Partitioning the power set of $[n]$ into C_k-free parts

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Abstract

We show that for $n \geq 3, n \neq 5$, in any partition of $\mathcal{P}(n)$, the set of all subsets of $[n] = \{1, 2, \dots, n\}$, into $2^{n-2} - 1$ parts, some part must contain a triangle — three different subsets $A, B, C \subseteq [n]$ such that $A \cap B$, $A \cap C$, and $B \cap C$ have distinct representatives. This is sharp, since by placing two complementary pairs of sets into each partition class, we have a partition into 2^{n-2} triangle-free parts. We also address a more general Ramsey-type problem: for a given graph G, find (estimate) $f(n, G)$, the smallest number of colors needed for a coloring of $\mathcal{P}(n)$, such that no color class contains a Berge-G subhypergraph. We give an upper bound for $f(n, G)$ for any connected graph G which is asymptotically sharp (for fixed k) when $G = C_k, P_k, S_k$, a cycle, path, or star with k edges. Additional bounds are given for $G = C_4$ and $G = S_3$.

1 Introduction and results

Hypergraph Ramsey problems usually address the existence of large monochromatic structures in colorings of the edges of K_n^r, the complete r-uniform hypergraph. It is rare that monochromatic structures are sought in colorings of hypergraphs containing *all subsets* of $[n]$, $\mathcal{P}(n)$. An exception is the Finite Unions Theorem of Folkman, Rado, Sanders [6]. A more recent research in this direction is by Axenovich and Gyárfás [1], where Ramsey numbers of Berge-G hypergraphs were studied for several graphs G in colorings of $\mathcal{P}(n)$. Ramsey numbers of Berge-G hypergraphs in the *uniform case* have been investigated also in [5, 9].

A hypergraph $H = (V, F)$ is called Berge-G if $G = (V, E)$ is a graph and there exists a bijection $g : E(G) \mapsto E(H)$ such that for $e \in E(G)$ we have $e \subseteq g(e)$. Note that for a given graph G there are many Berge-G-hypergraphs. Berge-G hypergraphs were defined by Gerbner and Palmer [4] to extend the notion of paths and cycles in hypergraphs introduced

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by Berge in [3]. In particular, a Berge-C_3 hypergraph consists of three subsets $A, B, C \subseteq [n]$ such that $A \cap B, A \cap C, B \cap C$ have distinct representatives. When there is no confusion, we will often refer to a Berge-G hypergraph simply as 'a G.' The graphs C_k, P_k, S_k denote cycle, path, and star with k edges, respectively. It is customary to use the names *triangle* and *claw* for the graphs C_3 and S_3, respectively.

A hypergraph H with vertex set $[n]$ and whose edges are sets from $\mathcal{P}(n)$ is called *G-free*, if it does not contain any subhypergraph isomorphic to a Berge-G hypergraph. The *intersection graph* of a hypergraph H is a graph G whose vertices represent edges of H and where there is an edge in G if and only if the corresponding edges of H have non-empty intersection. Note that if the intersection graph of H has no subgraph isomorphic to the intersection graph of G (that is, the line graph of G), then H is G-free. The reverse statement is not true: the intersection graph of the hypergraph H with edges $\{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}$ is a triangle but H is triangle-free.

To define the Ramsey-type problem we address here, let $f(n, G)$ be the smallest number of colors in a coloring of $\mathcal{P}(n)$ such that all color classes are G-free. In other words, in every coloring of $\mathcal{P}(n)$ with $f(n, G) - 1$ colors, there is a Berge-G subhypergraph in some color class. We use the terms *coloring, partitioning* of $\mathcal{P}(n)$ in the same sense. Since the presence of empty sets and singleton sets do not influence whether a coloring is G-free, we usually construct colorings of $\mathcal{P}^*(n)$, what we define to be $\mathcal{P}(n)$ with the empty set and the singletons removed. However, the following natural partition of the whole power set of $[n]$ is useful. For every $A \subseteq [n-1]$, the part defined by A is

$$\{X_1(A) = A, X_2(A) = [n] \setminus X_1(A), X_3(A) = A \cup \{n\}, X_4(A) = [n] \setminus X_3(A)\}.$$

Since A and $[n-1] \setminus A$ define the same part, we have 2^{n-2} parts (each of size four). This partition was used in [1] to show that $f(n, C_3) \leq 2^{n-2}$. Observing that

$$X_1(A) \cap X_2(A) = X_3(A) \cap X_4(A) = X_1(A) \cap X_4(A) = \emptyset,$$

these parts are C_3-free, C_4-free and S_3-free. Thus we have a natural upper bound for three small graphs:

Proposition 1. *$f(n, G) \leq 2^{n-2}$ for $G \in \{C_3, C_4, S_3\}$.*

How sharp is this upper bound for the three small graphs involved? The easiest lower bound comes for the claw.

