arXiv:1907.07479v1 [math.DS] 17 Jul 2019

LEE-YANG ZEROS OF THE ANTIFERROMAGNETIC ISING MODEL

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Abstract. We investigate the location of zeros for the partition function of the anti-ferromagnetic Ising Model, focusing on the zeros lying on the unit circle. We give a precise characterization for the class of rooted Cayley trees, showing that the zeros are nowhere dense on the most interesting circular arcs. In contrast, we prove that when considering all graphs with a given degree bound, the zeros are dense in a circular sub-arc, implying that Cayley trees are in this sense not extremal. The proofs rely on describing the rational dynamical systems arising when considering ratios of partition functions on recursively defined trees.

1. Introduction

Partition functions play a central role in statistical physics. The distribution of zeros of the partition functions are instrumental in describing phase changes in a variety of contexts. More recently there has been a second motivation for studying the zeros of partition functions, arising from a computational complexity perspective. Since the 1990's there has been significant interest in whether the values of partition functions can be approximated, up to an arbitrarily small multiplicative error, by a polynomial time algorithm. For graphs of bounded degrees this is known to be the case on open connected subsets of the zero free locus [Bar16, PR17]. In recent work of the last author with Regts [PR19, PR18], the zero free locus was successfully described by first considering a specific subclass of graphs, the Cayley trees, for which the location of zeros can be described by studying iteration properties of a rational function.

A common theme in the papers [PR19, PR18] was that the Cayley trees turned out to be extremal within the larger class of bounded degree graphs, in the sense that a maximal zero free locus for Cayley trees proved to be zero-free in the larger class as well. This observation is the main motivation for our studies here, where we investigate to which extend the extremality of the class Cayley trees holds for the antiferromagnetic Ising Model.

Let $G = (V, E)$ denote a simple graph and let $\lambda, b \in \mathbb{C}$. The *partition function of the Ising model* $Z_G(\lambda, b)$ is defined as

$$Z_G(\lambda) = Z_G(\lambda, b) = \sum_{U \subseteq V} \lambda^{|U|} \cdot b^{|\delta(U)|},$$

Date: July 18, 2019.

[†] The research leading to these results has received funding from the European Research Council under the European UnionâĂŹs Seventh Frame-work Programme (FP7/2007-2013) / ERC grant agreement n° 617747.

[‡] Supported by NWO TOP grant 613.001.851.

[§] Supported by NWO TOP grant 614.001.506.

where $\delta(U)$ denotes the set of edges with one endpoint in U and one endpoint in $U \setminus V$. In this paper we fix $b > 0$ and consider the partition function $Z_G(\lambda)$ as a polynomial in λ. The case $b < 1$ is often referred to as the *ferromagnetic case*, while $b > 1$ is referred to as the *anti-ferromagnetic* case.

For $d \geq 2$ let $\mathcal{G}_{d+1}$ be the set of all graphs of maximum degree at most $d+1$. Given a set of graphs $\mathcal{H}$, we write

$$\mathcal{Z}_{\mathcal{H}} = \mathcal{Z}_{\mathcal{H}}(b) = \{\lambda : Z_G(\lambda) = 0 \text{ for some } G \in \mathcal{H}\}.$$

When $b < 1$, the Lee–Yang Circle TheoremÂă [LY52a, LY52b] states that for any graph G, the zeros of Z_G are contained in the unit circle $\mathbb{S}^1$. The zeros in the ferromagnetic case have subsequently been known as the Lee–Yang zeros. To study the zeros of Z_G for all $G \in \mathcal{G}_{d+1}$ one can consider the subset of finite rooted Cayley trees with down degree d, which we denote by $\mathcal{C}_{d+1}$. The Lee–Yang zeros of Cayley trees are studied in [MHZ75, MH77, BM97, BG01, CHJR19] amongst other papers. In all of these papers some variation of the following rational function plays a important role:

$$f_\lambda(z) = f_{\lambda,d}(z) = \lambda \cdot \left(\frac{z+b}{bz+1}\right)^d, \tag{1}$$

where f_λ is viewed as a function on the Riemann sphere. The significance of f_λ in relation to the Cayley trees is explained by the following lemma.

Lemma 1.1 (e.g. [CHJR19, Proposition 1.1])**.** *Let $b \in \mathbb{R}$ and $d \geq 2$, then*

$$\mathcal{Z}_{\mathcal{C}_{d+1}} = \{\lambda : f_\lambda^n(\lambda) = -1 \text{ for some } n \in \mathbb{Z}_{\geq 0}\}.$$

Thus complex dynamical systems can be used to study the zeros of the partition function of Cayley trees. The following result from [PR18] shows that while the Cayley trees from a relatively small subset of the class of all graphs of bounded maximal degree, the zero free loci of these two classes are identical in the ferromagnetic case:

Theorem 1.2. *Let $d \geq 2$. If $0 < b \leq \frac{d-1}{d+1}$ then*

$$\overline{\mathcal{Z}_{\mathcal{C}_{d+1}}} = \overline{\mathcal{Z}_{\mathcal{G}_{d+1}}} = \mathbb{S}^1.$$

If $\frac{d-1}{d+1} < b < 1$ then

$$\overline{\mathcal{Z}_{\mathcal{C}_{d+1}}} = \overline{\mathcal{Z}_{\mathcal{G}_{d+1}}} = \mathrm{Arc}[\lambda_1, \overline{\lambda_1}],$$

where $\lambda_1 = \lambda_1(b) \in \mathbb{S}^1$ is the unique parameter in the upper half plane for which f_λ has a parabolic fixed point.

Given α, β on the unit circle, we will use notation $\mathrm{Arc}[\alpha, \beta]$ for the closed circular arc from α to β, traveling counter clockwise, and similarly for open and half-open circular arcs.

When $b > 1$ the Lee-Yang Circle Theorem fails, and the set of zeros of the partition function is considerably more complicated. Consider for example Figure 1, illustrating the location of zeros for Cayley trees and for the larger class of *spherically symmetric trees*, defined in Definition 1.7 below, both for maximal down-degree $d = 2$ and maximal depth 11. The pictures are symmetric with

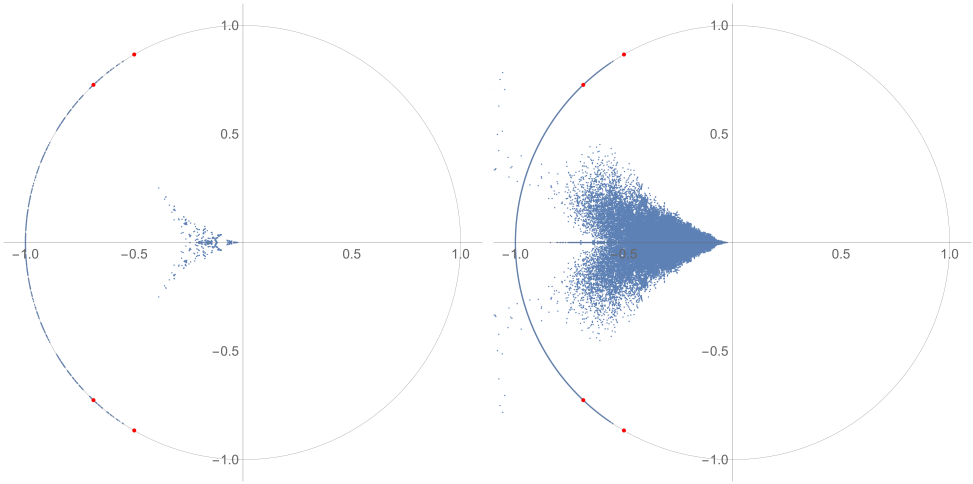

FIGURE 1. Comparing zeros of Cayley trees (left) and spherically symmetric trees (right) for $d = 2$ and depth at most 11.

respect to reflection in the unit circle, but only few zeros outside of the unit disk are depicted because of space concerns.

The pictures clearly demonstrate the appearance of zero parameters both on and off the unit circle. In this paper we focus on describing the set of zeros on the unit circle. Our main result show that, contrary to the ferromagnetic case, the zero free locus for the Cayley trees is strictly larger than that of the class of all bounded degree graphs.

We recall the following result from [PR18]:

Theorem 1.3. *Let $\lambda_0 = e^{i\theta_0} \in \mathbb{S}^1$ be the parameter with the smallest positive angle θ_0 for which $f_{\lambda_0}(\lambda_0) = 1$. Then*

$$\mathcal{Z}_{\mathcal{G}_{d+1}} \cap \mathbb{R}_+ \cdot \mathrm{Arc}[\overline{\lambda_0}, \lambda_0] = \emptyset,$$

but

$$\lambda_0, \overline{\lambda_0} \in \overline{\mathcal{Z}_{\mathcal{C}_{d+1}}}.$$

Note that in Figure 1 λ_0 and $\overline{\lambda_0}$ are depicted by the conjugate pair of red points with smallest absolute argument. The other conjugate pair of red points corresponds to λ_1 and $\overline{\lambda_1}$, having the same definition as in the ferromagnetic case.

Our main result is the following:

Theorem 1.4. *Let $d \geq 2$. If $b \geq \frac{d+1}{d-1}$ then*

$$\overline{\mathcal{Z}_{\mathcal{C}_{d+1}} \cap \mathbb{S}^1} = \overline{\mathcal{Z}_{\mathcal{G}_{d+1}} \cap \mathbb{S}^1} = \mathbb{S}^1.$$

If $1 < b < \frac{d+1}{d-1}$ then

(1) **Density for Cayley trees.**

$$\mathrm{Arc}[\lambda_1, \overline{\lambda_1}] \cup \{\lambda_0, \overline{\lambda_0}\} \subset \overline{\mathcal{Z}_{\mathcal{C}_{d+1}} \cap \mathbb{S}^1}.$$

(2) **Nowhere density for Cayley trees.** *The set*
$$\overline{\mathcal{Z}_{\mathcal{C}_{d+1}}} \cap \mathrm{Arc}[\lambda_0, \lambda_1]$$
is a nowhere dense subset of $\mathrm{Arc}[\lambda_0, \lambda_1]$.

(3) **Density for arbitrary graphs.** *There exists* $\lambda_3 \in \mathrm{Arc}(\lambda_0, \lambda_1)$ *such that*
$$\overline{\mathcal{Z}_{\mathcal{G}_{d+1}} \cap \mathrm{Arc}[\lambda_0, \lambda_3]} = \mathrm{Arc}[\lambda_0, \lambda_3].$$

Case (3) will be proved in section 6, building upon results from earlier sections. Cases (1) and (2) will be proved respectively in sections 4 and 3.

Remark 1.5. The fact that the closure of $\mathcal{Z}_{\mathcal{C}_{d+1}}$ is strictly smaller than the closure of $\mathcal{Z}_{\mathcal{G}_{d+1}}$ also holds outside of the unit circle, a statement that is considerably easier to prove. For example, the solution to the 1-dimensional Ising Model gives the density of zeros in a real interval $[-\alpha^{-1}, -\alpha]$, for some $\alpha \in (0, 1)$. On the other hand, using Corollary 3.10 one can prove the existence of a neighborhood of $\lambda = -1$ where all accumulation points of $\mathcal{Z}_{\mathcal{C}_{d+1}}$ must lie on the unit circle.

We prove case (3) for the subclass in $\mathcal{G}_{d+1}$ given by the spherically symmetric trees. These trees have the advantage that dynamical methods can be used to describe the location of zeros, as indicated by the following lemma, whose proof will be given later in this section.

Lemma 1.6. *Let* $d \geq 2$ *and* $\lambda, b \in \mathbb{C}$*, then there exists a spherically symmetric tree* T *with down degree at most* d *for which* $Z_T(\lambda, b) = 0$ *if and only if*
$$g(\lambda) = -1$$
for some $g \in H_{\lambda,d+1}$.

Here $H_{\lambda,d+1} = H_{\lambda,d+1}(b)$ is the rational semigroup generated by $f_{\lambda,1}, \ldots, f_{\lambda,d}$, i.e.
$$H_{\lambda,d} = \langle f_{\lambda,1}, \ldots, f_{\lambda,d} \rangle .$$
We will prove that a specific sub-semigroup of $H_{\lambda,d}$ is hyperbolic for all $\lambda \in \mathrm{Arc}[\lambda_0, \lambda_2)$, for some $\lambda_2 \in \mathrm{Arc}[\lambda_3, \lambda_1]$, i.e. on an arc $\mathrm{Arc}[\lambda_0, \lambda_2]$ that contains $\mathrm{Arc}[\lambda_0, \lambda_2]$. Moreover, we obtain uniform bounds on the expansion rate on compact subsets of $\mathrm{Arc}[\lambda_0, \lambda_2)$. We also show that for any $\lambda \in \mathrm{Arc}[\lambda_0, \lambda_1]$, there exists a sequence in the sub-semigroup for which $+1$ lies on the Julia set. Combining these two statements we obtain uniform expansion along an orbit of $+1$. The density of zero-parameters is a consequence for λ sufficiently close to λ_0.

We emphasize that in statement (3) of Theorem 1.4 we only consider zero parameters in $\mathbb{S}^1$. Alternatively we can consider $\overline{\mathcal{Z}_{\mathcal{G}_{d+1}}} \cap \mathbb{S}^1$, which is a priori a larger set. We prove in section 6 that this closure contains the circular arc
$$\mathrm{Arc}[\lambda_0, \lambda_2),$$
which is the arc where the earlier discussed sub-semigroup of $H_{\lambda,d}$ acts hyperbolically. The parameter λ_2 can be explicitly calculated. Computer evidence in fact suggests that
$$\overline{\mathcal{Z}_{\mathcal{G}_{d+1}}} \cap \mathbb{S}^1 = \mathrm{Arc}[\lambda_0, \overline{\lambda_0}].$$

In section 2 we prove basic results regarding the attracting intervals of the maps f_λ, to be used in later sections. In section 3 we consider the hyperbolic components

in the parameter space of the maps f_λ, and prove case (2) of Theorem 1.4. In section 4 we consider only parameters λ on the unit circle and prove case (1).

In the remainder of this introduction we recall the relationship between partition functions on Cayley trees and spherically symmetric trees on the one hand, and respectively iteration and semi-group actions on the other hand. In particular we give a short proof of Lemmas 1.1 and 1.6. In the rest of the paper we will only consider the two dynamical systems, with few references to partition functions.

1.1 Iterates and semigroups arising from trees Let us recall from [PR18] (but see also [CHJR19]) how the zeros of the Ising partition function $Z_G(\lambda)$ on some recursively defined trees can be studied using iterations or compositions of rational functions.

Let v be a marked node of a graph $G = (V, E)$. Note that

$$Z_G = Z^{in}_{G,v} + Z^{out}_{G,v},$$

where $Z^{in}_{G,v}$ sums only over $U \subset V$ with $v \in U$, and $Z^{out}_{G,v}$ sums only over $U \subset V$ with $v \notin U$. It follows that

$$Z_G = 0 \;\Leftrightarrow\; R_{G,v} := \frac{Z^{in}_{G,v}}{Z^{out}_{G,v}} = -1 \text{ or } Z^{in}_{G,v} = Z^{out}_{G,v} = 0.$$

Suppose now that $G = T$ is a tree. Denote the neighbors of v by $v_1, \ldots, v_k$, and the corresponding connected components of $G - v$ by $T_1, \ldots, T_k$. Then it follows that

$$R_{T,v} = \lambda \prod_{i=1}^{k} \frac{R_{T_i,v_i} + b}{bR_{T_i,v_i} + 1}.$$

Hence when all the rooted trees (T_i, v_i) are isomorphic, one obtains

$$R_{T,v} = \lambda \left(\frac{R_{T_i,v_i} + b}{bR_{T_i,v_i} + 1} \right)^k.$$

Definition 1.7. Let $d \geq 2$ and let $\omega = (k_1, k_2, \ldots) \in \{1, \ldots, d\}^{\mathbb{N}}$. Let T_0 be the rooted graph with a single vertex. Recursively define the trees $T_1, \ldots$ by letting T_n consist of a root vertex v of degree k_n, with each edge incident to v connected to the root of a copy of T_{n-1}. We say that the rooted trees T_n are *spherically symmetric* of degree at most d. Equivalently a rooted tree, with root v, is said to be spherically symmetric if all leaves have the same depth n, and all vertices of depth $1 \leq j < n$ have down-degree k_j. When all degrees k_n are equal to d the tree T_n is said to be a *(rooted) Cayley tree* of degree d.

