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# Heat Transfer by Numerical Solution for a Class of Radiating Fins

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# HEAT TRANSFER BY NUMERICAL SOLUTION FOR A CLASS OF RADIATING FINS

BY

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HEAT TRANSFER BY NUMERICAL SOLUTION

FOR A CLASS OF RADIATING FINS

This thesis is approved as a creditable and independent investigation by a candidate for the degree, Master of Science, and is acceptable as meeting the thesis requirements for this degree, but without implying that the conclusions reached by the candidate are necessarily the conclusions of the major department.

Thesis Adviser Date

Head, Mechanical Engineering Department Date

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## Nomenclature

| | |
|---|---|
| A | Cross-sectional area of fin |
| $B_1$,m | Geometric parameters pertaining to area of fin |
| $B_2$,n | Geometric parameters pertaining to perimeter of fin |
| $C_p$ | Specific heat |
| H | Convection coefficient of the surface and tip of fin |
| h | Convection coefficient of the base |
| k | Conductivity |
| L | Coordinate at the tip of the fin |
| P | Perimeter or the derivative of surface area with respect to x of fin |
| T | Temperature |
| t | Time |
| x | Coordinate at any point of the fin |
| $x_o$ | Coordinate at the base |
| $\epsilon$ | Emissivity |
| $\eta$ | Fin efficiency |
| $\rho$ | Density |
| $\sigma$ | Stefan-Boltzmann constant |

Subscripts:

| | |
|---|---|
| o | Ambient fluid |
| 1,f | Heating fluid |

Dimensionless Parameters:

| | |
|---|---|
| $\xi$ | Coordinate |
| $\theta$ | Temperature |
| $\omega$ | Ratio of the coordinates |
| $\tau$ | Time |
| $N_B$ | Biot number of the fin surface |
| $N_{BA}$ | Biot number of the fin base |
| $N_r$ | Radiation parameter |
| F | Fin parameter |

# CHAPTER I

## INTRODUCTION

It has long been known that the heat transfer from a solid body to an ambient fluid can be increased by increasing the surface area of the solid body. Extended surfaces, or fins, are indispensable for compact heat exchangers. Geometrically, fins may be classified as straight fins, annular fins, and rod fins or spines. In most applications of fins, a fluid is circulating inside the fin-supporting pipe while the outside is exposed to another ambient fluid. The purpose of fin analysis is to find the temperature distribution in the fin and the heat transfer from the fin to the ambient fluid, i.e., the fin efficiency, which is the basis of comparing various fin designs. Conduction is the heat transfer mechanism in the fin, and convection and radiation occur at the surface. The amount of radiation heat transfer, in accordance with Stefan-Boltzmann's law, is proportional to the difference between the fourth power of the temperature of the fin and the ambient fluid. The fourth-power term makes the fin equation non-linear and difficult to solve analytically. Earlier researchers linearized the radiation term by replacing the fourth-power law by an equivalent convection coefficient times the difference of the temperature in order to obtain an analytical solution. Thus, by linearization, Gardner (1)* derived a general equation for the

*Numbers in parentheses refer to the bibliography given at the end of the text.

temperature gradient and fin efficiency for a class of one-dimensional extended surfaces.

Radiating fins have become very important as a result of space exploration. For space vehicles at high altitude, radiation is the dominating heat transfer mechanism. Chamber and Somer (2), in solving an annular radiating fin by numerical methods, showed that the error in fin efficiency introduced by linearizing the radiation term can be as much as 60 percent. Thus, for high-temperature operation of the fin, the non-linear term must be included. Cobble (3) solved the steady-state problem of a one-dimensional constant-area fin with a fixed base temperature and insulated tip analytically by reducing the fourth power through the application of Newton's forward difference. Subsequently, Shouman (4,5) found the exact solution to the same problem. Numerical solutions are also available for steady-state heat transfer of some specific types of fins (6,7,8).

The present thesis deals with the question of transient efficiency of radiating and convecting extended surfaces. The results are compared with data given in some recent publications (3,9) for the case of steady operation of constant-area fins with a constant root temperature and insulated tip. Transient temperature and efficiency curves for various types of extended surfaces are presented for design purposes.

The topic of transient temperatures and efficiencies of convecting and radiating fins was covered, for the first time, in a recent paper by Lumsdaine and Hwang (10).

# CHAPTER II

# ANALYSIS

## 1. Fin Equation

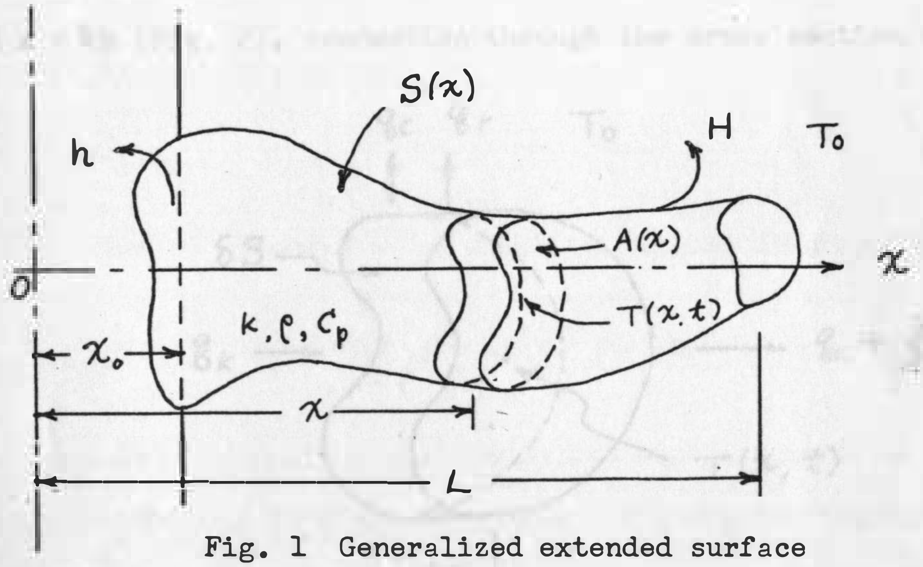


Fig. 1 Generalized extended surface

Consider an extended surface with variable cross section (Fig. 1) with the following assumptions:

1. The fin thickness is so small compared to the width that the problem can be considered as being one-dimensional. The temperature distribution is T(x,t).