Proposition 2. *$2^{n-2} - n/2 \leq f(n, S_3)$. In general, $\frac{2^{n-1}}{k-1} - O\left(n^{k-2}\right) \leq f(n, S_k)$.*

Proof. Consider a partition Q of $\mathcal{P}(n)$ into S_k-free parts. Let $H = (V, E)$ be the subhypergraph of $\mathcal{P}(n)$ determined by the edges of size at least k. Then Q partitions H into S_k-free parts $H_i = (V, E_i)$, for $i = 1, \ldots, t$. Since k edges of size at least k cannot have common intersection by the S_k-free property, each hypergraph H_i has maximum degree at most $k-1$. Therefore

$$n2^{n-1} - \left(n + 2\binom{n}{2} + \cdots + (k-1)\binom{n}{k-1}\right) = \sum_{v\in V} d_H(v) = \sum_{i=1}^{t}\sum_{v\in V} d_{H_i}(v) \le (k-1)nt,$$

implying $t \ge \frac{2^{n-1}}{k-1} - \frac{1}{k-1}\left(1 + \binom{n-1}{1} + \cdots + \binom{n-1}{k-2}\right) = \frac{2^{n-1}}{k-1} - O\left(n^{k-2}\right)$. For $k = 3$, this calculation gives $2^{n-2} - n/2 \le f(n, S_3)$. □

The discrepancy of $-n/2$ between Proposition 1 and 2 for $f(n, S_3)$ is the consequence of the fact that three edges of $\mathcal{P}^*(n)$ intersecting in a vertex v do not define a claw in the special case when the three edges are $\{v, x, y\}, \{v, x\}, \{v, y\}$. Utilizing this with several different designs, we have small examples in Section 5 showing that the upper bound for $f(n, S_3)$ in Proposition 1 can sometimes be lowered (in particular, we show that $f(6, S_3) \le 15$ and $f(9, S_3) \le 126$). It is unclear whether one can use this phenomenon to decrease the upper bound for infinitely many n.

For the case of the triangle, the upper bound of Proposition 1 is tight. For odd $n \ge 7$ this was shown with a simple proof in [1]. Somewhat surprisingly, this remains true for the even n case as well (but not for $n = 5$).

Theorem 1. *For $n \ge 3, n \ne 5$, $f(n, C_3) = 2^{n-2}$. Additionally, $f(5, C_3) = 7$.*

In case of $G = C_4$ we improve the upper bound of Proposition 1 by a constant factor and slightly improve the lower bound $\frac{2^{n-1}}{3}(1 - o(1))$ from [1].

Theorem 2. *For even n, we have $f(n, C_4) = \frac{2^{n-1}}{3}\left(1 + \Theta\left(\frac{1}{\sqrt{n}}\right)\right)$. Additionally, for all $n \ge 27$, we have $\frac{2^{n-1}}{3} \le f(n, C_4) \le \frac{2^{n-1}}{3}\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$.*

While our lower bound for $f(n, C_4)$ for even n is asymptotically larger than our lower bound for odd n, we have no reason to believe that the lower bound for odd n cannot be improved. We suspect that a better bound for odd n would follow from a similar proof as that with even n, just with more work involved.

For the upper bound on $f(n, C_4)$ we combine designs to include almost all sets in $\mathcal{P}(n)$. In fact, we do this to provide an upper bound for $f(n, G)$ *for any connected graph G*. The construction is based on *asymptotically optimal packings*, $D(n, m, r)$, which is a large subset $S \subseteq \binom{[n]}{m}$ with the property that every r-element subset of $[n]$ is contained in *at most one* member of S. The existence of such packings was proved in a breakthrough paper of Rödl [8]. For our purposes only a special case is needed, $D(n, m, m-1)$, where constructions were known earlier, for example in [7].

Theorem 3. *Let G be a connected graph with k edges, where $k \ge 2$ is fixed. Then $f(n, G) \le \frac{2^n}{2(k-1)}\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$.*

The upper bound of Theorem 3 gives the upper bound in Theorem 2. In fact, it also matches the corresponding asymptotic lower bound $\frac{2^{n-1}}{|E(G)|-1}(1 - o(1))$ in [1] when G is a cycle or path, and the asymptotic lower bound of Proposition 2, implying

Corollary 1. $\frac{2^n}{2(k-1)}(1 - o(1)) \le f(n, C_k), f(n, P_k), f(n, S_k) \le \frac{2^n}{2(k-1)}(1 + o(1))$.

