Weierstrass type approximation by weighted polynomials in $\mathbb{R}^{d*}$

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Abstract

In this paper we consider weighted polynomial approximation on *unbounded multidimensional domains* in the spirit of the weighted version of the Weierstrass trigonometric theorem according to which every continuous function on the real line with equal finite limits at $\pm\infty$ is a uniform limit on $\mathbb{R}$ of weighted algebraic polynomials of degree $2n$ with varying weights $(1+t^2)^{-n}$. We will verify a similar statement in the multivariate setting for a general class of *convex* weights.

We also consider the similar problem of multivariate polynomial approximation with varying weights for some typical *non convex* weights. In case of non convex weights of the form $w_\alpha(\mathbf{x}) := (1+|x_1|^\alpha + ... + |x_d|^\alpha)^{\frac{1}{\alpha}}, 0 < \alpha < 1$ in order for weighted polynomial approximation to hold for a given continuous function it is necessary that the function vanishes on a certain exceptional set consisting of all coordinate hyperplanes and ∞. Moreover, in case of rational α this condition is also sufficient for weighted polynomial approximation to hold.

1 Introduction

Let us start by recalling the *trigonometric* version of the classical Weierstrass approximation theorem which states that any 2π periodic continuous function $f(x)$ is a uniform limit on $[-\pi, \pi]$ of trigonometric polynomials of degree n as $n \to \infty$. Clearly the substitution $t = \tan\frac{x}{2}, x \in (-\pi, \pi)$ transforms 2π periodic continuous functions into continuous functions on $\mathbb{R}$ which have equal finite limits at $\pm\infty$, while the trigonometric polynomials of degree n transform into rational functions $(1+t^2)^{-n}p_{2n}(t)$ with $p_{2n}(t)$ being an algebraic polynomial of degree at most $2n$.

This results in the following equivalent version of the trigonometric Weierstrass theorem:

Every $f \in C(\mathbb{R})$ with equal finite limits at $\pm\infty$ is a uniform limit on $\mathbb{R}$ of weighted algebraic polynomials

$$w(t)^{-2n}p_{2n}(t), \ \ w(t) := \sqrt{1+t^2}, \ \deg p_{2n} \le 2n.$$

Naturally this leads to the question of uniform approximation by general weighted polynomials of the form

$$w(t)^{-n}p_n(t), \ \deg p_n \le n.$$

This question received a considerable attention in the recent years in case when the even weight $w(t)$ grows at ∞ faster than t, i.e., $\frac{w(t)}{t} \to \infty$ as $t \to \infty$, see for instance monographs [5] and [6]. The assumption $\frac{w(t)}{t} \to \infty$ obviously implies that $w(t)^{-n}p_n(t) \to 0, t \to \pm\infty$ for all polynomials of degree at most n. Thus, naturally under condition $\frac{w(t)}{t} \to \infty, t \to \infty$ weighted polynomials can not provide uniform approximation on all of the real line. Indeed, various results given in [5] and [6] show that only continuous functions with *finite support* can be approximated by weighted polynomials in this case, and in fact this finite domain of approximation which depends on w can be determined by methods of potential theory.

In this paper we intend to consider weighted approximation on *unbounded domains* in the spirit of the weighted version of Weierstrass trigonometric approximation theorem. As mentioned above if $\frac{w(t)}{t}$ tends to infinity as $t \to \infty$ only functions with finite support can be approximated, i.e. approximation on the whole real line is not possible in this case. On the other hand if $\frac{w(t)}{t} \to 0$ as $t \to \infty$ then weighted polynomials $w(t)^{-n}p_n(t)$ are unbounded as $t \to \infty$

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which also makes the approximation on unbounded domains impossible. Therefore we will require that the weights have **linear growth** at infinity. Moreover, this problem will be investigated below in the *multivariate* setting.

Let us denote by S^{d-1} the Euclidian unit sphere in $\mathbb{R}^d$ and consider the space $C_0(\mathbb{R}^d)$ of continuous functions on $\mathbb{R}^d$ which have equal continuous limits at infinity along lines passing through the origin , i.e.,

$$C_0(\mathbb{R}^d) := \{f \in C(\mathbb{R}^d) : \exists r_f \in C(S^{d-1}) \text{ such that } \lim_{|t| \to \infty} f(t\mathbf{x}) = r_f(\mathbf{x}), \text{ uniformly for } \mathbf{x} \in S^{d-1}\}.$$

Note that for every $f \in C_0(\mathbb{R}^d)$ and $\mathbf{x} \in \mathbb{R}^d \setminus \{0\}$ we have that

$$\lim_{t \to \pm\infty} f(t\mathbf{x}) = r_f\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right)$$

and therefore it is natural to extend $r_f \in C(S^{d-1})$ to $\mathbb{R}^d \setminus \{0\}$ by relation

$$r_f(\mathbf{x}) := r_f\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right), \mathbf{x} \in \mathbb{R}^d \setminus \{0\}.$$

Let $w \in C(\mathbb{R}^d), w(-\mathbf{x}) = w(\mathbf{x}), \forall \mathbf{x} \in \mathbb{R}^d$ be a positive even weight function on $\mathbb{R}^d$. We will consider uniform approximation of functions $f \in C_0(\mathbb{R}^d)$ by weighted polynomials $w^{-n}p_n$ on $\mathbb{R}^d$, where $p_n \in P_n^d$ are multivariate polynomials of d variables of degree at most n. We will assume throughout this paper that w is a positive continuous even weight on $\mathbb{R}^d$ such that $tw(\frac{\mathbf{x}}{t})$ is monotone increasing for $t > 0$ for every fixed $\mathbf{x} \in \mathbb{R}^d$, and has a continuous positive limit as $t \to 0$. This means that with some even positive continuous function

$$\tilde{w} : S^{d-1} \mapsto \mathbb{R}^+$$

we have by the Dini theorem that

$$|t|w\left(\frac{\mathbf{x}}{t}\right) \to \tilde{w}(\mathbf{x}), \quad t \to 0, \quad \text{uniformly for } \mathbf{x} \in S^{d-1}. \tag{1}$$

Hence in particular, $w(t\mathbf{x}) \sim |t|w(\mathbf{x}), t \to \infty$, i.e. the weight is of order $|t|$ at infinity. Evidently, extending $\tilde{w} \in C(S^{d-1})$ to $\mathbb{R}^d \setminus \{0\}$ according to the relation $\tilde{w}(\alpha\mathbf{x}) = |\alpha|\tilde{w}(\mathbf{x}), \alpha \in \mathbb{R}$ preserves (1) for every $\mathbf{x} \in \mathbb{R}^d$. In particular, when $d = 1$ we have that $\tilde{w}(t) \equiv c|t|, t \in \mathbb{R}$ with some $c > 0$.

In what follows positive continuous even weights w on $\mathbb{R}^d$ for which $tw(\frac{\mathbf{x}}{t})$ is monotone increasing with respect to $t > 0$ for every fixed $\mathbf{x} \in \mathbb{R}^d$ and (1) is satisfied with certain positive continuous function $\tilde{w}$ will be called **admissible**. It is important to observe that for admissible weights w we have $w^{-2n}p_{2n} \in C_0(\mathbb{R}^d)$ for any $p_{2n} \in P_{2n}^d$ and $n \in \mathbb{N}$.

For an admissible weight w we consider its **homogenization** defined for every $(\mathbf{x}, t) \in \mathbb{R}^{d+1} \setminus \{0\}$ by

$$w^*(\mathbf{x}, t) := \begin{cases} |t|w\left(\frac{\mathbf{x}}{t}\right), & \mathbf{x} \in \mathbb{R}^d, t \in \mathbb{R} \setminus \{0\} \\ \tilde{w}(\mathbf{x}), & t = 0. \end{cases} \tag{2}$$

It is easy to see that this function $w^* : \mathbb{R}^{d+1} \mapsto \mathbb{R}$ possesses the homogeneous property

$$w^*(c\mathbf{x}, ct) = |c|w^*(\mathbf{x}, t), \forall c \in \mathbb{R}, (\mathbf{x}, t) \in \mathbb{R}^{d+1}.$$

In addition, $w^*(\mathbf{x}, t)$ is positive everywhere in $\mathbb{R}^{d+1} \setminus \{0\}$ and we may set preserving continuity $w^*(0) = 0$. Evidently, it follows from (1) that the above relation defines a continuous weight $w^*(\mathbf{x}, t) \in C(\mathbb{R}^{d+1})$ since both w and $\tilde{w}$ are continuous. Also note that $w^*(\mathbf{x}, t)$ is even both in $\mathbf{x} \in \mathbb{R}^d$ and $t \in \mathbb{R}$.

