The Open Set Condition and Neighbor Maps in Fractal Geometry

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Declaration

The work in this thesis is my own except where otherwise stated.

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Abstract

Neighbor maps are a new development in fractal geometry that can be used to determine if an iterated function system (IFS) obeys the open set condition (OSC). Neighbor maps also describe the topology of the fractal attractor. In this thesis, the definition of the set of proper neighbor maps is generalised to any IFS comprising contractive similitudes. It is proven, an IFS of similitudes obeys the OSC if and only if the identity map is not in the closure of the set of proper neighbor maps. The extended definition of the set of proper neighbor maps is used to calculate several neighbor graphs of the generalised Sierpinski triangles. It is then proven, if a generalised Sierpinski triangle has scaling factors that obey the algebraic condition, then its neighbor graph is of finite type. The converse is proven to only hold for the Sierpinski triangle and the Steemson triangle. Neighbor maps are also applied to fractal tiling theory to discuss the prototile set.

Contents

Chapter 1

Introduction and Preliminary Theory

1.1   Introduction

Fractals are mathematical objects that duplicate scaling properties inherit in well-known geometry such as lines and triangles but without the restriction of smoothness. The fractals considered in this thesis are self-similar sets generated by an iterated function system (IFS). An IFS, $\mathcal{F} = \{\mathbb{R}^m; f_1, \ldots f_N\}$, is a finite collection of continuous functions on $\mathbb{R}^m$. We will be considering the case where f_i are similitudes which means that they are compositions of rotations, reflections, translations and uniform scalings of $\mathbb{R}^m$. The fractal attractor, denoted A, is the unique compact subset of $\mathbb{R}^m$ such that $A = \bigcup_{i=1}^{N} f_i(A)$. For an example of a fractal attractor see Figure 2.5 on page 22.

An important task in the field of fractal geometry is to determine the topology of fractal attractors. The question we ask is: does the IFS that generated the fractal attractor satisfy the open set condition (OSC)? An IFS $\mathcal{F}$ satisfies the OSC if there exists a nonempty open set $\mathcal{O}$ such that $f_i(\mathcal{O}) \subset \mathcal{O}$ and $f_i(\mathcal{O}) \cap f_j(\mathcal{O}) = \emptyset$ for all $i, j \in \{1, \ldots, N\}$ with $i \neq j$. When a fractal attractor satisfies the OSC it means that the self-similar pieces of which it consists do not overlap 'too much', this is made precise in Chapter 2.

The motivation to study fractal geometry comes from modelling problems. The natural world is rough and bumpy, so instead of using smooth models from Euclidean geometry or calculus, we can create non-smooth models using fractal

geometry. To model accurately it is required that we understand the fractal attractors being used, specifically, the Hausdorff dimension (Def 1.11) of the fractal attractor? The Hausdorff dimension is useful when applying fractal geometry to modelling problems, but often cannot be calculated. However, if the IFS satisfies the OSC then we can easily evaluate the Hausdorff dimension of the fractal attractor using the Moran-Hutchinson Theorem (Thm 1.14) [PMor46, Hut81]. Therefore, determining if an IFS satisfies the OSC is often a necessity for modelling problems but due to its difficulty, it has remained an open area of study in the field.

The main goal of this thesis is to provide an in-depth discussion of the OSC and an investigation into the recently developed theory of neighbor maps, including proving some original results. The remainder of this chapter is dedicated to introducing the preliminary theory for the discussion to follow. This includes formally defining contractive similitudes, which are the type of functions that we will consider and the basics of IFS theory. This is followed by an explanation of the Hausdorff measure and how it is used to calculate the Hausdorff dimension of fractal attractors.

Chapter 2 is a literature review of the work done up until now on the OSC. The historical overview starts in 1946 when P.A.P. Moran defined the OSC [PMor46]. Later, Lalley [Lal88] provided an alternative topological condition called the strong open set condition (SOSC). However, in $\mathbb{R}^m$ the SOSC is equivalent to the OSC as proved by Schief [Sch94]. Another equivalence to the OSC was discovered by Schief and is known as the combinatorial condition. Loosely speaking, it says that the OSC is satisfied if the self-similar pieces that comprise the fractal attractor only intersect with finitely many other fractal attractor pieces that are of an appropriate size. Chapter 2 finishes with a discussion of a proposed theorem by M. Morán [MMor99], it was stated that the OSC is satisfied if and only if the attractor is not equal to its dynamical boundary (Def 1.7). However, the proof that M. Morán offered was incorrect and so his mistake will be explained. In doing so we will discuss why the proposed theorem remains open.

Chapter 3 introduces neighbor maps from Bandt and collaborators [BG92, BHR05, BM09, BM18]. Neighbor maps are a new development in investigating if an IFS satisfies the OSC and in describing the topology of a fractal attractor. Consider the Sierpinski triangle (Fig: 2.5 on Pg 22), it contains many little

Sierpinski triangles of different sizes in various locations. For any pair of small Sierpinski triangles we can consider the similitude that maps one of those little triangles to the original Sierpinski triangle. This map will take the other little Sierpinski triangle 'elsewhere' in $\mathbb{R}^2$. The neighbor map between these two little Sierpinski triangles is the similitude that maps the original Sierpinski triangle to the image of the other little Sierpinski triangle that is sitting 'elsewhere'. In other words, neighbor maps give the relative relationships between the little self-similar pieces that make up the fractal attractor when one of the little self-similar pieces is treated as the original fractal attractor. Bandt showed that the OSC is satisfied if and only if the identity map is not in the closure of the set of neighbor maps [BG92]. In the case where all functions in the IFS have equal scaling factors, Bandt defined a subset of neighbor maps called the proper neighbor maps, which map the attractor to an isometric copy that has a nonempty intersection with the attractor [BM09]. The set of proper neighbor maps is used as the vertex set to construct a directed graph called the neighbor graph that describes the topology of the fractal attractor in the simplest possible way [BM18]. In this thesis, a stronger theorem than Bandt's is proven: for an IFS with uniform scaling factors the OSC is satisfied if and only if the identity map is not in the closure of the set of proper neighbor maps.

Chapter 4 is the application of Bandt's neighbor map theory to the recently discovered generalised Sierpinski triangles [SW18]. When the functions in the IFS do not have equal scaling factors but do have scaling factors that are integer powers of a common scaling factor, Grant [Gra18] proposed an extension to the definition of the set of proper neighbor maps. This allowed us to calculate the set of proper neighbor maps and neighbor graphs for specific cases of the generalised Sierpinski triangles. In this thesis, Grant and Bandt's discussion of proper neighbor maps is extended to the case where the similitudes in the IFS have arbitrary scaling factors. The new theorem from Chapter 3 is extended first to Grant's case and then to the arbitrary case. We now have, an IFS comprising similitudes of arbitrary scaling factors satisfies the OSC if and only if the identity map is not in the closure of the set of proper neighbor maps.

Chapter 5 provides a discussion of finite type neighbor graphs that involves proving original results. A neighbor graph is of finite type if the set of proper neighbor maps, which each correspond to a vertex in the graph, is finite. In this thesis, we prove that if the scaling factors of a generalised Sierpinski triangle are

integer powers of a common scaling factor then the neighbor graph is of finite type. The converse is also proven true for two types of the generalised Sierpinski triangles, the Sierpinski and Steemson triangles. However, the converse is shown not to be true for the other two types of generalised Sierpinski triangles, the Williams and Pedal triangles. Therefore, we are able to construct an example where the algebraic condition is not satisfied but the neighbor graph is of finite type. We conclude with a discussion of how neighbor maps can be applied to fractal tiling theory. A neighbor map gives the relationship between different tiles in the fractal tiling and the proper neighbor maps describe the fractal tiles that can be adjacent to each other in the tiling. Lastly, for each of the IFSs considered in this thesis we determine the minimum number of different sized fractal tiles needed in the tiling.

1.2 Functions

For the purposes of this thesis we restrict our attention to $(\mathbb{R}^m, d)$ where d is the usual Euclidean metric on $\mathbb{R}^m$. More generally we could consider an arbitrary complete metric space $(\mathbb{X}, d_{\mathbb{X}})$ however we do not find it necessary for the results proven here.

In fractal geometry, we are often concerned with affine maps. An affine map $f : \mathbb{R}^m \to \mathbb{R}^m$ is a transformation of the form $f(x) = Lx + b$, where $L \in \mathbb{R}^m \times \mathbb{R}^m$ is a linear transformation (i.e. a matrix) and $b \in \mathbb{R}^m$ is a vector. For this thesis we are concerned with a specific type of affine map known as a similitude.

Definition 1.1. A similitude is a function $f : \mathbb{R}^m \to \mathbb{R}^m$ of the form $f(x) = \lambda O x + b$, where $\lambda \in \mathbb{R}$, O is an isometry of $\mathbb{R}^m$ and $b \in \mathbb{R}^m$.

Remark 1.2. An isometry is a function $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ that preserves distance between points such that $d(\varphi(x), \varphi(y)) = d(x, y)$ for all $x, y \in \mathbb{R}^m$. Isometries of $\mathbb{R}^2$ are any composition of rotations around the origin, reflections about a straight line through the origin and translations. The isometries of $\mathbb{R}^3$ are the same as $\mathbb{R}^2$, except with rotation and reflection defined with respect to a line passing through the origin and a plane containing the origin. The natural extensions of rotation and reflection continue to give the isometries of $\mathbb{R}^m$ to be any composition of rotations, reflections and translations. For simplicity, the isometry O only needs to be considered as a composition of rotations and reflections which fix the origin since any translation can be absorbed into $b \in \mathbb{R}^m$.

A visual interpretation of a similitude is a transformation that can rotate, reflect, translate and scale the space uniformly. The more general affine transformations also allow the space to be scaled differently in different directions.

A function $f : \mathbb{R}^m \to \mathbb{R}^m$ is contractive if there exists $\lambda \in [0, 1)$ such that $d\left(f\left(x\right), f\left(y\right)\right) \leq \lambda d\left(x, y\right)$ for all $x, y \in \mathbb{R}^m$. If f is a similitude then it is contractive precisely when its scaling factor, λ, satisfies $0 < |\lambda| < 1$.

1.3   IFS Theory

A set of functions known as an iterated function system (IFS) are used to construct fractal attractors.

Definition 1.3. Iterated Function System (IFS)
If $f_n : \mathbb{R}^m \to \mathbb{R}^m$, $n \in \{1, 2, ..., N\}$, are a finite number of contractive similitudes then $\mathcal{F} := \{\mathbb{R}^m; f_1, \ldots, f_N\}$ is called an iterated function system.

Remark 1.4. An IFS, $\mathcal{F} = \{\mathbb{R}^m; f_1, \ldots, f_N\}$, is called contractive if all functions, f_i, in the IFS are contractive.

The Hausdorff subsets, denoted $\mathbb{H}\left(\mathbb{R}^m\right)$, are the nonempty compact sets of $(\mathbb{R}^m, d)$, i.e. the closed and bounded subsets of $\mathbb{R}^m$. Overloading the notation we can treat the IFS as one function that acts on the Hausdorff subsets, that is, $\mathcal{F} : \mathbb{H}\left(\mathbb{R}^m\right) \to \mathbb{H}\left(\mathbb{R}^m\right)$. We define the operation of an IFS on a 'point' $S \in \mathbb{H}\left(\mathbb{R}^m\right)$ as

$$\mathcal{F}(S) = \bigcup_{i=1}^{N} f_i\left(S\right) = f_1\left(S\right) \cup \cdots \cup f_N\left(S\right). \tag{1.1}$$

Let $\mathcal{F}^k$ represented repeated composition, $\mathcal{F}^k\left(S\right) = \mathcal{F}\left(\cdots \mathcal{F}(S) \cdots\right)$ (k times).

Definition 1.5. The attractor of an IFS $\mathcal{F}$ is $A \in \mathbb{H}\left(\mathbb{R}^m\right)$ if $\mathcal{F}\left(A\right) = A$.

Banach's contraction mapping theorem states that every contractive function in a complete metric space has a unique fixed point. The following theorem is a specific case of this by considering $\mathcal{F}$ acting on the complete metric space $\left(\mathbb{H}\left(\mathbb{R}^m\right), d_{\mathbb{H}}\right)$, where $d_{\mathbb{H}}$ is the Hausdorff metric defined for $X, Y \in \mathbb{H}(\mathbb{R}^2)$ as $d_{\mathbb{H}}(X, Y) = \max\left\{\max_{x \in X} \min_{y \in Y}\left\{d(x,y)\right\}, \max_{y \in Y} \min_{x \in X}\left\{d(x,y)\right\}\right\}.$

Theorem 1.6. *Hutchinson's Theorem [Hut81]*
If the IFS $\mathcal{F}$ is contractive on the complete metric space $\left(\mathbb{R}^m, d\right)$, then $\mathcal{F}$ has a unique attractor A and for any $S \in \mathbb{H}\left(\mathbb{R}^m\right)$, $\lim_{k \to \infty} \mathcal{F}^k(S) = A$, where the limit is taken with respect to the Hausdorff metric.

The critical set, also referred to as the set of overlap, is defined by $\mathcal{C} = \bigcup_{i \neq j,} f_i(A) \cap f_j(A)$. The critical set is used to define the dynamical boundary and to classify different types of IFSs.

Definition 1.7. The Dynamical Boundary
For an IFS $\mathcal{F} = \{\mathbb{R}^m; f_1, \ldots, f_N\}$ with attractor A and critical set $\mathcal{C}$ we define the dynamical boundary to be

$$\partial_M A = \overline{\bigcup_{k=1}^{\infty} \mathcal{F}^{-k}(C) \cap A},$$

where $\overline{S}$ denotes the closure of the set S.

Remark 1.8. The subscript M is in reference to M. Morán who defined it and to distinguish it from Bandt's definition (Def 2.16).

Remark 1.9. The reader may find the definition of $\partial_M A$ ambiguous due to not knowing the order of operations between the infinite union and the intersection. However, it is straightforward to see $\bigcup_{k=1}^{\infty} \left(\mathcal{F}^{-k}(\mathcal{C}) \cap A \right) = \left(\bigcup_{k=1}^{\infty} \mathcal{F}^{-k}(\mathcal{C}) \right) \cap A$ by the following argument (note that $\mathbb{N} = \{1, 2, 3, \ldots\}$):

$$
\begin{aligned}
x \in \bigcup_{k=1}^{\infty} \left(\mathcal{F}^{-k}(\mathcal{C}) \cap A \right) \iff & \; \exists j \in \mathbb{N} \text{ s.t. } x \in \mathcal{F}^{-j}(\mathcal{C}) \cap A \\
\iff & \; \exists j \in \mathbb{N} \text{ s.t. } x \in \mathcal{F}^{-j}(\mathcal{C}) \text{ and } x \in A \\
\iff & \; x \in \bigcup_{k=1}^{\infty} \mathcal{F}^{-k}(\mathcal{C}) \text{ and } x \in A \\
\iff & \; x \in \left(\bigcup_{k=1}^{\infty} \mathcal{F}^{-k}(\mathcal{C}) \right) \cap A.
\end{aligned}
\tag{1.2}
$$

An IFS is classified into one of three types using its critical set and dynamical boundary. If the critical set is empty then the IFS is called totally disconnected. If $\mathcal{C} \neq \emptyset$ and $\partial_M A \neq A$ then the IFS is just touching. If there exists a nonempty open set $U \subset \mathbb{R}^m$ such that $U \subset f_i(A) \cap f_j(A)$ for some $i \neq j$ then the IFS is said to be overlapping.

1.4   Hausdorff Measure and Fractal Dimension

In fractal geometry we use the s-dimensional Hausdorff measure to define the dimension of fractal attractors. First, let us define the diameter of a set $S \subset \mathbb{R}^m$,

$|S|$, to be the least upper bound for the distance between two points of the set. That is, $|S| = \sup\{d(x,y) : x, y \in S\}$. If $\{U_i\}$ is a countable collection of sets of diameter at most δ that cover the set F, i.e. $F \subset \bigcup_{i=1}^{\infty} U_i$ with $0 \leq |U_i| \leq \delta$ for each i, then it is called a δ-cover of F.
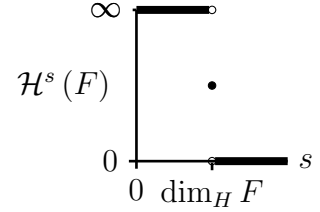
Definition 1.10. s-Dimensional Hausdorff Measure [Fal04]
Let $F \subset \mathbb{R}^m$ and $s \geq 0$. For any $\delta > 0$ we define

$$\mathcal{H}_\delta^s(F) = \inf\left\{\sum_{i=1}^{\infty}|U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F\right\}. \tag{1.3}$$

As δ decreases the class of permissible covers of F is reduced. Therefore, the infimum of $\mathcal{H}_\delta^s(F)$ increases, and so limits as $\delta \to 0$. This limit exists in $\mathbb{R} \cup \{\infty\}$, and therefore we define the s-dimensional Hausdorff measure as,

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(F). \tag{1.4}$$

Definition 1.11. Hausdorff Dimension [Fal04]
The Hausdorff dimension of a set $F \subset \mathbb{R}^m$, denoted $\dim_H F$, is the value at which $\mathcal{H}^s(F)$ 'jumps' from ∞ to 0. Formally we have

$$\dim_H F = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}. \tag{1.5}$$

In fractal geometry we have more than one measurement of dimension. Along with the Hausdorff dimension we also have the similarity dimension.

Definition 1.12. Similarity Dimension [Hut81]
If $\mathcal{F} = \{\mathbb{R}^m; f_1, \ldots, f_N\}$ is an IFS where each f_i is a similitude with scaling factor $0 < \lambda_i < 1$, then the similarity dimension is the unique positive solution D to the equation $\sum_{i=1}^{N} \lambda_i^D = 1$.

Remark 1.13. From [Hut81] we have that the solution D is unique. We see this by considering $\gamma(t) := \sum_{i=1}^{N} \lambda_i^t$. Since $0 < \lambda_i < 1$ then we have that γ is a monotonically decreasing continuous function with $\gamma(0) = N$ and $\gamma(t) \to 0$ as $t \to \infty$. Therefore there is a unique positive value t such that $\gamma(t) = 1$.

The Hausdorff dimension and similarity dimension of a fractal attractor are equal when the IFS generating the fractal attractor satisfies the OSC which was introduced in Section 1.1 and is formally defined in Definition 2.1.

Theorem 1.14. *The Moran-Hutchinson Theorem [Hut81, PMor46]*
If an IFS comprising similitudes satisfies the open set condition, then the Hausdorff dimension of the attractor is equal to the similarity dimension.

The following proposition is a nontrivial relation between the similarity dimension and the Hausdorff dimension when the OSC condition is not necessarily satisfied. It is taken from Hutchinson [Hut81] and is proven here in a way that will be useful to us later.

Proposition 1.15. *[Hut81](c.f. 5.1.(4)(i))*
If $\mathcal{F}$ is an IFS constructed only of similitudes with attractor A, similarity dimension D and Hausdorff dimension $\dim_H A$, then $\mathcal{H}^D(A) < \infty$ and $\dim_H A \leq D$.

Proof. Let $\mathcal{F} = \{\mathbb{R}^m; f_1, \ldots, f_N\}$ be an IFS of contractive similitudes each with a contraction factor λ_i. As A is the attractor of $\mathcal{F}$ then $A = \bigcup_{i=1}^N A_i$ where $A_i = f_i(A)$. This union describes a way of breaking up A into smaller pieces. Given any $\delta > 0$, there exists a finite integer p such that the process of breaking up A can be repeated so that $A = \bigcup_{\boldsymbol{i}} A_{\boldsymbol{i}}$, where $A_{\boldsymbol{i}} = A_{i_1,\ldots,i_p} = f_{i_1} \circ \cdots \circ f_{i_p}(A)$ and $|A_{\boldsymbol{i}}| < \delta$ for length p strings of integers $\{1, \ldots, N\}$. Therefore given a $\delta > 0$ we have a δ-cover of the attractor by using the pieces $A_{\boldsymbol{i}}$. This implies,

$$
\begin{aligned}
\mathcal{H}^D_\delta(A) &= \inf\left\{\sum_{i=1}^\infty |U_i|^D : \{U_i\} \text{ is a } \delta\text{-cover of } F\right\} \\
&\leq \sum_{\boldsymbol{i}} |A_{\boldsymbol{i}}|^D = \sum_{\boldsymbol{i}} |f_{\boldsymbol{i}}(A)|^D = \sum_{\boldsymbol{i}} \lambda_{\boldsymbol{i}}^D |A|^D = |A|^D .
\end{aligned}
\tag{1.6}
$$

The last equality comes from recalling that by the definition of the similarity dimension we have $\sum_{i=1}^N \lambda_i^D = 1$ and then expanding the summation. Therefore for any given $\delta > 0$ we have $\mathcal{H}^D_\delta(A) \leq |A|^D$. Taking the limit as $\delta \to 0$ gives us that $\mathcal{H}^D(A) \leq |A|^D$. As $A \in \mathbb{H}(\mathbb{R}^m)$ then A is bounded and so it has finite diameter and therefore $|A|^D < \infty$. Thus, it has been shown that for an IFS of similitudes with attractor A and similarity dimension D we have $\mathcal{H}^D(A) < \infty$. This says that the jump of $H^s(A)$ from ∞ to 0 occurs before or at D which implies that $\dim_H A \leq D$. $\qquad\square$

Chapter 2

The Open Set Condition

In an effort to understand the topology of a fractal attractor we have the following question: given an IFS, how can we prove whether or not it satisfies the open set condition? Before we can discuss this question we must properly understand the OSC.

2.1 The Open Set Condition

The open set condition was defined by P.A.P. Moran [PMor46] as a way of classifying if the critical set of an IFS is small.

Definition 2.1. The Open Set Condition (OSC)
The IFS $\mathcal{F} = \{\mathbb{R}^m; f_1, \ldots f_N\}$ with obeys the open set condition if there exists a nonempty open subset $\mathcal{O}$ of $\mathbb{R}^m$ such that

1. $f_i(\mathcal{O}) \subset \mathcal{O}$ for all $i \in \{1, \ldots, N\}$, and

2. $f_i(\mathcal{O}) \cap f_j(\mathcal{O}) = \emptyset$ for all $i, j \in \{1, \ldots, N\}$ with $i \neq j$.

It is important to realise that it is the IFS that obeys the OSC not the attractor. By Hutchinson's Theorem (Thm 1.6) each contractive IFS on a complete metric space has a unique attractor, however the converse is not true. Example 2.2 provides two distinct IFSs which have the same attractor where one obeys the OSC while the other does not.

Example 2.2. Let us define two IFSs, $\mathcal{F} = \left\{\mathbb{R}; f_1(x) = \frac{1}{2}x, f_2(x) = \frac{1}{2}x + \frac{1}{2}\right\}$ and $\mathcal{G} = \left\{\mathbb{R}; g_1(x) = \frac{3}{5}x, g_2(x) = \frac{3}{5}x + \frac{2}{5}\right\}$. For $A = [0,1]$ we have $\mathcal{F}(A) = A$ and $\mathcal{G}(A) = A$ by Equations 2.1 and 2.2. By Hutchinson's Theorem we have that

$A = [0, 1]$ is the unique attractor for both $\mathcal{F}$ and $\mathcal{G}$. A graphical representation of the IFSs is displayed in Figure 2.1.

$$\mathcal{F}([0, 1]) = f_1([0, 1]) \cup f_2([0, 1]) = \left[0, \frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right] = [0, 1] \qquad (2.1)$$

$$\mathcal{G}([0, 1]) = g_1([0, 1]) \cup g_2([0, 1]) = \left[0, \frac{3}{5}\right] \cup \left[\frac{2}{5}, 1\right] = [0, 1] \qquad (2.2)$$

Figure 2.1: The two IFSs $\mathcal{F}$ and $\mathcal{G}$, have $[0, 1]$ as the attractor but only $\mathcal{F}$ obeys the OSC.

In order to show that $\mathcal{F}$ obeys the OSC we can simply provide a sufficient nonempty open set such as $\mathcal{O} = (0, 1)$. We have that,

$$f_1(\mathcal{O}) = f_1((0, 1)) = \left(0, \frac{1}{2}\right) \subset (0, 1) = \mathcal{O},$$

$$f_2(\mathcal{O}) = f_2((0, 1)) = \left(\frac{1}{2}, 1\right) \subset (0, 1) = \mathcal{O}, \qquad (2.3)$$

$$\text{and} \quad f_1(\mathcal{O}) \cap f_2(\mathcal{O}) = \left(0, \frac{1}{2}\right) \cap \left(\frac{1}{2}, 1\right) = \emptyset.$$

Therefore $\mathcal{F}$ satisfies the OSC.

It will now be shown that $\mathcal{G}$ does not satisfy the open set condition. If we were to try and show this directly from the definition of the OSC then it would be nontrivial and inelegant. This is because it would involve showing that there does not exist any suitable open set $\mathcal{O}$. Initially, one might think that we could simply take an arbitrary open interval (a, b) and show that there does not exist any $a, b \in \mathbb{R}$ such that the OSC is satisfied. However, we must also consider arbitrary open sets such as those constructed from infinite unions of open intervals because there is no requirement that $\mathcal{O}$ must be convex or connected [Ban91]. Therefore, the task of showing that all nonempty open sets do not satisfy the OSC can be extremely difficult depending on the IFS. Note that for $\mathcal{F}$ above we showed that it satisfies the OSC by simply producing a suitable $\mathcal{O}$. If we were

not able to find the needed $\mathcal{O}$, then although an IFS could satisfy the OSC it could be equally difficult to show that it does satisfy the OSC.

Now, in order to show that $\mathcal{G}$ does not satisfy the OSC we will assume that it does and gain a contradiction. Assuming $\mathcal{G}$ satisfies the OSC then by the Moran-Hutchinson Theorem (Thm 1.14) we have that the Hausdorff dimension is equal to the similarity dimension. The similarity dimension is the unique positive solution D to the equation,

$$1 = \sum_{i=1}^{N} \lambda_i^D = 2 \left(\frac{3}{5}\right)^D.$$

By a simple calculation we have $D = \frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{3}{5}\right)} \approx 1.36$. It is a contradiction to have a Hausdorff dimension of 1.36 since the IFS is defined on $\mathbb{R}$. Therefore it was incorrect to assume that $\mathcal{G}$ satisfies the OSC.

2.2 The Strong Open Set Condition

Lalley [Lal88] strengthened the open set condition to the strong open set condition by adding the extra requirement that the nonempty open set must have a nonempty intersection with the attractor.

Definition 2.3. The Strong Open Set Condition (SOSC)
The IFS $\mathcal{F} = \{\mathbb{R}^m; f_1, \ldots f_N\}$ with attractor A obeys the strong open set condition if there exists a nonempty open subset $\mathcal{O}$ of $\mathbb{R}^m$ such that

1. $f_i(\mathcal{O}) \subset \mathcal{O}$ for all $i \in \{1, \ldots, N\}$,

2. $f_i(\mathcal{O}) \cap f_j(\mathcal{O}) = \emptyset$ for all $i, j \in \{1, \ldots, N\}$ with $i \neq j$, and

3. $\mathcal{O} \cap A \neq \emptyset$.

Remark 2.4. From the definitions, an IFS that satisfies the SOSC must also satisfy the OSC (SOSC $\implies$ OSC).

A natural question that arises from the definition of the SOSC is: what does an IFS look like that would satisfy the OSC but not the SOSC? Schief [Sch94] proved that in $\mathbb{R}^m$ the OSC is equivalent to the SOSC. This will be discussed further in Section 2.3. Therefore to construct an IFS that satisfies the OSC but

fails the SOSC requires leaving $(\mathbb{R}^m, d)$, and so we do not pursue this question. Thus, to show that an IFS on $\mathbb{R}^m$ does not satisfy the OSC via the direct method the type of nonempty open sets to consider has been restricted but there is still infinitely many. In order to understand the situation better let us now present an example of an IFS that satisfies the OSC (and equivalently the SOSC). It will be shown that there exists a nonempty open set which shows that the IFS satisfies the OSC but the same nonempty open set is not sufficient to show that the IFS satisfies the SOSC.