2 Proof of Theorem 1

It was shown in [1] that $f(n, C_3) = 2^{n-2}$ for any odd $n \geq 3, n \neq 5$. The following C_3-free partition of $\mathcal{P}^*(5)$ shows that $n = 5$ is indeed exceptional. (Here and later we represent sets of small numbers without commas and brackets.)

$$X_1 = \{[5], [4]\}, Y_1 = \{124, 234, 245\}, Y_2 = \{123, 134, 135\}, Z_1 = \{12, 35, 1235, 345\},$$

$$Z_2 = \{23, 45, 2345, 145\}, Z_3 = \{34, 15, 1345, 125\}, Z_4 = \{14, 25, 1245, 235\}. \tag{1}$$

In fact, (1) is the only partition of $\mathcal{P}^*(5)$ into at most seven C_3-free parts (up to permutations), implying $f(5, C_3) = 7$. For $n \neq 5$, three sets of size at least $\lfloor n/2 \rfloor + 1$ always form a triangle (this is proven for odd n in [1] and generalized for even n in Lemma 1). This is indeed not true for $n = 5$, as witnessed by the 'crowns' Y_1 and Y_2 in (1).

Let $\mathcal{L}$ be the set of all subsets of $[n]$ of size at least $\lfloor n/2 \rfloor + 1$ (these are the 'large' subsets). For even n let $\mathcal{M}$ be the set of all subsets of $[n]$ of size $n/2$ (these are the 'medium' subsets). Note that $2|\mathcal{L}| + |\mathcal{M}| = 2^n$.

Lemma 1. *For every even $n \geq 6$, we have the following:*

1. *For any distinct $M_1, M_2, M_3, M_4, M_5 \in \mathcal{M}$, some three form a triangle.*
2. *For any distinct $M_1, M_2, M_3 \in \mathcal{M}, L \in \mathcal{L}$, some three form a triangle.*
3. *Any distinct $M \in \mathcal{M}, L_1, L_2 \in \mathcal{L}$ form a triangle.*
4. *Any distinct $L_1, L_2, L_3 \in \mathcal{L}$ form a triangle.*

Proof of Theorem 1 from Lemma 1. For odd n the theorem was proved in [1]. By Proposition 1, we have to prove that $f(n, C_3) \geq 2^{n-2}$ for even n. Let $n \geq 6$, and let Q be a partition of $\mathcal{P}(n)$ into the minimum number of triangle-free parts. (For $n = 4$ a similar lemma and argument works.) Let there be a parts of Q with exactly two sets of $\mathcal{L}$, let there be b parts of Q with exactly one set of $\mathcal{L}$, and let there be c parts of Q with no sets of $\mathcal{L}$. Lemma 1 implies that these account for all the parts, so that $a + b + c = f(n, C_3)$. Lemma 1 also implies that $|\mathcal{M}| \leq 2b + 4c$, and since $|\mathcal{L}| = 2a + b$, we have

$$f(n, C_3) = a + b + c \geq \frac{1}{4}(2|\mathcal{L}| + |\mathcal{M}|) = \frac{1}{4}(2^n) = 2^{n-2}.$$

□

Proof of Lemma 1. Since a set of $\mathcal{L}$ always contains as a subset a set of $\mathcal{M}$, it is clear that statement 3 implies statement 4. Thus we only need to prove statements 1, 2, and 3.

Let's first note some basic intersection properties of sets from $\mathcal{M} \cup \mathcal{L}$. Let $L_1, L_2 \in \mathcal{L}$ and $M_1, M_2, M_3 \in \mathcal{M}$ be arbitrary. It is clear that $|L_1 \cap L_2| \geq 2$, $|L_1 \cap M_1| \geq 1$, and either $|M_1 \cap M_2| \geq 1$ or $|M_2 \cap M_3| \geq 1$.

In any of the three cases of the lemma, we first want to find three pairwise intersecting sets. In the first case, WLOG M_1 intersects with M_2, M_3, and M_4, and again WLOG M_2 intersects with M_3. In the second case, L intersects M_1, M_2, and M_3, and WLOG M_1 intersects M_2. In the third case, every pair of sets intersect.

Let $A, B, C \in \mathcal{M} \cup \mathcal{L}$ be three distinct pairwise intersecting sets, in any case, and suppose they do not form a triangle. By Hall's theorem as applied to distinct representatives, there are only a few cases where they may not form a triangle. WLOG, either $|(A \cap B) \cup (A \cap C)| \leq 1$ or $|(A \cap B) \cup (A \cap C) \cup (B \cap C)| \leq 2$. In the first case, it cannot be that $|A \cap B \cap C| = 0$, since the sets are pairwise intersecting, so we must have $|A \cap B \cap C| = 1$ and $|A \cap B \setminus C| = |A \cap C \setminus B| = 0$. In the second case, it likewise cannot be that $|A \cap B \cap C| = 0$. The case where $|A \cap B \cap C| = 1$ falls into the previous case, so this case reduces to $|A \cap B \cap C| = 2$ and $|A \cap B \setminus C| = |B \cap C \setminus A| = |C \cap A \setminus B| = 0$.

Define $\delta_A = |A| - n/2$, and likewise for B and C. Furthermore let $\delta = \delta_A + \delta_B + \delta_C$.

