Spring 2010

Modeling with bivariate geometric distributions

Jing Li
New Jersey Institute of Technology

ABSTRACT

MODELING WITH BIVARIATE GEOMETRIC DISTRIBUTIONS

by
Jing Li

This dissertation studied systems with several components which were subject to different types of failures. Systems with two components having frequency counts in the domain of positive integers, and the survival time of each component following geometric or mixture geometric distribution can be classified into this category. Examples of such systems include twin engines of an airplane and the paired organs in a human body. It was found that such a system, using conditional arguments, can be characterized as multivariate geometric distributions. It was proved that these characterizations of the geometric models can be achieved using conditional probabilities, conditional failure rates, or probability generating functions. These new models were fitted to real-life data using the maximum likelihood estimators, Bayes estimators, and method of moment estimators. The maximum likelihood estimators were obtained by solving score equations. Two methods of moments estimators were compared in each of the several bivariate geometric models using the estimated bias vectors and the estimated variance-covariance matrices. This comparison was done through a Monte-Carlo simulation for increasing sample sizes. The Chi-square goodness-of-fit tests were used to evaluate model performance.

MODELING WITH BIVARIATE GEOMETRIC DISTRIBUTIONS

by
Jing Li

A Dissertation
Submitted to the Faculty of
New Jersey Institute of Technology and
Rutgers, The State University of New Jersey – Newark
in Partial Fulfillment of the Requirements for the Degree of
Doctor of Philosophy in Mathematical Sciences

Department of Mathematical Sciences, NJIT
Department of Mathematics and Computer Science, Rutgers-Newark

May 2010

APPROVAL PAGE

MODELING WITH BIVARIATE GEOMETRIC DISTRIBUTIONS

Jing Li

Sunil Dhar, Dissertation Advisor — Date
Associate Professor, Department of Mathematical Sciences, NJIT

Manish Bhattacharjee, Committee Member — Date
Professor, Department of Mathematical Sciences, NJIT

Sundar Subramanian, Committee Member — Date
Associate Professor, Department of Mathematical Sciences, NJIT

Wenge Guo, Committee Member — Date
Assistant Professor, Department of Mathematical Sciences, NJIT

Aridaman K. Jain, Committee Member — Date
Senior University Lecturer, Department of Mathematical Sciences, NJIT

Ganesh K. Subramaniam, Committee Member — Date
Researcher, Research/AT&T Labs

BIOGRAPHICAL SKETCH

Author: Jing Li

Degree: Doctor of Philosophy

Date: May 2010

Undergraduate and Graduate Education:

- Doctor of Philosophy in Mathematical Sciences,
  New Jersey Institute of Technology, Newark, NJ, 2010

- Bachelor of Communication Engineering in Computer Science,
  Shanghai Maritime University, Shanghai, China, 2005

Major: Mathematical Sciences

Presentations and Publications:

Jing Li, Sunil Dhar, "Modeling with bivariate geometric distributions," *Communications in Statistics,* in preparation.

Jing Li, Jason Wang, Dongrong Wen, Bruce Shapiro, Katherine Herbert, and Kaushik Ghosh, "Toward an integrated RNA motif database," *Data Integration in the Life Sciences*, Volume 4544, pp.27–36, 2007

Jing Li, "Modeling with bivariate geometric distributions," *Fifth Annual Graduate Student Research Day*, Department of Mathematical Sciences, NJIT, November 11, 2009.

Jing Li, "Modeling with bivariate geometric distributions," *Summer Seminar Series*, Department of Mathematical Sciences, NJIT, July 11, 2009.

Jing Li, Sunil Dhar "Modeling with bivariate geometric distributions," *NJIT ADVANCE Research Showcase*, Department of Mathematical Sciences, NJIT, March 31, 2009.

This dissertation is dedicated to my wonderful parents. Thank you for all the unconditional love, guidance, and support that you have given me. Thank you for everything.

ACKNOWLEDGMENT

I would like to express my gratitude to my advisor Dr. Sunil K. Dhar, who not only introduced me to this problem, but also gave me countless ideas and suggestions. Without his constant encouragement and help, I would not have been able to complete this work. I would also like to thank all the members of my dissertation committee. Their advice and patience is greatly appreciated.

Special thanks go to Dr. Chung Chang and Dr. Wenge Guo who greatly enriched my knowledge with their exceptional insights into different areas of modern statistics. I would also like to acknowledge Dr. Manish Bhattacharjee for his invaluable advice, both with technical and non-technical issues. I am grateful to Dr. Aridaman Jain for his constant support, encouragement, and advice.

I am also thankful to the Department of Mathematical Sciences and the University for supporting my work, and for providing computing and other facilities during the course of my work.

Lastly, I would like to thank my family, especially my uncles and aunts for their love, support, and prayers.

I dedicate this dissertation to my parents.

TABLE OF CONTENTS

TABLE OF CONTENTS
(Continued)

LIST OF TABLES

CHAPTER 1

INTRODUCTION

1.1 Motivation

Most statistical models and methods for lifetime data are used to describe continuous nonnegative lifetime variables. However, it is sometimes more appropriate or convenient to measure lifetime using discrete random variables, as for instance, the incubation period of diseases such as AIDS, the remission time of cancers, and the time-to-failure of engineering systems. Discrete lifetimes should be used when either the clock time is not the best scale for measuring lifetime or the lifetime is measured discretely. In most cases, the lifetime data under study is not determined by a univariate distribution. Discrete multivariate distributions provide a natural answer to measure lifetime data. In particular, bivariate discrete distributions can be a useful way to study lifetime data involving a mixture of two discrete random variables. When a bivariate study is of sufficient duration so that multiple events may occur, within-subject correlation may be present and require special statistical consideration. Bivariate geometric distributions are such models which can retain within-subject correlation, while the marginal distributions are simple geometric or mixture geometric distributions.

1.2 Background

A variety of bivariate models have been proposed in statistics to represent lifetime data. Freund (1961) constructed his model as a bivariate extension of two exponential distributions. Marshall and Olkin (1985) studied a family of bivariate distributions generated by the bivariate Bernoulli distributions. Nair and Nair (1988) studied the characterizations of the bivariate exponential and geometric distributions. Basu and Dhar (1995) proposed a bivariate geometric model (BGD (B&D)) which is analog to the bivariate distribution of Marshall and Olkin (1967). Dhar (1998) derived a new bivariate geometric model (BGD (F)) which is a discrete analog to Freund's model.

In this dissertation, the bivariate fatal shock model derived by Basu and Dhar (1995) has been studied. However, this model is a reparameterized version of the bivariate geometric model of Hawkes (1972) and in contrast the BGD (B&D) random vector takes values in the set of cross-product of positive integers with itself. The other bivariate geometric model studied is the BGD (F) which deserves more explorations. Thus, this research derives several characterizations of BGD (F) and the other models.

Some of these characterizations are through conditionally specified distributions. The characterization for BGD (F) is studied in Chapter 3, while that of the BGD (B&D), has been done by Sreehari (2005) through Hawkes' model. Cox (1972) introduced conditional failure rate (CFR) in the area of reliability. Sun and Basu (1995) derived the characterization result based on this CFR for the BGD (B&D) model. Sreehari (2005) used a revised version of conditional failure rate to derive the characterization theorem for the BGD (B&D) through Hawkes' (1972) model. The present research derives characterization for the BGD (F) using CFR from Sreehari (2005) in Chapter 4. In Chapter 5, the joint probability generating function (p.g.f.) of the random variables (X, Y) from BGD (F) is derived, and verified, using the relationship between joint probability mass function and p.g.f. in terms of repeated partial derivatives (Kocherlakota & Kocherlakota, 1992).

In the context of reliability, we use the concept of CFR function introduced by Cox (1972). Two methods are discussed in the process of characterizing BGD (B&D) model, and BGD (F) model through CFR. The characterization theorem for BGD (B&D) model was derived by Sreehari (2005) through BGD (H) model. Kotz and Johnson (1991) gave a new definition of CFR. The relation between these two conditional failure rates will be examined as future work. Using the new conditional failure rate, more characterization results will be explored. Roy (1993) considered another bivariate failure rate and bivariate mean residual life function. Their usage will also be explored to derive characterization results for the various other bivariate geometric models.

In Chapter 6, application of modeling is performed to a real data set based on the BGD (B&D) model. The three estimation methods described in Chapter 2 are applied and compared on this data set. A Monte Carlo simulation is generated to study and compare different estimation methods. Bias vector and variance covariance matrix of the estimated parameter vector are estimated from this simulation. Chi-square goodness-of-fit tests are used to obtain the best fitted model based on different estimation methods.

CHAPTER 2

ESTIMATION METHODS USED IN BGD'S

In this chapter, the bivariate geometric model BGD (B&D) derived by Basu and Dhar (1995), and BGD (F) model derived by Dhar (1998) have been studied, and three methods of estimation are described, respectively, in the context of these models.

2.1 Estimation Methods Used in BGD (B&D)

2.1.1 Maximum Likelihood Estimation

The bivariate geometric distribution derived by Basu and Dhar (1995) (BGD (B&D)) is recalled in this section. A two-component system fails due to three types of failures: failure of component 1 only, failure of component 2 only, and simultaneous failure of the two components. The three processes are treated to be independent binomial with different failure rates. Let $B(x, 1-p_1)$, $B(y, 1-p_2)$, and $B(x \vee y, 1-p_{12})$ denote the three failure binomial processes. Suppose X and Y have discrete lifetime distributions of components 1 and 2 with support on $\mathbf{Z}^+ \times \mathbf{Z}^+$, respectively. Then the lifetime of the system is represented in terms of bivariate random variables. The system survival function is given by:

$$\begin{aligned} P(X > x, Y > y) &= P(B(x, p_1) = 0, B(y, p_2) = 0, B(x \vee y, p_{12}) = 0) \\ &= p_1^x p_2^y p_{12}^{x \vee y}, \end{aligned}$$

where $1 \leq x, y \in \mathbf{Z}^+$, $0 < p_1 < 1$, $0 < p_2 < 1$, and $0 < p_{12} \leq 1$. Here, $x \vee y=$ max (x,y) and $\mathbf{Z}^+$ is the set of positive integers. It is seen that the survival function satisfies the loss of memory property without any additional parameter restrictions, namely,

$$P(X > s_1 + t, Y > s_2 + t | X > s_1, Y > s_2) = P(X > t, Y > t) = (p_1 p_2 p_{12})^t,$$

$1 \leq s_1, s_2, t \in \mathbf{Z}^+$.

From the survival function, we see that

$$\begin{aligned} P(X = x, Y = y) = &P(X > x-1, Y > y-1) - P(X > x, Y > y-1) \\ &- P(X > x-1, Y > y) + P(X > x, Y > y), \end{aligned}$$

i.e,

$$P(X = x, Y = y) = \begin{cases} p_1^{x-1}(p_2p_{12})^{y-1}q_1(1-p_2p_{12}), & \text{if } x < y, \\ (p_1p_2p_{12})^{x-1}(1-p_1p_{12}-p_2p_{12}+p_1p_2p_{12}), & \text{if } x = y, \\ p_2^{y-1}(p_1p_{12})^{x-1}q_2(1-p_1p_{12}), & \text{if } x > y, \end{cases} \tag{2.1}$$

where $1 \leq x, y \in \mathbf{Z}^+$, $q_i = 1 - p_i$, $i = 1, 2$.

