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Conformal entropy for generalized gravity theories as a consequence of horizon properties

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We show that a microscopic entropy formula based on Virasoro algebra follows from properties of stationary Killing horizons for Lagrangians with arbitrary dependence on Riemann tensor. The properties used are a consequence of regularity of invariants of Riemann tensor on the horizon. Eventual generalization of these results to Lagrangians with derivatives of Riemann tensor, as suggested by an example treated in the paper, relies on assuming regularity of invariants involving derivatives of Riemann tensor. This assumption however leads also to new interesting restrictions on metric functions near the horizon.

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I. INTRODUCTION

One of the outstanding problems in gravity is to understand the nature of black hole entropy and, in particular, its microscopic interpretation. Besides being an important problem by itself, one can also hope that its solution would help to build the theory of quantum gravity. The problem of microscopic description of black hole entropy was approached by different methods like string theory which treated extremal black holes [1] or loop quantum gravity [2–4].

An interesting line of approach is based on conformal field theory and Virasoro algebra. One particular formulation for Einstein gravity was due to Solodukhin, who reduced the problem of D-dimensional black holes to effective two-dimensional theory with fixed boundary conditions on horizon. The effective theory was found to admit Virasoro algebra near horizon. Calculation of its central charge allows then to compute the entropy [5]. The result was later generalized for D-dimensional Gauss-Bonnet gravity [6]. An independent formulation for two-dimensional dilaton gravity and D-dimensional Einstein gravity is due to Carlip [7,8], who has shown that under certain simple assumptions on boundary conditions near the black hole horizon one can identify Virasoro algebra as a subalgebra of algebra of diffeomorphisms. The fixed boundary conditions give rise to central extensions of this algebra. The entropy is then calculated from the Cardy formula [9]

$$S_c = 2\pi\sqrt{\left(\frac{c}{6} - 4\Delta_g\right)\left(\Delta - \frac{c}{24}\right)}. \quad (1)$$

Here Δ is the eigenvalue of Virasoro generator L_0 for the state for which we calculate the entropy and Δ_g is the smallest eigenvalue.

In that way the well known Bekenstein-Hawking formula for Einstein gravity was reproduced. Explicitly,

$$S = \frac{A}{4}. \quad (2)$$

Here S denotes black hole entropy and A area of its horizon. Later these results were generalized to include D-dimensional Gauss-Bonnet gravity [10] and higher curvature Lagrangians [11]. For such a case, one can reproduce the generalized entropy formula [12]

$$S = \frac{\hat{A}}{4} = -2\pi \int_{\mathcal{H}} \hat{\epsilon}_{a_1\ldots a_{n-2}} E^{abcd} \eta_{ab}\eta_{cd}. \quad (3)$$

Here $\mathcal{H}$ is a cross section of the horizon, η_{ab} denotes binormal to $\mathcal{H}$, and $\hat{\epsilon}_{a_1\ldots a_{n-2}}$ is the induced volume element on $\mathcal{H}$. The tensor E^{abcd} is given by

$$E^{abcd} = \frac{\partial L}{\partial R_{abcd}}. \quad (4)$$

These derivations, however, included some additional plausible assumptions on boundary conditions near horizon. This includes, in particular, assumptions on behavior of the so-called spatial derivatives assumed in Appendix A of Ref. [8].

These assumptions have been even more crucial in the generalizations [10,11] of original procedure. Important progress in understanding these assumptions can be made due to the following observations [13] for stationary Killing horizons:

(i) The regularity of curvature invariants on horizon has strong implications on the behavior of metric functions near horizon.

(ii) The transverse components of the stress energy tensor have properties which suggest near-horizon conformal symmetry.

In fact using these results it was shown for four-dimensional Einstein gravity [14] that the microscopic black hole entropy formula based on the Virasoro algebra approach can be derived from properties of stationary Killing horizons. The above-mentioned additional assumptions are shown to be fulfilled.

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In this paper we would like to show that this is true not only for Einstein gravity but also for a generic class of Lagrangians which depend arbitrarily on Riemann tensor but do not depend on its covariant derivatives. Eventual exceptions which do not fall in this generic class will be defined more precisely in the text.

While in a previous case the results were obtained by explicit calculations, this is not possible for the generic case and thus we shall use a new method based on power counting.

We are interested in generalizing the result from Einstein gravity to more general cases because if that were possible it would indicate that near-horizon conformal symmetries and corresponding Virasoro algebras are characteristic of any diffeomorphism invariant Lagrangian and are independent of properties of particular Lagrangians and specific solutions. The additional interest in generalized Lagrangians is due to recent attempts to explain acceleration of the universe by considering modifications of the Einstein-Hilbert action that become important only in regions of extremely low spacetime curvature [15]. For a more complete list of references see [16]. In particular, terms proposed to add to Einstein-Hilbert action have been of the type R^{-n}, $n > 0$, and also inverse powers of $P = R_{ab}R^{ab}$ and $Q = R_{abcd}R^{abcd}$. In this moment, a proof valid also for Lagrangians involving derivatives of Riemann tensor is still missing. However, we present an indication of possible similar results in Appendix B considering one specific example. One finds that additional restrictions on behavior of metric functions near horizon are needed. However, these restrictions are very natural because it is found that they are the consequence of regularity of invariants, this time involving derivatives of Riemann tensor. Thus, requesting regularity of invariants with derivatives of Riemann tensor gives new restrictions for metric functions near horizon in addition to those obtained by Medved, Martin, and Visser [13].