Example 2.5. Let $\mathcal{F} = \left\{ \mathbb{R}; f_1(x) = \frac{1}{3}x, f_2(x) = \frac{1}{3}x + \frac{2}{3} \right\}$, it has the Cantor set, $\mathscr{C}$, as its attractor. The Cantor set has the following construction, let $C_0 = [0, 1]$ and construct C_1 by removing the open middle third to give $C_1 = \left[0, \frac{1}{3} \right] \cup \left[\frac{2}{3}, 1 \right]$. The first five iterations of this process are illustrated in Figure 2.2. The Cantor set is given by $\mathscr{C} = \bigcap_{n=0}^{\infty}$.

Figure 2.2: The first five iterations of the Cantor set which is the attractor of $\mathcal{F} = \left\{ \mathbb{R}; f_1(x) = \frac{1}{3}x, f_2(x) = \frac{1}{3}x + \frac{2}{3} \right\}$.

Let us consider $\mathcal{O} = (0, 1)$ which is a nonempty open set and we see that

$$f_1\left(\mathcal{O} \right) = f_1\left((0, 1) \right) = \left(0, \frac{1}{3} \right) \subset \mathcal{O},$$

$$f_2\left(\mathcal{O} \right) = f_2\left((0, 1) \right) = \left(\frac{2}{3}, 1 \right) \subset \mathcal{O},$$

$$\text{and} \quad f_1\left(\mathcal{O} \right) \cap f_2\left(\mathcal{O} \right) = \left(0, \frac{1}{3} \right) \cap \left(\frac{2}{3}, 1 \right) = \emptyset.$$

Therefore $\mathcal{F}$ satisfies the OSC and since $\mathcal{O} \cap A \neq \emptyset$ then $\mathcal{O}$ also shows that $\mathcal{F}$ satisfies the SOSC. Let us now construct a nonempty open set $\mathcal{U}$ which shows that $\mathcal{F}$ satisfies the OSC but $\mathcal{U} \cap A = \emptyset$ and so $\mathcal{U}$ is not sufficient to show that $\mathcal{F}$ satisfies the SOSC (if we were to forget about their equivalence in $\mathbb{R}^m$). Let $\mathcal{U} = (0, 1) \setminus A = (0, 1) \cap A^c$. As A is a Hausdorff subset it is compact in $\mathbb{R}$ which is to say that it is closed and bounded. Therefore its compliment, A^c, is a open set. As $\mathcal{U}$ is the intersection of two open sets it is also an open set. Thus $\mathcal{U}$ is open and since $A \neq (0, 1)$ then it is also nonempty. We have that $f_1\left(\left(\frac{1}{3}, \frac{2}{3} \right) \right) = \left(\frac{1}{9}, \frac{2}{9} \right)$

which is the middle interval that was from $\left[0, \frac{1}{3}\right]$ in creating C_2. Similarly, each of the disjoint open intervals comprising $\mathcal{U}$ are mapped by f_1 to another interval that is removed from $[0, 1]$ during the construction of the Cantor set. Therefore we have $f_1(\mathcal{U}) \subset \mathcal{U}$. Similarly we have $f_2(\mathcal{U}) \subset \mathcal{U}$. Since $\mathcal{U} \subset (0, 1)$ and from above we have $f_1((0, 1)) \cap f_2((0, 1)) = \emptyset$, then we have that $f_1(\mathcal{U}) \cap f_2(\mathcal{U}) = \emptyset$. Therefore $\mathcal{U}$ satisfies the OSC; however, $\mathcal{U} \cap A = ((0, 1) \setminus A) \cap A = \emptyset$, which is to say that $\mathcal{U}$ is not sufficient to show that $\mathcal{F}$ satisfies the SOSC (disregarding the equivalence of the OSC and SOSC in $\mathbb{R}^m$).

2.3   Schief's Equivalences of the OSC

Schief published a paper in 1994 [Sch94] that presented two important theorems relating to the OSC. The first, Theorem 2.6, proved the equivalence of the OSC and the SOSC in $\mathbb{R}^m$, relating them to the Hausdorff measure of the attractor and its dimension. Schief's second theorem provided a new and equivalent combinatorial condition to the OSC as seen in Theorem 2.8.

Theorem 2.6. *Schief's Equivalences [Sch94]*
If $\mathcal{F} = \{\mathbb{R}^m; f_1, \ldots, f_N\}$ is an IFS where each f_i is a contractive similitude with scaling factor λ_i and A is the attractor, then we have the following,

$$SOSC \iff OSC \iff \mathcal{H}^D(A) > 0 \implies \dim_H A = D. \tag{2.4}$$

Here D is the similarity dimension of $\mathcal{F}$, $\dim_H A$ is the Hausdorff dimension of the attractor A, and $\mathcal{H}^D$ is the D-dimensional Hausdorff measure.

The proof of Theorem 2.6 will not be presented but can be found in [Sch94]. Instead, we offer a discussion to support Schief's work. Let us note that three of these implications were already well-known. The first was that SOSC $\Rightarrow$ OSC as stated in Remark 2.4. The implication OSC $\Rightarrow \mathcal{H}^D(A) > 0$ was also known as P.A.P. Moran stated and proved it in 1946 [PMor46](c.f. Thm II) and Hutchinson [Hut81] (c.f 5.3.(1).(ii)) offered a different proof in 1981. The last implication that Schief already knew was $\left(\mathcal{H}^D(A) > 0 \implies \dim_H A = D\right)$. This is true since for an IFS of contractive similitudes we have that $\mathcal{H}^D(A) < \infty$ by Proposition 1.15. Thus, assuming $0 < \mathcal{H}^D(A)$ is equivalent to assuming $0 < \mathcal{H}^D(A) < \infty$ which by definition gives $\dim_H A = D$.

Schief [Sch94] stated that $\dim_H A = D \nRightarrow \mathcal{H}^D(A) > 0$ due to a counterexample credited to Mattila but with no reference. For that reason we present the

counterexample in explicit detail in Example 2.7.

The contribution that Schief [Sch94] supplied in proving Theorem 2.6 was to prove $\mathcal{H}^D(A) > 0 \implies$ SOSC. This turned the first and second implications into equivalences. Therefore we have that the SOSC is equivalent to the OSC in $\mathbb{R}^m$. Theorem 2.6 also provides a computation method, in that the IFS satisfies the OSC if and only if $\mathcal{H}^D(A) > 0$. However, the process of calculating $\mathcal{H}^D(A)$ is not necessarily any easier than the direct proof method of determining if the OSC is satisfied.

Example 2.7. A counter example will now be provided to show that $\dim_H A = D$ does not imply $\mathcal{H}^D(A) > 0$. To do so let us define the following IFS which has a disconnected Sierpinski triangle A as its attractor as shown in Figure 2.3,

$$\mathcal{F} = \left\{ \mathbb{R}^2;\ f_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \boldsymbol{b_i} \ \bigg|\ i \in \{1, 2, 3\},\right.$$
$$\left. \boldsymbol{b_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \boldsymbol{b_2} = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}, \boldsymbol{b_3} = \begin{pmatrix} \frac{1}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix} \right\}. \tag{2.5}$$
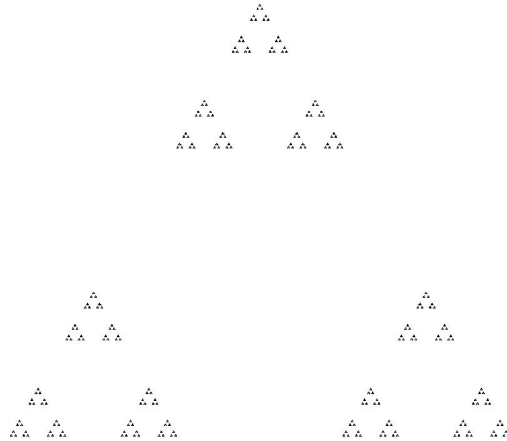
Figure 2.3: The attractor of the IFS given in Equation 2.5 is a disconnected Sierpinski triangle with side lengths of size 1.

It is clear to see that $\mathcal{F}$ obeys the OSC since $\mathcal{O} = (0,1) \times (0,1)$ is a suitable nonempty open set. Equivalently, it is a known fact that if the IFS is totally disconnected then it will obey the OSC since there exists $\epsilon > 0$ such that the attractor dilated by ϵ is a suitable nonempty open set.

The similarity dimension, D, of $\mathcal{F}$ is given by $\sum_{i=1}^{3} \lambda_i^D = 1$ with $\lambda_i = \frac{1}{3}$. The equation simplifies to $3\left(\frac{1}{3}\right)^D = 1$, and so $D = 1$. As the IFS satisfies the OSC then by the Moran-Hutchinson Theorem (Thm 1.14) the Hausdorff dimension of A is equal to its similarity dimension and so $\dim_H A = 1$.

Let L_θ be the straight line in $\mathbb{R}^2$ that passes through the origin and makes an angle θ with the positive x-axis measured in a counter-clockwise direction. The set A and the IFS $\mathcal{F}$ can be projected onto the 1-dimensional subspace of $\mathbb{R}^2$ described by L_θ. Let A_θ be the projection of A onto the line L_θ. The set A_θ is the attractor of IFS $\mathcal{F}_\theta$, where the maps in $\mathcal{F}_\theta$ are the maps in $\mathcal{F}$ composed with the projection map onto L_θ.
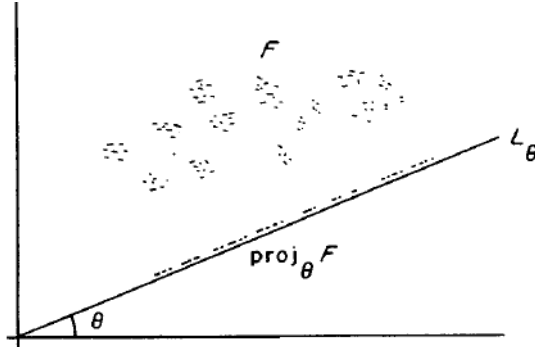

Figure 2.4: The projection of an arbitrary set F onto L_θ [Fal04].

For example let $\theta = 0$ so that A is being projected onto the x-axis. We have $\mathcal{F}_0 = \left\{\mathbb{R}; f_1(x) = \frac{1}{3}x, f_2(x) = \frac{1}{3}x + \frac{2}{3}, f_3(x) = \frac{1}{3}x + \frac{1}{3}\right\}$ which has an attractor $A_0 = [0, 1]$.

Similitudes contract the space uniformly in all directions and therefore the three maps in $\mathcal{F}_\theta$ will be three similitudes of scaling factor $\frac{1}{3}$ but the translation factors of the maps will depend on the value of $\theta \in [0, \pi)$. The similarity dimension for all $\mathcal{F}_\theta$ and corresponding A_θ will be $D = 1$ due to the IFS always having three maps of scaling one third.

Falconer [Fal04](c.f. Thm 6.1) states that if $\dim_H A = 1$ then $\dim_H A_\theta = \dim_H A$ for almost all $\theta \in [0, \pi)$. Therefore $\dim_H A_\theta = 1 = D$ for almost all $\theta \in [0, \pi)$, and so in order to have A_θ be our desired contradiction all that remains to show is $\mathcal{H}^D(A_\theta) = 0$.

According to Falconer [Fal04](c.f. Thm 6.4), since $A \in \mathbb{R}^2$ is the attractor of a totally disconnected IFS with Hausdorff dimension 1 then the 1-dimensional Lebesgue measure of the projection set A_θ is 0 for almost all $\theta \in [0, \pi)$. It is a known property for a set $F \subset \mathbb{R}^n$ that the n-dimensional Hausdorff measure and the n-dimensional Lebesgue measure differ by a nonzero multiplication constant that is dependent on n. Therefore the 1-dimensional Hausdorff measure of the projection set A_θ is 0 for almost all $\theta \in [0, \pi)$. Therefore, almost all projections sets A_θ satisfiy $\dim_H A_\theta = D$ and $\mathcal{H}^D (A_\theta) = 0$, and thus the implication $\dim_H A = D \Rightarrow \mathcal{H}^D (A) > 0$ is not true and we have our desired counterexample.

Prior to discussing Schief's second theorem and the combinatorial condition for the OSC we must first introduce some notation. Let $\mathcal{F} = \{\mathbb{R}^m; f_1 \ldots, f_N\}$ be a contractive IFS and let the scaling factor of f_i be $\lambda_i \in [0, 1)$. Let us consider finite length vectors $\boldsymbol{i} = (i_1, \ldots, i_n)$ and $\boldsymbol{j} = (j_1, \ldots, j_m)$ of length n and m respectively, where each element of $\boldsymbol{i}$ and $\boldsymbol{j}$ is from $\{1, \ldots N\}$. Let us denote the concatenation of $\boldsymbol{i}$ and $\boldsymbol{j}$ as $\boldsymbol{ij} = (i_1, \ldots, i_n, j_1, \ldots j_m)$. We say that $\boldsymbol{i}$ and $\boldsymbol{j}$ are incomparable if there does not exist any finite length vector $\boldsymbol{k}$ such that $\boldsymbol{i} = \boldsymbol{jk}$ or $\boldsymbol{j} = \boldsymbol{ik}$. In other words, $\boldsymbol{i}$ and $\boldsymbol{j}$ are incomparable if neither is an initial sub-vector of the other. Two pieces of a fractal attractor $A_{\boldsymbol{i}}$ and $A_{\boldsymbol{j}}$ are incomparable, one is not a subset of the other, if $\boldsymbol{i}$ and $\boldsymbol{j}$ are incomparable. Let us also define the following notational abbreviations: $f_{\boldsymbol{i}} = f_{i_1} \circ \cdots \circ f_{i_n}$, $A_{\boldsymbol{i}} = f_{\boldsymbol{i}} (A)$ and $\lambda_{\boldsymbol{i}} = \lambda_{i_1} \cdots \lambda_{i_n}$. We sometimes call $\boldsymbol{i}$ an address since it describes a particular 'location' in the fractal attractor, specifically the compact subset $A_{\boldsymbol{i}}$. If $\boldsymbol{i}$ was an infinite string then it would refer to a single point in the attractor.

For a given $0 < b \le 1$ we can define the following set,

$$I_b = \left\{ \boldsymbol{i} = (i_1, \ldots, i_n) \, | \, \lambda_{\boldsymbol{i}} < b \le \lambda_{(i_1, \ldots, i_{n-1})} \right\}, \tag{2.6}$$

where $\lambda_{(i_1, \ldots, i_{n-1})}$ is the scaling factor for the map $f_{(i_1, \ldots, i_{n-1})} = f_{i_1} \circ \cdots \circ f_{n-1}$ which is one iteration less into the fractal attractor then $f_{\boldsymbol{i}}$. The set I_b is the collection of addresses $\boldsymbol{i}$ that are of minimum length and correspond to attractor peices, $A_{\boldsymbol{i}}$, that has been scaled more than $b \in (0, 1]$. Therefore the elements of I_b are incomparable and satisfy $A = \bigcup_{\boldsymbol{i} \in I_b} A_{\boldsymbol{i}}$. Now fix $\epsilon \in \left(0, \frac{1}{3}\right)$ [Sch94] and let $\mathcal{B}(x, r)$ be the open ball centred at x with radius r. Using this notation we define the set $G_{\boldsymbol{k}}$ to be the dilated version of $A_{\boldsymbol{k}}$,

$$G_{\boldsymbol{k}} = \mathcal{B} (A_{\boldsymbol{k}}, \epsilon \lambda_{\boldsymbol{k}}) = \bigcup \left\{ \mathcal{B} (x, \epsilon \lambda_{\boldsymbol{k}}) \, | \, x \in A_{\boldsymbol{k}} \right\}. \tag{2.7}$$

Recalling that $|G_{\boldsymbol{k}}|$ is the diameter of $G_{\boldsymbol{k}}$. Without loss of generalisation we can assume that the $|G_{\boldsymbol{k}}| \leq 1$ by a change of coordinates. Thus, we have the following,

$$I(\boldsymbol{k}) = \left\{ \boldsymbol{i} \in I_{|G_{\boldsymbol{k}}|} \mid A_{\boldsymbol{i}} \cap G_{\boldsymbol{k}} \neq \emptyset \right\}. \tag{2.8}$$

$I(\boldsymbol{k})$ is the set of finite length vectors which describe the incomparable pieces $A_{\boldsymbol{i}}$ which intersect the $(\epsilon\lambda_{\boldsymbol{k}})$-neighborhood of $A_{\boldsymbol{k}}$ and are of size at least $\lambda_{min}|G_{\boldsymbol{k}}|$. Then we define $\gamma = \sup_{\boldsymbol{k}} \#I(\boldsymbol{k})$, where $\#I(\boldsymbol{k})$ denotes the cardinality of $I(\boldsymbol{k})$. Therefore γ is the least upper bound for the number of intersecting pieces for any $\boldsymbol{k}$. We are now equiped to understand Schief's [Sch94] second theorem.

Theorem 2.8. *Schief's Combinatorial Equivalence of the OSC [Sch94]*
If $\mathcal{F} = \{\mathbb{R}^m; f_1, \ldots, f_N\}$ is an IFS of contractive similitudes with scaling factors λ_i, similarity dimension D and attractor A the following are equivalent conditions.

1. *SOSC.*

2. *OSC.*

3. $\mathcal{H}^D(A) > 0$.

4. *For each $\epsilon > 0$, $\gamma < \infty$ holds.*

5. *There exists $\epsilon > 0$ such that $\gamma < \infty$.*

6. *SOSC holds with an open set U such that $\mu(U) = 1$, see below.*

Remark 2.9. Condition (6) tells us that U contains almost all of A with respect to an appropriate measure μ which is explicitly defined in Schief [Sch94] but we omit the technicalities here because the real points of interest for us are Conditions (4) and (5) since they relate to the combinatorial description of the open set condition.

Theorem 2.8 tells us that if there exists an $\epsilon > 0$ such that γ is finite then γ is finite for all ϵ which is equivalent to the OSC being satisfied. Let us now calculate γ for a simple example to show how this method may be used to establish that the OSC is satisfied.

Example 2.10. Let us take the IFS, $\mathcal{F} = \left\{\mathbb{R}; f_1(x) = \frac{1}{2}x, f_2(x) = \frac{1}{2}x + \frac{1}{2}\right\}$ from Example 2.2 which has attractor $[0, 1]$. Let us fix $\epsilon = \frac{1}{5}$. We begin by taking several values for $\boldsymbol{k}$ and determining the cardinality of $I(\boldsymbol{k})$.

Let $\boldsymbol{k} = (1)$. Then $G_{\boldsymbol{k}} = G_{(1)} = \mathcal{B}(A_1, \epsilon\lambda_1) = \mathcal{B}\left(A_1, \frac{1}{10}\right)$. Therefore

$\left|G_{(1)}\right| = \lambda_1 |A| + 2\epsilon\lambda_1 = \frac{1}{2} \cdot 1 + 2 \cdot \frac{1}{5} \cdot \frac{1}{2} = \frac{7}{10}$. Now note that $I_{|G_{\boldsymbol{k}}|} = I_{\frac{7}{10}} = \{(1), (2)\}$. We have that $G_{(1)}$ intersects both A_1 and A_2. Thus $\#I((1)) = 2$. Also, by symmetry we get that $\#I((2)) = 2$.

Let $\boldsymbol{k} = (1, 1)$. Then $G_{\boldsymbol{k}} = G_{(1,1)} = \mathcal{B}(A_{11}, \epsilon\lambda_{11}) = \mathcal{B}\left(A_{11}, \frac{1}{20}\right)$. Therefore $|G_{\boldsymbol{k}}| = \lambda_{11} |A| + 2\epsilon\lambda_{11} = \frac{1}{4} \cdot 1 + 2 \cdot \frac{1}{5} \cdot \frac{1}{4} = 0.35$. Now note that $I_{|G_{(1,1)}|} = I_{0.35} = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Only A_{11} and A_{12} intersect $G_{(1,1)}$. Thus $\#I((1, 1)) = 2$. By symmetry we get that $\#I((2, 2)) = 2$.

Let $\boldsymbol{k} = (1, 2)$. Then $G_{\boldsymbol{k}} = G_{(1,2)} = \mathcal{B}(A_{12}, \epsilon\lambda_{12}) = \mathcal{B}\left(A_{12}, \frac{1}{20}\right)$. Therefore $|G_{\boldsymbol{k}}| = \lambda_{12} |A| + 2\epsilon\lambda_{12} = \frac{1}{4} \cdot 1 + 2 \cdot \frac{1}{5} \cdot \frac{1}{4} = 0.35$. Now note that $I_{|G_{(1,2)}|} = I_{0.35} = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$. We have that A_{11}, A_{12} and A_{21} intersect $G_{(1,2)}$. Thus $\#I((1, 2)) = 3$. By symmetry we get that $\#I((2, 1)) = 3$.

Let $\boldsymbol{k} = (1, 2, 2)$. Then $G_{\boldsymbol{k}} = G_{(1,2,2)} = \mathcal{B}(A_{122}, \epsilon\lambda_{122}) = \mathcal{B}\left(A_{122}, \frac{1}{40}\right)$. Therefore $|G_{\boldsymbol{k}}| = \lambda_{122} |A| + 2\epsilon\lambda_{122} = \frac{1}{8} \cdot 1 + 2 \cdot \frac{1}{5} \cdot \frac{1}{8} = 0.175$. Now note that $I_{|G_{(1,2,2)}|} = I_{0.175} = \{\text{strings of length 3}\}$. We have that A_{121}, A_{122} and A_{211} intersect $G_{(1,2,2)}$. Thus $\#I((1, 2, 2)) = 3$. By symmetry we get that $\#I((1, 1, 2)) = \#I((1, 2, 1)) = \#I((2, 1, 1)) = \#I((2, 1, 2)) = \#I(2, 2, 1)) = 3$.

From the above values of $\boldsymbol{k}$ it is straightforward to see that for any $\boldsymbol{k}$ the cardinality of $I(\boldsymbol{k})$ is at most 3 since the piece $G_{\boldsymbol{k}}$ intersects $A_{\boldsymbol{k}}$ and can intersect at most one attractor piece $A_{\boldsymbol{i}}$ ($i \neq j$) near its left endpoint and the same for its right endpoint. The cases where we get $\#I(\boldsymbol{k}) < 3$ is when $A_{\boldsymbol{k}}$ includes either $\{0\}$ or $\{1\}$ and so we do not have both end point intersections. Therefore we have that $\gamma = 3$ for this IFS and value of ϵ.

As there exists an $\epsilon > 0$ where γ is finite, we have that γ is finite for all $\epsilon > 0$ and $\mathcal{F}$ has again been shown to satisfy the open set condition. We can also ask what is γ for other ϵ values. We find that if $\epsilon < 1$ (which it is because we assumed $0 < \epsilon < \frac{1}{3}$) then $\gamma = 3$ for this IFS. The proof of this follows identical reasoning to Example 2.11.

Example 2.10 is instructive in how to calculate the value of γ for an IFS but it was made easier by the use of symmetry, and so let us now do an example where the scaling factors are not equal.

Example 2.11. Let us consider the IFS $\mathcal{F} = \left\{\mathbb{R}; f_1(x) = \frac{1}{3}x, f_2(x) = \frac{2}{3}x + \frac{1}{3}\right\}$, which has attractor $[0, 1]$. We consider some fixed $0 < \epsilon < \frac{1}{3}$ and an arbitrary attractor piece given by $\boldsymbol{k}$. The dilated set around $A_{\boldsymbol{k}}$ is $G_{\boldsymbol{k}} = \mathcal{B}(A_{\boldsymbol{k}}, \epsilon\lambda_{\boldsymbol{k}})$. The diameter of $G_{\boldsymbol{k}}$ is $|G_{\boldsymbol{k}}| = \lambda_{\boldsymbol{k}}|A| + 2\epsilon\lambda_{\boldsymbol{k}} = \lambda_{\boldsymbol{k}}(1 + 2\epsilon)$. Now, $I_{|G_{\boldsymbol{k}}|}$ describes the addresses of the attractor pieces $A_{\boldsymbol{i}}$ whose size satisfy $\lambda_{\boldsymbol{i}} < |G_{\boldsymbol{k}}| \leq \lambda_{\lambda_1,\dots,\lambda_{n-1}}$. Therefore we have that the minimum size attractor piece that we would consider

for the given $\boldsymbol{k}$ is $\lambda_{min}\lambda_{\boldsymbol{k}}(1+2\epsilon)$, where $\lambda_{min} = \frac{1}{3}$ for the given IFS.

Now we claim that for the given IFS we have $\gamma = 3$. First note that the attractor piece A_{12} has nonempty intersection with itself, A_{11} and A_{21}. Therefore the dilated set $G_{(1,2)}$ will intersect at least three attractor pieces, namely those above. Therefore we have that $\gamma = \sup_{\boldsymbol{k}} \#I(\boldsymbol{k}) \geq 3$.

Now, let us prove that γ is not bigger than 3 by contradiction. Assume that there exists $\boldsymbol{j}$ such that $\#I(\boldsymbol{j}) \geq 4$. In order for this to be the case the amount $G_{\boldsymbol{j}}$ dilates must be at least as big as one of the adjacent attractor pieces. We have that $G_{\boldsymbol{j}}$ dilates by $\epsilon\lambda_{\boldsymbol{j}}$ and that the smallest possible attractor piece that we are considering is size $\frac{1}{3}\lambda_{\boldsymbol{j}}(1+2\epsilon)$. Thus we must have $\epsilon\lambda_{\boldsymbol{j}} \geq \frac{1}{3}\lambda_{\boldsymbol{j}}(1+2\epsilon)$. This implies $\epsilon \geq 1$, and so we have a contradiction as it was assumed $0 < \epsilon < \frac{1}{3}$ [Sch94]. Therefore, there does not exist $\boldsymbol{j}$ such that $\#I(\boldsymbol{j}) \geq 4$ and so for the given IFS we have that $\gamma = 3$.

Bandt, Hung and Rao [BHR05] describe Schief's combinatorial condition for the OSC as: there exists an integer N such that at most N incomparable pieces $A_{\boldsymbol{j}}$ intersect the ϵ-neighborhood of a piece $A_{\boldsymbol{i}}$ where $|A_{\boldsymbol{i}}|, |A_{\boldsymbol{j}}| \geq \epsilon$. This description of the combinatorial condition suggests some links between the OSC and fractal tiling theory. In that, the combinatorial condition is describing the number of fractal attractor pieces that are near to each other at different parts of the attractor which sounds like it is describing how many fractal tiles are in a region. The topic of fractal tiling is briefly considered in Chapter 5.

2.4   M. Morán and the Dynamical Boundary

M. Morán [MMor99] claimed that $A \neq \partial_M A$ was an equivalent to the OSC, where A is the attractor and $\partial_M A$ is Morán's definition of the dynamical boundary stated in Definition 1.7. Morán's original proof contained a mistake and his conjecture has remained an open question. During his work, Morán introduced a new version of the OSC.

Definition 2.12. The Restricted Open Set Condition (ROSC)
If $\mathcal{F} = \{\mathbb{R}^m; f_1, \ldots, f_N\}$ is an IFS with attractor A and (A, d) is the restricted topology, then $\mathcal{F}$ satisfies the ROSC if there exists a nonempty open set $\mathcal{O} \subset (A, d)$ such that

1. $f_i(\mathcal{O}) \subset \mathcal{O}$ for all $i \in \{1, \ldots, N\}$, and

2. $f_i(\mathcal{O}) \cap f_j(\mathcal{O}) = \emptyset$ for all $i \neq j \in \{1, \ldots, N\}$.

Remark 2.13. The ROSC is the OSC where the openness is with respect to the restricted topology (A, d) instead of the original (or extended) topology $(\mathbb{R}^m, d)$. Also note that M. Morán [MMor99] stated Condition (2) as $f_i(\mathcal{O}) \cap \mathcal{C} = \emptyset$ for all $i \in \{1, \ldots, N\}$, where $\mathcal{C}$ is the critical set from page 6.

M. Morán posed an impressive theorem regarding the OSC [MMor99] in 1999; however, the proof that Morán offered was later realised to be incorrect by Bandt, Hung and Rao in 2005 [BHR05].

Conjecture 1. *M. Morán's (open) Theorem*
For the IFS $\mathcal{F} = \{\mathbb{R}^m; f_1, \ldots, f_N\}$ with attractor A the following conditions are equivalent.