The problem of approximation of $f \in C_0(\mathbb{R}^d)$ by weighted polynomials $w^{-2n}p_{2n}$ uniformly on $\mathbb{R}^d$ is closely related to approximation by multivariate **homogeneous** polynomials

$$h \in H_n^d := \left\{\sum_{|\mathbf{k}|=n} a_\mathbf{k}\mathbf{x}^\mathbf{k} : a_\mathbf{k} \in \mathbb{R}\right\}$$

of degree n on boundaries of 0-symmetric convex bodies.

In this respect the following conjecture has been widely investigated in the past decade (see, for instance [2], [8], [3], [7]).

Conjecture 1. *For any 0-symmetric convex body $K \subset \mathbb{R}^d$ and every even $f \in C(\partial K)$ there exist homogeneous polynomials $h_{2n} \in H_{2n}^d$ such that $f = \lim_{n \to \infty} h_{2n}$ uniformly on ∂K.*

It is easy to see that the above conjecture is equivalent to the claim that every continuous function on ∂K with K as above can be uniformly approximated by the sum of 2 homogeneous polynomials, i.e., for every $f \in C(\partial K)$ we have that $f = \lim_{n \to \infty} h_n$ with certain $h_n \in H_{2n}^d + H_{2n+1}^d$ uniformly on ∂K.

This conjecture has been verified in the following three cases:

(i) when $d = 2$ (see [2, 8]);

(ii) for any 0-symmetric convex *polytope* in $\mathbb{R}^d, d \geq 2$ (see [8]);

(iii) for any 0-symmetric *regular* convex body in $\mathbb{R}^d, d \geq 2$ possessing a unique supporting hyperplane at every point on its boundary (see [3]).

Above results on homogeneous approximation will play an important role in our considerations below.

The outline of the paper is as follows. First a general result on density of weighted polynomials in $C_0(\mathbb{R}^d)$ in case of convex admissible weights will be given (Theorem 1). Then we proceed by exhibiting some model examples of convex weights for which density holds. Finally, Theorems 3 and 4 will deal with analogues questions for certain non convex weights. It will be shown that in non convex situation the density holds only for functions vanishing at a certain exceptional set related to the weight.

New results

Our first main result provides a Weierstrass type result asserting that uniform approximation on $\mathbb{R}^d$ by weighted polynomials $w^{-2n}p_{2n}$ is possible when the admissible weight is **convex** and, if $d > 1$ is also **piecewise** C^1. The piecewise C^1 property of the weight w is meant in the sence that w is a maximum of finite number of C^1 weights.

Theorem 1 *Let w be a convex admissible weight on $\mathbb{R}^d, d \geq 1$. In addition, if $d > 1$ assume that w is piecewise C^1, i.e., with some $s \in \mathbb{N}$ we have $w = \max\{w_j : 1 \leq j \leq s\}$ where each w_j is admissible convex and $w_j^* \in C^1(\mathbb{R}^{d+1} \setminus \{0\}), 1 \leq j \leq s$. Then for every $f \in C_0(\mathbb{R}^d)$ there exist polynomials $p_{2n} \in P_{2n}^d$ so that*

$$w^{-2n}p_{2n} \to f, \ n \to \infty$$

uniformly on $\mathbb{R}^d$.

Thus when $d = 1$ the **convexity** of the admissible weight yields the density of weighted polynomials $w^{-2n}p_{2n}$ in the space $C_0(\mathbb{R})$. If $d > 1$ we need in addition the piecewise C^1 smoothness of weights in order for the density to hold. It is plausible that the convexity of admissible weights suffices for the density of weighted polynomials $w^{-2n}p_{2n}$ in $C_0(\mathbb{R}^d), d > 1$, as well. Thus we would like to offer the next conjecture which would provide a full analogue of weighted Weierstrass approximation theorem in $\mathbb{R}^d$.

Conjecture 2. *For any convex admissible weight on $\mathbb{R}^d, d \geq 1$ and $f \in C_0(\mathbb{R}^d)$ there exist polynomials $p_{2n} \in P_{2n}^d$ so that*

$$w^{-2n}p_{2n} \to f, \ n \to \infty$$

uniformly on $\mathbb{R}^d$.

Let us now give some model examples of weights for which the above theorem is applicable.

Example 1. For any $0 < \alpha \leq \infty$ the l_α norm of $\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d$ is given by the relations

$$|\mathbf{x}|_\alpha := \begin{cases} (|x_1|^\alpha + ... + |x_d|^\alpha)^{\frac{1}{\alpha}}, \ 0 < \alpha < \infty \\ \max_{1 \leq j \leq d} |x_j|, \ \alpha = \infty. \end{cases}$$

Furthermore, for $\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d$ consider the weight

$$w_\alpha(\mathbf{x}) := |(1, \mathbf{x})|_\alpha = \begin{cases} (1 + |x_1|^\alpha + ... + |x_d|^\alpha)^{\frac{1}{\alpha}}, \ 0 < \alpha < \infty \\ \max\{1, |x_1|, ..., |x_d|\}, \ \alpha = \infty. \end{cases} \tag{3}$$

Now it easily follows that for this weight and $t > 0$

$$w_\alpha^*(\mathbf{x}, t) = tw_\alpha\left(\frac{\mathbf{x}}{t}\right) = |(t, \mathbf{x})|_\alpha, \ (t, \mathbf{x}) \in \mathbb{R}^{d+1}$$

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is monotone increasing for every fixed $\mathbf{x} \in \mathbb{R}^d$. Moreover, clearly (1) holds with

$$\tilde{w}_\alpha(\mathbf{x}) = \lim_{t \to 0} |t| w_\alpha \left(\frac{\mathbf{x}}{t} \right) = |\mathbf{x}|_\alpha, \quad \mathbf{x} \in \mathbb{R}^d.$$

Hence $w_\alpha(\mathbf{x})$ is an admissible weight for every $0 < \alpha \leq \infty$. Furthermore, $w_\alpha(\mathbf{x})$ is also convex whenever $1 \leq \alpha \leq \infty$. In addition, when $1 < \alpha < \infty$ these weights are C^1, i.e., $w_\alpha^* \in C^1(\mathbb{R}^{d+1} \setminus \{0\})$ while for $\alpha = 1, \infty$ the weights are piece wise C^1. Therefore for every $1 \leq \alpha \leq \infty$ weights $w_\alpha(\mathbf{x})$ defined in (3) provide a model of weights for which conditions of Theorem 1 hold. Hence we obtain the next

Corollary 2 *Let $1 \leq \alpha \leq \infty, d \geq 1$. Then for every $f \in C_0(\mathbb{R}^d)$ there exist polynomials $p_{2n} \in P_{2n}^d$ so that*

$$w_\alpha^{-2n} p_{2n} \to f, \ n \to \infty$$

uniformly on $\mathbb{R}^d$.

Example 2. It should be noted that the sufficient conditions imposed on the weight in Theorem 1 allow performing certain operations with the weights. In particular, the summation of the weights, or taking maximum of weights preserves the required properties. Therefore based on the weights (3) it follows for instance that Theorem 1 will also hold for weights

$$w(\mathbf{x}) := \sum_{1 \leq j \leq d} (1 + |x_j|^{\alpha_j})^{\frac{1}{\alpha_j}},$$

and

$$w(\mathbf{x}) := \max_{1 \leq j \leq d} (1 + |x_j|^{\alpha_j})^{\frac{1}{\alpha_j}},$$

whenever $1 \leq \alpha_j < \infty$, $1 \leq j \leq d$. Note that in contrast to weights (3) the above weights are in general *asymmetric* with respect to the variables $x_j, 1 \leq j \leq d$.

Example 3. Another example of piece wise smooth weights for which Theorem 1 can be applied is provided by *linear splines*. Namely, we can take w to be an even convex linear spline function on $\mathbb{R}^d$ in which case all requirements of Theorem 1 are fulfilled again.

While the admissibility of the weight appears to be a natural requirement for approximating every function in $C_0(\mathbb{R}^d)$ one may ask if the convexity of weight is also necessary for this approximation to hold, in general. A model example of non convex weights is given by weights (3) when $0 < \alpha < 1$. It turns out that convexity of the weight is crucial in order for the weighted approximation to hold for every $f \in C_0(\mathbb{R}^d)$.

Indeed, in a recent paper [4] the authors showed that in case $d = 1$ and $0 < \alpha < 1$ *there exist univariate weighted polynomials $w_\alpha^{-2n} p_{2n}$, $p_n \in P_{2n}^1$, $n \in \mathbb{N}$, converging to $f \in C(\mathbb{R})$ uniformly on $\mathbb{R}$ if and only if $f(0) = f(\infty) = f(-\infty) = 0$.* Here $f(\infty), f(-\infty)$ stand for the corresponding limits at infinity. Thus in case of these non convex admissible weights some additional restrictions need to be imposed on the functions which admit a univariate weighted polynomial approximation. Namely the function must vanish at a certain *exceptional set*. It is natural to expect that this phenomena will be preserved in multivariate setting, as well. The most interesting question here consists in finding the proper *multidimensional exceptional sets*.

