Polynomial inequalities with asymmetric weights

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Abstract

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1 Introduction

Consider the space P_n of real algebraic polynomials of degree at most n. Let $K \subset \mathbb{R}$ be any compact set and $\|p\|_K := \sup_{\mathbf{x}\in K} |p(\mathbf{x})|$ the usual supremum norm

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on K. The classical Bernstein problem consists in estimating the derivative of the polynomial $p'(x)$ for a given $p \in P_n$, $\|p\|_K = 1$ and $\mathbf{x} \in \mathrm{Int}K$. Typically, this estimate is given in terms of the degree n of the polynomials and the distance of point $\mathbf{x} \in \mathrm{Int}K$ to the boundary ∂K of the compact K. This problem goes back to Bernstein [**?**] who showed that when $K = [a, b]$ we have the estimate

$$|p'(x)| \leq \frac{n}{\sqrt{(x-a)(b-x)}} \|p\|_{[a,b]}, \quad x \in (a, b). \tag{1}$$

This estimate is sharp, in general. It is attained at certain points by the Chebyshev polynomial.

The classical Markov inequality provides a uniform upper bound

$$\|p'\|_K \leq \frac{2n^2}{b-a} \|p\|_{[a,b]}, \quad p \in P_n \tag{2}$$

which also turns into equality for the Chebyshev polynomial.

Various extensions of the Bernstein and Markov type inequalities for more general domains, norms and in multivariate case have been widely investigated in the past decades. In this paper we will be concerned with this question in case of *weighted* uniform norm on the interval. In a recent paper [4] Mastroianni and Totik established a rather general weighted versions of (1) and (1) for the class of so called $A^\star$ weights. Let $A^\star$ denote the set of integrable weights $w \geq 0$ satisfying the inequality

$$w(x) \leq \frac{C}{|E|} \int_E w(t)\, dt \qquad \text{for all} \quad x \in E \subseteq I := [-1, 1]\,. \tag{3}$$

Then it is shown in [4], p. 69 that for any $w \in A^\star$ and $p \in P_n$

$$\|\phi w p'\|_I \leq cn\|p\|_I, \quad \|wp'\|_I \leq cn^2 \|wp\|_{[a,b]}, \tag{4}$$

where $\phi(x) := \sqrt{1-x^2}$ and the constants above depend only on w.

The above condition $A^\star$ imposed on the weights is rather general, in particular it includes all Jacobi type weights $\prod_j |x - x_j|^{\gamma_j}$ which allow the weight to vanish as a power of x. In a very recent paper [**?**] the authors extended (4) to a wider class of weights which may vanish exponentially. However, all above classes of weights require that the weight has certain symmetry, that is it vanishes to the left and to the right of the given point with equal speed. In this paper we initiate the study of the Bernstein-Markov type inequalities for the so called **asymmetric weights** which may vanish at a given point with different rates. A typical asymmetric weight is given by

$$w_{\alpha,\beta}(x) = \begin{cases} |x|^\alpha, & \text{if} \quad -1 \le x \le 0, \\ x^\beta, & \text{if} \quad 0 < x \le 1, \end{cases} \qquad 0 \le \alpha \le \beta\,. \tag{5}$$

First we show that this weight does not belong to $A^\star$ if $\alpha < \beta$. Let $0 < h < 1$ and $E = \left[-h^{\frac{\beta+1}{\alpha+1}}, h\right]$. Then

$$\frac{1}{|E|}\int_E w_{\alpha,\beta}(x)\,dx \le \frac{h^{\beta+1} + h^{\beta+1}}{h + h^{\frac{\beta+1}{\alpha+1}}} < 2h^\beta\,.$$

Thus if we had (3) with $x = -h^{\frac{\beta+1}{\alpha+1}}$, this would mean $h^{\alpha\frac{\beta+1}{\alpha+1}} \le 2Ch^\beta$, a contradiction for a small h, since $\dfrac{\beta+1}{\alpha+1} < \dfrac{\beta}{\alpha}$.

In this paper we will give some new Bernstein type inequalities for such asymmetric Jacoby type weights. In contrast to the estimates provided previously for the symmetric weights in the asymmetric case the resulting bounds for the derivatives of n-th degree polynomials are typically of order $n^\gamma, \gamma > 1$, see Section 3 below. First in Section 2 we will derive some Remez type estimates for asymmetric weights needed in the sequel. Section 3 contains our main new results on Bernstein type inequalities for asymmetric Jacoby type weights. We will also provide some converse estimates showing that the increase of the rate of derivatives in non symmetric case is in general unavoidable.