2. The fin material is homogeneous and isotropic.

3. There is no heat source in the fin.

4. The thermal conductivity of the fin is constant.

5. The convection coefficient and the emissivity are constant over the fin surface.

In figure 1, the cross-sectional area A(x) normal to x for heat conduction is a function of x and so is the surface area S(x) for convection and radiation to the ambient fluid.

At an element of the fin contained between the cross sections x and x + $\delta$x (Fig. 2), conduction through the cross section at x is,

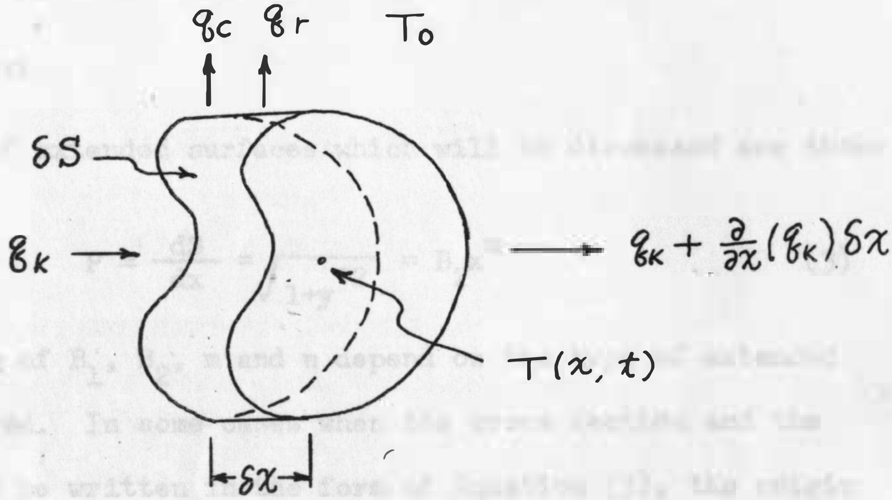


Fig. 2 Heat transfer at a fin element

from Fourier's law, $q_k = -kA\frac{\partial T}{\partial x}$; convection and radiation from the surface $\delta S$ are, from Newton's law and Stefan-Boltzmann's law, $q_c = h\delta S(T-T_o)$ and $q_r = \sigma\epsilon\delta S(T^4-T_o^4)$ respectively. An energy balance on this element gives the fin equation

$$\rho C_p \frac{\partial T}{\partial t}\delta x + q_k + q_r + \frac{\partial q_k}{\partial x}\delta x = 0$$

or

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p} \cdot \frac{1}{A} \frac{\partial}{\partial x}\left(A\frac{\partial T}{\partial x}\right) - \frac{H}{\rho C_p} \frac{P}{A}(T-T_o) - \frac{\sigma\epsilon}{\rho C_p} \frac{P}{A}(T^4-T_o^4) \tag{1}$$

where P = dS/dx is the perimeter. The boundary conditions are obtained by equating the heat transfer rates at the root and tip respectively, i.e.,

1. $\frac{\partial T}{\partial x}(x_o,t) = \frac{h}{k}\{T(x_o,t) - T_1(t)\}$

2. $\frac{\partial T}{\partial x}(L,t) = -\frac{H}{k}\{T(L,t) - T_o\} - \frac{\sigma\epsilon}{k}\{T^4(L,t) - T_o^4\}$ (2)

and the initial condition is

3. $T(x,o) = Z(x)$

The class of extended surfaces which will be discussed are those with

$$A = B_1 x^m \qquad P = \frac{dS}{dx} = \frac{}{\sqrt{1+y'^2}} = B_2 x^n \tag{3}$$

where the values of $B_1$, $B_2$, m and n depend on the type of extended surface considered. In some cases when the cross section and the perimeter cannot be written in the form of Equation (3), the origin of the coordinate can be taken at the tip of the fin by simply rearranging the boundary conditions. Some examples of extended surfaces which satisfy Equation (3) are shown in Appendix I.

Upon substitution of Equation (3) into Equation (1), the result is

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{m}{x}\frac{\partial T}{\partial x}\right) - \frac{H}{\rho C_p}\frac{P}{A}(T-T_o) - \frac{\sigma\epsilon}{\rho C_p}\frac{P}{A}(T^4-T_o^4) \tag{4}$$

The introduction of the dimensionless parameters

$$\xi = \frac{x-x_o}{L-x_o} \qquad \omega = \frac{x_o}{L} \qquad \theta = \frac{T}{T_o} \tag{5}$$

and

$$x = (L-x_o) + x_o \qquad \text{or} \qquad \frac{x}{L} = (1-\omega)\xi + \omega$$

into Equation (4) yields

$$T_o \frac{\partial \theta}{\partial t} = \frac{k}{\rho C_p} \left[ \frac{1}{(L-x_o)\xi + x_o} \frac{T_o}{L-x_o} \frac{\partial \theta}{\partial \xi} + \frac{T_o}{(L-x_o)^2} \frac{\partial^2 \theta}{\partial \xi^2} \right]$$

$$- \frac{HT_o}{\rho C_p} \frac{P}{A} (\theta - 1) - \frac{\sigma \epsilon T_o^4}{C_p} \frac{P}{A} (\theta^4 - 1)$$

The multiplication by $\rho C_p L(L-x_o)/kT_o$ on both sides gives

$$\frac{\rho C_p L^2}{k} (1-\omega) \frac{\partial \theta}{\partial t} = \left[ \frac{1}{(1-\omega)\xi + \omega} \frac{\partial \theta}{\partial \xi} + \frac{1}{1-\omega} \frac{\partial^2 \theta}{\partial \xi^2} \right]$$

$$- \frac{PL}{A}(1-\omega) \left[ \frac{HL}{k}(\theta - 1) + \frac{\sigma \epsilon L T_o^3}{k} (\theta^4 - 1) \right] \qquad (6)$$

