that reflects our beliefs about the latter; except for models with conjugate priors, solutions for Bayesian warranties in a closed form will be generally rare, relying instead on numerical solutions via computational methods. More realistic Bayesian warranty models will involve hierarchical modeling

and computationally intensive simulation using a Markov Chain Monte Carlo (MCMC) approach.

In the other case, when the lifetime distribution is unknown, one could imagine the possibility of exploiting a nonparametric Bayesian approach; which is methodologically an area of current active research and, to the best of our knowledge has not yet been used for warranty analysis. The development and application of Bayesian nonparametric methods in the context of warranties are therefore still in their infancy and awaits future research.

APPENDIX A

A SAMPLE PROGRAM

MATLAB program used to compute the optimal parameters K_y^* , L_y^* and the minimal expected cost $J^* \equiv J(K_y^*, L_y^*)$ for models in Chapter 3, Analysis of a 2-D Warranty Servicing Strategy with a Brown-Proschan Repair Option.

1. Constant Probability of Repair (p)

(% To compute the Weibull hazard rate function $h(x; \alpha(y))$.) function[h] = wei_haz_fn(b, g, y, x)

b1 = b * g; $y1 = y^{b1};$ y2 = y1 * b; b1 = (b - 1); $x1 = x^{b1};$ h = y2 * x1;

(% To compute the Weibull survival function $\overline{F}(x; \alpha(y))$.) $function[FbarX] = wei_surv_fn(b, g, y, x)$ $H = wei_cumhaz_fn(b, g, y, x);$ H1 = -H;FbarX = exp(H1);

(% To compute the Weibull cumulative hazard function $H(x; \alpha(y))$.) $function[H] = wei_cumhaz_fn(b, g, y, x)$ b1 = b * g; $y1 = y^{b1};$ $x1 = x^{b};$ (% To evaluate the equation g(x) = H(Wy) - H(x) - H(Wy - x).) $function[gx] = g_fn_value(Wy, x, b, g, y)$ $HWy = wei_cumhaz_fn(b, g, y, Wy)$; $HX = wei_cumhaz_fn(b, g, y, x)$; Wy1 = Wy - x; $HWyX = wei_cumhaz_fn(b, g, y, Wy1)$; gx = HWy - HX - HWyX;

(% To find the optimal L_y^* for a given usage rate y.) function[s] = find_opt_L(Cr, Cm, W, U, y, b, g) rho = Cr/Cm; rho1 = rho - 1; if (y <= U/W) Wy = W; else Wy = U/y; end Wy; syms x; g1 = g_fn_value(Wy, x, b, g, y); cond = rho1 - g1; s = solve(cond);

(% Given usage rate y and corresponding L_y^* , to find the optimal K_y^* .) $function[C1] = I_fn_root(p, y, L)$ Cr = 2; Cm = 1; W = 2; U = 2; b = 2; g = 2; rho = Cr/Cm; $if(y \le U/W)$ Wy = W;elseWy = U/y;end $FbarL = wei_surv_fn(b, q, y, L);$ $HWy = wei_{cumhaz} fn(b, q, y, Wy);$ $HL = wei_cumhaz_fn(b, g, y, L);$ A2 = HWy - HL;term5 = FbarL * A2;x = [0:.001:L];for i = 1 : length(x) $FbarX(i) = wei_surv_fn(b, q, y, x(i));$ $HX(i) = wei_cumhaz_fn(b, q, y, x(i));$ A1(i) = HWy - HX(i);term3(i) = A1(i) * FbarX(i); $gf(i) = g_{-}fn_{-}value(Wy, x(i), b, g, y);$ term1(i) = p * (gf(i) + 1 - rho) * FbarX(i); $term2(i) = int_IKyfun1(Wy, x(i), L, b, g, y, p);$ $term4(i) = int_IKyfun2(Wy, x(i), L, b, q, y);$ I1(i) = term1(i) + term2(i) - term3(i) + term4(i) + term5;I2(i) = 0;endI1;C = [x; I1];C1 = C';plot(x, I1, x, I2)