2 Some auxiliary Remez type estimates for asymmetric weights

Mastroianni and Totik [4] established a rather general weighted version of the classic Remez inequality for trigonometric polynomials which is valid for $A^\star$ weights $w \ge 0$. Namely, for any trigonometric polynomial t_n of degree at most n and any $w \in A^\star$ we have

$$\|wt_n\|_{[-\pi,\pi]} \le e^{Cn|E|}\|wt_n\|_{[-\pi,\pi]\setminus E} \qquad \text{for all} \qquad E \subset [-\pi,\pi], \tag{6}$$

where $C > 0$ is a constant depending only on w (see [4]). By a standard substitution this inequality yields a Remez inequality for algebraic polynomials $p_n \in \mathcal{P}_n$

$$\|wp_n\|_I \le e^{Cn|E|}\|wp_n\|_{I\setminus E} \qquad \text{for all} \qquad w \in A^\star, \qquad E \subset [-1/2, 1/2]. \quad (7)$$

We will need in the sequel a certain Remez type inequality for asymmetric Jacobi type weights. For any $n \in \mathbb{N}$ and $\gamma \ge 1$ let

$$I_n = [a_n, b_n], \quad \text{where} \quad a_n = -\frac{1}{4n^\gamma},\ b_n = \frac{1}{4n}\,.$$

Theorem 1. *For any* $0 \le \alpha \le \beta,\ 1 \le \gamma \le \dfrac{\beta+1}{\alpha+1}$ *and* $p_n \in \mathcal{P}_n$ *we have*

$$\|w_{\alpha,\beta}p_n\|_I \le (2 + 4^{\beta-\alpha})n^{\beta+1-\gamma(\alpha+1)}\|w_{\alpha,\beta}p_n\|_{I\setminus I_n}.$$

Proof of Theorem 1. Introducing the notations

$$A_n = \|w_{\alpha,\beta}p_n\|_{I_n} \qquad \text{and} \qquad B_n = \|w_{\alpha,\beta}p_n\|_{I\setminus I_n}\,,$$

the statement of the theorem will follow from the inequality

$$A_n \le (2 + 4^{\beta-\alpha})n^{\beta+1-\gamma(\alpha+1)}B_n\,. \quad (8)$$

Without loss of generality we may assume that $\|p_n\|_I = 1$. Let $d_n \in I$ be a point such that $|p_n(d_n)| = 1$. By $\gamma \le \dfrac{\beta+1}{\alpha+1} \le \dfrac{\beta}{\alpha}$ we have

$$A_n \le \max(w_{\alpha,\beta}(a_n), w_{\alpha,\beta}(b_n)) = \max((4n^\gamma)^{-1}, (4n)^{-1}) = w_{\alpha,\beta}(a_n)\,. \quad (9)$$

According to the position of d_n, we distinguish three cases.

Case 1: $d_n \in I_n$. By the mean value theorem and the Bernstein inequality

$$\frac{|p_n(d_n) - p_n(a_n)|}{d_n - a_n} = |p_n'(\xi_n)| \le \frac{n}{\sqrt{1-\xi_n^2}} \le \frac{4}{3}n, \qquad \xi_n \in I_n\,,$$

whence

$$|p_n(a_n)| \ge 1 - \frac{4}{3}n(d_n - a_n) \ge \frac{1}{3}\,.$$

But then by (9),

$$B_n \geq w_{\alpha,\beta}(a_n)|p_n(a_n)| \geq \frac{1}{3} w_{\alpha,\beta}(a_n) \geq \frac{1}{3} A_n \,.$$

Case 2: $d_n \in [-1, a_n]$. Then again by (9),

$$B_n \geq w_{\alpha,\beta}(d_n)|p_n(d_n)| = w_{\alpha,\beta}(d_n) \geq w_{\alpha,\beta}(a_n) \geq \frac{1}{3} A_n \,.$$

Case 3a: $d_n \in [b_n, 1]$ and $\lambda_n \in [0, b_n]$ where $A_n = w_{\alpha,\beta}(\lambda_n)|p_n(\lambda_n)|$. Then

$$B_n \geq w_{\alpha,\beta}(d_n)|p_n(d_n)| = w_{\alpha,\beta}(d_n) \geq w_{\alpha,\beta}(b_n) \geq w_{\alpha,\beta}(\lambda_n) \geq A_n \,.$$

Case 3b: $d_n \in [b_n, 1]$ and $\lambda_n \in [a_n, 0]$. Then by the mean value theorem and Bernstein inequality

$$|p_n(\lambda_n) - p_n(a_n)| \leq (\lambda_n - a_n)|p_n'(\xi_n)| \leq 4n|a_n| \leq n^{1-\gamma}, \qquad \xi_n \in (a_n, \lambda_n) \,,$$

whence

$$A_n = w_{\alpha,\beta}(\lambda_n)|p_n(\lambda_n)| \leq w_{\alpha,\beta}(a_n)|p_n(a_n)| + w_{\alpha,\beta}(a_n) n^{1-\gamma} \leq B_n + 4^{-\alpha} n^{1-\gamma(1+\alpha)} \,.$$