The likelihood function for this model is:

$$\begin{aligned} L(\mathbf{x}, \mathbf{y}|p_1, p_2, p_{12}) = &p_1^{\sum_{i=1}^n x_i - n} p_2^{\sum_{i=1}^n y_i - n} p_{12}^{\sum_{i=1}^n ((y_i-1)I[x_i<y_i]+(x_i-1)I[y_i \leq x_i])} q_1^{\sum_{i=1}^n I[x_i<y_i]} \\ &(1-p_2p_{12})^{\sum_{i=1}^n I[x_i<y_i]} q_2^{\sum_{i=1}^n I[y_i<x_i]} (1-p_1p_{12})^{\sum_{i=1}^n I[y_i<x_i]} \\ &(1-p_1p_{12}-p_2p_{12}+p_1p_2p_{12})^{\sum_{i=1}^n I[x_i=y_i]}, \end{aligned} \tag{2.2}$$

and the log-likelihood function is:

$$\begin{aligned} logL = &(\sum_{i=1}^n x_i - n)logp_1 + (\sum_{i=1}^n y_i - n)logp_2 + \{\sum_{i=1}^n ((y_i - 1)I[x_i < y_i] + \\ &(x_i - 1)I[y_i \leq x_i])\}logp_{12} + \sum_{i=1}^n I[x_i < y_i]logq_1 + \sum_{i=1}^n I[x_i < y_i]log(1 - p_2p_{12}) \\ &+ \sum_{i=1}^n I[y_i < x_i]logq_2 + \sum_{i=1}^n I[y_i < x_i]log(1 - p_1p_{12}) + \\ &\sum_{i=1}^n I[x_i = y_i]log(1 - p_1p_{12} - p_2p_{12} + p_1p_2p_{12}). \end{aligned} \tag{2.3}$$

To obtain the maximum likelihood estimators (MLE), the following score equations are to be solved:

$$\frac{\partial \log L}{\partial p_1} = \frac{\sum_{i=1}^n x_i - n}{p_1} - \frac{\sum_{i=1}^n I[x_i < y_i]}{1-p_1} + \frac{(-p_{12})\sum_{i=1}^n I[y_i < x_i]}{1-p_1p_{12}} + \left(\frac{\sum_{i=1}^n I[x_i = y_i]}{1-p_1p_{12}-p_2p_{12}+p_1p_2p_{12}}\right)(-p_{12}+p_2p_{12}) = 0\,,$$

$$\frac{\partial \log L}{\partial p_2} = \frac{\sum_{i=1}^n y_i - n}{p_2} - \frac{\sum_{i=1}^n I[y_i < x_i]}{1-p_2} + \frac{(-p_{12})\sum_{i=1}^n I[x_i < y_i]}{1-p_2p_{12}} + \left(\frac{\sum_{i=1}^n I[x_i = y_i]}{1-p_1p_{12}-p_2p_{12}+p_1p_2p_{12}}\right)(-p_{12}+p_1p_{12}) = 0\,,$$

$$\frac{\partial \log L}{\partial p_{12}} = \frac{(-p_1)\sum_{i=1}^n I[y_i < x_i]}{1-p_1p_{12}} + \frac{(-p_2)\sum_{i=1}^n I[x_i < y_i]}{1-p_2p_{12}} + \frac{\sum_{i=1}^n((y_i-1)I[x_i < y_i] + (x_i-1)I[y_i \le x_i])}{p_{12}} + \frac{(-p_1-p_2+p_1p_2)\sum_{i=1}^n I[x_i = y_i]}{1-p_1p_{12}-p_2p_{12}+p_1p_2p_{12}} = 0\,.$$

In order to simplify the expression, let $a = \sum_{i=1}^n x_i - n$, $b = \sum_{i=1}^n y_i - n$, $c = \sum_{i=1}^n I[y_i < x_i]$, $d = \sum_{i=1}^n I[x_i < y_i]$, $e = \sum_{i=1}^n((y_i-1)I[x_i < y_i] + \sum_{i=1}^n(x_i - 1)I[y_i \le x_i])$, and $g = \sum_{i=1}^n I[x_i = y_i]$.

Then the score equation set is rewritten as:

$$\frac{a}{p_1} - \frac{d}{1-p_1} - \frac{cp_{12}}{1-p_1p_{12}} + \frac{g(-p_{12}+p_2p_{12})}{1-p_1p_{12}-p_2p_{12}+p_1p_2p_{12}} = 0\,, \tag{2.4}$$

$$\frac{b}{p_2} - \frac{c}{1-p_2} - \frac{dp_{12}}{1-p_2p_{12}} + \frac{g(-p_{12}+p_1p_{12})}{1-p_1p_{12}-p_2p_{12}+p_1p_2p_{12}} = 0\,, \tag{2.5}$$

$$\frac{e}{p_{12}} - \frac{cp_1}{1-p_1p_{12}} - \frac{dp_2}{1-p_2p_{12}} + \frac{g(-p_1-p_2+p_1p_2)}{1-p_1p_{12}-p_2p_{12}+p_1p_2p_{12}} = 0\,. \tag{2.6}$$

The equation set is not easy to solve in a closed form due to the complex calculation. However, it is possible to obtain the explicit expression for the equation set if a data set $(\mathbf{x}, \mathbf{y})$ is given, since the values of a, b, c, d, e, and g are only related to the data set itself. Thus the equation set becomes easy, and obtaining the MLE's for p_1, p_2 and p_{12} is less challenging. An example will be given to illustrate this method in Chapter 6.

2.1.2 Bayes Estimation

The likelihood function (2.2) for the model expressed using notation "a–g" is given by:

$$L(\mathbf{x}, \mathbf{y}|p_1, p_2, p_{12}) = p_1^a p_2^b p_{12}^e q_1^d q_2^c (1 - p_1 p_{12})^c (1 - p_2 p_{12})^d \\ \cdot (1 - p_1 p_{12} - p_2 p_{12} + p_1 p_2 p_{12})^g \,. \tag{2.7}$$

Krishna & Pundir (2009) did Bayes estimation for some bivariate geometric distribution using a bivariate Dirichlet distribution (BDD) as the prior distribution whose posterior distribution is also BDD. However, using BDD as prior for BGD (B&D) to solve the Bayes estimator is not used here because the posterior distribution for BGD (B&D) is not BDD any more.

Note that this method of using uniform prior on the BGD (B&D) would be different from that of Krishna & Pundir's Bayes estimation using uniform prior on the BGD of Hawkes's (1972) model, due to its different parameters.

Accordingly, instead of using BDD as prior, a uniform prior distribution on (p_1, p_2, p_{12}) is considered here with probability density function (pdf)

$$f(p_1, p_2, p_{12}) = \begin{cases} 1, & \text{if } 0 < p_1, p_2 < 1,\ 0 < p_{12} \leq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.8}$$

Thus, the posterior distribution of (p_1, p_2, p_{12}) is in the form of

$$\pi(p_1, p_2, p_{12}|\mathbf{x}, \mathbf{y}) = \frac{1}{C} p_1^a p_2^b p_{12}^e q_1^d q_2^c (1 - p_1 p_{12})^c (1 - p_2 p_{12})^d \cdot (1 - p_1 p_{12} - p_2 p_{12} + p_1 p_2 p_{12})^g , \tag{2.9}$$

where $0 < p_1, p_2 < 1$, $0 < p_{12} \leq 1$, and C is the constant obtained by integrating the likelihood function (2.7) with respect to p_1, p_2, p_{12} each from zero to one.

However, it is again not easy to obtain a closed form of the Bayes estimators of p_1, p_2, p_{12} due to the complex form of the posterior distribution (2.9). Nevertheless, the expression of (2.9) can be simplified because the posterior distribution is similar to the likelihood function in MLE computation. The process to obtain Bayes estimator is now described as follows.

Using mean square error with Euclidean norm as the risk function, namely, $E\|\hat{\mathbf{p}} - \mathbf{p}\|^2$, where vector $\hat{\mathbf{p}}$ is the estimator of the vector of parameters $\mathbf{p} = (p_1, p_2, p_{12})$, the Bayes estimate of the unknown parameter is simply the conditional mean of the posterior distribution.

$$p_i^* = E(p_i|\mathbf{x}, \mathbf{y}) = \int_0^1 p_i f(p_i|\mathbf{x}, \mathbf{y}) dp_i , \tag{2.10}$$

where $p_i = p_1, p_2, p_{12}$, and $f(p_i|\mathbf{x}, \mathbf{y})$ is the marginal posterior distribution of p_1, p_2, p_{12}, respectively, i.e,

$$f(p_i|\mathbf{x}, \mathbf{y}) = \int_0^1 \int_0^1 \pi(p_1, p_2, p_{12}|\mathbf{x}, \mathbf{y}) dp_j dp_k , \tag{2.11}$$

where $i, j, k = 1, 2, 12$, and $i \neq j \neq k$. The application of this method is also illustrated in Chapter 6.

2.1.3 Method of Moments

Method of moments was used in Dhar (1998) to estimate the parameters by fitting a discrete bivariate geometric distribution to practical data sets. In this subsection, this method is used to evaluate the parameters for the BGD (B&D) model as defined in equality (2.1). Let us first derive the marginal survival functions of X and Y for model (2.1):

$$P(X > x) = P(X > x, Y > 0) = (p_1 p_{12})^x,$$

$$P(Y > y) = P(X > 0, Y > y) = (p_2 p_{12})^y,$$

where x, $y = 0, 1, 2....$

Then the marginal distributions of X and Y are:

$$P(X = x) = P(X > x - 1) - P(X > x) = (1 - p_1 p_{12})(p_1 p_{12})^{x-1}, \tag{2.12}$$

$$P(Y = y) = P(Y > y - 1) - P(Y > y) = (1 - p_2 p_{12})(p_2 p_{12})^{y-1}, \tag{2.13}$$

where x, $y = 1, 2....$

In order to apply method of moments, we have to find moments.

$$E(X) = \sum_{x=1}^{\infty} x P(X = x) = (1 - p_1 p_{12})^{-1},$$

and

$$E(Y) = \sum_{x=1}^{\infty} y P(Y = y) = (1 - p_2 p_{12})^{-1}.$$

Three parameters need to be estimated in this case. Thus, a second moment for X and Y is chosen as:

$$E(XY) = \frac{(1 - p_1 p_2 p_{12}^2)}{(1 - p_1 p_{12})(1 - p_2 p_{12})(1 - p_1 p_2 p_{12})}.$$

Replacing the population moments by their sample equivalents, we have

$$(1 - p_1 p_{12})^{-1} = \overline{x}, \tag{2.14}$$

$$(1 - p_2 p_{12})^{-1} = \overline{y}, \tag{2.15}$$

$$\frac{(1 - p_1 p_2 p_{12}^2)}{(1 - p_1 p_{12})(1 - p_2 p_{12})(1 - p_1 p_2 p_{12})} = \overline{z}, \tag{2.16}$$

where $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$, and $\overline{z} = \frac{1}{n}\sum_{i=1}^{n} x_i y_i$ represent the sample moments. These equations are solved to yield the method of moments estimators for p_1, p_2, and p_{12}. Let's denote this method as MOM1.

$$\tilde{p}_1 = \frac{\overline{y}(\overline{z} - \overline{x} - \overline{y} + 1)}{\overline{z}(\overline{y} - 1)}, \tag{2.17}$$

$$\tilde{p}_2 = \frac{\overline{x}(\overline{z} - \overline{x} - \overline{y} + 1)}{\overline{z}(\overline{x} - 1)}, \tag{2.18}$$

$$\tilde{p}_{12} = \frac{\overline{z}(\overline{x} - 1)(\overline{y} - 1)}{\overline{x} \cdot \overline{y}(\overline{z} - \overline{x} - \overline{y} + 1)}. \tag{2.19}$$

An alternative moment considered here is $E[min(X, Y)]$ instead of $E(XY)$, which is found to be $(1 - p_1 p_2 p_{12})^{-1}$. Then, an equation set can be constructed as:

$$(1 - p_1 p_{12})^{-1} = \overline{x},$$

$$(1 - p_2 p_{12})^{-1} = \overline{y},$$

$$(1 - p_1 p_2 p_{12})^{-1} = \overline{w},$$

where $\overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}$, $\overline{y} = \frac{\sum_{i=1}^{n} y_i}{n}$, and $\overline{w} = \frac{\sum_{i=1}^{n} min(x_i, y_i)}{n}$.

Then the alternative set of method of moments estimators are

$$\tilde{p}_1 = \frac{\overline{y} - \overline{y} \cdot \overline{w}}{\overline{w} - \overline{y} \cdot \overline{w}}, \tag{2.20}$$

$$\tilde{p}_2 = \frac{\overline{x} - \overline{x} \cdot \overline{w}}{\overline{w} - \overline{x} \cdot \overline{w}}, \tag{2.21}$$

$$\tilde{p}_{12} = \frac{\overline{w} - \overline{x} \cdot \overline{w} - \overline{y} \cdot \overline{w} + \overline{x} \cdot \overline{y} \cdot \overline{w}}{\overline{x} \cdot \overline{y} \cdot \overline{w} - \overline{x} \cdot \overline{y}}. \tag{2.22}$$

where $\overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}$, $\overline{y} = \frac{\sum_{i=1}^{n} y_i}{n}$, and $\overline{w} = \frac{\sum_{i=1}^{n} min(x_i, y_i)}{n}$. Let us denote this method as MOM2.

2.2 Estimation Methods Used in BGD (F)

The BGD (F) model discussed here was derived and motivated by Dhar (1998, 2003). Part of this bivariate geometric distribution was motivated based on the idea of Freund reliability models for the continuous case. The model is given by:

$$P(X = m, Y = n) = \begin{cases} \frac{q_1 q_4}{p_1 p_4} \left[\frac{p_1 p_2}{p_4}\right]^m p_4^n, & \text{if } m < n,\ m, n = 1, 2, \cdots, \\ \frac{q_1 q_2 q_{12}}{1 - p_1 p_2} p_{12}^{m-1}, & \text{if } m = n,\ m, n = 1, 2, \cdots, \\ \frac{q_2 q_3}{p_2 p_3} \left[\frac{p_1 p_2}{p_3}\right]^n p_3^m, & \text{if } m > n,\ m, n = 1, 2, \cdots. \end{cases} \tag{2.23}$$

where $0 < p_i < 1$, $p_i + q_i = 1$, $i = 1, 2, 3, 4$, $0 < p_{12} < 1$, $p_{12} + q_{12} = 1$, $p_1 p_2 < p_3$, and $p_1 p_2 < p_4$.