 $function[area1] = int_I Ky fun1(Wy, K, L, b, q, y, p)$ rho = 2;h = (L - K)/10000;for j = 1:10000x(j) = K + (j * h); $FbarX(j) = wei_surv_fn(b, g, y, x(j));$ $hazx(j) = wei_haz_fn(b, q, y, x(j));$ f1(j) = hazx(j) * FbarX(j) $;gfn(j) = g_fn_value(Wy, x(j), b, g, y);$ f2(j) = rho - 1 - qfn(j);f3(j) = p * f2(j) * f1(j);endf3;x;fsum = sum(f3);area1 = h * fsum;

$$\begin{split} function[area2] &= int_IKyfun2(Wy, K, L, b, g, y) \\ HWy &= wei_cumhaz_fn(b, g, y, Wy); \\ h &= (L - K)/10000; \\ for \quad j = 1:10000 \\ x(j) &= K + (j * h); \\ FbarX(j) &= wei_surv_fn(b, g, y, x(j)); \\ hazx(j) &= wei_haz_fn(b, g, y, x(j)); \\ f1(j) &= hazx(j) * FbarX(j); \\ HX(j) &= wei_cumhaz_fn(b, g, y, x(j)); \\ h1(j) &= 1 + HWy - HX(j); \end{split}$$

h2(j) = h1(j) * f1(j);end hsum = sum(h2);area2 = h * hsum;

(% To find the minimum cost J^* for a given usage rate y and corresponding optimal K_{y}^{*} and L_{y}^{*} .) $function[minco] = bpoptco_final(p, y, K, L)$ W = 2; U = 2; b = 2; g = 2; $if(y \le U/W)$ Wy = W: elseWy = U/y;end $FbarL = wei_surv_fn(b, q, y, L);$ $FbarK = wei_surv_fn(b, q, y, K);$ $HWy = wei_cumhaz_fn(b, g, y, Wy);$ $HK = wei_cumhaz_fn(b, g, y, K);$ $HL = wei_cumhaz_fn(b, g, y, L);$ A2 = HWy - HL;term1 = HK * FbarK; $term2 = int_I Ky fun1(Wy, K, L, b, q, y, p);$ $term3 = int_IKyfun2(Wy, K, L, b, g, y);$ term4 = FbarL * A2;brac = term1 + term2 + term3 + term4;minco = brac/FbarK;

2. Age-dependent Probability of Repair (p(t))

(% To evaluate probability function $p_1(t) = 1/(c+t)$, c = Cr - Cm.) function[p1] = bbscase1(Cr, Cm, t) c = Cr - Cm; c1 = c + t;p1 = 1/c1;

(% To evaluate probability function $p_2(t) = 1 - exp(-t)$) function[p2] = bbscase2(t) t1 = -t; e = exp(t1);p2 = 1 - e;

(% To evaluate the equation g(x) = H(Wy) - H(x) - H(Wy - x).) $function[gx] = g_fn_value(Wy, x, b, g, y)$ $HWy = wei_cumhaz_fn(b, g, y, Wy)$; $HX = wei_cumhaz_fn(b, g, y, x)$; Wy1 = Wy - x; $HWyX = wei_cumhaz_fn(b, g, y, Wy1)$; gx = HWy - HX - HWyX;

(% To find the optimal L_y^* for a given usage rate y.) $function[s] = find_opt_L(Cr, Cm, W, U, y, b, g)$ rho = Cr/Cm; rho1 = rho - 1; $if(y \le U/W)$ Wy = W;else

$$Wy = U/y;$$

end
$$Wy;$$

syms x;
$$g1 = g_{-}fn_{-}value(Wy, x, b, g, y);$$

cond = rho1 - g1;
$$s = solve(cond);$$

 $function[Area_int] = int_IKyfun1_BBS(Wy, K, L, b, g, y)$ Cr = 2; Cm = 1;rho = 2;h = (L - K)/10000;for j = 1:10000x(j) = K + (j * h); $FbarX(j) = wei_surv_fn(b, g, y, x(j));$ $hazx(j) = wei_haz_fn(b, q, y, x(j));$ f1(j) = hazx(j) * FbarX(j); $gfn(j) = g_{-}fn_{-}value(Wy, x(j), b, g, y);$ f2(j) = rho - 1 - gfn(j);p1(j) = bbscase1(Cr, Cm, x(j));p2(j) = bbscase2(x(j));F11(j) = p1(j) * f2(j) * f1(j);F22(j) = p2(j) * f2(j) * f1(j);end F11;F22;x;F11sum = sum(F11);