II. NEAR-HORIZON BEHAVIOR AND DERIVATION OF THE ENTROPY

As mentioned in the introduction, we want explore if one can define Virasoro algebra at horizon its central charge and the corresponding value for entropy for higher curvature Lagrangians of the type

$$L = L(g_{ab}, R_{abcd}). \tag{5}$$

As explained in previous references [10], the central charge is given by

$$\{J[\xi_1], J[\xi_2]\}^* = J[\{\xi_1, \xi_2\}] + K[\xi_1, \xi_2], \tag{6}$$

where ξ_1, ξ_2 are diffeomorphisms generated by vector fields

$$\xi^a = T\chi^a + R\rho^a, \tag{7}$$

where χ is Killing vector which is null on the horizon

$$\chi^2 = 0, \tag{8}$$

and ρ is defined by

$$\nabla_a \chi^2 = -2\kappa\rho_a. \tag{9}$$

Diffeomorphism functions T and R are restricted by conditions

$$R = -\frac{1}{\kappa}\frac{\chi^2}{\rho^2}\nabla_\chi T, \tag{10}$$

$$\rho^a \nabla_a T = 0. \tag{11}$$

In such a way, surface $\chi^2 = 0$ will remain fixed under these diffeomorphisms and also

$$\frac{\delta\chi^2}{\chi^2} = 0 \tag{12}$$

will be valid. One more condition on diffeomorphisms is required:

$$\delta \int_{\partial C} \hat{\epsilon}\left(\hat{\kappa} - \frac{\rho}{|\chi|}\kappa\right) = 0. \tag{13}$$

Here, $\hat{\kappa}^2 = -\frac{a^2}{\chi^2}$ and $a^a = \chi^b \nabla_b \chi^a$ is the acceleration of an orbit of χ^a. For a more complete definition of diffeomorphisms see [8]. This last condition leads to orthogonality relations for the one parameter group of diffeomorphisms.

Now, the Dirac bracket of boundary terms $J[\xi]$ in Hamiltonian $\{J[\xi_1], J[\xi_2]\}^*$ is given by [see Eq. (27) of [10]]

$$\begin{aligned}\{J[\xi_1], J[\xi_2]\}^* = \int_{\mathcal{H}} \epsilon_{apa_1\cdots a_{n-2}}\{&2(\xi_2^p E^{abcd}\nabla_d \delta_1 g_{bc} \\ &- \xi_1^p \nabla_d E^{abcd}\delta_2 g_{bc} - (1 \leftrightarrow 2)) \\ &- \xi_2 \cdot (\xi_1 \cdot \mathbf{L})\}.\end{aligned} \tag{14}$$

The information about Lagrangian is given with quantities E^{abcd}. We introduce the following abbreviations:

$$X^{(12)}_{abcd} = \xi_1^p \eta_{ap} \nabla_d \delta_2 g_{bc} - (1 \leftrightarrow 2), \tag{15}$$

$$\tilde{X}^{(12)}_{abc} = \xi_1^p \eta_{ap} \delta_2 g_{bc} - (1 \leftrightarrow 2). \tag{16}$$

In such a way (14) becomes

$$\begin{aligned}\{J[\xi_1], J[\xi_2]\}^* = -\int_{\mathcal{H}} \hat{\boldsymbol{\epsilon}}\{&2(X^{(12)}_{abcd}E^{abcd} - \tilde{X}^{(12)}_{abc}\nabla_d E^{abcd}) \\ &- \xi_2^a \xi_1^b \eta_{ab} L\}.\end{aligned} \tag{17}$$

We are interested in evaluating this expression on horizon. The third term is immediately seen to be zero due to (7)–(9) and regularity of the Lagrangian on horizon. It will be shown that the first two terms are given as follows:

$$\lim_{n\to 0}(X^{(12)}_{abcd}E^{abcd}) = \lim_{n\to 0}\bigg(-\frac{1}{4}\eta_{ab}\eta_{cd}E^{abcd} \times \bigg[\bigg(\frac{1}{\kappa}T_1\ddot{T}_2 - 2\kappa T_1\dot{T}_2\bigg) - (1\leftrightarrow 2)\bigg]\bigg), \tag{18}$$

$$\lim_{n\to 0}(\tilde{X}^{(12)}_{abc}\nabla_d E^{abcd}) = 0. \tag{19}$$

This was shown in [14] for the Einstein Lagrangian and for Lagrangians including quadratic terms in curvature. Here we want to extend it to Lagrangians of general form given with (5).

The main properties which we shall use in this paper will be the properties of stationary horizon. In particular, we shall use the results of [13] where it was shown that absence of curvature singularities implies explicit restrictions on Taylor series of metric functions near horizon. For a basis, we use two Killing vectors of axially symmetric black holes,

$$t^a = \bigg(\frac{\partial}{\partial t}\bigg)^a, \qquad \phi^a = \bigg(\frac{\partial}{\partial \phi}\bigg)^a, \tag{20}$$

with corresponding coordinates t, ϕ. In addition, in the equal time hypersurface we choose Gauss normal coordinate n ($n = 0$ on the horizon) and the remaining coordinate z such that the metric has the form

$$ds^2 = -N(n,z)^2dt^2 + g_{\phi\phi}(n,z)(d\phi - \omega(n,z)dt)^2 + dn^2 + g_{zz}(n,z)dz^2. \tag{21}$$

The mentioned properties imply that near-horizon metric coefficients have the following Taylor expansions:

$$\begin{aligned} N(n,z) &= \kappa n + \frac{1}{3!}\kappa_2(z)n^3 + O(n^4) \\ g_{\phi\phi}(n,z) &= g_{H\phi\phi}(z) + \frac{1}{2}g_{2\phi\phi}(z)n^2 + O(n^3) \\ g_{zz}(n,z) &= g_{Hzz}(z) + \frac{1}{2}g_{2zz}(z)n^2 + O(n^3) \\ \omega(n,z) &= \Omega_H + \frac{1}{2}\omega_2(z)n^2 + O(n^3). \end{aligned} \tag{22}$$

In the following, we will use the basis ${e_\mu}^a$ where ${e_1}^a \equiv \chi^a$, ${e_2}^a \equiv \rho^a$, ${e_3}^a \equiv \phi^a$, and ${e_4}^a \equiv z^a$.