2.2.1 Maximum Likelihood Estimation

The likelihood function for the BGD (F) under the assumption $p_{12} = p_1 p_2$ in the region $\{(x, y) : x, y = 1, 2, \cdots\}$ can be written as:

$$L(\mathbf{x}, \mathbf{y} | p_1, p_2, p_3, p_4) = p_1^{d+e-b+\phi-c} q_1^{b+c} p_2^{d+e-a+\phi-c} q_2^{a+c} p_3^{g-d-a} q_3^a p_4^{h-e-b} q_4^b, \tag{2.24}$$

where $a = \sum_{i=1}^{n} I\{x_i > y_i\}$, $b = \sum_{i=1}^{n} I\{x_i < y_i\}$, $c = \sum_{i=1}^{n} I\{x_i = y_i\}$, $d = \sum_{i=1}^{n} y_i I\{x_i > y_i\}$, $e = \sum_{i=1}^{n} x_i I\{x_i < y_i\}$, $g = \sum_{i=1}^{n} x_i I\{x_i > y_i\}$, $h = \sum_{i=1}^{n} y_i I\{x_i < y_i\}$, and $\phi = \sum_{i=1}^{n} x_i I\{x_i = y_i\}$.

Under the assumption $p_{12} = p_1p_2$, the same process was executed to derive the MLE of p_1, p_2, p_3, and p_4.

$$\hat{p}_1 = \frac{d+e+\phi-b-c}{d+e+\phi} \tag{2.25}$$

$$\hat{p}_2 = \frac{d+e+\phi-a-c}{d+e+\phi} \tag{2.26}$$

$$\hat{p}_3 = \frac{g-a-d}{g-d} \tag{2.27}$$

$$\hat{p}_4 = \frac{h-b-e}{h-e} \tag{2.28}$$

2.2.2 Bayes Estimation

Likewise, the Bayes estimation was applied on the BGD (F) using the uniform prior distribution on (p_1, p_2, p_3, p_4) with the additional assumption $p_{12} = p_1p_2$.

$$f(p_1, p_2, p_3, p_4) = \begin{cases} 18/11, & \text{if } 0 < p_i < 1,\ p_1p_2 < p_3,\ p_1p_2 < p_4, \\ 0, & \text{otherwise.} \end{cases} \tag{2.29}$$

The posterior distribution of (p_1, p_2, p_3, p_4) is in the form of

$$\pi(p_1, p_2, p_3, p_4|\mathbf{x}, \mathbf{y}) = \frac{1}{C} p_1^{d+e+\phi-b-c} p_2^{e+d+\phi-a-c} p_3^{g-d-a} p_4^{h-e-b} q_1^{b+c} q_2^{a+c} q_3^{a} q_4^{b}, \tag{2.30}$$

where $0 < p_i < 1$, $i = 1, 2, 3, 4$, $p_1p_2 < p_3$, $p_1p_2 < p_4$, and C is the constant obtained by integrating the likelihood function (2.24) with respect to p_1, p_2, p_3, p_4 each from zero to one. Using the same risk function, the Bayes estimators can be obtained as follows:

$$p_1^* = \frac{d+e+\phi-b-c+1}{d+e+\phi+2}, \tag{2.31}$$

$$p_2^* = \frac{d+e+\phi-a-c+1}{d+e+\phi+2}, \tag{2.32}$$

$$p_3^* = \frac{g-d-a+1}{g-d+2}, \tag{2.33}$$

$$p_4^* = \frac{h-b-e+1}{h-e+2}. \tag{2.34}$$

2.2.3 Method of Moments

For method of moments estimation on BGD (F) under the assumption $p_{12} = p_1p_2$, the moments EX, EY, EX^2, and EZ were considered, where $Z = min(X,Y)$. Then,

$$EX = \frac{p_3 - p_1}{(p_3 - p_1p_2)(1-p_1p_2)} + \frac{p_1q_2}{(p_3 - p_1p_2)(1-p_3)}, \tag{2.35}$$

$$EY = \frac{p_4 - p_2}{(p_4 - p_1p_2)(1-p_1p_2)} + \frac{p_2q_1}{(p_4 - p_1p_2)(1-p_4)}, \tag{2.36}$$

$$EX^2 = \frac{(p_3 - p_1)(1+p_1p_2)}{(p_3 - p_1p_2)(1-p_1p_2)^2} + \frac{p_1q_2(1+p_3)}{(p_3 - p_1p_2)(1-p_3)^2}, \tag{2.37}$$

$$EZ = (1-p_1p_2)^{-1}. \tag{2.38}$$

Equating the four equations $EX = \bar{x}$, $EY = \bar{y}$, $EX^2 = \bar{x}_2 = \frac{\Sigma_{i=1}^n x_i^2}{n}$, and $EZ = \bar{z} = \frac{\Sigma_{i=1}^n min(x_i, y_i)}{n}$, one can generate the method of moments estimators:

$$\hat{p}_1 = -\frac{2\bar{x}\bar{z}^2 - 2\bar{x}^2 + \bar{x}\bar{z} + \bar{x} - 2\bar{z}^2 - \bar{x}_2\bar{z} + \bar{x}_2}{\bar{x}_2\bar{z} + \bar{x}\bar{z} - 2\bar{x}\bar{z}^2}, \tag{2.39}$$

$$\hat{p}_2 = \frac{\bar{x}_2 + \bar{x} - \bar{x}_2\bar{z} - 3\bar{x}\bar{z} + 2\bar{x}\bar{z}^2}{2\bar{x}\bar{z}^2 - 2\bar{x}^2 + \bar{x}\bar{z} + \bar{x} - 2\bar{z}^2 + \bar{x}_2\bar{z} + \bar{x}_2}, \tag{2.40}$$

$$\hat{p}_3 = \frac{\bar{x}_2 - \bar{x} + 2\bar{z} - 2\bar{x}\bar{z}}{\bar{x}_2 + \bar{x} - 2\bar{x}\bar{z}}, \tag{2.41}$$

and

$$\hat{p}_4 = (\bar{x}_2\bar{y} - \bar{x} - \bar{x}_2 + \bar{x}\bar{y} - 2\bar{x}\bar{z} - 2\bar{x}^2\bar{y} + \bar{x}_2\bar{z}^2 + \bar{x}\bar{z}^2 - 2\bar{x}\bar{z}^3 -$$
$$2\bar{y}\bar{z}^2 + 2\bar{x}^2 + 2\bar{z}^2 + 2\bar{x}\bar{y}\bar{z}^2 - \bar{x}_2\bar{y}\bar{z} + \bar{x}\bar{y}\bar{z})/(\bar{x}_2\bar{y} + \bar{x}\bar{y} -$$
$$\bar{x}_2\bar{z} - \bar{x}\bar{z} - 2\bar{x}^2\bar{y} + \bar{x}_2\bar{z}^2 - \bar{x}\bar{z}^2 + 2\bar{x}^2\bar{z} - 2\bar{x}\bar{z}^3 - 2\bar{y}\bar{z}^2$$
$$+ 2\bar{z}^3 + 2\bar{x}\bar{y}\bar{z}^2 - \bar{x}_2\bar{y}\bar{z} + \bar{x}\bar{y}\bar{z}). \tag{2.42}$$

An alternative method of moments was developed by using EX, EY, EY^2, and EZ, where

$$EY^2 = \frac{(p_4 - p_2)(1 + p_1p_2)}{(p_4 - p_1p_2)(1 - p_1p_2)^2} + \frac{p_2q_1(1 + p_4)}{(p_4 - p_1p_2)(1 - p_4)^2}. \tag{2.43}$$

Equating the four equations, $EX = \bar{x}$, $EY = \bar{y}$, $EY^2 = \bar{y}_2 = \frac{\Sigma_{i=1}^{n} y_i^2}{n}$, and $EZ = \bar{z} = \frac{\Sigma_{i=1}^{n} min(x_i, y_i)}{n}$, one can solve for the alternative set of method of moments estimators.

$$\hat{p}_1 = \frac{\bar{y}_2 + \bar{y} - \bar{y}_2\bar{z} - 3\bar{y}\bar{z} + 2\bar{y}\bar{z}^2}{-2\bar{y}^2 + 2\bar{y}\bar{z}^2 + \bar{y}\bar{z} + \bar{y} - 2\bar{z}^2 - \bar{y}_2\bar{z} + \bar{y}_2}, \tag{2.44}$$

$$\hat{p}_2 = \frac{-2\bar{y}^2 + 2\bar{y}\bar{z}^2 + \bar{y}\bar{z} + \bar{y} - 2\bar{z}^2 - \bar{y}_2\bar{z} + \bar{y}_2}{\bar{y}_2\bar{z} + \bar{y}\bar{z} - 2\bar{y}\bar{z}^2}, \tag{2.45}$$

$$\hat{p}_3 = (\bar{x}\bar{y}_2 - \bar{y} - \bar{y}_2 + \bar{x}\bar{y} - 2\bar{y}\bar{z} - 2\bar{x}\bar{y}^2 - 2\bar{x}\bar{z}^2 + \bar{y}_2\bar{z}^2 + \bar{y}\bar{z}^2 - 2\bar{y}\bar{z}^3$$
$$+ 2\bar{y}^2 + 2\bar{z}^2 + 2\bar{x}\bar{y}\bar{z}^2 - \bar{x}\bar{y}_2\bar{z} + \bar{x}\bar{y}\bar{z})/(\bar{x}\bar{y}_2 + \bar{x}\bar{y} - \bar{y}_2\bar{z} - \bar{y}\bar{z} - 2\bar{x}\bar{y}^2 - 2\bar{x}\bar{z}^2$$
$$+ \bar{y}_2\bar{z}^2 - \bar{y}\bar{z}^2 + 2\bar{y}^2\bar{z} - 2\bar{y}\bar{z}^3 + 2\bar{z}^3 + 2\bar{x}\bar{y}\bar{z}^2 - \bar{x}\bar{y}_2\bar{z} + \bar{x}\bar{y}\bar{z}), \tag{2.46}$$

and

$$\hat{p}_4 = \frac{\bar{y}_2 - \bar{y} + 2\bar{z} - 2\bar{y}\bar{z}}{\bar{y}_2 + \bar{y} - 2\bar{y}\bar{z}}. \tag{2.47}$$

In this chapter, three estimation methods have been explained under both the BGD (B&D) and the BGD (F) models. Some expressions do not have closed forms and thus, the procedures to derive these estimators are explained step by step. Moreover, to study their behavior in terms of unbiasedness and efficiency, the expectation and variance should be investigated. In view of the complex nature of bivariate geometric distributions, Monte Carlo simulation method is used to achieve this goal and is illustrated in Chapter 6.

CHAPTER 3

CHARACTERIZATIONS BY CONDITIONALLY SPECIFIED DISTRIBUTIONS

The problem of characterizing a bivariate distribution of two random variables by properties of its conditional distributions was studied in Arnold, Castillo, and Sarabia (2001). In this chapter, some sufficient conditions to characterize two bivariate geometric models using their corresponding conditional distributions are considered.

3.1 Characterization of BGD (F) Model

Using the conditional distributions $g(m|n)$ of X given $Y = n$ and $h(n|m)$ of Y given $X = m$ obtained from the BGD (F) model (2.23) and loss of memory property of the model mentioned in Dhar (1998). The following characterization result for bivariate geometric model BGD (F) can be established.