Note that for a spherically symmetric tree

$$Z^{in}_n(\lambda) = \lambda(Z^{in}_{n-1}(\lambda) + bZ^{out}_{n-1}(\lambda))^{k_n} \in \mathbb{R}[\lambda], \text{ and}$$
$$Z^{out}_n(\lambda) = (bZ^{in}_{n-1}(\lambda) + Z^{out}_{n-1}(\lambda))^{k_n} \in \mathbb{R}[\lambda].$$

Since we will work with $b \notin \{-1, +1\}$ it follows by induction that $Z^{in}_n(\lambda)$ and $Z^{in}_n(\lambda)$ cannot both be equal to zero, from which it follows that

$$Z_G = 0 \;\Leftrightarrow\; R_{G,v} = -1.$$

Noting that $R_{T_0,v} = \lambda$, it follows for Cayley trees that

$$R_{T_n,v} = f_d^n(\lambda) = f_d^{n+1}(+1),$$

where

$$f(z) = f_d(z) = \lambda \left(\frac{z+b}{bz+1} \right)^d,$$

while for spherically symmetric trees we obtain

$$R_{T_n,v} = f_\omega^n(\lambda) := f_{k_n} \circ \cdots \circ f_{k_1}(\lambda).$$

Hence we have proved lemmas 1.1 and 1.6.

Motivated by this discussion we introduce the notations

$$\mathcal{Z}_f := \{\lambda \in \mathbb{C} : f^n(1) = -1 \text{ for some } n \in \mathbb{N}\}.$$

and

$$\mathcal{Z}_H := \{\lambda \in \mathbb{C} : f_\omega^n(1) = -1 \text{ for some } n \in \mathbb{N}, \omega \in \{1, \dots, d\}^{\mathbb{N}}\},$$

where H again refers to the semi-group $\langle f_1, \dots, f_d \rangle$. Thus $\lambda \in \mathcal{Z}_f$ if and only if $Z_G(\lambda) = 0$ for a Cayley tree G, while $\lambda \in \mathcal{Z}_H$ if and only if $Z_G(\lambda) = 0$ for a spherically symmetric tree G.

2. J-stable components

Given a family of rational maps f_λ parameterized by a complex manifold Λ, the set of J-stable parameters is the set of parameters for which the Julia set J_λ moves continuously with respect to the Hausdorff topology. The concept of J-stability plays a central role in the study of rational functions. We refer the interested reader to [MnSS83, McM94, Slo91] for a more detailed description of J-stability.

Given a positive integer $d \geq 2$ and $b > 1$, let f_λ be the family of rational functions given by (1) parameterized by $\Lambda = \widehat{\mathbb{C}}$. We will write Λ^{stb} for the set of J-stable parameters and Λ^{hyp} for the set of hyperbolic parameters, i.e. the values for which f_λ has no critical points nor parabolic cycles on J_λ. Recall that Λ^{stb} is a dense open set and that the set Λ^{hyp} is an open and closed subset of Λ^{stb}. Whether the equality $\Lambda^{stb} = \Lambda^{hyp}$ holds for the family given by (1) is a natural question, though not directly relevant for our purposes.

Given $\lambda \in \Lambda^{stb}$ we will write Λ_λ^{stb} for the connected component of Λ^{stb} containing the parameter λ.

Theorem 2.1. *There exists a holomorphic motion of J_λ over $(\Lambda_\lambda^{stb}, \lambda)$ which respects the dynamics, i.e. there exists a continuous map $\varphi : \widehat{\mathbb{C}} \times \Lambda_\lambda^{stb} \to \hat{\mathbb{C}}$ satisfying*

1. *φ_μ is the identity at the base point λ, i.e. $\varphi_\lambda(z) = z$,*
2. *for every $z \in J_\lambda$ the map $\varphi_\mu(z)$ is holomorphic in $\mu \in \Lambda_\lambda^{stb}$,*
3. *for every $\mu \in \Lambda_\lambda^{stb}$ the map $z \mapsto \varphi_\mu(z)$ is injective and can be extended to a quasi-conformal map $\varphi_\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.*

4. *for every $\mu \in \Lambda_\lambda^{stb}$ the map $\varphi_\mu : J_\lambda \to J_\mu$ is a homeomorphism. Furthermore, the following diagram commutes*

$$\begin{array}{ccc} J_\lambda & \xrightarrow{f_\lambda} & J_\lambda \\ {\scriptstyle \varphi_\mu}\downarrow & & \downarrow{\scriptstyle \varphi_\mu} \\ J_\mu & \xrightarrow{f_\mu} & J_\mu \end{array}$$

Such map φ satisfies the following two additional properties

5. *Given $z \in \widehat{\mathbb{C}}$ the map $\varphi_\mu^{-1}(z)$ is continuous with respect to μ,*
6. *Let $z_n \to z$ be a convergent sequence and assume that $\mu \mapsto \varphi_\mu(z)$ is not constant. Then there exists a subsequence n_k and $\mu_k \to \lambda$ so that*

$$\varphi_{\mu_k}(z_{n_k}) = z.$$

Remark 2.2. The existence of a continuous map φ satisfying properties $1-4$ was proven by [MnSS83, Slo91], while the properties 5 and 6 follow immediately from continuity of φ. The holomorphic motion is unique on the Julia set J_λ, in the sense that any other continuous map $\widetilde{\varphi}$ which satisfies the properties $1-4$ has to agree with φ on the set $J_\lambda \times \Lambda_\lambda^{stb}$.

Define the two sets

$$\mathcal{F} := \{\lambda \in \widehat{\mathbb{C}} \,|\, 1 \in F_\lambda\}, \qquad \mathcal{J} := \{\lambda \in \widehat{\mathbb{C}} \,|\, 1 \in J_\lambda\}.$$

Given a connected component $U \subset \Lambda^{stb}$ we further write $\mathcal{F}_U = \mathcal{F} \cap U$ and $\mathcal{J}_U = \mathcal{J} \cap U$. Since the Julia set J_λ moves continuously for $\lambda \in U$ it follows that $\mathcal{F}_U$ is open while $\mathcal{J}_U$ is closed with respect the intrinsic topology of U.

Definition 2.3. Let U be a connected component of Λ^{stb}. We say that U is *exceptional* if there exists $\lambda \in \mathcal{J}_U$ so that $\mu \mapsto \varphi_\mu(1)$ is constant, where φ_μ denotes the holomorphic motion of J_λ over (U, λ).

Remark 2.4. Suppose that U is an exceptional component of Λ^{stb} and let $\lambda \in U$ be so that the map $\mu \mapsto \varphi_\mu(1)$ is constant. Given another $\widetilde{\lambda} \in U$ we have that $\varphi_{\widetilde{\lambda}}(1) = \varphi_\lambda(1) = 1$, and therefore that $1 \in J_{\widetilde{\lambda}}$. Let $\widetilde{\varphi}_\mu$ be the holomorphic motion of $J_{\widetilde{\lambda}}$ over $(U, \widetilde{\lambda})$. Then we have

$$\widetilde{\varphi}_\mu(1) = \varphi_\mu \circ \varphi_{\widetilde{\lambda}}^{-1}(1) = 1, \qquad \forall \mu \in U.$$

This shows that if U is an exceptional component then $\mathcal{J}_U = U$ and for every $\lambda \in U$ the map $\mu \mapsto \varphi_\mu(1)$ is constant.

Proposition 2.5. *Let U be a connected component of Λ^{stb}. Then the set $\mathcal{J}_U$ is perfect with respect to the intrinsic topology of U.*

Proof. We already know that $\mathcal{J}_U$ is closed in U, thus we only have to show that $\mathcal{J}_U$ contains no isolated points. If U is an exceptional hyperbolic component, then according to Remark 2.4 we have $\mathcal{J}_U = U$ and the result follows immediately. Assume instead that U is not exceptional, let $\lambda \in \mathcal{J}_U$ and φ_μ be the holomorphic motion of J_λ over (U, λ).

Since the Julia set of a rational map is perfect, it follows that we may take $z_n \in J_\lambda$ which converges to 1, and that is not identically equal to 1. By Theorem 2.1 we may therefore find a sequence $n_k \geq 0$ and $\mu_k \to \lambda$ so that $\varphi_{\mu_k}(z_{n_k}) = 1$ for every k. Since $\varphi_\mu(J_\lambda) = J_\mu$ we conclude that $\mu_k \in \mathcal{J}_U$, proving that λ is not an isolated point of $\mathcal{J}_U$, and that $\mathcal{J}_U$ is perfect. □

The definition of active parameters is classical [McM00, DF08], and was inspired by [Lev81, Lyu83]. In all these works activity is always defined in terms of the family $\{f_\lambda^n \circ c(\lambda)\}$, where $c(\lambda)$ is the parameterization of a critical point. For our purpose it is natural to replace $c(\lambda)$ with the point 1, even though the point 1 is never critical.

Definition 2.6. A parameter $\lambda \in \widehat{\mathbb{C}}$ is *passive* if the family $\{\lambda \mapsto f_\lambda^n(1)\}_{n\in\mathbb{N}}$ is normal in some neighborhood of λ, and is *active* otherwise.

We further remark that, given a marked point $a(\lambda)$ and the corresponding family $\{f_\lambda^n \circ a(\lambda)\}$, it would be more accurate to say that *the marked point $a(\lambda)$ is passive/active at λ*. However since in our case the marked point is always 1, we will refer to passive/active parameters instead.

Lemma 2.7. *Every active parameter is in $\overline{\mathcal{Z}_{\mathcal{C}_{d+1}}}$.*

Proof. This is a standard normality argument. Assume first that $d \neq 2$ or that $\lambda \neq -1$. Let λ be an active parameter and choose $\alpha_0, \beta_0 \in f_\lambda^{-1}(\{-1\})$ so that $\{-1, \alpha_0, \beta_0\}$ are all distinct. Since -1 is never a critical value of f_λ we can define two holomorphic map α_μ, β_μ so that $f_\mu(\{\alpha_\mu, \beta_\mu\}) = -1$ in a neighborhood of λ. By conjugating with a holomorphically varying family of Möbius transformation we may assume that $\{-1, \alpha_\mu, \beta_\mu\} = \{-1, 0, \infty\}$. Since the family $\{f_\mu^n(1)\}$ is not normal at λ, by Montel's Theorem we conclude that it cannot avoid the three points $\{-1, 0, \infty\}$ in a neighborhood of λ. Since $\{0, \infty\}$ are both mapped to -1, the orbit $f_\lambda^n(1)$ cannot miss the point -1 near λ, proving that $\lambda \in \overline{\mathcal{Z}_{\mathcal{C}_{d+1}}}$. When $d = 2$ and $\lambda = -1$ the point -1 is fixed and has only two preimages $\{1, -1\}$. In this case we fix $\alpha_0 = 1$ and we choose γ_0 as one of the two preimages of 1, the proof is then the same as above. □

Lemma 2.8. *Let U be a non-exceptional component of Λ^{stb}. Then every $\lambda \in \mathcal{F}_U$ is passive and every $\lambda \in \mathcal{J}_U$ is active.*

Proof. Given $\lambda \in U$ let φ_μ be the holomorphic motion of J_λ over (U, λ). If $\lambda \in \mathcal{F}_U$ then the orbit $f_\lambda^n(1)$, avoids the Julia set J_λ and in particular it avoids three distinct points $\{a, b, c\} \subset J_\lambda$. Since the set $\mathcal{F}_U$ is open we have that $1 \in F_\mu$ for every μ sufficiently close to λ, and therefore the orbit of $f_\mu^n(1)$ avoids the set $\varphi_\mu(\{a, b, c\}) \subset J_\mu$. Using the normality argument from the proof of the previous lemma, we may therefore conclude that $\{f_\mu^n(1)\}$ is normal in a neighborhood of λ, showing that λ is passive.

Suppose that there exists $\lambda \in \mathcal{J}_U$ which is passive. Given any $0 < \varepsilon < \operatorname{diam}(J_\lambda)/2$ by equicontinuity we can find $\delta > 0$ so that

$$\left| f_\mu^n(1) - f_\lambda^n(1) \right| < \varepsilon/2, \qquad \forall \mu \in B(\lambda, \delta) \text{ and } n \in \mathbb{N}.$$

Given any open neighborhood $U \ni 1$ there exists $N \in \mathbb{N}$ so that $f_\lambda^N(U \cap J_\lambda) = J_\lambda$. Given $w \in J_\lambda$ with distance at least ε from $f_\lambda^N(1)$ we can therefore find $z \in U \cap J_\lambda$ so that $f_\lambda^N(z) = w$. We conclude that we can construct a sequence $z_k \in J_\lambda$ converging to the point 1 and a sequence of positive integers N_k so that

$$\left| f_\lambda^{N_k}(z_k) - f_\lambda^{N_k}(1) \right| > \varepsilon. \tag{2}$$

By Theorem 2.1, up to taking a subsequence of z_k if necessary, we may assume that there exists a sequence $\mu_k \in B(\lambda, \delta)$ so that $\mu_k \to \lambda$ and so that $\varphi_{\mu_k}(z_k) = 1$. This implies that

$$\left| \varphi_{\mu_k} \circ f_\lambda^{N_k}(z_k) - f_\lambda^{N_k}(1) \right| = \left| f_{\mu_k}^{N_k}(1) - f_\lambda^{N_k}(1) \right| < \varepsilon/2.$$

By continuity of the holomorphic motion, we may further assume that whenever k is sufficiently large we have

$$\left| \varphi_{\mu_k} \circ f_\lambda^{N_k}(z_k) - f_\lambda^{N_k}(z_k) \right| = \left| \varphi_{\mu_k} \circ f_\lambda^{N_k}(z_k) - \varphi_\lambda \circ f_\lambda^{N_k}(z_k) \right| < \varepsilon/2.$$

In combination with the previous inequality we conclude that for every k sufficiently large we have

$$\left| f_\lambda^{N_k}(z_k) - f_\lambda^{N_k}(1) \right| < \varepsilon, \qquad \forall n \in \mathbb{N},$$

contradicting the definition of the sequence z_k. Thus every $\lambda \in \mathcal{J}_U$ is active. □

3. Dynamics of the map f_λ

For given $d \geq 2$ and $b > 1$, we are interested in the dynamics of the map f_λ under the assumption that $\lambda \in \mathbb{S}^1$. In this case we have

$$f_\lambda : B(0,1) \to \widehat{\mathbb{C}} \setminus \overline{B(0,1)} \to B(0,1),$$
$$f_\lambda : \mathbb{S}^1 \to \mathbb{S}^1,$$

and the restriction of f to $\mathbb{S}^1$ is orientation reversing. If we write F_λ and J_λ for the Fatou and the Julia set of the map f_λ we conclude that

$$F_\lambda \supset \hat{\mathbb{C}} \setminus \mathbb{S}^1, \qquad J_\lambda \subset \mathbb{S}^1, \qquad \forall \lambda \in \mathbb{S}^1. \tag{3}$$

When $\lambda \in \mathbb{S}^1$, we further have

$$|f_\lambda'(z)| = \frac{d(b^2-1)}{1 + b^2 + 2b \operatorname{Re} z}, \qquad \forall z \in \mathbb{S}^1. \tag{4}$$

Therefore the value of $|f_\lambda'(z)|$ increases as $\operatorname{Re} z$ decreases. Recall that a rational map f is *expanding* on an invariant set K if f locally increases distances, while it is *uniformly expanding* if distances are locally increased by a multiplicative factor, bounded below by a constant strictly greater than 1.

Lemma 3.1 ([PR18, Lemma 9]). *If $b > \frac{d+1}{d-1}$ and $\lambda \in \mathbb{S}^1$ then the map f_λ is uniformly expanding on $\mathbb{S}^1$. If $b = \frac{d+1}{d-1}$ then the map f_λ is expanding on $\mathbb{S}^1$.*

Definition 3.2 ([PR18, Lemma 12]). Given $1 < b < \frac{d+1}{d-1}$ we write $\lambda_1 \in \mathbb{S}^1$ for the unique parameter satisfying $0 < \arg(\lambda_1) < \pi$ and for which f_{λ_1} has a parabolic fixed point.

