

Ramsey-Turán numbers for semi-algebraic graphs

Jacob Fox*

Department of Mathematics
Stanford University
Stanford, CA, U.S.A.
jacobfox@stanford.edu

János Pach†

Chair of Combinatorial Geometry DCG
École Polytechnique Fédérale de Lausanne
Lausanne, Switzerland
pach@cims.nyu.edu

Andrew Suk‡

Department of Mathematics
University of California San Diego
La Jolla, CA, U.S.A.
asuk@ucsd.edu

Submitted: Jul 7, 2018; Accepted: Nov 26, 2018; Published: Dec 21, 2018

© The authors. Released under the CC BY-ND license (International 4.0).

Abstract

A *semi-algebraic graph* $G = (V, E)$ is a graph where the vertices are points in \mathbb{R}^d , and the edge set E is defined by a semi-algebraic relation of constant complexity on V . In this note, we establish the following Ramsey-Turán theorem: for every integer $p \geq 3$, every K_p -free semi-algebraic graph on n vertices with independence number $o(n)$ has at most $\frac{1}{2} \left(1 - \frac{1}{|p/2|-1} + o(1)\right) n^2$ edges. Here, the dependence on the complexity of the semi-algebraic relation is hidden in the $o(1)$ term. Moreover, we show that this bound is tight.

Mathematics Subject Classifications: 05D10, 52C10

1 Introduction

Over the past decade, several authors have shown that many classical theorems in extremal graph theory can be significantly improved if we restrict our attention to semi-algebraic graphs, that is, graphs whose vertices are points in Euclidean space, and edges are defined by a semi-algebraic relation of constant complexity [1, 5, 8, 11, 9, 4]. In this note, we

*Supported by a Packard Fellowship and by NSF CAREER award DMS 1352121.

†Supported by Swiss National Science Foundation Grants 200020-162884 and 200021-175977.

‡Supported an NSF CAREER award and an Alfred Sloan Fellowship.

continue this sequence of works by studying Ramsey-Turán numbers for semi-algebraic graphs.

More formally, a graph $G = (V, E)$ is a *semi-algebraic graph* with *complexity* at most t , if its vertex set V is an ordered set of points in \mathbb{R}^d , where $d \leq t$, and if there are at most t polynomials $g_1, \dots, g_s \in \mathbb{R}[x_1, \dots, x_{2d}]$, $s \leq t$, of degree at most t and a Boolean formula Φ such that for vertices $u, v \in V$ such that u comes before v in the ordering,

$$(u, v) \in E \quad \Leftrightarrow \quad \Phi(g_1(u, v) \geq 0; \dots; g_s(u, v) \geq 0) = 1.$$

At the evaluation of $g_\ell(u, v)$, we substitute the variables x_1, \dots, x_d with the coordinates of u , and the variables x_{d+1}, \dots, x_{2d} with the coordinates of v . Here, we assume that the complexity t is a fixed parameter, and $n = |V|$ tends to infinity.

The classical theorem of Turán gives the maximum number of edges in a K_p -free graph on n vertices.

Theorem 1 (Turán, [13]). *Let $G = (V, E)$ be a K_p -free graph with n vertices. Then*

$$|E| \leq \frac{1}{2} \left(1 - \frac{1}{p-1} + o(1) \right) n^2.$$

The only graph for which this bound is tight is the complete $(p-1)$ -partite graph whose parts are of size as equal as possible. This graph can easily be realized as an intersection graph of segments in the plane, which is a semi-algebraic graph with complexity at most four. Therefore, Turán's theorem cannot be improved by restricting it to semi-algebraic graphs.

Let H be a fixed graph. The *Ramsey-Turán number* $\mathbf{RT}(n, H, \alpha)$ is defined as the maximum number of edges that an n -vertex graph of independence number at most α can have without containing H as a (not necessarily induced) subgraph. Ramsey-Turán numbers were introduced by Andrásfai [2] and were motivated by the classical theorems of Ramsey and Turán and their connections to geometry, analysis, and number theory. According to one of the earliest results in Ramsey-Turán theory, which appeared in [7], for every $p \geq 2$, we have

$$\mathbf{RT}(n, K_{2p-1}, o(n)) = \frac{1}{2} \left(1 - \frac{1}{p-1} \right) n^2 + o(n^2). \quad (1)$$

For excluded K_4 , a celebrated result of Szemerédi [12] and Bollobás-Erdős [3] states that

$$\mathbf{RT}(n, K_4, o(n)) = \frac{1}{8} n^2 + o(n^2).$$

This was generalized by Erdős, Hajnal, Sós, and Szemerédi [6] to all cliques of even size. For every $p \geq 2$, we have

$$\mathbf{RT}(n, K_{2p}, o(n)) = \frac{1}{2} \cdot \frac{3p-5}{3p-2} n^2 + o(n^2). \quad (2)$$

For more results in Ramsey-Turán theory, consult the survey of Simonovits and Sós [10].