So let us consider now approximation by weighted polynomials $w_\alpha^{-2n} p_{2n}, p_{2n} \in P_{2n}^d$ when $0 < \alpha < 1$, that is the weight is not convex.

Denote by

$$L_j^d := \{\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d : x_j = 0\}, 1 \leq j \leq d$$

the coordinate planes in $\mathbb{R}^d$, and let

$$L^d := \cup_{1 \leq j \leq d} L_j^d = \{\mathbf{x} = (x_1, ..., x_d) \in K_\alpha^d : x_1 \cdot ... \cdot x_d = 0\}$$

be the union of all coordinate planes. Furthermore, we will write $f(\infty) = 0$ if $f(\mathbf{y}) \to 0$ whenever $|\mathbf{y}| \to \infty$.

The next theorem is a multivariate extension of the result given in [4]. It identifies the exceptional zero set for the functions which admit weighted polynomial approximation on $\mathbb{R}^d$ with the non convex weights $w_\alpha, 0 < \alpha < 1$. Essentially our result shows that in multivariate case the exceptional zero set consists of the union of all coordinate planes L^d and the infinity.

Theorem 3 *Let $0 < \alpha < 1$ and $d \geq 2$. Then in order for $f \in C_0(\mathbb{R}^d)$ to be a uniform limit on $\mathbb{R}^d$ of weighted polynomials $w_\alpha^{-2n} p_{2n}, p_{2n} \in P_{2n}^d$ it is necessary that $f = 0$ on $L^d \cup \{\infty\}$. Moreover, if $0 < \alpha < 1$ is rational then any $f \in C(\mathbb{R}^d)$ which vanishes on $L^d \cup \{\infty\}$ is a uniform limit on $\mathbb{R}^d$ of weighted polynomials $w_\alpha^{-2n} p_{2n}, p_{2n} \in P_{2n}^d$.*

As mentioned above uniform approximation in $C_0(\mathbb{R}^d)$ by weighted polynomials $w^{-2n} p_{2n}$ is closely related to the approximation by multivariate homogeneous polynomials on the boundaries of 0-symmetric star like domains. So we will include now a companion to Theorem 3 related to homogeneous polynomial approximation.

Consider the L_α sphere in $\mathbb{R}^d$ given by

$$K_\alpha^d := \{\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d : |x_1|^\alpha + ... + |x_d|^\alpha = 1\} = \{|\mathbf{x}|_\alpha = 1\}, \qquad \alpha > 0.$$

When $0 < \alpha < 1$ the set K_α^d is not convex so the problem of homogeneous polynomial approximation on this set is not covered by the results mentioned in the Introduction. In Kroó-Totik [4] it is shown that if $d = 2$ and $0 < \alpha < 1$ then in order that even function $f(x, y) \in C(K_\alpha^2)$ be a uniform limit on K_α^2 of homogeneous polynomials $h_{2n} \in H_{2n}^2$ it is necessary and sufficient that $f(\pm 1, 0) = f(0, \pm 1) = 0$, i.e., the function must vanish at all the vertices of K_α^2. Evidently, when $d > 2$ the set of all *non smooth* points of K_α^d is given by

$$K_\alpha^d \cap L^d = \{\mathbf{x} = (x_1, ..., x_d) \in K_\alpha^d : x_1 \cdot ... \cdot x_d = 0\},$$

where as above L^d is the union of all coordinate planes in $\mathbb{R}^d$. It turns out that only functions that vanish at these non smooth points admit homogeneous polynomial approximation on K_α^d. Thus when $d > 2$ the exceptional zero set consisting of all non smooth points is essentially wider than the set of all vertices of K_α^d.

Theorem 4 *Let $d > 2$, $0 < \alpha < 1$. Then in order for an even function $f \in C(K_\alpha^d)$ to be a uniform limit on K_α of homogeneous polynomials $h_{2n} \in H_{2n}^d$ it is necessary that $f = 0$ on $K_\alpha^d \cap L^d$. Moreover, if $0 < \alpha < 1$ is rational then the condition $f(\mathbf{x}) = 0, \mathbf{x} \in K_\alpha^d \cap L^d$ is also sufficient for this homogeneous polynomial approximation to hold.*

Remark. It should be noted that Theorems 3 and 4 provide matching necessary and sufficient conditions for multivariate weighted and homogeneous approximation for *rational* $0 < \alpha < 1$. In [4] analogous results are given in the univariate case for *every* $0 < \alpha < 1$. In fact two proofs of sufficiency in Theorem 3 are given in [4] when $d = 1$: one using potential theoretic methods for every $0 < \alpha < 1$ and another one based on a explicit construction of approximating polynomials which works for rational $0 < \alpha < 1$. When $d \geq 2$ the needed potential theoretic methods are not available anymore, but nevertheless it will be shown below that for rational $0 < \alpha < 1$ the explicit construction of multivariate weighted approximating polynomials can still be accomplished even if $d \geq 2$. It appears to be plausible that sufficiency in Theorems 3 and 4 remains valid for irrational $0 < \alpha < 1$, as well. This remains an interesting open problem.

The next two sections contain the proofs of Theorems 1,3 and 4. We will need to verify several auxiliary lemmas some of which are of independent interest. In particular, a general duality between weighted and homogeneous polynomial approximation will be exhibited, see Lemmas 2 and 5 below.

Proof of Theorem 1

We will need first to verify some auxiliary lemmas. The first lemma asserts that w^* is convex on $\mathbb{R}^{d+1}$ for every admissible convex weight w, i.e., the homogenization of weights preserves convexity.

Lemma 1 *For any admissible convex weight $w \in C(\mathbb{R}^d)$ it follows that $w^*(\mathbf{x}, t)$ is also a convex continuous weight on $\mathbb{R}^{d+1}$.*

Proof. Since the weight w is even it follows that $w^*(\mathbf{x}, t) = |t| w\left(\frac{\mathbf{x}}{t}\right), \mathbf{x} \in \mathbb{R}^d$ is even in variable t. Therefore for every $(\mathbf{x}, a), (\mathbf{y}, b) \in \mathbb{R}^{d+1}$ with some reals $a, b \neq 0, a + b \neq 0$ we have by the convexity of w

$$w^*(\mathbf{x}, a) + w^*(\mathbf{y}, b) = |a| w\left(\frac{\mathbf{x}}{|a|}\right) + |b| w\left(\frac{\mathbf{y}}{|b|}\right) =$$

$$(|a| + |b|) \left(\frac{|a|}{|a| + |b|} w\left(\frac{\mathbf{x}}{|a|}\right) + \frac{|b|}{|a| + |b|} w\left(\frac{\mathbf{y}}{|b|}\right)\right) \geq (|a| + |b|) w\left(\frac{\mathbf{x} + \mathbf{y}}{|a| + |b|}\right).$$

Furthermore, recalling that $tw\left(\frac{\mathbf{x}}{t}\right)$ is a monotone increasing function of $t > 0$ for any $\mathbf{x} \in \mathbb{R}^d$ yields

$$w^*(\mathbf{x}, a) + w^*(\mathbf{y}, b) \geq (|a| + |b|)w\left(\frac{\mathbf{x} + \mathbf{y}}{|a| + |b|}\right) \geq |a + b|w\left(\frac{\mathbf{x} + \mathbf{y}}{|a + b|}\right) = w^*(\mathbf{x} + \mathbf{y}, a + b).$$

The above inequality was derived for any $(\mathbf{x}, a), (\mathbf{y}, b) \in \mathbb{R}^{d+1}$ with $a, b \neq 0, a + b \neq 0$. But since $w^*(\mathbf{x}, t)$ is continuous on $\mathbb{R}^{d+1}$ it follows that

$$w^*(\mathbf{x}, a) + w^*(\mathbf{y}, b) \geq w^*(\mathbf{x} + \mathbf{y}, a + b), \quad \forall (\mathbf{x}, a), (\mathbf{y}, b) \in \mathbb{R}^{d+1}.$$

In addition, we evidently have $w^*(c\mathbf{x}, ct) = |c|w^*(\mathbf{x}, t), c \in \mathbb{R}$ which together with the above inequality verifies the convexity of $w^*(\mathbf{x}, t)$. $\square$

Our next lemma provides an important duality between weighted polynomial approximation on $\mathbb{R}^d$ and homogeneous polynomial approximation on star like domains in $\mathbb{R}^{d+1}$ associated with the corresponding weight.