More dimensionless parameters are defined as

$$\tau = \frac{kt}{\rho C_p L^2}, \qquad N_B = \frac{HL}{k} \qquad N_r = \frac{\sigma \epsilon L T_o^3}{k} \qquad (7)$$

It can easily be realized that PL/A is dimensionless and

$$\frac{PL}{A} = \frac{B_2 x^n}{B_1 x^m} L = \frac{B_2 L^{n-m+1}}{B_1} \left(\frac{x}{L}\right)^{n-m} = F\left(\frac{x}{L}\right)^{n-m} \qquad (8)$$

where $F = (B_2/B_1)L^{n-m+1}$ is dimensionless also. Equation (6) can be rewritten in terms of $\tau$, $N_B$, $N_r$, and F as

$$(1-\omega)\frac{\partial \theta}{\partial t} = \left[ \frac{m}{(1-\omega)\xi + \omega} \frac{\partial \theta}{\partial \xi} + \frac{1}{1-\omega} \frac{\partial^2 \theta}{\partial \xi^2} \right]$$

$$-F \left[ (1-\omega)\xi + \omega \right]^{n-m} (1-\omega) \left[ N_B(\theta - 1) + N_r(\theta^4 - 1) \right] \qquad (9)$$

Boundary and initial conditions are now

$$1. \quad \frac{\partial \theta}{\partial \xi}(0,\tau) = N_{BA}\,(1-\omega)\left[\theta(0,\tau) - \theta_1(\tau)\right], \quad N_{BA} = \frac{hL}{k}$$

$$2. \quad \frac{\partial \theta}{\partial \xi}(1,\tau) = -(1-\omega)\left\{N_B\left[\theta(1,\tau) - 1\right] + N_r\left[\theta^4(1,\tau) - 1\right]\right\} \qquad (10)$$

and $\theta(\xi, 0) = Z(\xi)$, $Z(\xi) = z(x)/T_o$.

Thus, the solution of Equations (9) and (10) is of the form

$$\theta = G(\xi, \tau, \omega, F, m, n, N_B, N_r, N_{BA}) \qquad (11)$$

## 2. Finite Difference Formulation of the Fin Equation

If the fin is divided into q - 2 parts (Fig. 3), with the end points designated by the subscripts 2 and q, then the equivalent formulation of Equations (9) and (10) in finite difference form is

$$(1-\omega)\frac{\theta_i' - \theta_i}{\Delta\tau} = \left[\frac{m}{(1-\omega)\xi_i + \omega}\,\frac{\theta_{i+1} - \theta_{i-1}}{2\cdot\Delta\xi} + \frac{1}{1-\omega}\,\frac{\theta_{i+1} + \theta_{i-1} - 2\theta_i}{(\Delta\xi)^2}\right]$$

$$-F\left[(1-\omega)\xi_i + \omega\right]^{n-m}(1-\omega)\left[N_B(\theta_i - 1) + N_r(\theta_i^4 - 1)\right] \qquad (12)$$

$$i = 3, 4, \ldots (q-1)$$

where $\xi_i = (i-2)\cdot\Delta x$ and

$$1. \quad \frac{\theta_3 - \theta_2}{\Delta\xi} = N_{BA}\,(1-\omega)\cdot(\theta_2 - \theta_1)$$

$$2. \quad \frac{\theta_q - \theta_{q-1}}{\Delta\xi} = -(1-\omega)\left[N_B(\theta_q - 1) + N_r(\theta_q^4 - 1)\right] \qquad (13)$$