Theorem 3.1.1 *Suppose that the conditional distributions $g(m|n) = P(X = m|Y = n)$ of X given $Y = n$ and the conditional distributions $h(n|m) = P(Y = n|X = m)$ of Y given $X = m$ are given by*

$$g(m|n) = \begin{cases} \dfrac{(p_4 - p_1p_2)\, q_1q_4p_2^m p_1^{m-1} p_4^{n-m-1}}{(p_2 - p_1p_2)(1-p_4)p_4^{n-1} + (p_4 - p_2)(1 - p_1p_2)(p_1p_2)^{n-1}}, & \text{if } m < n, \\ \dfrac{(p_4 - p_1p_2)\, q_1q_2p_{12}^{m-1}}{(p_2 - p_1p_2)(1-p_4)p_4^{m-1} + (p_4 - p_2)(1 - p_1p_2)(p_1p_2)^{m-1}}, & \text{if } m = n, \\ \dfrac{(p_4 - p_1p_2) q_2q_3p_1^n p_2^{n-1} p_3^{m-n-1}}{(p_2 - p_1p_2)(1-p_4)p_4^{n-1} + (p_4 - p_2)(1 - p_1p_2)(p_1p_2)^{n-1}}, & \text{if } m > n, \end{cases}$$

and

$$h(n|m) = \begin{cases} \dfrac{(p_3 - p_1p_2)\, q_2q_3p_1^np_2^{n-1}p_3^{m-n-1}}{(p_1 - p_1p_2)(1-p_3)p_3^{m-1} + (p_3 - p_1)(1-p_1p_2)(p_1p_2)^{m-1}}, & \text{if } n < m, \\ \dfrac{(p_3 - p_1p_2)\, q_1q_2p_{12}^{n-1}}{(p_1 - p_1p_2)(1-p_3)p_3^{n-1} + (p_3 - p_1)(1-p_1p_2)(p_1p_2)^{n-1}}, & \text{if } n = m, \\ \dfrac{(p_3 - p_1p_2)q_1q_4p_2^mp_1^{m-1}p_4^{n-m-1}}{(p_1 - p_1p_2)(1-p_3)p_3^{m-1} + (p_3 - p_1)(1-p_1p_2)(p_1p_2)^{m-1}}, & \text{if } n > m, \end{cases}$$

where $0 < p_2 < p_4 < 1,\ 0 < p_1 < p_3 < 1,$ *and* $p_{12} = p_1p_2,\ m, n = 1, 2...$ *Then the joint distribution of (X, Y) is BGD (F).*

Proof : Let $P(X = m) = f_1(m)$ and $P(Y = n) = f_2(n)$. Then the fact

$$P(X = m|Y = n)f_2(n) = P(Y = n|X = m)f_1(m) \tag{3.1}$$

gives us for $m > n$,

$$\frac{(p_4 - p_1p_2)q_2q_3p_2^{n-1}p_3^{m-n-1}p_1^n}{(p_2 - p_1p_2)(1-p_4)p_4^{n-1} + (p_4 - p_2)(1-p_1p_2)(p_1p_2)^{n-1}} f_2(n) =$$

$$\frac{(p_3 - p_1p_2)\, q_2q_3p_1^np_2^{n-1}p_3^{m-n-1}}{(p_1 - p_1p_2)(1-p_3)p_3^{m-1} + (p_3 - p_1)(1-p_1p_2)(p_1p_2)^{m-1}} f_1(m).$$

In the above equation, isolate $f_2(n)$ and then sum over $n = 1, 2, ...$ to get that for $m = 1, 2, ...,$

$$f_1(m) = \frac{(p_1 - p_1p_2)(1-p_3)p_3^{m-1} + (p_3 - p_1)(1-p_1p_2)(p_1p_2)^{m-1}}{p_3 - p_1p_2}. \tag{3.2}$$

Therefore,

$$\begin{aligned} P(X = m, Y = n) &= P(Y = n|X = m)f_1(m) \\ &= \frac{q_2q_3}{p_2p_3}\left[\frac{p_1p_2}{p_3}\right]^n p_3^m, \qquad \text{if } m > n. \end{aligned}$$

In case $m < n$, by similarly summing equation (3.1) over $m = 1, 2, ...,$ we can obtain $f_2(n)$. Therefore,

$$\begin{aligned} P(X = m, Y = n) &= P(X = m | Y = n) f_2(n) \\ &= \frac{q_1 q_4}{p_1 p_4} \left[\frac{p_1 p_2}{p_4} \right]^m p_4^n, \qquad \text{if } m < n. \end{aligned}$$

In case $m = n$, substitute $f_1(m)$ as shown in (3.2) into the following

$$\begin{aligned} P(X = m, Y = m) &= P(Y = m | X = m) f_1(m) \\ &= q_1 q_2 p_{12}^{m-1}. \end{aligned}$$

It can be then observed that the joint distribution of (X, Y) is BGD (F) with $p_{12} = p_1 p_2$.

3.2 Characterization of BGD (B&D) Model

Characterization statement similar to that made in Section 3.1 can be made for BGD (B&D) model. Suppose that the following conditional distributions are given:

$$P(X = x | Y = y) = \begin{cases} p_1^{x-1} q_1, & \text{if } x < y, \\ \frac{p_1^{x-1}(1 - p_1 p_{12} - p_2 p_{12} + p_1 p_2 p_{12})}{1 - p_2 p_{12}}, & \text{if } x = y, \\ \frac{p_1^{x-1} p_{12}^{x-y} q_2 (1 - p_1 p_{12})}{(1 - p_2 p_{12})}, & \text{if } x > y, \end{cases} \tag{3.3}$$

$$P(Y = y | X = x) = \begin{cases} p_2^{y-1} q_2, & \text{if } y < x, \\ \frac{p_2^{y-1}(1 - p_1 p_{12} - p_2 p_{12} + p_1 p_2 p_{12})}{1 - p_1 p_{12}}, & \text{if } y = x, \\ \frac{p_2^{y-1} p_{12}^{y-x} q_1 (1 - p_2 p_{12})}{(1 - p_1 p_{12})}, & \text{if } y > x, \end{cases} \tag{3.4}$$

$1 \le x, y \in \mathbf{Z}^+$, $0 < p_1 < 1$, $0 < p_2 < 1$, $0 < p_{12} \le 1$, and $1 - p_1 p_{12} - p_2 p_{12} + p_1 p_2 p_{12} > 0$. Then, the joint distribution of (X, Y) is BGD (B&D).

However, this work has been done by Sreehari (2005) for Hawkes bivariate geometric distribution (BGD (H)) which is theoretically equivalent to BGD (B&D) except for the domain and parameters.

Sreehari (2005) proved that if the conditional distribution $g(m|n)$ of X given $Y = n$ and conditional distribution $h(n|m)$ of Y given $X = m$ are given by:

$$g(m|n) = \begin{cases} \alpha^{m-n}\beta^{n}(1-\alpha)(1-\delta)/(1-\gamma), & \text{if } m > n, \\ \beta^{m}[1-\alpha-\gamma(1-\beta)]/(1-\gamma), & \text{if } m = n, \\ \beta^{m}(1-\beta), & \text{if } m < n, \end{cases}$$

and

$$h(n|m) = \begin{cases} \delta^{n}(1-\delta), & \text{if } m > n, \\ \delta^{n}[1-\gamma-\alpha(1-\delta)]/(1-\alpha), & \text{if } m = n, \\ \delta^{m}\gamma^{n-m}(1-\gamma)(1-\beta)/(1-\alpha), & \text{if } m < n, \end{cases}$$

where $m, n = 0, 1, 2...$, $\alpha\delta = \beta\gamma$, $0 < \alpha, \beta, \gamma, \delta < 1$, and $\alpha + \gamma < 1 + \gamma\beta$. Then the joint distribution of (X, Y) is BGD (H).

Letting $\alpha = p_1 p_{12}$, $\beta = p_1$, $\gamma = p_2 p_{12}$, and $\delta = p_2$, one gets exactly the same conditional distributions as shown in equation (3.3) and equation (3.4). The parameter constrains:

- $\alpha\delta = \beta\gamma$ implies $p_1 p_{12} \cdot p_2 = p_1 \cdot p_2 p_{12}$;
- $0 < \alpha, \beta, \gamma, \delta < 1$ implies $0 < p_1 p_{12} < 1$, $0 < p_1 < 1$, $0 < p_2 p_{12} < 1$, and $0 < p_2 < 1$;
- $\alpha + \gamma < 1 + \gamma\beta$ implies $1 - p_1 p_{12} - p_2 p_{12} + p_1 p_2 p_{12} > 0$.

Note: Hawkes bivariate geometric model takes value starting with zero, while BGD (B&D) model takes value starting with one.

CHAPTER 4

CHARACTERIZATIONS BY CONDITIONAL FAILURE RATE

In this chapter characterizations for both of the bivariate geometric models based on the conditional failure rates are studied. The characterization theorem generated by Sreehari (2005) regarding BGD (H) is considered. Some slight changes in the parameters and domains are made to this theorem to accommodate the BGD (B&D) model. Also, we state the results of Sun and Basu (1995) who derived the characterization results based on conditional failure rates for BGD (H) model in order to derive the same for BGD (B&D) model.

4.1 Conditional Failure Rate for BGD (F) Model

Several versions of CFRs have been used to characterize different bivariate geometric distributions. One of these, defined by Cox (1972), is given by:

$$r_1(m|n) = \frac{P(M=m, N=n)}{P(M \geq m, N=n)}, \qquad \text{for } m > n,$$

$$r_2(n|m) = \frac{P(M=m, N=n)}{P(M=m, N \geq n)}, \qquad \text{for } n > m, \tag{4.1}$$

$$r(t) = P(min(M,N)=t)/P(M \geq t, N \geq t),$$

with (M, N) taking values in the set $\{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\}$, and $t \in \{0, 1, 2, \ldots\}$. Notice that $r(t)$ is the failure rate of $min(M, N)$ and $r_1(m|n)$ is the conditional failure rate of M, given $N = n$ for $m > n$. The quantity $r_2(n|m)$ can be similarly interpreted.

A characterization of BGD (F) can be found in Dhar and Balaji (2006). Asha and Nair (1998) considered this CFR as given in (4.1) and discussed its roles in characterizing the Hawkes model (BGD (H)), and extended the domain of $r_1(m|n)$

and $r_2(n|m)$ to the entire region $\{0,1,2,...\} \times \{0,1,2,...\}$ which includes the region $m = n, m, n = 0,1,2,....$ Using different constant CFRs with loss of memory property (Dhar, 1998), the specified geometric nature of the BGD (F) density at (n,n), $n \geqslant 2$, as the sufficient conditions, the BGD (F) is derived.

Theorem 4.1.1 *Suppose X and Y are random variables with probability mass function f(m,n) satisfying the following conditions:*

$$r_1(m|n) = q_3, \qquad \textit{for } m > n, m, n = 1,2,...,$$

$$r_2(n|m) = q_4, \qquad \textit{for } n > m, m, n = 1,2,...,$$

$$r_1(m|m) = q_1, \qquad \textit{for } m = 1,2,...,$$

$$r_2(n|n) = q_2, \qquad \textit{for } n = 1,2,...,$$

and

$$f(n,n) = (p_1p_2)^{n-1}f(1,1), \qquad \textit{for } n = 2,3,...,$$

where $0 < q_i < 1$, $p_i + q_i = 1$, $i = 1,2,3,4$, *then the joint distribution of (X, Y) is BGD (F) with* $p_{12} = p_1 \cdot p_2$.

Proof : Using the given $r_1(m|n)$ and induction on k,

$$f(n+k,n) = \frac{p_1q_3}{q_1}p_3^{k-1}f(n,n), \qquad k = 1,2,..., \tag{4.2}$$

is derived first as follows. Suppose

$$r_1(m|n) = \begin{cases} q_3, & m > n, m, n = 1,2,..., \\ q_1, & m = n, m, n = 1,2,..., \end{cases} \tag{4.3}$$

where $0 < q_1, q_3 < 1$. The equality in (4.3) implies that

$$r_1(n|n) = q_1 = \frac{f(n,n)}{P(X \geq n, Y = n)}$$
$$= \frac{f(n,n)}{f(n,n) + P(X \geq n+1, Y = n)},$$

$$\frac{1}{q_1} = 1 + \frac{P(X \geq n+1, Y = n)}{f(n,n)}.$$

Thus,

$$P(X \geq n+1, Y = n) = \frac{1-q_1}{q_1} f(n,n).$$

In the region $m > n$, when $m = n+1$,

$$r_1(n+1|n) = q_3 = \frac{f(n+1,n)}{P(X \geq n+1, Y = n)},$$

$$\begin{aligned} f(n+1,n) &= q_3 P(X \geq n+1, Y = n) \\ &= q_3 \frac{1-q_1}{q_1} f(n,n) \\ &= \frac{p_1 q_3}{q_1} f(n,n), \end{aligned}$$

if $m = n+2$,

$$r_1(n+2|n) = q_3 = \frac{f(n+2,n)}{P(X \geq n+2, Y = n)},$$

$$
\begin{aligned}
f(n+2,n) &= q_3 P(X \geq n+2, Y=n) \\
&= q_3[P(X \geq n+1, Y=n) - P(X=n+1, Y=n)] \\
&= q_3[\frac{1-q_1}{q_1} f(n,n) - f(n+1,n)] \\
&= q_3[\frac{1-q_1}{q_1} f(n,n) - q_3 \frac{1-q_1}{q_1} f(n,n)] \\
&= q_3 \frac{1-q_1}{q_1} f(n,n)(1-q_3) \\
&= \frac{p_1 q_3 p_3}{q_1} f(n,n),
\end{aligned}
$$

Using induction on k, the following equation is shown to be true. The equation

$$
f(n+k-1,n) = \frac{p_1 q_3}{q_1} p_3^{k-2} f(n,n)
$$

is now obviously true for $k = 2, 3$. Now assume that the above equation is true for k. It can be shown that the above equality holds true for $k+1$, $k = 1, 2, 3...$,

$$
\begin{aligned}
f(n+k,n) &= q_3 P(X \geq n+k, Y=n) \\
&= q_3[P(X \geq n+k-1, Y=n) - P(X=n+k-1, Y=n)] \\
&= q_3[\frac{f(n+k-1,n)}{q_3} - f(n+k-1,n)] \\
&= q_3(\frac{1}{q_3} - 1) f(n+k-1,n) \\
&= q_3 \frac{1-q_3}{q_3} \frac{p_1 q_3}{q_1} p_3^{k-2} f(n,n) \\
&= \frac{p_1 q_3}{q_1} p_3^{k-1} f(n,n).
\end{aligned}
$$

Thus, one proves that

$$
f(m,n) = \frac{p_1 q_3}{q_1} p_3^{m-n-1} f(n,n), \qquad m > n, m, n = 1, 2, ..., \tag{4.4}
$$

by induction. Similarly, suppose that

$$r_2(n|m) = \begin{cases} q_4, & n > m, m, n = 1, 2, ..., \\ q_2, & m = n, m, n = 1, 2, ..., \end{cases} \tag{4.5}$$

where $0 < q_2, q_4 < 1$, we can show that

$$f(m,n) = \frac{p_2 q_4}{q_2} p_4^{n-m-1} f(m,m), \qquad n > m, m, n = 1, 2, ... \tag{4.6}$$

In addition, from supposition

$$f(m,m) = (p_1 p_2)^{m-1} f(1,1), \qquad m = 2, 3, ... \tag{4.7}$$

Since $f(m,n)$ must add over $\mathbf{Z}^+ \times \mathbf{Z}^+$ up to 1, gives $f(1,1) = P(X = 1, Y = 1) = q_1 q_2$. Substituting the equation (4.7) into equations (4.4) and (4.6), one can see that the joint distribution of (X, Y) is BGD (F) with $p_{12} = p_1 \cdot p_2$ as defined in equation (2.23).