The leading terms of nonvanishing products of basis vectors are

$$\begin{aligned} \chi\cdot\chi &= -\kappa^2 n^2 + O(n^4) \\ \chi\cdot\phi &= -\frac{1}{2}g_{H\phi\phi}(z)\omega_2(z)n^2 + O(n^3) \\ \phi\cdot\phi &= g_{H\phi\phi}(z) + O(n^2) \\ \rho\cdot\rho &= \kappa^2 n^2 + O(n^4) \\ \rho\cdot z &= O(n^4) \\ z\cdot z &= g_{Hzz}(z) + O(n^2), \end{aligned} \tag{23}$$

and all other products are zero:

$$\chi\cdot\rho = \chi\cdot z = \phi\cdot\rho = \phi\cdot z = 0. \tag{24}$$

It will be convenient to use ${e_\perp}^a$ for χ^a or ρ^a when equations hold for both χ^a and ρ^a, and similarly ${e_\parallel}^a$ for z^a and ϕ^a.

In the evaluation of (18) and (19), it is important to realize that tensors $X^{(12)}_{abcd}$ and $\tilde{X}^{(12)}_{abc}$ depend only on details of the black hole and its symmetry properties but their form does not depend on the form of the Lagrangian. Also the derivation of (18) depends only on symmetry properties of tensor E and not on its particular form. For that reason Eq. (18) can be calculated as in [14].

However, the proof of statement (19) for Einstein gravity and Lagrangians quadratic in Riemann tensor was based on explicit calculations. These are of course not possible for the generic class of Lagrangians of the type (5). Thus we need a new approach.

The derivation in this case will be based on properties (22) of metric functions near horizon and power counting for quantities we need to establish the relation (19).

The main aim is to derive the leading term of Taylor expansion of the scalar,

$$(\tilde{X}^{(12)}_{abc}\nabla_d E^{abcd}). \tag{25}$$

For that purpose we need to describe how to count the powers of various quantities. In particular, the leading power of Taylor expansion of some scalar will be called the order of that scalar. Having that in mind and also having in mind relations (23), we can give definitions for the order of basis vectors as

$$\mathrm{order}(e_\perp) = 1, \qquad \mathrm{order}(e_\parallel) = 0. \tag{26}$$

For arbitrary tensor T, we first expand it in the basis ${e_\mu}^a$:

$$T^{a_1\ldots a_m}{}_{b_1\ldots b_n} = \sum_{\mu_1\ldots\mu_m,\nu_1\ldots\nu_n} T^{\mu_1\ldots\mu_m,\nu_1\ldots\nu_n}{e_{\mu_1}}^{a_1}\ldots{e_{\mu_m}}^{a_m} \times e_{\nu_1 a_1}\ldots e_{\nu_n b_n}. \tag{27}$$

Then we calculate the order for each term in the sum as a sum of orders of its factors. The order of T is defined as the order of its leading term (i.e. of the term with the lowest order):

$$\text{order}\,(T^{a_1\ldots a_m}{}_{b_1\ldots b_n}) = \min_{\mu_1\ldots\mu_m,\nu_1\ldots\nu_n}\Bigg[\text{order}(T^{\mu_1\ldots\mu_m,\nu_1\ldots\nu_n}) + \sum_{i=0}^{m}\text{order}(e_{\mu_i}{}^{a_i}) + \sum_{i=0}^{n}\text{order}(e_{\nu_i b_i})\Bigg]. \tag{28}$$

Note that, by definition, $e_{\mu a}$ and $e_\mu{}^a$ are of the same order. The definition (28) implies that for basis vectors $e_\mu{}^a$ and $e_\nu{}^b$:

$$\text{order}\,(e_{\mu a}e_\nu{}^b) = \text{order}(e_{\mu a}) + \text{order}(e_\nu{}^b), \tag{29}$$

and when we contract indices we get [from (23) and (24)]

$$\text{order}\,(e_{\mu a}e_\nu{}^a) \geq \text{order}(e_{\mu a}) + \text{order}(e_\nu{}^a). \tag{30}$$

[For example from (23) we have order$(\chi\cdot\phi) = 2$, while order$(\chi) = 1$ and order$(\phi) = 0$.] For products of tensors we have an analogous situation. For tensors T_1 and T_2 we have, of course, order$(T_1 \otimes T_2) =$ order(T_1) + order(T_2) [i.e. when there are no contractions, the leading term is the tensor product of two leading terms], and also in the case of arbitrary contractions we have from (28) and (30) that

$$\text{order}\,(T_{1\ldots}^{\ldots}\ldots T_{n\ldots}^{\ldots}) \geq \sum_{i=1}^{n}\text{order}(T_{i\ldots}^{\ldots}). \tag{31}$$

The right-hand side of (31) gives the lower bound for the order of the arbitrary product of tensors $T_{1\ldots}^{\ldots}\ldots T_{n\ldots}^{\ldots}$ (with possible arbitrary contractions of indices) which is suitable for our purpose of showing (58).

The fact that $e_{\mu a}$ and $e_\mu{}^a$ are of the same order is consistent with (31) and the fact that g_{ab} and g^{ab} are of order 0.