F22sum = sum(F22); Area11 = h * F11sum; Area22 = h * F22sum; $Area_int = [Area11, Area22];$

 $function[area2] = int_I Ky fun2(Wy, K, L, b, g, y)$ $HWy = wei_cumhaz_fn(b, g, y, Wy);$ h = (L - K)/10000; $for \quad j = 1:10000$ x(j) = K + (j * h); $FbarX(j) = wei_surv_fn(b, g, y, x(j));$ $hazx(j) = wei_haz_fn(b, g, y, x(j));$ f1(j) = hazx(j) * FbarX(j); $HX(j) = wei_cumhaz_fn(b, g, y, x(j));$ h1(j) = 1 + HWy - HX(j); h2(j) = h1(j) * f1(j); end hsum = sum(h2); area2 = h * hsum;

(% Given y and corresponding L_y^* , to find the optimal K_y^* for both $p_1(t)$ and $p_2(t)$.) $function[C1] = I_fn_root_BBS(y, L)$ Cr = 2; Cm = 1; W = 2; U = 2; b = 2; g = 2; rho = Cr/Cm; $if(y \le U/W)$ Wy = W;else

Wy = U/y;end $FbarL = wei_surv_fn(b, q, y, L);$ $HWy = wei_cumhaz_fn(b, q, y, Wy);$ $HL = wei_cumhaz_fn(b, q, y, L);$ A2 = HWy - HL;term5 = FbarL * A2;x = 0 : .001 : L;for i = 1 : length(x) $area = int_IKyfun1_BBS(Wy, x(i), L, b, g, y);$ $FbarX(i) = wei_surv_fn(b, q, y, x(i));$ $HX(i) = wei_cumhaz_fn(b, q, y, x(i));$ A1(i) = HWy - HX(i);term3(i) = A1(i) * FbarX(i); $qf(i) = q_fn_value(Wy, x(i), b, q, y);$ p1(i) = bbscase1(Cr, Cm, x(i));p2(i) = bbscase2(x(i));% case1term11(i) = p1(i) * (gf(i) + 1 - rho) * FbarX(i);term21(i) = area(1); $term4(i) = int_IKyfun2(Wy, x(i), L, b, q, y);$ I11(i) = term11(i) + term21(i) - term3(i) + term4(i) + term5;% case2term12(i) = p2(i) * (qf(i) + 1 - rho) * FbarX(i);term22(i) = area(2);I12(i) = term12(i) + term22(i) - term3(i) + term4(i) + term5;I2(i) = 0;end

I11; I12; C = [x; I11; I12]; C1 = C'; plot(x, I11, x, I12, x, I2)

(% Given y, K_y^* and L_y^* , to find the minimum cost J^* for both $p_1(t)$ and $p_2(t)$.) $function[minco] = BBSoptco_final(y, K1, K2, L)$ W = 2; U = 2; b = 2; q = 2; $if(y \le U/W)$ Wy = W;elseWy = U/y;end $FbarL = wei_surv_fn(b, q, y, L);$ $FbarK = [wei_surv_fn(b, q, y, K1), wei_surv_fn(b, q, y, K2)];$ $HWy = wei_cumhaz_fn(b, g, y, Wy);$ $HK = [wei_cumhaz_fn(b, g, y, K1), wei_cumhaz_fn(b, g, y, K2)];$ $HL = wei_cumhaz_fn(b, q, y, L);$ A2 = HWy - HL; $area = [int_IKyfun1_BBS(Wy, K1, L, b, g, y);$ $int_IKyfun1_BBS(Wy, K2, L, b, q, y)$] %Case1 when p(t) = 1/(c+t), c = Cr - Cm. term11 = HK(1) * FbarK(1);term21 = area(1,1) $term31 = int_IKyfun2(Wy, K1, L, b, q, y);$ term41 = FbarL * A2;brac1 = term11 + term21 + term31 + term41;

$$\begin{split} minco1 &= brac1/FbarK(1); \\ \% \text{Case2 when } p(t) &= 1 - exp(-t) \\ term12 &= HK(2) * FbarK(2); \\ term22 &= area(2,2) \\ term32 &= int_IKyfun2(Wy,K2,L,b,g,y); \\ term42 &= FbarL * A2; \\ brac2 &= term12 + term22 + term32 + term42; \\ minco2 &= brac2/FbarK(2); \\ minco &= [minco1, minco2]; \end{split}$$

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