The proof of the above characterization of BGD (F) model is analogous to that of Sreehari (2005).

4.2 Conditional Failure Rate for BGD (B&D) Model

Using the CFR definition in (4.1), some characterization results for BGD (B&D) are also obtained here.

It is known that the BGD (B&D) is the reparameterized version of BGD (H). Sun and Basu (1995) already proved a characterization result for BGD (H). We thus easily see the analog statement for BGD (B&D):

$$r(k) = P[min(X,Y) = k | X \geq k, Y \geq k] = \frac{P[min(X,Y) = k]}{P[X \geq k, Y \geq k]} = 1 - p_1 p_2,$$

$$r_1(x|y) = \frac{p_2^{y-1} q_2 (p_1 p_{12})^{x-1} (1 - p_1 p_{12})}{p_2^{y-1} q_2 (p_1 p_{12})^{x-1}} = 1 - p_1 p_{12}, \quad \text{for } x > y,$$

$$r_2(y|x) = \frac{p_1^{x-1} q_1 (p_2 p_{12})^{y-1} (1 - p_2 p_{12})}{p_1^{x-1} q_1 (p_2 p_{12})^{y-1}} = 1 - p_2 p_{12}, \quad \text{for } x < y.$$

where $x, y = 1, 2, \ldots$. Total failure rate ($1 - p_1 p_2$,$1 - p_1 p_{12}$,$1 - p_2 p_{12}$) in this case is a constant, and the marginal distributions of X and Y are geometric, as given in equation (2.12) and equation (2.13). Then (X, Y) is BGD (B&D).

Sreehari (2005) proved a characterization result of BGD (H) from a different point of view. Likewise, this result can also be translated to BGD (B&D) since they are equivalent except for the domain and the parameters. We thus have the following result.

Suppose X and Y are random variables with probability mass function $f(m, n)$ satisfying the following conditions:

$$r_1(m|n) = 1 - p_1 p_{12}, \qquad \text{for } m > n,$$

$$r_2(n|m) = 1 - p_2 p_{12}, \qquad \text{for } m < n,$$

$$r_1(m|m) = \frac{1 - p_2 p_{12} - p_1 p_{12} + p_1 p_2 p_{12}}{1 - p_2 p_{12}}, \qquad \text{for } m = 1, 2, \ldots,$$

$$r_2(n|n) = \frac{1 - p_2 p_{12} - p_1 p_{12} + p_1 p_2 p_{12}}{1 - p_1 p_{12}}, \qquad \text{for } n = 1, 2, \ldots,$$

and

$$f(n, n) = (p_1 p_2 p_{12})^{n-1} f(1, 1), \qquad \text{for } n = 2, 3, \ldots,$$

then the joint distribution of (X, Y) is BGD (B&D).

CHAPTER 5

CHARACTERIZATIONS VIA PROBABILITY GENERATING FUNCTION

In this chapter probability generating functions are developed for both forms of these bivariate geometric models. The probability generating functions are used to determine the probability mass functions.

5.1 Probability Generating Function for BGD (F) Model

The joint probability generating function (p.g.f.) of the paired random variables (X, Y) for the BGD (F) can be derived as follows:

$$\begin{aligned}
\pi(t_1, t_2) = E[t_1^X t_2^Y] &= \sum_{(x,y)\in T} t_1^x t_2^y f(x, y) \\
&= \sum_{x<y} t_1^x t_2^y f(x, y) + \sum_{x=y} t_1^x t_2^y f(x, y) + \sum_{x>y} t_1^x t_2^y f(x, y) \\
&= \sum_{x<y} t_1^x t_2^y \frac{q_1 q_4}{p_1 p_4} (\frac{p_1 p_2}{p_4})^x p_4^y + \sum_{x=y} t_1^x t_2^y \frac{q_1 q_2 q_{12}}{1 - p_1 p_2} p_{12}^{x-1} \\
&+ \sum_{x>y} t_1^x t_2^y \frac{q_2 q_3}{p_2 p_3} (\frac{p_1 p_2}{p_3})^y p_3^x \\
&= \frac{q_1 q_4 p_2 t_1 t_2^2}{(1 - t_2 p_4)(1 - t_1 t_2 p_1 p_2)} + \frac{q_2 q_3 p_1 t_1^2 t_2}{(1 - t_1 p_3)(1 - t_1 t_2 p_1 p_2)} \\
&+ \frac{q_1 q_2 q_{12} t_1 t_2}{(1 - p_1 p_2)(1 - t_1 t_2 p_{12})}. \qquad (5.1)
\end{aligned}$$

Here $0 < p_i < 1$, $i = 1, 2, 3, 4$, $0 < p_{12} < 1$, $p_1 p_2 < p_3$, $p_1 p_2 < p_4$, $| t_1 | \leq min\{\frac{p_4}{p_1 p_2}, \frac{1}{p_3}\}$, $| t_2 | \leq min\{\frac{p_3}{p_1 p_2}, \frac{1}{p_4}\}$, and $| t_1 t_2 | \leq min\{\frac{1}{p_1 p_2}, \frac{1}{p_{12}}\}$.

It is well known that the probability generating function has a one-to-one relationship to the probability mass function. In order to determine the probability mass function $f(m, n)$ of the BGD (F), as given in equation (2.23) using the p.g.f., it is required that one differentiate $\pi(t_1, t_2)$ partially with respect to t_1, x times, and with respect to t_2, y times at $(0, 0)$.

In the remaining part of Section 5.1, equation (5.1) is validated using the fact:

$$f(x,y) = \frac{1}{x!}\frac{1}{y!}\frac{\partial^{x+y}}{\partial t_1^x \partial t_2^y}\pi(t_1, t_2) \mid_{t_1=0,t_2=0} .$$

Note that BGD (F) given in equation (2.23) can be characterized by equation (5.1). The equation (5.1) can be verified by expressing $\pi(t_1, t_2) = A(t_1, t_2) + B(t_1, t_2) + C(t_1, t_2)$, where A, B, and C are the first three terms on the right hand side of equation (5.1), respectively. Thus,

$$A = \frac{q_1 q_4 p_2 t_1 t_2^2}{(1 - t_2 p_4)(1 - t_1 t_2 p_1 p_2)},$$

$$B = \frac{q_2 q_3 p_1 t_1^2 t_2}{(1 - t_1 p_3)(1 - t_1 t_2 p_1 p_2)},$$

and

$$C = \frac{q_1 q_2 q_{12} t_1 t_2}{(1 - p_1 p_2)(1 - t_1 t_2 p_{12})}.$$

Then,

$$\begin{aligned}\frac{1}{x!}\frac{1}{y!}\frac{\partial^{x+y}}{\partial t_1^x \partial t_2^y}\pi(t_1, t_2)|_{t_1=0,t_2=0} &= \frac{1}{x!}\frac{1}{y!}\frac{\partial^{x+y}A}{\partial t_1^x \partial t_2^y}|_{t_1=0,t_2=0} + \frac{1}{x!}\frac{1}{y!}\frac{\partial^{x+y}B}{\partial t_1^x \partial t_2^y}|_{t_1=0,t_2=0} \\ &+ \frac{1}{x!}\frac{1}{y!}\frac{\partial^{x+y}C}{\partial t_1^x \partial t_2^y}|_{t_1=0,t_2=0}.\end{aligned}$$

The expressions A, B, and C can be written as geometric power series. Also, using the fact that the derivatives of the power series can be obtained by term-by-term differentiations within the summation sign, the following expressions can be derived. In the domain $|t_2 p_4| < 1$ and $|t_1 t_2 p_1 p_2| < 1$, A can be rewritten as

$$A = \frac{q_1 q_4}{p_1 p_4}\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}(p_1 p_2)^j p_4^m t_1^j t_2^{j+m}.$$

Differentiating A partially with respect to t_1, x times, and evaluate it at $t_1 = 0$, results in

$$\frac{\partial^x A}{\partial t_1^x}|_{t_1=0} = \frac{q_1 q_4}{p_1 p_4}(p_1 p_2)^x \cdot x! \sum_{m=1}^{\infty} p_4^m t_2^{x+m}.$$

This partial derivative is further differentiated partially with respect to t_2, y times and evaluated at $t_2 = 0$, giving

$$\frac{\partial^{x+y} A}{\partial t_1^x \partial t_2^y}|_{t_1=0,t_2=0} = \begin{cases} \frac{q_1 q_4}{p_1 p_4}[\frac{p_1 p_2}{p_4}]^x p_4^y \cdot x! \cdot y!, & \text{if } y > x, \\ 0, & \text{elsewhere.} \end{cases}$$

Thus,

$$\frac{1}{x!}\frac{1}{y!}\frac{\partial^{x+y} A}{\partial t_1^x \partial t_2^y}|_{t_1=0,t_2=0} = \begin{cases} \frac{q_1 q_4}{p_1 p_4}[\frac{p_1 p_2}{p_4}]^x p_4^y, & \text{if } y > x, \\ 0, & \text{elsewhere.} \end{cases} \tag{5.2}$$

Analogously, B can be derived for $|t_1 p_3| < 1$ and $|t_1 t_2 p_1 p_2| < 1$.

$$B = \frac{q_2 q_3}{p_2 p_3} \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} (p_1 p_2)^j p_3^m t_1^{j+m} t_2^j.$$

Differentiating B partially with respect to t_2, y times, and evaluating its result at $t_2 = 0$, yields

$$\frac{\partial^y B}{\partial t_2^y}|_{t_2=0} = \frac{q_2 q_3}{p_2 p_3}(p_1 p_2)^y \cdot y! \sum_{m=1}^{\infty} p_3^m t_1^{y+m}.$$

Partial derivatives are again applied to the above expression with respect to t_1 x times and evaluated at $t_1 = 0$, yielding

$$\frac{\partial^{x+y} B}{\partial t_1^x \partial t_2^y}|_{t_1=0,t_2=0} = \begin{cases} \frac{q_2 q_3}{p_2 p_3}[\frac{p_1 p_2}{p_3}]^y p_3^x \cdot x! \cdot y!, & \text{if } y < x, \\ 0, & \text{elsewhere.} \end{cases}$$

Thus,

$$\frac{1}{x!}\frac{1}{y!}\frac{\partial^{x+y} B}{\partial t_1^x \partial t_2^y}|_{t_1=0,t_2=0} = \begin{cases} \frac{q_2 q_3}{p_2 p_3}[\frac{p_1 p_2}{p_3}]^y p_3^x, & \text{if } y < x, \\ 0, & \text{elsewhere.} \end{cases} \tag{5.3}$$

Also, C can be rewritten for $|t_1 t_2 p_{12}| < 1$ as:

$$C = \frac{q_1 q_2 q_{12}}{(1 - p_1 p_2) p_{12}} \sum_{m=1}^{\infty} (t_1 t_2 p_{12})^m.$$

Differentiating C partially with respect to t_1, x times, and evaluating it at $t_1 = 0$, yields

$$\frac{\partial^x C}{\partial t_1^x}|_{t_1=0} = \frac{q_1 q_2 q_{12}}{(1 - p_1 p_2) p_{12}} \cdot p_{12}^x \cdot t_2^x \cdot x!.$$

This partial derivative is further differentiated partially with respect to t_2, y times, and evaluated at $t_2 = 0$, to give

$$\frac{\partial^{x+y} C}{\partial t_1^x \partial t_2^y}|_{t_1=0,t_2=0} = \begin{cases} \frac{q_1 q_2 q_{12}}{(1-p_1 p_2) p_{12}} \cdot p_{12}^x \cdot x! \cdot x!, & \text{if } y = x, \\ 0, & \text{elsewhere.} \end{cases}$$