The important role will have

$$\begin{aligned}
\nabla_a\chi_b &= \frac{1}{\kappa n^2}(-\chi_a\rho_b + \rho_a\chi_b) + \text{order} \geq 1 \text{ terms}\\
\nabla_a\rho_b &= \frac{1}{\kappa n^2}(\rho_a\rho_b - \chi_a\chi_b) + \text{order} \geq 1 \text{ terms}\\
\nabla_a\phi_b &= \frac{A(z)}{n^2}(-\chi_a\rho_b + \rho_a\chi_b) + B(z)(-\phi_a z_b + z_a\phi_b)\\
&\quad + \text{order} \geq 1 \text{ terms}\\
\nabla_a z_b &= C(z)(\phi_a\phi_b + z_a z_b) + \text{order} \geq 1 \text{ terms},
\end{aligned}$$

and we can summarize them as

$$\begin{aligned}
\nabla_a e_{\perp b} &\sim \frac{1}{n^2}e_{\perp a}e_{\perp b} + \text{order} \geq 1 \text{ terms}\\
\nabla_a e_{\parallel b} &\sim \frac{1}{n^2}e_{\perp a}e_{\perp b} + e_{\parallel a}e_{\parallel b} + \text{order} \geq 1 \text{ terms}.
\end{aligned} \tag{32}$$

Also,

$$\begin{aligned}
\nabla_a t &= -\frac{1}{\kappa^2 n^2}\chi_a + \text{order} \geq 0 \text{ terms}\\
\nabla_a n &= \frac{1}{\kappa n}\rho_a + \text{order} \geq 1 \text{ terms}\\
\nabla_a \phi &= -\frac{\Omega_H}{\kappa^2 n^2}\chi_a + \text{order} \geq 0 \text{ terms}\\
\nabla_a z &= \frac{1}{g_{Hzz}(z)}z_a + \text{order} \geq 1 \text{ terms}.
\end{aligned} \tag{33}$$

The derivative lowers the order (28) at most by one. That can be seen from (32) and (33)

$$\text{order}\,(\nabla T) \geq \text{order}(T) - 1. \tag{34}$$

For a function $f(z)$ it follows from (33) that

$$\nabla_a f(z) = \frac{\partial f}{\partial z}\frac{1}{g_{Hzz}(z)}z_a + \text{nonleading terms}, \tag{35}$$

so that in this case

$$\text{order}\,(\nabla_a f(z)) \geq \text{order}(f(z)). \tag{36}$$

From (23) and (24) we see that $e_\mu\cdot\chi = O(n^2)$ for $\mu = 1, 2$ and $e_\mu\cdot\chi = 0$ for $\mu = 3, 4$, so we can write

$$\text{order}\,(e_\mu\cdot\chi) \geq 2, \tag{37}$$

where, since we are interested in Taylor expansion around $n = 0$, we can formally treat 0 as $O(n^\infty)$, and that is why there is a $\geq$ sign. There is an analogous relation for ρ:

$$\text{order}\,(e_\mu\cdot\rho) \geq 2, \tag{38}$$

so we can write

$$\text{order}\,(e_\mu\cdot e_\perp) \geq 2. \tag{39}$$

We also note that following relations hold:

$$\text{order}\,(\nabla_a e_{\parallel b}) \geq 0, \tag{40}$$

$$\text{order}\left(\nabla_a\left(\frac{1}{n^2}\chi_{[b}\rho_{c]}\right)\right) > -1. \tag{41}$$

These relations will enable us later to raise the lower bound calculated by the right-hand side of (31).

Since the Lagrangian is of the form (5), it can be expressed as a function of scalar invariants I_n:

$$L = L'(I_1, I_2, \ldots) \tag{42}$$

(e.g., we can take $I_1 = R$, $I_2 = R_{abcd}R^{abcd}$, $I_3 = R_{ab}R^{ab}$, $I_4 = R^2$, ...). Since L does not contain derivatives of Riemann tensors E^{abcd} is given by (4)

$$E^{abcd} = \frac{\partial L}{\partial R_{abcd}} = \frac{\partial L'}{\partial I_n}\frac{\partial I_n}{\partial R_{abcd}} \equiv \frac{\partial L'}{\partial I_n}E^{abcd}_{I_n}. \tag{43}$$

Since order$(g_{ab}) = 0$ and order$(R_{abcd}) = 0$ [by explicit calculation, see (A4) and (A6)], from (31) it follows that for $E_{I_n}^{abcd}$ defined in Eq. (43) we have

$$\text{order}\,(E_{I_n}^{abcd}) \geq 0 \tag{44}$$

because I_n consists only of tensors R_{abcd} and g_{ab}. If in addition we require that for each scalar invariant I_n

$$\frac{\partial L'}{\partial I_n} = \text{finite on the horizon,} \tag{45}$$

then (45) implies that

$$\text{order}\,(E^{abcd}) \geq 0. \tag{46}$$

Now we write E^{abcd} using components $E^{\mu\nu\rho\sigma}$ in basis $e_\mu{}^a$

$$E^{abcd} = \sum_{\mu\nu\rho\sigma} E^{\mu\nu\rho\sigma} e_\mu{}^a e_\nu{}^b e_\rho{}^c e_\sigma{}^d. \tag{47}$$

In the same way, we also expand derivative $\nabla^e E^{abcd}$:

$$\nabla^e E^{abcd} = \sum_{\mu\nu\rho\sigma\lambda} K^{\mu\nu\rho\sigma\lambda} e_\mu{}^a e_\nu{}^b e_\rho{}^c e_\sigma{}^d e_\lambda{}^e, \tag{48}$$

and contraction

$$\nabla_d E^{abcd} = \sum_{\mu\nu\rho} C^{\mu\nu\rho} e_\mu{}^a e_\nu{}^b e_\rho{}^c. \tag{49}$$

Note that components $E^{\mu\nu\rho\sigma}$, $K^{\mu\nu\rho\sigma\lambda}$, and $C^{\mu\nu\rho}$ have symmetries which follow from symmetries of Riemann tensor, and also, in this basis, these components are functions of n and z only.