The following proposition describes the set of hyperbolic parameters on the unit circle

Proposition 3.3. *We have*

$$\mathbb{S}^1 \cap \Lambda^{hyp} = \begin{cases} \mathbb{S}^1 & \text{if } b > \frac{d+1}{d-1}, \\ \mathbb{S}^1 \setminus \{1\} & \text{if } b = \frac{d+1}{d-1}, \\ \mathbb{S}^1 \setminus \{\lambda_1, \overline{\lambda}_1\} & \text{if } 1 < b < \frac{d+1}{d-1}. \end{cases}$$

Proof. When $b > \frac{d+1}{d-1}$ then by Lemma 3.1 the map f_λ is uniformly expanding and therefore hyperbolic. When $b = \frac{d+1}{d-1}$ and $\lambda \in \mathbb{S}^1 \setminus \{1\}$ then for every $z \in \mathbb{S}^1$ either z or $f(z)$ is uniformly bounded away from 1. By (4) we obtain again that the map f_λ^2 is uniformly expanding, proving that f_λ is hyperbolic. On the other hand when $\lambda = 1$ the map f_λ has a parabolic fixed point, and therefore it is not hyperbolic.

Given $1 < b < \frac{d+1}{d-1}$, then $\lambda_1, \overline{\lambda}_1$ are the unique parameters on the unit circle for which f_λ has a parabolic fixed point. Suppose that there exists $\lambda \in \mathbb{S}^1 \setminus \{\lambda_1, \overline{\lambda}_1\}$ which is not hyperbolic. By (3) the set $\widehat{\mathbb{C}} \setminus \mathbb{S}^1$ is contained in the Fatou set, and therefore the critical points of f_λ are also contained in the Fatou set. It follows that the map f_λ must have a parabolic cycle with period at least 2. Since there are at most two Fatou components we conclude that the period of the parabolic cycle is exactly 2.

Notice that for every $\lambda \in \mathbb{S}^1$ we have that $f_\lambda(1/\overline{z}) = 1/\overline{f_\lambda(z)}$, and therefore that

$$f_\lambda^n(-1/b) = 1/\overline{f_\lambda^n(-b)}$$

Let $z_1, z_2 \in \mathbb{S}^1$ be the parabolic cycle of f_λ. These two points are parabolic fixed points for f_λ^2 and both of them have an immediate basin that must coincide with a Fatou component of f_λ. By replacing z_1 with z_2 if necessary, we may therefore assume that $B(0,1)$ is the attracting basin of z_1, while $\widehat{\mathbb{C}} \setminus \overline{B}(0,1)$ is the attracting fixed point of z_2. This shows that $f_\lambda^{2n}(-1/b) \to z_1$ and that $f_\lambda^{2n}(-b) \to z_2$. And therefore that $z_1 = 1/\overline{z}_2 = z_2$, contradicting the fact that the period of the cycle is 2. □

We notice that -1 is always a hyperbolic parameter, therefore the set Λ_{-1}^{hyp}, i.e., the connected component of Λ^{hyp} containing -1, is always well defined. On the other hand the set Λ_1^{hyp} is defined for $b \neq \frac{d+1}{d-1}$ and does not coincide with Λ_{-1}^{hyp} if and only if $1 < b < \frac{d+1}{d-1}$.

Proposition 3.4. *Let $b > 1$. Then for every $\lambda \in \Lambda_{-1}^{hyp}$ the Julia set J_λ is a quasi-circle, while the Fatou set F_λ contains exactly two components which are the attracting basin of a (super)attracting 2-cycle. If we further assume that $|\lambda| = 1$ then $J_\lambda = \mathbb{S}^1$.*

Let $1 < b < \frac{d+1}{d-1}$. Then for every $\lambda \in \Lambda_1^{hyp}$ the Julia set J_λ is a Cantor set, while the Fatou set F_λ coincides with the attracting component of an attracting fixed point.

Proof. The function $g(z) = f_{-1}^2(z) - z$ satisfies $g(0) < 0$ and $g(-1/b) > 0$. Since g is a real map and f_{-1} maps the disk to the complement of its closure, we conclude that f_{-1} has a periodic point of order 2 in $B(0,1)$. By (3) it is clear that $J_{-1} = \mathbb{S}^1$. The holomorphic motion of J_{-1} over $(\Lambda_{-1}^{hyp}, -1)$ given by Theorem 2.1 now implies

that the Julia set J_λ is a quasi-circle for every $\lambda \in \Lambda_{-1}^{hyp}$. The two components of F_{-1} are mapped into each other, and by continuity the same holds for F_λ. Hyperbolicity of f_λ implies that they are the basin of a (super)attracting 2-cycle.

When $1 < b < \frac{d+1}{d-1}$ the map f_1 has an attracting fixed point at 1. It is well known that the Julia set of a rational function with a single invariant attracting basin containing all the critical points is a Cantor set (see also [Mil00, Theorem B.1]). Proceeding as above we obtain that for every $\lambda \in \Lambda_1^{hyp}$ the set J_λ is a Cantor set and that F_λ coincides with the attracting basin of a (super)attracting fixed point. Since the critical point of f_λ are not fixed point, we conclude that the fixed point is attracting. □

Remark 3.5. A *bicritical* rational map is a rational with two distinct critical points (counted without multiplicity). The space of bicritical rational map of degree d was studied by Milnor [Mil00], where he shows that its Moduli space (the space of holomorphic conjugacy classes) is biholomorphic to $\mathbb{C}^2$. In this paper he constructs explicit conjugacy invariants $f \mapsto (X, Y)$. In our case the invariants associated to the map f_λ are given by

$$X = \frac{b^2}{1-b^2}, \qquad Y = \left(\lambda + \frac{1}{\lambda}\right) \frac{b^{d-1}}{(1-b^2)^d}.$$

A bicritical rational map is *real* if its invariants are real, or equivalently if there exists an antiholomorphic involution α which commutes with the map. When $b \in \mathbb{R} \setminus \{0, 1\}$, the map f_λ is real if and only if $\lambda \in \mathbb{S}^1$, and the corresponding involution is $\alpha = 1/\overline{z}$. The results obtained by Milnor for real maps are sufficient to conclude that given $\lambda \in \mathbb{S}^1$ the Julia set is either a Cantor set or the whole circle.

The following definition follows from the proposition above. Recall that when $1 < b < \frac{d+1}{d-1}$ by Proposition 3.3 we have $\mathrm{Arc}(\overline{\lambda_1}, \lambda_1) = \Lambda_1^{hyp} \cap \mathbb{S}^1$ and that when $\lambda \in \mathbb{S}^1$ all fixed point of f_λ are on the unit circle.

Definition 3.6. Given $1 < b < \frac{d+1}{d-1}$ and $\lambda \in \mathrm{Arc}(\overline{\lambda_1}, \lambda_1)$, we write $R_\lambda \in \mathbb{S}^1$ for the attracting fixed point of f_λ and I_λ for the connected component of $F_\lambda \cap \mathbb{S}^1$ containing R_λ. Notice that the map f_λ is an orientation reversing bijection $f_\lambda : I_\lambda \to I_\lambda$.

Theorem 3.7. *Let $1 < b < \frac{d+1}{d-1}$. Then there exist unique parameters $\lambda_0 \in \mathbb{S}^1$ with $0 < \arg(\lambda_0) < \arg(\lambda_1) < \pi$, so that when $\lambda \in \mathrm{Arc}[1, \lambda_1)$*

$$\begin{cases} \mathrm{Arc}(1, \lambda) \Subset I_\lambda, & \text{for } \lambda \in \mathrm{Arc}[1, \lambda_0), \\ I_\lambda = \mathrm{Arc}(1, \lambda), & \text{for } \lambda = \lambda_0, \\ I_\lambda \Subset \mathrm{Arc}(1, \lambda), & \text{for } \lambda \in \mathrm{Arc}(\lambda_0, \lambda_1). \end{cases}$$

Similar inclusions hold for $\lambda \in (\overline{\lambda}_1, 1]$.

The existence of λ_0 and the first two inclusions follows from [PR18, Theorem 5]. Since the dynamics of f_λ is conjugate to the dynamics of $f_{\overline{\lambda}}$ it will be sufficient to prove that $I_\lambda \Subset \mathrm{Arc}(1, \lambda)$ for $\lambda \in \mathrm{Arc}(\lambda_0, \lambda_1)$.

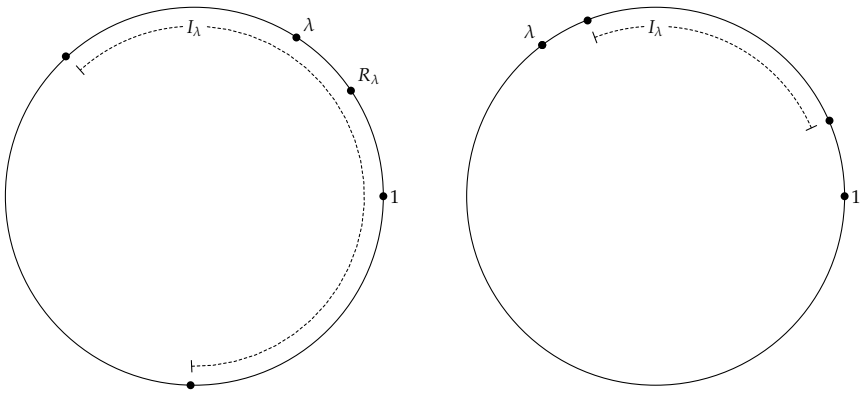

FIGURE 2. The position of I_λ for $\lambda \in \mathrm{Arc}(1, \lambda_0)$ and for $\lambda \in \mathrm{Arc}(\lambda_0, \lambda_1)$.

By the implicit function theorem the point R_λ moves holomorphically in a neighborhood of $\mathrm{Arc}(\overline{\lambda_1}, \lambda_1)$, furthermore by (4) it satisfies

$$\operatorname{Re} R_\lambda > \frac{b^2(d-1)-(d+1)}{2b} > -1. \tag{5}$$

Lemma 3.8. *Let* $1 < b < \frac{d+1}{d-1}$. *Then for every* $\lambda \in \mathrm{Arc}(1, \lambda_1)$ *we have* $\operatorname{Im} R_\lambda > 0$.

Proof. A simple calculation shows that 1 is an attracting fixed point for f_λ if and only if $\lambda = 1$. This fact together with (5) imply that $R_1 = 1$ and that $R_\lambda \neq \pm 1$ as $\lambda \in \mathrm{Arc}(1, \lambda_1)$. If we differentiate both sides of the equation $R_\lambda = f_\lambda(R_\lambda)$ with respect to λ, and then we evaluate at $\lambda = 1$, we obtain

$$\partial_\lambda R_\lambda|_{\lambda=1} = \frac{R_\lambda}{\lambda(1 - f'_\lambda(R_\lambda))}\Big|_{\lambda=1} = \frac{1}{1 - d\frac{1-b}{1+b}} > 0.$$

Therefore for $\lambda \in \mathrm{Arc}(1, \lambda_1)$ sufficiently close to 1 the point R_λ lies in the upper half plane. However as λ varies within $\mathrm{Arc}(1, \lambda_1)$, the point R_λ moves on $\mathbb{S}^1$ without intersecting $\{-1, 1\}$. Therefore $\operatorname{Im} R_\lambda > 0$ on the whole $\mathrm{Arc}(1, \lambda_1)$. □

Proof of Theorem 3.7. Let $z_\lambda, w_\lambda \in \mathbb{S}^1$ so that $I_\lambda = \mathrm{Arc}(z_\lambda, w_\lambda)$. Since the map $f_\lambda : I_\lambda \to I_\lambda$ is an orientation reversing bijection, we have

$$f_\lambda(z_\lambda) = w_\lambda, \qquad f_\lambda(w_\lambda) = z_\lambda,$$

showing that z_λ, w_λ are fixed points for f_λ^2. The Fatou set is connected, therefore there can be only one attracting or parabolic fixed point for f_λ^2, which is R_λ. This shows that the cycle z_λ, w_λ is repelling. By the implicit function theorem the points z_λ, w_λ move holomorphically and without collisions on some neighborhood $U \supset \mathrm{Arc}(\lambda_0, \lambda_1)$.

By the previous lemma and (5) we have

$$R_\lambda \in \left\{ x + iy \,\middle|\, y > 0,\ x > \frac{b^2(d-1)-(d+1)}{2b} > -1 \right\}.$$

Suppose now that for some $\lambda \in \mathrm{Arc}(\lambda_0, \lambda_1)$ we have $z_\lambda \in \mathrm{Arc}(1, R_\lambda)$. By (4) the map f_λ is a contraction on $\mathrm{Arc}[1, R_\lambda]$. As the point z moves counterclockwise on $\mathrm{Arc}[1, R_\lambda]$, its image $f_\lambda(z)$ moves clockwise on $\mathbb{S}^1$ starting at λ and ending at R_λ. Since f_λ is a contraction and $\mathrm{Im} R_\lambda > 0$, this is possible only if $f_\lambda : \mathrm{Arc}[1, R_\lambda] \to \mathrm{Arc}[R_\lambda, \lambda]$ is an orientation reversing bijection. We conclude that $w_\lambda = f_\lambda(z_\lambda) \in \mathrm{Arc}(R_\lambda, \lambda)$ and thus that $I_\lambda \Subset \mathrm{Arc}(1, \lambda)$.

If we differentiate both sides of $z_\lambda = f_\lambda^2(z_\lambda)$ we obtain the equation

$$\partial_\lambda z_\lambda \left(1 - (f_\lambda^2)'(z_\lambda)\right) = \lambda^{-1}\left(z_\lambda + f_\lambda'(w_\lambda) w_\lambda\right). \tag{6}$$

Since $I_{\lambda_0} = \mathrm{Arc}(1, \lambda)$ it follows that $z_{\lambda_0} = 1$ is a repelling fixed point for $f_{\lambda_0}^2$, furthermore since $|f_{\lambda_0}'(1)| < 1$ we must have $|f_{\lambda_0}'(\lambda_0)| > 1$ and $|(f_{\lambda_0}^2)'(1)| > 1$. If we evaluate the expression above at $\lambda = \lambda_0$ we obtain that

$$\partial_\lambda z_\lambda|_{\lambda=\lambda_0} = \frac{1}{\lambda_0} \frac{1 - |f_{\lambda_0}'(\lambda_0)|}{1 - |(f_{\lambda_0}^2)'(1)|} = \frac{C}{\lambda_0},$$

for some positive constant $C > 0$. If we write $\lambda(\varepsilon) = \lambda_0 e^{i\varepsilon}$ then we obtain that

$$z_{\lambda(\varepsilon)} = 1 + iC\varepsilon + O(\varepsilon^2)$$

therefore as $\lambda \in \mathrm{Arc}(\lambda_0, \lambda_1)$ is sufficiently close to λ_0, we must have $z_\lambda \in \mathrm{Arc}(1, R_\lambda)$ and thus that $I_\lambda \Subset \mathrm{Arc}(1, \lambda)$.

This also proves that the point z_λ moves counterclockwise as λ is close to λ_0. We will show that z_λ moves counterclockwise on the whole arc between λ_0 and λ_1. Assume otherwise, then there is some $\mu \in \mathrm{Arc}(\lambda_0, \lambda_1)$ such that

$$0 = \partial_\lambda z_\lambda|_{\lambda=\mu} = \frac{1}{\mu}\left(z_\mu + f_\mu'(w_\mu) w_\mu\right).$$

Note that, since $z_\lambda \neq R_\lambda$ for any $\lambda \in \mathrm{Arc}(\lambda_0, \lambda_1)$, it follows that $\mu \in \mathrm{Arc}(1, R_\mu)$. As a result we must have $|f_\mu(z_\mu)| < 1$, and therefore

$$|f_\mu'(z_\mu)| \cdot |f_\mu'(w_\mu)| = |f_\mu'(z_\mu)| \cdot \left|\frac{z_\mu}{w_\mu}\right| < 1,$$

which contradicts the fact that z_μ is a repelling fixed point of f_μ^2. This shows that $z_\lambda \in \mathrm{Arc}(1, R_\lambda)$ for every $\lambda \in \mathrm{Arc}(\lambda_0, \lambda_1)$ and therefore $I_\lambda \Subset \mathrm{Arc}(1, \lambda)$, concluding the proof of the proposition. □

Recall that for $b > 1$ the point -1 is a hyperbolic parameter and that Λ^{hyp} is an open and closed subset of Λ^{stb}. Therefore the connected component Λ_{-1}^{hyp} is a connected component of Λ^{stb}. The same is clearly true for Λ_1^{hyp} when 1 is a hyperbolic parameter.