Let $K \subset \mathbb{R}^d$ be a compact 0-symmetric set with nonempty interior which is *starlike* with respect to the origin, that is for every $\mathbf{x} \in K$ we have that $(-\mathbf{x}, \mathbf{x}) \subset \text{Int} K$.

When K is a 0-symmetric star like domain in $\mathbb{R}^d$ its Minkowski functional is defined by the relation

$$\phi_K(\mathbf{x}) := \inf\{\alpha > 0 : \frac{\mathbf{x}}{\alpha} \in K\}, \ \mathbf{x} \in \mathbb{R}^d.$$

Note that $\phi_K(\alpha\mathbf{x}) := |\alpha|\phi_K(\mathbf{x}), \alpha \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^d$ and

$$K = \{\mathbf{x} \in \mathbb{R}^d : \phi_K(\mathbf{x}) \leq 1\}, \ \partial K := \{\mathbf{x} \in \mathbb{R}^d : \phi_K(\mathbf{x}) = 1\}.$$

Let $w \in C(\mathbb{R}^d), d \geq 1$ be an admissible weight on $\mathbb{R}^d$. With this weight we associate a 0-symmetric star like domain defined by

$$K_w := \{(\mathbf{x}, t) \in \mathbb{R}^{d+1} : w^*(\mathbf{x}, t) \leq 1\},$$

where $w^*(\mathbf{x}, t)$ is given by the relation (2). Since $w^*(\mathbf{x}, t)$ is even, both in $\mathbf{x} \in \mathbb{R}^d$ and $t \in \mathbb{R}$, it follows that K_w is symmetric with respect to $\mathbf{x} \in \mathbb{R}^d$ and $t \in \mathbb{R}$. Thus K_w is a 0-symmetric star like domain in $\mathbb{R}^{d+1}$. In addition it follows from Lemma 1 that K_w is convex whenever w is convex. Moreover, it is also easy to see that $w^*(\mathbf{x}, t)$ is the Minkowski functional of K_w.

Conversely, assume that K is a 0-symmetric star like set of points $(\mathbf{x}, t) \in \mathbb{R}^{d+1}$ which is also symmetric with respect to $\mathbf{x} \in \mathbb{R}^d$ for every fixed $t \in \mathbb{R}$, i.e., $(\mathbf{x}, t) \in K \Leftrightarrow (-\mathbf{x}, t) \in K, \forall t$. Then it is easy to see that its Minkowski functional $\phi_K(\mathbf{x}, t)$ is even both in $\mathbf{x} \in \mathbb{R}^d$ and $t \in \mathbb{R}$. Now we associate this set K with an even positive weight on $\mathbb{R}^d$ defined by the relation

$$w_K(\mathbf{x}) := \phi_K(\mathbf{x}, 1), \ \mathbf{x} \in \mathbb{R}^d.$$

Lemma 2 *(i) Let $w \in C(\mathbb{R}^d), d \geq 1$ be an admissible weight on $\mathbb{R}^d$. Assume that for each $g \in C_0(\mathbb{R}^d)$ there exist polynomials $p_{2n} \in P_{2n}^d$ so that $w^{-2n}p_{2n} \to g, n \to \infty$ uniformly on $\mathbb{R}^d$. Then for every even function $f \in C(\partial K_w)$ there exist homogeneous polynomials $h_{2n} \in H_{2n}^{d+1}$ for which $f = \lim h_{2n}$ uniformly on ∂K_w.*

(ii) Conversely, let K be any 0-symmetric star like set of points $(\mathbf{x}, t) \in \mathbb{R}^{d+1}$ which is symmetric with respect to $\mathbf{x} \in \mathbb{R}^d$ for every fixed $t \in \mathbb{R}$. Assume that for each even function $f \in C(\partial K)$ there exist homogeneous polynomials $h_{2n} \in H_{2n}^{d+1}$ such that $f = \lim h_{2n}$ uniformly on ∂K. Then for every $g \in C_0(\mathbb{R}^d)$ there exist polynomials $p_{2n} \in P_{2n}^d$ so that $w_K^{-2n}p_{2n} \to g, n \to \infty$ uniformly on $\mathbb{R}^d$.

Proof. (i) By definition (2) it follows that for every $\mathbf{x} \in \mathbb{R}^d$

$$w^*\left(\frac{\mathbf{x}}{w(\mathbf{x})}, \frac{1}{w(\mathbf{x})}\right) = 1.$$

Since

$$\partial K_w := \{(\mathbf{x}, t) \in \mathbb{R}^{d+1} : w^*(\mathbf{x}, t) = 1\}$$

this means that

$$\left(\frac{\mathbf{x}}{w(\mathbf{x})}, \frac{1}{w(\mathbf{x})}\right) \in \partial K_w, \ \mathbf{x} \in \mathbb{R}^d.$$

Now given any even function $f \in C(\partial K_w)$ set

$$g(\mathbf{x}) := f\left(\frac{\mathbf{x}}{w(\mathbf{x})}, \frac{1}{w(\mathbf{x})}\right), \quad \mathbf{x} \in \mathbb{R}^d.$$

Since $f \in C(\partial K_w)$ and w is positive and continuous this implies $g \in C(\mathbb{R}^d)$. Moreover, let us show that $g \in C_0(\mathbb{R}^d)$. Indeed using that f is even it follows by (1) that as $|t| \to \infty$

$$g(t\mathbf{x}) = f\left(\frac{t\mathbf{x}}{w(t\mathbf{x})}, \frac{1}{w(t\mathbf{x})}\right) \to f\left(\frac{\mathbf{x}}{\tilde{w}(\mathbf{x})}, 0\right) \quad \text{uniformly for } \mathbf{x} \in S^{d-1}.$$

Thus $g \in C_0(\mathbb{R}^d)$ and hence there exist polynomials $p_{2n} \in P_{2n}^d$ so that $w^{-2n}p_{2n} \to g$, $n \to \infty$ uniformly on $\mathbb{R}^d$. Now set

$$h_{2n}(\mathbf{x}, t) := t^{2n} p_{2n}\left(\frac{\mathbf{x}}{t}\right) \in H_{2n}^{d+1}$$

and

$$\partial K_w^+ := \{\mathbf{x} = (x_1, ..., x_{d+1}) \in K_w, \ x_{d+1} > 0\}.$$

Note that for any $(\mathbf{x}, t) \in \partial K_w^+$, we have $tw(\mathbf{x}/t) = 1, t > 0$. Thus using substitution $\mathbf{y} = \mathbf{x}/t \in \mathbb{R}^d$ and relation $t = 1/w(\mathbf{y})$ we arrive at

$$f(\mathbf{x}, t) - h_{2n}(\mathbf{x}, t) = f(\mathbf{x}, t) - t^{2n}p_{2n}\left(\frac{\mathbf{x}}{t}\right) = f\left(\frac{\mathbf{y}}{w(\mathbf{y})}, \frac{1}{w(\mathbf{y})}\right) - w^{-2n}(\mathbf{y})p_{2n}(\mathbf{y}) = g(\mathbf{y}) - w^{-2n}(\mathbf{y})p_{2n}(\mathbf{y}). \quad (4)$$

Since $w^{-2n}p_{2n} \to g$, $n \to \infty$ uniformly on $\mathbb{R}^d$ we obtain that $f = \lim h_{2n}$ uniformly on ∂K_w^+. Since both f and h_{2n} are even continuous functions the last statement clearly extends to all of ∂K_w $\square$.

(ii) For any $g \in C_0(\mathbb{R}^d)$ set

$$f(\mathbf{x}, t) := \begin{cases} g\left(\frac{\mathbf{x}}{t}\right), \ \mathbf{x} \in \mathbb{R}^d, \ t \in \mathbb{R} \setminus \{0\} \\ r_g(\mathbf{x}), \ t = 0 \end{cases}$$

where by the definition of the space $C_0(\mathbb{R}^d)$

$$\lim_{|t| \to \infty} g(t\mathbf{x}) = r_g(\mathbf{x}) \text{ uniformly for } \mathbf{x} \in S^{d-1}.$$

This easily yields that f is continuous on $\mathbb{R}^{d+1} \setminus \{0\}$. In addition, it is clear that f is even. Then by the assumption (ii) there exist homogeneous polynomials $h_{2n} \in H_{2n}^{d+1}$ such that $f = \lim_{n \to \infty} h_{2n}$ uniformly on ∂K. Recall that for $(\mathbf{x}, t) \in \partial K \subset \mathbb{R}^{d+1}$ we have by the definition of w_K

$$1 = \phi_K(\mathbf{x}, t) = |t|\phi_K\left(\frac{\mathbf{x}}{t}, 1\right) = |t|w_K\left(\frac{\mathbf{x}}{t}\right),$$

i.e., $|t| = w_K^{-1}\left(\frac{\mathbf{x}}{t}\right)$. Now we set $p_{2n}(\mathbf{y}) := h_{2n}(\mathbf{y}, 1) \in P_{2n}^d$ and use again substitution $\mathbf{y} = \mathbf{x}/t \in \mathbb{R}^d$ yielding

$$f(\mathbf{x}, t) - h_{2n}(\mathbf{x}, t) = g\left(\frac{\mathbf{x}}{t}\right) - t^{2n}h_{2n}\left(\frac{\mathbf{x}}{t}, 1\right) = g(\mathbf{y}) - w_K^{-2n}(\mathbf{y})p_{2n}(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^d. \quad (5)$$

Since $f = \lim_{n \to \infty} h_{2n}$ uniformly on ∂K it follows that $w_K^{-2n}p_{2n} \to g$, $n \to \infty$ uniformly on $\mathbb{R}^d$. This completes the proof of the lemma.