3. $\theta_i = Z_i$ at $\tau = 0$, with $i = 2, 3, \ldots q$

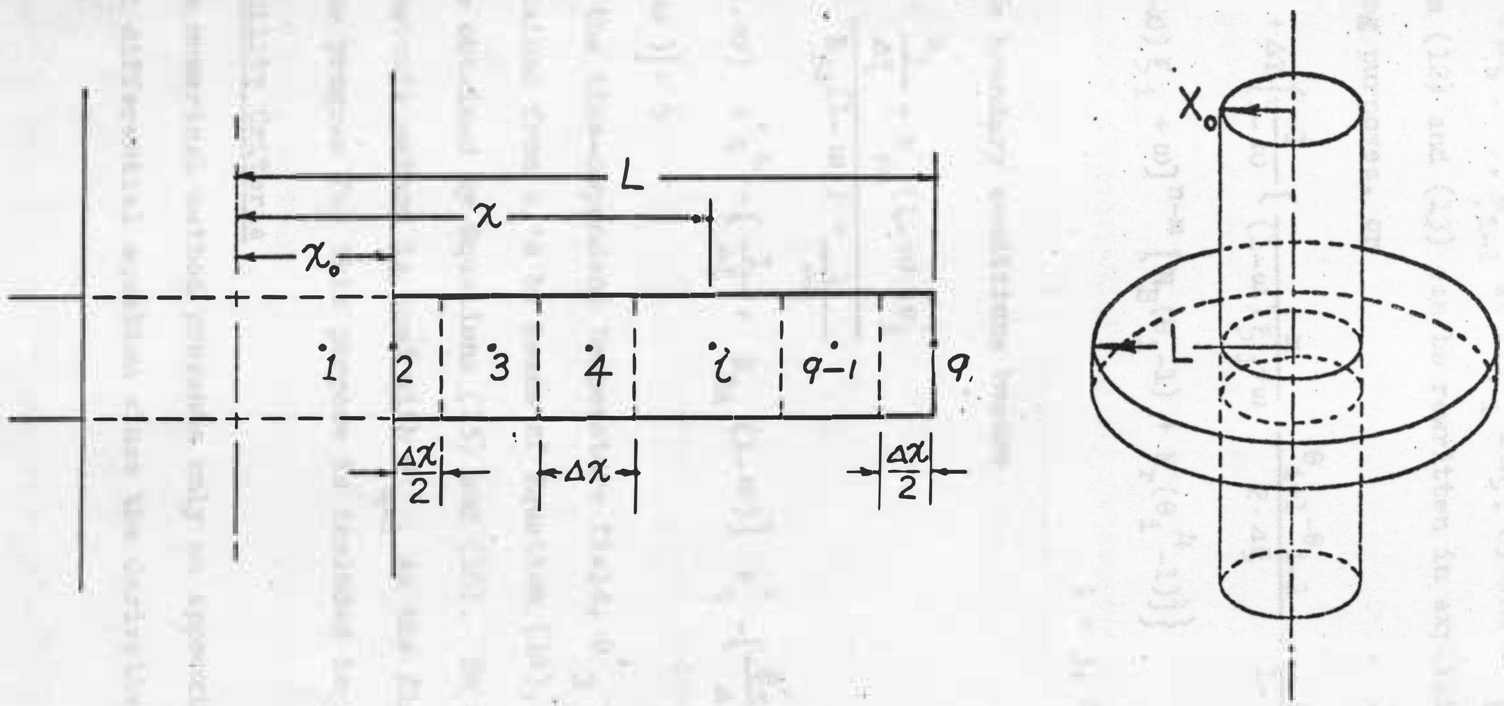


Fig. 3  Coordinate system and subdivision for an annular fin

In the formulation above, $\theta_i$ is the value of $\theta$ at $(\xi_i, \tau)$, $\theta_{i+1}$ at $(\xi_i + \Delta\xi, \tau)$, $\theta_{i-1}$ at $(\xi_i - \Delta\xi, \tau)$ and $\theta'_i$ at $(\xi_i, \tau + \Delta\tau)$. Equations (12) and (13) can be rewritten in explicit form for programing purposes, or

$$\theta'_i = \theta_i + \Delta\tau \left\{ \frac{1}{1-\omega} \left[ \frac{m}{(1-\omega)\xi_i + \omega} \frac{\theta_{i+1} - \theta_{i-1}}{2 \cdot \Delta\xi} + \frac{1}{1-\omega} \frac{\theta_{i+1} + \theta_{i-1} - 2\theta_i}{(\Delta\xi)^2} \right] \right.$$

$$\left. - F \left[ (1-\omega)\xi_i + \omega \right]^{n-m} \left[ N_B(\theta_i - 1) + N_r(\theta_i^4 - 1) \right] \right\} \quad (14)$$

$$i = 3, 4, \ldots. (q-1)$$

Also, the boundary conditions become

1. $$\theta'_2 = \frac{\frac{\theta'_3}{\Delta\xi} + N_{BA}(1-\omega)\theta'_i}{N_{BA}(1-\omega) + \frac{1}{\Delta\xi}} \quad (15)$$

2. $$N_r(1-\omega)\ \theta'^4_q + \left[ \frac{1}{\Delta\xi} + N_{BA}(1-\omega) \right] \theta'_q - \left[ \frac{\theta'_{q-1}}{\Delta\xi} + (N_B + N_r)(1-\omega) \right] = 0 \quad (16)$$

To find the time-dependent temperature field, $\theta'_3$ through $\theta'_{q-1}$ were obtained from $\theta_i$'s by means of Equation (14), then $\theta'_2$ and $\theta'_q$ were obtained by Equations (15) and (16). In solving Equation (16), Newton's method is used with $\theta'_{q-1}$ as the first estimate for $\theta'_q$. The program for this purpose is included in Appendix II.

3. Stability Criteria

The numerical method presents only an approximate solution to the original differential equation since the derivatives are replaced by finite differences. A truncation error is introduced by the use of finite subdivision, and the numerical error is due mainly to the accumulation of round-off errors. As the increments are taken smaller and smaller, the numerical results approach the corresponding exact values more closely.

For non-steady numerical problems solved explicitly, the stability condition of the differential equation must be considered. Stability is the condition under which the truncation and numerical errors introduced at one point in time either damp out or increase in amplitude with time.

From physical reasoning the higher the $\theta_i$, the higher the $\theta'_i$, thus the derivative of $\theta'_i$ with respect to $\theta_i$ must be positive. From Equation (14),

$$\frac{d\theta'_i}{d\theta_i} = 1 + \Delta\tau\left\{\frac{-2}{(1-\omega)^2}\,\frac{1}{(\Delta\xi)^2} - F\left[(1-\omega)\xi_i + \omega\right]^{n-m}\left[N_B + 4N_r\theta_i^{\,3}\right]\right\} > 0 \quad (17)$$

Notice the following inequalities,

$$\theta_1 > \theta_i > 1 \qquad (18)$$

$$0 < \omega < (1-\omega)\xi_i + \omega < 1 \qquad i = 2, 3, \ldots\ldots q$$

and depending upon the values of m and n, one of the following conditions will hold

$$\text{If } n < m \text{, then} \qquad \omega^{n-m} > \left[(1-\omega)\xi_i + \omega\right]^{n-m} > 1$$

$$\text{If } n = m \text{, then} \qquad \left[(1-\omega)\xi_i + \omega\right]^{n-m} = 1 \qquad (19)$$

$$\text{If } n > m \text{, then} \qquad \omega^{n-m} < \left[(1-\omega)\xi_i + \omega\right]^{n-m} < 1$$

For n < m,

$$1 + \Delta\tau\left\{\frac{-2}{(1-\omega)^2}\ \frac{1}{\Delta\xi} - F\left[(1-\omega)\xi_i + \omega\right]^{n-m}(N_B + 4N_r\theta_i^3)\right\}$$

$$> 1 + \Delta\tau\left\{\frac{-2}{(1-\omega)^2}\ \frac{1}{\Delta\xi} - F\omega^{n-m}(N_B + 4N_r\theta_i^3)\right\} > 0$$

Therefore, the stability condition can be established from the last inequality as

$$\Delta\tau < \frac{1}{\frac{2}{(1-\omega)^2}\ \frac{1}{(\Delta\xi)^2} + F\omega^{n-m}(N_B + 4N_r\theta_1^3)} \tag{20}$$

and similarly, for $n \geq m$

$$\Delta\tau < \frac{1}{\frac{2}{(1-\omega)^2}\ \frac{1}{(\Delta\xi)^2} + F(N_B + 4N_r\theta_1^3)} \tag{21}$$