On the other hand,

$$B_n \geq w_{\alpha,\beta}(d_n)|p_n(d_n)| = w_{\alpha,\beta}(d_n) \geq w_{\alpha,\beta}(b_n) = \frac{1}{(4n)^\beta} \,,$$

and thus

$$A_n \leq B_n + 4^{-\alpha} n^{1-\gamma(1+\alpha)} B_n (4n)^\beta \leq (1 + 4^{\beta-\alpha} n^{\beta+1-\gamma(\alpha+1)}) B_n$$

which completes the proof. $\square$

Remarks. 1. In the special case $\alpha = \beta$ (i.e., $\gamma = 1$) Theorem 1 yields for $I_n = \left[-\frac{1}{4n}, \frac{1}{4n}\right]$,

$$\|w_{\alpha,\alpha} p_n\|_I \leq 3 \|w_{\alpha,\alpha} p_n\|_{I \setminus I_n} \qquad \text{for all} \quad p_n \in \mathcal{P}_n \,.$$

which is of course consistent with (7) and $w_{\alpha,\alpha} \in A^\star$.

Similarly, if $\gamma = \dfrac{\alpha+1}{\beta+1}$ Theorem 1 yields

$$\|w_{\alpha,\beta}p_n\|_I \le (2+4^{\beta-\alpha})\|w_{\alpha,\beta}p_n\|_{I\setminus I_n} \qquad \text{for all} \quad p_n \in \mathcal{P}_n\,, 0 \le \alpha \le \beta\,.$$

The last two estimates correspond to the cases when deleting a proper set can change the norm of the polynomial only by a constant factor. In contrast to this when $\gamma = 1$ Theorem 1 yields with $I_n = [-1/(4n), 1/(4n)]$

$$\|w_{\alpha,\beta}p_n\|_I \le (2+4^{\beta-\alpha})n^{\beta-\alpha}\|w_{\alpha,\beta}p_n\|_{I\setminus I_n}.$$

Here the asymmetry of the weight causes and increase of the norm estimate by a factor $n^{\beta-\alpha}$. We will show now that apart from a log-factor, this upper bound is sharp.

Proposition 1. *Let $0 \le \alpha \le \beta$ and $I_n = [-9\beta \log n/n, 0]$. Then there exist polynomials $p_n \in \mathcal{P}_n$ such that*

$$\|w_{\alpha,\beta}p_n\|_{I_n} \ge c\left(\frac{n}{\sqrt{\log n}}\right)^{\beta-\alpha} \|w_{\alpha,\beta}p_n\|_{I\setminus I_n}$$

where $c > 0$ is a constant depending only on α and β.

(Here and in what follows, $c > 0$ will denote unspecified constants independent of n, not necessary the same at each occurrences.)

Proof. We will make use of the so-called "needle" polynomials

$$q_{n,h}(x) := \frac{T_n^2(1+h^2-x^2)}{T_n^2(1+h^2)} \in \mathcal{P}_{4n}, \qquad 0 < h \le 1\,,$$

where $T_n(x) = \cos(n \arccos x)$ is the Chebyshev polynomial (see [3]). It satisfies the following lower and upper estimates:

$$\frac{1}{4}\exp\left(-\frac{8nx^2}{h}\right) \le q_{n,h}(x) \le 4\exp\left(-\frac{nx^2}{9h}\right), \qquad |x| \le h \le \frac{1}{4}\,. \tag{10}$$

To show these inequalities, we use the formula

$$T_n(x) = \frac{1}{2}[(x+\sqrt{(x^2-1)^2-1})^n + (x-\sqrt{(x^2-1)^2-1})^n]\,. \tag{11}$$

We obtain

$$q_{n,h}(x) \le 4\left(\frac{1+h^2-x^2+\sqrt{(1+h^2-x^2)^2-1}}{1+h^2+\sqrt{(1+h^2)^2-1}}\right)^{2n}$$

$$= 4\left(1-\frac{x^2+h\sqrt{2+h^2}-\sqrt{(1+h^2-x^2)^2-1}}{1+h^2+h\sqrt{2+h^2}}\right)^{2n}$$

$$\le 4\left(1-\frac{x^2+h\sqrt{2+h^2}-\sqrt{(1+h^2-x^2)^2-1}}{2+\sqrt{3}}\right)^{2n}$$

$$\le 4\left(1-\frac{2x^2}{(2+\sqrt{3})(1+2\sqrt{3})h}\right)^{2n}$$

$$\le 4\left(1-\frac{x^2}{18h}\right)^{2n} \le 4\exp\left(-\frac{nx^2}{9h}\right) \qquad (|x| \le h \le 1)\,.$$

