Pseudoscalar $q\bar{q}$ mesons and effective QCD coupling enhanced by $\langle A^2 \rangle$ condensate

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Recent developments provided evidence that the dimension 2 gluon condensate $\langle A^2 \rangle$ is important for the nonperturbative regime of Yang-Mills theories (quantized in the Landau gauge). We show that it may be relevant for the Dyson-Schwinger approach to QCD. In order that this approach leads to a successful hadronic phenomenology, an enhancement of the effective quark-gluon interaction seems to be needed at intermediate ($p^2 \sim 0.5$ GeV^2) momenta. It is shown that the gluon condensate $\langle A^2 \rangle$ provides such an enhancement. It is also shown that the resulting effective strong running coupling leads to the sufficiently strong dynamical chiral symmetry breaking and successful phenomenology at least in the light sector of pseudoscalar mesons.

I. INTRODUCTION AND SURVEY

Dyson-Schwinger (DS) equations provide a prominent approach to physics of strong interactions. One of its aspects, reviewed [1] and exemplified recently by Refs. [1,2], consists of *ab initio* studies of DS equations for Green's functions of QCD, typically in the Landau gauge (LG). The other aspect consists of phenomenological DS studies (also typically in LG) of hadrons as quark bound states, but relying more on modeling; e.g., see a recent review [3]. Such phenomenological studies have mostly been relying on the rainbow-ladder approximation (RLA), where generation of dynamical chiral symmetry breaking (DχSB) is well understood [1,3–6]. As it has been stressed in, e.g., Refs. [3,7], RLA is the leading-order term of a procedure [8,9] that can be systematically improved towards less severe truncations of DS equations. This general procedure provides a means to identify *a priori* the channels in which RLA is likely to work well [9]. Pseudoscalar mesons are the most notable among those channels because of the correct chiral QCD behavior due to DχSB and Goldstone theorem, since (almost) massless pseudoscalar mesons are reproduced in the (vicinity of) chiral limit not only by the exact QCD treatment but also by all consistent truncations such as RLA [10].

Consistent RLA implies [1,3–6] that *Ansätze* of the form

$$[K(p)]_{ef}^{hg} = -4\pi\alpha_{\text{eff}}(p^2) D_{\mu\nu}^{ab}(p)_0 \left[\frac{\lambda^a}{2}\gamma_\mu\right]_{eg} \left[\frac{\lambda^b}{2}\gamma_\nu\right]_{hf} \tag{1}$$

must be used for the interactions between quarks in both the gap equation $S^{-1} = S_0^{-1} - \Sigma$ for the full quark propagator S (S_0 is the *free* one) and the Bethe-Salpeter (BS) equation for the meson (M) bound-state vertex Γ_{M}; i.e.,

$$[\Sigma]_{ef} = \int S_{gh}[K]_{ef}^{hg}, \qquad [\Gamma_{\text{M}}]_{ef} = \int [S\Gamma_{\text{M}}S]_{gh}[K]_{ef}^{hg}, \tag{2}$$

where, writing *schematically*, integrations are meant over loop momenta while e, f, g, h in Eqs. (1) and (2) represent spinor, color, and flavor indices [3]. In LG, the free gluon propagator $D_{\mu\nu}^{ab}(p)_0 \equiv \delta^{ab}(\delta_{\mu\nu} - p_\mu p_\nu/p^2)/p^2$.

In Eq. (1), $\alpha_{\text{eff}}(p^2)$ is an effective running coupling. It is only partially known from the fact that at large spacelike momenta (our convention is $p^2 > 0$ for spacelike p), $\alpha_{\text{eff}}(p^2)$ must reduce to $\alpha_{\text{pe}}(p^2)$, the well-known running coupling of perturbative QCD. However, for $p^2 \lesssim 1$ GeV^2, where nonperturbative QCD applies, the interaction is still not known. Thus, in phenomenological DS studies, $\alpha_{\text{eff}}(p^2)$ must be modeled for $p^2 \lesssim 1$ GeV^2; e.g., see Refs. [1,3–6]. There one can see that the most successful (phenomenologically) of these modeled interactions have a rather large bump at intermediate momenta, around $p^2 \sim 0.5$ GeV^2 [e.g., in Fig. 1 see $\alpha_{\text{eff}}(p^2)$ used by Jain and Muncek (JM) [4] and by Maris, Roberts, and Tandy (MRT) [3,5,6]]. In any case, successful DS phenomenology demands that the modeled part of the interaction (1) be fairly strong; regardless of details of the form of the interaction, its *integrated strength* (for $p^2 \lesssim 1$ GeV^2) must be fairly high to achieve acceptable description of hadrons, notably mass spectra and DχSB [3,11]. Theoretical explanations of the origin of such a strong nonperturbative part of the phenomenologically required interaction are obviously much needed, either from the *ab initio* DS studies or from somewhere outside the DS approach. This is the main motivation for the present paper.