4. Fin Efficiency

The fin efficiency $\eta$ is defined as the ratio of the actual heat transfer from the extended surface to the heat transfer from the extended surface if the whole surface were at base temperature, or mathematically,

$$\eta = \frac{\int_S \left[H(T-T_o) + \sigma\epsilon(T^4-T_o^4)\right] dS}{\left\{H\left[T(x_o,t) - T_o\right] + \sigma\epsilon\left[T^4(x_o,t) - T_o^4\right]\right\} S} \tag{22}$$

If the surface is divided in the way shown by dotted lines in Figure 3, the finite difference form of Equation (22) is

$$\eta = \Big\{ P_2 \frac{\Delta x}{2} \left[ H\,(T_2 - T_o) + \sigma\epsilon (T_2^{\,4} - T_o^{\,4}) \right] + \sum_{i=3}^{q-1} P_i \Delta x \left[ H(T_i - T_o) \right.$$

$$\left. + \sigma\epsilon\,(T_i^{\,4} - T_o^{\,4}) \right] + (P_q \frac{x}{2} + A_q) \left[ H\,(T_q - T_o) + \sigma\epsilon\,(T_q^{\,4} - T_o^{\,4}) \right] \Big\} \quad (23)$$

$$\Big\{ \left[ H(T_2 - T_o) + \sigma\epsilon (T_2^{\,4} - T_o^{\,4}) \right] \left[ P_2 \frac{\Delta x}{2} + \sum_{i=3}^{q-1} P_i \Delta x + P_q \frac{\Delta x}{2} + A_q \right] \Big\}^{-1}$$

When both denominator and numerator are divided by $kT_o \Delta x B_2 L^{n-1}$ and with

$$\frac{A_2}{\Delta x B_2 L^n} = \frac{B_1 L^m}{B_2 L^n} \frac{1}{\Delta x} = \frac{B_1}{B_2 L^{n-m+1}} \frac{L}{\Delta x} = \frac{q-2}{F} \quad (24)$$

Equation (23) can be rewritten in terms of dimensionless parameters as

$$\eta = \Big\{ \frac{\omega^n}{2} \left[ N_B(\theta_2 - 1) + N_r(\theta_2^{\,4} - 1) \right] + \sum_{i=3}^{q-1} \left[ (1-\omega)\xi_i + \omega \right]^n \left[ N_B(\theta_i - 1) \right.$$

$$\left. + N_r(\theta_i^{\,4} - 1) \right] + \left( \frac{1}{2} + \frac{q-2}{F} \right) \left[ N_B(\theta_q - 1) + N_r(\theta_q^{\,4} - 1) \right] \Big\} \quad (25)$$

$$\Big\{ \left[ N_B(\theta_2 - 1) + N_r(\theta_q^{\,4} - 1) \right] \left[ \frac{\omega^n}{2} + \sum_{i=3}^{q-1} \overline{(1-\omega\xi_i} + \omega)^n + \frac{1}{2} + \frac{q-2}{F} \right] \Big\}^{-1}$$

Since it is impractical to store the data for the time-dependent temperature, the program for efficiency is included in the program for temperature so that both results can be obtained simultaneously.

# CHAPTER III

## RESULTS AND DISCUSSION

To check the accuracy of the present solution, comparisons were made with two recent publications for steady-state heat transfer in a constant-area straight fin with convection and radiation at the surface. Figure 4 shows the comparison with the analytical and experimental results of Cobble (3) and Figure 5 is a comparison with the work of Sparrow and Nierwith (6). The discrepancy between the present numerical results and the numerical results of reference (6) is not large and certainly within engineering accuracy. Figure 6 gives the steady-state efficiency of the same fin with and without tip heat transfer. Because of the numerous combinations possible, a parametric study will be quite involved but should be done at a future date. It was decided to present curves for a typical range of values of the dimensionless parameters.

Figures 7 to 22 give the transient temperature and efficiency for annular fins and Figures 23 to 30 for straight fins. The effect of changing geometry on transient temperature and efficiency is given in Figures 31 to 36. The values are selected for typical applications in engineering practice.

A constant ambient temperature was assumed in Chapter II. If it were time-dependent, θ would be defined as

$$\theta = T(x,t) / T_o(0)$$

$$\theta_o = T_o(t) / T_o(0)$$

and ( $\theta - \theta_o$ ) would take the place of ( $\theta - 1$ ) and ( $\theta^4 - \theta_o^4$ ) for ( $\theta^4 - 1$ ) in all the formulas.

Since curves on Figure 7 through 36 were run for comparison purposes, the temperatures of the circulating fluid and the ambient fluid were assumed to be constant. In practical applications, the values of time-dependent temperatures should be substituted during calculation.