Case 1: $|A \cap B \cap C| = 2$ and $|A \cap B \setminus C| = |B \cap C \setminus A| = |C \cap A \setminus B| = 0$. Here we count

$$n \geq |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \geq \frac{3}{2}n + \delta - 2 - 2 - 2 + 2$$

implying

$$n + 2\delta \leq 8.$$

Case 1a: Suppose $n = 8$. Then $\delta = 0$ and so $A, B, C \in \mathcal{M}$ and WLOG the configuration is isomorphic to $A = 1234$, $B = 1256$, and $C = 1278$. A fourth set $D \in \mathcal{M} \cup \mathcal{L}$ must meet two of A, B, C in a vertex not in $\{1, 2\}$, forming a triangle with them.

Case 1b: Suppose $n = 6$. If $\delta = 0$, then WLOG $A = 123$, $B = 124$, and $C = 125$. The only pairs of vertices a fourth set $D \in \mathcal{M} \cup \mathcal{L}$ may contain without forming a triangle are those pairs containing 6 and the pair 12. Thus unless $D = 126$, we have a triangle. If $D = 126$, then we must be in the first case of the lemma, and so we may take a fifth set $E \in \mathcal{M}$. Since $\{1, 2\} \not\subseteq E$, we have that E must contain two vertices of $\{3, 4, 5, 6\}$, forming a triangle.

If $\delta = 1$, then WLOG we are in the second case of the lemma and $A = 123$, $B = 124$, and $C = 1256$. Since the fourth set $D \in \mathcal{M}$ contains some pair of vertices other than 12 and 56, we have a triangle.

Case 2: $|A \cap B \cap C| = 1$ and $|A \cap B \setminus C| = |A \cap C \setminus B| = 0$. Here we count

$$n \geq |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \geq \frac{3}{2}n + \delta - |B \cap C| - 1,$$

therefore

$$\frac{1}{2}n + \delta - 1 \leq |B \cap C| \leq n - |A| \quad \text{(since } B \text{ and } C \text{ are distinct)},$$

implying

$$2\delta_A + \delta_B + \delta_C \leq 1.$$

Thus $\delta_A = 0$, and at most one of δ_B and δ_C is 1, meaning that WLOG $A, B \in \mathcal{M}$, so we are not in the third case of the lemma. This means that the third case of the lemma was proved in case 1, so we are free to use it to finish the proof here. Let $D \in \mathcal{M}$ be a set distinct from A, B, and C.

If $D \subseteq B \cup C$, then B, C, and D are three sets of size at least $(n-2)/2+1$ contained within a set of size $n/2+1 \leq n-2$. We may apply the third case of the lemma (with $n-2$ for n) to see that B, C, D form a triangle.

Otherwise, let $x \in A \cap B \cap C$, let $y \in D \cap A \setminus \{x\}$, and let $z \in D \cap (B \cup C) \setminus \{x\}$. Note that x, y, z necessarily exist and are distinct, so either A, D, B or A, D, C form a triangle.

$\square$

3 Proof of Theorem 2

The upper bound of Theorem 2 follows from Theorem 3 (with $k = 4$). So we prove the lower bound. As in Section 2, we need a lemma concerning sets of $\mathcal{M} \cup \mathcal{L}$. We also give the corresponding lemma for odd n, which we prove from Lemma 2.

Lemma 2. *For every even $n \geq 26$, we have the following:*

1. *For any distinct $M_1, M_2, M_3, M_4, M_5 \in \mathcal{M}$, some four form a C_4.*
2. *For any distinct $M_1, M_2, M_3, M_4 \in \mathcal{M}$, $L_1 \in \mathcal{L}$, some four form a C_4.*
3. *Any distinct $M_1, M_2 \in \mathcal{M}$, $L_1, L_2 \in \mathcal{L}$ form a C_4.*
4. *Any distinct $M_1 \in \mathcal{M}$, $L_1, L_2, L_3 \in \mathcal{L}$ form a C_4.*

Lemma 3. *For every odd $n \geq 27$, any distinct $L_1, L_2, L_3, L_4 \in \mathcal{L}$ form a C_4.*

Proof of Theorem 2 from Lemmas 2 and 3. Let $n \geq 26$ be even, and let Q be a partition of $\mathcal{P}(n)$ into the minimum number of C_4-free parts. Say Q has a parts with three sets in $\mathcal{L}$, b parts with two sets in $\mathcal{L}$, c with one, and d with no sets in $\mathcal{L}$. Lemma 2 implies that these account for all the parts of Q, so $a+b+c+d = f(n, C_4)$. Moreover, Lemma 2 implies the relations $|\mathcal{L}| = 3a + 2b + c$ and $|\mathcal{M}| \leq b + 3c + 4d$. Since $|\mathcal{M}| = \binom{n}{n/2} = \Theta(2^n/\sqrt{n})$, this gives us (by $b + 3c + 4d \leq \frac{3b}{2} + 3c + \frac{9d}{2}$) that

$$f(n, C_4) = a+b+c+d \geq \frac{1}{6}\left(2|\mathcal{L}| + \frac{4}{3}|\mathcal{M}|\right) = \frac{1}{6}\left(2^n + \frac{1}{3}|\mathcal{M}|\right) = \frac{2^{n-1}}{3}\left(1 + \Theta\left(\frac{1}{\sqrt{n}}\right)\right).$$