The *ab initio* DS studies showed that, in LG, the effects of ghosts are crucial for the intermediate-momenta enhancement of the effective quark-gluon interaction. This is obvious in the expression for the strong running coupling $\alpha_{\text{s}}(p^2)$ in these LG studies [1,2],

$$\alpha_{\text{s}}(p^2) = \alpha_{\text{s}}(\mu^2) Z(p^2) G(p^2)^2, \tag{3}$$

where $\alpha_{\text{s}}(\mu^2) = g^2/4\pi$ and $Z(\mu^2)G(\mu^2)^2 = 1$ at the renormalization point $p^2 = \mu^2$. The ghost and gluon renormalization functions $G(p^2)$ and $Z(p^2)$ define the full ghost
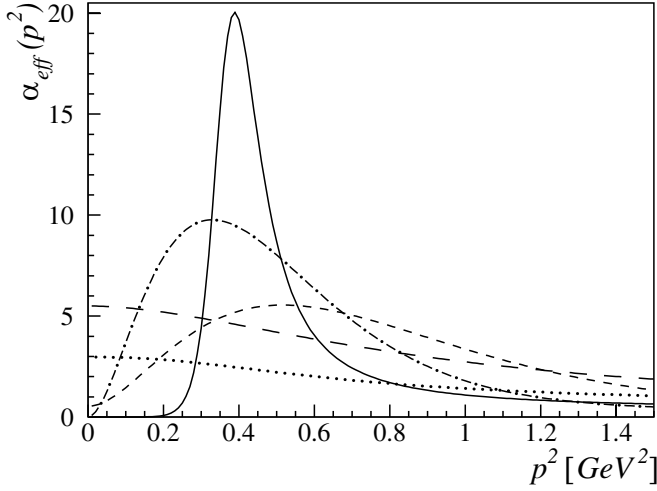
FIG. 1.   The momentum dependence of various strong running couplings mentioned in the text. JM [4] and MRT [3,6] $\alpha_{\rm eff}(p^2)$ are depicted by, respectively, dashed and dash-dotted curves. The effective coupling (18) proposed and analyzed in the present paper is depicted by the solid curve, and $\alpha_s(p^2)$ (3) of Fischer and Alkofer [2] (fit A) by the dotted curve. The long-dashed curve is the fit (extrapolated all the way to $p^2 = 0$) of the lattice results of Ref. [29].

propagator $D_G^{ab}(p) = -\delta^{ab} G(p^2)/p^2$ and the full LG gluon propagator $D_{\mu\nu}^{ab}(p) = Z(p^2)D_{\mu\nu}^{ab}(p)_0$.

While the *ab initio* DS studies [1,2] do find significant enhancement of $\alpha_s(p^2)$, Eq. (3), until recently this seemed still not enough to yield a sufficiently strong DχSB (e.g., see Sec. 5.3 in Ref. [1]) and a successful phenomenology. Nevertheless, going beyond the ladder truncation and so getting additional interaction strength from dressed vertices, for carefully constructed dressed quark-gluon vertex *Ansätze*, Fischer and Alkofer [2] have recently obtained good results for constituent quark masses and pion decay constant f_π, although not simultaneously also for the chiral quark-antiquark $\langle \bar{q}q \rangle$ condensate, which then came out somewhat too large. Thus, the overall situation is that there is progress [2,11], but that further investigation and elucidation of the origin of phenomenologically successful effective interaction kernels remains one of the primary challenges in DS studies [3]. Here, we will point out that such an interaction kernel for DS studies results from combining the DS ideas on $\alpha_s(p^2)$ of the form (3) [1,2] and the ideas [12–16] on the dimension 2 gluon condensate $\langle A_\mu^a A^{a\mu} \rangle \equiv \langle A^2 \rangle$ *in LG.*

II. CONDENSATES IN GLUON AND GHOST PROPAGATORS

Many years ago Refs. [17–20] found in the operator product expansion (OPE) the $\langle A^2 \rangle$ contributions to QCD propagators, recently confirmed by Kondo [15]. For LG (adopted throughout this paper), number of colors $N_c = 3$ and space-time dimensions $D = 4$, their results for gluon

and ghost propagators amount to

$$Z(p^2) = \frac{1}{1 + \frac{m_A^2}{p^2} + \frac{\mathcal{O}_A(1/p^2)}{p^2}}, \qquad (4)$$

$$G(p^2) = \frac{1}{1 + \frac{m_G^2}{p^2} + \frac{\mathcal{O}_G(1/p^2)}{p^2}}, \qquad (5)$$

$$m_A^2 = \frac{3}{32} g^2 \langle A^2 \rangle = -m_G^2, \qquad (6)$$

where m_A and m_G are, respectively, dynamically generated effective gluon (A) and ghost (G) mass. The later references [21,22] also worked out the perturbative QCD corrections inducing the logarithmic p^2 dependence of these dynamically generated masses, i.e. $m_A^2(p^2)$, to which we will return and comment on in the next section.

These now well-established propagator contributions (4)–(6) then suggest the importance of $\langle A^2 \rangle$ for the DS approach to hadrons, where propagators, usually in LG, are used to get solutions for quark bound states and calculate observable quantities. Notably, see Ref. [10] for gauge-parameter independent expressions for f_π and a generalization of the Gell-Mann-Oakes-Renner relation (GMOR) that demonstrates gauge-parameter independence of the meson mass. Still, how can $\langle A^2 \rangle$ influence these observable quantities, since this condensate is *not* gauge invariant? It turns out [13,14,23] that in LG, $\langle A^2 \rangle$ equals a nonlocal, but gauge-invariant quantity: the minimal (with respect to the choice of gauge) value of $A_\mu^a A^{\mu a}$ integrated over the space-time, indicating that $\langle A^2 \rangle$ in LG may have a physical meaning. Outside LG, besides $\langle A^2 \rangle$ other (ghost) condensates of dimension 2 appear [15]. They very likely cancel the variation which $\langle A^2 \rangle$ suffers in going to another gauge, since the physics behind all these different dimension 2 condensates in different gauges must be the same: gluon-ghost condensation lowers the QCD vacuum energy E, which is a physical, gauge-invariant quantity, to a stable ("$E < 0$") vacuum [23].