From (34) and (46) we have for each μ, ν, ρ, σ, and λ,

$$\text{order}\,(E^{\mu\nu\rho\sigma} e_\mu{}^a e_\nu{}^b e_\rho{}^c e_\sigma{}^d) \geq 0, \tag{50}$$

$$\text{order}\,(K^{\mu\nu\rho\sigma\lambda} e_\mu{}^a e_\nu{}^b e_\rho{}^c e_\sigma{}^d e_\lambda{}^e) \geq -1, \tag{51}$$

$$\text{order}\,(C^{\mu\nu\rho} e_\mu{}^a e_\nu{}^b e_\rho{}^c) \geq -1. \tag{52}$$

That implies

$$\begin{gathered} \text{order}(E^{\perp\perp\perp\perp}) \geq -4 \\ \text{order}(E^{\perp\perp\perp\parallel}) \geq -3 \\ \text{order}(E^{\perp\perp\parallel\parallel}) \geq -2 \\ \text{etc.} \end{gathered} \tag{53}$$

$$\begin{gathered} \text{order}(K^{\perp\perp\perp\perp\perp\perp}) \geq -6 \\ \text{order}(K^{\perp\perp\perp\perp\perp\parallel}) \geq -5 \\ \text{order}(K^{\perp\perp\perp\parallel\perp}) \geq -5 \\ \text{order}(K^{\perp\perp\perp\parallel\parallel}) \geq -4 \\ \text{etc.} \end{gathered} \tag{54}$$

$$\begin{aligned} &\text{order}\,(C^{\perp\perp\perp}) \geq -4 && \text{order}(C^{\perp\perp\parallel}) \geq -3 \\ &\text{order}(C^{\perp\parallel\perp}) \geq -3 && \text{order}(C^{\perp\parallel\parallel}) \geq -2 \\ &\text{order}(C^{\parallel\parallel\perp}) \geq -2 && \text{order}(C^{\parallel\parallel\parallel}) \geq -1. \end{aligned} \tag{55}$$

The coefficients $K^{\mu\nu\rho\sigma\lambda}$ and $E^{\mu\nu\rho\sigma}$ are related with

$$\sum_{\mu\nu\rho\sigma\lambda} K^{\mu\nu\rho\sigma\lambda} e_\mu{}^a e_\nu{}^b e_\rho{}^c e_\sigma{}^d e_\lambda{}^e = \nabla^e \sum_{\mu\nu\rho\sigma} (E^{\mu\nu\rho\sigma} e_\mu{}^a e_\nu{}^b e_\rho{}^c e_\sigma{}^d). \tag{56}$$

The coefficients $C^{\mu\nu\rho}$ and $K^{\mu\nu\rho\sigma\lambda}$ are related with

$$C^{\mu\nu\rho} = \sum_{\sigma\lambda} K^{\mu\nu\rho\sigma\lambda} e_\sigma \cdot e_\lambda. \tag{57}$$

Now we prove that (19) holds for Lagrangians of type (5). That will be the case if

$$\text{order}\,(\nabla_d E^{abcd} \tilde{X}^{(12)}_{abc}) > 0. \tag{58}$$

Explicit calculation (A5) of $\tilde{X}^{(12)}_{abc}$ (whose form does not depend on the Lagrangian) tells us that the leading terms in it are of order 1:

$$\text{order}\,(\tilde{X}^{(12)}_{abc}) = 1, \tag{59}$$

and these are

$$\chi_a\rho_b\chi_c O\left(\frac{1}{n^2}\right) + \chi_a\rho_b\rho_c O\left(\frac{1}{n^2}\right) + +\rho_a\chi_b\chi_c O\left(\frac{1}{n^2}\right) + \rho_a\chi_b\rho_c O\left(\frac{1}{n^2}\right), \tag{60}$$

and also that

$$\text{order}\,(\text{other terms in } \tilde{X}^{(12)}_{abc}) \geq 2. \tag{61}$$

On the other hand, we see from (34) and (46) that

$$\text{order}\,(\nabla_d E^{abcd}) \geq -1. \tag{62}$$

If we show that leading terms (of order -1) in $\nabla_d E^{abcd}$ cancel when contracted with leading terms (60) (of order 1) in $\tilde{X}^{(12)}_{abc}$, then (58) will follow.

We look at terms (60) contracted with (49), and using (31) we count the order to be at least 0:

$$\text{order}\left[\sum_{\mu\nu\lambda} C^{\mu\nu\lambda}\left(e_\mu \cdot \chi e_\nu \cdot \rho e_\lambda \cdot \chi O\left(\frac{1}{n^2}\right) + e_\mu \cdot \chi e_\nu \cdot \rho e_\lambda \cdot \rho O\left(\frac{1}{n^2}\right)\right)\right] \geq 0, \tag{63}$$

where we used $C^{\mu\nu\rho} = C^{[\mu\nu]\rho}$. To prove (58) we need to prove

$$\text{order}\Big[C^{\mu\nu\lambda}\Big(e_\mu \cdot \chi e_\nu \cdot \rho e_\lambda \cdot \chi O\Big(\frac{1}{n^2}\Big) + e_\mu \cdot \chi e_\nu \cdot \rho e_\lambda \cdot \rho O\Big(\frac{1}{n^2}\Big)\Big)\Big] > 0, \quad (64)$$

for each μ, ν, and λ. Using (39) and (55), we see that (64) will follow if

$$\text{order}\,(C^{\perp\perp\perp}) > -4. \quad (65)$$

On the other hand, from (31) and (57) we see that

$$\text{order}\,(C^{\perp\perp\perp}) \geq \text{order}\Big(\sum_{\lambda\sigma} K^{\perp\perp\perp\lambda\sigma} e_\lambda \cdot e_\sigma\Big). \quad (66)$$