The lower estimate of $q_{n,h}(x)$ in (10) can be shown similarly. The monotonicities of $q_{n,h}(x)$ in the intervals $h \le |x| \le 1$ also imply

$$\frac{1}{4}\exp(-8nh) \le q_{n,h}(x) \le 4\exp(-nh/9), \qquad h \le |x| \le 1\,. \tag{12}$$

After these preliminaries let

$$p_n(x) := q_{n,h}(x) \qquad \text{with} \quad h = \frac{9\beta\log n}{n}\,.$$

Using the lower estimate in (10) we obtain with $x_0 = -\dfrac{\sqrt{\beta\log n}}{n} \in I_n$,

$$\|w_{\alpha,\beta}p_n\|_{I_n} \ge w_{\alpha,\beta}(x_0)p_n(x_0) \ge \frac{1}{4}\left(\frac{\sqrt{\beta\log n}}{n}\right)^{\alpha}.$$

On the other hand, using the upper estimate in (12),

$$\|w_{\alpha,\beta}p_n\|_{\{h\le|x|\le 1\}} \le 4e^{-\beta\log n} = 4n^{-\beta}\,,$$

and

$$\|w_{\alpha,\beta}p_n\|_{[0,h]} = \left\|x^{\beta}\exp\left(-\frac{n^2x^2}{9\beta\log n}\right)\right\|_{[0,h]} \le \left(\frac{3\sqrt{\beta\log n}}{n}\right)^{\beta}.$$

Comparint the last three inequalities, we obtain the statement. □

3 Bernstein-type inequalities for asymmetric weights

In order to state our Bernstein type inequality, we need the following Schur type result. Denote

$$\mu(E) := \int_E \frac{dx}{\sqrt{1-x^2}}$$

the Chebyshev measure of a set $E \subset I$. Let $\phi(x) \le 1$ be a bounded, a.e. positive function on I, and for any $\delta > 0$ denote

$$\psi(\delta) := \sup\{c > 0 : \mu(x \in I : \phi(x) \le c) \le \delta\}\,.$$

Lemma (Kroó [2], Lemma 1). *For any weight $w \in A^\star$ and $p_n \in \mathcal{P}_n$ we have*

$$\|wp_n\|_I \le \frac{c}{\psi(1/n)} \|w\phi p_n\|_I\,. \tag{13}$$

With the functions ϕ and ψ defined above, we now state a Bernstein type inequality.

Theorem 2. *Let $W(x) = w(x)\phi(x)$, where $w \in A^\star$ and $0 < \phi(x) \le 1$ a.e. on I. Then we have*

$$\|\varphi W p_n'\|_I \le \frac{cn}{\psi(1/n)} \|W p_n\|_I \qquad \text{for all } p_n \in \mathcal{P}_n$$

where $\varphi(x) = \sqrt{1-x^2}$.

Remark. Since ϕ need not be symmetric, W can be asymmetric, too.

Proof. Since $w \in A^\star$, we have

$$\|\varphi w p_n'\|_I \le cn\|wp_n\|_I$$

(see Mastroianni-Totik [4], (7.28)). Using this and the Lemma we obtain

$$\|\varphi W p_n'\|_I \le \|\phi\|_I \cdot \|\varphi w p_n'\|_I \le cn\|wp_n\|_I$$
$$\le \frac{cn}{\psi(1/n)} \|\phi w p_n\|_I = \frac{cn}{\psi((1/n)} \|W p_n\|_I\,.$$

□

Theorem 3. *We have*

$$\|\varphi w_{\alpha,\beta} p_n'\|_I \le c n^{\gamma} \|w_{\alpha,\beta} p_n\|_I$$

for all $p_n \in \mathcal{P}_n$, where $c > 0$ is a constant depending only on α, β, and

$$\gamma = \begin{cases} 1+\beta-\alpha & if \quad 0 \le \alpha \le \beta \le \alpha + \dfrac{\alpha+1}{2\alpha+1}, \\ 1 + \dfrac{\beta+1}{2(\alpha+1)} & if \quad \alpha + \dfrac{\alpha+1}{2\alpha+1} \le \beta \le 2\alpha+1, \\ 2, & if \quad \beta \ge 2\alpha+1\,. \end{cases}$$