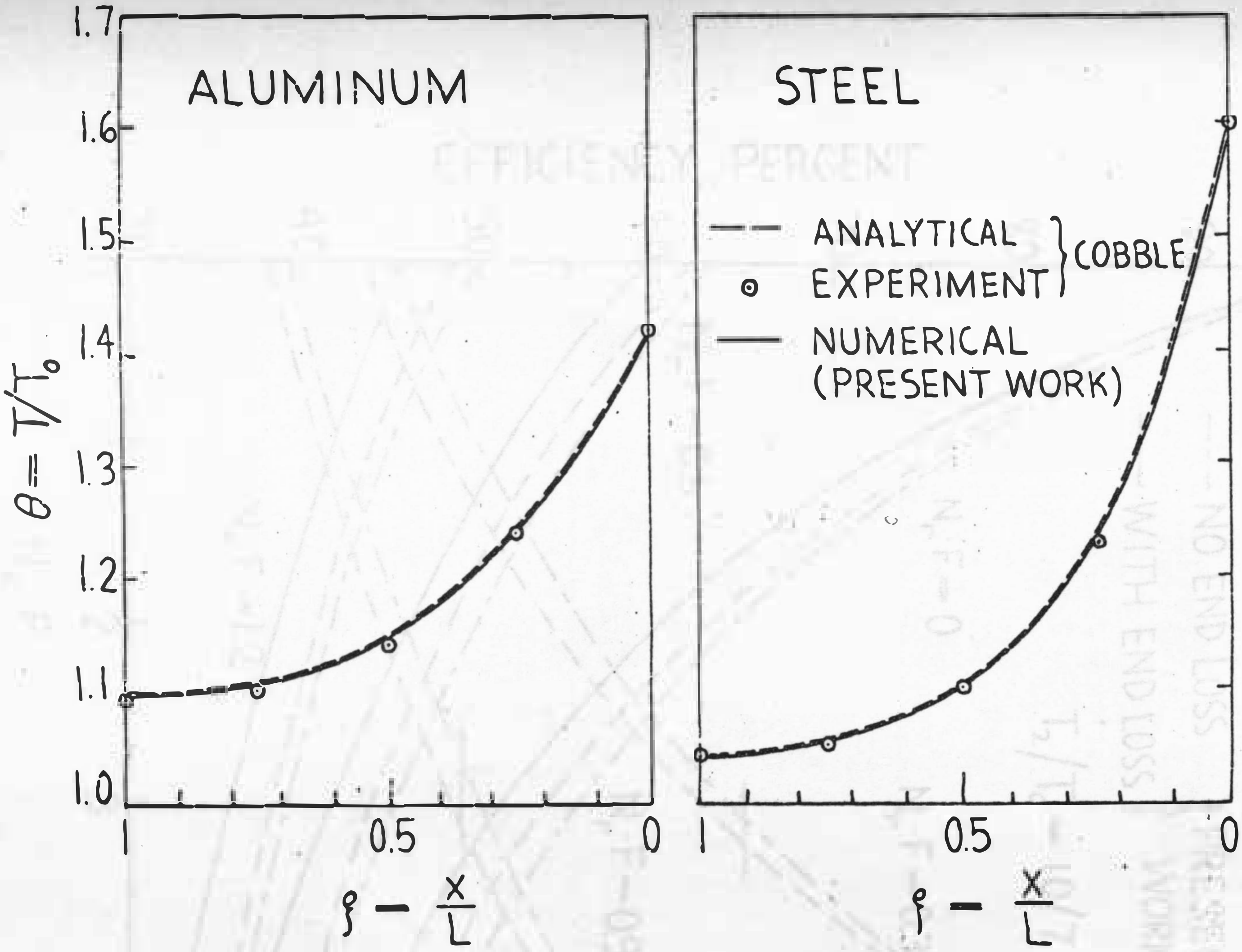


Fig. 4 Steady-state temperature distribution for straight steel and aluminum fins of constant area

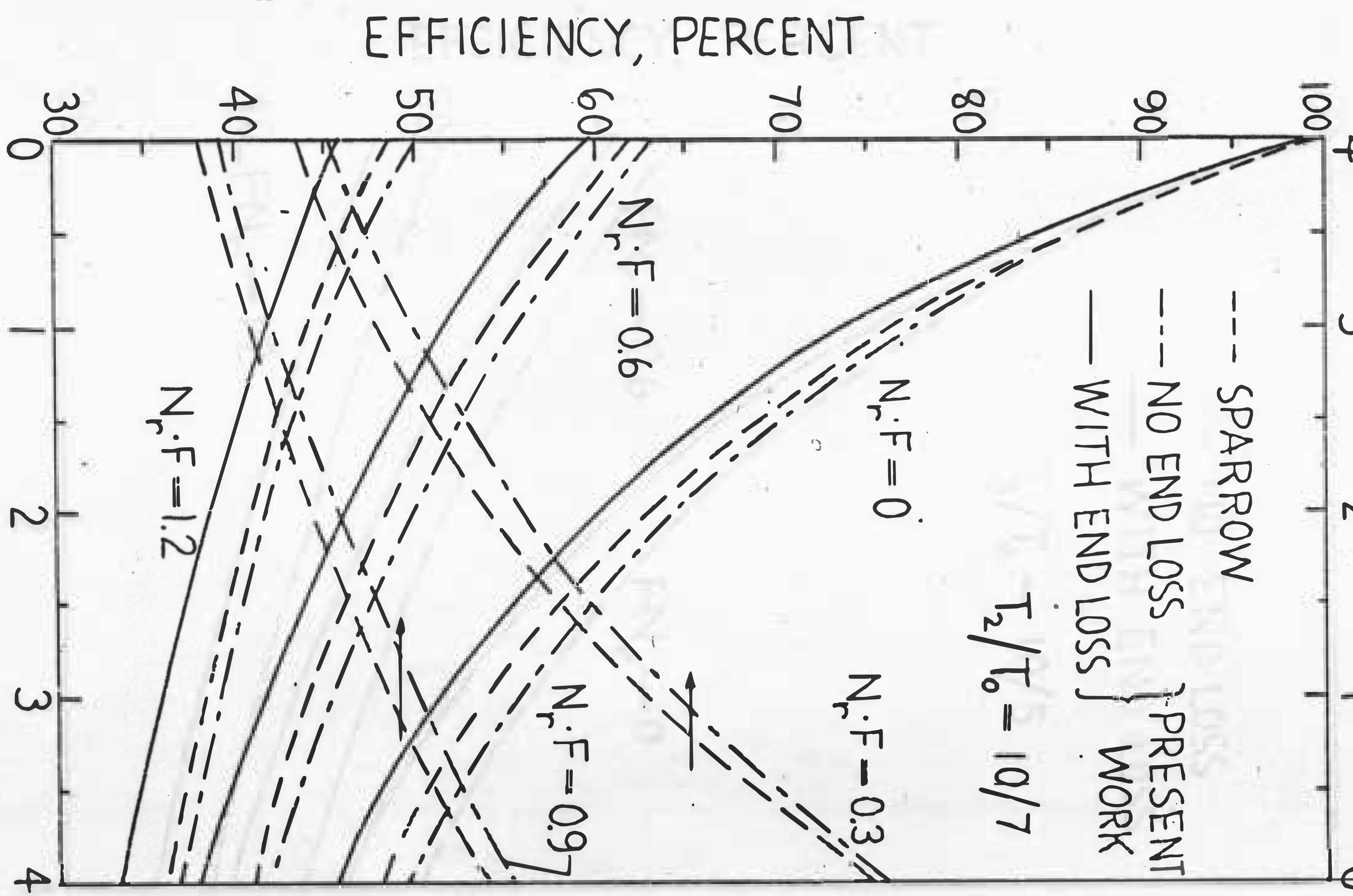


Fig. 5 Steady-state efficiency of convecting and radiating straight constant-area fins, with $\omega = 0$