Writing in terms of the lower bound of the right-hand side, we get

$$\text{order}\,(C^{\perp\perp\perp}) \geq \min_{\lambda\sigma}\text{order}(K^{\perp\perp\perp\lambda\sigma} e_\lambda \cdot e_\sigma). \quad (67)$$

Expanding λ and σ, we get

$$\begin{aligned}\text{order}(C^{\perp\perp\perp}) \geq \min\{&\text{order}(K^{\perp\perp\perp\perp\perp} e_\perp \cdot e_\perp),\\ &\text{order}(K^{\perp\perp\perp\perp\|} e_\perp \cdot e_\|),\\ &\text{order}(K^{\perp\perp\perp\|\perp} e_\| \cdot e_\perp),\\ &\text{order}(K^{\perp\perp\perp\|\|} e_\| \cdot e_\|)\}. \end{aligned} \quad (68)$$

Explicitly,

$$\begin{aligned}\text{order}\,(C^{\perp\perp\perp}) &\geq \min\{(-6) + 2, (-5) + 2, (-5) + 2, (-4) + 0\}\\ &\geq \min\{-4, -3, -3, -4\}.\end{aligned} \quad (69)$$

So if we prove that

$$\text{order}\,(K^{\perp\perp\perp\perp\perp}) > -6 \quad (70)$$

and

$$\text{order}\,(K^{\perp\perp\perp\|\|}) > -4, \quad (71)$$

then (65) will hold.

From (32) and (56) we see that $K^{\perp\perp\perp\perp\perp}$ can only get contribution from 4 terms in the sum on the right-hand side of (56) which are of the form

$$E^{\perp\perp\perp\perp} e_\perp{}^a e_\perp{}^b e_\perp{}^c e_\perp{}^d \quad (72)$$

and $4 \cdot 8 = 32$ terms of the form

$$\begin{matrix} E^{\|\perp\perp\perp} e_\|{}^a e_\perp{}^b e_\perp{}^c e_\perp{}^d & E^{\perp\|\perp\perp} e_\perp{}^a e_\|{}^b e_\perp{}^c e_\perp{}^d \\ E^{\perp\perp\|\perp} e_\perp{}^a e_\perp{}^b e_\|{}^c e_\perp{}^d & E^{\perp\perp\perp\|} e_\perp{}^a e_\perp{}^b e_\perp{}^c e_\|{}^d \end{matrix} \quad (73)$$

when the derivative acts as

$$(\nabla_e E^{\perp\perp\perp\perp}) e_\perp{}^a e_\perp{}^b e_\perp{}^c e_\perp{}^d \quad (74)$$

or as

$$\begin{matrix} E^{\perp\perp\perp\perp} (\nabla_e e_\perp{}^a) e_\perp{}^b e_\perp{}^c e_\perp{}^d \\ E^{\perp\perp\perp\perp} e_\perp{}^a (\nabla_e e_\perp{}^b) e_\perp{}^c e_\perp{}^d \\ E^{\perp\perp\perp\perp} e_\perp{}^a e_\perp{}^b (\nabla_e e_\perp{}^c) e_\perp{}^d \\ E^{\perp\perp\perp\perp} e_\perp{}^a e_\perp{}^b e_\perp{}^c (\nabla_e e_\perp{}^d) \end{matrix} \quad (75)$$

on (72), and as

$$\begin{matrix} E^{\|\perp\perp\perp} (\nabla_e e_\|{}^a) e_\perp{}^b e_\perp{}^c e_\perp{}^d & E^{\perp\|\perp\perp} e_\perp{}^a (\nabla_e e_\|{}^b) e_\perp{}^c e_\perp{}^d \\ E^{\perp\perp\|\perp} e_\perp{}^a e_\perp{}^b (\nabla_e e_\|{}^c) e_\perp{}^d & E^{\perp\perp\perp\|} e_\perp{}^a e_\perp{}^b e_\perp{}^c (\nabla_e e_\|{}^d) \end{matrix} \quad (76)$$

on (73). We also see that $K^{\perp\perp\perp\|\|}$ can only get a contribution from 8 terms of the form

$$E^{\perp\perp\perp\|} e_\perp{}^a e_\perp{}^b e_\perp{}^c e_\|{}^d \quad (77)$$

when the derivative acts as

$$E^{\perp\perp\perp\|} e_\perp{}^a e_\perp{}^b e_\perp{}^c (\nabla_e e_\|{}^d) \quad (78)$$

which is included in (76).

The sum of (74) and (75) is

$$\begin{aligned}&\nabla_e (E^{\perp\perp\perp\perp} e_\perp{}^a e_\perp{}^b e_\perp{}^c e_\perp{}^d)\\ &\quad = 4\nabla_e (E^{1212}(n, z) e_{[1}{}^a e_{2]}{}^b e_{[1}{}^c e_{2]}{}^d)\\ &\quad = 4\nabla_e \Big(\frac{f(z)}{n^4} e_{[1}{}^a e_{2]}{}^b e_{[1}{}^c e_{2]}{}^d + \text{order} > 0 \text{ terms}\Big), \end{aligned} \quad (79)$$

where $f(z) \equiv \lim_{n\to 0}[n^4 E^{1212}(n, z)]$, and has order ≥ 0 because of (53). The leading term in (79) is