  1. *$\mathcal{F}$ satisfies the ROSC.*

  2. *The open set $V = A \backslash \partial_M A$ is nonempty.*

  3. *$\mathcal{F}$ satisfies the SOSC.*

M. Morán [MMor99] correctly proves (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1). However, his proof for (2) $\Rightarrow$ (3) contains a mistake. Before discussing the mistake, let us first discuss something good that comes from M. Morán's work. By combining the third and first of the correctly proved implications we have (3) $\Rightarrow$ (2) which is that if $\mathcal{F}$ satisfies the SOSC then $V = A \backslash \partial_M A \neq \emptyset$. This is equivalent to, if $\mathcal{F}$ satisfies the SOSC then $A \neq \partial_M A$. Taking the contrapositive of this provides the following corollary.

Corollary 2.14. *If $\mathcal{F} = \{\mathbb{R}^m; f_1, \ldots, f_N\}$ is an IFS with attractor A and $A = \partial_M A$ then the SOSC is not satisfied and so the OSC is also not satisfied.*

Now we discuss the openness of V. From Definition 1.7 we have that $\partial_M A$ is a closed set and so in the restricted topology of (A, d) the set $V = A \backslash \partial_M A$ is the complement of a closed set and therefore is an open set. In fact, if V is nonempty then V satisfies the ROSC. In order to see this let us show how the IFS operates on the dynamical boundary:

$$
\begin{aligned}
\mathcal{F}(\partial_M A) = \mathcal{F}\left(\overline{\bigcup_{k=1}^{\infty} \mathcal{F}^{-k}(\mathcal{C})} \cap A\right) &= \overline{\mathcal{F}\left(\bigcup_{k=1}^{\infty} \mathcal{F}^{-k}(\mathcal{C}) \cap A\right)} \\
&= \overline{\mathcal{F}\left(\bigcup_{k=1}^{\infty} \mathcal{F}^{-k}(\mathcal{C})\right)} \cap \mathcal{F}(A) = \overline{\left(\bigcup_{k=1}^{\infty} \mathcal{F}^{-k}(\mathcal{C}) \cup \mathcal{C}\right)} \cap A
\end{aligned}
\tag{2.9}
$$

$$= \overline{\left(\bigcup_{k=1}^{\infty} \mathcal{F}^{-k}\left(\mathcal{C}\right) \cap A\right) \cup \left(\mathcal{C} \cap A\right)} = \overline{\left(\bigcup_{k=1}^{\infty} \mathcal{F}^{-k}\left(\mathcal{C}\right) \cap A\right)} \cup \mathcal{C}$$

$$= \overline{\left(\bigcup_{k=1}^{\infty} \mathcal{F}^{-k}\left(\mathcal{C}\right) \cap A\right)} \cup \mathcal{C} = \partial_M A \cup \mathcal{C}.$$

Therefore we have that $\partial_M A \subset \mathcal{F}\left(\partial_M A\right)$. Taking the complement reverses the direction of the set inclusion to give $\left(\mathcal{F}\left(\partial_M A\right)\right)^c \subset \left(\partial_M A\right)^c$. This implies $\mathcal{F}\left(\left(\partial_M A\right)^c\right) \subset \left(\partial_M A\right)^c$ which is equal to $\mathcal{F}\left(V\right) \subset V$ and thus $f_i(V) \subset V$ for all $i \in \{1, \dots, N\}$. We also have $f_i(V) \cap f_j(V) = \emptyset$ for all $i \neq j$ by construction of $\partial_M A$. Therefore V satisfies the ROSC.

For the implication $(2) \Rightarrow (3)$ of Morán's conjecture, we assume that V is nonempty and thus we have that $V \cap A$ is nonempty and therefore it might seem like V would satisfy the SOSC. However, the SOSC (or equivalently the OSC) requires a nonempty open set in $(\mathbb{R}^m, d)$ while the ROSC only requires a nonempty open set in (A, d). It is not the case that since V is open in (A, d) then V is open in $(\mathbb{R}^m, d)$.

To demonstrate this point let us now provide an example where extending V to the full topology results in V not being an open set. Let $\mathcal{F}$ be the following IFS,

$$\mathcal{F} = \left\{ \mathbb{R}^2; \ f_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \boldsymbol{b}_i \ \middle| \ i \in \{1, 2, 3\}, \right.$$
$$\left. \boldsymbol{b_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \boldsymbol{b_2} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \boldsymbol{b_3} = \begin{pmatrix} \frac{1}{4} \\ \frac{\sqrt{3}}{4} \end{pmatrix} \right\}. \tag{2.10}$$

The attractor of $\mathcal{F}$ is the Sierpinski triangle with side lengths of one and it is shown in Figure 2.5. Through a simple calculation we get the critical set of $\mathcal{F}$ and Morán's dynamical boundary to be

$$\mathcal{C} = \left\{ \left(\frac{1}{2}, 0 \right), \ \left(\frac{1}{4}, \frac{\sqrt{3}}{4} \right), \ \left(\frac{3}{4}, \frac{\sqrt{3}}{4} \right) \right\}, \quad \text{and}$$

$$\partial_M A = \left\{ (0, 0), \ (1, 0), \ \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right) \right\}.$$

Therefore in (A, d) we have that $V = A \backslash \partial_M A$ is an open set since it is the complement of the closed set $\partial_M A$. However, in $(\mathbb{R}^2, d)$ we have that $V = A \backslash \partial_M A$
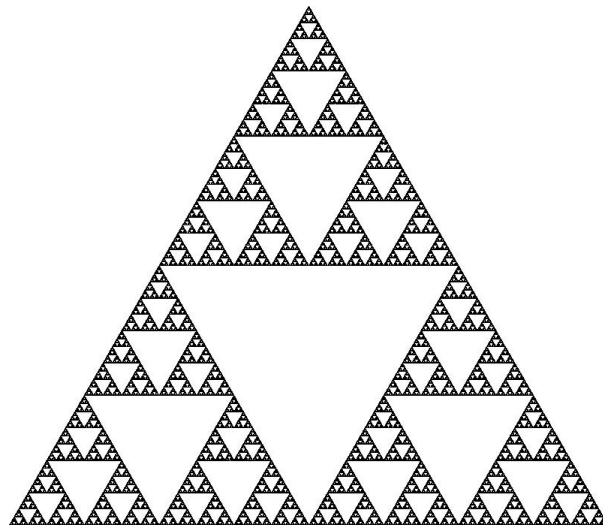
Figure 2.5: The Sierpinski triangle

is the Sierpinski triangle with its vertices removed. This is not an open set (nor is it closed). Therefore, V is not necessarily an open set in the extended topology.

It has been shown that in general V does not satisfy the SOSC (or the OSC) since V may not be open. The natural fix to this is to dilate V by a minuscule amount so that V becomes an open set. Therefore, let $\mathcal{U} := \bigcup_{x \in V} \mathcal{B}(x, \epsilon)$. This was the approach that M. Morán [MMor99] took to prove his conjecture but it does not work.

The mistake M. Morán made was to say that since $f_i(V) \cap f_j(V) = \emptyset$ for $i \neq j \in \{1, \ldots, N\}$ and $\mathcal{U}$ is an ϵ dilation of V then $f_i(\mathcal{U}) \cap f_j(\mathcal{U}) = \emptyset$ for $i \neq j \in \{1, \ldots, N\}$. Let us now explain why we cannot assert that $f_i(\mathcal{U}) \cap f_j(\mathcal{U}) = \emptyset$ for $i \neq j \in \{1, \ldots, N\}$. Recall from Definition 1.7 and remark 1.9 we have $\partial_M A = \overline{\bigcup_{k=1}^{\infty} (\mathcal{F}^{-k}(\mathcal{C}) \cap A)}$. Therefore, at each stage in the calculation of the preimages of $\mathcal{C}$, only the points in the attractor are taken. But, what if there were a point not in A whose preimage under $\mathcal{F}$ was in A? This would mean that $\partial_M A$ would not include all points which could map to the critical set. However, as the next proposition shows once a point escapes A then it never returns under the operation of the inverse maps.

Proposition 2.15. *For an IFS $\mathcal{F} = \{\mathbb{R}^m; f_1, \ldots, f_N\}$ with attractor A. If $x \notin A$ then $f_i^{-1}(x) \notin A$ for all $i \in \{1, \ldots, N\}$.*

Proof. Let $x \notin A$ and let $i \in \{1, \ldots N\}$. In order to gain a contradiction let us assume that $f_i^{-1}(x) \in A$. This implies that $x \in f_i(A) \subset A$, so $x \in A$. This is a contradiction and therefore $f_i^{-1}(x) \notin A$ for all $i \in \{1, \ldots, N\}$. $\square$

Therefore, once the point $x \in A$ leaves A under the operation of $\mathcal{F}^{-1}$, it can never return. However, the orbit could get arbitrarily close to A which means that when V is dilated to create $\mathcal{U}$ we cannot guarantee that $\mathcal{U} \cap \overline{\bigcup_{k=1}^{\infty} \mathcal{F}^{-k}(\mathcal{C})}$ is equal to the empty set. Therefore U does not imply that the IFS satisfies the SOSC (or OSC). This concludes the discussion of how M. Morán's proof was incorrect.

In an effort to fix the proof, we give the following revised definition of the dynamical boundary that was motivated by Bandt [BHR05] and so we have the subscript B.

Definition 2.16. For an IFS $\mathcal{F} = \{\mathbb{R}^m; f_1, \ldots, f_N\}$ with attractor A and critical set $\mathcal{C}$ we define Bandt's dynamical boundary to be

$$\partial_B A = \overline{\bigcup_{k=1}^{\infty} \mathcal{F}^{-k}(\mathcal{C})} \cap A.$$

It is helpful to note that for arbitrary sets A and B we cannot assert $\overline{A \cap B} = \overline{A} \cap \overline{B}$ and therefore we do not necessarily have $\partial_B A = \partial_M A$.

Using the revised dynamical boundary we could replace M. Morán's conjecture with "$\mathcal{F}$ satisfies the OSC if and only if $A \backslash \partial_B A \neq \emptyset$" which is equivalent to "$\mathcal{F}$ satisfies the OSC if and only if $A \not\subset \partial_B A$". This is not proven here and if it were to be then it may not be of much use for determining if $\mathcal{F}$ satisfies the OSC considering it would involve calculating $\overline{\bigcup_{k=1}^{\infty} \mathcal{F}^{-k}(\mathcal{C})}$. Consider the simplest situation where $\mathcal{C}$ only contains a single point and the IFS is only made of two maps, then the kth preimage would contain 2^k points. Thus, in the case where $\mathcal{C}$ is nontrivial this would be an extreme computation. As an alternative to M. Morán's approach, Bandt introduced some new theory called neighbor maps. Neighbor maps are the feature of the next chapter and it will be seen that there exists a set H such that the OSC is satisfied if and only if $A \not\subset \overline{H}$ (Cor 3.4).

Chapter 3

Bandt's Neighbor Maps

3.1 Neighbor Maps

In order to show that an IFS $\mathcal{F}$ satisfies the OSC, a particular nonempty open set $\mathcal{O}$ must be found. If we were unable to find a suitable $\mathcal{O}$ it by no means proves that the OSC is not satisfied. To show that the IFS does not satisfy the OSC we would need to prove that there does not exist any such $\mathcal{O}$ which is a much stronger statement then simply failing to find one. As a result, the following question is still open: "does an arbitrary IFS obey the OSC?". Bandt and various collaborator [BG92, BHR05, BM09, BM18] developed the theory of neighbor maps and neighbor graphs in an attempt to answer this question. The concept comes from the idea that if the OSC is satisfied that means that a certain group of isometries are acting discontinuously and the set $\mathcal{O}$ is a fundamental domain of that group [BG92].

Let us now introduce the notation for neighbor maps which we take from Bandt and Graf [BG92]. Let the IFS $\mathcal{F} = \{\mathbb{R}^m; f_1, \ldots, f_N\}$ comprise similitudes with scaling factors λ_i and have attractor A. We define $S := \{1, \ldots, N\}$ to be the alphabet set. Let $S^k := \{1, \ldots, N\}^k$ denote the set of words of length k and if $k = 0$ then S^0 is the empty word. The set of all words is given by $S^* = \bigcup \{S^n \mid n \in \mathbb{N}_0\}$ where $\mathbb{N}_0 = \{0, 1, 2 \ldots\}$. Let $\boldsymbol{i} = i_1 i_2 \ldots i_p$ and $\boldsymbol{j} = j_1 j_2 \ldots j_q$ be two words from S^* of length p and q, denoted $|\boldsymbol{i}|$ and $|\boldsymbol{j}|$ respectively. We say that $\boldsymbol{i}$ is an initial word of $\boldsymbol{j}$, denoted $\boldsymbol{i} \sqsubseteq \boldsymbol{j}$, if $p \leq q$ and $i_m = j_m$ for all $m \in \{1, \ldots, p\}$. The words $\boldsymbol{i}$ and $\boldsymbol{j}$ are incomparable if $\boldsymbol{i} \not\sqsubseteq \boldsymbol{j}$ and $\boldsymbol{j} \not\sqsubseteq \boldsymbol{i}$. Note that this definition of incomparable agrees with the one given on page 16. Recall that we have the following notation abbreviations: $f_{\boldsymbol{i}} = f_{i_1} \circ \cdots \circ f_{i_p}$,

$\lambda_{\boldsymbol{i}} = \lambda_{i_1} \cdots \lambda_{i_p}$ and $A_{\boldsymbol{i}} = f_{\boldsymbol{i}}(A)$. If $\boldsymbol{i} \sqsubseteq \boldsymbol{j}$ then $A_{\boldsymbol{j}} \subseteq A_{\boldsymbol{i}}$ and so if $\boldsymbol{i}$ and $\boldsymbol{j}$ are comparable then $A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}} \neq \emptyset$. However, if $\boldsymbol{i}$ and $\boldsymbol{j}$ are incomparable we cannot make a similar general statement because $A_{\boldsymbol{i}}$ and $A_{\boldsymbol{j}}$ may or may not overlap.

For any attractor pieces $A_{\boldsymbol{i}}$ and $A_{\boldsymbol{j}}$ we can define a similitude $f_{\boldsymbol{j}} f_{\boldsymbol{i}}^{-1}$ that maps one onto the other. These functions are commonly called household maps and describe the relationship between the two attractor pieces. It would be nice to be able to say that $A_{\boldsymbol{i}}$ and $A_{\boldsymbol{j}}$ are non-overlapping (their intersection does not contain a nonempty open set) if $f_{\boldsymbol{j}} f_{\boldsymbol{i}}^{-1}$ is far from the identity map. However, this is not quite the case. In order to see why, let us recall the Sierpinski triangle shown in Figure 2.5 on page 22. The household map $f_{21} f_{12}^{-1}$ from A_{12} onto A_{21}, which are non-overlapping, is a horizontal translation by $\frac{1}{4}$. Note that we are using the notation f_{12}^{-1} to mean the inverse of f_{12} and so $f_{12}^{-1} = (f_1 f_2)^{-1} = f_2^{-1} f_1^{-1}$. Similarly we get that $f_{211} f_{122}^{-1}$ is a horizontal translation of $\frac{1}{8}$ between non-overlapping attractor pieces. We see that as the attractor pieces get smaller the household map between them can approach the identity map despite the attractor pieces being non-overlapping. For this reason we consider the functions of the form $f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}}$ (the order has swapped). The attractor pieces $A_{\boldsymbol{i}}$ and $A_{\boldsymbol{j}}$ have the same geometric relationship as A and $f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}}(A)$ but now all the different maps can be compared as they are on the same scale. Providing that $\boldsymbol{i}$ and $\boldsymbol{j}$ are incomparable then $h := f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}}$ is a neighbor map. From Bandt and Graf [BG92] we have that the set of all neighbor maps, denoted $\mathcal{N}$, is

$$\mathcal{N} = \left\{ f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}} \mid \boldsymbol{i}, \boldsymbol{j} \in S^*, \ \boldsymbol{i}, \boldsymbol{j} \text{ incomparable} \right\}. \tag{3.1}$$

This definition can be simplified by noting that if $\boldsymbol{i}$ and $\boldsymbol{j}$ have their first m letters the same for $m < \min\{p, q\}$ (note that $m \neq \min\{p, q\}$ since the words are incomparable) then the neighbor map $h = f_{i_p}^{-1} \cdots f_{i_1}^{-1} f_{j_1} \cdots f_{j_q}$ simplifies to $h = f_{i_p}^{-1} \cdots f_{i_{m+1}}^{-1} f_{j_{m+1}} \cdots f_{j_q}$. This map is already included $\mathcal{N}$ since $i_{m+1} \cdots i_p$ and $j_{m+1} \cdots j_q$ are incomparable with distinct first letters. Therefore the set of neighbor maps can be more simply given by

$$\mathcal{N} = \left\{ h = f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}} \mid \boldsymbol{i}, \boldsymbol{j} \in S^*, i_1 \neq j_1 \right\}. \tag{3.2}$$

This simplification was first noted by Bandt, Hung and Rao [BHR05]. Bandt and Graf [BG92] proved the following theorem where id is the identity map (i.e. $f(x) = x$). It provides a way that neighbor maps can be used to determine if an IFS obeys the OSC.

Theorem 3.1. *If $\mathcal{F} = \{\mathbb{R}^m; f_1, \ldots, f_N\}$ is an IFS comprised of similitudes and $\mathcal{N}$ is the set of neighbor maps of $\mathcal{F}$, then $\mathcal{F}$ obeys the OSC if and only if $id \notin \overline{\mathcal{N}}$.*

Remark 3.2. We have that $\overline{\mathcal{N}}$ denotes the closure of $\mathcal{N}$ and that all neighbor maps are similitudes, $h = f_i^{-1} f_j : \mathbb{R}^m \to \mathbb{R}^m$, with scaling factor $\frac{\lambda_j}{\lambda_i}$. Theorem 3.1 is requiring that for $\mathcal{F}$ to satisfy the OSC the identity map is not a neighbor map and not a limit point of a sequence of neighbor maps. For a limit point to be well-defined we must note that we are in the space of similitudes from $\mathbb{R}^m$ to $\mathbb{R}^m$ which we denote $\mathbb{S}(\mathbb{R}^m)$ and let d_s measure the distance between two similitudes f and g of $\mathbb{S}(\mathbb{R}^m)$ as

$$d_s(f, g) = \sup_{x \in \mathbb{R}^m, \|x\|=1} \{\|f(x) - g(x)\|\}. \tag{3.3}$$

We have that $(\mathbb{S}(\mathbb{R}^m), d_s)$ is a complete metric space.

Remark 3.3. When Bandt and Graf presented the theorem [BG92], Schief was yet to produce his work of Theorem 2.6 [Sch94] and therefore it was originally presented with reference to the attractor having a positive D dimensional Hausdorff measure.

Bandt, Hung and Rao [BHR05] provided an alternative condition for relating the OSC to neighbor maps.

Corollary 3.4. *An IFS $\mathcal{F}$ satisfies the OSC if and only if A is not contained in $\overline{H}$, where A is the attractor, $H = \bigcup \{h(A) \mid h \in \mathcal{N}\}$ and we call $h(A)$ a neighbor set of A.*

By Theorem 3.1 we now have a new way of determining if an IFS satisfies the OSC. It involves calculating the set of neighbor maps, $\mathcal{N}$, and determining if the identity map is or is not contained in the closure of $\mathcal{N}$. Calculating the set of all neighbor maps is a big task and so we ask, can we consider only a subset of the neighbor maps?

Bandt and Mesing [BM09] only consider the neighbor maps $h = f_i^{-1} f_j$ such that A_i and A_j have nonempty intersection and are of 'approximately' the same size. Also, the only IFSs they consider comprise of similitudes with equal scaling factors. Their motivation for only considering these neighbor maps is to construct neighbor graphs as discussed in Section 3.2. Neighbor maps which contribute to the neighbor graphs are called proper neighbor maps, the set of which is denoted $\mathcal{N}^*$ [BM18]. Following the example of Bandt [BM09, BM18] we will only consider

IFSs comprising of similitudes with equal scaling factors for the remainder of this chapter.

If $\mathcal{F} = \{\mathbb{R}^m; f_1, \ldots, f_N\}$ is an IFS of similitudes where each functions has the same scaling factor $\lambda \in [0, 1)$ then two attractor pieces, $A_{\boldsymbol{i}}$ and $A_{\boldsymbol{j}}$ are of 'approximately' the same size only when $|\boldsymbol{i}| = |\boldsymbol{j}|$. We see this because the difference is sizes is given by the scaling factor of h, that is $\frac{\lambda_{\boldsymbol{j}}}{\lambda_{\boldsymbol{i}}}$ as stated in Remark 3.2. The scaling factor of h simplifies to $\lambda^{|\boldsymbol{j}|-|\boldsymbol{i}|}$. Thus, if the scaling of h is in $(1 - \lambda, 1 + \lambda)$ then $|\boldsymbol{i}| = |\boldsymbol{j}|$. Therefore, for an IFS with uniform scaling factors we define the set of proper neighbor maps as,

$$\mathcal{N}^* = \left\{ h = f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}} \mid \boldsymbol{i}, \boldsymbol{j} \in S^*, i_1 \neq j_1, |\boldsymbol{i}| = |\boldsymbol{j}|, A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}} \neq \emptyset \right\}. \qquad (3.4)$$

The set of proper neighbor maps only includes the neighbor maps which are isometries (since $|\boldsymbol{i}| = |\boldsymbol{j}|$) and do not have 'too large' of a translation factor since $A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}} \neq \emptyset$. As the following theorem states, when checking if the identity map is contained in the closure of the neighbor maps we only need to consider the proper neighbor maps since they represent the only neighbor maps which are 'near' to the identity. To the best the author's knowledge the following result is new.

Theorem 3.5. *If $\mathcal{F}$ is an IFS comprised of similitudes with equal scaling factors λ, $\mathcal{N}$ is the set of neighbor maps and $\mathcal{N}^*$ is the set of proper neighbor maps, then $id \notin \overline{\mathcal{N}}$ if and only if $id \notin \overline{\mathcal{N}^*}$.*

Proof. Let $\mathcal{F}$ be an IFS on $\mathbb{R}^m$ consisting of similitudes with equal scaling factors λ. Without loss of generality we may assume that the attractor A contains the origin by a simple change of coordinates. Let us now prove each direction independently.

($\Rightarrow$) By definition we have that the set of proper neighbor maps is a subset of the set of all neighbor maps and therefore we have $\overline{\mathcal{N}^*} \subset \overline{\mathcal{N}}$. Therefore if $id \notin \overline{\mathcal{N}}$ then we have $id \notin \overline{\mathcal{N}^*}$.

($\Leftarrow$) Let us now assume that $id \notin \overline{\mathcal{N}^*}$ and it will be shown that $id \notin \overline{\mathcal{N}}$. In order for the identity map to be in $\overline{\mathcal{N}}$ when not in $\overline{\mathcal{N}^*}$ it must be the case that the identity map is a limit point of a sequence of non-proper neighbor maps. Therefore, if we show that $h \in \mathcal{N} \backslash \mathcal{N}^*$ cannot be arbitrarily close to the identity then we have $id \notin \overline{\mathcal{N}}$. Let $h = f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}} \in \mathcal{N} \backslash \mathcal{N}^*$ then we have $\boldsymbol{i}, \boldsymbol{j} \in S^*$ and $i_1 \neq j_1$ as h is a neighbor map but we must have that either $|\boldsymbol{i}| \neq |\boldsymbol{j}|$ or $A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}} = \emptyset$ since h is not a proper neighbor map. All maps in the IFS are similitudes with equal

scaling factor $0 < \lambda < 1$ and so they take the form $f_i(x) = \lambda O_i x + b_i$, where O_i is an isometry of $\mathbb{R}^m$ and $b_i \in \mathbb{R}^m$ is a translation vector. As h is a composition of neighbor maps we have $h(x) = \lambda^k O x + b$ where $k \in \mathbb{Z}$, O is an isometry and $b \in \mathbb{R}^m$. By Remark 1.2 we may assume that the origin is a fixed point of the isometry O. Now that we understand the form of h it will be shown that h is not arbitrarily close to the identity map. Recalling the metric on similitudes from Remark 3.2 we have that the distance between h and the identity is given by,

$$
\begin{aligned}
d_s(h, id) &= \sup_{x \in \mathbb{R}^m, \|x\|=1} \{\|h(x) - id(x)\|\} \\
&= \sup_{x \in \mathbb{R}^m, \|x\|=1} \{\|\lambda^n O x + b - x\|\}.
\end{aligned}
\tag{3.5}
$$

Let us assume that $\|b\| = 0$ and therefore $b = 0$. The origin is in A and is a fixed point of O so it is also a fixed point of h which means that $A \cap h(A) = A \cap f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}}(A) \neq \emptyset$ which implies that $A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}} \neq \emptyset$. Therefore we must have that $|\boldsymbol{i}| \neq |\boldsymbol{j}|$ since $h \in \mathcal{N} \backslash \mathcal{N}^*$. This implies that the scaling factor of h cannot be one or equivalently $k \in \mathbb{Z} \backslash \{0\}$. Now we continue to determine the distance between h and the identity,

$$
\begin{aligned}
d_s(h, id) &= \sup_{x \in \mathbb{R}^m, \|x\|=1} \{\|\lambda^k O x - x\|\} \geq \sup_{x \in \mathbb{R}^m, \|x\|=1} \{\|\|\lambda^k O x\| - \|x\|\|\} \\
&= \sup_{x \in \mathbb{R}^m, \|x\|=1} \{\|\lambda^k \|Ox\| - \|x\|\|\} = \|\lambda^k - 1\| \\
&= |\lambda^k - 1| \geq 1 - \lambda.
\end{aligned}
\tag{3.6}
$$

Where the last inequality comes from noting that as $0 < \lambda < 1$ then $\lambda \geq \lambda^k$ for all $k \geq 1$. Therefore $1 - \lambda^k \geq 1 - \lambda$ for all $k \geq 1$. Also, if k is negative then $k = -K$ for $K \geq 1$ and so

$$
|\lambda^k - 1| = |\lambda^{-K} - 1| = \left|\frac{1}{\lambda^K} - 1\right| = \left|\frac{1 - \lambda^K}{\lambda^K}\right| = \frac{1 - \lambda^K}{\lambda^K} \geq 1 - \lambda^K \geq 1 - \lambda.
$$

Therefore $|\lambda^k - 1| \geq 1 - \lambda$ for any $k \in \mathbb{Z} \backslash \{0\}$. Thus it has been shown that, in the case of $\|b\| = 0$, h cannot be arbitrarily close to the identity.

Let us now assume that $\|b\| > 0$. Let L be the line in $\mathbb{R}^m$ that passes through the origin and b. Let $\pm z$ be the intersection points between L and the unit (m-dimensional) ball centred at the origin, where z is the point closer to b by the Euclidean metric. Let $\pm y$ with $\| \pm y\| = 1$ denote the image of $\pm z$ under the isometry O. The situation is illustrated in Figure 3.1. Now, in the case of
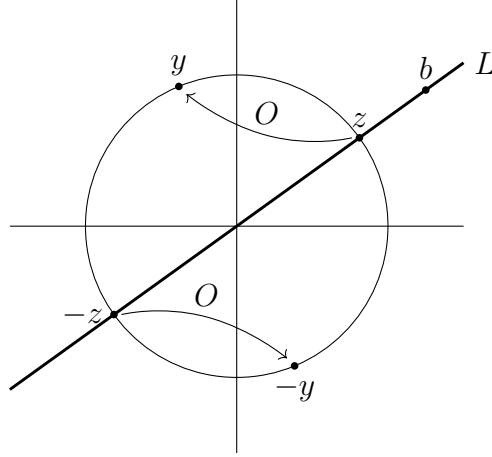
Figure 3.1: The situation described for $\|b\| > 0$ drawn in a plane of $\mathbb{R}^m$ that contains $\pm z$ and $\pm y := O(\pm z)$ (the plane is not necessarily unique).