For odd $n \geq 27$, again take such a minimal C_4-free partition of $\mathcal{P}(n)$. Each part has at most three sets in $\mathcal{L}$, so $f(n, C_4) \geq \frac{1}{3}|\mathcal{L}| = \frac{2^{n-1}}{3}$.

$\square$

In order to prove Lemma 2, we need the following definition:

Definition. Assume $n \geq 4$ is even. We say that four distinct sets $A, B, C, D \in \mathcal{M} \cup \mathcal{L}$ form a Ψ-*configuration* if there exists some x such that $A \cap B, A \cap C, A \cap D \subseteq \{x\}$. In such a configuration we call A a *stem*.

Let us elaborate on the structure of a Ψ-configuration A, B, C, D. Suppose A is a stem and $A \cap (B \cup C \cup D) \subseteq \{x\}$. Since A, B, C, D are distinct sets in $\mathcal{M} \cup \mathcal{L}$, we have the inequalities $|A| \geq \frac{n}{2}$ and $|B \cup C \cup D| \geq \frac{n}{2} + 1$. But also,

$$n + 1 \geq |A \cup (B \cup C \cup D)| + |A \cap (B \cup C \cup D)| = |A| + |B \cup C \cup D| \geq \frac{n}{2} + (\frac{n}{2} + 1) = n + 1.$$

So in fact, $|A| = \frac{n}{2}$ and $|B \cup C \cup D| = \frac{n}{2} + 1$. That is to say, $A \in \mathcal{M}$, and B, C, D are $\frac{n}{2}$- or $(\frac{n}{2} + 1)$-subsets of the $(\frac{n}{2} + 1)$-set $([n] \setminus A) \cup \{x\}$. Based on this, it is easy to see that a stem of a Ψ-configuration is unique.

Also note that in this Ψ-configuration we have

$$|B \cap C| = |B| + |C| - |B \cup C| \geq \frac{n}{2} + \frac{n}{2} - (\frac{n}{2} + 1) = \frac{n}{2} - 1,$$

and similarly $|B \cap D|, |C \cap D| \geq \frac{n}{2} - 1$. Thus, the non-stem sets of a Ψ-configuration pairwise intersect in at least $\frac{n}{2} - 1$ elements. Finally, observe that a Ψ-configuration does not form a C_4.

Proof of Lemma 2. Suppose $n \geq 26$ is even. We first prove the following claim:

Claim: Any four distinct $A, B, C, D \in \mathcal{M} \cup \mathcal{L}$ form either a C_4 or a Ψ-configuration.

Proof of Claim. Suppose that A, B, C, D do not form a Ψ-configuration. We wish to show that A, B, C, D form a C_4

First assume that two of the sets, say A and C, are complementary. Since the complement of any set is unique and our sets are in $\mathcal{M} \cup \mathcal{L}$, the intersections $A \cap B$ and $A \cap D$ are nonempty. Moreover, because $A \cap C = \emptyset$ and A, B, C, D do not form a Ψ-configuration, $A \cap B$ and $A \cap D$ cannot be the same singleton set. Thus, there exist distinct representatives $x_1 \in A \cap B$, $x_2 \in A \cap D$. Similarly, there exist distinct representatives $x_3 \in B \cap C$, $x_4 \in C \cap D$. Clearly x_1 and x_2 are distinct from x_3 and x_4, since the first two are contained in A while the second two are contained in $C = [n] \setminus A$. Thus A, B, C, D form a C_4.

Now assume that A, B, C, D are pairwise intersecting. Consider all perfect matchings $\{X_1, X_2\}, \{X_3, X_4\}$ (that is, partitions into sets of size 2) of $\{A, B, C, D\}$, and let $\{A, C\}, \{B, D\}$ be the one that minimizes $|(X_1 \cap X_2) \cup (X_3 \cap X_4)|$. We will show that if A, B, C, D do not form a C_4 in that cyclic order, then there is another cyclic order of A, B, C, D that forms a C_4. To do this, we use Hall's theorem on distinct representatives as we did in Lemma 1. WLOG, the following are the only cases in which A, B, C, D may fail to form a C_4 in that cyclic order:

Case 1: $|(A \cap B) \cup (B \cap C) \cup (C \cap D)| \leq 3$. (Note that this case covers when $|(A \cap B) \cup (B \cap C) \cup (C \cap D)| \leq 2$ and when $|(A \cap B) \cup (B \cap C) \cup (C \cap D) \cup (D \cap A)| \leq 3$.) The intersections in this union must each be a subset of a 3-set $\{x_1, x_2, x_3\}$. By minimality of $|(A \cap C) \cup (B \cap D)|$, $A \cap C$ and $B \cap D$ are subsets of a 3-set $\{y_1, y_2, y_3\}$. It follows that the sets $X \setminus \{x_1, x_2, x_3, y_1, y_2, y_3\}$ for $X \in \{A, B, C\}$ are pairwise disjoint. Counting the number of elements in $[n]$ outside of $\{x_1, x_2, x_3, y_1, y_2, y_3\}$, we get the inequality

$$3\left(\frac{n}{2} - 6\right) \leq n - 6,$$

from which it follows that $n \leq 24$. Since we assumed that $n \geq 26$, this is impossible.

Case 2: $|(A\cap B)\cup(C\cap D)| \leq 1$. Since our sets are pairwise intersecting, $A\cap B = C\cap D = \{x\}$ for some x. Then $x \in A\cap C$ and $x \in B \cap D$. Since we assumed that $|(A\cap C)\cup(B\cap D)|$ is minimal, it follows that $(A\cap C)\cup(B\cap D) = \{x\}$. But then

$$(A\cup D)\cap(B\cup C) = (A\cap B)\cup(C\cap D)\cup(A\cap C)\cup(B\cap D) = \{x\},$$

from which we get that

$$n+1 \geq |(A\cup D)\cup(B\cup C)| + |\{x\}| = |A\cup D| + |B\cup C| \geq \left(\frac{n}{2}+1\right) + \left(\frac{n}{2}+1\right) = n+2,$$

a contradiction.

Case 3: $|(A\cap B)\cup(A\cap D)| \leq 1$. Similar to case 3, $A\cap B = A\cap D = \{x_1\}$ for some x_1. Since A, B, C, D do not form a Ψ-configuration, there must be some $x_2 \in A\cap C$ different from x_1. Now, $C\cap D$ cannot be $\{x_1\}$ because otherwise $A\cap B = C\cap D = \{x_1\}$, which we showed is an impossible circumstance in Case 2. Moreover, $C\cap D$ cannot contain x_2 because otherwise $x_2 \in A\cap D = \{x_1\}$. Thus, there must be some $x_3 \in C\cap D$ different from x_1 and x_2. Finally, note that

$$\begin{aligned}
n &\geq |A\cup B\cup D| \\
&= |A| + |B| + |D| - |A\cap B| - |A\cap D| - |B\cap D| + |A\cap B\cap D| \\
&\geq \frac{n}{2} + \frac{n}{2} + \frac{n}{2} - 1 - 1 - |B\cap D| + 1 \\
&= \frac{3n}{2} - 1 - |B\cap D|,
\end{aligned}$$

from which we get that $|B\cap D| \geq \frac{n}{2} - 1 \geq 4$. So there must be some $x_4 \in B\cap D$ different from each of x_1, x_2, x_3. It follows that A, C, D, B form a C_4 in that cyclic order.

This concludes the proof of the claim.

□

Now we prove the statements of Lemma 2. Observe that, similar to Lemma 1, statement 2 follows from statement 1, and statement 4 follows from statement 3. So we prove statements 1 and 3.

Statement 3 follows immediately from the observations about Ψ-configurations, specifically that their stem must be an $\frac{n}{2}$-set, and their non-stem sets must be subsets of an $(\frac{n}{2}+1)$-set, say X. There is only one set in $\mathcal{L}$ that could be part of such a configuration, namely X itself. Thus, it is impossible for distinct $M_1, M_2 \in \mathcal{M}, L_1, L_2 \in \mathcal{L}$ to form a Ψ-configuration. By the claim, they must form a C_4.

For statement 1, first consider the sets M_1, M_2, M_3, M_4. If they form a C_4, then we are done; otherwise, they must form a Ψ-configuration. Say that M_1 is the stem. Next consider the sets M_1, M_2, M_3, M_5. Again we are done if they form a C_4; otherwise, they must form another Ψ-configuration. M_1 must again be the stem because $M_1\cap M_2$, $M_1\cap M_3$ have at most one element, and we have seen that the non-stem sets of Ψ-configurations must pairwise intersect in at least $\frac{n}{2} - 1$ elements. So finally consider the sets M_2, M_3, M_4, M_5. They are all non-stem sets in our previous two configurations, so $|M_i \cap M_j| \geq \frac{n}{2} - 1$ for all distinct $i, j \in \{2, 3, 4, 5\}$. Thus M_2, M_3, M_4, M_5 form a C_4.

□

Note that in case 1 of the proof of the claim, the required lower bound on n of 26 is not tight because the x_i's and y_i's considered in the proof may not all be distinct. This bound can definitely be reduced, but doing so requires extra casework.

Now we prove Lemma 3 from Lemma 2.