For $g^2 \langle A^2 \rangle$, LG lattice studies of Boucaud *et al.* [12] yield the value 2.76 GeV^2, compatible with the bound resulting from the discussions of Gubarev *et al.* [13,14] on the physical meaning of $\langle A^2 \rangle$ and its importance for confinement. This value gives (6) $m_A = 0.845$ GeV, which will turn out to be a very good initial estimate for $m_{A,G}$.

As for the contributions $\mathcal{O}_A(1/p^2)$ and $\mathcal{O}_G(1/p^2)$ in Eqs. (4) and (5), one expects a prominent role of the dimension 4 gluon condensate $\langle F_{\mu\nu}^a F^{a\mu\nu} \rangle \equiv \langle F^2 \rangle$, which, contrary to $\langle A^2 \rangle$, is gauge invariant [24]. References [18,19] showed that the OPE contributions of dimension 4 condensates were far more complicated [20] than found previously [17]: not only many kinds of condensates contributed to terms $\propto 1/p^2$, but for many of them (gauge-dependent gluon, ghost, and mixed ones) there has been no assignment of any kind of values yet. Terms $\propto (1/p^2)^n$ $(n > 1)$ were not considered at all. Thus, at this

point, the only practical approach is that the contributions $\mathcal{O}_A(1/p^2)$ and $\mathcal{O}_G(1/p^2)$ in Eqs. (4) and (5) are approximated by the terms $\propto 1/p^2$ and parametrized, i.e.,

$$\mathcal{O}_A(1/p^2) \approx \frac{C_A}{p^2}, \qquad \mathcal{O}_G(1/p^2) \approx \frac{C_G}{p^2}, \qquad (7)$$

where both C_A and C_G are, in principle, free parameters to be fixed by phenomenology. Still, we should mention that the effective gluon propagator advocated by Lavelle [25] would indicate $C_A \approx (0.640 \text{ GeV})^4$ for the following reason: for LG and $D = 4$, the contribution which this gluon propagator receives from the so-called "pinch diagrams" vanishes, and its [25] $\mathcal{O}_A(1/p^2)$ contribution

$$\Pi_A^{\langle F^2 \rangle}(p^2) = \frac{34 N_c \pi \alpha_s \langle F^2 \rangle}{9(N_c^2 - 1)p^2} = \frac{(0.640 \text{ GeV})^4}{p^2} \qquad (8)$$

stems entirely from the gluon polarization function in Ref. [18], provided one invokes some fairly plausible assumptions, like using equations of motion, to eliminate all condensates except $\langle F^2 \rangle$. (The quark condensate $\langle \bar{q}q \rangle$ could also be neglected [25].) Since Ref. [26] indicates that the true value of $\alpha_s \langle F^2 \rangle$ is still rather uncertain, and since Refs. [19,20] make clear that Lavelle's [25] propagator misses some (unknown) three- and four-gluon contributions, we do not attach too much importance to the precise value $C_A = (0.640 \text{ GeV})^4$ [24,25] in Eq. (8), but just use it as an inspired initial estimate. Fortunately, the corresponding variations of C_A still permit good phenomenological fits, since we will find below that our results are not very sensitive to C_A.

There is no similar estimate for C_G, but one may suppose that it would not differ from C_A by orders of magnitude. We thus try $C_G = C_A = (0.640 \text{ GeV})^4$ as an initial guess. It will turn out, *a posteriori*, that this value of C_G leads to a remarkably good fit to phenomenology.

III. COUPLING ENHANCED BY THE GLUON CONDENSATES

Having set the stage, we are now ready to propose that $m_A^2 = -m_G^2 \propto \langle A^2 \rangle$ leads to the enhancement of $\alpha_{\text{eff}}(p^2)$ at intermediate p^2. To derive the running coupling exhibiting this property, let us first recall the aforementioned perturbative corrections to OPE results (4)–(6). In Eqs. (4), (5), and (7), gluon and ghost renormalization functions are parametrized by the coefficients m_A, C_A, and C_G, which are constants at the tree level but develop momentum dependence through the perturbative corrections. To see this, we note that the generic forms of the ghost and gluon renormalization functions including OPE contributions *and* perturbative QCD corrections [21,22,27] can be written as

$$Z(p^2) = \frac{1}{r_0^A(p^2) + \frac{r_2^A(p^2)}{p^2} + \frac{r_4^A(p^2)}{p^4} + \cdots}, \qquad (9)$$

$$G(p^2) = \frac{1}{r_0^G(p^2) + \frac{r_2^G(p^2)}{p^2} + \frac{r_4^G(p^2)}{p^4} + \cdots}, \qquad (10)$$

where $r_0^A(p^2)$, $r_2^A(p^2)$, $r_4^A(p^2)$, etc., are the terms of mass dimension 0, 2, 4, etc., for the gluon case, and $r_0^G(p^2)$, $r_2^G(p^2)$, $r_4^G(p^2)$, etc., are the terms of mass dimension 0, 2, 4, etc., for the ghost case. For example, the terms of the dimension zero, up to one loop, are