$\|b\| > 0$, we will calculate the distance between a non-proper neighbor map h and the identity,

$$\begin{aligned}
d_s(h, id) &= \sup_{x \in \mathbb{R}^m, \|x\|=1} \left\{ \left\| \lambda^k Ox + b - x \right\| \right\} \\
&\geq \sup_{x=\pm z} \left\{ \left\| \lambda^k Ox + b - x \right\| \right\} \\
&= \max \left\{ \left\| \lambda^k Oz + b - z \right\|, \ \left\| \lambda^k O(-z) + b - (-z) \right\| \right\} \\
&= \max \left\{ \left\| \lambda^k y + b - z \right\|, \ \left\| -\lambda^k y + b + z \right\| \right\}.
\end{aligned} \tag{3.7}$$

We have that both $b - z$ and $b + z$ lie on line L and that $\|b - z\| \geq 0$ while $\|b + z\| \geq 1$. Therefore

$$\left\| \lambda^k y + b - z \right\| \geq \left\| \lambda^k y \right\| = \lambda^k \|y\| = \lambda^k \qquad \text{and}$$

$$\left\| -\lambda^k y + b + z \right\| = \left\| b + z - \lambda^k y \right\| \geq \left| \|b + z\| - \left\| \lambda^k y \right\| \right| = \left\| 1 - \lambda^k \right\|.$$

Therefore we have that the distance between h and the identity has the following lower bound,

$$d_s(h, id) \geq \max \left\{ \lambda^k, \ \left| 1 - \lambda^k \right| \right\}. \tag{3.8}$$

If $k = 0$ then $d_s(h, id) \geq 1$ and as before, if $k \neq 0$ then $d_s(h, id) \geq \left| 1 - \lambda^k \right| \geq 1 - \lambda$. Therefore, it has been shown that for $h \in \mathcal{N} \backslash \mathcal{N}^*$ we have $d_s(h, id) \geq 1 - \lambda$ which implies that the distance from a non-proper neighbor map to the identity cannot be arbitrarily small and therefore any such h cannot contribute to the tail of a sequence which limits to the identity map and therefore $id \notin \overline{\mathcal{N}}$. $\qquad \square$

Applying Theorem 3.5 to Theorem 3.1 we get the following result, which we have not found in the literature.

Corollary 3.6. *If $\mathcal{F} = \{\mathbb{R}^m; f_1, \ldots, f_N\}$ is an IFS comprised of similitudes with equal scaling factors λ and $\mathcal{N}^*$ is the set of proper neighbor maps, then $\mathcal{F}$ obeys the OSC if and only if $\mathrm{id} \notin \overline{\mathcal{N}^*}$.*

Remark 3.7. Corollary 3.6 is a much stronger statement than Theorem 3.1 from Bandt and Graf [BG92] but it has only been shown for when all similitudes in the IFS have equal scaling factors. In Chapter 4 this will be generalised to other types of IFSs (Cor 4.4 and 4.16).

Now that the set of proper neighbor maps have been introduced let us provide some basic examples of calculating them to determine if an IFS obeys the OSC.

Example 3.8. Let us consider the IFS $\mathcal{F} = \left\{ \mathbb{R}; f_1(x) = \frac{1}{2}x, f_2(x) = \frac{1}{2}x + \frac{1}{2} \right\}$ from Example 2.2 with its attractor $A = [0, 1]$ which is shown in Figure 2.1. We already know that $\mathcal{F}$ obeys the OSC with $\mathcal{O} = (0, 1)$, but as an exercise in calculating neighbor maps we will now determine that that $\mathcal{F}$ obeys the OSC by using Corollary 3.6.

All similitudes in $\mathcal{F}$ have the same scaling factor and therefore we have that $|\boldsymbol{i}| = |\boldsymbol{j}|$ for the proper neighbor maps $h = f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}}$. Let us first look at words of length one. This only gives us two neighbor maps, $f_1^{-1} f_2$ and $f_2^{-1} f_1$. Explicitly calculating these maps gives us $f_1^{-1} f_2(x) = x + 1$ and $f_2^{-1} f_1(x) = x - 1$. Thus we have determined two of the proper neighbor maps. Now we look at words of length two. Recalling that we also require $i_1 \neq j_1$ and $A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}} \neq \emptyset$ then the only proper neighbor maps of length two are due to the nonempty intersection of $A_{\mathbf{12}}$ and $A_{\mathbf{21}}$ as seen in Figure 3.2. Thus, our proper neighbor maps due to words of length two are $f_2^{-1} f_1^{-1} f_2 f_1$ and $f_1^{-1} f_2^{-1} f_1 f_2$. Calculating the proper neighbor maps of length two we get $f_2^{-1} f_1^{-1} f_2 f_1(x) = x + 1$ and $f_1^{-1} f_2^{-1} f_1 f_2(x) = x - 1$. Comparing the proper neighbor maps of length two to those already calculated of length one we observe that,

$$f_2^{-1} f_1^{-1} f_2 f_1 = f_1^{-1} f_2 \qquad \text{and} \qquad f_1^{-1} f_2^{-1} f_1 f_2 = f_2^{-1} f_1.$$

We see that this pattern continues because as the length of the words increase from n to $(n + 1)$ the only proper neighbor maps that will be considered are associated to the intersection occurring at 0.5 between $A_{1(2)^n}$ and $A_{2(1)^n}$. This will produce the neighbor map $\left(f_2^{-1} \right)^{\circ n} f_1^{-1} f_2 \left(f_1 \right)^{\circ n}$ and its inverse. Since we have that $f_2^{-1} f_1^{-1} f_2 f_1 = f_1^{-1} f_2$ then from a straightforward induction argument

we have that $\left(f_2^{-1}\right)^{\circ n} f_1^{-1} f_2 \left(f_1\right)^{\circ n} = f_1^{-1} f_2$. Therefore we have found all proper neighbor maps for $\mathcal{F}$ to be

$$\mathcal{N}^* = \left\{f_1^{-1} f_2,\ f_2^{-1} f_1\right\} = \{x+1, x-1\}.$$

Now, as the identity map is not contained in the closure of the set of proper neighbor maps we have, by Corollary 3.6, that $\mathcal{F}$ obeys the OSC.
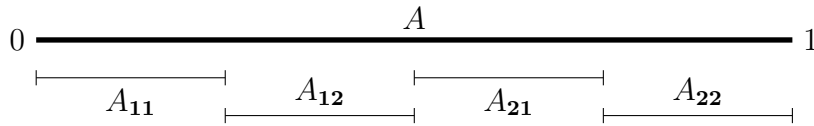

Figure 3.2: The pieces of the attractor given by words of length two for the IFS $\mathcal{F} = \left\{\mathbb{R}; f_1(x) = \frac{1}{2}x, f_2(x) = \frac{1}{2}x + \frac{1}{2}\right\}$.

Remark 3.9. For Example 3.8 the following neighbor maps are not proper neighbor maps.

- The map $f_1^{-1} f_1^{-1} f_2 f_2(x) = x + 3$ is not a proper neighbor map since $A_{11} \cap A_{22} = \emptyset$. This corresponds to $f_1^{-1} f_1^{-1} f_2 f_2(A) \cap A = \emptyset$ and so the neighbor set $h(A)$ is not a 'close' neighbor and need not be considered when trying to determine if the identity map is in the closure of $\mathcal{N}$.

- The map $f_1^{-1} f_2 f_1(x) = \frac{1}{2}x + 1$ is not a proper neighbor map since $|\boldsymbol{i}| \neq |\boldsymbol{j}|$. As all maps in the IFS have the same scaling factor then if a neighbor map $h = f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}}$ has $|\boldsymbol{i}| \neq |\boldsymbol{j}|$ then the scaling factor on the x term cannot approach 1 in the limit and so need not be considered when determining if the identity map is in the closure of $\mathcal{N}$.

Example 3.10. Let $\mathcal{F} = \left\{\mathbb{R}; f_1(x) = \frac{1}{3}x, f_2(x) = \frac{1}{3}x + \frac{2}{3}\right\}$ be the IFS from Example 2.5 with the Cantor set as its attractor. We already know that $\mathcal{F}$ obeys the OSC with $\mathcal{O} = (0, 1)$ but let us discuss the set of proper neighbor maps. The Cantor set is a totally disconnected set and therefore we have that for all $\boldsymbol{i}, \boldsymbol{j} \in S^*$, if $i_1 \neq j_1$ then $A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}} = \emptyset$. Therefore the set of proper neighbor maps is empty and so $id \notin \overline{\mathcal{N}^*}$ which implies that $\mathcal{F}$ obeys the OSC as previously stated in Example 2.5. In fact we have something stronger here. Any totally disconnected IFS has an empty proper neighbor map set and therefore obeys the OSC.

Let us now consider an example where calculating the proper neighbor maps shows that the IFS does not obey the OSC.

Example 3.11. Let us consider the IFS $\mathcal{F} = \{\mathbb{R}; f_1(x) = \phi x, f_2(x) = \phi x + (1 - \phi)\}$, where ϕ is the inverse golden ratio given by $\left(0.5\left(\sqrt{5} + 1\right)\right)^{-1} \approx 0.618$. The attractor of $\mathcal{F}$ is $[0, 1]$ and it can be shown to not satisfy the OSC through a similar argument to that presented in Example 2.2. However, here we will use neighbor maps to show that it does not satisfy the OSC.
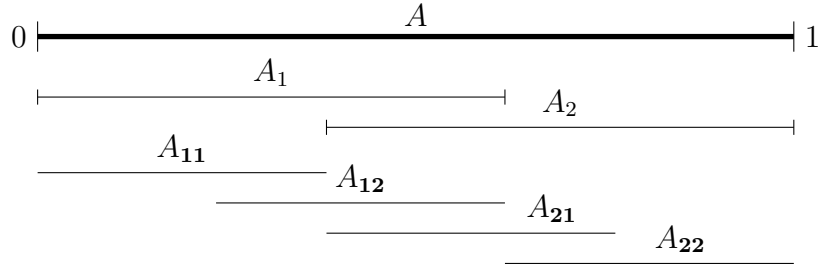

Figure 3.3: The pieces of the attractor for words of length at most two for the IFS $\mathcal{F} = \{\mathbb{R}; f_1(x) = \phi x, f_2(x) = \phi x + (1 - \phi)\}$ with $\phi = \left(0.5\left(\sqrt{5} + 1\right)\right)^{-1} \approx 0.618$.

Figure 3.3 shows the first two iterations of attractor pieces and it can already be seen that the proper neighbor map set will be much more complicated than in Example 3.8 because $A_{11} \cap A_{21} \neq \emptyset \neq A_{12} \cap A_{22}$. Figure 3.3 does not show the attractor pieces for words of length 3 or more as it would become too messy, but we will now consider the intersection between A_{122} and A_{211}. Note that the inverse functions are given by $f_1^{-1}(x) = \frac{x}{\phi}$, $f_2^{-1}(x) = \frac{x}{\phi} - \phi$ and we have the property $\frac{1-\phi}{\phi} = \phi$. The proper neighbor map from the intersection of A_{122} and A_{211} is given by,

$$
\begin{aligned}
f_2^{-1} f_2^{-1} f_1^{-1} f_2 f_1 f_1(x) = f_2^{-1} f_2^{-1} f_1^{-1} f_2 f_1 \left(\phi x\right) &= f_2^{-1} f_2^{-1} f_1^{-1} f_2 \left(\phi^2 x\right) \\
&= f_2^{-1} f_2^{-1} f_1^{-1} \left(\phi^3 x + (1 - \phi)\right) = f_2^{-1} f_2^{-1} \left(\phi^2 x + \phi\right) \quad (3.9) \\
&= f_2^{-1} \left(\phi x + (1 - \phi)\right) = x.
\end{aligned}
$$

Therefore, we have just shown that the proper neighbor map $f_2^{-1} f_2^{-1} f_1^{-1} f_2 f_1 f_1$ is the identity map. Therefore we have that $id \in \overline{\mathcal{N}}$ which implies, by Theorem 3.1, that $\mathcal{F}$ does not satisfy the OSC.

3.2 Neighbor Graphs

The proper neighbor maps do not only determine if an IFS satisfies the OSC but they are also the vertex set for a directed graph, called a neighbor graph,

which "describes the topology of A in the simplest possible way" [BM18]. The neighbor graph allows the calculation of various geometric parameters of the fractal attractor. These include but are not limited to: the connectedness of the attractor, if it encloses holes and the Hausdorff dimension of the topological boundary [BM18]. Bandt and Mesing [BM09] describe the construction of the neighbor graph as follows. For the neighbor graph, each proper neighbor map $h = f_i^{-1} f_j$ is denoted by a vertex. Between these vertices there is a directed edge labelled (i', j') from h to $\bar{h}$ if $\bar{h} = f_{i'}^{-1} h f_{j'}$ for some $i', j' \in S$. The identity map is the root vertex of the neighbor graph but by convention it is not drawn in the graph (unless it is a neighbor map) and all edges from the root to the other vertices have no initial vertex. An IFS, $\mathcal{F}$, is of finite type if there are only finitely many proper neighbor maps and therefore finitely many vertices in the neighbor graph [BM18]. Let us now present an example of how to construct a neighbor graph.

Example 3.12. We will now calculate the neighbor graph for the IFS from Example 3.8 that had the unit interval as its attractor. Recall that $\mathcal{N}^* = \left\{ f_1^{-1} f_2, \ f_2^{-1} f_1 \right\} = \{x + 1, x - 1\}$ and we have the following behaviour of the proper neighbor maps,

$$f_2^{-1} f_1^{-1} f_2 f_1 = f_1^{-1} f_2 \qquad \text{and} \qquad f_1^{-1} f_2^{-1} f_1 f_2 = f_2^{-1} f_1.$$

Let us denote the proper neighbor maps (and hence the vertices of the neighbor graph) by $a := f_1^{-1} f_2$ and $a^- := f_2^{-1} f_1$. Vertex a has an edge going to it from the unmarked identity vertex labelled $(1, 2)$. Then, by the left-hand side of the above equation, we see that there is a loop on vertex a marked $(2, 1)$. Identical reasoning gives the edges for vertex a^-. The neighbor graph is shown in Figure 3.4.
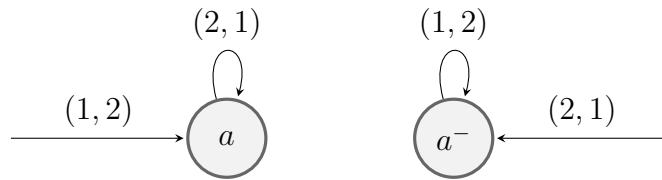

Figure 3.4: Neighbor Graph for $\mathcal{F} = \left\{ \mathbb{R}; f_1(x) = \frac{1}{2} x, f_2(x) = \frac{1}{2} x + \frac{1}{2} \right\}$ with attractor $[0, 1]$. Where $a := f_1^{-1} f_2$ and $a^- := f_2^{-1} f_1$.

Note that the proper neighbor maps $f_1^{-1} f_2$ and $f_2^{-1} f_1$ are inverses of each other and so that is why there vertices have been named a and a^- respectively.

We could simplify the neighbor graph of Example 3.12 by noting that the graph is symmetrical about a vertical line through the centre. This will always be the case for neighbor graphs because the inverse of every neighbor map is also a neighbor map and for any edge (i, j) between h_1 and h_2, there is always an edge (j, i) between h_1^{-1} and h_2^{-1}. Therefore, if the neighbor graph becomes large and there are no edges between the two sides of the graph then only one side of the neighbor graph needs to be drawn.

The classical Cantor set as given in Example 3.10 has an empty set of proper neighbor maps and hence there are no vertices in its neighbor graph; i.e the neighbor graph of the classical Cantor set is the empty graph.

3.3 Neighbor Maps of the Sierpinski Triangle

The Sierpinski triangle is a fundamental object in fractal geometry due to its significance in history and ubiquity in pedagogy. Therefore it seems appropriate to determine its proper neighbor maps and its associated neighbor graph. The results of this calculation have been presented by Grant [Gra18], however the calculation was not included and so we will now do that here. Let us define the IFS for the Sierpinski triangle as $\mathcal{F} = \{\mathbb{R}^2; f_1, f_2, f_3\}$ with maps

$$f_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad f_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

$$\text{and} \quad f_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ \frac{\sqrt{3}}{4} \end{pmatrix}. \tag{3.10}$$

The attractor of $\mathcal{F}$ is an equilateral Sierpinski triangle with sides of unit length. In order to calculate the proper neighbor maps let us start with words of length 1. By a straightforward calculation this gives the following six proper neighbor maps:

$$f_1^{-1}f_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad f_2^{-1}f_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$f_1^{-1}f_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \qquad f_3^{-1}f_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{-1}{2} \\ \frac{-\sqrt{3}}{2} \end{pmatrix} \tag{3.11}$$

$$f_2^{-1}f_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{-1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \qquad f_3^{-1}f_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{-\sqrt{3}}{2} \end{pmatrix}.$$

Note that maps in the same row are inverses of each other. We have exhausted all possible proper neighbor maps with length 1. Now we consider words of length 2 recalling the requirements that $i_1 \neq j_1$ and $A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}} \neq \emptyset$. Thus the only neighbor maps to consider for length 2 are associated with the following intersections: $A_{12} \cap A_{21} = \left\{ \left(\frac{1}{2}, 0 \right) \right\}$, $A_{13} \cap A_{31} = \left\{ \left(\frac{1}{4}, \frac{\sqrt{3}}{4} \right) \right\}$ and $A_{23} \cap A_{32} = \left\{ \left(\frac{3}{4}, \frac{\sqrt{3}}{4} \right) \right\}$. For each of these intersections there are two corresponding neighbor maps. Let us first look at the intersection $A_{12} \cap A_{21}$ which gives the proper neighbor map $f_{12}^{-1} f_{21} = f_2^{-1} f_1^{-1} f_2 f_1$ and its inverse $f_1^{-1} f_2^{-1} f_1 f_2$. By a straightforward calculation we see that $f_2^{-1} f_1^{-1} f_2 f_1 = f_1^{-1} f_2$ and similarly $f_1^{-1} f_2^{-1} f_1 f_2 = f_2^{-1} f_1$. Therefore the two neighbor maps given by the intersection of $A_{12} \cap A_{21}$ are already included in $\mathcal{N}^*$. Now considering the proper neighbor maps $(f_{13})^{-1} f_{31}$, $(f_{23})^{-1} f_{32}$ and their inverses we obtain similar identities as follows:

$$f_3^{-1} f_1^{-1} f_3 f_1 = f_1^{-1} f_3, \qquad f_1^{-1} f_3^{-1} f_1 f_3 = f_3^{-1} f_1,$$
$$f_3^{-1} f_2^{-1} f_3 f_2 = f_2^{-1} f_3, \qquad f_2^{-1} f_3^{-1} f_2 f_3 = f_3^{-1} f_2. \tag{3.12}$$

Therefore, all the proper neighbor maps of length 2 are equal to a proper neighbor map of length 1 and thus already contained in $\mathcal{N}^*$. We must now consider proper neighbor maps $f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}}$ for $|\boldsymbol{i}| = |\boldsymbol{j}| > 2$. The only $\boldsymbol{i}$ and $\boldsymbol{j}$ such that $i_1 \neq j_1$ and $A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}}$ will be of the form $(f_{ij^n})^{-1} f_{ji^n}$ with $i, j \in \{1, 2, 3\}$ and distinct. Note that the identities shown above give that $f_j^{-1} f_i^{-1} f_j f_i = f_i^{-1} f_j$ and so by a simple induction argument we have that $f_j^{-n} f_i^{-1} f_j f_i^n = f_i^{-1} f_j$. Therefore all proper neighbor maps of any length are equal to one of the six proper neighbor maps calculated for $|\boldsymbol{i}| = |\boldsymbol{j}| = 1$ which gives,

$$\mathcal{N}^* = \left\{ f_1^{-1} f_2, f_2^{-1} f_1, f_1^{-1} f_3, f_3^{-1} f_1, f_2^{-1} f_3, f_3^{-1} f_2 \right\}.$$

For all proper neighbor maps we have that the Sierpinski triangle is mapped to an isometric copy that intersects with the original at its vertices. Figure 3.5 from Grant [Gra18] shows the set $h(A)$ for each $h \in \mathcal{N}^*$ with the centre triangle shaded red to indicate that it is the original.

Now that the set of proper neighbor maps has been determined, the neighbor graph can be drawn. Let us denote the vertices of the neighbor graph by

$$a := f_1^{-1} f_2 \qquad\qquad a^- := f_2^{-1} f_1$$
$$b := f_1^{-1} f_3 \qquad\qquad b^- := f_3^{-1} f_1$$
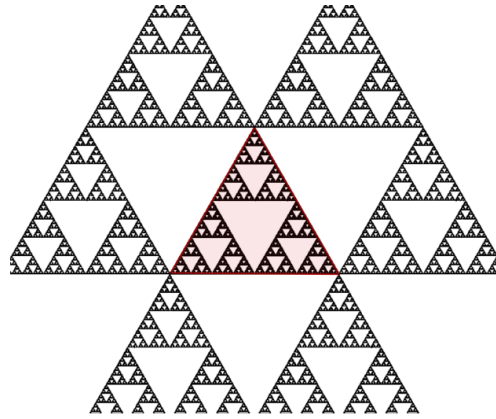$$c := f_2^{-1} f_3 \qquad\qquad c^- := f_3^{-1} f_2.$$

Figure 3.5: The images of the proper neighbor maps of the Sierpinski triangle [Gra18].

Each proper neighbor map $f_i^{-1} f_j$ corresponds to a vertex in the neighbor graph. On that vertex there will be a directed edge coming into it with no initial vertex (the identity vertex is not drawn) that will be labelled by (i, j) and there will also be a loop on it labelled (j, i) due to the property $f_j^{-1} f_i^{-1} f_j f_i = f_i^{-1} f_j$. The neighbor graph for the Sierpinski triangle is drawn in Figure 3.6. We have chosen to draw the entire neighbor graph instead of taking advantage of the symmetry property because it has allowed us to use the location of each vertex to illustrate the translation of each neighbor map and therefore which isometric copy it is responsible for in Figure 3.5.
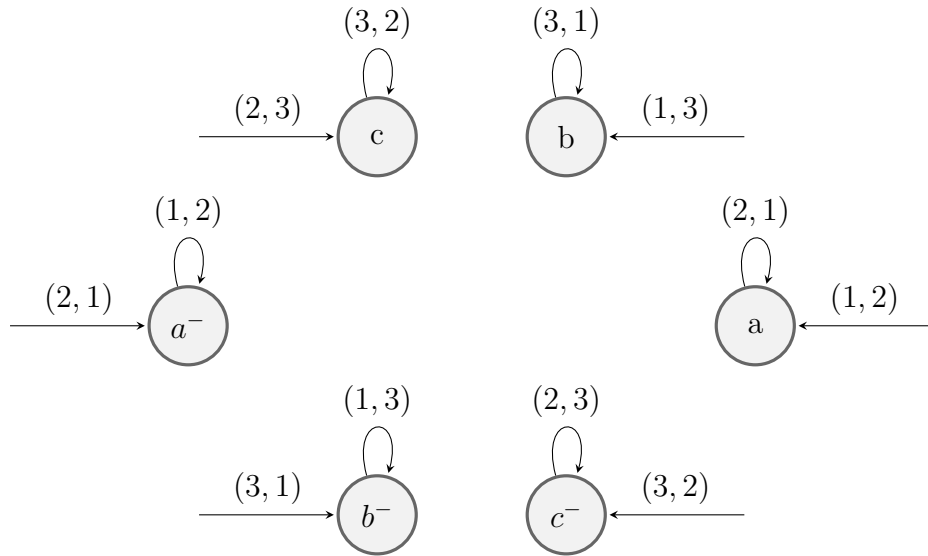

Figure 3.6: Neighbor Graph for the classical Sierpinski Triangle IFS.

We have calculated the set of proper neighbor maps and the neighbor graph for the IFS given in Equation 3.10 whose attractor is the equilateral Sierpinski triangle. We have found that the identity map is not contained in $\mathcal{N}^*$, moreover as $\mathcal{N}^*$ is a finite set $\mathcal{N}^* = \overline{\mathcal{N}^*}$ and thus $id \notin \overline{\mathcal{N}^*}$. Therefore, by Corollary 3.6, we have that the IFS that generated the equilateral Sierpinski triangle with side length one obeys the OSC. This could have been determined by noting that the open set $\mathcal{O} = (0,1) \times \left(0, \frac{\sqrt{3}}{2}\right)$ satisfies the OSC; however, it is good to note that we have obtained the same answer using neighbor maps.

If it were desired to calculate the set of all neighbor maps, not just the set of proper neighbor maps, it could be done but the task would be cumbersome. The approach to determining all neighbor maps is as follows. A neighbor map is of the form $f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}}$ with $i_1 \neq j_1$ and a proper neighbor map has the additional requirements that $|\boldsymbol{i}| = |\boldsymbol{j}|$ and $A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}} \neq \emptyset$. Removing the restriction of $|\boldsymbol{i}| = |\boldsymbol{j}|$, we get the collection of neighbor maps given by

$$h \begin{pmatrix} x \\ y \end{pmatrix} = f_j^{-n} f_i^{-1} f_j f_i^k \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2^{-k+n} & 0 \\ 0 & 2^{-k+n} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{b_k},$$

$$\text{with } \mathbf{b_1} = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}, \ \mathbf{b_2} = \begin{pmatrix} \pm C_x \\ \pm C_y \end{pmatrix}, \ \mathbf{b_3} = \begin{pmatrix} \pm \left(-2^{-k+n} + Cx \right) \\ \pm C_y \end{pmatrix},$$

where $i, j \in \{1, 2, 3\}$, $i \neq j$ and the $\pm$ has been included in the translation vectors to account for the inverse maps. Removing the restriction $A_i \cap A_j \neq \emptyset$ gives neighbor maps of the following forms $f_{1^n}^{-1} f_{2^k}$, $f_{13^n}^{-1} f_{21^k}$, $f_{231^n}^{-1} f_{31^k}$ and infinitely many other forms. Therefore, there is an infinite number of forms that a neighbor map can take and each of these gives rise to an infinite number of neighbor maps. However, there may not be infinitely many proper neighbor maps as already seen in Example 3.8 and above for the Sierpinski triangle. Therefore, the new result, Theorem 3.6 which allows us to only consider the proper neighbor maps when determining if the IFS satisfies the OSC, is useful.

Chapter 4

The Generalised Sierpinski Triangles

The generalised Sierpinski triangles considered here are a recent discovery in the field of fractal geometry [SW18]. The family of generalised Sierpinski triangles has four members: the classic Sierpinski triangle ($\triangle NNN$), the Pedal triangle ($\triangle FFF$), and the two newly discovered triangles denoted ($\triangle FNN$) and ($\triangle FFN$). Grant [Gra18] named the new fractal triangles the Steemson triangle and the Williams triangle respectively after their discoverers. For examples of the generalised Sierpinski triangles glance forward to Figure 4.3 on page 42.

An arbitrary triangle $\triangle ABC$ in $\mathbb{R}^2$ is fully described by the coordinates of its three vertices. That is, the side lengths, interior angles, location and orientation of the triangle in the plane is determined by the vertex positions. Note that if the vertices are all collinear then the triangle degenerates to a line segment. Without loss of generalisation we assume that A lies at the origin, $B = (1, 0)$ and $C = (C_x, C_y)$ is in the top half of the $\mathbb{R}^2$ plane. This can be done without changing the geometric properties of the triangle because side length ratios and interior angles are unchanged by rotation, translation and uniform scaling. Now, as vertices A and B are fixed we choose to describe the triangle by the side lengths $a = |BC|$ and $b = |AC|$ as illustrated in Figure 4.1a. Note $c = |AB| = 1$ and the top vertex is given by $C_x = \frac{1}{2}\left(b^2 - a^2 + 1\right)$ and $C_y = \frac{1}{2}\sqrt{4b^2 - \left(b^2 - a^2 + 1\right)^2}$.