Proof. *First estimate.* Choose

$$w(x) = w_{\alpha,\alpha}(x) \in A^{\star} \qquad \text{and} \qquad \phi(x) = w_{0,\beta-\alpha}(x)\,,$$

then clearly $W(x) = w_{\alpha,\beta}(x)$ and $\psi(\delta) = \delta^{\beta-\alpha},\ 0 < \delta < 1$. Thus Theorem 3 yields

$$\|\varphi w_{\alpha,\beta} p_n'\|_I \le c n^{1+\beta-\alpha} \|w_{\alpha,\beta} p_n\|_I\,.$$

Second estimate. Using the classic Bernstein inequality on the interval $[-1.-1/2]$ we get

$$\|\varphi w_{\alpha,\beta} p_n'\|_{[-1,-3/4]} \le 8\|(1+x)|1/2+x|p_n'\|_{[-1,-3/4]}$$

$$\le 8\|(1+x)|1/2+x|p_n'\|_{[-1,-1/2]} \le cn\|w_{\alpha,\beta}p_n\|_{[-1,-1/2]} \le cn\|w_{\alpha,\beta}p_n\|_I\ .$$

Similarly,

$$\|\varphi w_{\alpha,\beta} p_n'\|_{[3/4,1]} \le cn\|w_{\alpha,\beta}p_n\|_I\,.$$

It remains to estimate $\|\varphi w_{\alpha,\beta} p_n'\|_{[-1/2,1/2]}$. Using Theorem 1 on the interval $J = [-1/2, 1/2]$ (with $J_n = \dfrac{1}{2} I_n$) we obtain

$$\|\varphi w_{\alpha,\beta} p_n'\|_{[-1/2,1/2]} \le \|w_{\alpha,\beta} p_n'\|_J \le c\|w_{\alpha,\beta} p_n'\|_{J\setminus J_n}$$

$$\le \frac{c}{\sqrt{|a_n|}}\|\sqrt{|x|(1+x)}w_{\alpha,\beta}p_n'\|_{[-1,0]} + \frac{c}{\sqrt{b_n}}\|\sqrt{x(1-x)}w_{\alpha,\beta}p_n'\|_{[0,1]}\,.$$

Since $w_{\alpha,\beta}(x)$ is an $A^{\star}$ weight on the intervals $[-1,0]$ and $[0,1]$, we can apply the Bernstein inequality from [4] (see (7.28) there) to get

$$\|\varphi w_{\alpha,\beta} p_n'\|_{[-1/2,1/2]} \le c n^{1+\frac{\beta+1}{2(\alpha+1)}} \|w_{\alpha,\beta} p_n\|_I\,.$$

Third estimate. When $\beta \geq 2\alpha+1$, the Bernstein factor becomes n^2 which can be seen by applying (7.30) in [4] separately for $[-1,0]$ and $[0,1]$. $\square$

Next we give an example which shows that the Bernstein factor indeed can be of higher order than $O(n)$ in some cases.

Example 2. *Let the weight $w(x) \geq 0$ $(x \in I)$ satisfy*

$$\mu := \sup_{-1/2 \leq x < 0} \frac{\log w(x)}{\log \log \frac{1}{|x|}} < \frac{1}{3} \qquad \text{and} \qquad \inf_{0 < x \leq 1/2} \frac{\log w(x)}{\log x} > 0\,. \tag{14}$$

Then for any $\lambda \in (\mu, 1/3)$ there exists a polynomial $p_n \in \mathcal{P}_n$ such that

$$\|\varphi w p_n'\|_I \geq cn \log^{\lambda-\mu} n \|wp_n\|_I\,. \tag{15}$$

Remark. For example, the weight in $[-1,0]$ can be chosen as $\log^{-\mu} \dfrac{2}{|x|}$, and in $[0,1]$ as $x^\alpha \log^\beta \dfrac{2}{x}$ $(\alpha, \beta > 0)$, or $\exp(-1/x)$.

Proof. In constructing our polynomial, we use two well-known polynomials. The first is the needle polynomial introduced in the proof of Example 1. The other tool we use is a so-called fast decreasing polynomial $r_n \in \mathcal{P}_n$ which is even and has the properties

$$r_n(0) = 1, \qquad \text{and} \qquad 0 \leq r_n(x) \leq C \exp(-nf(x)) \quad (|x| \leq 1) \tag{16}$$

if and only if $\displaystyle\int_0^1 \frac{f(x)}{x^2} dx < \infty$ (cf. Ivanov and Totik [1]). Here we choose

$$f(x) = \frac{|x|}{\log^{\frac{3}{2}(1-\lambda)} \frac{2}{|x|}}, \qquad (\mu < \lambda < 1/3;$$

then the above integral condition is obviously satisfied.