Proof of Lemma 3. Let $n \geq 27$ be odd, and let $L_1, L_2, L_3, L_4 \in \mathcal{L}$, meaning that $|L_i| \geq \frac{n+1}{2}$. We break into two cases.

Suppose there exists $j \in [n]$ such that j is in at most two of the L_i. Without loss of generality, $j \notin L_3, L_4$. This means that L_3 and L_4 are sets of size at least $\frac{n+1}{2} = \frac{n-1}{2} + 1$ contained in a set of size $n - 1$ (namely, $[n] \setminus \{j\}$). Let $M_1 \subseteq L_1 \setminus \{j\}$ and $M_2 \subseteq L_2 \setminus \{j\}$ be distinct sets of size $\frac{n-1}{2}$, which are necessarily contained in the same set of size $n-1$ as before (namely, $[n] \setminus \{j\}$). We may then consider L_3 and L_4 to be in $\mathcal{L}$ and M_1 and M_2 to be in $\mathcal{M}$ in the sense that Lemma 2(3) applies in $[n] \setminus \{j\}$: since these sets are distinct, they form a C_4.

Otherwise, suppose every $j \in [n]$ is in at least three of the L_i. This implies $\sum |L_i| \geq 3n$. No three of the L_i can have size exactly $\frac{n+1}{2}$, since this would imply that the fourth set has size at least $3n - 3\frac{n+1}{2} = \frac{3n}{2} - \frac{3}{2} > n$, an impossibility. Thus at most two of the L_i have size $\frac{n+1}{2}$, meaning that (as in the preceding paragraph) upon the removal of any vertex there are at most two sets of size $\frac{n-1}{2}$. In a similar fashion as the previous paragraph, Lemma 2(3) implies that these sets form a C_4.

□

4 Proof of Theorem 3

Here we construct a partition of $\mathcal{P}(n)$ where almost all of the sets are in parts of size $2(k-1)$. In fact, these parts of size $2(k-1)$ consist of $k-1$ sets of size less than $n/2$, and $k-1$ sets of size at least $n/2$, in such a way that all of the larger sets are disjoint from the smaller sets. The sets not in parts of size $2(k-1)$ can be placed arbitrarily in parts of size at most $k-1$. This partition is G-free since the intersection graph of any partition class has connected components with at most $k-1$ vertices. We assume that $n \geq 2(k-1)$.

Define $A_{m,r} := \{A \in \binom{[n]}{m} : \sum_{a \in A} a \equiv r \pmod{n}\}$. Since $\sum_{r=0}^{n-1} |A_{m,r}| = \binom{n}{m}$, there exists some r_m such that $|A_{m,r_m}| \geq \frac{1}{n}\binom{n}{m}$. Fix these r_m for $k-1 \leq m < n/2$. We construct a part in our partition from each $A \in A_{m,r_m}$ for $k-1 \leq m < n/2$.

Let $A \in A_{m,r_m}$ and enumerate $A = \{a_0, \ldots, a_{m-1}\}$ and $B = [n] \setminus A = \{b_0, \ldots, b_{n-m-1}\}$. For integers i with $0 \leq i \leq \lfloor \frac{m}{k-1} \rfloor - 1$, construct the part consisting of the sets

$$A \setminus \{a_{(k-1)i}\}, A \setminus \{a_{(k-1)i+1}\}, \ldots, A \setminus \{a_{(k-1)i+(k-2)}\},$$

$$B \setminus \{b_{(k-1)i}\}, B \setminus \{b_{(k-1)i+1}\}, \ldots, B \setminus \{b_{(k-1)i+(k-2)}\}.$$

The sets of the form $A \setminus \{a_j\}$ (in the first line) are all different. Indeed, suppose $A \setminus \{a_j\} = A' \setminus \{a'_{j'}\}$, so necessarily $|A| = |A'|$. Also, $\sum_{a_i \in A \setminus \{a_j\}} a_i \equiv \sum_{a'_i \in A' \setminus \{a'_{j'}\}} a'_i \pmod{n}$. This implies $-a_j + \sum_{a_i \in A} a_i \equiv -a'_{j'} + \sum_{a'_i \in A'} a'_i \pmod{n}$. By construction, this is equivalent to $r_{|A|} - a_j \equiv r_{|A'|} - a'_{j'} \pmod{n}$, and thus $a_j \equiv a'_{j'} \pmod{n}$. This means that $a_j = a'_{j'}$, which together with $A \setminus \{a_j\} = A' \setminus \{a'_{j'}\}$ implies that $A = A'$. Thus the two sets were the same. Analogous reasoning concludes that the sets appearing in the second line are also all different. Finally, for any m, sets in the first line have size less than $\frac{n}{2} - 1$, while in the second line the sets have size at least $\frac{n}{2} - 1$. Therefore the constructed sets are all different in any part.