Historically, the term 'Pedal triangle' referred to a sub-triangle that exists inside of another triangle created from a selected point. The geometric construction is illustrated in Figure 4.1b and goes as follows. Take a point P in the
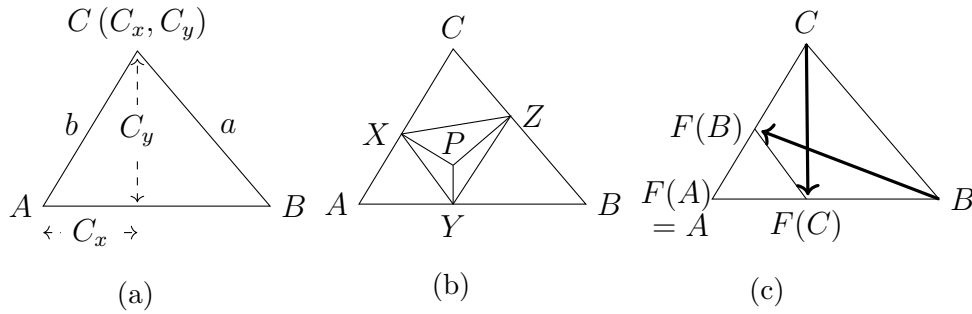
Figure 4.1: (a) The side lengths a and b fully describe $\triangle ABC$.
(b) The Pedal triangle $\triangle XYZ$ generated by P from $\triangle ABC$ [SW18].
(c) The operation of a flip map, F, on $\triangle ABC$.

interior of the triangle $\triangle ABC$. Let X be the point on the line AC such that PX is perpendicular to AC. Similarly define the points Y and Z on the lines AB and BC respectively. The triangle $\triangle XYZ$ was referred to as the Pedal triangle generated by P. This Pedal triangle can be used to construct a fractal attractor by considering an IFS consisting of three maps that take $\triangle ABC$ to $\triangle AXY$, $\triangle ZBY$ and $\triangle ZXC$. That is, the large triangle is mapped into each of the three smaller corner triangles. For any point P generated Pedal triangle this can be done using affine maps in a variety of ways. A specific interior Pedal triangle can be constructed by joining the three feet of the altitudes of $\triangle ABC$ as described in [ZHWD08] and in this case the functions that map the large triangle into the smaller corner triangles are similitudes. These similitudes involve a reflection and a rotation so we call them flip maps. The operation of a flip map, F, that takes the big triangle $\triangle ABC$ to the lower left triangle $\triangle AYX$ is illustrated in Figure 4.1c. The map F fixes the vertex A as that is the corner it is mapping to and it performs a reflection along the x-axis and a rotation so that the other two vertices exchange the lines that they lie on. This can be more easily thought of as flipping the triangle over, hence the name. We will be using the term 'Pedal triangle' for the triangular shaped fractal attractor generated by three flip maps. The Pedal triangles and their Hausdorff dimension have been well researched in [CM13, DL09, DT10, ZHWD08].

The Sierpinski triangle is not a singular triangle. Rather it describes the fractal nature that can occurs inside of an arbitrary triangle. Similarly, the other types of generalised Sierpinski triangles describe the fractal nature inside a triangular hull. Therefore, we can use any arbitrary triangle with side lengths a

and b as the convex hull for a generalised Sierpinski triangle. Note that there is one exception to this: a Pedal triangle is only defined for an acute triangular hull as otherwise the constructed $\triangle XYZ$ does not sit inside of the triangular hull.

The Sierpinski triangle is the attractor of an IFS with three non-flip maps (N) and the Pedal triangle is the attractor of an IFS with three flip maps (F). The combinations of non-flip and flip maps is what gives the Steemson ($\triangle FNN$) and the Williams ($\triangle FFN$) fractal triangles. When considering the Steemson triangle, without loss of generality we can assume that the flip map is mapping to the lower left and so fixes the origin. Similarly we can assume that for the Williams triangle the flip maps are the lower left and the lower right maps. These assumptions can be made because when the arbitrary triangle was taken to have vertices at $(0,0)$, $(1,0)$ and with the third vertex in the top half of the plane there is a total of six ways of doing this using similitudes and so preserving the geometry of the triangle. The full derivation of the Steemson and Williams triangles is included in [SW18] and various examples are displayed in Figure 4.3. The IFSs that generate each of the generalised Sierpinski triangles are given in Equation 4.1 where α, β, γ are the scaling factors of f_1, f_2, f_3 which fix the origin, $(1,0)$ and (C_x, C_y) respectively. For each of the generalised Sierpinski triangles the scaling factors are given in Figure 4.2 [SW18].

$$
\begin{aligned}
\mathcal{F}_{NNN} &= \left\{\mathbb{R}^2; f_{1,N}, f_{2,N}, f_{3,N}\right\} & \mathcal{F}_{FNN} &= \left\{\mathbb{R}^2; f_{1,F}, f_{2,N}, f_{3,N}\right\} \\
\mathcal{F}_{FFN} &= \left\{\mathbb{R}^2; f_{1,F}, f_{2,F}, f_{3,N}\right\} & \mathcal{F}_{FFF} &= \left\{\mathbb{R}^2; f_{1,F}, f_{2,F}, f_{3,F}\right\}
\end{aligned}
\tag{4.1}
$$

$\triangle$	α	β	γ
$\triangle NNN$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\triangle FNN$	$\frac{b}{b^2+1}$	$\frac{1}{b^2+1}$	$\frac{b^2}{b^2+1}$
$\triangle FFN$	$\frac{b}{a^2+b^2}$	$\frac{a}{a^2+b^2}$	$\frac{a^2+b^2-1}{a^2+b^2}$
$\triangle FFF$	$\frac{1}{2b}\left(-a^2+b^2+1\right)$	$\frac{1}{2a}\left(a^2-b^2+1\right)$	$\frac{1}{2ab}\left(a^2+b^2-1\right)$

Figure 4.2: The scaling factors for the generalised Sierpinski triangles.

The explicit formulas for the maps are stated below [SW18]:

$$
f_{1,N}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad f_{2,N}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}
$$

$$f_{3,N}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{C_x}{2} \\ \frac{C_y}{2} \end{pmatrix} \qquad f_{1,F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{b}C_x & \frac{\alpha}{b}C_y \\ \frac{\alpha}{b}C_y & -\frac{\alpha}{b}C_x \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$f_{2,F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\beta}{a}(1-C_x) & -\frac{\beta}{a}C_y \\ -\frac{\beta}{a}C_y & -\frac{\beta}{a}(1-C_x) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 - \frac{\beta}{a}(1-C_x) \\ \frac{\beta}{a}C_y \end{pmatrix} \qquad (4.2)$$

$$f_{3,F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \left(\frac{\alpha}{b}-\frac{\beta}{a}\right)C_x + \left(\frac{\beta}{a}-1\right) & \left(\frac{\alpha}{b}-\frac{\beta}{a}\right)C_y \\ \left(\frac{\alpha}{b}-\frac{\beta}{a}\right)C_y & -\left(\frac{\alpha}{b}-\frac{\beta}{a}\right)C_x - \left(\frac{\beta}{a}-1\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 - \frac{\beta}{a}(1-C_x) \\ \frac{\beta}{a}C_y \end{pmatrix}.$$

(a) $\triangle NNN$ (b) $\triangle FNN$ (c) $\triangle FFN$ (d) $\triangle FFF$

(e) $\triangle NNN$ (f) $\triangle FNN$ (g) $\triangle FFN$ (h) $\triangle FFF$

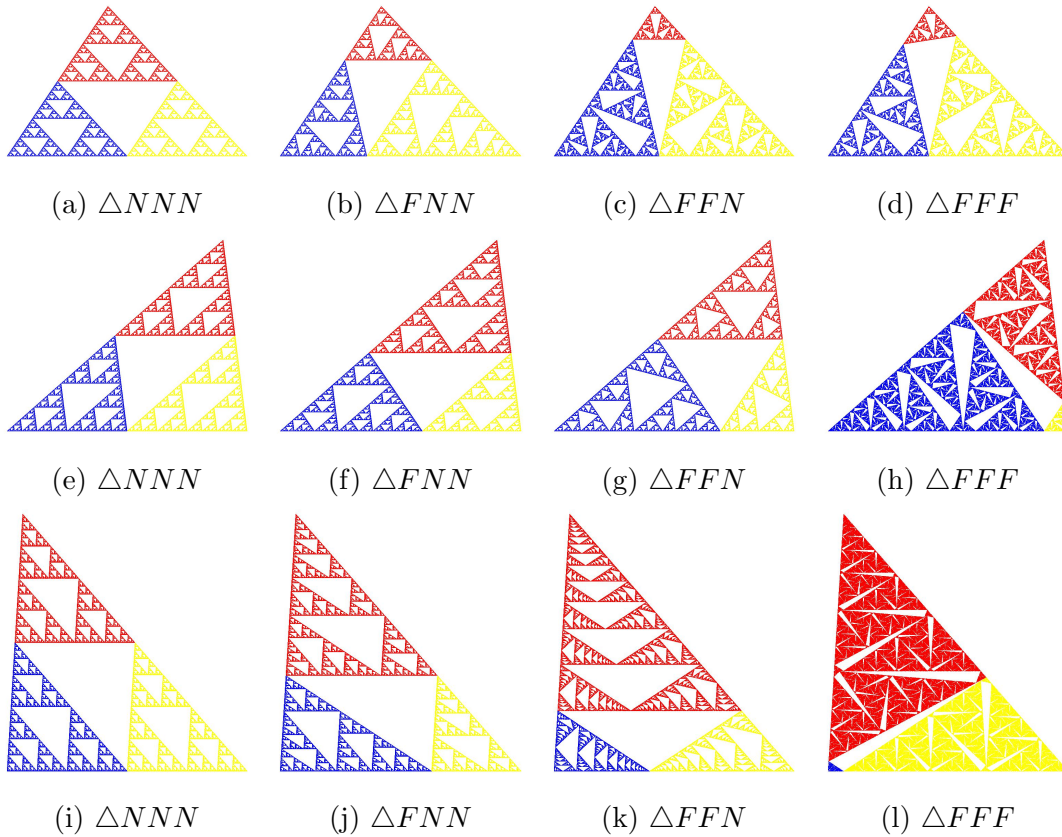(i) $\triangle NNN$ (j) $\triangle FNN$ (k) $\triangle FFN$ (l) $\triangle FFF$

Figure 4.3: The generalised Sierpinski triangles are shown across the columns with each row for a different pair of side lengths (a, b). From top to bottom the side lengths are $(0.85, 0.75)$, $(0.8, 1.2)$ and $(1.6, 1.3)$. For each of the triangles f_1, f_2, f_3 are shown in blue, yellow and red respectively.

4.1   Neighbor Maps of $\triangle NNN$

For the generalised Sierpinski triangle ($\triangle NNN$) the set of proper neighbor maps has already been calculated in the case of $(C_x, C_y) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, see Section 3.3.

When repeating the process for the arbitrary $\triangle NNN$ case we find that $\mathcal{N}$, $\mathcal{N}^*$ and the neighbor graph do not change when the notation $f_i^{-1} f_j$ is used for the neighbor maps. This is not surprising because moving the top vertex has no effect on the fractal structure or on the way attractor pieces intersect. Note that if the maps are expressed in $\mathbb{R}^2$ notation then the neighbor map formulas depend on (C_x, C_y).

4.2 Neighbor Maps of $\triangle FNN$

The similitudes in the IFS that generate the Steemson triangle ($\triangle FNN$) have scaling factors $\frac{b}{b^2+1}$, $\frac{1}{b^2+1}$ and $\frac{b^2}{b^2+1}$. Assuming $b \neq 1$ then the maps have different scaling factors. This does not prevent us from calculating the set of neighbor maps as defined by Equation 3.2. However, we are currently unable to calculate the set of proper neighbor maps as defined by Equation 3.4 because the similitudes in the IFS that generates the Steemson triangle do not have equal scaling factors. Since we do not yet have a definition of proper neighbor maps when the scaling factors are not equal, we will calculate the neighbor maps and use Theorem 3.1 to show that the IFS, $\mathcal{F}_{FNN}$, that generates the Steemson triangle satisfies the OSC.

We will now calculate the maps $h = f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}}$ for $\boldsymbol{i}$, $\boldsymbol{j} \in S^*$, $i_1 \neq j_1$ and $A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}} \neq \emptyset$. The set of these maps is a subset of $\mathcal{N}$ defined by the addition requirement $A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}} \neq \emptyset$. In the case of uniform scaling factors, this restriction was included in the definition of the set of proper neighbor maps because it prevented us from considering maps which have large translation factors. The scaling factors of the similitudes in the IFS do not affect the validity of this argument and therefore if the identity is not in the closure of this translation restricted set of neighbor maps then it is not in the closure of the set of neighbor maps.

Let us consider the neighbor maps that satisfy the condition $A_1 \cap A_2 \neq \emptyset$. The first such neighbor maps are $f_1^{-1} f_2$ and its inverse. The map $f_1^{-1} f_2$ gives rise to infinitely many additional neighbor maps of the form $f_{\boldsymbol{i}}^{-1} f_1^{-1} f_2 f_{\boldsymbol{j}}$ for $\boldsymbol{i}, \boldsymbol{j} \in S^*$ and $A_{1\boldsymbol{i}} \cap A_{2\boldsymbol{j}}$. Noting the flip nature of f_1 and the non-flip nature of f_2 we see that the only such $\boldsymbol{i}, \boldsymbol{j}$ are of the form $\boldsymbol{i} = 3^n$ and $\boldsymbol{j} = 1^m$ for $n, m \in \mathbb{N}$. The neighbor maps $f_{13^n}^{-1} f_{21^m} = (f_3)^{-n} f_1^{-1} f_2 (f_1)^m$ for all $n, m \in \mathbb{N}$ can be explicitly calculated. Note that f_1 is a flip map and when it is composed on itself it becomes a non-flip map, that is, f_1^2 does not flip and it does scale by α^2. Therefore the calculation of

these neighbor maps simplifies by considering the parity of m. Letting $m = 2k+1$ and $m = 2k$ for some $k \in \mathbb{N}$, we have:

$$(f_3)^{-n} f_1^{-1} f_2 (f_1)^{2k+1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{b^{2k-2n}}{(b^2+1)^{2k+2n+1}} & 0 \\ 0 & \frac{b^{2k-2n}}{(b^2+1)^{2k-2n+1}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} C_x \\ C_y \end{pmatrix}$$

$$(f_3)^{-n} f_1^{-1} f_2 (f_1)^{2k} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{b^{2k-2n-1}}{(b^2+1)^{2k+2n}} C_x & \frac{b^{2k-2n-1}}{(b^2+1)^{2k+2n}} C_y \\ \frac{b^{2k-2n-1}}{(b^2+1)^{2k+2n}} C_y & -\frac{b^{2k-2n-1}}{(b^2+1)^{2k+2n}} C_x \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} C_x \\ C_y \end{pmatrix}.$$

In both cases we cannot get the identity map or approach it in a limit since the translation factors are fixed and nonzero. Let us also note that the Steemson triangle is fully described by its vertices $\{(0,0),\ (1,0),\ (C_x, C_y)\}$ and so we can determine the neighbor set $h(A)$ for each of the above maps by determining the image of the vertices. We have

$$(f_3)^{-n} f_1^{-1} f_2 (f_1)^{2k+1} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} C_x \\ C_y \end{pmatrix} \right\} = \left\{ \begin{pmatrix} C_x \\ C_y \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} \frac{b^{2k-2n}}{(b^2+1)^{2k+2n+1}} + C_x \\ C_y \end{pmatrix}, \begin{pmatrix} \left(1 + \frac{b^{2k-2n}}{(b^2+1)^{2k+2n+1}}\right) C_x \\ \left(1 + \frac{b^{2k-2n}}{(b^2+1)^{2k+2n+1}}\right) C_y \end{pmatrix} \right\} \quad \text{and}$$

$$(f_3)^{-n} f_1^{-1} f_2 (f_1)^{2k} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} C_x \\ C_y \end{pmatrix} \right\} = \left\{ \begin{pmatrix} C_x \\ C_y \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} \left(1 + \frac{b^{2k-2n-1}}{(b^2+1)^{2k+2n}}\right) C_x \\ \left(1 + \frac{b^{2k-2n-1}}{(b^2+1)^{2k+2n}}\right) C_y \end{pmatrix}, \begin{pmatrix} \frac{b^{2k-2n-1}}{(b^2+1)^{2k+2n}} + C_x \\ C_y \end{pmatrix} \right\}.$$

The image of the vertices show that the above neighbor maps give a collection of neighbor sets which intersect the attractor only at the top vertex and whose adjacent sides are horizontal and a continuation of side b. That is, these maps give the equivalent neighbor as the top right neighbor set in Figure 3.5 for the Sierpinski triangle. Note that the inverse of these neighbor maps give the left and bottom left neighbors of the Steemson depending on the parity of m.

This process can be repeated for the neighbor maps associated to the intersections $A_1 \cap A_3$ and $A_2 \cap A_3$. The collection of neighbor maps $f_i^{-1} f_j$ such that $A_i \cap A_j \neq \emptyset$ are given by:

$$f_{12^n}^{-1} f_{31^m} = (f_2)^{-n} f_1^{-1} f_3 (f_1)^m, \qquad f_{23^n}^{-1} f_{32^m} = (f_3)^{-n} f_2^{-1} f_3 (f_2)^m,$$

and their inverses. For the neighbor maps involving f_1 we are again concerned parity. Therefore we express the above neighbor maps as follows for $n, m, k \in \mathbb{N}$:

$$(f_2)^{-n} f_1^{-1} f_3 (f_1)^{2k+1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{b^{2(k+1)}}{(b^2+1)^{2k-n+1}} & 0 \\ 0 & \frac{b^{2(k+1)}}{(b^2+1)^{2k-n+1}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$(f_2)^{-n} f_1^{-1} f_3 (f_1)^{2k} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{b^{2k}}{(b^2+1)^{2k-n}} C_x & \frac{b^{2k}}{(b^2+1)^{2k-n}} C_y \\ \frac{b^{2k}}{(b^2+1)^{2k-n}} C_y & -\frac{b^{2k}}{(b^2+1)^{2k-n}} C_x \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$(f_3)^{-n} f_2^{-1} f_3 (f_2)^m \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{b^{2-2n}}{(b^2+1)^{k-n}} & 0 \\ 0 & \frac{b^{2-2n}}{(b^2+1)^{k-n}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} C_x - \frac{b^{2-2n}}{(b^2+1)^{k-n}} \\ C_y \end{pmatrix}.$$

All neighbor maps $f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}}$ such that $A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}} \neq \emptyset$ have now been determined. As all maps have non-zero translation factors that do not limit to zero then the identity is not contained in the closure of the set of neighbor maps. Therefore by Theorem 3.1 we have that the Steemson triangles given by the IFS $\mathcal{F}_{FNN}$ obey the OSC. Alternatively, taking $\mathcal{O}$ as the open convex hull of the attractor satisfies the OSC [SW18], yet is it good to see that Bandt's theory does again agree with previously established results.

Thus far we have avoided defining the set of proper neighbor maps for an IFS with nonequal scaling factors. However, we would now like to construct a neighbor graph for a Steemson triangle and so we must find an appropriate replacement condition for $|\boldsymbol{i}| = |\boldsymbol{j}|$. All of Bandt's published work to date has only considered the case where all similitudes in the IFS have equal scaling factors. However, from Grant [Gra18] we have the following definition for the case when the scaling factors are integer powers of a common scaling factor.

Definition 4.1. [Gra18] If $\mathcal{F} = \{\mathbb{R}^2; f_1, \ldots f_N\}$ is an IFS of contractive similitudes with scaling factors $\lambda_i = \lambda^{a_i}$ where $0 < \lambda < 1$ is a fixed scaling factor base, $a_i \in \mathbb{N} = \{1, 2, 3, \ldots\}$, $i \in \{1, \ldots N\}$ and $gcd(a_1, \ldots a_N) = 1$, then we define the set of proper neighbor maps to be

$$\left\{ h = f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}} \mid \boldsymbol{i}, \boldsymbol{j} \in S^*, i_1 \neq j_1, A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}} \neq \emptyset, |\xi(\boldsymbol{i}) - \xi(\boldsymbol{j})| < a_{max} \right\}.$$

Here $a_{max} = \max\{a_1, \ldots a_N\}$ and $\xi : S^* \to \mathbb{N}_0$ is the function from Barnsley and Vince [BV18] such that $\xi(\emptyset) = 0$ and for $\boldsymbol{i} = i_1 i_2 \cdots i_p \in S^*$ we have

$$\xi(\boldsymbol{i}) = a_{i_1} + a_{i_2} + \cdots + a_{i_p}.$$

Thus, $\xi(\boldsymbol{i})$ sums the integer powers for the scaling factors of each map in the composition $f_{\boldsymbol{i}}$.

Remark 4.2. When the scaling factors of $\mathcal{F}$ satisfy $\lambda_i = \lambda^{a_i}$ for $i \in \{1, \dots N\}$ we say that the scaling factors satisfy the algebraic condition. Note that we assume $gcd(a_1, \dots a_N) = 1$, where gcd denotes the greatest common divisor, so that if the scaling factors satisfy the algebraic condition then they do so for a unique base scaling factor.

The set of proper neighbor maps, $\mathcal{N}^*$, includes all neighbor maps which have scaling factors 'close' to one. It would be reasonable to think that $< a_{max}$ could be changed (to $\leq a_{max}$ for example) and a different but equally reasonable set of proper neighbor maps would be defined. However, we argue that this is the best condition since in the case where all scaling factors are equal, $\lambda_i = \lambda^1$, we have $a_{max} = 1$. Therefore the condition simplifies to $|\boldsymbol{i}| = |\boldsymbol{j}|$, and so the new definition encompasses the initial definition of the set of proper neighbor maps [Gra18]. Note that the motivation for Grant [Gra18] to construct this definition came from fractal tiling theory. Importantly, Theorem 3.5 generalises for IFSs which satisfy the algebraic condition as we will now show.

Theorem 4.3. *If $\mathcal{F} = \{\mathbb{R}^m; f_1, \dots, f_N\}$ is an IFS with scaling factors λ^{a_i} for fixed $0 < \lambda < 1$, $a_i \in \mathbb{N}$ and $i \in \{1, \dots N\}$, then $id \notin \overline{\mathcal{N}}$ if and only if $id \notin \overline{\mathcal{N}^*}$, where $\mathcal{N}$ is the set of neighbor maps by Equation 3.2 and $\mathcal{N}^*$ is the set of proper neighbor maps given by Definition 4.1.*

Proof. The proof of this theorem is a straightforward generalisation of the proof supplied for Theorem 3.5; in fact, the proof of the forward direction is identical. The changes in the proof of the reverse direction come from noting that the non-proper neighbor map $h = f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}}$ must either satisfy $A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}} = \emptyset$ or $|\xi(\boldsymbol{i}) - \xi(\boldsymbol{j})| \geq a_{max}$. We have that h is of the form $h(x) = \lambda^k Ox + b \in \mathcal{N} \backslash \mathcal{N}^*$ for $k \in \mathbb{Z}$, O is a rotation reflection isometry and $b \in \mathbb{R}^m$ is a translational vector. When $\|b\| = 0$ is assumed this implies that $k \in \mathbb{Z} \backslash \{-a_{max} + 1, \dots, a_{max} - 1\}$ and so we get $d_s(h, id) \geq 1 - \lambda^{a_{max}}$. In the case of $\|b\| > 0$ the proof follows that of Theorem 3.5 and so we get $d_s(h, id) \geq 1 - \lambda$. Therefore any non-proper neighbor map h cannot be arbitrarily close to the identity so we have that $id \notin \overline{\mathcal{N}^*}$ implies $id \notin \overline{\mathcal{N}}$. $\qquad\square$

Applying Theorem 4.3 to Theorem 3.1 we get the following result.

Corollary 4.4. *If $\mathcal{F} = \{\mathbb{R}^m; f_1, \dots, f_N\}$ is an IFS of contractive similitudes with scaling factors that satisfy the algebraic condition, then $\mathcal{F}$ obeys the OSC if and only if $id \notin \overline{\mathcal{N}^*}$.*

If the scaling factors of the IFS satisfy the algebraic condition then the possible scaling factors for the neighbor maps are discrete and given by λ^k for $k \in \mathbb{Z}$. Therefore consider the following subset of $\mathcal{N}^*$,

$$\mathcal{N}^\dagger = \left\{ h = f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}} \mid \boldsymbol{i}, \boldsymbol{j} \in S^*, i_1 \neq j_1, A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}} \neq \emptyset, \xi(\boldsymbol{i}) = \xi(\boldsymbol{j}) \right\}. \tag{4.3}$$

Theorem 4.3 remains true if $\mathcal{N}^*$ is replaced with $\mathcal{N}^\dagger$ and so we get the following result.

Corollary 4.5. *If $\mathcal{F} = \{\mathbb{R}^m; f_1, \ldots, f_N\}$ is an IFS of contractive similitudes with scaling factors that satisfy the algebraic condition, then $\mathcal{F}$ obeys the OSC if and only if $id \notin \overline{\mathcal{N}^\dagger}$.*

Remark 4.6. We do not call $\mathcal{N}^\dagger$ the set of proper neighbor maps even though it is a 'smaller' set of neighbor maps which are sufficient to check for the OSC. This is because the set of proper neighbor maps, as Bandt and Mesing [BM09] define them, generate a neighbor graph that describes the topology of the fractal attractor, see Remark 4.9. We claim that $\mathcal{N}^\dagger$ does not generate a neighbor graph that sufficiently describes the topology of the fractal attractor. This claim is not proved here but represents an exciting area of the topic to be further researched. In Section 5.2 it will be discussed how the neighbor graph illustrates the topology by describing how different sub-attractor pieces relate to each other. However, $\mathcal{N}^\dagger$ only describes the relationship between sub-attractor pieces of equal size.

Now that the set of proper neighbor maps have been defined when the scaling factors satisfy the algebraic condition we would like to know about the set of proper neighbor maps for arbitrary scaling factors. This will be discussed in detail in Section 4.5; however, for the majority of our work we restrict ourselves to IFSs for which the scaling factors satisfy the algebraic condition. This is because, by Theorem 5.1, the algebraic condition guarantees that the neighbor graph of the generalised Sierpinski triangles will be of finite type. However, Example 5.5 provides a counterexample to the converse of this statement, that is, a situation where the algebraic condition is not satisfied yet the neighbor graph is finite.

Returning to our goal of constructing a neighbor graph of the Steemson triangle, we will find a particular Steemson triangle whose scaling factors obey the algebraic condition. This involves solving the system of linear equations that relate the scaling factors to the side lengths, namely

$$\alpha b + \beta = 1, \qquad \alpha + \gamma b = b, \qquad \beta a + \gamma a = a,$$

for $\alpha = \lambda^i, \beta = \lambda^j, \gamma = \lambda^k$ with $i, j, k \in \mathbb{Z}_+$. In Mathematica a triple 'For' loop was written to iterate through the integer values of i, j, k and for each case it attempted to solve for the side lengths a and b and the scaling factor λ. We find that there does not exist a solution for all integer combinations and that the side length a is a free variable. One particular solution is $(i, j, k) = (2, 1, 3)$ with $b \approx 0.68$ and $\lambda \approx 0.65$. This Steemson triangle is shown in Figure 4.4 and its neighbor graph is calculated in Example 4.8.

In order to present an example of determining the set of proper neighbor maps and then calculating the neighbor graph for an IFS which satisfies the algebraic condition, we must first introduce some notation from Grant [Gra18].

Definition 4.7. The set of symbolic neighbor pairs is given by:

$$P = \left\{ (\boldsymbol{i}, \boldsymbol{j}) \in S^* \times S^* \mid \xi(\boldsymbol{i}), \xi(\boldsymbol{j}) \leq a_{max} \right\}. \tag{4.4}$$

The set of symbolic neighbor pairs is the starting point for determining the set of proper neighbor maps. The maps $h = f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}}$ such that $(\boldsymbol{i}, \boldsymbol{j}) \in P$, $i_1 \neq j_1$ and $A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}} \neq \emptyset$ are the proper neighbor maps which come from the identity. Then the map $\tilde{h} = f_{\tilde{\boldsymbol{i}}}^{-1} f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}} f_{\tilde{\boldsymbol{j}}}$ for $\left(\tilde{\boldsymbol{i}}, \tilde{\boldsymbol{j}} \right) \in P$ is a proper neighbor map providing that $\left| \xi \left(\boldsymbol{i}\tilde{\boldsymbol{i}} \right) - \xi \left(\boldsymbol{j}\tilde{\boldsymbol{j}} \right) \right| < a_{max}$ and $A_{\boldsymbol{i}\tilde{\boldsymbol{i}}} \cap A_{\boldsymbol{j}\tilde{\boldsymbol{j}}} \neq \emptyset$. This process can be continually repeated to find more proper neighbor maps. However, this will not necessarily find all proper neighbor maps, as will be seen later. The examples provided in the remainder of this chapter illustrate a systematic way of determining all proper neighbor maps, even those which are not connected to the identity. This method also determines the directed edges of the neighbor graph.