After these preparations our polynomial is defined as

$$p_n(x) := q_{n,h}(x) r_n(x) T_m(2x+1) > 0 \qquad (|x| \leq 1\,, m = [\delta \sqrt{n} \log^{\lambda/2} n]),$$

where

$$h = \frac{\log^{3-2\lambda} n}{n}\,,$$

and the constant $\delta > 0$ will be determined later.

Let $x_0 = -1/n^2$ and $Q_n(x) = r_n(x)q_{n,h}(x)$, then $\|Q_n\|_I \le c$. Then by (14) $w(x_0) \ge \dfrac{1}{\log^\mu \frac{2}{|x_0|}}$ and we obtain

$$\|\varphi w p_n'\|_I \ge c w(x_0) p_n'(x_0) \ge \frac{c}{\log^\mu n}[T_m'(2x_0+1)Q_n(x_0) - Q_n'(x_0)]\,. \qquad (17)$$

Since the argument $2x_0 + 1$ is at a distance $O(m^{-3})$ from the endpoint 1, evidently $T_m'(2x_0+1) \ge cm^2 \ge cn\log^\lambda n$. By the mean value theorem and using that $\|Q_n\|_I \le c$ implies $\|Q_n'\|_{I/2} \le cn$,

$$1 - Q_n(x_0) = Q_n(0) - Q_n(x_0) \le cn^{-2}|Q_n'(\xi_n)| \le c/n \qquad (\xi_n \in (x_0, 0))\,,$$

whence $Q_n(x_0) \ge c > 0$. On the other hand, since $\|Q_n''\|_{I/4} \le cn^2$, we obtain

$$Q_n'(x_0) = Q_n'(x_0) - Q_n'(0) \le |Q_n''(\eta_n)|n^{-2} \le c \qquad (\eta_n \in (x_0, 0))\,.$$

Thus we get from (17),

$$\|\varphi w p_n'\|_I \ge cn\log^{\lambda-\mu} n - \frac{c}{\log^\mu n} \ge cn\log^{\lambda-\mu} n\,.$$

In order to show (15) we have to prove that $\|wp_n\|_I \le c$. Obviously, $\|wp_n\|_{\{-1\le x\le 0\}} \le c$. For the case $0 < x \le 1$ we distinguish three cases.

Case 1: $0 < x \le \dfrac{\log^{2-\lambda} n}{n}$. Then, using the estimate $0 < T_m(2x+1) \le \exp(cm\sqrt{x})$ (which follows from (11)), as well as the inequality $w(x) \le x^\varepsilon$ with some $\varepsilon > 0$ (which follows from (14)),

$$w(x)p_n(x) \le x^\varepsilon T_m(2x+1) \le x^\varepsilon e^{cm\sqrt{x}} \le \frac{\log^{\varepsilon(2-\lambda)} n}{n^\varepsilon} \cdot e^{c\delta \log n} \le 1$$

provided $\delta < \varepsilon/c$.

Case 2: $\dfrac{\log^{2-\lambda} n}{n} < x \le h = \dfrac{\log^{3-2\lambda} n}{n}$. Then, instead of the weight, we use the needle polynomial and its upper estimate (10):

$$w(x)p_n(x) \le q_{n,h}(x)T_m(2x+1) \le 4\exp\left(-\frac{nx^2}{9h} + cm\sqrt{x}\right)\,.$$

Here the exponent is negative if

$$x > \left(\frac{9chm}{n}\right)^{2/3} \geq (9c\delta)^{2/3}\frac{\log^{2-\lambda} n}{n},$$

which holds in the interval in question if $\delta < 1/(9c)$. Thus $w(x)p_n(x) \leq 1$ in this interval.

Case 3: $h = \dfrac{\log^{3-2\lambda} n}{n} < x \leq 1$. Then we use the fast decreasing polynomial and its property (10):

$$w(x)p_n(x) \leq r_n(x)T_m(2x+1) \leq \exp\left(-c\frac{nx}{\log^{\frac{3}{2}(1-\lambda)} n} + cm\sqrt{x}\right).$$

Here the exponent is negative if

$$x > \left(\frac{cm}{n}\right)^2 \log^{3(1-\lambda)} n = (c\delta)^2\frac{\log^{3-2\lambda} n}{n},$$

which holds in the interval in question if $\delta \leq 1/c$. Thus $w(x)p_n(x) \leq 1$ in this interval as well. $\square$

The next theorem shows that with a proper weight the Bernstein factor can be arbitrarily close to $O(n^2)$.