Lemma 3.9. *For $b > 1$ the component Λ_{-1}^{hyp} is not exceptional. For $1 < b < \frac{d+1}{d-1}$ the component Λ_1^{hyp} is not exceptional.*

Proof. When $1 < b < \frac{d+1}{d-1}$ the component Λ_1^{hyp} is not exceptional since 1 is an attracting fixed point for the map f_1.

Suppose now that Λ_{-1}^{hyp} is exceptional for some $b > 1$ and let φ_μ be the holomorphic motion of J_{-1} over $(\Lambda_{-1}^{hyp}, -1)$. Since the holomorphic motion respects the dynamics we obtain that for every $\mu \in \Lambda_{-1}^{hyp}$

$$f_\mu^n(1) = f_\mu^n \circ \varphi_\mu(1) = \varphi_\mu \circ f_{-1}^n(1). \tag{7}$$

This shows that when the degree d is even the function f_μ maps 1 to a fixed point of f_μ, proving that $f_\mu^2(1) = f_\mu(1)$. On the other hand, when d is odd the point 1 is periodic with period two, and therefore $f_\mu^2(1) = 1$. Once the values of b and d are fixed there are only finitely many values of $\mu \in \Lambda_{-1}^{hyp}$ for which equation (7) is satisfied, giving a contradiction. □

Corollary 3.10. *Suppose that $b > 1$ and $U = \Lambda_{-1}^{hyp}$, or alternatively that $1 < b < \frac{d+1}{d-1}$ and $U = \Lambda_1^{hyp}$. Then the set of accumulation points of $\mathcal{Z}_{\mathcal{C}_{d+1}}$ in U equals $\mathcal{J}_U$. Moreover, if the degree d is even then*

$$\mathcal{J}_U = \overline{\mathcal{Z}_{\mathcal{C}_{d+1}}} \cap U. \tag{8}$$

Proof. We first prove the statement for general degrees d. By Lemmas 2.7 and 2.8 we obtain the inclusion

$$\mathcal{J}_U \cup (\mathcal{Z}_{\mathcal{C}_{d+1}} \cap U) \subset \overline{\mathcal{Z}_{\mathcal{C}_{d+1}}} \cap U.$$

Therefore it suffices to show that every $\lambda \in \mathcal{F}_U$ is either an isolated point of $\mathcal{Z}_{\mathcal{C}_{d+1}}$ or is not contained in $\overline{\mathcal{Z}_{\mathcal{C}_{d+1}}}$. The map f_λ is hyperbolic, therefore the orbit of 1 converges to an attracting periodic point Q_λ of period N. By Proposition 3.4 when $U = \Lambda_1^{hyp}$ the Fatou set F_λ is connected and $N = 1$. Similarly, when $U = \Lambda_{-1}^{hyp}$, the set F_λ is the union of two distinct connected components, and $N = 2$.

The parameter λ is passive, therefore $f_\mu^{2n+k}(1) \to f^k(Q_\mu)$ uniformly on some small ball $B(\lambda, \varepsilon)$, where $k = 0, 1$ and Q_μ is the holomorphic continuation of the periodic point Q_λ . The point -1 cannot be an attracting periodic point of order 1 or 2, thus $Q_\mu, f_\mu(Q_\mu) \neq -1$. We conclude that whenever n is sufficiently large the point $f_\mu^n(1)$ is bounded away from -1. Therefore the intersection $B(\lambda, \varepsilon) \cap \mathcal{Z}_U$ only contains isolated points.

Now suppose that d is even and let $\lambda \in \mathcal{Z}_{\mathcal{C}_{d+1}} \cap U$. Let $N > 0$ be the first integer so that $f_\lambda^N(1) = -1$. Since d is even the point -1 is periodic with period N. If $N > 2$ the point -1 is a repelling periodic point, since attracting fixed points in U have period 1 or 2. On the other hand -1 cannot be an attracting periodic point with order 1 or 2. This shows that $\mathcal{Z}_{\mathcal{C}_{d+1}} \cap U \subset \mathcal{J}_U$, which proves (8). □

Proposition 3.11. *Let $1 < b < \frac{d+1}{d-1}$. Then the set $\mathcal{J}_{\Lambda_1^{hyp}}$ is a Cantor set, with respect to the intrinsic topology of Λ_1^{hyp}.*

Proof. Proposition 2.5 states that $\mathcal{J}_{\Lambda_1^{hyp}}$ is a perfect set. Therefore we only need to show that every connected component K of this set consists of a single point. Let φ_μ be the holomorphic motion of J_1 over $(\Lambda_1^{hyp}, 1)$. By Theorem 2.1 the map $\varphi_\mu^{-1}(1)$ is continuous and sends K inside J_1. Since J_1 is a Cantor set we conclude

that $\varphi_\mu^{-1}(1)$ is constant on K, and therefore that $\varphi_\mu(c) = 1$ for some $c \in \widehat{\mathbb{C}}$ and every $\mu \in K$.

If K contains more then one point then by the identity principle we must have $\varphi_\mu(c) = 1$ for all $\mu \in U$, and in particular $c = \varphi_1(c) = 1$, thus showing that Λ_1^{hyp} is an exceptional component, contradicting Lemma 3.9. We conclude that K is a single point. □

Combining the previous proposition with Corollary 3.10 we conclude the proof of claim (2) in Theorem 1.4.

4. Restriction to the unit circle

Throughout this section it will be assumed that $b > 1$.

Lemma 4.1. *Let $\lambda \in \mathbb{S}^1 \cap \Lambda^{hyp}$, and let φ_μ be the holomorphic motion of J_λ over $(\Lambda_\lambda^{hyp}, \lambda)$. Assume that $1 \in J_\lambda$ and that one of the two following conditions is satisfied:*

1. *the partial derivative $\partial_\mu \varphi_\mu(1)|_{\mu=\lambda} \neq 0$,*
2. *there exist sequences $z_n, w_n \in J_\lambda$ both converging to 1 and satisfying $1 \in \mathrm{Arc}(z_n, w_n)$.*

Then there exists a sequence $\lambda_n \in \mathbb{S}^1$ converging to λ so that 1 is a repelling periodic point for each map f_{λ_n}.

Proof. Given $\lambda \in \mathbb{S}^1$ the Julia set J_λ is contained in the unit circle. Therefore for every $\varepsilon > 0$ there exists $\delta > 0$ so that the map $\mu \mapsto \varphi_\mu(1)$ sends $I_\delta = B(\lambda, \delta) \cap \mathbb{S}^1$ inside a relatively compact subset of $I_\varepsilon = B(1, \varepsilon) \cap \mathbb{S}^1$. By continuity of the holomorphic motion we may assume that the same is true for the map $\mu \mapsto \varphi_\mu(z)$ whenever $z \in J_\lambda$ is sufficiently close to 1. The map $\varphi_\mu(z) : I_\delta \to I_\varepsilon$ can be interpreted as a map between intervals. We will denote its graph as $\Gamma(z) \subset I_\delta \times I_\varepsilon$.

Assume first that condition (1) holds. Let $z_n \in J_\lambda$ be a sequence of repelling periodic points so that $z_n \to 1$ and $z_n \neq 1$. Since $\partial_\mu \varphi_\mu(1)|_{\mu=\lambda} \neq 0$ the graph $\Gamma(1)$ intersects the line $w = 1$ transversally at the point $(\lambda, 1)$. By Theorem 2.1, when n is sufficiently large the graph $\Gamma(z_n)$ is uniformly close to $\Gamma(1)$ and therefore it intersects the line $w = 1$ in $(\lambda_n, 1)$ for some $\lambda_n \in I_\delta \setminus \{\lambda\}$ close to λ. It follows that $\lambda_n \to \lambda$ and that $1 = \varphi_{\lambda_n}(z_n)$ is a repelling point for f_{λ_n}

Assume now that condition (2) holds. Since repelling fixed points are dense in the Julia set, we may assume from the beginning that the z_n and w_n are repelling fixed points for f_λ. When n is sufficiently large the graphs $\Gamma(z_n)$ and $\Gamma(w_n)$ are both close to $\Gamma(1)$. Furthermore, since the map $z \mapsto \varphi_\mu(z)$ is injective, we conclude that

$$\varphi_\mu(1) \in \mathrm{Arc}\left(\varphi_\mu(z_n), \varphi_\mu(w_n)\right), \qquad \forall \mu \in I_\delta,$$

meaning that the graph $\Gamma(1)$ lies in between $\Gamma(z_n)$ and $\Gamma(w_n)$.

It follows that when n is sufficiently large there exists $\lambda_n \in I_\delta \setminus \{\lambda\}$ close to λ so that either $\varphi_{\lambda_n}(z_n) = 1$ or $\varphi_{\lambda_n}(w_n) = 1$. As in the previous case we obtain that 1 is a repelling fixed point for f_{λ_n} and that $\lambda_n \to \lambda$, concluding the proof of the proposition. □

Proposition 4.2. *Let $\lambda \in \mathbb{S}^1$ be so that 1 is a repelling periodic point for f_λ. Then $\lambda \in \overline{\mathcal{Z}_{\mathcal{C}_{d+1}} \cap \mathbb{S}^1}$.*

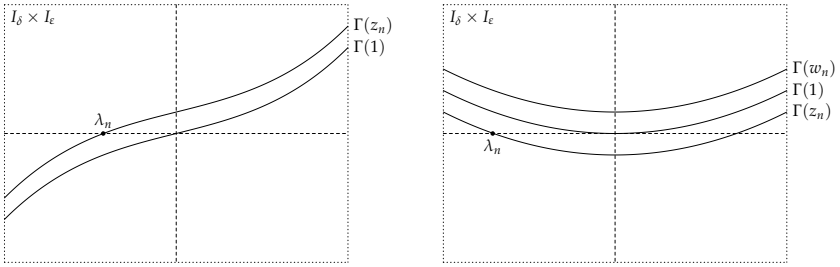

FIGURE 3. The graphs of the holomorphic motions respectively in case (1) and (2).

This proposition is proved after Lemma 4.5. We claim that is enough to prove the statement for hyperbolic parameters. When $b > \frac{d+1}{d-1}$ by Proposition 3.3 all points in the circle are hyperbolic, and the claim is certainly true. Assume instead that $1 < b \leq \frac{d+1}{d-1}$ and that the Proposition holds for hyperbolic parameters. Given $\lambda \in \Lambda_{-1}^{hyp} \cap \mathbb{S}^1$ by Corollary 3.4 the Julia set J_λ coincides with the unit circle, and by Lemma 4.1 we may find parameters in $\Lambda_{-1}^{hyp} \cap \mathbb{S}^1$ arbitrarily close to λ for which 1 is a repelling periodic point. It follows that $\lambda \in \overline{\mathcal{Z}_{\mathcal{C}_{d+1}} \cap \mathbb{S}^1}$ and that

$$\overline{\Lambda_{-1}^{hyp} \cap \mathbb{S}^1} \subseteq \overline{\mathcal{Z}_{\mathcal{C}_{d+1}} \cap \mathbb{S}^1}.$$

This shows that the proposition holds also when $\lambda \in \mathbb{S}^1$, proving the claim (by Proposition 3.3, non-hyperbolic point on the circle are in $\overline{\Lambda_{-1}^{hyp} \cap \mathbb{S}^1}$).

From now on we will fix $\lambda \in \mathbb{S}^1 \cap \Lambda^{hyp}$ so that 1 is a repelling point for f_λ. Given such λ we will write N for the period of the point 1, and φ_μ for the holomorphic motion of J_λ over $(\Lambda_\lambda^{hyp}, \lambda)$.

Lemma 4.3. *Suppose that $b \geq \frac{d+1}{d-1}$ or that $N \geq 3$. Then there exist sequences $z_n, w_n \in J_\lambda$ both converging to 1 and satisfying $1 \in \mathrm{Arc}(z_n, w_n)$.*

Proof. Suppose first that $b \geq \frac{d+1}{d-1}$ or that $1 < b < \frac{d+1}{d-1}$ and $\lambda \in \mathrm{Arc}(\lambda_1, \overline{\lambda}_1)$. In this case $J_\lambda = \mathbb{S}^1$, and the result follows immediately.

When instead $1 < b < \frac{d+1}{d-1}$ and $\lambda \in \mathrm{Arc}(\overline{\lambda}_1, \lambda_1)$, then by Proposition 3.4 the Julia set $J_\lambda \subset \mathbb{S}^1$ is a Cantor set. Suppose for the purpose of a contradiction that the Lemma were false, so that $1 \in \partial \widehat{I}$, where $\widehat{I}$ is a connected component of $F_\lambda \cap \mathbb{S}^1$. The connected components of $F_\lambda \cap \mathbb{S}^1$ are open arcs which are mapped one to another by f_λ and are eventually mapped into the invariant arc I_λ containing the unique attracting fixed point R_λ.

If $1 \in \partial \widehat{I}$ it follows that for some $n > 0$ we must have $f_\lambda^n(1) \in \partial I_\lambda$. But I_λ is invariant and we are assuming that 1 is periodic, therefore $1 \in \partial I_\lambda$. However this is not possible when $N \geq 3$ since the boundary points of I_λ are repelling periodic points with period 2, giving a contradiction. □

The point 1 is a fixed point if and only if $\lambda = 1$. If $1 < b < \frac{d+1}{d-1}$ then the point 1 is an attracting fixed point for f_1, thus if 1 is a repelling periodic point for f_λ we must have $N \geq 2$.

Lemma 4.4. *Suppose that $1 < b < \frac{d+1}{d-1}$ and that $N = 2$. Then $\partial_\mu \varphi_\mu(1)|_{\mu=\lambda} \neq 0$.*

Proof. Suppose instead that $\partial_\mu \varphi_\mu(1)|_{\mu=\lambda} = 0$ and write $f_\lambda(z) = \lambda g(z)$. Since the point $\varphi_\mu(1)$ is a repelling periodic point of period 2 for f_μ it follows that

$$\begin{aligned} 0 &= \partial_\mu \left(f_\mu^2 \circ \varphi_\mu(1) \right) \Big|_{\mu=\lambda} \\ &= g(\lambda) + f_\lambda'(\lambda). \end{aligned}$$

For $\lambda \in \mathbb{S}^1$ we then have $|f_\lambda'(\lambda)| = |g(\lambda)| = 1$, and therefore $|(f_\lambda^2)'(1)| = |f_\lambda'(1)| < 1$, contradicting the fact that 1 is a repelling fixed point with period 2. □

Since λ is a hyperbolic parameter, there exists an integer $j \geq 1$, an open neighborhood $U \supset J_\lambda$ and $\kappa > 1$ so that whenever $z, w \in U$ are sufficiently close we have

$$|f_\lambda^{jN}(z) - f_\lambda^{jN}(w)| \geq \kappa |z - w|.$$

Given $\delta > 0$ sufficiently small we may further assume that the same is true for f_μ when we take $\mu \in I_\delta = B(\lambda, \delta) \cap \mathbb{S}^1$. Since the map $\mu \mapsto J_\mu$ is continuous with respect to the Hausdorff distance, we may further assume that $J_\mu \subset U$ for every $\mu \in I_\delta$. We therefore obtain the following:

Lemma 4.5. *There exists $\varepsilon > 0$ so that for every $\mu \in I_\delta$ and $z, w \in J_\mu$ distinct we can find $k \geq 0$ so that*

$$|f_\mu^{kN}(z) - f_\mu^{kN}(w)| \geq 2\varepsilon.$$

Proof of Proposition 4.2. Let $\lambda \in \Lambda^{hyp} \cap \mathbb{S}^1$ be a parameter for which 1 is a repelling periodic point of period $N \geq 2$. We will assume first that $b \geq \frac{d+1}{d-1}$ or that $N \geq 3$.

By Lemma 4.3 there exist two sequences in J_λ converging to 1; one contained in the upper half plane and one in the lower half plane. Since the backward images of the point -1 accumulates on the Julia set J_λ, and thus on every point in such sequences, we may find two preimages $\alpha_\lambda, \beta_\lambda$ of the point -1 contained in $I_\varepsilon = B(1, \varepsilon) \cap \mathbb{S}^1$ so that $1 \in \mathrm{Arc}(\alpha_\lambda, \beta_\lambda)$. Write $M_1, M_2 > 0$ for the two positive integers so that $f^{M_1}(\alpha_\lambda) = f^{M_2}(\beta_\lambda) = -1$.