In the present note, we establish asymptotically tight bounds on Ramsey-Turán numbers for semi-algebraic graphs. We define $\mathbf{RT}_t(n, K_p, o(n))$ as the maximum number of edges that n -vertex K_p -free semi-algebraic graphs with complexity at most t can have, if their independence number is $o(n)$. Strictly speaking, this definition and all above results apply to *sequences* of graphs with n vertices, as n tends to infinity.

It turns out that if the size of the excluded clique is *even*, then the answer to the Ramsey-Turán question significantly changes when the graphs are required to be semi-algebraic. However, in the odd case, we obtain the same asymptotics for the Ramsey-Turán function as in (1). More precisely, we have

Theorem 2. *For any fixed integers $t \geq 5$ and $p \geq 2$, we have*

$$\mathbf{RT}_t(n, K_{2p-1}, o(n)) = \mathbf{RT}_t(n, K_{2p}, o(n)) = \frac{1}{2} \left(1 - \frac{1}{p-1} \right) n^2 + o(n^2).$$

2 Proof of Theorem 2

The aim of this section is to prove Theorem 2. One of the main tools used in the proof is the following regularity lemma for semi-algebraic graphs. Given a graph $G = (V, E)$, a vertex partition is called *equitable* if any two parts differ in size by at most one. Given two disjoint subsets $V_i, V_j \subset V$, we say that the pair (V_i, V_j) is *homogeneous* if $V_i \times V_j \subset E$ or $(V_i \times V_j) \cap E = \emptyset$.

Lemma 3 ([9]). *For any positive integer t , there exists a constant $c = c(t) > 0$ with the following property. Let $0 < \varepsilon < 1/2$ and let $G = (V, E)$ be a semi-algebraic graph with complexity at most t . Then V has an equitable partition $V = V_1 \cup \dots \cup V_K$ into K parts, where $1/\varepsilon < K < (1/\varepsilon)^c$, such that all but an ε -fraction of the pairs of parts are homogeneous.*

The upper bound in Theorem 2 follows from

Theorem 4. *Let $\varepsilon > 0$ and let $G = (V, E)$ be an n -vertex semi-algebraic graph with complexity at most t . If G is K_{2p} -free and $|E| > \frac{1}{2} \left(1 - \frac{1}{p-1} + \varepsilon \right) n^2$, then G has an independent set of size γn , where $\gamma = \gamma(t, p, \varepsilon)$.*

Proof. We apply Lemma 3 with parameter $\varepsilon/4$ to obtain an equitable partition $\mathcal{P} : V = V_1 \cup \dots \cup V_K$ such that $\frac{4}{\varepsilon} \leq K \leq \left(\frac{4}{\varepsilon}\right)^c$, where $c = c(t)$ and all but an at most $\frac{\varepsilon}{4}$ -fraction of all pairs of parts in \mathcal{P} are homogeneous (complete or empty with respect to E). If $n \leq 10K$, then G has an independent set of size one, and the theorem holds trivially. So, we may assume $n > 10K$.

By deleting all edges inside each part, we have deleted at most

$$K \binom{\lceil n/K \rceil}{2} \leq \frac{4n^2}{5K} \leq \varepsilon \frac{n^2}{5}$$

edges. Deleting all edges between non-homogeneous pairs of parts, we lose an additional at most

$$\left\lceil \frac{n}{K} \right\rceil^2 \frac{\varepsilon K^2}{4} \leq \varepsilon \frac{n^2}{5}$$

edges. In total, we have deleted at most $2\varepsilon n^2/5$ edges of G . The only edges that remain in G are edges between homogeneous pairs of parts, and we have at least $\frac{1}{2} \left(1 - \frac{1}{p-1} + \varepsilon/5\right) n^2$ edges. By Turán's theorem (Theorem 1), there is at least one remaining copy of K_p , and its vertices lie in p distinct parts $V_{i_1}, \dots, V_{i_p} \in \mathcal{P}$ that form a complete p -partite subgraph. If any of the parts V_{i_j} forms an independent set in G , then there is an independent set of order $|V_{i_j}| \geq \lfloor n/K \rfloor \geq \gamma n$, where $\gamma = \gamma(t, \varepsilon, p)$, and we are done. Otherwise, there is an edge in each of the p parts, and the endpoints of these p edges form a K_{2p} in G , a contradiction. \square

The lower bound on $\mathbf{RT}(n, K_{2p-1}, o(n))$ and $\mathbf{RT}(n, K_{2p}, o(n))$ in Theorem 2 is constructive and is based on the following result of Walczak.