Thus,

$$\frac{1}{x!} \frac{1}{y!} \frac{\partial^{x+y} C}{\partial t_1^x \partial t_2^y}|_{t_1=0,t_2=0} = \begin{cases} \frac{q_1 q_2 q_{12}}{1-p_1 p_2} \cdot p_{12}^{x-1}, & \text{if } y = x, \\ 0, & \text{elsewhere.} \end{cases} \tag{5.4}$$

The three equations (5.2), (5.3), and (5.4) added together yield the joint probability function of the BGD (F) as given in equation (2.23).

$$\frac{1}{x!} \frac{1}{y!} \frac{\partial^{x+y}}{\partial t_1^x \partial t_2^y} \pi(t_1, t_2)|_{t_1=0,t_2=0} = \begin{cases} \frac{q_1 q_4}{p_1 p_4} \left[\frac{p_1 p_2}{p_4}\right]^x p_4^y, & \text{if } y > x, \\ \frac{q_1 q_2 q_{12}}{1-p_1 p_2} \cdot p_{12}^{x-1}, & \text{if } y = x, \\ \frac{q_2 q_3}{p_2 p_3} \left[\frac{p_1 p_2}{p_3}\right]^y p_3^x, & \text{if } y < x. \end{cases}$$

5.2 Probability Generating Function for BGD (B&D) Model

The probability generating function of the paired random variables (X, Y) for the BGD (B&D) is given below:

$$
\begin{aligned}
E[t_1^X t_2^Y] &= \sum_{(x,y)\in T} t_1^x t_2^y f(x,y) \\
&= \sum_{x<y} t_1^x t_2^y f(x,y) + \sum_{x>y} t_1^x t_2^y f(x,y) + \sum_{x=y} t_1^x t_2^y f(x,y) \\
&= \sum_{x<y} t_1^x t_2^y p_1^{x-1} (p_2p_3)^{y-1} q_1 (1 - p_2p_3) + \sum_{x>y} t_1^x t_2^y p_2^{y-1} (p_1p_3)^{x-1} q_2 (1 - p_1p_3) \\
&+ \sum_{x=y} t_1^x t_2^y (p_1p_2p_3)^{x-1} (1 - p_1p_3 - p_2p_3 + p_1p_2p_3) \\
&= \frac{t_1t_2q_1(1 - p_2p_3)(t_2p_2p_3)}{(1 - t_1t_2p_1p_2p_3)(1 - t_2p_2p_3)} + \frac{t_1t_2q_2(1 - p_1p_3)(t_1p_1p_3)}{(1 - t_1t_2p_1p_2p_3)(1 - t_1p_1p_3)} \\
&+ \frac{t_1t_2(1 - p_1p_3 - p_2p_3 + p_1p_2p_3)}{(1 - t_1t_2p_1p_2p_3)}.
\end{aligned} \tag{5.5}
$$

Here $0 < p_i < 1$, $i = 1, 2, 3$, $p_3 = p_{12}$, $\mid t_1 \mid < 1/p_1$, $\mid t_2 \mid < 1/p_2$, and $\mid t_1t_2 \mid < 1/p_1p_2p_3$. The same p.g.f. is obtained from the most natural generalization of the geometric distribution of Hawkes (1972, equation 3). Using an analogous method as in Section 5.1, we derived equation (5.5). Let (X, Y) be a bivariate random vector in the support of $\mathbf{Z}^+ \times \mathbf{Z}^+$ and the probability generating function of the pair of the random variables (X, Y) is in the form of equation (5.5). Then, (X, Y) has the BGD (B&D) given in (2.1).

CHAPTER 6

DATA ANALYSIS WITH MODELING

The applications of bivariate geometric models can be widely used in the analysis of sports data, engineering systems, and biostatistics data. For instance, the following set of sports data was taken from Dhar (2003) to exemplify the usage of the BGD (B&D) model.

The estimation methods introduced in Chapter 2 are applied to this data set. In the meanwhile, Monte-Carlo simulations are also generated in order to identify the best estimation method.

6.1 A Real Data Example

In this section, the BGD (B&D) model is fitted to a real data set from Dhar (2003) for demonstration purposes. This data set consists of scores given by seven judges from seven different countries in the form of a video recording. The score given by each judge is a discrete random variable taking positive integer values and also the midpoints of consecutive integers between zero and ten. The data given in Table 1 displays the scores which have been converted into integer valued random variable. The score corresponding to the dive of Michael Murphy Of Australia (item number 3) was not displayed by NBC sports during the recording.

In this case, one would be interested in comparing the scores given by the judges from different regions. In other words, we need to find out the probabilities $P(X > Y)$ and $P(X < Y)$. These probabilities can be obtained from their joint distribution as given in (2.1) with respect to the corresponding domains.

$$\begin{aligned}
P(X > Y) &= \sum_{j=1}^{\infty}\sum_{y=1}^{\infty} P(X = y + j, Y = y) \\
&= \sum_{j=1}^{\infty}\sum_{y=1}^{\infty} p_2^{y-1}(p_1p_{12})^{y+j-1}q_2(1-p_1p_{12}) \\
&= \sum_{j=1}^{\infty}(p_1p_{12})^j q_2(1-p_1p_{12})/(1-p_1p_2p_{12})\sum_{y=1}^{\infty}p_2^{y-1} \\
&= \frac{p_1p_{12}q_2}{1-p_1p_2p_{12}} \qquad \text{for } x > y,\ y = 1, 2, 3...
\end{aligned} \tag{6.1}$$

and

$$\begin{aligned}
P(Y > X) &= \sum_{j=1}^{\infty}\sum_{x=1}^{\infty} P(X = x, Y = x + j) \\
&= \sum_{j=1}^{\infty}\sum_{x=1}^{\infty} p_1^{x-1}(p_2p_{12})^{x+j-1}q_1(1-p_2p_{12}) \\
&= \sum_{j=1}^{\infty}(p_2p_{12})^j q_1(1-p_2p_{12})/(1-p_1p_2p_{12})\sum_{x=1}^{\infty}p_1^{x-1} \\
&= \frac{p_2p_{12}q_1}{1-p_1p_2p_{12}} \qquad \text{for } y > x,\ x = 1, 2, 3...
\end{aligned} \tag{6.2}$$

where $0 < p_1 < 1$, $0 < p_2 < 1$, and $0 < p_{12} \leq 1$.

Item	Diver	X: max score, Asian & Caucasus	Y: max score, West
1	Sun Shuwei, China	19	19
2	David Pichler, USA	15	15
3	Jan Hempel, Germany	13	14
4	Roman Volodkuv, Ukraine	11	12
5	Sergei Kudrevich, Belarus	14	14
6	Patrick Jeffrey, USA	15	14
7	Valdimir Timoshinin, Russia	13	16
8	Dimitry Sautin, Russia	7	5
9	Xiao Hailiang, China	13	13
10	Sun Shuwei, China	15	16
11	David Pichler, USA	15	15
12	Jan Hempel, Germany	17	18
13	Roman Volodkuv, Ukraine	16	16
14	Sergei Kudrevich, Belarus	12	13
15	Patrick Jeffrey, USA	14	14
16	Valdimir Timoshinin, Russia	12	13
17	Dimitry Sautin, Russia	17	18
18	Xiao Hailiang, China	9	10
19	Sun Shuwei, China	18	18

Table 6.1 Scores taken from a video recorded during the summer of 1995 relayed by NBC sports TV, IX World Cup diving competition, Atlanta, Georgia.

It is reasonable to assume that the maximum scores (X, Y) follow the BGD (B&D) model since the marginal distribution of the scores from either region can be considered as following a univariate geometric distribution. Using the estimation methods discussed in Chapter 2, one can calculate the estimators of parameters p_1, p_2, and p_{12} using Mathematica. The results are summarized in Table 6.2 below.

Estimator	MLE	Bayes	MOM1	MOM2
p_1	0.961605	0.957713	0.9343997	0.9968594
p_2	0.985481	0.979849	0.9365146	0.9991156
p_{12}	0.940199	0.939019	0.9934730	0.9312265

Table 6.2 Estimated parameters by fitting the BGD (B&D) model to the data set shown in Table 6.1.

After comparing the results in Table 6.2, we see that the estimators obtained through maximum likelihood estimation, Bayes method, and MOM2 are close, while the MOM1 estimators are slightly off by about 0.06. Furthermore, the above estimates helped us to identify the judges from which particular region tend to give higher scores than the other region. By substituting the above estimates into equations (6.1) and (6.2), we find the following results.

Method	MLE	Bayes	MOM1	MOM2
$\hat{\mathrm{P}}(X > Y)$	0.1204	0.1525	0.4511	0.0113
$\hat{\mathrm{P}}(X < Y)$	0.3263	0.3275	0.4672	0.0403

Table 6.3 Comparisons of probabilities reflecting which group tends to give higher scores for data set given in Table 6.1.

From Table 6.3, it can be seen that the probability $\hat{\mathrm{P}}(X < Y)$ is higher than the probability $\hat{\mathrm{P}}(X > Y)$ for each column (estimation method), which shows that judges from West tend to give higher scores than judges from Asia and Caucasus. This conclusion is consistent with the empirical estimates $\tilde{\mathrm{P}}(X > Y) = 2/19 = 0.1053$ and $\tilde{\mathrm{P}}(X < Y) = 9/19 = 0.4737$.

6.2 Simulation Results

A Monte Carlo simulation study is performed by generating 500 simulations each from the BGD (B&D) of sizes $n = 20$, $n = 50$, and $n = 100$. To simulate observations from this model, the marginal distribution of Y and the conditional distribution of X given Y are used.

An observation from the marginal distribution of the random variable Y is generated using the inverse-transformation method. Using this realization of y, a value of X is generated using the inverse-transformation method again based on the conditional distribution of X given $Y{=}y$ as given in equation (3.3). The R codes to realize this simulation are attached in Appendix A.1.

In view of the complicated nature of the probability function for the BGD (B&D), only two methods of moments as introduced in Section 2.1.3 are examined here for comparison through simulation. Using

$$p_1 = 0.95,\ p_2 = 0.96,\ p_{12} = 0.97,$$

as the true values, the following 500 simulations each for different sample sizes (20, 50, 100) are generated. From these simulated observations, the mean of the 500 estimated vectors of the parameter $\mathbf{p} = (p_1, p_2, p_{12})$ and the estimated variance-covariance matrices based on these 500 vectors are computed. The performances of the two estimation methods are assessed based on estimated expected value of the estimator vector and their estimated variance covariance matrix for different sample sizes. To measure and compare the magnitudes of the estimated mean vectors $\hat{\mathbf{p}}$ and the estimated variance covariance matrices of $\hat{\mathbf{p}}$, the Euclidean norm $\|A\| = \sqrt{\Sigma_{i=1}^{m}\Sigma_{j=1}^{n} \mid a_{ij} \mid^2}$ is used. The results of this simulation for sample size $n = 20$, $n = 50$, and $n = 100$ are described as follows.

	MOM1		MOM2	
True parameters	bias	variance	bias	variance
$p_1 = 0.95$	-0.011958	0.0008984997	-0.0051045	0.0005899290
$p_2 = 0.96$	-0.0092005	0.0008798356	-0.0022724	0.0005287179
$p_{12} = 0.97$	0.0067059	0.0007443059	-0.0007186	0.0004115899

Table 6.4 Estimated bias and variances of $\hat{\mathbf{p}}$ for the samples from BGD (B&D) using methods of moments when n=20.

The estimated mean vector and the estimated variance-covariance matrix of $\hat{\mathbf{p}}$ for sample data with sample size $n = 20$ using MOM1 are:

$$\hat{E}(\hat{p}_1,\ \hat{p}_2,\ \hat{p}_{12}) = [0.9380420,\ 0.9507995,\ 0.9767059],$$

and

$$\hat{cov}(\hat{p}_1, \hat{p}_2, \hat{p}_{12}) = \begin{bmatrix} 0.0008984997 & 0.0006347615 & -0.0006245523 \\ 0.0006347615 & 0.0008798356 & -0.0006878916 \\ -0.0006245523 & -0.0006878916 & 0.0007443059 \end{bmatrix},$$

where the estimated mean vector of $\hat{\mathbf{p}}$ contains the arithmetic averages of the estimates of three parameters when $n = 20$. Using the difference between the mean vector and the vector of true values of the parameters, the corresponding bias is estimated as shown in Table 6.4. The Euclidean norm of the estimated bias vector and that of the estimated variance-covariance matrix are reported in Table 6.7.

The estimated mean vector and the estimated variance-covariance matrix of $\hat{\mathbf{p}}$ for the sample data when $n = 20$ using MOM2 are given by:

$$\hat{E}(\hat{p}_1,\ \hat{p}_2,\ \hat{p}_{12}) = [0.9448955,\ 0.9577276,\ 0.9692814],$$

and

$$c\hat{o}v(\hat{p}_1, \hat{p}_2, \hat{p}_{12}) = \begin{bmatrix} 0.0005899290 & 0.0003041345 & -0.0003029366 \\ 0.0003041345 & 0.0005287179 & -0.0003439391 \\ -0.0003029366 & -0.0003439391 & 0.0004115899 \end{bmatrix}.$$

The Euclidean norm of the estimated bias vector and that of the estimated variance-covariance matrix of $\hat{\mathbf{p}}$ are reported in Table 6.7.