Theorem 4. *Let $\{\Psi_n\}_{n=1}^\infty$ be an arbitrary sequence of positive numbers monotone increasing to ∞ as $n \to \infty$. Then there exist a weight $w \in C(I)$, $w(0) = 0$, $0 < w(x) \leq 1$, $0 < |x| \leq 1$, and polynomials $p_n \in \mathcal{P}_n, n \in \mathbb{N}$, such that*

$$\|\varphi w p_n'\|_I \geq \frac{cn^2}{\Psi_n}\|wp_n\|_I, \quad n \in \mathbb{N}.$$

Proof. We may assume that $\Psi_n = o(n^2)$, otherwise the statement is trivial. We shall again apply Chebyshev polynomials $T_m(2x+1)$ and needle polynomials $q_{n-m,h}(x)$ with

$$m := \left[\frac{n}{\psi_n}\right], \quad h := \frac{a}{\psi_n}$$

where $\psi_n := \sqrt[3]{\Psi_n}$ and the constant $a > 0$ will be specified below. We have that with certain positive absolute constants $c_2 < 1 < c_1$

$$T_m(2x+1) \leq e^{c_1 m\sqrt{x}}, \quad 0 \leq x \leq 1, \quad 0 < q_{n-m,h}(x) \leq e^{-c_2(n-m)h}, \quad h \leq x \leq 1.$$

Now we set $p_n := T_m q_{n-m,h} \in \mathcal{P}_n$ with $a := \dfrac{2c_1}{c_2}$.

Let $w \in C[-1,1],\ w(0) = 0,\ 0 < w(x) \le 1,\ 0 < |x| \le 1$, be such that

$$w\left(\frac{a}{\psi_n}\right) = e^{-2a^2 n \psi_n^{-3/2}}, \quad w(-n^{-2}) = \psi_n^{-1}, \quad n \in \mathbb{N}\,,$$

and let w be linear between the adjacent points where the values are prescribed.

Let us show first that $\|wp_n\|_I \le 1$. By the above estimates we have

$$\|wp_n\|_I = \max\left(\max_{0\le x\le h} |wp_n|(x),\ \max_{h\le x\le 1} |wp_n|(x),\ \max_{-1\le x\le 0} |wp_n|(x)\right)$$

$$\le \max\left(w(h)e^{c_1 m\sqrt{h}},\ e^{c_1 m - c_2(n-m)h},\ 1\right)$$

$$\le \max\left(\exp(-2a^2 + c_1\sqrt{a})n\psi_n^{-3/2},\ \exp\frac{c_1 n}{\psi_n^2}(-\psi_n + 2),\ 1\right) = 1.$$

Now we can get a lower bound for $|\varphi w p_n'|$ as follows

$$2\|\varphi w p_n'\|_I \ge |wp_n'|(-n^{-2}) \ge |wT_m' q_{n-m,h}|(-n^{-2}) - |wT_m q_{n-m,h}'|(-n^{-2}) \ge$$

$$\frac{1}{2}\psi_n^{-1}T_m'(1-2n^{-2}) - |q_{n-m,h}'|(-n^{-2}) \ge c\psi_n^{-1}m^2 - O(1) \ge \frac{cn^2}{\psi_n^3} = \frac{cn^2}{\Psi_n}. \ \Box$$

These results naturally lead to the following

Question 1. *Consider an arbitrary a.e. positive weight w. Is it true that*

$$\|\varphi w p_n'\|_I = o(n^2)\|wp_n\|_I$$

for all polynomials $p_n \in \mathcal{P}_n$?

Question 2. *Consider an arbitrary nonnegative weight w. Is it true that*

$$\|\varphi w p_n'\|_I = O(n^2)\|wp_n\|_I$$

for all polynomials $p_n \in \mathcal{P}_n$?

The example

$$w(x) = \begin{cases} 1 & \text{if} \quad -1 \le x \le 0, \\ 0 & \text{if} \quad 0 < x \le 1, \end{cases} \qquad p_n(x) = T_n(2x+1)$$

shows that the $O(n^2)$ Bernstein factor can be attained, since $p_n'(0) = 2n^2$.

Finally, we show that for a wide class of weights (including asymmetric weights), if we perform a slight change in the weight, namely add a properly chosen quantity to it which goes to zero as n goes to infinity, then the classic Bernstein inequality holds.

Theorem 5. *Let $w(x)$ be an $r \geq 0$ times continuously differentiable positive weight in I except that $w(0) = 0$. Further let*

$$w_n(x) := w(x) + \frac{C}{n^r}\omega\left(w^{(r)}, \frac{1}{n}\right) \qquad (|x| \leq 1,\ n = 1, 2, \dots)$$

where ω is the modulus of continuity of the corresponding function, and $c > 0$ is an arbitrary constant. In case $r \geq 1$, also assume that

$$\sup_{0<|x|\leq 1} \frac{|xw'(x)|}{w(x)} < \infty\,. \tag{18}$$

Then for all polynomials $p_n \in \mathcal{P}_n$ we have

$$\|\varphi w_n p_n'\|_I \leq cn\|w_n p_n\|_I\,. \tag{19}$$

Proof. We distinguish two cases.