$$\begin{aligned}\nabla_e \Big(\frac{f(z)}{n^4} e_{[1}{}^a e_{2]}{}^b e_{[1}{}^c e_{2]}{}^d\Big) &= (\nabla_e f(z)) \frac{1}{n^4} e_{[1}{}^a e_{2]}{}^b e_{[1}{}^c e_{2]}{}^d\\ &= +f(z)\Big(\nabla_e \frac{1}{n^2} e_{[1}{}^a e_{2]}{}^b\Big)\\ &\quad \times \Big(\frac{1}{n^2} e_{[1}{}^c e_{2]}{}^d\Big)\\ &= +f(z)\Big(\frac{1}{n^2} e_{[1}{}^a e_{2]}{}^b\Big)\\ &\quad \times \Big(\nabla_e \frac{1}{n^2} e_{[1}{}^c e_{2]}{}^d\Big). \end{aligned} \quad (80)$$

$$= (\nabla_e f(z)) \frac{1}{n^4} e^a_{[1} e^b_{2]} e^c_{[1} e^d_{2]}$$
$$= + f(z)\left(\nabla_e \frac{1}{n^2} e^a_{[1} e^b_{2]}\right)\left(\frac{1}{n^2} e^c_{[1} e^d_{2]}\right)$$
$$= + f(z)\left(\frac{1}{n^2} e^a_{[1} e^b_{2]}\right)\left(\nabla_e \frac{1}{n^2} e^c_{[1} e^d_{2]}\right)$$

Using (31), (35), and (41) and inserting (80) into (79), we get

$$\text{order}\,(\nabla_e(E^{\perp\perp\perp\perp} e_\perp{}^a e_\perp{}^b e_\perp{}^c e_\perp{}^d)) > -1. \quad (81)$$

Also from (31) and (40) we get the bound for the order of Eqs. (76):

$$\text{order}\,(E^{\|\perp\perp\perp}(\nabla_e e_\|{}^a) e_\perp{}^b e_\perp{}^c e_\perp{}^d) > -1. \quad (82)$$

Using (56) we see that (70) and (71) follow, which completes the proof of (19). Now we are able to use properties (18) and (19) to calculate the central charge

$$\frac{c}{12} = \frac{\hat{A}}{8\pi}, \quad (83)$$

and entropy formula (3). The last derivation is analogous to one in Refs. [8,11].

III. CONCLUSION

The derivation of black hole entropy [8,10,11] which used ideas of conformal symmetry and Virasoro algebras has been based on additional plausible assumptions. It is important to find examples of theories where these assumptions are fulfilled. It is also important to understand if they depend on properties of interactions or instead on the properties of horizons. In this paper we show that the latter is the case. In fact using the properties of horizons of the stationary black hole which follow from regularity of curvature invariants, one can derive the mentioned result. This was done for Einstein gravity and quadratic Lagrangians in the previous reference [14] by explicit calculation. This is not possible for the generic case, and thus we have used here a new method based on power counting. In such a way, we have been able to generalize the result to Lagrangians with arbitrary dependence on the Riemann tensor.

In particular, inverse powers of invariants are allowed terms. They are restricted with condition (45). As mentioned in the introduction, such Lagrangians are also of interest due to the present effort to investigate if they could accommodate acceleration of the universe. Of course an investigation valid for Lagrangians involving derivatives of Riemann tensor is still missing. However, in Appendix B we present a special case of such a Lagrangian where these results are again true. A by-product of this investigation is the conclusion that requiring the regularity of invariants involving derivatives of Riemann tensor gives even more restrictions on metric functions.

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APPENDIX A

In this text we have used relation (46) stating that

$$\text{order}\,(E^{abcd}) \geq 0. \quad (A1)$$

That relation was in turn the consequence of properties

$$\text{order}\,(g^{ab}) = 0, \qquad \text{order}(R_{abcd}) = 0 \quad (A2)$$

which are the consequence of Taylor expansions (22) for metric functions. We have also used relation (59)

$$\text{order}\,(\tilde{X}^{(12)}_{abc}) = 1, \quad (A3)$$

based also on (22). In this appendix we give decomposition of tensors g_{ab}, $\tilde{X}^{(12)}_{abc}$, and R_{abcd} in the basis $\{\chi^a, \rho^a, \phi^a, z^a\}$ introduced in the text. These decompositions are a result of lengthy but straightforward calculations which can be done e.g. with the help of MATHEMATICA. We give these expressions:

$$g_{ab} = \chi_a\chi_b\left(-\frac{1}{\kappa^2 n^2} + O(n^{-1})\right) + \rho_a\rho_b\left(\frac{1}{\kappa^2 n^2} + O(n^{-1})\right) + \phi_a\phi_b\left(\frac{1}{g_{H\phi\phi}} + O(n)\right) + z_a z_b\left(\frac{1}{g_{Hzz}} + O(n)\right) + \text{terms of order} \geq 1, \quad (A4)$$

$$\tilde{X}^{(12)}_{abc} = \chi_a\rho_b\chi_c\left(-\frac{\dot{T}_1\ddot{T}_2 - \dot{T}_2\ddot{T}_1}{2\kappa^4 n^2} + O(n^{-1})\right) + \chi_a\rho_b\rho_c\left(\frac{\ddot{T}_1 T_2 - \ddot{T}_2 T_1}{2\kappa^3 n^2} + O(n^{-1})\right) + \rho_a\chi_b\chi_c\left(\frac{\dot{T}_1\ddot{T}_2 - \dot{T}_2\ddot{T}_1}{2\kappa^4 n^2} + O(n^{-1})\right) + \rho_a\chi_b\rho_c\left(-\frac{\ddot{T}_1 T_2 - \ddot{T}_2 T_1}{2\kappa^3 n^2} + O(n^{-1})\right) + \text{terms of order} \geq 2, \quad (A5)$$