The next lemma appears to be of independent interest. It shows that if weighted polynomial approximation on $\mathbb{R}^d$ holds for the admissible weights $w_j, 1 \le j \le s$ then it also holds for their maximum $w := \max\{w_j, 1 \le j \le s\}$. The proof of this statement will be based on Lemma 2 and an elegant result of Varjú [8]. This latter states that given any two 0-symmetric star-like domains $K_1, K_2 \in \mathbb{R}^d$ satisfying the homogeneous polynomial density Conjecture 1 formulated in the Introduction it follows that the same holds true for their intersection $K_1 \cap K_2$, as well.

Lemma 3 *Let $w_j \in C(\mathbb{R}^d), d \ge 1, 1 \le j \le s$ be admissible weights on $\mathbb{R}^d$ such that each $g \in C_0(\mathbb{R}^d)$ is a uniform limit on $\mathbb{R}^d$ of some weighted polynomials $w_j^{-2n}p_{2n,j}$, $p_{2n,j} \in P_{2n}^d$, $1 \le j \le s$ as $n \to \infty$. Then the same holds true with the weight $w := \max\{w_j, 1 \le j \le s\}$.*

Proof. Assume that each $g \in C_0(\mathbb{R}^d)$ is a uniform limit on $\mathbb{R}^d$ of weighted polynomials $w_j^{-2n} p_{2n,j}$, $p_{2n,j} \in P_{2n}^d$, $1 \le j \le s$ as $n \to \infty$. Then by Lemma 2 (i) every even function $f \in C(\partial K_{w_j})$ is a uniform limit on ∂K_{w_j} of some homogeneous polynomials $h_{2n,j} \in H_{2n}^{d+1}$, $1 \le j \le s$, where K_{w_j} are the 0-symmetric star like domains defined by $K_{w_j} := \{(\mathbf{x}, t) \in \mathbb{R}^{d+1} : w_j^*(\mathbf{x}, t) \le 1\}$, with $w_j^*(\mathbf{x}, t)$ given by relation (2) being the Minkowski functionals of K_{w_j}. Then by the result of Varjú [8] mentioned above the homogeneous polynomial density will hold for the 0-symmetric star like domain $K := \cap_{1 \le j \le s} K_{w_j}$, as well. It is easy to see that the corresponding Minkowski functionals satisfy the relation

$$\phi_K = \max\{\phi_{K_j}, 1 \le j \le s\} = \max\{w_j^*, 1 \le j \le s\}.$$

Furthermore, since for each even function $f \in C(\partial K)$ there exist homogeneous polynomials $h_{2n} \in H_{2n}^{d+1}$ such that $f = \lim_{n \to \infty} h_{2n}$ uniformly on ∂K it follows by Lemma 2 (ii) that for every $g \in C_0(\mathbb{R}^d)$ there exist polynomials $p_{2n} \in P_{2n}^d$ so that $w_K^{-2n} p_{2n} \to g$, $n \to \infty$ uniformly on $\mathbb{R}^d$, where by the previous relation and (2)

$$w_K(\mathbf{x}) = \phi_K(\mathbf{x}, 1) = \max\{w_j^*(\mathbf{x}, 1), 1 \le j \le s\} = \max\{w_j(\mathbf{x}), 1 \le j \le s\}, \quad \mathbf{x} \in \mathbb{R}^d,$$

i.e., weighted polynomial approximation on $\mathbb{R}^d$ also holds with the weight $w := \max\{w_j, 1 \le j \le s\}$. $\square$

Now we are in position to verify Theorem 1. First let us note that based on Lemma 3 it suffices to prove the theorem for the case $s = 1$. Thus consider a convex admissible weight w on $\mathbb{R}^d, d \ge 1$ such that in addition, w^* is C^1 on $\mathbb{R}^{d+1} \setminus \{0\}$ if $d > 1$. Consider the 0-symmetric star like domain associated with w given by

$$K_w := \{(\mathbf{x}, t) \in \mathbb{R}^{d+1} : w^*(\mathbf{x}, t) \le 1\}.$$

Here as before $w^*(\mathbf{x}, t)$ given by relation (2) is the Minkowski functional of K_w which is even in both $\mathbf{x} \in \mathbb{R}^d$ and $t \in \mathbb{R}$. Note that since w is convex it follows by Lemma 1 that $w^*(\mathbf{x}, t)$ is convex too, and therefore K_w is a 0-symmetric convex domain in $\mathbb{R}^{d+1}$. Moreover, if $d > 1$, i.e., $d + 1 > 2$ then the Minkowski functionals of K_w given by w^* is C^1 which implies that K_w is a *regular* 0-symmetric convex domain in $\mathbb{R}^{d+1}$ when $d + 1 > 2$. Then by the results on homogeneous polynomial approximation proved in [2],[8] and [3] for the case of convex bodies of dimension 2, and in case of any 0-symmetric regular convex body, respectively (see the introduction above), it follows that for each even function $f \in C(\partial K_w)$ there exist homogeneous polynomials $h_{2n} \in H_{2n}^{d+1}$ such that $f = \lim_{n \to \infty} h_{2n}$ uniformly on ∂K_w. Then by Lemma 2 (ii) for every $g \in C_0(\mathbb{R}^d)$ there exist polynomials $p_{2n} \in P_{2n}^d$ so that $w_{K_w}^{-2n} p_{2n} \to g$, $n \to \infty$ uniformly on $\mathbb{R}^d$. Here w_{K_w} is the weight associated with K_w which satisfies relations

$$w_{K_w}(\mathbf{x}) := \phi_{K_w}(\mathbf{x}, 1) = w^*(\mathbf{x}, 1) = w(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d.$$

Thus the required weighted polynomial approximation holds true for the weight w. This completes the proof of Theorem 1. $\square$

Proof of Theorems 3 and 4

Proof of necessity in Theorem 3. Assume now that $f \in C_0(\mathbb{R}^d)$ is a uniform limit on $\mathbb{R}^d$ of weighted polynomials $w_\alpha^{-2n} p_{2n}, p_{2n} \in P_{2n}^d$. In particular, this implies that for every $\mathbf{y} \in K_\alpha^d$, $|\mathbf{y}|_\alpha = 1$ the univariate function $g(t) := f(t\mathbf{y})$ is a uniform limit on $\mathbb{R}$ of univariate weighted polynomials $(1 + |t|^\alpha)^{-2n/\alpha} p_{2n}, p_{2n} \in P_{2n}^1$. Then by [4], Theorem 1 $g(0) = g(\infty) = g(-\infty) = 0$. Since $f \in C_0(\mathbb{R}^d)$ and thus $\lim_{|t| \to \infty} f(t\mathbf{x}) = r_f(\mathbf{x})$ uniformly on S^{d-1} it follows that $r_f(\mathbf{x}) = 0, \forall \mathbf{x} \in S^{d-1}$, i.e., $f(\infty) = 0$. Furthermore, for any $1 \le j \le d$ and any $\mathbf{x} = (x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_d) \in L_j^d$ set $a := (1 + |\mathbf{x}|_\alpha^\alpha)^{1/\alpha}$. Now consider the univariate function

$$g(t) := f(x_1, ..., x_{j-1}, at, x_{j+1}, ..., x_d).$$

Since $f = \lim_{n \to \infty} w_\alpha^{-2n} p_{2n}$ uniformly on $\mathbb{R}^d$ it obviously follows that

$$g(t) = \lim_{n \to \infty} (1 + |t|^\alpha)^{-2n/\alpha} q_{2n}, \quad q_{2n} := a^{-2n} p_{2n}(x_1, ..., x_{j-1}, at, x_{j+1}, ..., x_d) \in P_{2n}^1.$$