The results when n=50 are given below.

	MOM1		MOM2	
True parameters	bias	variance	bias	variance
$p_1 = 0.95$	-0.0035407	0.0003825174	-0.0012051	0.0001947459
$p_2 = 0.96$	-0.0039660	0.0003583178	-0.0016015	0.0001758509
$p_{12} = 0.97$	0.0026561	0.0003086988	0.0000670	0.0001277123

Table 6.5 Estimated bias and variances of $\hat{\mathbf{p}}$ for the samples from BGD (B&D) using methods of moments when n=50.

The estimated mean vector and the estimated variance-covariance matrix of $\hat{\mathbf{p}}$ for the sample data when $n = 50$ using MOM1 are given by:

$$\hat{E}(\hat{p}_1,\ \hat{p}_2,\ \hat{p}_{12}) = [0.9464593,\ 0.9560340,\ 0.9726561],$$

and

$$c\hat{o}v(\hat{p}_1, \hat{p}_2, \hat{p}_{12}) = \begin{bmatrix} 0.0003825174 & 0.0002876939 & -0.0002828341 \\ 0.0002876939 & 0.0003583178 & -0.0002796300 \\ -0.0002828341 & -0.0002796300 & 0.0003086988 \end{bmatrix}.$$

The Euclidean norm of the estimated bias vector and that of the estimated variance-covariance matrix of $\hat{\mathbf{p}}$ are reported in Table 6.7.

The estimated mean vector and the estimated variance-covariance matrix of $\hat{\mathbf{p}}$ for the sample data when $n = 50$ based on MOM2 are:

$$\hat{E}(\hat{p}_1,\ \hat{p}_2,\ \hat{p}_{12}) = [0.9487949,\ 0.9583985,\ 0.9700670],$$

and

$$\hat{cov}(\hat{p}_1, \hat{p}_2, \hat{p}_{12}) = 10^{-4} * \begin{bmatrix} 1.947459 & 1.024981 & -0.9800245 \\ 1.024981 & 1.758509 & -0.9806184 \\ -0.9800245 & -0.9806184 & 1.277123 \end{bmatrix}.$$

The Euclidean norm of the estimated bias vector and that of the estimated variance-covariance matrix of $\hat{\mathbf{p}}$ are reported in Table 6.7, both of which are less than those obtained by using MOM1.

The results for $n = 100$ are given below.

	MOM1		MOM2	
True parameters	bias	variance	bias	variance
$p_1 = 0.95$	-0.0020813	0.0001763506	-0.0000126	0.00008880826
$p_2 = 0.96$	-0.0021517	0.0001768563	-0.0000633	0.00008399345
$p_{12} = 0.97$	0.0016712	0.0001439446	-0.0005366	0.00005659574

Table 6.6 Estimated bias and variances for the samples from BGD (B&D) using methods of moments when n=100.

Using MOM1, the estimated mean vector and the estimated variance-covariance matrix of $\hat{\mathbf{p}}$ for the sample data when $n = 100$ are:

$$\hat{E}(\hat{p}_1,\ \hat{p}_2,\ \hat{p}_{12}) = [0.9479187,\ 0.9578483,\ 0.9716712],$$

and

$$\hat{cov}(\hat{p}_1, \hat{p}_2, \hat{p}_{12}) = \begin{bmatrix} 0.0001763506 & 0.0001291313 & -0.0001243538 \\ 0.0001291313 & 0.0001768563 & -0.0001348726 \\ -0.0001243538 & -0.0001348726 & 0.0001439446 \end{bmatrix}.$$

The Euclidean norm of the estimated bias vector and that of the estimated variance-covariance matrix of $\hat{\mathbf{p}}$ are reported in Table 6.7.

Finally, the estimated mean vector and the estimated variance-covariance matrix of $\hat{\mathbf{p}}$ for the sample data when $n = 100$ using MOM2 are:

$$\hat{E}[\hat{p}_1,\ \hat{p}_2,\ \hat{p}_{12}) = [0.9499874,\ 0.9599367,\ 0.9694634],$$

and

$$\hat{cov}(\hat{p}_1, \hat{p}_2, \hat{p}_{12}) = 10^{-5} * \begin{bmatrix} 8.880826 & 3.888380 & -3.698442 \\ 3.888380 & 8.399345 & -4.470513 \\ -3.698442 & -4.470513 & 5.659574 \end{bmatrix}.$$

The Euclidean norm of the estimated bias vector and that of the estimated variance-covariance matrix of $\hat{\mathbf{p}}$ are reported in Table 6.7. In all cases, the norms computed using MOM2 are less than those computed using MOM1.

	MOM1		MOM2	
	ENEB	ENVC	ENEB	ENVC
n=20	0.016510970	0.002160502	0.005633481	0.001184099
n=50	0.005943112	0.000922994	0.002005382	0.000380280
n=100	0.003428488	0.0004286696	0.0005404676	0.0001670376

Table 6.7 Summary of Euclidean norms of the estimated bias vectors (ENEB) and that of the estimated variance-covariance matrices (ENVC).

The above results in Table 6.7 show that the Euclidean norms of the estimated bias vectors using MOM2 are less than those computed using MOM1 with respect to different sample sizes. This is also true for the norms of the estimated variance-covariance matrices. Hence, MOM2 (using score equation corresponding to $E[min(X, Y)]$ instead of $E[XY]$) provides more accurate estimations for the parameters than MOM1 (using score equation corresponding to $E[XY]$ instead of $E[min(X, Y)]$). Also, as should be the case, the magnitude of the estimated bias and that of the estimated variance-covariance matrix decrease as sample size increases.

6.3 A Random Sample Example

A chi-square goodness-of-fit test was performed to assess the performances of different estimation methods using one random sample from BGD (B&D).

X	Y	X	Y	X	Y	X	Y
41	38	14	6	2	24	9	3
12	12	1	10	4	2	65	3
14	21	14	14	50	10	2	10
2	11	42	12	11	11	23	81
2	2	7	7	10	19	5	5

Table 6.8 A randomly simulated sample from BGD (B&D).

Using the estimation methods introduced in Chapter 2, the following results in Table 6.9 were obtained through Mathematica. Let H_o: the data follows BGD (B&D) distribution; H_1: negation H_o.

The degree of freedom of a chi-square goodness-of-fit test is one less than the number of classes under a given multinomial distribution. Thus, considering there

Method	p_1	p_2	p_{12}
MLE	0.9647230	0.9606100	0.9746760
Bayes	0.9616000	0.9573000	0.9728000
MOM1	0.9595252	0.9535610	0.9790196
MOM2	0.9514900	0.9455758	0.9872872

Table 6.9 Estimated parameters for the BGD (B&D) for the sample data given in Table 6.8.

are three parameters in the BGD (B&D) distribution, it seems plausible to divide the region $\mathbf{Z}^+ \times \mathbf{Z}^+$ into seven cells:

1. $0 < x \leq 5$, $0 < y \leq 5$,
2. $0 < x \leq 5$, $5 < y \leq 15$,
3. $5 < x \leq 15$, $0 < y \leq 5$,
4. $5 < x \leq 20$, $5 < y \leq 15$,
5. $0 < x \leq 20$, $15 < y \leq 25$,
6. $20 < x \leq 65$, $0 < y \leq 15$,
7. otherwise.

The chi-square goodness-of-fit test value is calculated by

$$\chi^2 = \sum_{i=1}^{7} \frac{(o_i - e_i)^2}{e_i},$$

where o_i is the number of the observations in region i and e_i is the expected observations in the region i. The maximum likelihood estimators, Bayes estimators, and the methods of moments estimators shown in Table 6.9 are assumed to be the true values of the parameters then the corresponding χ^2 goodness-of-fit statistics and $p-values$

are computed and compared with degree of freedom 7-1=6 and 7-1-3=3, respectively. The R codes to implement the χ^2 goodness-of-fit tests using one of the estimation methods (MLE) are attached in Appendix A.2.

For example, the χ^2 goodness-of-fit statistic using maximum likelihood estimators is 5.834179 with 6 degrees of freedom having a p-value of 0.4420, and with 3 degrees of freedom having a p-value of 0.1199. Both of these $p - values$ are higher than an alpha value of 0.05. This suggests that the fit is good. These results for different estimators are summarized in Table 6.10.

Method	χ^2 goodness-of-fit statistic	p-value ($df = 6$)	p-value ($df = 3$)
MLE	5.834179	0.4420	0.1199
Bayes	4.704356	0.5822	0.1947
MOM1	4.863244	0.5614	0.1820
MOM2	4.633646	0.5916	0.2007

Table 6.10 Comparisons of Chi-square goodness-of-fit statistics and p-values using different estimates. (df: degree of freedom)

From this table, it is observed that the chi-square goodness-of-fit statistics calculated using Bayes estimation, methods of moments estimation are close. The result obtained using maximum likelihood estimators is slightly off by about 1.00 in the computed χ^2 test statistic value. All the results are consistent with each other since all the p-values are greater than the alpha value of 0.05. The best fit based on largest $p - value$ here is obtained from MOM2 for both degrees of freedom 6 and 3.

CHAPTER 7

CONCLUSION

In this dissertation, modeling of two bivariate geometric distributions have been developed to study discrete lifetime data. One of them is the bivariate geometric model BGD (B&D), which is a reparameterized version of the bivariate geometric model of Hawkes (1972). It can be used to study two-component systems with the lifetime of each component having frequency counts in the domain of positive integers, and the survival time of each component following geometric distribution. The other bivariate geometric model BGD (F) is a discrete analog to Freund's model (1961), which can be used to describe rare events of simultaneous failures. These bivariate geometric models have wide applications in the fields of medical and biological sciences.

The parameter estimations of the two bivariate geometric models have been formulated in Chapter 2. Maximum Likelihood Estimations, Bayes Estimations, and Methods of Moments Estimations are discussed for these two models, respectively. In view of the complexity arising from the parametric estimation of the BGD (B&D) model, the procedures to compute the maximum likelihood estimates and the Bayes estimates are explained step-by-step in this study. Two methods of moments are also derived for each of these two bivariate geometric models.

Conditional probabilities are used to characterize the BGD (F) model in this research. Similar characterization is stated for the BGD (B&D) model since this work has been done by Sreehari (2005) for Hawkes bivariate geometric distribution (BGD (H)) which is a reparameterized version of the BGD (B&D) except for the domain and the parameters. To further find the meaningful characterizations for these bivariate geometric models, specific conditional failure rates are considered and used to characterize these bivariate geometric models in Chapter 4. Probability generating functions can uniquely determine the probability mass functions. Using this relationship, the joint distribution of two random variables from the BGDs is validated in Chapter 5.

The BGD (B&D) model is used to fit a real data set taken from Dhar (2003) for demonstration purposes. The estimation methods developed in this study are applied on this sports data set to find meaningful probabilities arising from this problem. A Monte-Carlo simulation study is performed by generating simulations from the BGD (B&D) model for increasing sample sizes in order to compare two Methods of Moments estimations. MOM2 shows to be the more promising method compared to MOM1 based on the Euclidean norm of the estimated bias vector and that of the estimated variance-covariance matrix. Finally, Chi-square goodness-of-fit tests are used to obtain the best fitted model based on different estimation methods.

APPENDIX

R CODES

The R codes used for simulation studies and modeling are given in this appendix.