For each $A \in A_{m,r_m}$, there are $\left\lfloor \frac{m}{k-1} \right\rfloor$ possible values of i in the construction, that is, $\left\lfloor \frac{m}{k-1} \right\rfloor$ different parts that A generates. Since $\binom{n}{\lfloor n/2 \rfloor} = \Theta(2^n/\sqrt{n})$, this construction creates at least

$$\sum_{m=k-1}^{\lceil \frac{n}{2} \rceil - 1} \left\lfloor \frac{m}{k-1} \right\rfloor \frac{1}{n} \binom{n}{m} \geq \frac{2^n}{2(k-1)} \left(1 - \frac{c}{\sqrt{n}}\right)$$

parts of size $2(k-1)$, for some constant $c > 0$ not depending on n or k.

Since this construction yields at least $\frac{2^n}{2(k-1)} (1 - c/\sqrt{n})$ parts of size $2(k-1)$, there are at least $2^n (1 - c/\sqrt{n})$ sets placed in parts this way. Thus at most $2^n - 2^n(1 - c/\sqrt{n}) = 2^n (c/\sqrt{n})$ sets have not been placed into a part. We place these remaining sets arbitrarily into parts of size $k-1$ (with one possible smaller part). Partitioning the rest this way generates at most $\frac{2^n}{k-1} (c/\sqrt{n}) + 1$ additional parts. Thus, in total the partition will have at most $\frac{2^n}{2(k-1)} (1 - c/\sqrt{n}) + \frac{2^n}{k-1} (c/\sqrt{n}) + 1 = \frac{2^n}{2(k-1)} (1 + \Theta(1/\sqrt{n}))$ parts.

5 Bounds on $f(6, S_3)$ and $f(9, S_3)$

Proposition 3. $f(6, S_3) \leq 15$.

Proof. Let $X = \{123, 456, 12, 13, 23, 45, 46, 56\}$ be one (3-regular but claw-free) partition class. All other classes will be 2-regular (thus automatically claw-free). Let Y_1, Y_2, Y_3, Y_4 contain two pairs of complementary triples, not using the pair $123, 456$. Then define

$$Z_1 = \{14, 23456, 12356\}, Z_2 = \{25, 13456, 12346\}, Z_3 = \{36, 12456, 12345\}.$$

Let U_1, U_2, U_3, U_4, U_5 be defined as the complementary sets of the 1-factors in a 1-factorization of K_6. Then W is defined by the edges of the 6-cycle $1, 5, 3, 4, 2, 6, 1$ and R contains $[6]$ together with the one complementary pair of triples not used in X and in Y_i. Now we have 15 claw-free partition classes of $\mathcal{P}^*(6)$. $\square$

Proposition 4. $f(9, S_3) \leq 126$.

Proof. Take a partition Q of $\binom{[9]}{3}$ into 28 classes, each containing three pairwise disjoint triples - a very special case of Baranyai's theorem [2]. However, we need another property of Q: four of these classes X_1, X_2, X_3, X_4 must form a Steiner triple system. Then these can be extended by the nine pairs covered by their triples implying that $X_1 \cup X_2 \cup X_3 \cup X_4$ covers each pair of [9] exactly once. The existence of these X_i-s certainly follows from a much stronger result, stating that $\binom{[9]}{3}$ can be partitioned into seven Steiner triple systems (but probably there are easier ways to get them). Then the X_i-s provide four claw-free 3-regular partition classes. Partition classes Y_i can be defined by putting together 12 pairs of the remaining 24 classes of Q, they form a double cover of [9]. Next we can define 28 partition classes Z_i by the complements of the 28 classes of Q, each of them forms a double cover on [9].

Next we design 9 double covers of type $(5,5,8)$ and 18 double covers of type $(4,7,7)$. To prepare, set $A_i = \{i+1, i+2, i+3, i+6\}$, $B_i = \{i+4, i+5, i+7, i+8\}$ with arithmetic mod 9. Then 9 double covers of [9] are defined as $U_i = \{A_i \cup i, B_i \cup i, A_i \cup B_i\}$. Set

$$C_i = [9] \setminus \{i+1, i+2\}, D_i = [9] \setminus \{i+3, i+6\},$$

$$E_i = [9] \setminus \{i+4, i+8\}, F_i = [9] \setminus \{i+5, i+7\}.$$

Then 2×9 double covers of [9] are defined as $W_i = \{A_i, C_i, D_i\}$ and $R_i = \{B_i, E_i, F_i\}$. Note that U_i, W_i, R_i take care of 18 complementary pairs of sizes 4 and 5. The remaining $\binom{9}{4} - 18$ such pairs can be placed into 54 partition classes T_i forming double covers on [9]. Finally, [9] alone forms a partition class (leaving some hope of improvement).

Altogether we have $4 + 12 + 28 + 9 + 18 + 54 + 1 = 126$ claw-free partition classes of $\mathcal{P}^*(9)$. □

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