Then using again [4], Theorem 1 we obtain $g(0) = f(\mathbf{x}) = 0$. Thus in addition to $f(\infty) = 0$ we also have that $f = 0$ on L^d. $\square$

Proof of necessity in Theorem 4. When $0 < \alpha < 1$ the l_α sphere in $\mathbb{R}^d$ given by

$$K_\alpha^d = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}|_\alpha = 1\}$$

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is not convex. In this case it has been verified in [4], Corollary 2 that if an even function $f(x,y) \in C(K_\alpha^2), d = 2, 0 < \alpha < 1$ is a uniform limit on K_α^2 of even homogeneous polynomials $f = \lim_{n\to\infty} h_{2n}, h_{2n} \in H_{2n}^d$, then $f(\pm 1, 0) = f(0, \pm 1) = 0$, i.e., the function must vanish at all vertices of K_α^2. Consider now arbitrary

$$\mathbf{y} = (y_1, ... y_d) \in K_\alpha^d \cap L^d, \ |\mathbf{y}|_\alpha = 1, \ y_1 \cdot ... \cdot y_d = 0\}, \ d \geq 2.$$

Then we can assume without loss of generality that $\mathbf{y} = (0, y_2, ..., y_d)$, and hence $|y_2|^\alpha + ... + |y_d|^\alpha = 1$. Denote $\mathbf{a}_1 := (1, 0, ..., 0) \in \mathbb{R}^d$ and let $M :=\text{span}\{\mathbf{y}, \mathbf{a}_1\}$, be the 2 dimensional plane spanned by $\mathbf{y}, \mathbf{a}_1$. Then evidently

$$K_\alpha^d \cap M = \{t_1\mathbf{a}_1 + t_2\mathbf{y} \in K_\alpha^d : \ t_1, t_2 \in \mathbb{R}\} =$$

$$\{(t_1, t_2) \in \mathbb{R}^2 : \ |t_1|^\alpha + |t_2|^\alpha(|y_2|^\alpha + ... + |y_d|^\alpha) = |t_1|^\alpha + |t_2|^\alpha = 1\}$$

is a 2 dimensional l_α sphere. Moreover, if the even function $f \in C(K_\alpha^d)$ is a uniform limit on K_α^d of homogeneous polynomials $h_{2n} \in H_{2n}^d$ then the bivariate continuous function $g(t_1, t_2) := f(t_1\mathbf{a}_1 + t_2\mathbf{y})$ is a uniform limit on $K_\alpha^d \cap M$ of corresponding bivariate homogeneous polynomials. Then using the bivariate result from [4] cited above it follows that $g(\pm 1, 0) = g(0, \pm 1) = 0$. Hence in particular, $g(0, 1) = f(\mathbf{y}) = 0$. This verifies that any even function $f \in C(K_\alpha^d)$ which is a uniform limit on K_α^d of homogeneous polynomials $h_{2n} \in H_{2n}^d$ must vanish at every point of $K_\alpha^d \cap L^d$, i.e., the necessity in Theorem 4 follows. $\square$

Now we proceed to the more difficult task of verifying the sufficiency in Theorems 3 and 4.

First we present a lemma which establishes an equivalence between weighted polynomial approximation on $\mathbb{R}^d$ and homogeneous polynomial approximation on K_α^{d+1} in the presence of exceptional zero sets. This lemma and its proof is similar to the duality statement of Lemma 2 and therefore we only briefly outline its proof.

Lemma 5 *Let $d \geq 1, \alpha > 0$. Then the following statements are equivalent:*
(i) For any $g \in C(\mathbb{R}^d)$ such that $g = 0$ on $L^d \cup \{\infty\}$ there exist $p_{2n} \in P_{2n}^d$ so that

$$g = \lim_{n\to\infty} w_\alpha^{-2n}(\mathbf{y})p_{2n}(\mathbf{y}). \tag{6}$$

uniformly on $\mathbb{R}^d$.
(ii) Every even function $f \in C(K_\alpha^{d+1})$ such that $f = 0$ on $K_\alpha^{d+1} \cap L^{d+1}$ is a uniform limit on K_α^{d+1} of certain homogeneous polynomials $h_{2n} \in H_{2n}^{d+1}$ of degree $2n$.

Proof. $(i) \Rightarrow (ii)$ For any even function $f \in C(K_\alpha^{d+1})$ such that $f = 0$ on $K_\alpha^{d+1} \cap L^{d+1}$ set

$$g(\mathbf{y}) := f\left(\frac{\mathbf{y}}{w_\alpha(\mathbf{y})}, \frac{1}{w_\alpha(\mathbf{y})}\right) = f(\mathbf{x}), \ \mathbf{y} \in \mathbb{R}^d, \ \mathbf{x} \in \mathbb{R}^{d+1}.$$

As shown in the proof of Lemma 2 (i) it follows that $g \in C_0(\mathbb{R}^d)$. Now we need to show that in addition $g = 0$ on $L^d \cup \{\infty\}$. First note that if $\mathbf{y} = (y_1, ..., y_d) \in L^d$, i.e., $y_j = 0$ for some $1 \leq j \leq d$ then also $x_j = 0$ yielding that $\mathbf{x} \in K_\alpha^{d+1} \cap L^{d+1}$. Hence $g(\mathbf{y}) = f(\mathbf{x}) = 0$, i.e., $g = 0$ on L^d. Furthermore, since

$$w_\alpha(\mathbf{y}) \geq \max_{1\leq j\leq d} |y_j| \geq d^{-\frac{1}{2}}|\mathbf{y}|$$

it follows that $x_{d+1} = \frac{1}{w_\alpha(\mathbf{y})} \to 0$ whenever $|\mathbf{y}| \to \infty$. Thus using again that $f \in C(K_\alpha^{d+1})$ and $f(\mathbf{x}) = 0$ if $x_{d+1} = 0$ we obtain that $g(\mathbf{y}) = f(\mathbf{x}) \to 0$ if $|\mathbf{y}| \to \infty$. Thus summarizing above observations we obtain that $g \in C_0(\mathbb{R}^d)$ and $g = 0$ on $L^d \cup \{\infty\}$. Therefore by (i) with suitable $p_{2n} \in P_{2n}^d, n \in \mathbb{N}$ relation (6) must hold uniformly on $\mathbb{R}^d$. However, as shown in (4) for every $\mathbf{x} \in \partial K_\alpha^{d+1}, x_{d+1} > 0$

$$g(\mathbf{y}) - w^{-2n}(\mathbf{y})p_{2n}(\mathbf{y}) = f(\mathbf{x}) - h_{2n}(\mathbf{x}),$$

where h_{2n} is a homogeneous polynomial of $\mathbf{x} \in \mathbb{R}^{d+1}$ of degree $2n$. Thus since both f and h_{2n} are even continuous functions it follows that $f = \lim h_{2n}$ uniformly on ∂K_α^{d+1}.

$(ii) \Rightarrow (i)$ Consider any $g \in C(\mathbb{R}^d), d \geq 1$ such that $g = 0$ on $L^d \cup \{\infty\}$. Given $\mathbf{x} \in K_\alpha^{d+1}$ set

$$f(\mathbf{x}) := \begin{cases} g\left(\frac{x_1}{x_{d+1}}, ..., \frac{x_d}{x_{d+1}}\right), \ x_{d+1} \neq 0 \\ 0, \ x_{d+1} = 0. \end{cases}$$

Evidently, since $g \in C_0(\mathbb{R}^d)$ and $g(\infty) = 0$ it follows that $f \in C(K_\alpha^{d+1})$. Furthermore, f is even and since $g = 0$ on L^d we also have that $f = 0$ on $K_\alpha^{d+1} \cap L^{d+1}$. Then by assumption (ii) $f = \lim h_{2n}, h_{2n} \in H_{2n}^{d+1}$ uniformly on K_α^{d+1}. Setting $y_j := x_j/x_{d+1}, 1 \le j \le d$ for any $\mathbf{x} = (x_1, ..., x_{d+1}) \in K_\alpha^{d+1}, x_{d+1} > 0$ we obtain

$$x_{d+1} = \frac{1}{w_\alpha(\mathbf{y})}, \ x_j = \frac{y_j}{w_\alpha(\mathbf{y})}, 1 \le j \le d$$

and hence as in (5)

$$f(\mathbf{x}) - h_{2n}(\mathbf{x}) = g(\mathbf{y}) - w_\alpha^{-2n}(\mathbf{y})p_{2n}(\mathbf{y}) \to 0$$

uniformly on $\mathbb{R}^d$ $\square$.

The next lemma provides an explicit construction of certain multivariate polynomials which are used for multivariate weighted approximation in $\mathbb{R}^d, d \ge 2$.