$$r_0^A(p^2) = \left(\frac{\alpha_{\text{pe}}(p^2)}{\alpha_{\text{pe}}(\mu^2)} \right)^\gamma, \qquad (11)$$

$$r_0^G(p^2) = \left(\frac{\alpha_{\text{pe}}(p^2)}{\alpha_{\text{pe}}(\mu^2)} \right)^\delta, \qquad (12)$$

where γ and δ are, respectively, gluon and ghost anomalous dimensions. The perturbative corrections for the Wilson coefficients of dimension 2 have also been calculated for the pure Yang-Mills case by Boucaud *et al.* [27] (for the gluon propagator) and Kondo and collaborators [21,22] (for both the gluon and ghost propagators), and since they imply that

$$\frac{r_2^A(p^2)}{r_0^A(p^2)} = -\frac{r_2^G(p^2)}{r_0^G(p^2)} \equiv m_A^2(p^2) = -m_G^2(p^2), \qquad (13)$$

we write the renormalization functions as

$$Z(p^2) = \left(\frac{\alpha_{\text{pe}}(p^2)}{\alpha_{\text{pe}}(\mu^2)} \right)^{-\gamma} \frac{1}{1 + \frac{m_A^2(p^2)}{p^2} + \frac{C_A(p^2)}{p^4} + \cdots}, \qquad (14)$$

$$G(p^2) = \left(\frac{\alpha_{\text{pe}}(p^2)}{\alpha_{\text{pe}}(\mu^2)} \right)^{-\delta} \frac{1}{1 - \frac{m_A^2(p^2)}{p^2} + \frac{C_G(p^2)}{p^4} + \cdots}. \qquad (15)$$

The perturbative corrections for the Wilson coefficients of dimension 4 and higher have not been calculated yet, but we introduced also the notation

$$\frac{r_4^A(p^2)}{r_0^A(p^2)} = C_A(p^2), \qquad \frac{r_4^G(p^2)}{r_0^G(p^2)} = C_G(p^2), \qquad (16)$$

to point out the correspondence of Eqs. (14) and (15) with the relations (4)–(6) and (7). The latter differ from the former just by the absence of the slowly varying logarithmic p^2 dependence in $m_A(p^2)$, $C_A(p^2)$, $C_G(p^2)$, and $\alpha_{\text{pe}}(p^2)$, and of the dots denoting terms of dimension larger than 4.

Regarding the perturbatively generated prefactors $[\alpha_{\text{pe}}(p^2)/\alpha_{\text{pe}}(\mu^2)$ to the powers of $-\gamma$ and $-\delta]$, the forms (14) and (15) are consistent with the corresponding forms given by Eqs. (12) and (13) and also (14) in Ref. [28] and by Eq. (41) in Ref. [2]. For N_c colors and N_f quark flavors,

the anomalous dimensions of the gluon and ghost propagator are, respectively, given by $\gamma = (-13N_c + 4N_f)/(22N_c - 4N_f)$ and $\delta = -9N_c/(44N_c - 8N_f)$ (see, e.g., Ref. [2]). This ensures $\gamma + 2\delta = -1$. The definition of the strong running coupling constant, Eq. (3), together with Eqs. (14) and (15), thus gives

$$\alpha_s(p^2) = \alpha_{\text{pe}}(p^2) \frac{1}{1 + \frac{m_A^2(p^2)}{p^2} + \frac{C_A(p^2)}{p^4} + \cdots}$$

$$\times \left(\frac{1}{1 - \frac{m_A^2(p^2)}{p^2} + \frac{C_G(p^2)}{p^4} + \cdots} \right)^2. \qquad (17)$$

Neglecting the p^2 dependence in the coefficients m_A, C_A, and C_G, as well as the higher terms in the denominators, we finally get

$$\alpha_s(p^2) \approx \alpha_{\text{pe}}(p^2) \frac{1}{1 + \frac{m_A^2}{p^2} + \frac{C_A}{p^4}} \left(\frac{1}{1 - \frac{m_A^2}{p^2} + \frac{C_G}{p^4}} \right)^2$$

$$\equiv \alpha_{\text{eff}}(p^2)$$

$$= \alpha_{\text{pe}}(p^2) Z^{\text{Npe}}(p^2) G^{\text{Npe}}(p^2)^2, \qquad (18)$$

depicted in Fig. 1 by the solid line [for the parameter values (26), discussed below]. The suggestive abbreviations

$$Z^{\text{Npe}}(p^2) = \frac{1}{1 + \frac{m_A^2}{p^2} + \frac{C_A}{p^4}} \qquad (19)$$

and

$$G^{\text{Npe}}(p^2) = \frac{1}{1 - \frac{m_A^2}{p^2} + \frac{C_G}{p^4}} \qquad (20)$$

for the factors giving the deviation of Eq. (18) from the perturbative coupling $\alpha_{\text{pe}}(p^2)$, stress that our approximations amount to assuming that nonperturbative (Npe) effects are given by the OPE-based results of Refs. [15,17–20], which in our present case boil down to Eqs. (4)–(6), and by the parametrization (7).