Example 4.8. Consider the Steemson triangle with scaling factors satisfying the algebraic condition described by $(i, j, k) = (2, 1, 3)$ that is displayed in Figure 4.4. We will determine the proper neighbor maps and the neighbor graph for this Steemson triangle. Let us start by calculating the set of symbolic neighbor pairs,

$$P = \left\{ (\boldsymbol{i}, \boldsymbol{j}) \mid \boldsymbol{i}, \boldsymbol{j} \in \left\{ 1, 2, 3, 2^2, 12, 21, 2^3 \right\} \right\}.$$

Note that we have adopted the notation 2^3 for the string 222 in order to make the following working more readable. From P we see that the neighbor maps which come from the identity are given by the pairs; $(1, 2)$, $(1, 21)$, $(3, 1)$, $(3, 12)$, $(2, 3)$ and their inverses. From now we will omit discussion of the corresponding inverse maps and only look at the five just stated maps and those which connect to them
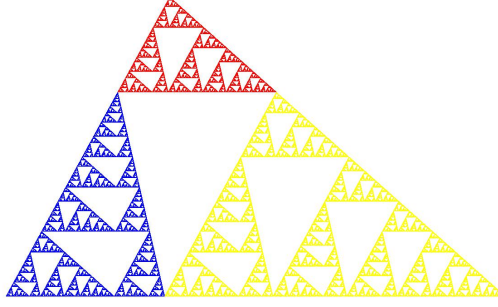
Figure 4.4: The Steemson triangle ($\triangle FNN$) with scaling factors $(\alpha, \beta, \gamma) = (\lambda^2, \lambda, \lambda^3)$ for $\lambda \approx 0.6477$. The side lengths are $(a, b) \approx (0.9, 0.68)$.

in the neighbor graph. We do this because the neighbor graph will be quite large and so we will take advantage of the symmetry property. Let us now begin our systematic method for determining all proper neighbor maps and first consider the maps which arise due to the intersection between A_1 and A_2. The maps which come from the identity at this intersection are $(1, 2)$ and $(1, 21)$ as stated above. Additional maps for this intersection will be of the form $(n_1 1, 2n_2)$ or $(n_3 1, 21n_4)$ for n_i being a string of length l_i, that is $n_i \in \{1, 2, 3\}^{l_i}$. Recalling the requirement that $A_i \cap A_j \neq \emptyset$ and noting that f_1 is a flip map and f_2 is a non-flip map then we have that n_1 and n_3 can only be strings of 3's and n_2 and n_4 can only be strings of 1's. Therefore all proper neighbor maps for the intersection $A_1 \cap A_2$ are of the form $\left(3^n 1, 21^k\right)$ with the additional restriction that $\left|\xi\left(3^n 1\right) - \xi\left(21^k\right)\right| < 3$. We are now in a position where we can simply write down all the different proper neighbor maps for $A_1 \cap A_2$. We do this in Figure 4.5 by fixing n for each column and varying k such that the ξ condition is satisfied.

$(1, 2)$	$(31, 21)$	$(3^2 1, 21^3)$	$(3^3 1, 21^4)$	$(3^4 1, 21^6) = (1, 2)$
$(1, 21)$	$(31, 21^2)$	$(3^2 1, 21^4)$	$(3^3 1, 21^5)$	$(3^4 1, 21^7) = (1, 21)$
	$(31, 21^3)$		$(3^3 1, 21^6)$	

Figure 4.5: The list of proper neighbor maps for the Steemson triangle with scaling factors $(\lambda^2, \lambda^1, \lambda^3)$ that are due to the intersection $A_1 \cap A_2$. Note that for each of these maps there is an inverse map which we are omitting.

The scaling factor of f_3 is λ^3 and the scaling factor of f_1 is λ^2, and so the maps f_3^2 and f_1^3 both have a scaling factor of λ^6. Note that the sets $f_3^2 f_1(A)$ and $f_1^3 f_2(A)$ do not have the same relationship as $f_1(A)$ and $f_2(A)$ since f_1 is a flip

map and has been applied an odd number of times. Thus, we double the exponents and observe that $f_3^4 f_1(A)$ and $f_1^6 f_2(A)$ have the same relationship as $f_1(A)$ and $f_2(A)$. Therefore $(3^4 1, 21^6) = (1, 2)$. Due to this equality we can conclude that Figure 4.5 includes all proper neighbor maps for $A_1 \cap A_2$. Since all proper neighbor maps for this intersection are of the form $(3^n 1, 21^k)$ and that 3 and 1 appear in P as only single digits then the only pair $\left(\tilde{\boldsymbol{i}}, \tilde{\boldsymbol{j}}\right) \in P$ to map between neighbor maps is $(3, 1)$. Therefore we have fully determined the neighbor graph for the intersection of $A_1 \cap A_2$. It is drawn in Figure 4.6 and note that the entire graph is mirrored for the inverse vertices but these are omitted from the graph.
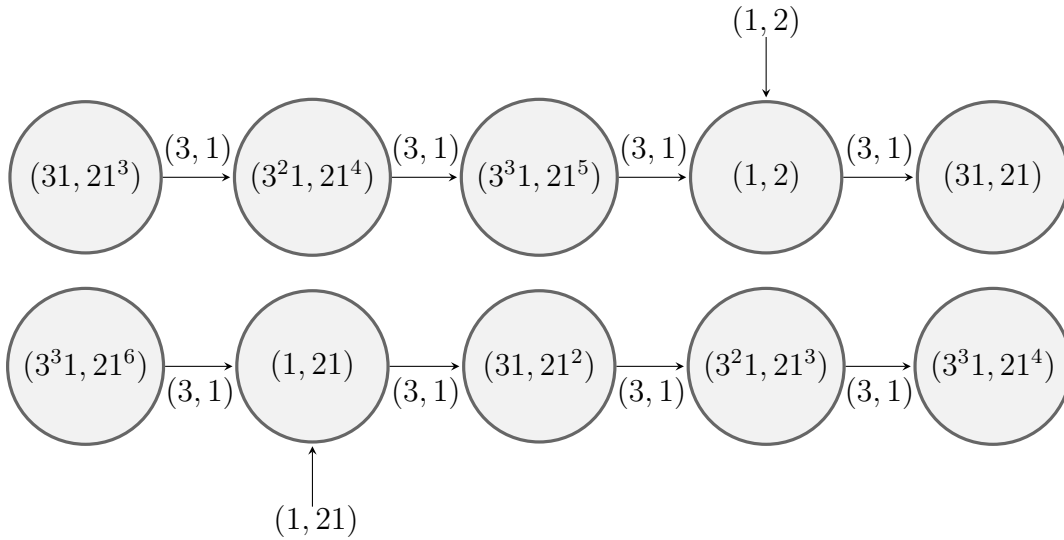

Figure 4.6: The neighbor graph of the Steemson triangle ($\triangle FNN$) with scaling factors $(\alpha, \beta, \gamma) = (\lambda^2, \lambda, \lambda^3)$ for the intersection between A_1 and A_2.

We now must repeat the process for the neighbor maps that are associated to the intersection between A_1 and A_3. The proper neighbor maps which come from the identity are $(3, 1)$ and $(3, 12)$. Accounting for the flip nature of f_1 and the non-flip nature of f_2 we have that all proper neighbor maps for this intersection will be of the form $\left(1^n 3, 12^k\right)$ with the ξ restriction. By similar reasoning to above we have that $(1^2 3, 12^4) = (3, 1)$. This allows an exhaustive list of the maps to be written out. For neighbor maps being mapped to other neighbor maps there exists two pairs $\left(\tilde{\boldsymbol{i}}, \tilde{\boldsymbol{j}}\right) \in P$ which are $(1, 2)$ and $(1, 2^2)$.

Lastly, we determine the proper neighbor maps for $A_2 \cap A_3$. Only $(2, 3)$ comes from the identity and all maps will be of the form $\left(3^n 2, 32^k\right)$ where the ξ condition

is satisfied. The only pairs which can take one neighbor map to another are $(3, 2)$ and $(3, 2^2)$.

The proper neighbor maps for the Steemson triangle with scaling factors $(\alpha, \beta, \gamma) = (\lambda^2, \lambda, \lambda^3)$ are denoted by letters of the alphabet in Figure 4.10 as is the convention for neighbor graphs. Figure 4.7 gives the proper neighbor map that corresponds to each vertex using symbolic neighbor pair notation.

$a = (31, 21^3)$	$b = (3^2 1, 21^4)$	$c = (3^3 1, 21^5)$	$d = (1, 2)$	$e = (31, 21)$
$f = (3^3 1, 21^6)$	$g = (1, 21)$	$h = (31, 21^2)$	$i = (3^2 1, 21^3)$	$j = (3^3 1, 21^4)$
$k = (13, 12^5)$	$l = (3, 12^2)$	$m = (13, 12^3)$	$n = (3, 1)$	$o = (13, 12)$
$p = (3, 12^3)$	$q = (13, 12^4)$	$r = (3, 12)$	$s = (13, 12^2)$	$t = (1^2 3, 12^3)$
$u = (2, 3)$	$v = (32, 32)$	$w = (3^2 2, 32^2)$	$x = (32, 32^2)$	$y = (32, 3)$

Figure 4.7: The list of proper neighbor maps for the Steemson triangle with scaling factors $(\lambda^2, \lambda^1, \lambda^3)$ that are seen in the Neighbor Graph of Figure 4.10.

The Steemson $(2, 1, 3)$ neighbor graph, Figure 4.10, is only half of the complete neighbor graph but the other half has been omitted because it contains no new information and doing so simplifies a relatively large graph. Note that in some cases the inverse half and the original half of the neighbor graphs interplay but it is not the case for this example which can be seen by explicitly calculating all proper neighbor maps and their inverses. Alternatively, an argument could be constructed from the geometry of the fractal attractor. However, for this case Mathematica was used because there were only 25 neighbor maps and their inverses to check.

Remark 4.9. It was stated in Remark 4.6 that $\mathcal{N}^*$ (Def 4.1) is the set of proper neighbor maps even though $\mathcal{N}^\dagger$ (Def 4.3) is a 'smaller' set of neighbor maps that is sufficient for checking that the IFS obeys the OSC. As a point of interest we calculate $\mathcal{N}^\dagger$ for the Steemson triangle with scaling factors $(\lambda^2, \lambda, \lambda^3)$. The proper neighbor maps in symbolic neighbor pair notation (omitting inverses) that satisfy $\xi(\boldsymbol{i}) = \xi(\boldsymbol{j})$ are $(31, 21^2)$, $(3^3 1, 21^5)$, $(3, 12)$, $(13, 12^3)$ and $(32, 32)$. These correspond to vertices h, c, r, m and v respectively. If we were to draw a neighbor graph of just these vertices then the only directed edge would be $(1, 2^2)$ between m and r. This does not prove the claim made in Remark 4.6, that $\mathcal{N}^\dagger$ does not generate a sufficient neighbor graph, but it provides a supporting example.

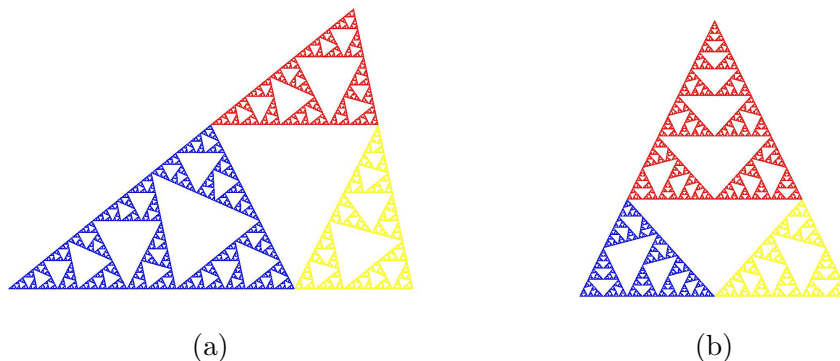(a)   (b)

Figure 4.8: (a) The Williams triangle ($\triangle FFN$) with scaling factors $(\alpha, \beta, \gamma) = (\lambda, \lambda^2, \lambda^2)$ for $\lambda \approx 0.6435$. The side lengths are $(a, b) \approx (0.71, 1.10)$.
(b) The Williams triangle ($\triangle FFN$) with scaling factors $(\alpha, \beta, \gamma) = (\lambda^2, \lambda^2, \lambda)$ for $\lambda \approx 0.6477$. The side lengths are $(a, b) \approx (1.19, 1.19)$.

4.3   Neighbor Maps of $\triangle FFN$

The goal of this section is to calculate two neighbor graphs for the Williams triangle. Calculating more neighbor graphs will help us understand them better and two examples for the same type of generalised Sierpinski triangle will allow us to compare neighbor graphs for similar fractal attractors. The introduction of a second flip map to change a Steemson triangle to a Williams triangle does not affect the process used to determine the set of proper neighbor maps or how to construct the neighbor graph. One difference between $\triangle FNN$ and $\triangle FFN$ is that the Williams triangle has 'more' solutions to the algebraic condition. It is not necessary to make explicit what is meant by 'more'; however, we can do a quick comparison to demonstrate the difference. Let the scaling factors for the triangles be $(\alpha, \beta, \gamma) = (\lambda^i, \lambda^j, \lambda^k)$ with $i, j, k \in \{1, 2, 3, 4\}$ and not all three integer powers equal. For the Steemson triangle there are four solutions and for the Williams triangle there are 39 solutions.

The Williams triangle with scaling factors given by $(i, j, k) = (1, 2, 2)$ can be seen in Figure 4.8a and Grant [Gra18] determined the associated neighbor graph. However, there was one vertex (and its inverse) missing from Grant's graph. The revised neighbor graph is included in Figure 4.11. Something interesting to note is that although we do have three disconnected sections of the neighbor graph, they are not each due to one of the intersection points. Instead $A_1 \cap A_2$ gives the vertices $a = (1, 2)$ and $b = (31, 2)$ which are not connected and so $(31, 2)$

represents a neighbor map which cannot be reached from the identity. We also see that the regions due to $A_1 \cap A_3$ and $A_2 \cap A_3$ are connected. This is because f_1 and f_2 have the same scaling factors. The vertices represent the following neighbor maps in Figure 4.11,

$$
\begin{aligned}
c &= (1,3) = (23, 21^3) & d &= (21, 31^2) = (3, 21) \\
e &= (21, 31) = (3, 2) & f &= (2^2 1, 31^3) = (23, 21^2) \qquad (4.5) \\
g &= (2^2 1, 31^2) = (23, 21) & h &= (21, 3) = (2^2 3, 21^3).
\end{aligned}
$$

For an interesting comparison the neighbor graph for the Williams triangle with $(i, j, k) = (2, 2, 1)$ was also calculated. This triangle can be seen in Figure 4.8b. The neighbor graph is seen in Figure 4.12 and the vertices denote the following proper neighbor maps:

$$
\begin{aligned}
t &= (1, 23) & u &= (1, 2) & v &= (31, 2) \\
& & w &= (1, 3) & x &= (21, 3) \qquad (4.6) \\
& & y &= (3, 2) & z &= (23, 21).
\end{aligned}
$$

Let us now compare the two neighbor graphs of the Williams with scaling factors $(1, 2, 2)$ and $(2, 2, 1)$ given in Figures 4.11 and 4.12 respectively. First note that both graphs have been drawn using the symmetry property. For both graphs $(3, 3)$ is a loop on vertices that are due to the intersection $A_1 \cap A_2$. But for the second graph, we have that $\xi(3) < a_{max}$ so the maps $(3^2, 3)$ and $(3, 3^2)$ are allowed and are the inverse of each other. Also, since for the second graph we have $\xi(1) = \xi(2)$ we get a third vertex in the section due to $A_1 \cap A_2$. For both graphs the map $(2, 1)$ is applicable to both the $(1, 3)$ and $(3, 2)$ section. But for the first case with $\xi(1) = 1$ and $\xi(2) = 2$ the map $(2, 1^2)$ is allowed when it is not for the second case as $\xi(1) = \xi(2) = 2$ which is the maximum. For the first case, we observe that the map $(2, 1^2)$ is its own inverse, that is, if you apply it twice you get back to where you started. However, for the second case the map $(2, 1)$ is its own inverse .

The two neighbor graphs for the Williams triangles are significantly simpler than the neighbor graph for the Steemson triangle. It would be incorrect to assume that this is due to the introduction of a second flip map but rather it is due to the smaller integer powers of the scaling factor λ and that two of the maps have the same scaling. To demonstrate this point the neighbor graph for the Williams triangle with scaling factors $(i, j, k) = (1, 2, 4)$ was calculated. For

this case the symbolic neighbor set is

$$P = \left\{ (\boldsymbol{i}, \boldsymbol{j}) \mid \boldsymbol{i}, \boldsymbol{j} \in \left\{ 1, 2, 3, 1^2, 1^3, 12, 21, 1^2 2, 21^2, 212, 1^4 \right\} \right\}.$$

Therefore when calculating the proper neighbor maps due to $A_1 \cap A_3$ we have that both $(1, 3)$ and $(1, 32)$ are from the identity. Possible neighbor pairs that can be applied to these maps involve putting 2's on the left and 1's on the right, these are $(2, 1), (2, 1^2), (2, 1^3), (2, 1^4), (2^2, 1), (2^2, 1^2), (2^2, 1^3), (2^2, 1^4)$. Continuing the working it was found that 14 distinct proper neighbor maps (and their inverses) came from this intersection and there were 92 directed edges between those 14 maps. Thus, this example is not included because the neighbor graph is so congested that it offers little in enlightening the reader about neighbor graphs except that when a_{max} is increased the complexity of the graph increases dramatically.

Additional neighbor graphs were calculated for the Pedal triangle ($\triangle FFF$) such as for the scaling factors $(i, j, k) = (1, 2, 3)$. But its presence does not add much to the understanding of neighbor graphs. Instead, we now progress to a situation where we can compare the neighbor graphs of each of the types of the generalised Sierpinski triangles.

4.4   Neighbor Graphs of Equilateral Generalised Sierpinski Triangles

The equilateral generalised Sierpinski triangles are found by setting the side lengths a and b equal to 1. This forces the scaling factors to satisfy $\alpha = \beta = \gamma = \frac{1}{2}$. Therefore, although each type of generalised Sierpinski triangle still has a distinct IFS we do have the same attractor. Calculating the neighbor graphs for the equilateral generalised Sierpinski triangles offers the interesting opportunity to compare the IFSs when the attractors are the same.

The equilateral Sierpinski triangle had its neighbor graph drawn in Figure 3.6 when it was first calculated but we redraw it in Figure 4.13 with a different structure so that it may be more easily compared to the other equilateral generalised Sierpinski triangles. The Steemson, Williams and Pedal equilateral neighbor graphs are drawn in Figures 4.14, 4.15 and 4.16 respectively. Note that the entire neighbor graphs are drawn because the 'inverse' half of the graph interplays with the 'original' half. The vertices are labelled as $a = (1, 2)$, $b = (1, 3)$
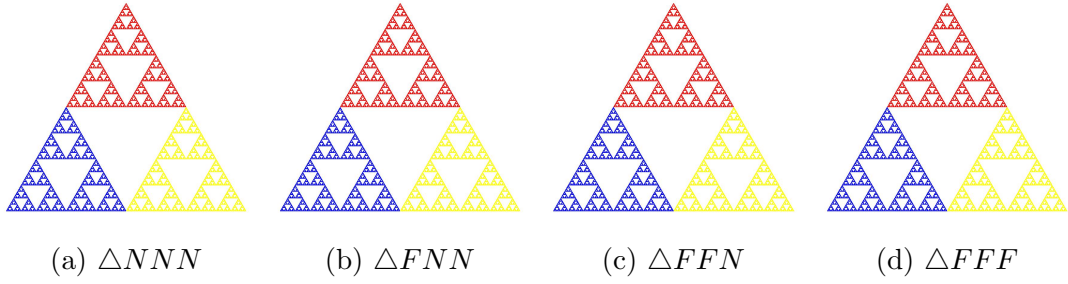
(a) $\triangle NNN$ (b) $\triangle FNN$ (c) $\triangle FFN$ (d) $\triangle FFF$

Figure 4.9: Equilateral generalised Sierpinski triangles are given for $a = b = 1$ and the maps f_1, f_2 and f_3 are shown in blue, yellow and red respectively. Refer to Figure 4.3 to see that the fractal attractors are different for the non-equilateral cases.

and $c = (2, 3)$ for all equilateral generalised Sierpinski triangles, with the addition of $x = (13, 21)$ and $y = (12, 31)$ for the Steemson triangle in Figure 4.14. Let us now make some observations about the neighbor graphs of the equilateral generalised Sierpinski triangles.

When going from the Sierpinski triangle to the Steemson triangle f_1 becomes a flip map. As a result the neighbor graph of the Steemson triangle now contains the vertices x and y which represent the parity of how many times the flip has been applied. It has already been noted that f_1^2 only scales the space because it is its own inverse with respect to its flipping nature. Thus when $(3, 1)$ or $(2, 1)$ has been applied twice (to $(1, 2)$ or $(1, 3)$ respectively) then you have returned to the initial vertex. Since it is only f_1 which has been changed then the section of the neighbor graph that is due to the intersection $A_2 \cap A_3$ has been left unchanged.

When the neighbor graph of the equilateral Williams triangle was originally calculated it looked similar to the neighbor graph of the equilateral Steemson triangle, in that a new vertex was introduced near c which was called $z = (12, 32)$ and the vertex x disappeared into vertex a which now has a loop on it. The disappearance of x and the appearance of z was due to f_2 now also being a flip map, and specifically a mirror image flip of f_1 due to the equilateral nature. However, the neighbor graph of the Williams triangle simplified significantly when it was noted that z^- equalled b. This resulted in c^- being connected to b and that is why it is now on the left of the neighbor graph.

All maps for the equilateral Pedal triangle are flip maps. This means that at

each intersection $A_i \cap A_j$ the only map that can be applied is (k, k) for $i, j, k \in \{1, 2, 3\}$ and all distinct. Since the map (k, k) is a flip map then instead of getting loops on each vertex as we did for the Sierpinski we get 2-cycles with (k, k) being its own inverse. The Pedal graph simplifies when it is noted that $(31, 23) = (2, 1)$ and similar for the other intersections. This results in a neighbor graph where the vertex and its inverse are connected, which is not uncommon in more complicated graphs but is the first example we have seen of it for such a simple fractal attractor.

4.5   Proper Neighbor Maps for an Arbitrary IFS

In order for the neighbor map $h = f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}}$ to be a proper neighbor map it needs to have a scaling factor 'near' to 1, so for an IFS with uniform scaling factors we had the requirement $|\boldsymbol{i}| = |\boldsymbol{j}|$ which was then replaced by $|\xi(\boldsymbol{i}) - \xi(\boldsymbol{j})| < a_{max}$ for an IFS with scaling factors that satisfy the algebraic condition. We are desire a new definition for the set of proper neighbor maps when the IFS has arbitrary scaling factors. It might be initially tempting to use the following restriction. For a fixed $\epsilon > 0$, suppose $h = f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}}$ is a proper neighbor map if and only if $A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}} \neq \emptyset$ and $|\prod_{k=1}^{p} \lambda_{i_k} - \prod_{k=1}^{q} \lambda_{j_k}| < \epsilon$ for $\boldsymbol{i} = i_1 i_2 \ldots i_p$ and $\boldsymbol{j} = j_1 j_2 \ldots j_q$. However, we find that this is not a good restriction as the following example demonstrates.

Example 4.10. Let $\mathcal{F} = \left\{ \mathbb{R}; f_1(x) = \frac{1}{2}x, f_2(x) = \frac{1}{2}x + \frac{1}{2} \right\}$ be the IFS from Examples 3.8 and 3.12 which has its neighbor graph drawn in Figure 3.4. The set of neighbor maps $f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}}$ which satisfy $i_1 \neq j_1$ and $A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}} \neq \emptyset$ are of the form $f_2^{-n} f_1^{-1} f_2 f_1^k$. Letting $n = 10$, $k = 15$ and $\epsilon = 2^{-10}$ we find that $\left| \prod_{k=1}^{l} \lambda_{i_k} - \prod_{k=1}^{m} \lambda_{j_k} \right| = |2^{-11} - 2^{-16}| < \epsilon$ so for $n = 10$ and $k = 15$ we do have a 'proper neighbor map' by this alternative definition. Recalling that $f_2^{-1} f_1^{-1} f_2 f_1 = f_1^{-1} f_2$ then $f_2^{-10} f_1^{-1} f_2 f_1^{15} = f_1^{-1} f_2 f_1^5$. Therefore, under this alternative neighbor map definition we have that $f_1^{-1} f_2 f_1^5(x) = 2^{-5}x + 1$ is a 'proper neighbor map'. This does not agree with the results previously established and we see that for any $\epsilon > 0$ there would exist infinite proper neighbor maps. Therefore this definition does not yield anything useful.

Instead we look to the ξ function from Definition 4.1 [BV18] to extend our definition of proper neighbor maps for arbitrary scaling factors. Consider the IFS $\mathcal{F} = \{\mathbb{R}^m; f_1, \ldots f_N\}$ of similitudes with scaling factors $\lambda_1, \ldots \lambda_N$ respectively. Let us fix $0 < \lambda < 1$ and let $a_i \in \mathbb{R}$ be such that $\lambda_i = \lambda^{a_i}$. Therefore $a_i = \frac{\log(\lambda_i)}{\log(\lambda)}$. It is important to note that the way of choosing the base λ is not unique but as

stated in Theorem 4.13 all choices of λ produce the same set of proper neighbor maps. Let $\widetilde{\xi} : S^* \to \mathbb{R}$ be the extended ξ function such that $\widetilde{\xi}(\emptyset) = 0$ and for $\boldsymbol{i} = i_1 i_2 \cdots i_p \in S^*$ we have

$$\widetilde{\xi}(\boldsymbol{i}) = a_{i_1} + a_{i_2} + \cdots + a_{i_p}.$$

Thus, $\widetilde{\xi}(\boldsymbol{i})$ sums the scaling factor powers of each map in the composition $f_{\boldsymbol{i}}$.

Definition 4.11. If $\mathcal{F} = \{\mathbb{R}^2; f_1, \ldots f_N\}$ is an IFS with scaling factors $\lambda_i = \lambda^{a_i}$ for fixed $0 < \lambda < 1$, $a_i \in \mathbb{R}$ and $i \in \{1, \ldots N\}$, then we define the set of proper neighbor maps to be

$$\left\{ h = f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}} \mid \boldsymbol{i}, \boldsymbol{j} \in S^*, i_1 \neq j_1, A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}} \neq \emptyset, \left| \widetilde{\xi}(\boldsymbol{i}) - \widetilde{\xi}(\boldsymbol{j}) \right| < a_{max} \right\}.$$

Here $a_{max} = \max\{a_1, \ldots a_N\}$ and let $\widetilde{\xi} : S^* \to \mathbb{R}$ be the extended version of the ξ function originally taken from Barnsley and Vince [BV18].

Remark 4.12. As a proposed convention we will select the scaling factor base as $\lambda = \min_i \{\lambda_i\}$. We do this so that $a_{max} = 1$ which makes working with the extended function $\widetilde{\xi}$ a little bit easier.