Case 1: $r = 0$. By the Jackson theorem, for a sufficiently large $c > 1$, there exist polynomials $q_n(x) \in \mathcal{P}_{cn}$ such that

$$\|w_n - q_n\|_I \leq \frac{1}{2}\omega\left(w, \frac{1}{n}\right)\,.$$

Since $w_n(x) \geq \omega\left(w, \frac{1}{n}\right)$, hence

$$\frac{1}{2}q_n(x) \leq w_n(x) \leq \frac{3}{2}q_n(x) \qquad (|x| \leq 1)\,. \tag{20}$$

Also,

$$\|\varphi q_n'\|_I \leq cn\omega\left(q_n, \frac{1}{n}\right) \tag{21}$$

$$\leq cn\left[\omega\left(q_n - w_n, \frac{1}{n}\right) + \omega\left(w_n, \frac{1}{n}\right)\right] \leq c\omega\left(w, \frac{1}{n}\right)\,.$$

Thus we obtain

$$\|\varphi w_n p_n'\|_I \le \frac{3}{2}\|\varphi q_n p_n'\|_I \le \frac{3}{2}\|\varphi (q_n p_n)'\|_I + \frac{3}{2}\|\varphi q_n' p_n\|_I \,. \tag{22}$$

We estimate the two terms on the right hand side separately. Concerning the first term, we use the ordinary Bernstein inequality for the polynomial $q_n p_n \in \mathcal{P}_{(c+1)n}$ to get

$$\|\varphi (q_n p_n)'\|_I \le cn\|q_n p_n\|_I \le cn\|w_n p_n\|_I \,, \tag{23}$$

where we used (20).

As for the second term, using (21) and $\omega\left(w, \frac{1}{n}\right) \le w_n(x)$ we get

$$\|\varphi q_n' p_n\|_I \le cn\omega\left(w, \frac{1}{n}\right)\|p_n\|_I \le cn\|w_n p_n\|_I \,.$$

Collecting these estimates, we obtain the statement of the theorem in Case 1.

Case 2: $r \ge 1$. Then there exist polynomials $q_n \in \mathcal{P}_{cn}$ such that

$$\|w_n^{(i)} - q_n^{(i)}\|_I \le \frac{1}{2n^{r-i}}\omega\left(w^{(r)}, \frac{1}{n}\right) \qquad (i = 0, \ldots, r)\,, \tag{24}$$

provided $c > 1$ is large enough. Using this estimate with $i = 0$ as well as the inequality $\frac{1}{n^r}\omega\left(w^{(r)}, \frac{1}{n}\right) \le w_n(x)$, we obtain (20). Next, using (24) with $i = 1$ as wells as (18) we get

$$|q_n'(x)| \le w'(x) + \frac{1}{2n^{r-1}}\omega\left(w^{(r)}, \frac{1}{n}\right)$$

$$\le c\frac{w(x)}{x} + cnw_n(x) \le cnw_n(x) \qquad (1/n \le |x| \le 1)\,.$$

Hence by the Remez inequality

$$\|\varphi q_n' p_n\|_I \le c\|q_n' p_n\|_I\{1/n \le |x| \le 1\} \le cn\|w_n p_n\|_I \,.$$

Using (22)-(23) together with this estimate, we can finish the proof as in Case 1. □

Remark. In particular, if

$$w_n(x) = w_{\alpha,\beta}(x) + \frac{c}{n^\alpha} \qquad (0 < \alpha \leq \beta)\,,$$

then we have (19). Namely, in this case

$$r = \begin{cases} [\alpha] & \text{if } \alpha > [\alpha], \\ \alpha - 1 & \text{if } \alpha = [\alpha] > 0\,, \end{cases}$$

$\omega(w^{(r)}, h) \leq h^{\alpha - r}$, and for $\alpha > 1$, (18) holds.

Moreover, if $\alpha = \beta > 0$, then we obtain by the classic Remez inequality

$$\|\varphi w_{\alpha,\alpha} p_n'\|_I \leq \|\varphi w_n p_n'\|_I \leq cn\|w_n p_n\|_I \leq \|w_{\alpha,\alpha} p_n\|_I + cn^{1-\alpha}\|p_n\|_I$$

$\leq cn\|w_n p_n\|_I \leq \|w_{\alpha,\alpha} p_n\|_I + cn^{1-\alpha}\|p_n\|_I + cn^{1-\alpha}\|p_n\|_{\{|x|\geq 1/n\}} \leq\leq cn\|w_{\alpha,\alpha} p_n\|_I$
which is just a special case of the inequality (7.28) in [4].

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