Lemma 5 ([14]). *For any pair of positive integers n and p , where n is a multiple of $p-1$, there is a collection S of $n/(p-1)$ segments in the plane whose intersection graph G_S is triangle-free and has no independent set of size $c_p n / \log \log n$. Here c_p is a suitable constant.*

The construction. Take $p-1$ dilated copies of a set S meeting the requirements in Lemma 5, and label them as S_1, \dots, S_{p-1} , so that S_i lies inside a ball with center $(i, 0)$ and radius $1/10$. Set $V = S_1 \cup \dots \cup S_{p-1}$. Note that $|S_i| = n/(p-1)$ so that $|V| = n$. Let $G = (V, E)$ be the graph whose vertices are the elements of V , and two vertices (that is, two segments) are connected by an edge if and only if they cross or their left endpoints are at least $1/2$ apart. The graph G consists of a complete $(p-1)$ -partite graph, where each part induces a copy of the triangle-free graph G_S . Clearly, G is K_{2p-1} -free and does not contain any independent set of size $c_p n / \log \log n$. Moreover,

$$|E(G)| \geq \frac{1}{2} \left(1 - \frac{1}{p-1}\right) n^2.$$

Every segment can be represented by a point in \mathbb{R}^4 , and whether or not two segments intersect can be determined by four polynomial inequalities of degree at most two (see [1]). Thus, counting the distance condition, we have 5 quadratic inequalities, showing that E is a semi-algebraic relation of complexity 5.

References

- [1] N. Alon, J. Pach, R. Pinchasi, R. Radoičić, and M. Sharir, Crossing patterns of semi-algebraic sets, *J. Combin. Theory Ser. A* **111** (2005), 310–326.
- [2] B. Andrásfai, Über ein Extremalproblem der Graphentheorie, *Acta Math. Hungar.* **13** (1962) 443–455.

- [3] B. Bollobás and P. Erdős, On a Ramsey-Turán type problem, *J. Combin. Theory Ser. B* **21** (1976), 166–168.
- [4] B. Bukh, J. Matoušek, Erdős-Szekeres-type statements: Ramsey function and decidability in dimension 1, *Duke J. Math.* **163** (2014), 2243–2270.
- [5] D. Conlon, J. Fox, J. Pach, B. Sudakov, A. Suk, Ramsey-type results for semi-algebraic relations, *Trans. Amer. Math. Soc.* **366** (2014), 5043–5065.
- [6] P. Erdős, A. Hajnal, V. Sós, and E. Szemerédi, More results on Ramsey-Turán type problems, *Combinatorica* **3** (1983), 69–81.
- [7] P. Erdős and V. Sós, Some remarks on Ramsey’s and Turan’s theorem, *Coll. Math. Soc. J. Bolyai* **4**, *Comb. Theory and its Appl.*, North-Holland (1969), 395–404.
- [8] J. Fox, J. Pach, A. Sheffer, A. Suk, J. Zahl, A semi-algebraic version of Zarankiewicz’s problem, *J. Eur. Math. Soc.* **19** (2017), 1785–1810.
- [9] J. Fox, J. Pach, and A. Suk, A polynomial regularity lemma for semi-algebraic hypergraphs and its applications in geometry and property testing, *SIAM J. Comput.* **45** (2016), 2199–2223.
- [10] M. Simonovits and V.T. Sós, Ramsey-Turán theory, *Discrete Math.* **229** (2001), 293–340.
- [11] A. Suk, Semi-algebraic Ramsey numbers, *J. Combin. Theory Ser. B* **116** (2016), 465–483.
- [12] E. Szemerédi, On graphs containing no complete subgraph with 4 vertices (Hungarian), *Mat. Lapok* **23** (1972), 113–116.
- [13] P. Turán, On an extremal problem in graph theory, *Matematikai és Fizikai Lapok* (in Hungarian) **48** (1941), 436–452.
- [14] B. Walczak, Triangle-free geometric intersection graphs with no large independent sets, *Discrete Comput. Geom.* **53** (2015), 221–225.