Theorem 4.13. *The set of proper neighbor maps is independent of the choice of scaling factor base λ.*

Proof. Let us consider the IFS $\mathcal{F} = \{\mathbb{R}^2; f_1, \ldots f_N\}$ with scaling factors λ_i. Let $0 < \lambda_a, \lambda_b < 1$ be two arbitrary scaling factor bases so that $\lambda_i = \lambda_a^{a_i}$ and $\lambda_i = \lambda_b^{b_i}$ for $a_i, b_i \in \mathbb{R}$ and $i \in \{1, \ldots, N\}$. Therefore we have that $a_i = \frac{\log(\lambda_i)}{\log(\lambda_a)}$, $a_{max} = \frac{\log(\lambda_{min})}{\log(\lambda_a)}$ and similar for λ_b. Let $\widetilde{\xi}_a, \widetilde{\xi}_b$ be the extended ξ functions which return the sum of the exponents for the bases λ_a and λ_b respectively. Let $\boldsymbol{i} = i_1 i_2 \ldots i_p$ and $\boldsymbol{j} = j_1 j_2 \ldots j_q$ in S^* be such that they satisfy the $\widetilde{\xi}$ condition for base λ_a. It will be shown that they also satisfy the $\widetilde{\xi}$ condition for base λ_b. Thus we have:

$$
\begin{aligned}
&\left| \widetilde{\xi}_a(\boldsymbol{i}) - \widetilde{\xi}_a(\boldsymbol{j}) \right| < a_{max} \\
\Leftrightarrow\ &\left| \left(a_{i_1} + \cdots + a_{i_p} \right) - \left(a_{j_1} + \cdots + a_{j_q} \right) \right| < a_{max} \\
\Leftrightarrow\ &\left| \sum_{k=1}^{p} a_{i_k} - \sum_{l=1}^{q} a_{j_l} \right| < a_{max} \\
\Leftrightarrow\ &\left| \sum_{k=1}^{p} \frac{\log(\lambda_{i_k})}{\log(\lambda_a)} - \sum_{l=1}^{q} \frac{\log(\lambda_{j_l})}{\log(\lambda_a)} \right| < \frac{\log(\lambda_{min})}{\log(\lambda_a)} \\
\Leftrightarrow\ &\frac{1}{|\log(\lambda_a)|} \left| \sum_{k=1}^{p} \log(\lambda_{i_k}) - \sum_{l=1}^{q} \log(\lambda_{j_l}) \right| < \frac{\log(\lambda_{min})}{\log(\lambda_a)}
\end{aligned}
$$

(4.7)

$$\Leftrightarrow \frac{\log\left(\lambda_a\right)}{\log\left(\lambda_b\right)}\frac{1}{\left|\log\left(\lambda_a\right)\right|}\left|\sum_{k=1}^{p}\log\left(\lambda_{i_k}\right)-\sum_{l=1}^{q}\log\left(\lambda_{j_l}\right)\right|<\frac{\log\left(\lambda_a\right)}{\log\left(\lambda_b\right)}\frac{\log\left(\lambda_{\min}\right)}{\log\left(\lambda_a\right)}$$

$$\Leftrightarrow \frac{1}{\left|\log\left(\lambda_b\right)\right|}\left|\sum_{k=1}^{p}\log\left(\lambda_{i_k}\right)-\sum_{l=1}^{q}\log\left(\lambda_{j_l}\right)\right|<\frac{\log\left(\lambda_{\min}\right)}{\log\left(\lambda_b\right)}$$

$$\Leftrightarrow \left|\sum_{k=1}^{p}\frac{\log\left(\lambda_{i_k}\right)}{\log\left(\lambda_b\right)}-\sum_{l=1}^{q}\frac{\log\left(\lambda_{j_l}\right)}{\log\left(\lambda_b\right)}\right|<\frac{\log\left(\lambda_{\min}\right)}{\log\left(\lambda_b\right)}$$

$$\Leftrightarrow \left|\sum_{k=1}^{p}b_{i_k}-\sum_{l=1}^{q}b_{j_l}\right|<b_{max}$$

$$\Leftrightarrow \left|\left(b_{i_1}+\cdots+b_{i_p}\right)-\left(b_{j_1}+\cdots+b_{j_q}\right)\right|<b_{max}$$

$$\Leftrightarrow \left|\widetilde{\xi}_b\left(\boldsymbol{i}\right)-\widetilde{\xi}_b\left(\boldsymbol{j}\right)\right|<b_{max}$$

Therefore it has been shown that for any $\boldsymbol{i},\boldsymbol{j}\in S^*$ then $\left|\widetilde{\xi}_a\left(\boldsymbol{i}\right)-\widetilde{\xi}_a\left(\boldsymbol{j}\right)\right|<a_{max}$ if and only if $\left|\widetilde{\xi}_b\left(\boldsymbol{i}\right)-\widetilde{\xi}_b\left(\boldsymbol{j}\right)\right|<b_{max}$. This implies that if $f_{\boldsymbol{i}}^{-1}f_{\boldsymbol{j}}$ is a proper neighbor map when the scaling factors are written in terms a particular base scaling then it will be a proper neighbor map for any scaling factor base. Therefore the set of proper neighbor maps is independent of the chosen base. $\qquad\square$

If we were to take any of the previous examples that involved calculating the set of proper neighbor maps and the neighbor graph then we would only have $a_{max}=1$ in the case of uniform scaling factors. For the situation where the algebraic condition was satisfied with non-uniform scaling factors we had $a_{max}\neq 1$. Therefore, Theorem 4.13 gives that the set of proper neighbor maps and the resulting neighbor graph are the same as if the algebraic condition for integer powers had never been solved and λ_{min} was chosen as the scaling factor base so that $a_{max}=1$. Therefore, the extension of ξ to $\widetilde{\xi}$ so that $\mathcal{N}^*$ could be defined for arbitrary scaling factors is consistent with previously determined results. We now note that Theorem 3.1 generalises to the case of arbitrary IFSs.

Theorem 4.14. *If $\mathcal{F}=\{\mathbb{R}^m;f_1,\ldots,f_N\}$ is an IFS of contractive similitudes with the set of neighbor maps, $\mathcal{N}$, given by Equation 3.2 and the set of proper neighbor maps, $\mathcal{N}^*$, given by Definition 4.11, then $\mathrm{id}\notin\overline{\mathcal{N}}$ if and only if $\mathrm{id}\notin\overline{\mathcal{N}^*}$.*

Proof. The proof of this theorem is a straightforward generalisation of the proofs supplied for Theorem 3.5 and then for Theorem 4.3, in fact the proof of the forward direction is identical. For the reverse direction let us note that given an IFS of similitudes the scaling factors λ_i can be written in base $\lambda=\min_i\{\lambda_i\}$ by $\lambda_i=\lambda^{a_i}$ for $a_i\in\mathbb{R}$ and $a_{max}=1$. The non-proper neighbor map $h=f_{\boldsymbol{i}}^{-1}f_{\boldsymbol{j}}$

must either satisfy $A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}} = \emptyset$ or $\left| \widetilde{\xi}(\boldsymbol{i}) - \widetilde{\xi}(\boldsymbol{j}) \right| \geq 1$. We have that h is of the form $h(x) = \lambda^k O x + b$ for $k \in \mathbb{R}$, O is a rotation reflection isometry and $b \in \mathbb{R}^m$ is a translation vector. Assuming $\|b\| = 0$ implies $k \in \mathbb{R} \setminus (-1, 1)$ and so we get $d_s(h, id) \geq 1 - \lambda$. In the case of $\|b\| > 0$ the proof follows as for Theorem 3.5 and so we get that $d_s(h, id) \geq 1 - \lambda$ and therefore any non-proper neighbor map h cannot be arbitrarily close to the identity so we have that $id \notin \overline{\mathcal{N}^*}$ implies $id \notin \overline{\mathcal{N}}$. $\qquad\square$

Remark 4.15. In Definition 4.11, we could replace a_{max} with $\frac{a_{max}}{n}$ for any finite $n \in \mathbb{N}$. This would define a subset of the proper neighbor maps which we would denote $\mathcal{N}^\dagger$. By a slight variation of the proof for Theorem 4.14 we would find that $h \in \mathcal{N} \setminus \mathcal{N}^\dagger$ could not be arbitrarily close to the identity map and therefore the IFS satisfies the OSC if and only if $id \notin \overline{\mathcal{N}^\dagger}$. As for the algebraic condition case, we do not call $\mathcal{N}^\dagger$ the set of proper neighbor maps because it does not describe the way sub-attractor pieces relate. The relationship between $\mathcal{N}^\dagger$ and neighbor graphs would be an exciting area for further research into this topic.

Applying Theorem 4.14 to Theorem 3.1 we get the following result.

Corollary 4.16. *If $\mathcal{F}$ is an IFS of similitudes then $\mathcal{F}$ obeys the OSC if and only if $id \notin \overline{\mathcal{N}^*}$.*

The generalisation of this result from the case of uniform scaling factors to the algebraic condition case and then to the case of arbitrary scaling factors allows us to only consider proper neighbor maps when determining if an IFS satisfies the OSC. This does not complete the research in the field since it is unknown to us if the extended definitions of proper neighbor maps generate neighbor graphs that describe the topology of the fractal attractor in the same way as Bandt and Mesing [BM09] initial definition does for IFSs with uniform scaling factors. Another area for future research is proper neighbor maps for IFSs comprising general affine maps (not similitudes) as no mention of these could be found in the literature.

In Section 3.1, Corollary 3.4 states that $\mathcal{F}$ satisfies the OSC if and only if A is not contained in $\overline{H}$ where $H = \bigcup \{ h(A) \mid h \in \mathcal{N} \}$ is the union of all neighbor sets. Considering the work that we have just done, we get the following conjecture.

Conjecture 2. *$\mathcal{F}$ satisfies the OSC if and only if A is not contained in $\overline{H^*}$, where $H^* = \bigcup \{ h(A) \mid h \in \mathcal{N}^* \}$.*

Since $\mathcal{N}^* \subset \mathcal{N}$ then $H^* \subset H$. Therefore, if $\mathcal{F}$ satisfies the OSC then $A \not\subset \overline{H}$ by Corollary 3.4 which implies $A \not\subset \overline{H^*}$, and so the forward direction is true. However, the reverse direction cannot yet be proven because we would be making the same mistake as M. Morán as discussed in Section 2.4. Note that we tried and could not construct a counterexample to disprove the reverse direction and so it has been labelled a conjecture.
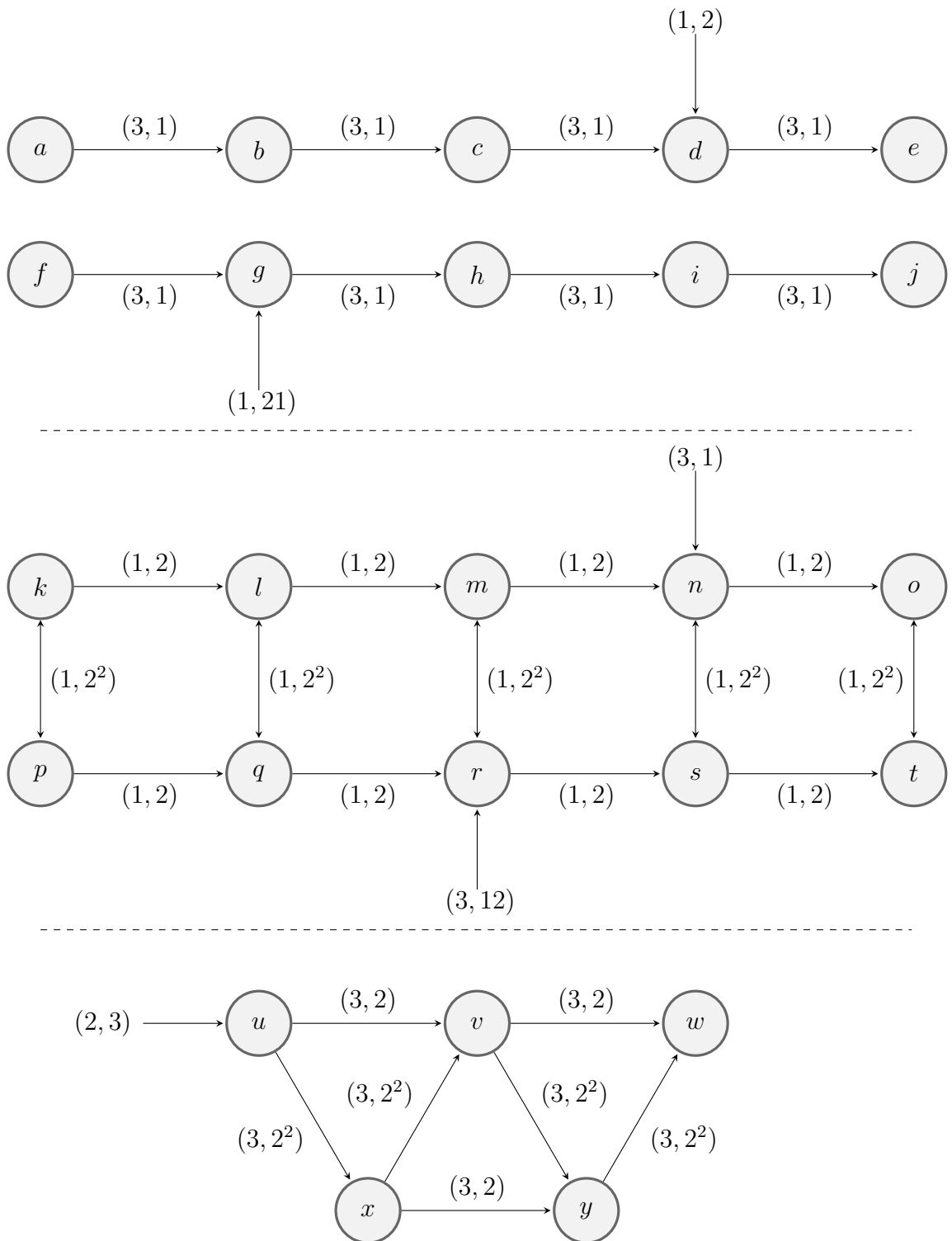
Figure 4.10: The neighbor graph of the Steemson triangle ($\triangle FNN$) with scaling factors $(\alpha, \beta, \gamma) = (\lambda^2, \lambda, \lambda^3)$ for $\lambda \approx 0.68$. For the side lengths we have $b \approx 0.68$ and a is a free variable so its value does not affect the neighbor graph.
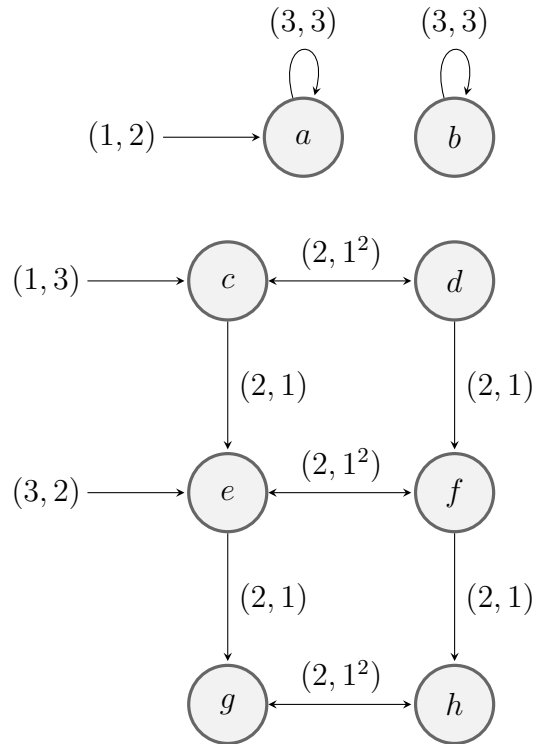
Figure 4.11: The neighbor graph of the Williams triangle ($\triangle FFN$) with scaling factors $(\alpha, \beta, \gamma) = (\lambda, \lambda^2, \lambda^2)$.
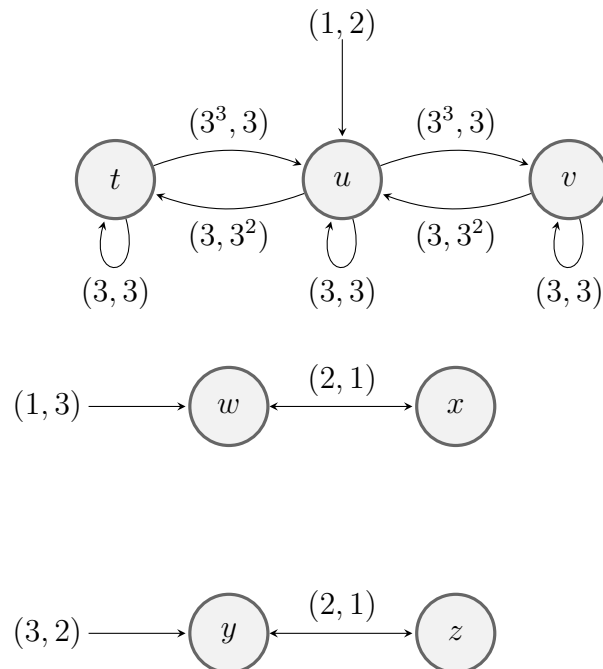

Figure 4.12: The neighbor graph of the Williams triangle ($\triangle FFN$) with scaling factors $(\alpha, \beta, \gamma) = (\lambda^2, \lambda^2, \lambda)$.
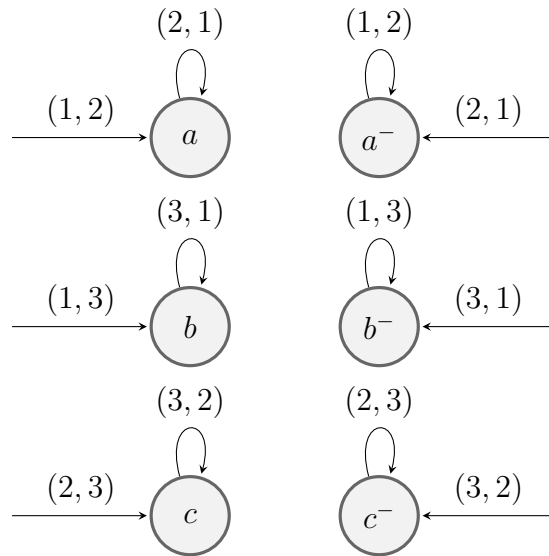
Figure 4.13: The neighbor graph of the equilateral Sierpinski triangle ($\triangle NNN$).
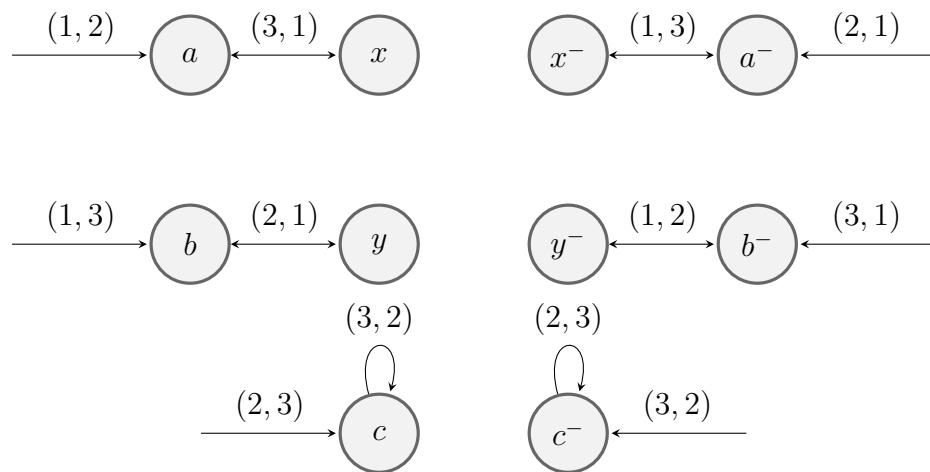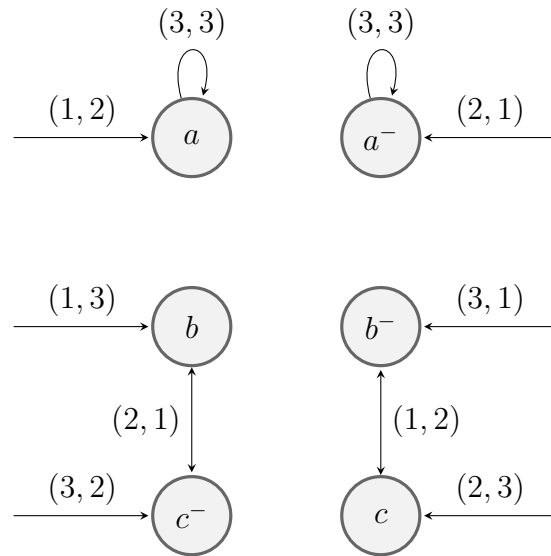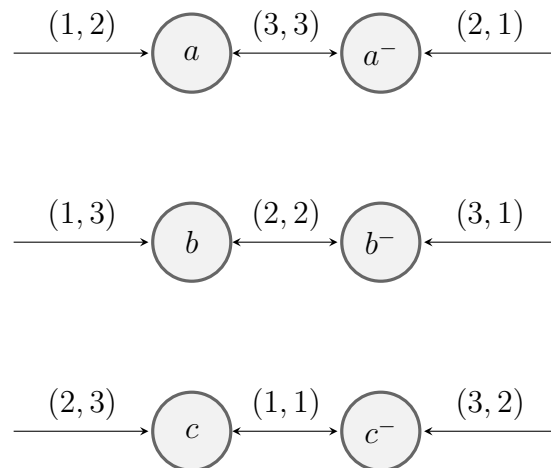

Figure 4.14: The neighbor graph of the equilateral Steemson triangle ($\triangle FNN$).

Figure 4.15: The neighbor graph of the equilateral Williams triangle ($\triangle FFN$).

Figure 4.16: The neighbor graph of the equilateral Pedal triangle ($\triangle FFF$).

Chapter 5

Finite Type Neighbor Graphs

The neighbor graphs considered in this thesis thus far have all been of finite type. Recall that this means that the number of vertices in the neighbor graph or equivalently the set of proper neighbor maps is finite. We are interested in determining if an IFS has a finite type neighbor graph because it can make it easier to determine if the identity map is in the closure of the set of proper neighbor maps and therefore easier to determine if the IFS satisfies the OSC.

The following chapter provides original results regarding when the neighbor graph of the generalised Sierpinski triangles is of finite type and offers a brief discussion about the connection between neighbor maps and fractal tiling theory.

5.1   Discussion of Finite Type Neighbor Graphs

For the generalised Sierpinski triangles, we have that if the IFS satisfies the algebraic condition (or more simply the scaling factors are uniform) then the neighbor graph is of finite type.

Theorem 5.1. *If a generalised Sierpinski triangle has scaling factors that can be written as $\lambda_i = \lambda^{a_i}$ with $a_i \in \mathbb{N}$ for $i = \{1, 2, 3\}$, then the associated neighbor graph is of finite type.*

Proof. First note that this theorem and its converse are trivially true for the generalised Sierpinski ($\triangle NNN$) triangle since it has all scaling factors equal to $\frac{1}{2}$ and has a finite type neighbor graph as calculated in Section 4.1.

Let us assume that the scaling ratios of the generalised Sierpinski triangle satisfy the algebraic condition, that is, $(\alpha, \beta, \gamma) = (\lambda^{a_1}, \lambda^{a_2}, \lambda^{a_3})$ for $0 < \lambda < 1$, $a_i \in \mathbb{N}$

and $a_{max} = \max_i \{a_i\}$. We define the set N_{ij} to be the set of neighbor maps that occur at the intersection between A_i and A_j for $i, j \in \{1, 2, 3\}$ and $i \neq j$. Explicitly that is,

$$N_{ij} = \left\{ h = f_x^{-n} f_i^{-1} f_j f_y^m, \ h^{-1} \ : \ x, y \in \{1, 2, 3\}, \ A_{ix} \cap A_{jy} \neq \emptyset, \ n, m \in \mathbb{N}_0 \right\}.$$

Let the scaling factors of f_x and f_y be λ^{a_x} and λ^{a_y} respectively with $a_x, a_y \in \{a_1, a_2, a_3\}$. Take $\hat{n} = 2a_y$ and $\hat{m} = 2a_x$. By Lemma 5.2 (see below) we have $f_x^{-\hat{n}} f_i^{-1} f_j f_y^{\hat{m}} = f_i^{-1} f_j$. We can now define a subset of N_{ij} that contains all the proper neighbor maps from the intersection $A_i \cap A_j$,

$$\widehat{N_{ij}} = \left\{ h, \ h^{-1} \in N_{ij} \ : \ 0 \leq n \leq (\hat{n} + a_{max}), \ |\xi(ix^n) - \xi(jy^m)| < a_{max} \right\}.$$

Note that we are taking $0 \leq n \leq (\hat{n} + a_{max})$ which does involve taking some proper neighbor maps twice but it guarantees that all proper neighbor maps are captured. To demonstrate this, consider if $n = 0$, $\xi(i) = 1$ and $a_{max} = 3$ then $\xi(ix^n) = \xi(i) = 1$ and so $\xi(jy^m) \in \{-1, 0, 1, 2, 3\}$ by the ξ condition but it cannot achieve all these values since $\xi(\boldsymbol{k})$ cannot be negative for any $\boldsymbol{k}$. Therefore, by increasing the maximum value of n we ensure that all proper neighbor maps due to the intersection $A_i \cap A_j$ are included in $\widehat{N_{ij}}$. Taking the union across the various intersection points we collect all proper neighbor maps and therefore,

$$\mathcal{N}^* \subset \left(\widehat{N_{12}} \cup \widehat{N_{13}} \cup \widehat{N_{23}} \right).$$

By construction of $\widehat{N_{ij}}$ we have that $(\hat{n} + a_{max})$ is finite and the number of allowable m values for each n is also finite due to the ξ restriction. Thus $\widehat{N_{ij}}$ is a finite set for each intersection which implies $\left(\widehat{N_{12}} \cup \widehat{N_{13}} \cup \widehat{N_{23}} \right)$ is finite. Therefore $\mathcal{N}^*$ is a finite set which is equivalent to saying that the neighbor graph has a finite number of vertices or that the neighbor graph is of finite type. $\qquad \square$

We will now prove the following lemma as it was invoked in the above proof.

Lemma 5.2. *If the generalised Sierpinski triangles have scaling factors $\lambda_i = \lambda^{a_i}$ for fixed $0 < \lambda < 1$ and $a_i \in \mathbb{N}$ then we have that $f_x^{-2a_y} f_i^{-1} f_j f_y^{2a_x} = f_i^{-1} f_j$ for $i, j, x, y \in \{1, 2, 3\}$, $i \neq j$ and $A_{ix} \cap A_{jy} \neq \emptyset$.*

Proof. We will provide a geometric argument for why this is true; however, it has also been checked computationally for each generalised Sierpinski triangle at each intersection point. For a flip map g with scaling factor κ we have that g^2 is a non-flip map with scaling factor κ^2 because the flipping component within the

map is its own inverse. Thus, f_x^2 and f_y^2 are non-flip maps with scaling factors λ^{2a_x} and λ^{2a_y} respectively, and translations such that $A_{ix^2} \cap A_{jy^2} = A_i \cap A_j$. As $a_x, a_y \in \mathbb{N}$ we can take the maps $(f_x^2)^{a_y}$ and $(f_y^2)^{a_x}$ which both have a scaling factor of $\lambda^{2a_x a_y}$ and translations such that $A_{ix^{2a_y}} \cap A_{jy^{2a_x}} = A_i \cap A_j$. Therefore we have that A_i and A_j have the same geometric relationship as $A_{ix^{2a_y}}$ and $A_{jy^{2a_x}}$. This implies that the attractor A has the same geometric relationship to the neighbor set $f_x^{-2a_y} f_i^{-1} f_j f_y^{2a_x}(A)$ as it does to $f_i^{-1} f_j(A)$. Therefore the functions $f_x^{-2a_y} f_i^{-1} f_j f_y^{2a_x}$ and $f_i^{-1} f_j$ are equal. $\qquad \square$

The converse of Theorem 5.1 is not always true. Before discussing this further, let us first note that the proper neighbor maps for each type of the generalised Sierpinski triangles take the following form depending on which point of intersection is of interest. We require that $n, m \in \mathbb{N}$ and that they satisfy the ξ condition.

Steemson	Williams	Pedal
$(1,2) \rightarrow_{(3,1)} (3^n 1, 21^m)$	$(1,2) \rightarrow_{(3,3)} (3^n 1, 23^m)$	$(1,2) \rightarrow_{(3,3)} (3^n 1, 23^m)$
$(1,3) \rightarrow_{(2,1)} (2^n 1, 31^m)$	$(1,3) \rightarrow_{(2,1)} (2^n 1, 31^m)$	$(1,3) \rightarrow_{(2,2)} (2^n 1, 32^m)$
$(2,3) \rightarrow_{(3,2)} (3^n 2, 32^m)$	$(2,3) \rightarrow_{(1,2)} (1^n 2, 32^m)$	$(2,3) \rightarrow_{(1,1)} (1^n 2, 31^m)$

The notation $(1,2) \rightarrow_{(3,1)} (3^n 1, 21^m)$ is being used to represent that $f_1^{-1} f_2$ is a proper neighbor map and all other proper neighbor maps due to the intersection $A_1 \cap A_2$ can be achieved by adding 3's on the left and 1's on the right to give $f_3^{-n} f_1^{-1} f_2 f_1^m$ with the ξ condition is satisfied.

For completeness note that the generalised Sierpinski has proper neighbor maps of the form $(i, j) \rightarrow_{(j,i)} (j^n i, j i^n)$ for $i, j \in \{1, 2, 3\}$ and $i \neq j$.