Lemma 6 *Let $\alpha = \frac{p}{q}, p, q \in \mathbb{N}$ and $t_k := e^{\frac{\pi k i}{q}}$, $1 \le k \le 2q$, be the $2q$-th roots of unity. Then for any $n \in \mathbb{N}$ and $\mathbf{x} \in \mathbb{R}^d$*

$$Q_n(\mathbf{x}) := |x_1|^\alpha ... |x_d|^\alpha \sum_{1 \le s_j \le 2q, 1 \le j \le d} t_{s_1} ... t_{s_d} (1 + t_{s_1}|x_1|^\alpha + ... + t_{s_d}|x_d|^\alpha)^{nq} \in P_{pn+d}^d. \tag{7}$$

Proof. We use the relation

$$\sum_{k=1}^{2q} t_k^l = 0,$$

which holds for any integer $l \ne 2rq$, $r \in \mathbb{N}$. To verify that, it suffices to note that, by the periodicity of roots of unity, $\sum_{k=1}^{2q} t_k^l = t_1^l \sum_{k=1}^{2q} t_k^l$. Hence if $t_1^l \ne 1$, then the above relation must hold. In addition, we evidently have $\sum_{k=1}^{2q} t_k^l = 2q$ whenever $l = 2rq, r \in \mathbb{N}$.

Using these relations and the multinomial formula we have for any $\mathbf{u} = (u_1, ... u_d) \in \mathbb{R}^d$

$$Q(\mathbf{u}) := u_1 ... u_d \sum_{1 \le s_j \le 2q, 1 \le j \le d} t_{s_1} ... t_{s_d} (1 + t_{s_1} u_1 + ... + t_{s_d} u_d)^{nq}$$

$$= u_1 ... u_d \sum_{1 \le s_j \le 2q, 1 \le j \le d} t_{s_1} ... t_{s_d} \sum_{m_1 + ... + m_d \le nq} b_{\mathbf{m}} (t_{s_1} u_1)^{m_1} ... (t_{s_d} u_d)^{m_d}$$

$$\sum_{m_1 + ... + m_d \le nq} b_{\mathbf{m}} u_1^{m_1+1} ... u_d^{m_d+1} \prod_{1 \le j \le d} \sum_{1 \le s_j \le 2q} t_{s_j}^{m_j+1}$$

with proper multinomial coefficients $b_{\mathbf{m}}$. Clearly, we have for the product

$$\prod_{1 \le j \le d} \sum_{1 \le s_j \le 2q} t_{s_j}^{m_j+1} = (2q)^d$$

whenever $m_j + 1 = 2qr_j, 1 \le j \le d$ for some integers $r_j \in \mathbb{N}$, and otherwise this product is always zero. Thus we obtain that

$$Q(\mathbf{u}) = \sum_{2r_1 + ... + 2r_d \le n + \frac{d}{q}} c_{\mathbf{r}} u_1^{2qr_1} ... u_d^{2qr_d}$$

with some $c_{\mathbf{r}} \in \mathbb{R}$. Hence setting $u_j := |x_j|^\alpha = |x|^{\frac{p}{q}}, 1 \le j \le d$ it follows that $u_j^{2qr_j} = x_j^{2pr_j}, 1 \le j \le d$ and therefore

$$Q_n(\mathbf{x}) := Q(|x_1|^\alpha, ..., |x_d|^\alpha) = \sum_{2r_1 + ... + 2r_d \le n + \frac{d}{q}} c_{\mathbf{r}} x_1^{2pr_1} ... x_d^{2pr_d} \in P_{pn+d}^d. \ \square$$

Lemma 7 *Let $0 < \alpha = \frac{p}{q}, p, q \in \mathbb{N}$ and $Q_n \in P_{pn+d}^d$ be the polynomial given by (7). Then for any polynomial $G \in P_p^d$ of degree at most p we have*

$$\left\| \frac{G(\mathbf{x})Q_n(\mathbf{x})}{w_\alpha(\mathbf{x})^{p(n+d+2)}} - \frac{G(\mathbf{x})|x_1 ... x_d|^\alpha}{w_\alpha(\mathbf{x})^{p(d+2)}} \right\|_{C(\mathbb{R}^d)} = O\left(\frac{1}{n}\right), \ n \in \mathbb{N}. \tag{8}$$

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Proof. We will need the following identity which holds for any given $a > 0$ and $t_k = e^{\frac{\pi k i}{q}}, 1 \leq k \leq 2q$

$$|1 + t_k a|^2 = (1 + a\Re t_k)^2 + a^2(\Im t_k)^2 = (1 + a)^2 + 2a\Re t_k - 2a = (1 + a)^2 - 4a \sin^2 \frac{\pi k}{2q}.$$

Evidently it implies that whenever $1 \leq k \leq 2q - 1$, i.e, $t_k \neq 1$

$$|1 + t_k a| \leq (1 + a)\sqrt{1 - \frac{4a}{(1 + a)^2} \sin^2 \frac{\pi k}{2q}} \leq 1 + a - \frac{2a}{1 + a} \sin^2 \frac{\pi}{2q} = 1 + a - \frac{\gamma_q a}{1 + a}, \quad \gamma_q := 2 \sin^2 \frac{\pi}{2q}. \tag{9}$$

Iterating the above estimate further consider the sum $|1 + t_{s_1} a_1 + ... + t_{s_d} a_d|, a_j > 0, 1 \leq s_j \leq 2q$ assuming that $t_{s_d} \neq 1$. Then by (9)

$$|1 + t_{s_1} a_1 + ... + t_{s_d} a_d| \leq a_1 + ... + a_{d-1} + |1 + t_{s_d} a_d| \leq 1 + a_1 + ... + a_d - \frac{\gamma_q a_d}{1 + a_d}. \tag{10}$$

Now we will apply relation (7) which yields using notation $|\mathbf{x}|_\alpha := (|x_1|^\alpha + ... + |x_d|^\alpha)^{\frac{1}{\alpha}}$

$$\frac{Q_n(x_1, ..., x_d)}{(1 + |\mathbf{x}|_\alpha^\alpha)^{qn+s}} = \frac{|x_1...x_d|^\alpha}{(1 + |\mathbf{x}|_\alpha^\alpha)^s} + R(x_1, ..., x_d) \tag{11}$$

where

$$R(x_1, ..., x_d) := \frac{|x_1...x_d|^\alpha \sum^* t_{s_1}...t_{s_d}(1 + t_{s_1}|x_1|^\alpha + ... + t_{s_d}|x_d|^\alpha)^{nq}}{(1 + |\mathbf{x}|_\alpha^\alpha)^{qn+s}}$$

with the symbol $\sum^*$ indicating summation over all $1 \leq s_j \leq 2q, 1 \leq j \leq d$ except the term corresponding to $s_1 = ... = s_d = 2q$. Thus in each term of $\sum^*$ at least one of t_j-s is distinct from 1. Therefore using relation (10) and the symmetry we obtain that for each term in $\sum^*$

$$\left\| \frac{|x_1...x_d|^\alpha(1 + t_{s_1}|x_1|^\alpha + ... + t_{s_d}|x_d|^\alpha)^{nq}}{(1 + |\mathbf{x}|_\alpha^\alpha)^{qn+s}} \right\|_{C(\mathbb{R}^d)} \leq \left\| \frac{|x_1...x_d|^\alpha(1 + |\mathbf{x}|_\alpha^\alpha - \frac{\gamma_q|x_d|^\alpha}{1+|x_d|^\alpha})^{nq}}{(1 + |\mathbf{x}|_\alpha^\alpha)^{qn+s}} \right\|_{C(\mathbb{R}^d)}$$

Thus for the remainder term in (11) we obtain the next estimate

$$\|R\|_{C(\mathbb{R}^d)} \leq (2q)^d \left\| \frac{|x_1...x_d|^\alpha(1 + |\mathbf{x}|_\alpha^\alpha - \frac{\gamma_q|x_d|^\alpha}{1+|x_d|^\alpha})^{nq}}{(1 + |\mathbf{x}|_\alpha^\alpha)^{qn+s}} \right\|_{C(\mathbb{R}^d)}.$$

We will now give an upper bound for the norm of the above function using notations

$$v(\mathbf{x}) = v(x_1, ..., x_d) := 1 + |\mathbf{x}|_\alpha^\alpha, \quad u(\mathbf{x}) = u(x_1, ..., x_d) := \frac{|x_d|^\alpha}{(1 + |x_d|^\alpha)v(\mathbf{x})}.$$