Our final expression (18) for the running coupling, to be used in phenomenological calculations below, does not depend on the renormalization scale μ^2 explicitly, although $Z(p^2)$ and $G(p^2)$, appearing in the intermediate steps, still do. Namely, the same approximation applied to the gluon and ghost renormalization functions (14) and (15) gives

$$Z(p^2) = \left(\frac{\alpha_{\text{pe}}(p^2)}{\alpha_{\text{pe}}(\mu^2)} \right)^{-\gamma} \frac{1}{1 + \frac{m_A^2}{p^2} + \frac{C_A}{p^4}}, \qquad (21)$$

$$G(p^2) = \left(\frac{\alpha_{\text{pe}}(p^2)}{\alpha_{\text{pe}}(\mu^2)} \right)^{-\delta} \frac{1}{1 - \frac{m_A^2}{p^2} + \frac{C_G}{p^4}}, \qquad (22)$$

plotted in Figs. 2 and 3 as our $G(p^2)$ and $Z(p^2)$ [again for the parameter values (26)].

In our figures we also plot some results of lattice and *ab initio* DS studies. Namely, before we turn to our main goal, i.e., exploring whether our running coupling (18)
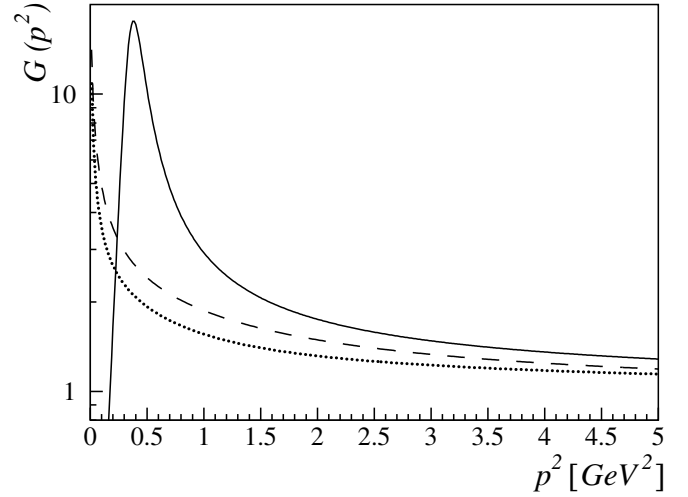

FIG. 2. The momentum dependence of the ghost renormalization function $G(p^2)$. The solid curve is our result (22). The dotted curve is a result of the *ab initio* DS study [2]. Concretely, it depicts the fit A in Eq. (41) of Ref. [2] evaluated at $\mu = 5$ GeV, for which value the dotted curve agrees rather well (in the displayed momentum range) with the long-dashed curve, representing Eq. (4) of the lattice Ref. [29], renormalized at $\mu = 5$ GeV. Our result (22) is therefore plotted for the same value of μ.
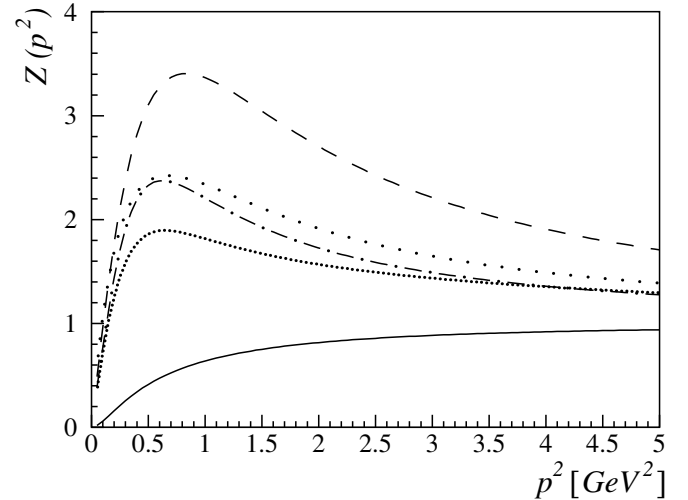

FIG. 3. The momentum dependence of the gluon renormalization function $Z(p^2)$. The solid curve is our result (21). The densely dotted curve is the *ab initio* DS result (fit A) from Refs. [2,32]. As in Fig. 2, both the *ab initio* DS result and our Eq. (21) are plotted for $\mu = 5$ GeV. The other curves pertain to some recent lattice results [29,33–35]. The ones that agree with the *ab initio* DS result [2,32] very well are those of Leinweber *et al.* [33] [their Eq. (5.14), depicted here by the sparsely dotted curve], and Eq. (3) of Iida *et al.* [34] (dash-dotted curve) fitting the lattice data of Ref. [35]. The third lattice result for $Z(p^2)$ is displayed by the long-dashed curve. It corresponds to Eq. (3) of Ref. [29], which is the fit (extrapolated all the way to $p^2 = 0$) to the lattice data of Ref. [29]. For all lattice results, the renormalization condition $Z(\mu^2) = 1$ is imposed at $\mu = 5$ GeV.