Take $\delta > 0$ and write $I_\delta = B(\lambda, \delta) \cap \mathbb{S}^1$. Then if δ is sufficiently small we have $\varphi_\mu(1) : I_\delta \to I_\varepsilon$ and we can find two continuous functions $\alpha_\mu, \beta_\mu : I_\delta \to I_\varepsilon$ so that $f^{M_1}(\alpha_\mu) = f^{M_2}(\beta_\mu) = -1$ and

$$\varphi_\mu(1) \in \mathrm{Arc}(\alpha_\mu, \beta_\mu) \qquad \forall \mu \in I_\delta.$$

Lemma 4.3 shows that condition (2) in Lemma 4.1 is satisfied. It follows that there exists $\lambda' \in I_\delta \setminus \{\lambda\}$ arbitrarily close to λ so that 1 is also a repelling periodic point for $f_{\lambda'}$, and therefore $1 \in J_{\lambda'}$. Furthermore by Lemma 3.9 the holomorphic map $\varphi_\mu(1)$ is not constant, therefore we may choose λ' so that $1 \neq \varphi_{\lambda'}(1)$.

By Lemma 4.5 we may find k so that $|f_{\lambda'}^{kN}(\varphi_{\lambda'}(1)) - f_{\lambda'}^{kN}(1)| \geq 2\varepsilon$, and since $\varphi_{\lambda'}(1) \in I_\varepsilon$ is a periodic point for $f_{\lambda'}$ of period N we conclude that

$$|f_{\lambda'}^{kN}(1) - 1| \geq |f_{\lambda'}^{kN}(1) - \varphi_{\lambda'}(1)| - |\varphi_{\lambda'}(1) - 1| \geq \varepsilon,$$

showing that $f_{\lambda'}^{kN}(1) \in \mathbb{S}^1 \setminus I_\varepsilon$.

The map $\mu \mapsto f_\mu^{kN}(1)$ is continuous and we have

$$f_\lambda^{kN}(1) = 1 \in \mathrm{Arc}(\alpha_\lambda, \beta_\lambda), \qquad f_{\lambda'}^{kN}(1) \notin \mathrm{Arc}(\alpha_{\lambda'}, \beta_{\lambda'}).$$

Therefore we may conclude that there exists $\lambda'' \in \mathrm{Arc}(\lambda, \lambda')$ so that either $f_{\lambda''}^{kN}(1) = \alpha_{\lambda''}$ or $f_{\lambda''}^{kN}(1) = \beta_{\lambda''}$. Since $\alpha_{\lambda''}$ and $\beta_{\lambda''}$ are preimages of the point -1 we conclude in both cases that $\lambda'' \in \mathcal{Z}_{\mathcal{C}_{d+1}} \cap \mathbb{S}^1$. Since λ', and thus λ'', can be chosen arbitrarily close to λ, we must have $\lambda \in \overline{\mathcal{Z}_{\mathcal{C}_{d+1}} \cap \mathbb{S}^1}$.

Suppose now that $1 < b < \frac{d+1}{d-1}$ and that $N = 2$. We notice that once b and d are fixed, this can happen only for finitely many values of λ. Combining Lemma 4.4 and Lemma 4.1 we can find $\lambda' \in \Lambda^{hyp} \cap \mathbb{S}^1$ arbitrarily close to λ for which 1 is a repelling fixed point for $f_{\lambda'}$ with period greater than 3. Since the proposition holds for λ' it must hold for λ as well, concluding the proof of the proposition. □

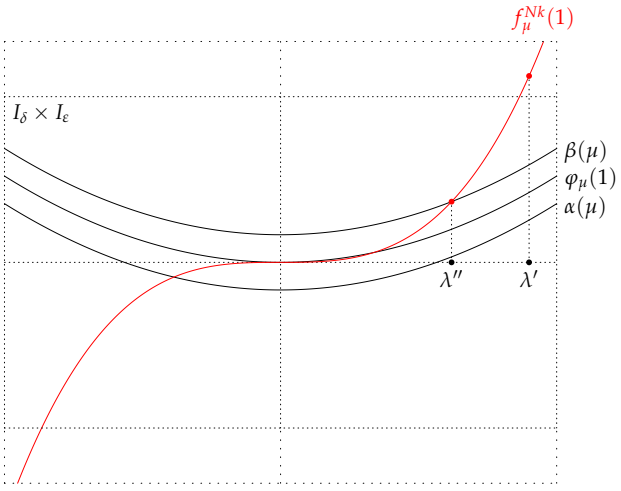

FIGURE 4. The position of the points λ' and λ''.

Proof of claim (1) *in Theorem 1.4.* We already showed that Lemma 4.1 together with Proposition 4.2 imply $\overline{\Lambda_{-1}^{hyp} \cap \mathbb{S}} \subset \overline{\mathcal{Z}_{\mathcal{C}_{d+1}} \cap \mathbb{S}}$. Therefore when $1 < b < \frac{d+1}{d-1}$ by Proposition 3.3 we have

$$\mathrm{Arc}[\lambda_1, \overline{\lambda_1}] \subset \overline{\mathcal{Z}_{\mathcal{C}_{d+1}} \cap \mathbb{S}}.$$

If $\lambda = \lambda_0$ or $\lambda = \overline{\lambda_0}$, then 1 is a repelling 2-cycle of f_{λ_0}, therefore by Proposition 4.2 we have $\{\lambda_0, \overline{\lambda_0}\} \subset \overline{\mathcal{Z}_{\mathcal{C}_{d+1}} \cap \mathbb{S}}$. □

5. HYPERBOLIC SEMIGROUPS AND EXPANDING ORBITS

All throughout this section we will assume that $d \in \mathbb{N}_{\geq 2}$ and $b \in (1, \frac{d+1}{d-1})$ are fixed.

Definition 5.1. Given $\lambda \in \hat{\mathbb{C}}$ we define the semigroup $H = \langle f_1, \dots, f_d \rangle$ as the semigroup generated by the maps

$$f_k(z) = \lambda \left(\frac{z+b}{bz+1} \right)^k, \qquad k = 1, \dots, d.$$

We will write F_k, J_k for the Fatou and Julia set of the map f_k and F_H, J_H for the Fatou and Julia set of the semigroup H.

The following characterization of the semigroup H will not be used later in the paper.

Proposition 5.2. *For all but countably many $\lambda \in \widehat{\mathbb{C}}$ the semigroup H is freely generated by $\{f_1, \dots, f_d\}$.*

Proof. Let S be the free group generated by $\{f_1, \dots, f_d\}$ and write $\Phi : S \to H$ for the homomorphism

$$\Phi[f_{i_1} \cdots f_{i_k}] = f_{i_k} \circ \cdots \circ f_{i_1}.$$

Given a word $s = f_{i_1} \cdots f_{i_k} \in S$ the map $\Phi[s]$ is a rational map in both z and λ. Its degrees with respect to z and λ are equal to

$$\begin{aligned} \deg_z(s) &= i_1 \cdots i_k, \\ \deg_\lambda(s) &= 1 + i_k + i_k \cdot i_{k-1} + \cdots + i_k \cdots i_1. \end{aligned}$$

We notice that the map $\Phi[s](1)$ is also a rational map in λ of degree

$$\deg'_\lambda(s) = 1 + i_k + i_k \cdot i_{k-1} + \dots i_k \cdots i_2.$$

The set S is countable, therefore it is sufficient to show that for every $s_1, s_2 \in S$ either $\Phi(s_1) = \Phi(s_2)$ for finitely many λ or $s_1 = s_2$.

Suppose instead that $s_1 = f_{i_1} \cdots f_{i_k}$ and $s_2 = f_{j_1} \cdots f_{j_l}$ satisfy $\Phi(s_1) = \Phi(s_2)$ for infinitely many λ. Using the identity principle, it is not hard to show that $\Phi(s_1)(z) = \Phi(s_2)(z)$ holds for all $\lambda \in \widehat{\mathbb{C}}$. This shows that the maps $\Phi(s_1)$ and $\Phi(s_2)$ coincide as rational maps in both variables z and λ, and therefore that

$$i_1 = \frac{\deg_z(s_1)}{\deg_\lambda(s_1) - \deg'_\lambda(s_1)} = \frac{\deg_z(s_2)}{\deg_\lambda(s_2) - \deg'_\lambda(s_2)} = j_1.$$

If we now write $s'_1 = f_{i_2} \cdots f_{i_k}$ and $s'_2 = f_{j_2} \cdots f_{j_l}$ we obtain that $\Phi(s'_1) = \Phi(s'_2)$ also holds for infinitely many λ and therefore $i_2 = j_2$. Iterating this procedure we obtain that $s_1 = s_2$, concluding the proof of the proposition. □

Notice that for $k \leq d$ we have $\frac{k+1}{k-1} \geq \frac{d+1}{d-1}$ and hence $b \in (1, \frac{k+1}{k-1})$.

Definition 5.3 (Notation)**.** In order to maintain the notation readable we are avoiding (where possible) the use of the subscript λ. As an example, notice that we are writing f_k instead of the more accurate $f_{k,\lambda}$. The function f_d will play an important role in the analysis of the semigroup H. Therefore we will write λ_0 and λ_1 for the parameters obtained by Theorem 3.7 applied to the map f_d, and I for the immediate attracting arc relative to the function f_d (if it exists). When it is necessary to distinguish between different values of the parameter λ we will write $f_{k,\lambda}$ and I_λ.

We will write Ω for the set of all possible sequences with entries in $\{1, \dots, d\}$. For every element $g \in H$ we can find $\omega \in \Omega$ and $n \in \mathbb{N}$ so that $g = f_\omega^n$ where we write

$$f_\omega^n = f_{\omega_n} \circ \cdots \circ f_{\omega_1}.$$

For $0 \leq m \leq n$ we will further write

$$f_\omega^{n,m} = f_{\omega_n} \circ \cdots \circ f_{\omega_m}.$$

Definition 5.4. We define the length of an element $g \in H$ as the minimum integer n for which $g = f_\omega^n$ for some $\omega \in \Omega$.

We proved in section 4 that for any λ in the closed arc $\mathrm{Arc}[\lambda_1, -1]$ there exists a (rooted) Cayley tree T_n of degree d and a parameter $\lambda' \in \mathbb{S}^1$ arbitrarily close to λ so that

$$Z_{T_n}(\lambda', b) = 0.$$

Recall from [PR18, Theorem B] on the other hand that for any $\lambda \in [1, \lambda_0)$, any $r \geq 0$ and any graph G of degree d we have

$$Z_G(r \cdot \lambda, b) \neq 0.$$

The situation for the arc $\mathrm{Arc}(\lambda_0, \lambda_1)$ appears to be more complicated. If we only consider Cayley trees of degree d it is possible to show that indeed there exist zeros of the partition function of some tree T_n on the arc $\mathrm{Arc}(\lambda_0, \lambda_1)$. However as shown in section 3, these zeros form a nowhere dense subset of the arc. Our purpose is to show that zeros of the partition for general bounded degree graphs are dense in $\mathrm{Arc}[\lambda_0, \lambda_3]$, where λ_3 is a parameter on the unit circle close to λ_0. In order to do so we will consider the class of rooted spherically symmetric trees of bounded degree for which, according to Lemma 1.6, the zero parameters can be understood by studying the semigroup dynamics of H.

In what follows we will study the semigroup dynamics (mostly) under the assumption that $\lambda \in \mathbb{S}^1$. Under these assumptions for any $g \in H_\lambda$ we have $g(\hat{\mathbb{C}} \setminus \mathbb{S}^1) = \hat{\mathbb{C}} \setminus \mathbb{S}^1$, and therefore that

$$\hat{\mathbb{C}} \setminus \mathbb{S}^1 \subset F_{H_\lambda}, \qquad J_{H_\lambda} \subset \mathbb{S}^1.$$

5.1 Hyperbolicity of the semigroup

Lemma 5.5. *There exists $\lambda_2 \in \mathrm{Arc}(\lambda_0, \lambda_1)$ so that for every $\lambda \in \mathrm{Arc}[\lambda_0, \lambda_2)$ and every $k \in \{1, \dots, d-1\}$ we have*

$$f_1 \circ f_k(I) \Subset I. \tag{9}$$

Proof. Assume first that $\lambda = \lambda_0$ and recall that by Theorem 3.7 we have $I = \mathrm{Arc}(1, \lambda_0)$. The Möbius transformation $\gamma(z) = (z + b)/(bz + 1)$ is a bijection of the unit circle into itself which revers the orientation, and $f_k(z) = \lambda_0 \gamma(z)^k$. The map $f_d : I \to I$ is an orientation reversing bijection and therefore satisfies $f_d(I) = I$. It follows immediately that, if we write ℓ for the length of the arc I, the image $f_k(I)$ is an arc of length $\ell^{k/d} < \ell$, therefore the map $f_k : I \to \mathbb{S}^1$ is not surjective.

Notice that when the point z move counterclockwise on the arc $I = \mathrm{Arc}(1, \lambda_0)$ starting at the point 1, its image $f_k(z)$ moves clockwise on the unit circle, starting at $\lambda_0 = f_k(1)$, until it reaches $f_k(\lambda_0)$. In principle it is possible that $f_k(z)$ rotates once or more times around the circle, however since f_k is not surjective on I this

does not happen. We conclude that $f_k : I \to \mathrm{Arc}(f_k(\lambda_0), \lambda_0)$ is also an orientation reversing bijection, and since the length of the arc $f_k(I)$ is less than the length of I, we must have $f_k(\lambda_0) \in I$. If we now compose with the map f_1 we find that

$$f_1 \circ f_k(I) = f_1\big(\mathrm{Arc}(f_k(\lambda_0), \lambda_0)\big) = \mathrm{Arc}\left(f_1(\lambda_0), f_1 \circ f_k(\lambda_0)\right) \Subset I.$$

Since I moves continuously in λ, the same holds sufficiently close λ_0, which implies the existence of λ_2. □

In the following we will denote by λ_2 the parameter with the maximal argument that satisfies the requirements of the previous lemma.

Definition 5.6. We define the semigroup $\widehat{H} \subset H$ as the semigroup generated by the maps

$$\widehat{H} = \langle \widehat{f}_1, \dots, \widehat{f}_d \rangle,$$

where $\widehat{f}_d = f_d$ and $\widehat{f}_k = f_1 \circ f_k$ for $k \in \{1, \dots, d-1\}$. We will write $F_{\widehat{H}}, J_{\widehat{H}}$ for the Fatou and the Julia set of the semigroup $\widehat{H}$.

Since $F_H \subset F_{\widehat{H}}$, it follows that $\hat{\mathbb{C}} \setminus \mathbb{S}^1 \subset F_{\widehat{H}}$. Furthermore by the previous lemma the interval I is invariant for every map in $\widehat{H}$ for $\lambda \in \mathrm{Arc}[\lambda_0, \lambda_2)$, and therefore it is contained in $F_{\widehat{H}}$, proving that for these λ

$$(\widehat{\mathbb{C}} \setminus \mathbb{S}^1) \cup I \subset F_{\widehat{H}}, \qquad J_{\widehat{H}} \subset \mathbb{S}^1 \setminus I, \tag{10}$$

and therefore that the Fatou set $F_{\widehat{H}}$ is connected.

Similarly to the case of the semigroup H, given $\omega \in \Omega$ and $0 \le m \le n$ we will write

$$\widehat{f}^n_\omega = \widehat{f}_{\omega_n} \circ \cdots \circ \widehat{f}_{\omega_1},$$
$$\widehat{f}^{n,m}_\omega = \widehat{f}_{\omega_n} \circ \cdots \circ \widehat{f}_{\omega_m}.$$

Also in this case every element of the semigroup can be written as $\widehat{f}^n_\omega$ for some $\omega \in \Omega$ and $n \in \mathbb{N}$.

By the previous lemma, given $\lambda \in \mathrm{Arc}[\lambda_0, \lambda_2)$ it follows that there exists a closed arc $J \subset I$ so that $\widehat{f}_k(J) \Subset J$ for every $k = 1, \dots, d$. Indeed, if we write $I = \mathrm{Arc}(z, w)$, this property is satisfied by $J = \mathrm{Arc}[\zeta, \eta]$, where ζ is sufficiently close to z and η lies in $\mathrm{Arc}(f_d(\zeta), f_d^{-1}(\zeta))$ (where the preimage is taken inside I). We may therefore define the family $\mathcal{C}$ and the closed arc K as follows:

$$\begin{aligned} &\mathcal{C} := \{J \subset I \mid J \neq \emptyset \text{ closed arc such that } \widehat{f}_k(J) \Subset \mathrm{int}(J)\ \forall k = 1, \dots, d\}, \\ &K := \bigcap_{J \in \mathcal{C}} J. \end{aligned} \tag{11}$$

In this definition $\mathrm{int}(J)$ refers to the interior of J with respect to the topology of the unit circle.