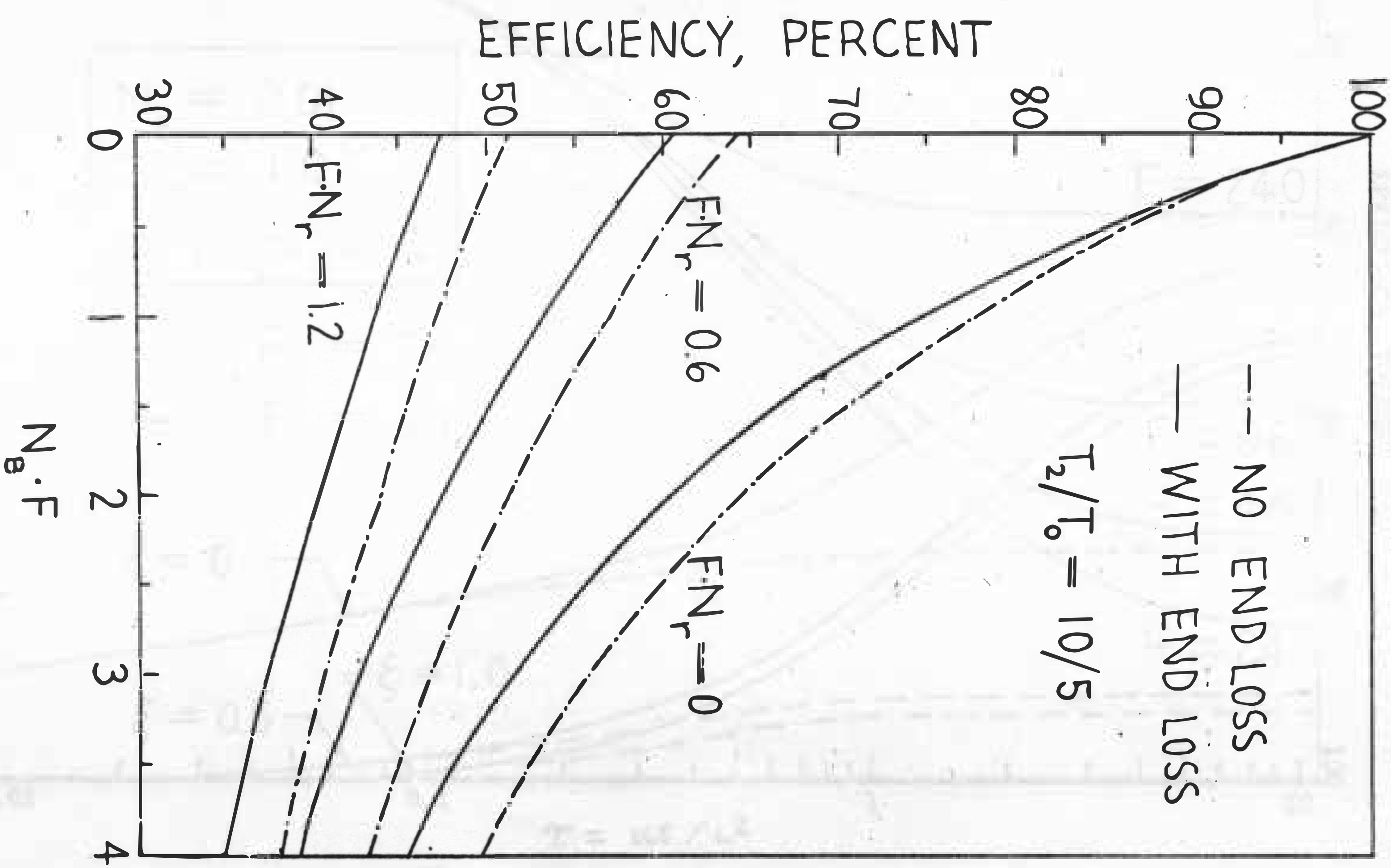


Fig. 6 Steady-state efficiency of convecting and radiating straight constant-area fins, with $\omega = 0$