A.1 R Codes for Generating Estimated Mean Vectors and Estimated Variance-Covariance Matrices of $\hat{\mathrm{p}}$ Using Two Methods of Moments

$p1 = 0.95$

$p2 = 0.96$

$p3 = 0.97$

$n = 2$ $m = 10$

$aa < -array(0, dim = c(m, 3, n))$

$bb < -array(0, dim = c(n, 3))$

$bbb < -array(0, dim = c(n, 3))$

$cc < -array(0, dim = c(n, 3))$ $ccc < -array(0, dim = c(n, 3))$

for (l in 1:n){

$u < -runif(m)$

$u < -array(u, dim = c(m, 1))$

$x < -rgeom(m, 1 - p1 * p3) + 1$

$x < -array(x, dim = c(m, 1))$

$y < -array(c(0), dim = c(m, 1))$

$min\_xy < -array(c(0), dim = c(m, 1))$

$a < -array(c(u, x, y), dim = c(m, 3))$

for (i in $1 : m$) {

if ($x[i] == 1$ && $u[i] < 1 - p2 * p3 * (1 - p1)/(1 - p1 * p3)$) $\{y[i] = 1\}$ else if $(x[i] == 1)$ {for (s in 2:1000) if $(u[i] >= 1 - (1-p1)*(p2*p3)^(s-1)/(1-p1*p3)$

$\&\&\ u[i] < 1 - (1-p1)*(p2*p3)^s/(1-p1*p3))$ $\{y[i] = s\}\}$

else

for (j in $1 : x[i] - 2$) if $(u[i] >= 1 - p2^j$ && $u[i] < 1 - p2^{(j+1)})$ $\{y[i] = j+1\}$ else

if $(u[i] >= 1 - p2^{(x[i]-1)}$ && $u[i] < 1 - (p2^{x[i]}) * p3 * (1-p1)/(1-p1*p3))$ $\{y[i] = x[i]\}$

else

if $(u[i] >= 1 - (p2^{x[i]}) * p3 * (1-p1)/(1-p1*p3)$ && $u[i] < 1 - (1-p1)*(p2^{(x[i]+1)}) *$
$p3^2/(1-p1*p3))$ $\{y[i] = x[i]+1\}$ else

if $(u[i] >= 1 - (1-p1)*(p2^{(x[i]+1)}) * p3^2/(1-p1*p3))$ {

for (k in $1 + x[i] : 1000$) if

$(u[i] >= 1 - (1-p1)*(p3^{(-x[i]+1)})*((p2*p3)^k)/(1-p1*p3)$ && $u[i] < 1-(1-p1)*$
$(p3^{(-x[i]+1)}) * ((p2*p3)^{(k+1)})/(1-p1*p3))$ $\{y[i] = k+1\}\}\}$

for (i in $1 : m$){

$min\_xy[i] < -min(x[i], y[i])$

}

$xbar < -mean(x)$

$ybar < -mean(y)$

$zbar < -mean(min\_xy)$

$pp1 < -(ybar - ybar * zbar)/(zbar - ybar * zbar)$

$pp2 < -(xbar - xbar * zbar)/(zbar - xbar * zbar)$

$pp3 < -(zbar - xbar * zbar - ybar * zbar + xbar * ybar * zbar)/(xbar * ybar * zbar -$
$xbar * ybar)$

$bias\_p1 < -pp1 - p1$

$bias\_p2 < -pp2 - p2$

$bias\_p3 < -pp3 - p3$

$bias\_p1$

$bias\_p2$

$bias\_p3$

$zbar1 < -mean(x * y)$

$pp1\_sec < -ybar * (zbar1 - xbar - ybar + 1)/(zbar1 * (ybar - 1))$

$pp2\_sec < -xbar * (zbar1 - xbar - ybar + 1)/(zbar1 * (xbar - 1))$

$pp3\_sec < -zbar1 * (xbar - 1) * (ybar - 1)/(xbar * ybar * (zbar1 - xbar - ybar + 1))$

$bias\_p1\_sec < -pp1\_sec - p1$

$bias\_p2\_sec < -pp2\_sec - p2$

$bias\_p3\_sec < -pp3\_sec - p3$

$bias\_p1\_sec$

$bias\_p2\_sec$

$bias\_p3\_sec$

for $(ii$ in $1{:}m)$

$aa[ii, 1, l] < -x[ii]$

$aa[ii, 2, l] < -y[ii]$

$aa[ii, 3, l] < -u[ii]$

}

$bb[l, 1] < -bias\_p1$

$bb[l, 2] < -bias\_p2$

$bb[l, 3] < -bias\_p3$

$bbb[l, 1] < -bias\_p1\_sec$

$bbb[l, 2] < -bias\_p2\_sec$

$bbb[l,3] < -bias\_p3\_sec$

$cc[l,1] < -pp1$

$cc[l,2] < -pp2$

$cc[l,3] < -pp3$

$ccc[l,1] < -pp1\_sec$

$ccc[l,2] < -pp2\_sec$

$ccc[l,3] < -pp3\_sec$

}

$mean\_pp1 < -mean(cc[,1])$

$mean\_pp2 < -mean(cc[,2])$

$mean\_pp3 < -mean(cc[,3])$

$mean\_pp1\_sec < -mean(ccc[,1])$

$mean\_pp2\_sec < -mean(ccc[,2])$

$mean\_pp3\_sec < -mean(ccc[,3])$

$var\_p1 = var(cc[,1])$

$var\_p2 = var(cc[,2])$

$var\_p3 = var(cc[,3])$

$var\_p1\_sec = var(ccc[,1])$

$var\_p2\_sec = var(ccc[,2])$

$var\_p3\_sec = var(ccc[,3])$

$cov\_p1p2 < -mean(cc[,1] * cc[,2]) - mean\_pp1 * mean\_pp2$

$cov\_p1p3 < -mean(cc[,1] * cc[,3]) - mean\_pp1 * mean\_pp3$

$cov\_p2p3 < -mean(cc[,2] * cc[,3]) - mean\_pp2 * mean\_pp3$

$cov\_p1p2\_sec < -mean(ccc[,1] * ccc[,2]) - mean\_pp1\_sec * mean\_pp2\_sec$

$cov\_p1p3\_sec < -mean(ccc[,1] * ccc[,3]) - mean\_pp1\_sec * mean\_pp3\_sec$

$cov\_p2p3\_sec < -mean(ccc[,2] * ccc[,3]) - mean\_pp2\_sec * mean\_pp3\_sec$

$cov\_mat < -array(c(var\_p1, cov\_p1p2, cov\_p1p3, cov\_p1p2, var\_p2, cov\_p2p3,$
$cov\_p1p3, cov\_p2p3, var\_p3), dim = c(3,3))$

$cov\_mat\_sec < -array(c(var\_p1\_sec, cov\_p1p2\_sec, cov\_p1p3\_sec, cov\_p1p2\_sec,$
$var\_p2\_sec, cov\_p2p3\_sec, cov\_p1p3\_sec, cov\_p2p3\_sec, var\_p3\_sec), dim = c(3,3))$

A.2 R Codes for Implementing Chi-square Goodness of Fit Test on the Simulated Sample from the BGD (B&D) Model by Using MLE

$p_1 = 0.964723$

$p_2 = 0.96061$

$p_3 = 0.974676$

$q1 = 1 - p1$

$q2 = 1 - p2$

$q3 = 1 - p3$

$N = 20$

$x < -seq(1, 100, by = 1)$

$x < -array(x, dim = c(100, 1))$

$y < -seq(1, 100, by = 1)$

$y < -array(y, dim = c(100, 1))$

$f < -array(0, dim = c(100, 100))$

$r1 < -array(0, dim = c(100, 100))$

$r2 < -array(0, dim = c(100, 100))$

$r3 < -array(0, dim = c(100, 100))$

$r4 < -array(0, dim = c(100, 100))$

$r5 < -array(0, dim = c(100, 100))$

$r6 < -array(0, dim = c(100, 100))$

for (i in 1:100){

for (j in 1:100){

if ($x[i] < y[j]$) { $f[i,j] = (p_1^{(}x[i]-1)) * (p_2 * p_3)^{(}y[j]-1) * q_1 * (1-p_2 * p_3)\}$ else

if ($x[i] > y[j]$) { $f[i,j] = (p_2^{(}y[j]-1)) * (p1 * p3)^{(}x[i]-1) * q2 * (1-p1 * p3)\}$ else

if ($x[i] == y[j]$) { $f[i,j] = ((p_1 * p_2 * p_3)^{(}x[i]-1)) * (1-p_1 * p_3 - p_2 * p_3 + p_1 * p_2 * p_3)\}\}\}$

sum(f)

for (i in 1:100){

for (j in 1:100){

if ($x[i] <= 5$ && $y[j] <= 5$){

$r1[i,j] < -f[i,j]\}\}\}$

$s1 < -sum(r1)$

$s1$

$ss1 < -N * s1$

$chi1 < -((3 - ss1)^2)/ss1$

$chi1$

for (i in 1:100){

for (j in 1:100){

if ($x[i] <= 5$ && $y[j] > 5$ && $y[j] <= 15$){

$r2[i,j] < -f[i,j]\}\}\}$

$s2 < -sum(r2)$

```
s2
ss2 <- N * s2
chi2 <- (3 - ss2)^2/ss2
chi2

for (i in 1:100){
for (j in 1:100){
if (x[i] > 5 && x[i] <= 20 && y[j] > 0 && y[j] <= 5){
r3[i,j] <- f[i,j]}}}
s3 <- sum(r3)
s3
ss3 <- N * s3
chi3 <- (1 - ss3)^2/ss3
chi3

for (i in 1:100){
for (j in 1:100){
if (x[i] > 5 && x[i] <= 20 && y[j] > 5 && y[j] <= 15){
r4[i,j] <- f[i,j]}}}
s4 <- sum(r4)
s4
ss4 <- N * s4
chi4 <- (5 - ss4)^2/ss4
chi4

for (i in 1:100){
for (j in 1:100){
```

```
if (x[i] > 0 && x[i] <= 20 && y[j] > 15 && y[j] <= 25){
r5[i, j] < -f[i, j]}}}
s5 < -sum(r5)
s5
ss5 < -N * s5
chi5 < -(3 - ss5)^2/ss5
chi5

for (i in 1:100){
for (j in 1:100){
if (x[i] > 20 && x[i] <= 65 && y[j] > 0 && y[j] <= 15){
r6[i, j] < -f[i, j]}}}
s6 < -sum(r6)
s6
ss6 < -N * s6
chi6 < -(3 - ss6)^2/ss6
chi6

sum1 < -sum(s1, s2, s3, s4, s5, s6)
s7 < -1 - sum1
ss7 < -N * s7
ss7
chi7 < -(2 - ss7)^2/ss7
chi7

c1 < -array(c(s1, s2, s3, s4, s5, s6, s7), dim = c(7, 1))
c2 < -array(c(ss1, ss2, ss3, ss4, ss5, ss6, ss7), dim = c(7, 1))
```

$c3 < -array(c(chi1, chi2, chi3, chi4, chi5, chi6, chi7), dim = c(7, 1))$

$cc < -array(c(c1, c2, c3), dim = c(7, 3))$

cc

$dd < -array(c(sum(c1), sum(c2), sum(c3)), dim = c(1, 3))$

dd

REFERENCES

[1] B. C. Arnold, E. Castillo, and J. Sarabia. Conditionally specified distributions: An introduction. *Statistical Science*, 16:249–274, 2001.

[2] G. Asha and N.U. Nair. *Quality Improvement Through Statistical Methods.* Brikhauser Publications, 1998.

[3] Asit P. Basu and Sunil Dhar. Bivariate Geometric Distribution. *Journal Applied Statistical Science*, 2(1):33–44, 1995.

[4] D. R. Cox. Regression models and life tables. *Journal of the Royal Statistical Society, Serious B: Statistical Methodology*, (34):187–220, 1972.

[5] Sunil Dhar. Modeling with a bivariate geometric distribution. In N.Balakrishnan, editor, *Advances on Methodological and Applied Aspects of Probability and Statistics*, volume 1, pages 101–109, London, 2003.

[6] Sunil Dhar and Srinivasan Balaji. On the characterization of a bivariate geometric distribution. *Journal of Communications in Statistics, Theory and Methods*, 35(5):759–766, 2006.

[7] Sunil K. Dhar. Data analysis with discrete analog of freund's model. *Journal of Applied Statistical Science*, 7:169–183, 1998.

[8] J. E. Freund. A bivariate extension of the exponential distribution. *Journal of the American Statistical Association*, 56:971–977, 1961.

[9] A. G. Hawkes. A bivariate exponential distribution with application to reliability. *Journal of the Royal Statistical Society, Serious B: Statistical Methodology*, (34):129–131, 1972.

[10] Subrahmaniam Kocherlakota and Kathleen Kocherlakota. *Bivariate Discrete Distributions.* Marcel Dekker, Inc, 1992.

[11] Samuel Kotz and Norman L. Johnson. A note on renewal (partial sums) distributions for discrete variables. *Statistics & Probability Letters*, 12:229–231, September 1991.

[12] Hare Krishna and Pramendra Pundir. A bivariate geometric distribution with applications to reliability. *Journal of Communications in Statistics, Theory and Methods*, 38:1079–1093, 2009.

[13] Albert W. Marshall and Ingram Olkin. A multivariate exponential distribution. *Journal of the American Statistical Association*, 62:30–44, 1967.

[14] Albert W. Marshall and Ingram Olkin. A family of bivariate distributions generated by the bivariate bernoulli distribution. *Journal of the American Statistical Association*, 80(390):332–338, 1985.

[15] K. R. Muraleedharan Nair and Unnikrishnan Nair. On characterizing the bivariate exponential and geometric distributions. *Annals of the Institute of Statistical Mathematics*, 40(2):267–271, 1988.

[16] Dilip Roy. Reliability measures in the discrete bivariate set-up and related characterization results for a bivariate geometric distribution. *Journal of Multivariate Analysis*, 46:362–373, 1993.

[17] M. Sreehari. Characterizations via conditional distributions. *Journal of the Indian Statistical Association*, 43(1):77–93,, 2005.

[18] Kai Sun and Asit P. Basu. A characterization of a bivariate geometric distribution. *Statistics and Probability Letters*, 23:307–311, 1995.