Lemma 5.7. *Let $\lambda \in \mathrm{Arc}[\lambda_0, \lambda_2)$. Then the set $K \subset I$ is a non-empty closed arc which is forward invariant under $\widehat{H}$. Furthermore the map $\lambda \mapsto K_\lambda$ is upper semi-continuous for $\lambda \in \mathrm{Arc}[\lambda_0, \lambda_2)$.*

Proof. Write $\mathcal{C}_{\mathbb{Q}}$ for the subset of all intervals $J \in \mathcal{C}$ whose extrema are rational angles. Notice that given $J \in \mathcal{C}$ there exists $J' \in \mathcal{C}_{\mathbb{Q}}$ for which $J' \subset J$, and therefore

$$K = \bigcap_{J \in \mathcal{C}_{\mathbb{Q}}} J.$$

Given $J \in \mathcal{C}$ we have that $f_d(J) \Subset J$ and $J \subset I$, proving that the arc J contains the unique attracting fixed point of f_d. This shows that for any pair $J_1, J_2 \in \mathcal{C}$ the intersection $J' = J_1 \cap J_2$ is non empty. Furthermore one can show that J' is again an element of $\mathcal{C}$. Similarly given $J_1, J_2 \in \mathcal{C}_{\mathbb{Q}}$ one has that $J_1 \cap J_2 \in \mathcal{C}_{\mathbb{Q}}$.

The set $\mathcal{C}_{\mathbb{Q}}$ is countable, hence we can enumerate its elements as $\{J_k\}_{k \geq 1}$. If we write $L_k = J_1 \cap \cdots \cap J_k$ it then follows that $L_k \in \mathcal{C}_{\mathbb{Q}}$, that $L_{k+1} \subset L_k$ and that

$$K = \bigcap_{k=1}^{\infty} L_k.$$

This shows that K is the intersection of a nested family of non-empty, compact and connected arcs, and therefore that K is a non-empty closed arc. Since every L_k is forward invariant for any $g \in \widehat{H}$, the same holds for K.

We will now write $\mathcal{C}_\mu, K_\mu$ in order to study parameter values close to λ. On the other hand we will write L_k for the set defined above for the parameter value λ. For every positive integer k we have $L_k \in \mathcal{C}_\mu$ for every $\mu \in \mathrm{Arc}[\lambda_0, \lambda_2)$ sufficiently close to λ, and for such parameters μ it then follows that $K_\mu \subset L_k$. Since L_k is a sequence of nested sets approximating K_λ we conclude immediately that the map $\mu \mapsto K_\mu$ is upper-semicontinuous at the point λ, concluding the proof of the lemma. □

Proposition 5.8. *Let $\lambda \in \mathrm{Arc}[\lambda_0, \lambda_2)$. Then every limit of a convergent sequence in $\widehat{H}$ with divergent length is constant on $F_{\widehat{H}}$ and contained in K.*

Proof. Let $g_k \in \widehat{H}$ be a sequence with divergent length that converges uniformly to $g_\infty : F_{\widehat{H}} \to \overline{F}_{\widehat{H}} = \hat{\mathbb{C}}$. Choose sequences $\omega_k \in \Omega$ and $n_k \in \mathbb{N}$ so that $g_k = \widehat{f}^{n_k}_{\omega_k}$, and notice that since the sequence g_k has divergent length we must have $n_k \to \infty$. Since $F_{\widehat{H}} \cap \mathbb{S}^1 \neq \emptyset$, it follows that $F_{\widehat{H}}$ is connected. Write ρ for the hyperbolic metric of $F_{\widehat{H}}$.

The Fatou set $F_{\widehat{H}}$ is forward invariant with respect to each element in the semigroup. In particular we have $\widehat{f}_1(F_{\widehat{H}}) \subset F_{\widehat{H}}$. Since the closed arc $K \subset F_{\widehat{H}}$ is forward invariant, it contains the unique attracting fixed point R_1 of $\widehat{f}_1$ (which coincides with the attracting fixed point of f_1). Assume now that $\widehat{f}_1(F_{\widehat{H}}) = F_{\widehat{H}}$. Since $\widehat{f}_1$ is invertible, it then follows that the Fatou set $F_{\widehat{H}}$ is completely invariant with respect to $\widehat{f}_1$, and therefore it contains the whole attracting basin of R_1. Since $\widehat{f}_1$ is a Möbius transformation, we conclude that the Julia set of the semigroup $J_{\widehat{H}}$ must consist of a single point, giving a contradiction. Therefore

$$\widehat{f}_1(F_{\widehat{H}}) \subsetneq F_{\widehat{H}}.$$

For every $k = 2, \ldots, d$ the map $\widehat{f}_k$ has two critical points $\{-b, -1/b\} \subset F_{\widehat{H}}$. It follows from the Schwarz-Pick Lemma that for every $k = 1, \ldots, d$ the map $\widehat{f}_k$ is a

contraction with respect to the metric ρ. This means that for every compact set $Q \subset F_{\widehat{H}}$ we may find a constant $c_Q < 1$ so that for any $z, w \in Q$ we have

$$\rho\left(\widehat{f}_k(w), \widehat{f}_k(z)\right) \leq c_Q \cdot \rho\,(w,z)\,, \qquad k = 1, \ldots, d.$$

Let K as in (11), $r \in \mathbb{R}_{>0}$ and define $Q = \overline{B}_\rho(K, r)$, where the ball is taken with respect to the hyperbolic metric of $F_{\widehat{H}}$. Given $z \in K$ and $w \in B_\rho(z, r)$, we know that for every $k, n \geq 0$ we have $\widehat{f}^n_{\omega_k}(z) \in K$ and

$$\rho\left(\widehat{f}^n_{\omega_k}(w), \widehat{f}^n_{\omega_k}(z)\right) \leq \rho(w, z).$$

This shows that $\widehat{f}^n_{\omega_k}(w), \widehat{f}^n_{\omega_k}(z) \in Q$, and thus that

$$\rho\left(\widehat{f}^{n_k}_{\omega_k}(w), \widehat{f}^{n_k}_{\omega_k}(z)\right) \leq c_Q^{n_k} \cdot \rho\,(w,z)\,,$$

which finally implies that $g_\infty(w) = g_\infty(z)$ for every $w \in B_\rho(z, r)$. We took r to be arbitrary and thus we can conclude that g_∞ is constant. For every $g \in \widehat{H}$ and $z \in K$ it follows that $g(z) \in K$, completing the proof. □

We recall the following definition of hyperbolicity for semigroups, introduced in [Sum97, Sum98].

Definition 5.9. Let G be a rational semigroup and consider the *postcritical set*

$$P_G := \overline{\bigcup_{g \in G} \{ \text{ critical values of } g\}}.$$

We say that the semigroup in *hyperbolic* if $P_G \subset F_G$.

Proposition 5.10. *The semigroup $\widehat{H}$ is hyperbolic.*

Proof. In our case the postcritical set can be written as

$$P_{\widehat{H}} = \overline{\bigcup_{\substack{\omega \in \Omega \\ n \in \mathbb{N}}} \widehat{f}^n_\omega(\{-b, -1/b\})}.$$

It is clear that $f^n_\omega(\{-b, -1/b\}) \in \widehat{\mathbb{C}} \setminus \mathbb{S}^1$ for every n and ω. Furthermore, by the previous theorem we know that every limit point belongs to K, showing that

$$P_{\widehat{H}} \subset (\widehat{\mathbb{C}} \setminus \mathbb{S}^1) \cup K \subset F_{\widehat{H}}$$

□

Given $\omega \in \Omega$ we write F_ω, J_ω for the Fatou and Julia sets of the family $\{\widehat{f}^n_\omega\}$. By (10) we have that

$$J_\omega \subset J_{\widehat{H}} \subset \mathbb{S}^1 \setminus I$$

and one can show that if $z \in J_\omega$ its orbit avoids the set I. On the other hand given $z \in F_\omega \cap \mathbb{S}^1$, by Proposition 5.8 we can find $n > 0$ so that $\widehat{f}^n_\omega(z) \in I$. We conclude that we can write

$$J_\omega = \bigcap_{n \in \mathbb{N}} (\widehat{f}^n_\omega)^{-1}(\mathbb{S}^1 \setminus I). \tag{12}$$

The following lemma shows expansivity of the dynamics on the Julia set J_ω of every sequence ω. The result corresponds to [Sum98, Theorem 2.6]; for the sake of completeness we provide a sketch the proof.

Lemma 5.11. *Let $\lambda \in \mathrm{Arc}[\lambda_0, \lambda_2)$ and $\kappa > 1$. Then there exists a positive integer $N \geq 1$ so that for any $\omega \in \Omega$ and $z \in J_\omega$ we have*

$$|(\widehat{f}_\omega^N)'(z)| \geq \kappa.$$

Sketch of the proof. We start by choosing an open simply connected neighborhood $V \subset (\widehat{\mathbb{C}} \setminus \mathbb{S}^1) \cup I$ containing K and two open simply connected neighborhoods $U' \Subset U$ containing $\mathbb{S}^1 \setminus I$ and disjoint from $P_{\widehat{H}}$ and V. Then choose $C > 1$ so that $d\rho_{U'} \geq C \cdot d\rho_U$, where we write $d\rho_U, d\rho_{U'}$ for the infinitesimal hyperbolic metric of the two sets.

By Proposition 5.8 there exists a positive integer N_0 so that

$$\widehat{f}_\omega^{N_0}(\overline{U} \setminus U') \subset V, \qquad \forall\, \omega \in \Omega,\, \forall n \geq N_0.$$

Given $\omega \in \Omega$ and $z \in J_\omega$, by (12), we have $\widehat{f}_\omega^n(z) \in U'$ for every n. Therefore we may define $U_0 \subset U'$ as the connected component of $(\widehat{f}_\omega^{N_0})^{-1}(U)$ containing z. Since U is simply connected and disjoint from the postcritical set, the map $\widehat{f}_\omega^{N_0}$ preserves the hyperbolic metrics of U_0 and U. Since $U_0 \subset U'$, this implies that

$$|(\widehat{f}_\omega^{N_0})'(z)|_U \geq C.$$

The hyperbolic metric of U and the Euclidean metric are comparable on U'. Therefore by taking an integer k sufficiently large, which does not depend on the choice of the sequence ω and z, we conclude that the value $N = kN_0$ satisfies the requirements of the Lemma. □

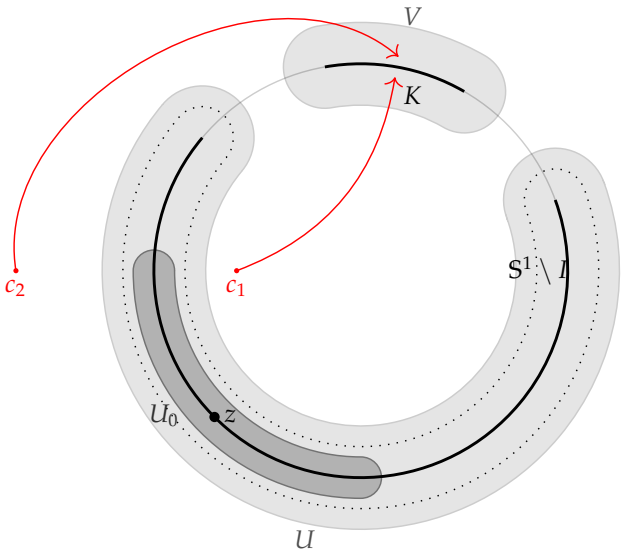

FIGURE 5. Sets used in the proof of Lemma 5.11

We will now show that, once we are bounded away from λ_2, the value of N can be chosen independently from λ. It will be convenient to reintroduce the subscript λ in order to distinguish between different parameter values. It is clear the set J_ω is dependent on λ, thus it will be denoted as $J_{\omega,\lambda}$.

Lemma 5.12. *Let $\lambda' \in \mathrm{Arc}[\lambda_0, \lambda_2)$ and $\kappa > 1$. Then there exists a positive integer $N \geq 1$ so that for any $\lambda \in \mathrm{Arc}[\lambda_0, \lambda']$, any $\omega \in \Omega$ and any $z \in J_\omega$ there exists $1 \leq n \leq N$ so that*

$$|(\widehat{f}^n_{\omega,\lambda})'(z)| \geq \kappa.$$

Proof. Given $\lambda \in \mathrm{Arc}[\lambda_0, \lambda_2)$ let N_λ be the minimum integer for which the previous lemma is valid. Suppose now that there exists a sequence $\lambda_k \in \mathrm{Arc}[\lambda_0, \lambda']$ such that $N_k = N_{\lambda_k} \to \infty$. It follows that we may find sequences $\omega_k \in \Omega$ and $z_k \in J_{\omega_k,\lambda_k}$, so that

$$|(\widehat{f}^n_{\omega_k,\lambda_k})'(z_k)| < \kappa, \qquad \forall\, 0 \leq n \leq N_k - 1.$$

By passing to a subsequence if necessary, we may assume that the three following conditions are satisfied

(1) the parameters λ_k converge to $\lambda_\infty \in \mathrm{Arc}[\lambda_0, \lambda_1]$;
(2) the points z_k converge to $z_\infty \in \mathbb{S}^1$;
(3) the sequences ω_k and ω_{k+1} agree on the first k elements.

Since the arc I_λ varies continuously in λ and by (12) every $z_k \in \mathbb{S}^1 \setminus I_{\lambda_k}$ it follows that $z_\infty \in \mathbb{S}^1 \setminus I_{\lambda_\infty}$. Let ω_∞ be the sequence given by $\omega_{\infty,k} = \omega_{k,k}$, where $\omega_{k,n}$ denotes the n-th element of the sequence ω_k. Then it is clear that ω_k and ω_∞ agree on the first k elements. Given $n \in \mathbb{N}$ we have

$$\widehat{f}^n_{\omega_\infty,\lambda_\infty}(z_\infty) = \lim_{k\to\infty} \widehat{f}^n_{\omega_k,\lambda_k}(z_k) \in \mathbb{S}^1 \setminus I_{\lambda_\infty},$$

proving that $z_\infty \in J_{\omega_\infty,\lambda_\infty}$. Furthermore, when k is sufficiently large we have $n \leq N_k$, therefore

$$|(\widehat{f}^n_{\omega_\infty,\lambda_\infty})'(z_\infty)| = \lim_{k\to\infty} |(\widehat{f}^n_{\omega_k,\lambda_k})'(z_k)| \leq \kappa,$$

contradicting the previous lemma. □

Proposition 5.13. *Let $\lambda' \in \mathrm{Arc}[\lambda_0, \lambda_2)$ and $\kappa > 1$. Then there exists a positive integer $N \geq 0$ so that for any $\lambda \in \mathrm{Arc}[\lambda_0, \lambda']$, any $\omega \in \Omega$ and any $z \in J_{\omega,\lambda}$ we have*

$$|(\widehat{f}^N_{\omega,\lambda})'(z)| \geq \kappa.$$

Proof. Let N as in the previous lemma. For any $\lambda \in \mathbb{S}^1$, any $\omega \in \Omega$ and any $n \in \mathbb{N}$ the rational map $\widehat{f}^n_{\omega,\lambda}$ has no critical points on the unit circle. We may therefore find a constant $\varepsilon > 0$ such that

$$|(\widehat{f}^n_{\omega,\lambda})'(z)| > \varepsilon, \qquad \forall \lambda, z \in \mathbb{S}^1, \quad \forall\, \omega \in \Omega, \quad \forall\, n \leq N.$$

Let $j \in \mathbb{N}$ so that $\varepsilon \cdot \kappa^j > \kappa$.