leads to successful phenomenology when used in quark gap DS and bound-state BS equations through Eq. (1), we will first comment on the comparison of our gluon and ghost renormalization functions with some other results. In particular, recent Tübingen results of SU(2) lattice gauge simulations [29–31] and of *ab initio* DS studies [2,32] agree with each other well for $G(p^2)$ and quite reasonably for $Z(p^2)$. This is seen in Figs. 2 and 3. (Also, the results in the most recent Tübingen lattice reference [31], which however does not give the corresponding fitting formulas, are after proper renormalization somewhat lower than the results of Ref. [29] plotted in Figs. 2 and 3, and thus agree somewhat better with the displayed *ab initio* DS results [2,32].) As can be seen in Fig. 3, there are some other lattice results [33–35] for the gluon renormalization function which agree with $Z(p^2)$ from the DS approach [2,32] even better, but they do not give the ghost renormalization function. Thus, Tübingen lattice results [29–31] are presently of particular interest, because they give both $Z(p^2)$ and $G(p^2)$. Admittedly, there is a caveat: while the *ab initio* DS studies [2,32] do not have problems with reaching low momenta and in fact make strong statements about the asymptotic behavior in the $p^2 \to 0$ limit, the lattice data [29] do not reach very low momenta. The lowest data point for $G(p)$ as well as $\alpha(p)$ in Ref. [29] is at $p^2 \sim 0.36$ GeV^2, so that one must keep in mind that for lower p^2 the corresponding lattice-data-fitting curves in Figs. 2 and 3, and therefore also in Fig. 1, are just extrapolations. Nevertheless, presently most important is that comparing the long-dashed and dotted curves in Fig. 1 shows that the respective running couplings (3) following from these "lattice" [29–31] and "*ab initio* DS" [2,32] renormalization functions typically do not differ by more than a factor of 2. Thus, in spite of the mentioned caveat, these lattice results [29–31] and the aforementioned DχSB scenario of Fischer and Alkofer [2] support each other. Still, the behavior of QCD propagators, especially the ghost ones, and the resulting running coupling, is not a closed issue yet, so that the presently proposed scenario should also be considered although it is not supported by lattice results.

The examples [36–39] of lattice results differing from the Tübingen lattice [29,31] and *ab initio* DS results [2,32] are not only the relatively old ones such as those of Suman and Schilling on the ghost renormalization function (which abruptly falls for the very smallest probed momenta, possibly indicating the infrared vanishing behavior) [36,37], but also some of the most recent ones, such as Ref. [38], where Landau gauge lattice calculations give the strong running coupling which, supposedly due to instanton effects, decreases[1] at small momentum roughly as p^4 [38,39].

Some other, quite independent methods, also give the QCD running coupling vanishing at small p^2 [40] although not so fast as our form (18). Now, we want to make clear that we do not argue that our results are another indication that the running coupling indeed vanishes as $p^2 \to 0$, because we are aware (as we comment in more detail below) that the behavior of our running coupling at very small p^2 is just an artifact of the way Eq. (18) was derived. Fortunately, however, it turns out that the small-p^2 behavior does not influence much our final, observable results (in contradistinction to intermediate p^2's). To see all this, let us discuss in detail the behavior of our form (18) and especially the possible objections to it. The first, less serious one is that $\alpha_{\mathrm{pe}}(p^2)$ ultimately hits the Landau pole as p^2 gets lower. However, we handle this as in other phenomenological DS studies [4–6,41–45], where this pole is shifted to timelike momenta in all logarithms: $\ln(p^2/\Lambda_{\mathrm{QCD}}^2) \to \ln(x_0 + p^2/\Lambda_{\mathrm{QCD}}^2)$. (Dynamical gluon mass can provide the physical reason for this [46]; i.e., $x_0 \propto m_A^2/\Lambda_{\mathrm{QCD}}^2 \sim 10$.) For $\alpha_{\mathrm{pe}}(p^2)$ we use the two-loop expression used before by JM [4] and our earlier DS studies, e.g., Refs. [41–45]. This means the infrared (IR) regulator (to which all results are almost totally insensitive) is $x_0 = 10$, and $\Lambda_{\mathrm{QCD}} = 0.228$ GeV. The parameters of $\alpha_{\mathrm{pe}}(p^2)$ are thereby fixed and do not belong among variable parameters such as C_A and C_G.

Back to the possible objections: the second, in the present context the more serious, one is that we cannot in advance give an argument that the factor $Z^{\mathrm{Npe}}(p^2)G^{\mathrm{Npe}}(p^2)^2$ in the proposed $\alpha_{\mathrm{eff}}(p^2)$ (18) indeed approximates well nonperturbative contributions at low p^2 (say, $p^2 < 1$ GeV^2), but can only hope that our results to be calculated will provide an *a posteriori* justification for using it as low as $p^2 \sim 0.3$ GeV^2 [since Eq. (18) takes appreciable values down to about $p^2 \sim 0.3$ GeV^2]. Of course, $Z^{\mathrm{Npe}}(p^2)$ and $G^{\mathrm{Npe}}(p^2)$ must be wrong in the limit $p^2 \to 0$, as the OPE-based results (4)–(6) of Refs. [15,17–20] certainly fail in that limit. Thus, the extreme suppression for small p^2 is an unrealistic artifact of the proposed form (18) when applied down to the $p^2 \to 0$ limit. Nevertheless, because of the integration measure in the integral equations in DS calculations, integrands at these small p^2 do not contribute much, at least not to the quantities (such as $\langle\bar{q}q\rangle$ condensate, meson masses, decay constants and amplitudes) calculated in phenomenological DS analyses. Hence, the form of $\alpha_{\mathrm{eff}}(p^2)$ at p^2 close to zero is not very important for the outcome of these phenomenological DS calculations.[2] This is because the most important for the success of phenomenological DS calculations seems the enhancement at somewhat higher values of p^2; e.g., see the humps at $p^2 \sim 0.4$ to 0.6 GeV^2 in the JM [4] or

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[1]Their strong running coupling becomes roughly 0.1 or smaller at $p^2 \sim 0.16$ GeV^2. Below this value of p^2, the lattice evaluation was found unreliable [38].