$$
\begin{aligned}
R_{abcd} = {}& \chi_a\rho_b\chi_c\rho_d\Big(-\frac{R_\perp}{2\kappa^4 n^4}+O(n^{-3})\Big)+\chi_a\rho_b\rho_c\phi_d\Big(\frac{3\omega_3}{2\kappa^4 n^3}+O(n^{-2})\Big)+\chi_a\rho_b\phi_c z_d\Big(-\frac{g_{H\phi\phi}\omega_2'+\omega_2 g_{H\phi\phi}'}{2g_{H\phi\phi}g_{Hzz}\kappa^3 n^2} \\
&+O(n^{-1})\Big)+\chi_a\phi_b\chi_c\phi_d\Big(\frac{\omega_2^2+\frac{2g_{2\phi\phi}\kappa^2}{g_{H\phi\phi}^2}}{4\kappa^4 n^2}+O(n^{-1})\Big)+\chi_a\phi_b\rho_c z_d\Big(-\frac{g_{H\phi\phi}\omega_2'+\omega_2 g_{H\phi\phi}'}{4g_{H\phi\phi}g_{Hzz}\kappa^3 n^2}+O(n^{-1})\Big) \\
&+\chi_a z_b\chi_c z_d\Big(\frac{g_{2zz}}{2g_{Hzz}^2\kappa^2 n^2}+O(n^{-1})\Big)+\chi_a z_b\rho_c\phi_d\Big(\frac{g_{H\phi\phi}\omega_2'+\omega_2 g_{H\phi\phi}'}{4g_{H\phi\phi}g_{Hzz}\kappa^3 n^2}+O(n^{-1})\Big) \\
&+\rho_a\phi_b\rho_c\phi_d\Big(-\frac{\omega_2^2+\frac{2g_{2\phi\phi}\kappa^2}{g_{H\phi\phi}^2}}{4\kappa^4 n^2}+O(n^{-1})\Big)+\rho_a z_b\rho_c z_d\Big(\frac{-g_{2zz}}{2g_{Hzz}^2\kappa^2 n^2}+O(n^{-1})\Big) \\
&+\phi_a z_b\phi_c z_d\Big(\frac{R_\parallel}{2g_{H\phi\phi}g_{Hzz}}+O(n)\Big)+\text{terms of order}\geq 1 \\
&+\text{terms related by symmetries of } R_{abcd},
\end{aligned}
\tag{A6}
$$

where ω_3 is defined as $\omega(n,z)=\Omega_H+\frac{1}{2}\omega_2(z)n^2+\omega_3(z)n^3+O(n^4)$, and

$$R_\perp=\frac{3\omega_2^2 g_{H\phi\phi}-4\kappa_2\kappa}{2\kappa^2}, \tag{A7}$$

$$R_\parallel=\frac{g_{Hzz}g_{H\phi\phi}'^2+g_{H\phi\phi}g_{H\phi\phi}'g_{Hzz}'-2g_{H\phi\phi}g_{Hzz}g_{H\phi\phi}''}{2g_{H\phi\phi}^2 g_{Hzz}^2}. \tag{A8}$$

From these expressions the properties (A1) and (A3) can be read.

APPENDIX B

An analysis which would include generic Lagrangians involving derivatives of the Riemann tensor is of course much more complex. Here we consider a special case where we add to Lagrangians (5) the term

$$(\nabla_a R)^2. \tag{B1}$$

Now, Dirac brackets

$$\{J[\xi_1],J[\xi_2]\}^*=\int_{\mathcal{H}}(\xi_2\cdot\theta_1-\xi_1\cdot\theta_2-\xi_2\cdot(\xi_1\cdot\mathbf{L})) \tag{B2}$$

change by the term

$$\int_{\mathcal{H}}\hat{\boldsymbol{\epsilon}}\{2(X^{(12)}_{abcd}E^{abcd}-\tilde{X}^{(12)}_{abc}F^{abc})\},$$

where

$$E^{abcd}=\frac{1}{2}(g^{ad}g^{bc}-g^{ac}g^{bd})\nabla^2 R \tag{B3}$$

and

$$
\begin{aligned}
F^{abc}={}&\nabla^b\nabla^c\nabla^a R-2g^{bc}\nabla^a\nabla^2 R+g^{ac}\nabla^b\nabla^2 R \\
&-g^{bc}R^{ae}\nabla_e R-R^{bc}\nabla^a R+2R^{ac}\nabla^b R \\
&-2R^{ab}\nabla^c R.
\end{aligned}
\tag{B4}
$$

A long but straightforward calculation shows that for special case (B1) usual results can be obtained provided we restrict the class of metric functions. The restrictions are

$$\omega_3=0 \tag{B5}$$

and

$$\frac{3g_{3zz}}{g_{Hzz}}+\frac{8\kappa_3}{\kappa}+\frac{3g_{3\phi\phi}}{g_{H\phi\phi}}=0 \tag{B6}$$

where $g_{3zz}(z)$, $g_{3\phi\phi}(z)$, and $\omega_3(z)$ are coefficients of n^3, and $\kappa_3(z)$ of n^4 in Taylor expansions (22).

These restrictions can be understood also by terms of regularity of scalar curvature invariants on horizon. Namely, if we require regularity of

$$(\nabla_a R_{bc})^2 \quad \text{and} \quad \nabla^2 R, \tag{B7}$$

we obtain relations (B5) and (B6).

From (B5) it follows that order$(\nabla_e R_{abcd})=0$ and so all polynomial invariants involving the Riemann tensor and its first derivatives will be regular on the horizon. This is in fact a generalization of results from [13] that regularity of invariants of the Riemann tensor has implications on metric functions near horizon. Here, we see that regularity of invariants involving derivatives of Riemann tensor has even stronger consequences on metric functions.

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