Suppose now that $\lambda \in \mathrm{Arc}[\lambda_0, \lambda']$, that $\omega \in \Omega$ and that $z \in J_{\omega,\lambda}$. Thanks to the previous lemma there exist positive integers $J \geq j$ and $n_1, \ldots, n_J \in \{1, \ldots, N\}$ which satisfy $(j-1) \cdot N < n_1 + \cdots + n_J \leq jN$ and so that the following holds: if we write $m_0 = 0$ and $m_i = n_1 + \ldots n_i$ then for $i \in \{1, \ldots, J\}$

$$|(\widehat{f}^{m_i,m_{i-1}}_{\omega,\lambda})'(\widehat{f}^{m_{i-1}}_{\omega,\lambda}(z))| \geq \kappa,$$

showing that

$$|(\widehat{f}^{jN}_{\omega,\lambda})'(z)| \geq |(\widehat{f}^{jN,jN-m_j}_{\omega,\lambda})'(\widehat{f}^{m_j}_{\omega,\lambda}(z))| \cdot \kappa^J \geq \varepsilon \cdot \kappa^j \geq \kappa.$$

By choosing jN instead of N, we conclude the proof of the proposition. □

5.2 Existence of expanding sequences

Lemma 5.14. *Given $d \in \mathbb{Z}_{\geq 2}$, $m \in \{1, \dots, d-1\}$, $t \in (0,1)$ and $s \in (0,t)$ there exists a $k \in \{1, \dots, d-1\}$ such that*

$$A_k := \left(\frac{2m-s}{d}\right) \cdot k + t$$

is an element of the interval $(1,2)$ when reduced modulo 2.

Proof. Note that either $m < (d+s)/2$ or $m > (d+s)/2$, since $s \notin \mathbb{Z}$. We consider these two cases separately.

If $m < (d+s)/2$ then $A_{k+1} - A_k < 1$ for all k. It follows that for any open interval $(a, a+1)$ contained in the interval $[A_0, A_d]$, there exists an integer $0 < k < d$ such that $A_k \in (a, a+1)$. Since $A_0 = t < 1$ and $A_d = 2m + t - s > 2$, there exists a k such that $A_k \in (1,2)$.

Now assume that $m > (d+s)/2$ and define

$$\tilde{A}_p = \left(\frac{2(d-m)+s}{d}\right) \cdot p + t - s.$$

Observe that $A_k - \tilde{A}_{d-k} = 2(k+m-d)$, and thus $A_k \equiv \tilde{A}_{d-k} \pmod 2$. Therefore it suffices to find a $p \in \{1, \dots, d-1\}$ for which $\tilde{A}_p \in (1,2)$. Such a p can be found by the same argument as above, because $\tilde{A}_0 = t - s < 1$, $\tilde{A}_d = 2(d-m) + t > 2$ and $\tilde{A}_{p+1} - \tilde{A}_p < 1$. □

Proposition 5.15. *Let $d \geq 2$ and $\lambda \in \mathrm{Arc}[\lambda_0, \lambda_1)$. Then for every $z \in \mathbb{S}^1 \setminus I$ there exists $\omega \in \Omega$ so that $z \in J_\omega$.*

Proof. Choose $0 < t < 1$ so that $\lambda = e^{i\pi t}$. The function f_d can be written as $f_d(z) = \lambda \cdot \gamma(z)^d$, where $\gamma(z) = (z+b)/(bz+1)$ is a Möbius transformation that fixes the unit circle. It follows that we can find d disjoint sets $J_0, \dots, J_{d-1}$ such that

$$f_d^{-1}(\mathrm{Arc}(1,\lambda)) = J_0 \cup \cdots \cup J_{d-1},$$

ordered in such a way that for all $m \in \{0, \dots, d-1\}$ we have

$$\gamma(J_m) = \mathrm{Arc}\left(\exp\left(i\pi\frac{2m-t}{d}\right), \exp\left(i\pi\frac{2m}{d}\right)\right).$$

Since f_d inverts the orientation of the unit circle and $f_d(1) = \lambda$, we have for $z \in \mathrm{Arc}(1,\lambda)$ close enough to 1 that $f_d(z) \in \mathrm{Arc}(1,\lambda)$. Furthermore by Theorem 3.7 we cannot have $f_d(\lambda) \in \mathrm{Arc}(1,\lambda)$. This shows that one of the connected component of $f_d^{-1}(\mathrm{Arc}(1,\lambda))$ is of the form $J = \mathrm{Arc}(1,z') \subset \mathrm{Arc}(1,\lambda)$. This component must contain the arc I. Since $\gamma(1) = 1$, we find that $J = J_0$.

Now let $J'_0, \dots, J'_{d-1}$ denote the inverse arcs of I under the map f_d in such a way that $J'_m \subset J_m$ for all m. Then we see that $J'_0 = I$. We now present a way to choose, given a $z \in \mathbb{S} \setminus I$, an integer k such that $\widehat{f}_k(z) \in \mathbb{S} \setminus I$.

- If $z \notin f_d^{-1}(I)$, we can choose $k = d$ because then $\widehat{f}_k(z) = f_d(z) \notin I$.
- If $z \in f_d^{-1}(I) = J_0' \cup \cdots \cup J_{d-1}'$ but $z \notin I = J_0$, we see that $z \in J_m$ for some $m \in \{1, \ldots, d-1\}$ and thus we can write

$$\gamma(z) = \exp\left(i\pi \frac{2m-s}{d}\right)$$

for some $0 < s < t$. We find that

$$\widehat{f}_k(z) = f_1\left(\lambda \cdot \gamma(z)^k\right) = f_1\left(\exp\left(i\pi\left(\frac{2m-s}{d} \cdot k + t\right)\right)\right).$$

According to Lemma 5.14, we can choose k in such a way that $\lambda \cdot \gamma(z)^k \in \mathrm{Arc}(-1,1)$. Since $f_1(\mathrm{Arc}(-1,1)) = \mathrm{Arc}(\lambda, -\lambda)$, we conclude that $\widehat{f}_k(z) \notin I$.

This procedure defines the first n steps of the sequence ω and satisfies $\widehat{f}_\omega^k(z) \in \mathbb{S}^1 \setminus I$ for all $k \leq n$. By iterating this procedure for the point $\widehat{f}_\omega^n(z)$ we find a sequence $\omega \in \Omega$ such that $\widehat{f}_\omega^n(z) \in \mathbb{S}^1 \setminus I$ for all $n \in \mathbb{N}$. By (12) we conclude that $z \in J_\omega$. □

Let $\lambda \in \mathrm{Arc}[\lambda_0, \lambda_1)$ and $z \in \mathrm{Arc}[\lambda, 1]$. Then we can modify the procedure described in the proof above to obtain the following:

Proposition 5.16. *Let $d \geq 2$ and $\lambda \in \mathrm{Arc}[\lambda_0, \lambda_1)$. Then for every $z \in \mathrm{Arc}[\lambda, 1]$ there exists $\omega \in \Omega$ so that $z \in J_\omega$ and*

$$\widehat{f}_\omega^n(z) \in \mathrm{Arc}[\lambda, 1], \qquad \forall n \in \mathbb{N}$$

Corollary 5.17. *For every $\lambda \in \mathrm{Arc}[\lambda_0, \lambda_2)$ we have*

$$\widehat{\mathbb{C}} \setminus (\mathbb{S}^1 \setminus I) = F_{\widehat{H}}, \qquad \mathbb{S}^1 \setminus I = J_{\widehat{H}}.$$

Proof. Given $z \in \mathbb{S}^1 \setminus I$ choose $\omega \in \Omega$ so that $z \in J_\omega$, which exists according to Proposition 5.15. For every positive integer $k \in \mathbb{N}$ by Proposition 5.13 we may find some N_k such that

$$|(\widehat{f}_\omega^{N_k})'(z)| \geq k,$$

showing that $|(\widehat{f}_\omega^{N_k})'(z)| \to \infty$ and therefore that $z \in J_{\widehat{H}}$. The two equalities follow from (10). □

6. Zeros for the semigroup

Our goal in this section is to give a more precise description of the zeros of $Z_G(\lambda)$ in $\mathrm{Arc}[\lambda_0, \lambda_1]$ for trees G that are spherically symmetric of bounded degree. We will prove case (3) of Theorem 1.4:

Theorem 6.1. *Let $d \in \mathbb{N}_{\geq 2}$. Then there exists $\lambda_3 \in \mathrm{Arc}(\lambda_0, -1]$ so that the set of zero parameters for spherically symmetric trees of degree d contains a dense subset of $\mathrm{Arc}[\lambda_0, \lambda_3]$.*

We will also prove the following weaker statement, which we will however prove for a considerably larger circular arc.

Theorem 6.2. *Let $d \in \mathbb{N}_{\geq 2}$. Then the closure of the set of zero parameters of spherically symmetric trees of degree d contains $\mathrm{Arc}[\lambda_0, \lambda_2]$.*

6.1 Proof of Theorem 6.1 Since we will repeatedly deal with distinct values of the parameter λ, we will always use the subscript λ in order to specify the map f_λ that we are using.

Choose a parameter value $\lambda' \in \mathrm{Arc}(\lambda_0, \lambda_2)$. By Proposition 5.13 there exists a positive integer N so that for every $\lambda \in \mathrm{Arc}[\lambda_0, \lambda']$, any $\omega \in \Omega$ and any $z \in J_{\omega,\lambda}$ we have

$$|(\widehat{f}^N_{\omega,\lambda})'(z)| > 3. \tag{13}$$

Having fixed N, it follows from the compactness of $\mathbb{S}^1$ that there exists a constant $C > 0$ so that for any $z \in \mathbb{S}^1$ and any $\omega \in \Omega$ we have

$$\left|\widehat{f}^N_{\omega,\lambda}(z) - \widehat{f}^N_{\omega,\mu}(z)\right| \leq C|\lambda - \mu| \qquad \forall\, \lambda, \mu \in \mathbb{S}^1.$$

Lemma 6.3. *There exist $\lambda_3 \in \mathrm{Arc}[\lambda_0, \lambda']$ and a positive integer M so that for every $\lambda \in \mathrm{Arc}[\lambda_0, \lambda_3]$ and $1 \leq m \leq M$ we have*

$$f^m_{d,\lambda}(1) \in \mathrm{Arc}[\lambda, 1] \quad \text{and} \quad |\partial_\lambda f^M_{d,\lambda}(1)| > 2C + 1. \tag{14}$$

Proof. The point 1 is a repelling periodic point of order 2 for the map f_{d,λ_0}, meaning that $|(f^2_{d,\lambda_0})'(1)| > 1$. Writing $\gamma(z) = \frac{z+b}{bz+1}$ we obtain

$$\begin{aligned}\left(\partial_\lambda f^{2m}_{d,\lambda}(1)\right)|_{\lambda=\lambda_0} &= \left(\partial_\lambda f^2_{d,\lambda}(1)\right)|_{\lambda=\lambda_0}\left(1 + (f^2_{d,\lambda_0})'(1) + \cdots + (f^2_{d,\lambda_0})'(1)^{m-1}\right)\\ &= \left(\gamma(\lambda_0)^d + f'_{d,\lambda_0}(1)\right)\frac{(f^2_{d,\lambda_0})'(1)^m - 1}{(f^2_{d,\lambda_0})'(1) - 1}.\end{aligned}$$

Observing that $\gamma(\lambda_0)^d + f'_{d,\lambda_0}(1) \neq 0$ it follows that when $M = 2m$ for m sufficiently large:

$$\left|\left(\partial_\lambda f^M_{d,\lambda}(1)\right)|_{\lambda=\lambda_0}\right| > 2C + 1.$$

By continuity there exists $\lambda_3 \in \mathrm{Arc}[\lambda_0, \lambda']$ so that

$$\left|\partial_\lambda f^M_{d,\lambda}(1)\right| > 2C + 1, \qquad \forall\, \lambda \in \mathrm{Arc}[\lambda_0, \lambda_3].$$

Given $\lambda \in \mathrm{Arc}(\overline{\lambda_1}, \lambda_1)$ we write z_λ, w_λ for the two boundary points of I_λ. These boundary points form a repelling 2-cycle, hence by the implicit function theorem they vary holomorphically in an open neighborhood of $\mathrm{Arc}(\overline{\lambda_1}, \lambda_1)$. By exchanging the order of z_λ and w_λ if necessary, we may assume that $z_{\lambda_0} = 1$. By Theorem 3.7 for every $\lambda \in \mathrm{Arc}(\lambda_0, \lambda_1)$ we have $z_\lambda \in \mathrm{Arc}(1, \lambda)$.

The point $f_{d,\lambda}(1) = \lambda$ clearly belongs to $\mathrm{Arc}[\lambda, 1]$. Recall that the map $f^2_{d,\lambda}$ preserves the orientation of the unit circle. Since z_λ is a repelling fixed point, by replacing λ_3 with a parameter in $\mathrm{Arc}(\lambda_0, \lambda_2)$ sufficiently close to λ_0 so that the point z_λ remains close to 1 for every $\lambda \in \mathrm{Arc}[\lambda_0, \lambda_3]$, we may assume that $f^2_{d,\lambda}(1) \in \mathrm{Arc}[\lambda, 1]$. Up to replacing at each step λ_3 with a parameter closer to λ_0, we may therefore assume that also $f^3_{d,\lambda}(1), \ldots, f^M_{d,\lambda}(1) \in \mathrm{Arc}[\lambda, 1]$, concluding the proof of the lemma. □

By Proposition 4.2 we know that zero parameters $\lambda \in \overline{\mathcal{Z}_{\mathcal{C}_{d+1}} \cap \mathbb{S}^1}$ accumulate on λ_0. From now until the end of the proof of Theorem 6.1 we will fix the value of the parameter $\lambda \in \mathrm{Arc}(\lambda_0, \lambda_3]$.

By Theorem 3.7 we know that $I_\lambda \Subset \mathrm{Arc}(1,\lambda)$. This fact together with (14) and (13) implies the existence of the following constant:

Definition 6.4 (Choice of the constant $\varepsilon > 0$)**.** There exists a sufficiently small constant $\varepsilon > 0$ so that the following three conditions are satisfied:

a. The distance between the attracting interval I_λ and $\mathrm{Arc}[\lambda,1]$ is at least ε,
$$\inf_{\substack{z\in I_\lambda \\ w\in \mathrm{Arc}[\lambda,1]}} |z-w| > \varepsilon.$$

b. For any $\mu \in B(\lambda,\varepsilon)$ we have
$$|f^M_{d,\lambda}(1) - f^M_{d,\mu}(1)| > 2C|\lambda-\mu|;$$

c. For any $\omega \in \Omega$, any $z \in J_{\omega,\lambda}$ and any $w \in B(z,\varepsilon)$ we have
$$|\widehat{f}^N_{\omega,\lambda}(z) - \widehat{f}^N_{\omega,\lambda}(w)| > 2|z-w|.$$

Definition 6.5 (Choice of the sequence $\sigma \in \Omega$)**.** we define a sequence of the form
$$\sigma = (\underbrace{d,\dots,d}_{M},\underbrace{\sigma_{M+1},\sigma_{M+2},\dots}_{\sigma^0}),$$
where $\sigma^0 \in \Omega$ is chosen such that $\widehat{f}^n_{\sigma,\lambda}(1) \in \mathrm{Arc}[\lambda,1]$ for all $n \in \mathbb{N}$. The existence of such sequence is guaranteed by Proposition 5.16, and we have $1 \in J_{\sigma,\lambda}$.

Lemma 6.6. *For every $\mu \in \mathbb{S}^1 \setminus \{\lambda\}$ there exists a positive integer n_μ so that*
$$|\widehat{f}^{n_\mu}_{\sigma,\lambda}(1) - \widehat{f}^{n_\mu}_{\sigma,\mu}(1)| \geq \varepsilon$$

Proof. Suppose on the other hand that there exists $\mu \in \mathbb{S}^1 \setminus \{\lambda\}$ so that $|\widehat{f}^n_{\omega,\lambda}(1) - \widehat{f}^n_{\omega,\mu}(1)| < \varepsilon$ for every $n \in \mathbb{N}$.

Since $\widehat{f}_{\sigma,\mu}(1) = \mu$, it follows in particular $|\lambda-\mu| < \varepsilon$. Therefore after the first M steps we obtain that
$$|\widehat{f}^M_{\sigma,\lambda}(1) - \widehat{f}^M_{\sigma,\mu}(1)| > 2C|\lambda-\mu|.$$

Write $T : \Omega \to \Omega$ for the left shift map, and define $\sigma^j = T^{M+jN}(\sigma)$ and $z_{j,\mu} = \widehat{f}^{M+jN}_{\sigma,\mu}(1) \in \mathbb{S}^1$. Thanks to the choice of the sequence σ^0 it follows that $z_{j,\lambda} \in \mathrm{Arc}[\lambda,1]$ and that
$$z_{j,\lambda} \in J_{\sigma^j,\lambda}.$$

We claim that for every $j \in \mathbb{N}$ we have $|z_{j,\lambda} - z_{j,\mu}| > \alpha_n C|\lambda-\mu|$, where $\alpha_0 = 2$ and $\alpha_{n+1} = 2\alpha_n - 1$. The claim certainly holds for $j = 0$. We will now assume it holds for $j \in \mathbb{N}$ and prove that it holds for $j+1$.