[2]This is supported by, e.g., Fischer [47], who found that nearly all dynamically generated mass is produced by the integration strength above $p^2 = 0.25$ GeV^2.

MRT interaction [5,6] in Fig. 1. Our $\alpha_{\mathrm{eff}}(p^2)$ (18), the solid curve in Fig. 1, exhibits such an enhancement centered around $p^2 \approx m_A^2/2$. This enhancement is readily understood when one notices that Eq. (18) has four poles,

$$(p^2)_{1,2} = \frac{1}{2}\left(m_A^2 \pm i\sqrt{4C_G - m_A^4}\right), \qquad (23)$$

$$(p^2)_{3,4} = \frac{1}{2}\left(-m_A^2 \pm i\sqrt{4C_A - m_A^4}\right), \qquad (24)$$

in the complex p^2 plane. For $\min\{C_G, C_A\} > m_A^4/4$ there are no poles on the real axis, but saddles between two complex conjugated poles. For the DS studies, which are almost exclusively carried out in Euclidean space, space-like p^2, i.e., $p^2 > 0$, is the relevant domain and is thus pictured in Fig. 1. There, the maximum of $\alpha_{\mathrm{eff}}(p^2)$ (18) at the real axis is at $p^2 \approx m_A^2/2$, i.e., the real part of its *double* poles $(p^2)_{1,2}$ coming from $G^{\mathrm{Npe}}(p^2)^2$. The height and the width of the peak is influenced by both C_G and m_A. The enhancement of $\alpha_{\mathrm{eff}}(p^2)$ (18) is thus determined by $\langle A^2 \rangle$ through Eq. (6), and by the manner this condensate contributes to the ghost renormalization function.

We are thus motivated to use this form (18) of $\alpha_{\mathrm{eff}}(p^2)$ for all p^2 to test its success in DS calculations. We are aware of the shortcomings due to its oversimplified character, but its study helps to answer whether the $\langle A^2 \rangle$ condensate, which has recently attracted so much attention, may be important for the enhancement of the effective interaction needed for successful DS phenomenology. The results presented below indicate that the $\langle A^2 \rangle$ condensate may indeed provide an important mechanism not considered so far.

IV. END RESULTS WITH DISCUSSION AND CONCLUSION

We solved the gap and BS equations (2) for quark propagators

$$S(p) = \frac{1}{i\gamma \cdot p\, \mathcal{A}(p^2) + \mathcal{B}(p^2)} \equiv \frac{\mathcal{A}(p^2)^{-1}}{i\gamma \cdot p + M(p^2)} \quad (25)$$

and for pseudoscalar meson $q\bar{q}$ ($q = u, d, s$) bound-state vertices (Γ_M) in the same way as in our previous phenomenological DS studies, e.g., Refs. [41–45]. This essentially means as in the JM approach [4], except that instead of JM's $\alpha_{\mathrm{eff}}(p^2)$, Eq. (18) is employed in the RLA interaction (1). We can thus immediately present the results because we can refer to Refs. [4,41–45] for all calculational details, such as procedures for solving DS and BS equations, all model details, as well as explicit expressions for calculated quantities and inputs such as the aforementioned IR-regularized $\alpha_{\mathrm{pe}}(p^2)$.

These calculations show that the initial m_A, C_A, and C_G estimates motivated and given above need only slight (a few %) modifications to provide a very good description of the light pseudoscalar sector. Concretely, now we will [both in the chiral limit (χ lim) and realistically away from it] quote only results obtained for the parameter set

$$C_A = (0.6060 \text{ GeV})^4 = C_G, \qquad m_A = 0.8402 \text{ GeV}, \tag{26}$$

while a broader investigation of parameter dependence shows the following. (i) The results are only weakly sensitive to moderate variations (up to the factors of 2 to 1/2) of C_A. (ii) Contrary to that, the results are very sensitive to m_A and C_G, since they determine the peak of our $\alpha_{\mathrm{eff}}(p^2)$ (18). However, between $C_G^{\mathrm{min}} \sim (0.6 \text{ GeV})^4$ and $C_G^{\mathrm{max}} \sim (0.9 \text{ GeV})^4$ there are many pairs of these quantities which give fits comparable (within a percent) to that resulting from the values (26), as long as they approximately satisfy the linear relation

$$(C_G)^{1/4} = 0.7742 m_A - 0.0442 \text{ GeV}. \tag{27}$$

Thus, the two parameters ruling the strength of $\alpha_{\mathrm{eff}}(p^2)$ are not independent.

Already the chiral-limit results are very satisfactory: $f_{\pi^\pm} = f_{\pi^0} \equiv f_\pi = 90.5$ MeV, the $\pi^0 \to \gamma\gamma$ chiral-limit amplitude $T_{\pi^0}^{\gamma\gamma}(\chi \text{ lim}) \equiv 1/(4\pi^2 f_\pi) = 0.280$ GeV^{-1}, and the chiral condensate $\langle \bar{q}q \rangle = (-217 \text{ MeV})^3$. We also get the correct QCD chiral-limit behavior: massless $q\bar{q}$ pseudoscalars and satisfied (within $\sim 4\%$) GMOR.