Applying these notations in the above estimate for $\|R\|_{C(\mathbb{R}^d)}$ and using that

$$|x_1...x_{d-1}|^\alpha \leq v(\mathbf{x})^{d-1}, \quad |x_d|^\alpha \leq v(\mathbf{x})^2 u(\mathbf{x})$$

clearly implies

$$\|R\|_{\mathbb{R}^d} \leq (2q)^d \left\| |x_1|...x_d|^\alpha \frac{(1 - \gamma_q u(\mathbf{x}))^{nq}}{v(\mathbf{x})^s} \right\|_{C(\mathbb{R}^d)} \leq (2q)^d \left\| \frac{u(\mathbf{x})(1 - \gamma_q u(\mathbf{x}))^{nq}}{v(\mathbf{x})^{s-d-1}} \right\|_{C(\mathbb{R}^d)}.$$

Now recalling that $s \geq d + 1$ and using that $v(\mathbf{x}) \geq 1$, $\mathbf{x} \in \mathbb{R}^d$ we obtain

$$\|R\|_{C(\mathbb{R}^d)} \leq \frac{(2q)^d}{\gamma_q} \|t(1 - t)^{nq}\|_{C(\mathbb{R})} \leq \frac{c(q, d)}{n}$$

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with a constant $c(q,d)$ depending only on q and d. Applying this estimate in (11) yields

$$\left| \frac{Q_n(\mathbf{x})}{(1+|\mathbf{x}|_\alpha^\alpha)^{qn+s}} - \frac{|x_1...x_d|^\alpha}{(1+|\mathbf{x}|_\alpha^\alpha)^s} \right| \le \frac{c(q,d)}{n}, \qquad \mathbf{x} = (x_1,...,x_d) \in \mathbb{R}^d, \ n \in \mathbb{N}. \tag{12}$$

Now we can observe that for any polynomial $G \in P_p^d$ of degree at most p we have

$$\left\| \frac{G}{w_\alpha^p} \right\|_{C(\mathbb{R}^d)} \le c.$$

Thus multiplying relation (12) by the bounded quantity $|\frac{G(\mathbf{x})}{w_\alpha(\mathbf{x})^p}|$ and setting $s := \frac{p(d+1)}{\alpha} > d+1$ in (12) yields

$$\left\| \frac{G(\mathbf{x})Q_n(\mathbf{x})}{w_\alpha(\mathbf{x})^{p(n+d+2)}} - \frac{G(\mathbf{x})|x_1...x_d|^\alpha}{w_\alpha(\mathbf{x})^{p(d+2)}} \right\|_{C(\mathbb{R}^d)} = O\left(\frac{1}{n}\right),$$

which is the statement of the lemma. $\square$

Proof of sufficiency in Theorems 3 and 4. Now the proof of the sufficiency in Theorem 3 can be finalized by using a Stone-Weierstrass-type argument. For any closed subalgebra of functions $A \subset C(K)$ with K being a compact Hausdorff space denote by $Z_A := \{x \in K : g(x) = 0, \text{for all } g \in A\}$ the zero set of A. Then the Stone-Weierstrass theorem (see e.g., [1, p. 13]) states that if A separates points in $K \setminus Z_A$, then any $f \in C(K)$ which vanishes on Z_A belongs to A, i.e., $A = \{f \in C(K) : f = 0 \text{ on } Z_A\}$. We will use this statement in the space

$$C^*(\mathbb{R}^d) := \{f \in C(\mathbb{R}^d) : f(\infty)\} = 0$$

of continuous functions vanishing at infinity which clearly provides the needed compactification of the domain.

Let $\alpha = \frac{p}{q}$, $p,q \in \mathbb{N}$, $0 < \alpha < 1$. The proof will be accomplished in two steps. First we will verify the need statement for the subsequence $\mathbb{N}_p := \{pn : n \in \mathbb{N}\}$.

Set

$$A_p := \{f \in C^*(\mathbb{R}^d) : f = \lim_{n\to\infty} w_\alpha^{-pn} g_{pn}, \ g_{pn} \in P_{pn}^d\}.$$

Obviously, A_p is a closed subalgebra of $C^*(\mathbb{R}^d)$, where by the already proven necessary condition in Theorem 3 we have $L^d \cup \{\infty\} \in Z_{A_p}$. Furthermore it follows by Lemma 7 that for any polynomial $G \in P_p^d$ of degree at most p we have

$$G(\mathbf{x})|x_1...x_d|^\alpha w_\alpha(\mathbf{x})^{-p(d+2)} \in A_p. \tag{13}$$

Since function $|x_1...x_d|^\alpha w_\alpha(\mathbf{x})^{-p(d+2)}$ vanishes only on $L^d \cup \{\infty\}$ it follows that $L^d \cup \{\infty\} = Z_{A_p}$. Moreover polynomials $G \in P_p^d, p \ge 1$ separate points in all of $\mathbb{R}^d$ and hence functions (13) separate points in $\mathbb{R}^d \setminus \{L^d \cup \{\infty\}\}$. Thus any function $f \in C(\mathbb{R}^d)$ which vanishes on $L^d \cup \{\infty\}$ is in A_p i.e., is a uniform limit on $\mathbb{R}^d$ of weighted polynomials $w_\alpha^{-pn} g_{pn}, \ g_{pn} \in P_{pn}^d$.

Now for any $0 \le j \le p-1$ and any polynomial $G \in P_1^d$ of degree at most 1 consider continuous functions

$$f_j(\mathbf{x}) := x_1...x_d G(\mathbf{x}) w_\alpha(\mathbf{x})^{-p-d+j-1}, \ \ 0 \le j \le p-1.$$

It can be easily verified that f_j vanishes on $L^d \cup \{\infty\} = Z_{A_p}$ for every $0 \le j \le p-1$. Thus by the previous step of the proof each f_j is a uniform limit on $\mathbb{R}^d$ of weighted polynomials $w_\alpha^{-pn} g_{j,n}, \ g_{j,n} \in P_{pn}^d, 0 \le j \le p-1$. Of course this is equivalent to

$$w_\alpha(\mathbf{x})^{-pn-j} g_{j,n}(\mathbf{x}) \to x_1...x_d G(\mathbf{x}) w_\alpha(\mathbf{x})^{-p-d-1} \in C^*(\mathbb{R}^d), \qquad 0 \le j \le p-1, \qquad n \to \infty.$$

Now for an $m \in \mathbb{N}$, $m \equiv j \pmod{p}$, say for $m = pn+j$ with some $n \in \mathbb{N}$ and $0 \le j \le p-1$, set

$$g_m := g_{j,n} \in P_{pn}^d \subset P_m^d.$$

Then the last relation means that we found a sequence of polynomials $g_m \in P_m^d$, $m = 1, 2, \ldots$, such that

$$w_\alpha(\mathbf{x})^{-m} g_m(\mathbf{x}) \to x_1...x_d G(\mathbf{x}) w_\alpha(\mathbf{x})^{-p-d-1} \in C^*(\mathbb{R}^d), \qquad 0 \le j \le p-1, \qquad n \to \infty.$$

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Therefore, if we define

$$A := \{f \in C^*(\mathbb{R}^d) : f = \lim_{m \to \infty} w_\alpha^{-m} g_m, \ g_m \in P_m^d\},$$

then it follows that $x_1...x_d G(\mathbf{x}) w_\alpha(\mathbf{x})^{-p-d-1} \in A$ for any polynomial $G \in P_1^d$ of degree at most 1. Just as above this means that $Z_A = L^d \cup \{\infty\}$ and we have suitable test functions in the closed subalgebra $A \subset C^*(\mathbb{R}^d)$ which separate points in $\mathbb{R}^d \setminus \{L^d \cup \{\infty\}\}$. Hence, repeated application of the Stone-Weierstrass theorem implies that any function $f \in C(\mathbb{R}^d)$ which vanishes on $L^d \cup \{\infty\}$ is in A i.e., is a uniform limit on $\mathbb{R}^d$ of weighted polynomials $w_\alpha^{-n} g_n, \ g_n \in P_n^d$. This completes the proof of the sufficiency in Theorem 3. Moreover, recalling Lemma 5 we obtain the sufficiency in Theorem 4, as well. $\square$

Conclusions. As it can be seen from the above results convexity of the weight plays a crucial role in the study of density of weighted polynomial approximation. In this respect it would be very important to verify that density holds for every convex admissible weight without the assumption of piecewise C^1 smoothness as stated in Conjecture 2 for $d > 1$. Another promising line of investigation could be related to the study of non convex weights. As it is indicated by Theorem 3 in non convex case certain exceptional zero sets come into play and finding these sets poses a rather nontrivial question. Furthermore, the study of weighted polynomial approximation in $C_0(\mathbb{R}^d)$ might lead to a further insight into the question of density of multivariate homogeneous polynomials.

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