By the assumption on μ we know that $|z_{j,\lambda} - z_{j,\mu}| < \varepsilon$ for every $j \in \mathbb{N}$. Therefore, thanks to the choice of the constants ε and C, we conclude that
$$\begin{aligned}
|z_{j+1,\lambda} - z_{j+1,\mu}| &= |\widehat{f}^{M+(j+1)N}_{\sigma,\lambda}(1) - \widehat{f}^{M+(j+1)N}_{\sigma,\mu}(1)| \\
&> |\widehat{f}^N_{\sigma_j,\lambda}(z_{j,\lambda}) - \widehat{f}^N_{\sigma^j,\lambda}(z_{j,\mu})| - |\widehat{f}^N_{\sigma^j,\lambda}(z_{j,\mu}) - \widehat{f}^N_{\sigma^j,\mu}(z_{j,\mu})| \\
&> 2|z_{j,\lambda} - z_{j,\mu}| - C|\lambda-\mu| \\
&> (2\alpha_n - 1)C|\lambda-\mu|.
\end{aligned}$$

It then follows that for every n we must have $\alpha_n C|\lambda - \mu| < \varepsilon$, which is possible only if $\lambda = \mu$, giving a contradiction. □

Lemma 6.7. *There exists a positive integer n_0 so that every point in $\mathbb{S}^1 \setminus I_\lambda$ lies at distance at most $\varepsilon/4$ from the set*

$$\Sigma := \bigcup_{\substack{\omega\in\Omega \\ 0\le n\le n_0}} \widehat{f}^{-n}_{\omega,\lambda}\left(\{-1\}\right).$$

Proof. By Corollary 5.17 we have $\mathbb{S}^1 \setminus I_\lambda \subset J_{\widehat{H}_\lambda}$. The semigroup $\hat{H}_\lambda$ has no exceptional points, therefore the collection of all preimages of a given point in $\hat{\mathbb{C}}$ accumulates on the whole Julia set, and therefore on $\mathbb{S}^1 \setminus I_\lambda$.

Writing $V = \bigcup_{g\in\widehat{H}_\lambda} g^{-1}\left(\{-1\}\right)$ we therefore conclude that

$$\mathbb{S}^1 \setminus I_\lambda \subset \overline{V} \subset \bigcup_{z\in K} B(z, \varepsilon/4).$$

By compactness of $\mathbb{S}^1 \setminus I_\lambda$ we may find $w_1, \dots, w_\nu \in V$ so that $B(w_1, \varepsilon/4) \cup \cdots \cup B(w_n, \varepsilon/4)$ still covers the set $\mathbb{S}^1 \setminus I_\lambda$. Every w_j is the preimage of some element $g_j \in \widehat{H}_\lambda$ of length $n_j < \infty$, meaning that there exists $\omega_j \in \Omega$ so that

$$\widehat{f}^{n_j}_{\omega_j,\lambda}(z_j) = -1.$$

By taking $n_0 = \max n_i$ we obtain

$$\{z_1, \dots, z_n\} \subset \bigcup_{\substack{\omega\in\Omega \\ 0\le n\le n_0}} \widehat{f}^{-n}_{\omega,\lambda}\left(\{-1\}\right),$$

concluding the proof of the lemma. □

Write $\Sigma = \{w_1, \dots, w_\nu\}$. For every $w_j \in \Sigma$ there exists $\omega_j \in \Omega$ and $0 \le n_j \le n_0$ so that

$$\widehat{f}^{\nu_j}_{\omega_j,\lambda}(w_j) = -1.$$

No element in the semigroup $\widehat{H}_\lambda$ can have critical points on the unit circle. Therefore by the implicit function Theorem there exists a holomorphic map $\mu \mapsto w_{j,\mu}$ defined in a neighborhood of λ so that

$$\widehat{f}^{n}_{\omega,\mu}(w_{j,\mu}) = -1, \qquad w_{j,\lambda} = w_j.$$

By taking $\delta_0 > 0$ sufficiently small we may assume that for every $j = 1, \dots, \nu$ the map $w_{j,\mu}$ is defined on $B(\lambda, \delta_0)$ and that

$$|w_{j,\mu} - w_{j,\lambda}| < \varepsilon/4, \qquad \forall\, \mu \in B(\lambda, \delta_0),\ \forall\, j = 1, \dots, \nu.$$

Given $\delta < \delta_0$ we choose $\mu' \in \mathbb{S}^1 \cap B(\lambda, \delta)^*$ in such a way that $\mathrm{Arc}[\lambda, \mu'] \subset B(\lambda, \delta)$. We note that this last condition is not necessary, but simplifies the proof.

By Lemma 6.6 we may choose $n' = n_{\mu'}$ so that $|\widehat{f}^{n'}_{\sigma,\lambda}(1) - \widehat{f}^{n'}_{\sigma,\mu}(1)| \ge \varepsilon$. Given $\mu \in \mathrm{Arc}[\lambda, \mu']$, write $z_\mu = \widehat{f}^{n'}_{\sigma,\mu}(1)$.

Proposition 6.8. *There exists $j \in \{1, \dots, \nu\}$ and $\mu \in \mathrm{Arc}[\lambda, \mu']$ so that $z_\mu = w_{j,\mu}$*

Proof. By the definition of σ we have that $z_\lambda \in \mathrm{Arc}[\lambda, 1]$ and by the choice of the constant ε we know that $d(I_\lambda, z_\lambda) > \varepsilon$. Therefore, by replacing μ' with another parameter closer to λ in such a way that the value of n' does not change, we may further assume that $z_\mu \in \mathbb{S}^1 \setminus I_\lambda$ for every $\mu \in \mathrm{Arc}[\lambda, \mu']$.

The image of the map $\mathrm{Arc}[\lambda, \mu'] \ni \mu \mapsto z_\mu$ contains either $\mathrm{Arc}[z_\lambda, z_{\mu'}]$ or $\mathrm{Arc}[z_{\mu'}, z_\lambda]$ (notice that one of the two possibility occurs, since the image does not intersect I_λ). We will prove the proposition assuming that the first case occurs, a similar proof works in the other case.

Choose a point $\zeta \in \mathrm{Arc}[z_\lambda, z_\mu]$ so that the distance of the point from both the extrema of the arc is bigger or equal to $\varepsilon/2$, which is possible since $|z_\lambda - z_\mu| \geq \varepsilon$. Let $j \in \{1, \dots, \nu\}$ so that $|w_{j\lambda} - \zeta| < \varepsilon/4$. It then follows that $w_{j,\lambda} \in \mathrm{Arc}[z_\lambda, z_{\mu'}]$ and that

$$|w_{j,\lambda} - z_\lambda| > \varepsilon/4, \qquad |w_{j,\lambda} - z_{\mu'}| > \varepsilon/4,$$

and thanks to the fact that $\delta < \delta_0$ we conclude that

$$w_{j,\mu} \in \mathrm{Arc}[z_\lambda, z_{\mu'}], \qquad \forall\, \mu \in \mathrm{Arc}[\lambda, \mu'],$$

and therefore that there exists $\mu \in \mathrm{Arc}[\lambda, \mu']$ so that $w_{j,\mu} = z_\mu$. □

Let $\mu \in \mathbb{S}^1 \cap B(\lambda, \delta)$ and j as in the previous lemma, and let $\omega_j \in \Omega$ and $n_j \in \mathbb{N}$ so that $\widehat{f}^{n_j}_{\omega_j,\mu}(w_{j,\mu}) = -1$. We conclude that

$$\begin{aligned} \widehat{f}^{n_j}_{\omega_j,\mu}(\widehat{f}^{n'}_{\sigma,\mu}(1)) &= \widehat{f}^{n_j}_{\omega_j,\mu}(z_\mu) \\ &= \widehat{f}^{n_j}_{\omega_j,\mu}(w_{j,\mu}) \\ &= -1, \end{aligned}$$

concluding the proof of Theorem 6.1.

6.2 Proof of Theorem 6.2 Let $\lambda' \in \mathrm{Arc}[\lambda_0, \lambda_2)$. By Proposition 5.13 there exists a positive integer $N \geq 0$ so that for any $\lambda \in \mathrm{Arc}[\lambda_0, \lambda']$ any $\omega \in \Omega$ and any $z \in J_{\omega,\lambda}$ we have

$$|(\widehat{f}^N_{\omega,\lambda})'(z)| > 3.$$

Once N is fixed we have the following:

Lemma 6.9. *There exist constants $\varepsilon, \delta > 0$ so that for any $\lambda \in \mathrm{Arc}[\lambda_0, \lambda']$, any $\omega \in \Omega$, and any $z \in J_{\omega,\lambda}$, there exists a holomorphic map $F_{z,\omega,\lambda} : B(\lambda, \delta) \to B(z, \varepsilon)$ with $F_{z,\omega,\lambda}(\lambda) = z$, satisfying:*

(1)
$$|\widehat{f}^{kN}_{\omega,\lambda}(z) - \widehat{f}^{kN}_{\omega,\mu}(F_{z,\omega,\lambda}(\mu))| < \varepsilon, \qquad \forall \mu \in B(\lambda, \delta),\ \forall k \geq 0, \tag{15}$$

(2) *given any $\mu \in B(\lambda, \delta)$ and any $w \in B(z, \varepsilon) \setminus \{F_{z,\omega,\lambda}(\mu)\}$ there exists a positive integer k so that*
$$|\widehat{f}^{kN}_{\omega,\mu}(F_{z,\omega,\lambda}(\mu)) - \widehat{f}^{kN}_{\omega,\mu}(w)| \geq 3\varepsilon.$$

Proof. Note that the second derivative of $\widehat{f}^N_{\omega,\lambda}$ is bounded in a neighborhood of $\mathbb{S}^1$. It follows that for $\epsilon > 0$ and $\delta > 0$ sufficiently small, the maps $\widehat{f}^N_{\omega,\lambda}$ are uniformly expanding in a given neighborhood of $J_{\omega,\lambda}$. The existence of the point $F_{z,\omega,\lambda}(\mu)$

follows immediately. The fact that $F_{z,\omega,\lambda}$ can be given as the limit of a sequence of contracting inverse branches implies the holomorphic dependency on μ. □

Proposition 6.10. *Let $\lambda' \in \mathrm{Arc}[\lambda_0, \lambda_2)$. Then the family of maps*

$$\mathcal{A} := \{\lambda \mapsto g_\lambda(1) \,|\, g_\lambda \in \widehat{H}_\lambda\}$$

is not normal near λ'.

Proof. Let $\varepsilon, \delta > 0$ as in the previous lemma. By Lemma 5.7 the map $\lambda \mapsto K_\lambda$ is upper semi-continuous in $\mathrm{Arc}[\lambda_0, \lambda_2)$, and $\lambda \mapsto I_\lambda$ is continuous. Therefore by compactness of $\mathrm{Arc}[\lambda_0, \lambda']$, and by taking smaller ε, δ if necessary, we may assume that

$$\inf_{\substack{z \in K_{\lambda_2} \\ w \in \mathbb{S}^1 \setminus I_{\lambda_1}}} |z - w| > 2\varepsilon.$$

Now assume for the purpose of a contradiction that the family of holomorphic functions $\mathcal{A} = \{\lambda \mapsto g_\lambda(1) \,|\, g_\lambda \in \widehat{H}_\lambda\}$ is normal near λ'. By Ascoli–Arzelá Theorem it follows that there exists $\delta' < \delta$ so that for every $\mu \in B(\lambda', \delta')$ we have

$$|\widehat{f}^n_{\omega,\lambda'}(1) - \widehat{f}^n_{\omega,\mu}(1)| < \varepsilon \qquad \forall \omega \in \Omega, \forall n \in \mathbb{N}.$$

By Proposition 5.16 we can fix $\omega \in \Omega$ so that $1 \in J_{\omega,\lambda}$ and $\widehat{f}^n_\omega(1) \in \mathrm{Arc}[\lambda, 1]$ for every positive integer n. It follows that $|\widehat{f}^{kN}_{\omega,\lambda'}(1) - \widehat{f}^{kN}_{\omega,\mu}(1)| < \varepsilon$, on the ball $B(\lambda', \delta')$. By the identity principle it follows that $F_{1,\omega,\lambda'}(\mu) = 1$ on the bigger ball $B(\lambda', \delta)$, where $F_{1,\omega,\lambda'}$ is the map defined in the previous lemma.

Now notice that given $\lambda'' \in B(\lambda', \delta) \cap \mathrm{Arc}[\lambda_0, \lambda']$ we have $1 \in J_{\omega,\lambda''}$. If this were not the case then by (10) we could find a positive integer n so that $\widehat{f}_{\omega,\lambda''}(1) \in I_{\lambda''} \subset F_{\widehat{\mathcal{G}}}$, and by Proposition 5.8 we conclude that $\widehat{f}^n_{\omega,\lambda''}(1) \to K''_\lambda$. In particular when k is sufficiently large the point $\widehat{f}^{kN}_{\omega,\lambda''}(1)$ lies at distance strictly less than ε from the set $K_{\lambda''}$. Since instead the point $\widehat{f}^{kN}_{\omega,\lambda'}(1)$ lies in $\mathbb{S}^1 \setminus I_{\lambda'}$ and the two sets $\mathbb{S}^1 \setminus I_{\lambda'}$ and $K_{\lambda''}$ have distance greater than 2ε, we conclude that for k sufficiently large

$$|\widehat{f}^{kN}_{\omega,\lambda'}(1) - \widehat{f}^{kN}_{\omega,\lambda''}(1)| \geq \varepsilon,$$

contradicting the fact that $F_{1,\omega,\lambda'}(\lambda'') = 1$.

On the intersection $U = B(\lambda', \delta) \cap B(\lambda'', \delta)$ the maps $F_1 = F_{1,\omega,\lambda'}$ and $F_2 = F_{1,\omega,\lambda''}$ are well defined. For every $\mu \in U$ and every positive integer k we have $F_1(\mu) = 1$ and

$$\begin{aligned} |\widehat{f}^{kN}_{\omega,\lambda'}(1) - \widehat{f}^{kN}_{\omega,\mu}(1)| &< \varepsilon \\ |\widehat{f}^{kN}_{\omega,\lambda'}(1) - \widehat{f}^{kN}_{\omega,\lambda''}(1)| &< \varepsilon \\ |\widehat{f}^{kN}_{\omega,\lambda''}(1) - \widehat{f}^{kN}_{\omega,\mu}(F_2(\mu))| &< \varepsilon, \end{aligned}$$

which implies that

$$|\widehat{f}^{kN}_{\omega,\mu}(1) - \widehat{f}^{kN}_{\omega,\mu}(F_2(\mu))| < 3\varepsilon, \qquad \forall k \in \mathbb{N},$$

and thus that $F_2(\mu) = 1$ on the open set U. By the identity principle it follows that $F_{1,\omega,\lambda''}(\mu) = 1$ on $B(\lambda'', \delta)$. By iterating this procedure we conclude that for

every $\lambda \in \mathrm{Arc}[\lambda_0, \lambda']$ we have $1 \in J_{\omega,\lambda}$ and $F_{1,\omega,\lambda}(\mu) = 1$ on the all ball $B(\lambda, \delta)$. It follows that $1 \in J_{\omega,\lambda_0}$.

Recall that by Theorem 3.7 we have $I_{\lambda_0} = \mathrm{Arc}(1, \lambda_0)$, therefore by Lemma 5.5 we have $1 \in J_{\omega,\lambda_0}$ if and only if $\omega = (d, d, d, \dots)$. Hence for every $\lambda \in \mathrm{Arc}[\lambda_0, \lambda']$ we must have

$$f_{d,\lambda}^n(1) \in \mathbb{S}^1 \setminus I_\lambda,$$

which implies that $1 \in J_{d,\lambda}$ for all $\lambda \in \mathrm{Arc}[\lambda_0, \lambda']$, which contradicts Lemma 3.9. This concludes the proof of the proposition. □

Theorem 6.2 now follows from Montel's Theorem as in Lemma 2.7.

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