In the chiral limit, where the quark mass is purely dynamically generated since the bare (and current) quark masses vanish, the only parameters are m_A, C_A, and C_G. Away from the chiral limit, chiral symmetry is explicitly broken by the nonvanishing bare mass parameters $\tilde{m}_q$ of light quarks ($q = u, d, s$) entering the quark-propagator gap equation and the $q\bar{q}$ BS equation. For the very light quarks u and d, the dynamically generated quark masses of u and d quarks away from the chiral limit are practically the same as in the chiral limit. In Fig. 4, the solid curve depicts our results for the momentum dependence of these dynamical masses $M(p^2) \equiv \mathcal{B}(p^2)/\mathcal{A}(p^2)$ of u and d quarks (in the isosymmetric limit). Figure 4 also presents the lattice results for the u- and d-quark dynamical masses of Refs. [48,49] which give their fits to the lattice-data points in terms of analytic, closed-form expressions. Our results for the dynamical quark mass $M(p^2)$ agree very well with both the *ab initio* DS results [2,32] and the lattice results [48,49].

As the first attempt to depart realistically from the chiral limit, we adopt without change the $\tilde{m}_q$ values obtained earlier by JM [4] in a very broad DS fit of the meson phenomenology (with *their* α_{eff} [4]), i.e., $\tilde{m}_u = \tilde{m}_d = 3.1$ MeV and $\tilde{m}_s = 73$ MeV. Already the corresponding results for the masses and decay constants of pions and kaons and the $\pi^0 \to \gamma\gamma$ decay amplitude $T_{\pi^0}^{\gamma\gamma}$, given in the first line of Table I, show a very good agreement with experimental values, except for the kaon mass. The second
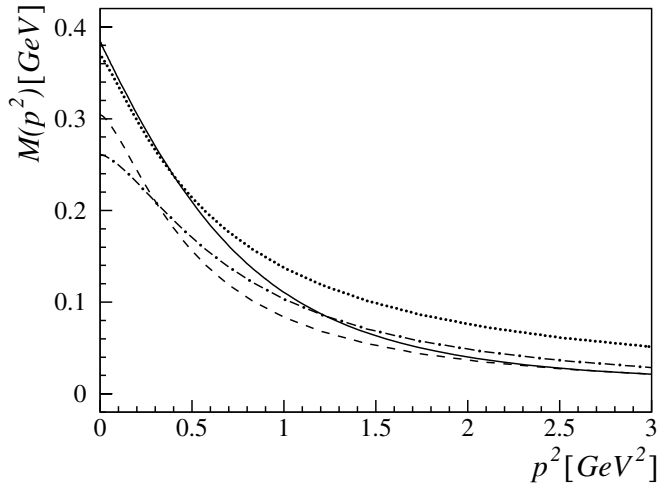
FIG. 4. The momentum dependence of the dynamically generated quark mass $M(p^2)$ for u and d quarks. The solid curve is our result for the parameters giving the second line of Table I, but our $M(p^2)$ depends in fact very little on the small explicit chiral symmetry breaking mass parameters $\tilde{m}_u$ and $\tilde{m}_d$ of the very light u and d quarks as long as their values are at all realistic. The dotted curve is the *ab initio* DS result [2,32]. The short-dashed curve is the $M(p^2)$ fit of Ref. [48] to the extrapolation of their lattice data to the chiral limit. The dash-dotted curve is the similar result from another lattice calculation, namely, the fit of $M(p^2)$ from Fig. 14 of Ref. [49].

line shows that just a slight readjustment of the quark masses, to $\tilde{m}_u = \tilde{m}_d = 3.046$ MeV, $\tilde{m}_s = 67.70$ MeV, is enough to get an almost perfect fit to the pion and kaon masses.

TABLE I. The masses and decay constants of pions and kaons and $\pi^0 \to \gamma\gamma$ decay amplitude $T_{\pi^0}^{\gamma\gamma}$, for the parameter values (26) and our $\alpha_{\mathrm{eff}}(p^2)$ (18). The JM [4] quark bare masses $\tilde{m}_u = \tilde{m}_d = 3.1$ MeV, $\tilde{m}_s = 73$ MeV give the first line. The second line are the results obtained with $\tilde{m}_u = \tilde{m}_d = 3.046$ MeV, $\tilde{m}_s = 67.70$ MeV. The last line gives the experimental values. (The distinction between neutral and charged mesons applies only to this line, as we calculate in the isosymmetric limit.) Everything is in MeV except $T_{\pi^0}^{\gamma\gamma}$.

M_{π^0}	f_{π^+}	M_{K^+}	f_{K^+}	$T_{\pi^0}^{\gamma\gamma}$ (GeV^{-1})
136.17	93.0	516.28	112.5	0.256
134.96	92.9	494.92	111.5	0.256
134.98	92.4 ± 0.3	493.68	113.0 ± 1.0	0.274 ± 0.010

The presented results allow us to conclude that the dimension 2 gluon condensate $\langle A^2 \rangle$ provides an enhanced effective interaction $\alpha_{\mathrm{eff}}(p^2)$ which leads to a sufficiently strong DχSB, pions and kaons as (quasi-)Goldstone bosons of QCD, and successful DS phenomenology at least in the light sector of pseudoscalar mesons. This opens the possibility that instead of modeling $\alpha_{\mathrm{eff}}(p^2)$, its enhancement at intermediate p^2 may be understood in terms of gluon condensates.

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