All proper neighbor maps are of the form $(x^n i, j y^m)$ with $i, j, x, y \in \{1, 2, 3\}$ and $i \neq j$. Note that we also have $i \neq x$ and $j \neq y$ due to the fractal nature of the generalised Sierpinski triangles. Interestingly, we can partition the set of proper neighbor maps by separating those such that $x \neq y$ from those with $x = y$. We find that it is proper neighbor maps with $x = y$ that prevent the converse of Theorem 5.1 from being true. Note that we always have $x \neq y$ for the Steemson triangle and so we can prove the converse.

Theorem 5.3. *If the scaling factors of the Steemson triangle can be written as $\lambda_i = \lambda^{a_i}$ with $a_i \in \mathbb{N}$ for $i = \{1, 2, 3\}$ if and only if its neighbor graph is of finite type.*

Proof. Due to Theorem 5.1 we are only required to prove the reverse direction. Let us assume that the neighbor graph is of finite type and so $\mathcal{N}^*$ is finite. Recall that in the general case, the set of proper neighbor maps is defined as:

$$\mathcal{N}^* = \left\{ h = f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}}, \mid \boldsymbol{i}, \boldsymbol{j} \in S^*, \ i_1 \neq j_1, \ A_{\boldsymbol{i}} \cap A_{\boldsymbol{j}} \neq \emptyset, \left| \widetilde{\xi}(\boldsymbol{i}) - \widetilde{\xi}(\boldsymbol{j}) \right| < a_{max} \right\}.$$

Let us define the set of proper neighbor maps which are due to each point of intersection between A_i and A_j for $i, j \in \{1, 2, 3\}$ and $i \neq j$,

$$\mathcal{N}_{ij}^* = \left\{ h = f_x^{-n} f_i^{-1} f_j f_y^m, h^{-1} \in \mathcal{N}^* \mid x, y \in \{1, 2, 3\} \right\},$$

where x, y is given depending on i, j but we have that $x \neq y$ since this is the case for the Steemson triangle. As $\mathcal{N}_{ij}^*$ is a subset of $\mathcal{N}^*$ then we also have $A_{ix^n} \cap A_{jy^m} \neq \emptyset$ and $\left| \widetilde{\xi}(ix^n) - \widetilde{\xi}(jy^m) \right| < a_{max}$. By construction we have that $\mathcal{N}^* = \mathcal{N}_{12}^* \cup \mathcal{N}_{13}^* \cup \mathcal{N}_{23}^*$ and it was assumed that $\mathcal{N}^*$ is a finite set and thus so is $\mathcal{N}_{ij}^*$ for each pair i, j. Since $\mathcal{N}_{ij}^*$ is finite then there exists a maximum value for n and m that define the maps $f_x^{-n} f_i^{-1} f_j f_y^m \in \mathcal{N}_{ij}^*$. Let $\bar{n}$ and $\bar{m}$ be integers which are larger than the maximum values of n and m respectively in $\mathcal{N}_{ij}^*$ and also satisfy the $\widetilde{\xi}$ condition. Therefore $f_x^{-\bar{n}} f_i^{-1} f_j f_y^{\bar{m}}$ is a proper neighbor map which is not contained in $\mathcal{N}_{ij}^*$ so it must be equal to another proper neighbor map in $\mathcal{N}_{ij}^*$. Thus for some $k, l \in \mathbb{N}_0$ we have, $f_x^{-\bar{n}} f_i^{-1} f_j f_y^{\bar{m}} = f_x^{-k} f_i^{-1} f_j f_y^l$. Applying f_x to the left and f_y^{-1} to the right does not change this equality so for $p = \bar{n} - k$ and $q = \bar{m} - l$ we have,

$$f_i^{-1} f_j = f_x^{-p} f_i^{-1} f_j f_y^q.$$

From this we see that the scaling ratio of f_x^p must equal the scaling ratio of f_y^q. Thus, $\lambda_x^p = \lambda_y^q$ and so $(\lambda^{a_x})^p = (\lambda^{a_y})^q$ which gives $a_x p = a_y q$. Equivalently we have $q = \frac{a_x}{a_y} p$ and as $p, q \in \mathbb{N}$ then we must have $a_x, a_y \in \mathbb{Q}$. Note that we have,

$$\frac{a_x}{a_y} = \frac{\log (\lambda_x) / \log(\lambda)}{\log(\lambda_y) / \log(\lambda)} = \frac{\log(\lambda_x)}{\log(\lambda_y)} = \log_{\lambda_y} (\lambda_x).$$

Which is equivalent to $\lambda_x = \lambda_y^{\frac{a_x}{a_y}}$. Therefore λ_x is a rational power of λ_y. By Lemma 5.4 (see below) we have that λ_x and λ_y can be written as integer powers of some common scaling ratio. This is true for all pairs $i, j \in \{1, 2, 3\}$ with $i \neq j$. Therefore from $(i, j) = (1, 2)$ we have $\lambda_1 = r_1^{b_1}, \lambda_2 = r_1^{b_2}$ and from $(i, j) = (1, 3)$ we have $\lambda_1 = r_2^{c_1}, \lambda_3 = r_2^{c_2}$ for $0 < r_1, r_2 < 1$ and $b_1, b_2, c_1, c_2 \in \mathbb{Z}_+$. Note that we have two expressions for λ_1 which gives $r_1^{b_1} = r_2^{c_1}$ and so $r_2 = r_1^{b_1/c_1}$. Therefore $\lambda_3 = \left(r_1^{b_1/c_1} \right)^{c_2}$ and so all scaling factors are now written in terms of the same

base. Let us define a new base scaling factor $0 < \lambda < 1$ such that $\lambda = r_1^{\frac{1}{c_1}}$ then $\lambda_1 = r_1^{b_1} = r_1^{\frac{b_1 c_1}{c_1}} = \lambda^{b_1 c_1}$, $\lambda_2 = r_1^{b_2} = r_1^{\frac{b_2 c_1}{c_1}} = \lambda^{b_2 c_1}$ and $\lambda_3 = r_1^{\frac{b_1 c_2}{c_1}} = \lambda^{b_1 c_2}$. Therefore we have expressed all scaling factors λ_i in terms of integer powers of a common scaling factor λ and thus the algebraic condition is satisfied. $\qquad\square$

Lemma 5.4. *Let $\lambda_1, \lambda_2 \in \mathbb{R}$ be such that $\lambda_2 = \lambda_1^k$ for $k \in \mathbb{Q}$. Then there exists $\lambda \in \mathbb{R}$ and $n_1, n_2 \in \mathbb{Z}$ such that $\lambda_1 = \lambda^{n_1}$ and $\lambda_2 = \lambda^{n_2}$.*

Proof. As $k \in \mathbb{Q}$ then it can be expressed as a reduced fraction $k = \frac{n_2}{n_1}$ where $n_1, n_2 \in \mathbb{Z}$ and $gcd(n_1, n_2) = 1$. Let us define the common factor to be the real number $\lambda = \lambda_1^{\frac{1}{n_1}}$. Then $\lambda_1 = \lambda_1^{\frac{n_1}{n_1}} = \lambda^{n_1}$ and $\lambda_2 = \lambda_1^k = \lambda_1^{\frac{n_2}{n_1}} = \lambda^{n_2}$. Therefore we have that λ_1 and λ_2 can be written as integer powers of a common base. $\qquad\square$

Theorem 5.3 is only true for the Steemson triangle (and the Sierpinski triangle) because we have $f_x \neq f_y$ but this is not the case for the Williams or Pedal triangles. As we can have $f_x = f_y$ then there are situations where the neighbor graph being of finite type does not imply that the algebraic condition is satisfied. Below, we provide an example for the Pedal triangle, and examples also exist for the Williams triangle using the same construction method.

Example 5.5. Let us consider the Pedal triangle ($\triangle FFF$) with scaling factors $(\alpha, \beta, \gamma) = \left(\frac{3}{4}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}} \right)$ displayed in Figure 5.1. First note that it is not possible to write these scaling factors as integer powers of a common scaling factor λ since α includes a factor of 3 while β and γ only include factors of 2. Thus, we are calculating the set of proper neighbor maps when the algebraic condition is not satisfied. As per the proposed convention we still write the scaling factors in terms of a common base $\lambda = \min \left\{ \frac{3}{4}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}} \right\} = \frac{1}{2\sqrt{2}}$. For $k = \frac{\log\left(\frac{4}{3}\right)}{\log(2\sqrt{2})} \approx 0.28$ we have $(\alpha, \beta, \gamma) = \left(\lambda^k, \lambda, \lambda \right)$, where k is irrational. The set of symbolic neighbor pairs for this Pedal triangle is $P = \{(\boldsymbol{i}, \boldsymbol{j}) \,|\, \boldsymbol{i}, \boldsymbol{j} \in \{1, 1^2, 1^3, 2, 3\}\}$ and we have $a_{max} = 1$. As all three maps of the Pedal triangle are flip maps we have that the proper neighbor maps will be of the forms $(3^n 1, 23^m)$, $(2^n 1, 32^m)$ and $(1^n 2, 31^m)$ where $n, m \in \mathbb{N}$ such that the $\widetilde{\xi}$ condition is satisfied. Note that we are omitting discussion of the inverse of these maps due to symmetry. The proper neighbor maps due to the intersection $A_1 \cap A_2$ are either of the form $(3^n 1, 23^n)$ or $(3^{n+1} 1, 23^n)$. We see this because we start with the map $(1, 2)$ which comes from the identity and the only symbolic neighbor map that we can apply to it is $(3, 3)$ which we can do repeatedly. Then note that $(21, 3)$ also satisfies the $\widetilde{\xi}$ condition which again can only be operated on by $(3, 3)$. Finally we note that $(3^2 1, 23^2) = (1, 2)$
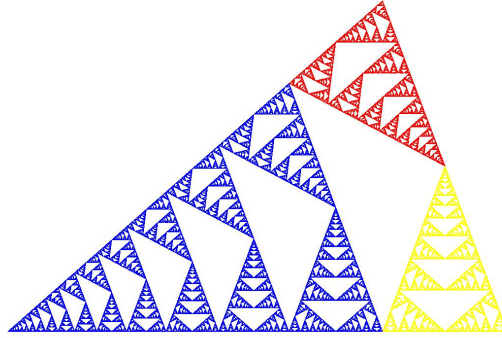
Figure 5.1:   The Pedal triangle $(\triangle FFF)$ with scaling factors $(\alpha, \beta, \gamma) = \left(\frac{3}{4}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$ and side lengths $(a, b) = \left(\frac{1}{\sqrt{2}}, 1\right)$.

which implies $(3^3 1, 23^2) = (31, 2)$. Therefore the only proper neighbor maps from $A_1 \cap A_2$ are $(1, 2), (31, 23)$ and $(31, 2), (3^2 1, 23)$, where each pair can map between each other using $(3, 3)$. Determining the set of proper neighbor maps due to the intersection $A_1 \cap A_3$ follows identical reasoning since $\widetilde{\xi}(2) = \widetilde{\xi}(3)$. Therefore the only proper neighbor maps are $(1, 3), (21, 32)$ and $(21, 3), (2^2 1, 32)$, where each pair can map between each other using $(2, 2)$. The neighbor graph for the intersections $A_1 \cap A_2$ and $A_1 \cap A_3$ both take the form shown in Figure 4.12 for the combined $(1, 3)$ and $(3, 2)$ section except we would only have one map from the identity. The neighbor graph for this example is not drawn though due to the complexity of the section of the neighbor graph that is associated to $A_2 \cap A_3$. Let us now determine the proper neighbor maps for the intersection $A_2 \cap A_3$ which are explicitly given in Figure 5.2. From the identity we get the map $(2, 3)$ and the set of symbolic neighbor maps that can be applied to this is $\left\{(1^a, 1^b)|a, b \in \{1, 2, 3\}\right\}$. We also note that $(1^2 2, 31^2) = (2, 3)$ and therefore we can list the set of proper neighbor maps for this intersection by looking at $(2, 31^k)$ and determining the k values that satisfy the $\widetilde{\xi}$ condition. Then looking at $(12, 31^k)$ for various k values and continue increasing the number of $1's$ on the left.

The first two columns of Figure 5.2 are complete because there does not exists any more non-negative k values for which the $\widetilde{\xi}$ condition is satisfied. For the remaining three columns we have that increasing the k value produces a map that has already been considered because of $(1^2 2, 31^2) = (2, 3)$ and continuing to increase k does not produce any new maps because of the $\widetilde{\xi}$ condition. Lastly, there is not a sixth column because $(1^5 2, 31^2)$ would be the first allowed map and

$\left(2, 31^k\right)$	$\left(12, 31^k\right)$	$\left(1^2 2, 31^k\right)$	$\left(1^3 2, 31^k\right)$	$\left(1^4 2, 31^k\right)$
$(2, 3)$	$(12, 3)$	$(1^2 2, 3)$	$(1^3 2, 3)$	
$(2, 31)$	$(12, 31)$	$(1^2 2, 31)$	$(1^3 2, 31)$	$(1^4 2, 31)$
$(2, 31^2)$	$(12, 31^2)$			
$(2, 31^3)$	$(12, 31^3)$			
	$(12, 31^4)$			

Figure 5.2: List of proper neighbor maps for the Pedal triangle with scaling factors $\left(\frac{3}{4}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$ for the intersection $A_2 \cap A_3$.

this simplifies to $(1^3 2, 3)$ which has already been considered. Therefore we have that this is a complete list of the proper neighbor maps from the intersection $A_2 \cap A_3$. A simple piece of code was written to explicitly calculate the identified proper neighbor maps and their inverses to see if we have double counted any maps but this was not the case. Between the 14 proper neighbor maps associated to $A_2 \cap A_3$ there is 48 directed edges. This is why the neighbor graph has not been drawn. Therefore, the set of proper neighbor maps has been determined to have exactly 44 $(2 \times (4 + 4 + 14))$ maps, and so the neighbor graph has 44 vertices. Thus, we have produced an example were the algebraic condition is not satisfied but the neighbor graph is still of finite type.

5.2 Fractal Tiling

An application of neighbor maps is that they can be applied to fractal tiling theory in order to explain how different tiles relate to each other. Barnsley and Vince [BV14, BV17b, BV18] provide an in-depth explanation and the technicalities of fractal tiling theory. However, for our purposes, a conceptual understanding of fractal tiling in $\mathbb{R}^2$ will suffice.

Let $\mathcal{F} = \{\mathbb{R}^2; f_1, \ldots f_N\}$ be an IFS with attractor A. By definition we have $A = \bigcup_{i=1}^N f_i(A)$ so when we 'zoom in' on the attractor we will 'see' smaller copies of it. If the IFS is comprised only of similitudes then zooming in reveals smaller copies of the attractor which have the same geometric proportions. The reverse of this concept creates what we call fractal blow ups and it is the fractal blow up that we tile. A tile is a nonempty compact subset of $\mathbb{R}^2$ and therefore all attractors are themselves tiles. A tiling of $\mathbb{R}^2$ is a union of tiles which all have equal Hausdorff dimension and are non-overlapping [BV14]. Two tiles are non-
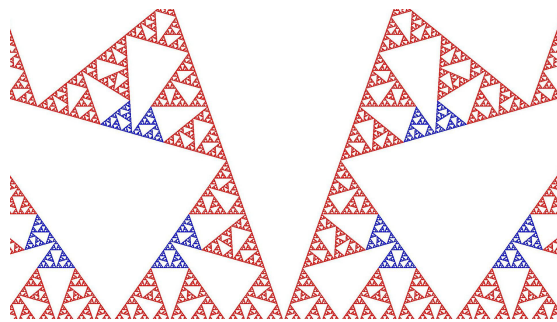
Figure 5.3: A fractal blow up of the Pedal triangle ($\triangle FFF$) with scaling factors $(\alpha, \beta, \gamma) = (\lambda, \lambda, \lambda^2)$. This image was created by zooming in at the point $(0.5, 0)$ [SW18].

overlapping if their intersection does not contain a nonempty open set. In the case where the tiles are fractal attractors, we can equivalently require the IFS that generated the fractal attractor to obey the OSC [BV14].

The following is a conceptual but not precise explanation of fractal blow ups and how they can be tiled. For a complete picture see [BV14, BV17b, BV18]. Consider the fractal attractor $A \subset \mathbb{R}^2$ of a similitude IFS. Pick a point $x \in A$ and zoom in on that point 'infinitely' so that when you stop zooming the fractal attractor extends in all directions further than the eye can see. Now, forget that you ever zoomed in and the coordinates of $x \in A$. In the region just near you, there will be many sections of the fractal blow up that look similar to the original fractal attractor. Find a section that has the same geometry as the original fractal and choose coordinates such that this is of size 'one'. Now, if you look around in the vast expanse surrounding you there will be infinitely many copies of the (new) attractor which have the same geometry but of varying sizes. Therefore the space can be tiled by different sized isometric copies, that is, copies that undergo an isometry and then a uniform scaling. Note that tiling the space by scaled isometric copies can always be done [BV14]. The prototile set $\mathcal{P}$ is a minimal set of tiles such that the space is tiled and that every tile involved in the tiling is an isometric copy of a tile in $\mathcal{P}$ [BV17b]. The question we are interested in is how many different sized tiles would you need to tile the space. Note that 'the space' is not the ambient space $\mathbb{R}^2$ that the attractor lives in but rather the infinite fractal blow up that is a subset of $\mathbb{R}^2$. The tiling of fractal blow ups links directly to Bandt's neighbor maps because the transformation from one tile to another is described by a neighbor map and the tiles are neighbor sets. More so,

when two tiles intersect the transformation from one to the other is given by a proper neighbor map. For the prototile set we have the following from Barnsley and Vince [BV17b] (c.f. Result 4).

Theorem 5.6. *If an IFS consisting of contractive similitudes has scaling factors that obey the algebraic condition, $\lambda_i = \lambda^{a_i}$ for $a_i \in \mathbb{N}$ and are coprime, then the prototile set is $\{\lambda A, \lambda^2 A, \dots, \lambda^{a_{max}} A\}$ [BV17b].*

All IFSs for which the neighbor graph has been considered in this paper, except for Example 5.5, obey the algebraic condition and therefore must have finite prototile sets. Therefore, let us use the neighbor maps and neighbor graphs for the previously considered examples to discuss their prototile sets.

First, let us consider the IFS $\mathcal{F} = \left\{ \mathbb{R}; f_1(x) = \frac{1}{2}x, f_2(x) = \frac{1}{2}x + \frac{1}{2} \right\}$ from Example 3.12 with attractor $A = [0, 1]$. The set of proper neighbor maps and the neighbor graph consisted of only two maps, namely $a = f_1^{-1} f_2$ and $a^- = f_2^{-1} f_1$. Both of these maps have a scaling factor of one and therefore the neighbor sets $f_1^{-1} f_2(A)$ and $f_2^{-1} f_1(A)$ are the same size as the attractor. Therefore the prototile set consists of only one tile, namely $\mathcal{P} = \{\lambda A\} = \left\{ \left[0, \frac{1}{2}\right] \right\}$. More generally, we have that for an IFS comprising similitudes with uniform scaling factors the proper neighbor maps take the form $h = f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}} \in \mathcal{N}^*$ with $|\boldsymbol{i}| = |\boldsymbol{j}|$. Therefore the scaling factor of h must be 1 which means that all neighbor sets are of equal size to the attractor and so the prototile set consists of only one tile which is again $\mathcal{P} = \{\lambda A\}$. This is not a new result as it is a special case of Theorem 5.6 but it is interesting that we could make the same observation using proper neighbor maps. The equilateral and generalised Sierpinski triangle ($\triangle NNN$) from Sections 3.3 and 4.1 respectively as well as the equilateral Steemson triangle ($\triangle FFN$), equilateral Williams triangle ($\triangle FFN$) and equilateral Pedal triangle ($\triangle FFN$) from Section 4.4 all have equal scaling factors and therefore have only one tile in their prototile set.

Now we look at the neighbor graph examples with non-uniform scaling factors. In Section 4.3 we calculated the neighbor graph for the Williams triangle ($\triangle FFN$) with scaling factors $(\alpha, \beta, \gamma) = (\lambda, \lambda^2, \lambda^2)$ and $(\lambda^2, \lambda^2, \lambda)$. The proper neighbor maps $h = f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}}$ for this situation must satisfy $|\xi(\boldsymbol{i}) - \xi(\boldsymbol{j})| < a_{max} = 2$. Therefore we have that $\xi(\boldsymbol{i}) - \xi(\boldsymbol{j}) \in \{-1, 0, 1\}$ are the allowed ξ condition values. Each of these situations correspond to proper neighbor maps of scaling λ^{-1}, $\lambda^0 = 1$ and λ. We can confirm that all three types of proper neighbor map scaling

factors are in $\mathcal{N}^*$ by computing the determinants of the proper neighbor maps, since $|det(L)|$ describes how the linear transformation L scales the m-dimension volume of a tile in $\mathbb{R}^m$. We do this computation and find that all three types of scaling are in $\mathcal{N}^*$. Therefore one might think that the size of the prototile set would be three but this should raise alarms bells since $a_{max} = 2$ for both Williams triangles being considered here and that would contradict Theorem 5.6. Instead let us view the situation by considering two tiles: t_a and t_b which intersect and so the transformation from one to the other is given by a proper neighbor map. Let h be the proper neighbor map such that $h(t_a) = t_b$. Then t_b is an isometric copy of t_a which has either been scaled by $\lambda^{-1}, 1$ or λ. However, note that scaling t_a by λ^{-1} or by λ produces the same situation with one little tile and one big tile and scaling t_a by 1 does not create a tile of a new size. Therefore we have that the prototile set has one big tile and one little tile. Therefore $\mathcal{P} = \{\lambda A, \lambda^2 A\}$ which agrees with Theorem 5.6.

More generally, for an IFS with scaling factors that satisfy the algebraic condition the proper neighbor maps $h = f_{\boldsymbol{i}}^{-1} f_{\boldsymbol{j}} \in \mathcal{N}^*$ must satisfy $|\xi(\boldsymbol{i}) - \xi(\boldsymbol{j})| < a_{max}$. By definition of $\xi : S^* \to \mathbb{N}_0$ we have that its output is a natural number and so the allowable values of $\xi(\boldsymbol{i}) - \xi(\boldsymbol{j})$ are in

$$\{-(a_{max} - 1), -(a_{max} - 2), \cdots, -1, 0, 1, \cdots, a_{max} - 2, a_{max} - 1\}.$$

Therefore the possible scaling factors of h are λ^b for b from the above set and so the maximum number of different scaling factors on proper neighbor maps is $2a_{max} - 1$. Now, using the symmetry between h_1 of scaling λ^b and h_2 of scaling λ^{-b} we find that the maximum size of a prototile set is a_{max} which again agrees with Theorem 5.6.

Now let us now consider the neighbor graph from Section 4.2 of the Steemson triangle ($\triangle FNN$) with scaling factors $(\lambda^2, \lambda, \lambda^3)$. As $a_{max} = 3$ we have that the maximum number of different scaling proper neighbor maps is five given by λ^b for $b \in \{-2, -1, 0, 1, 2\}$. By simply calculating the determinants for each of the proper neighbor maps we can confirm that all of the different scaling factors are present. Therefore by symmetry the prototile set has $a_{max} = 3$ tiles, namely, $\mathcal{P} = \{\lambda A, \lambda^2 A, \lambda^3 A\}$.

The last case to consider is Example 5.5 which was the Pedal triangle ($\triangle FFF$) with scaling factors $\left(\frac{3}{4}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$ which do not satisfy the algebraic condition.

Although the neighbor graph was never drawn we did calculate the set of proper neighbor maps and therefore using the absolute value of the determinants we can determine how many different scaling factors are used for $h \in \mathcal{N}^*$. We find that there is nine different scaling factors given by $\left\{1, s_1^{\pm 1}, s_2^{\pm 1}, s_3^{\pm 1}, s_4^{\pm 1}\right\}$ with $0 < s_i < 1$ being irrational numbers that are not integer powers of each other or a common scaling factor base. As the algebraic condition is not satisfied we do not know how many tiles are in the prototile set or their scaling factors. However, we claim that a finite number of different scaling for the proper neighbor maps implies a finite prototile set. This remains an open question and was not found in the literature and so would be an interesting area for future research.

For all the examples considered thus far we have that when the algebraic condition is satisfied there are $2a_{max} - 1$ different possible scaling factors for the proper neighbor maps and the prototile set is of size a_{max}. However, let us now construct an example to show that $h \in \mathcal{N}^*$ does not always achieve all possible scaling factors.

Example 5.7. Consider the IFS $\mathcal{F} = \left\{\mathbb{R}; f_1(x) = \delta^{10}x, \; f_2(x) = \delta^6 x + \delta^{10}, \right.$ $\left. f_3(x) = \delta^{15}x + \delta^{10} + \delta^6\right\}$ where $\delta^{10} + \delta^6 + \delta^{15} = 1$ and so $\delta \approx 0.89$. The attractor of $\mathcal{F}$ is $A = [0, 1]$ and the proper neighbor maps are either due to the intersection $A_1 \cap A_2$ or $A_2 \cap A_3$. Therefore, for $n, m, p, q \in \mathbb{N}_0$ such that the ξ condition is satisfied we have that the proper neighbor maps are given by:

$$f_3^{-n} f_1^{-1} f_2 f_1^m \qquad\qquad f_3^{-p} f_2^{-1} f_3 f_1^q$$

and the inverses of the above maps. The ξ condition for each of these maps is,

$$|\xi(13^n) - \xi(21^m)| = |\xi(1) + n\xi(3) - \xi(2) - m\xi(1)| = |10 + 15n - 6 - 10m|$$
$$= |15n - 10m + 4| \qquad \text{and}$$
$$|\xi(23^p) - \xi(31^q)| = |\xi(2) + p\xi(3) - \xi(3) - q\xi(1)| = |6 + 15p - 15 - 10q|$$
$$= |15p - 10q - 9|.$$
$$(5.1)$$

In order to gain a contradiction let us assume that there exits $n, m, p, q \in \mathbb{N}$ such the ξ conditions from Equation 5.1 equals 2. For the intersection $A_1 \cap A_2$ this gives $|15n - 10m + 4| = 2$ which implies $15n - 10m + 4 = \pm 2$. Taking the $\pm$ cases separately gives

$$\begin{array}{ll} 15n - 10m + 4 = 2 & \quad 15n - 10m + 4 = -2 \\ 15n = 10m - 2 & \quad 15n = 10m - 6. \end{array} \qquad (5.2)$$

For both these equations we see that $LHS \equiv 0 \mod 5$ and $RHS \not\equiv 0 \mod 5$, and so we have a contradiction. Therefore there does not exists $n, m \in \mathbb{N}$ such that the ξ condition is 2. Now for the intersection $A_2 \cap A_3$ the ξ condition is $|15p - 10q - 9| = 2$ which implies $15p - 10q - 9 = \pm 2$. Taking the $\pm$ cases separately gives,

$$
\begin{aligned}
15p - 10q - 9 &= 2 \\
15p &= 10q + 11
\end{aligned}
\quad \text{and} \quad
\begin{aligned}
15p - 10q - 9 &= -2 \\
15p &= 10q + 7.
\end{aligned}
\tag{5.3}
$$

For both these equations we see that $LHS \equiv 0 \mod 5$ and $RHS \not\equiv 0 \mod 5$, and so we have a contradiction. Therefore there does not exists $p, q \in \mathbb{N}$ such that the ξ condition is equal to 2. Therefore there does not exist any proper neighbor map with scaling $\delta^{\pm 2}$. This process can be repeated to check all possible cases $\{0, 1, \dots, 14\}$ as $a_{max} = 15$. We find that the possible scaling for the proper neighbor maps are $\delta^{\pm b}$ for $b \in \{1, 4, 6, 9, 11, 14\}$.

Therefore we have that the proper neighbor maps do not define all tile sizes. Note that we think the only way to construct examples where the proper neighbor maps do not achieve all possible scaling factors values is to have $gcd(a_i, a_j) \neq 1$ for every pair of scaling factors but $gcd(a_1, a_2, \dots, a_N) = 1$ so that the scaling factors are written in their simplest form.

In Chapter 3 and 4 we investigated the neighbor maps, the proper neighbor maps and the associated neighbor graph for IFSs of contractive similitudes. Here, in Chapter 5 we have offered a brief discussion about how neighbor map theory is related to the structure of a fractal blow up and its prototile set. This relationship is not fully understood and would be an interesting field of research to pursue, particularly with the new neighbor map theory results provided in this thesis.

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