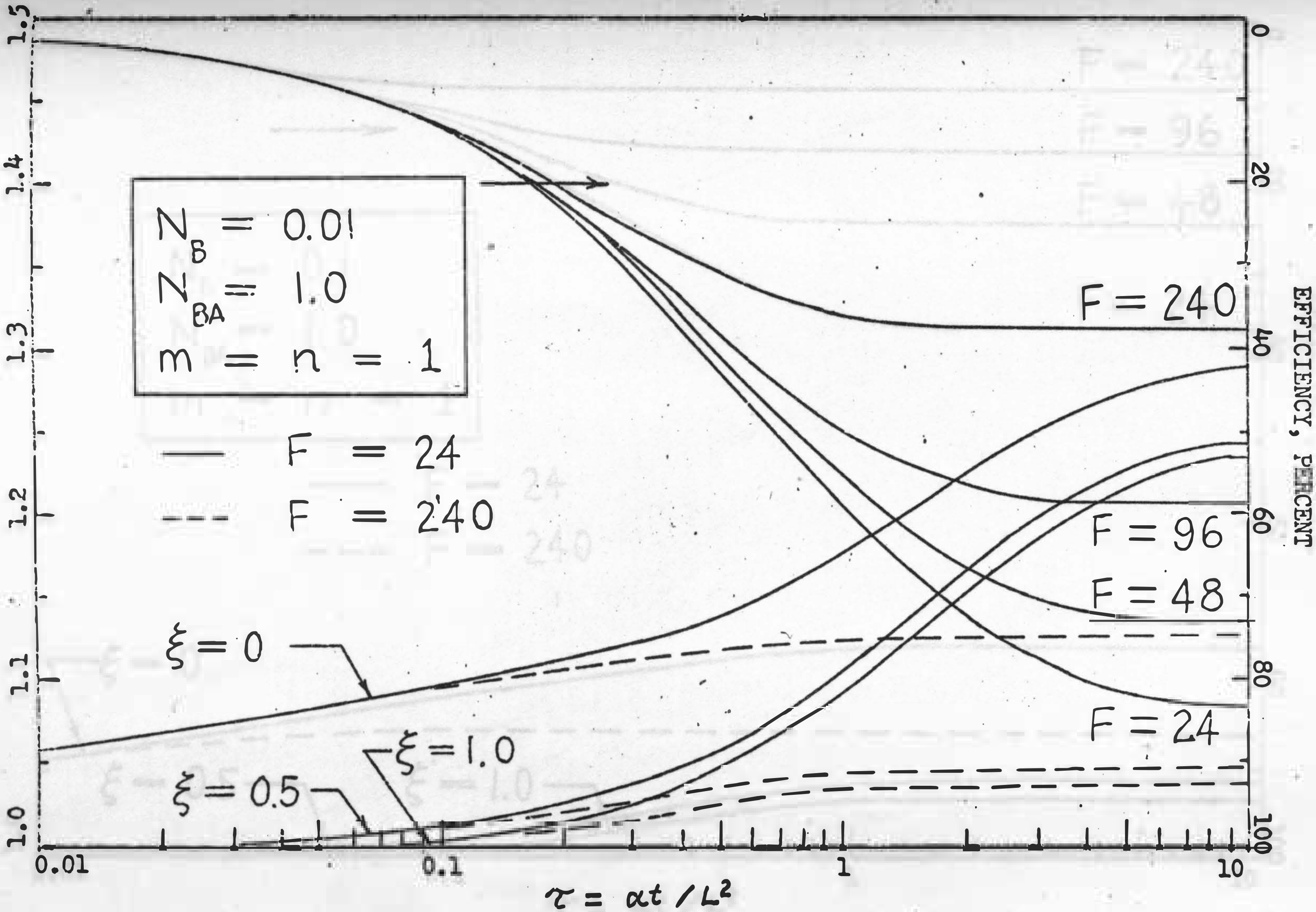


Fig. 7 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.1$, $N_r = 0.0001$

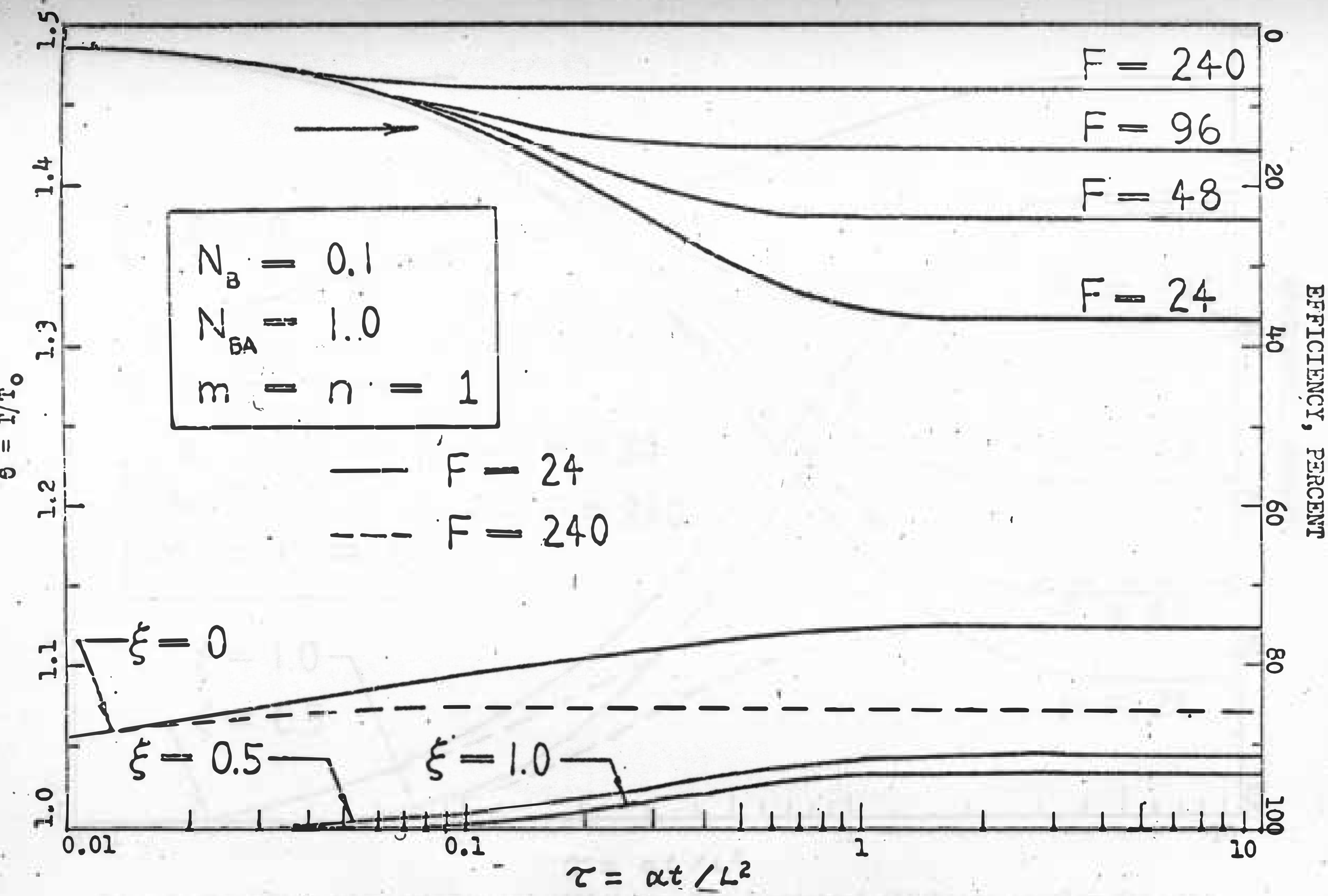


Fig. 8 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.1$, $N_r = 0.0001$

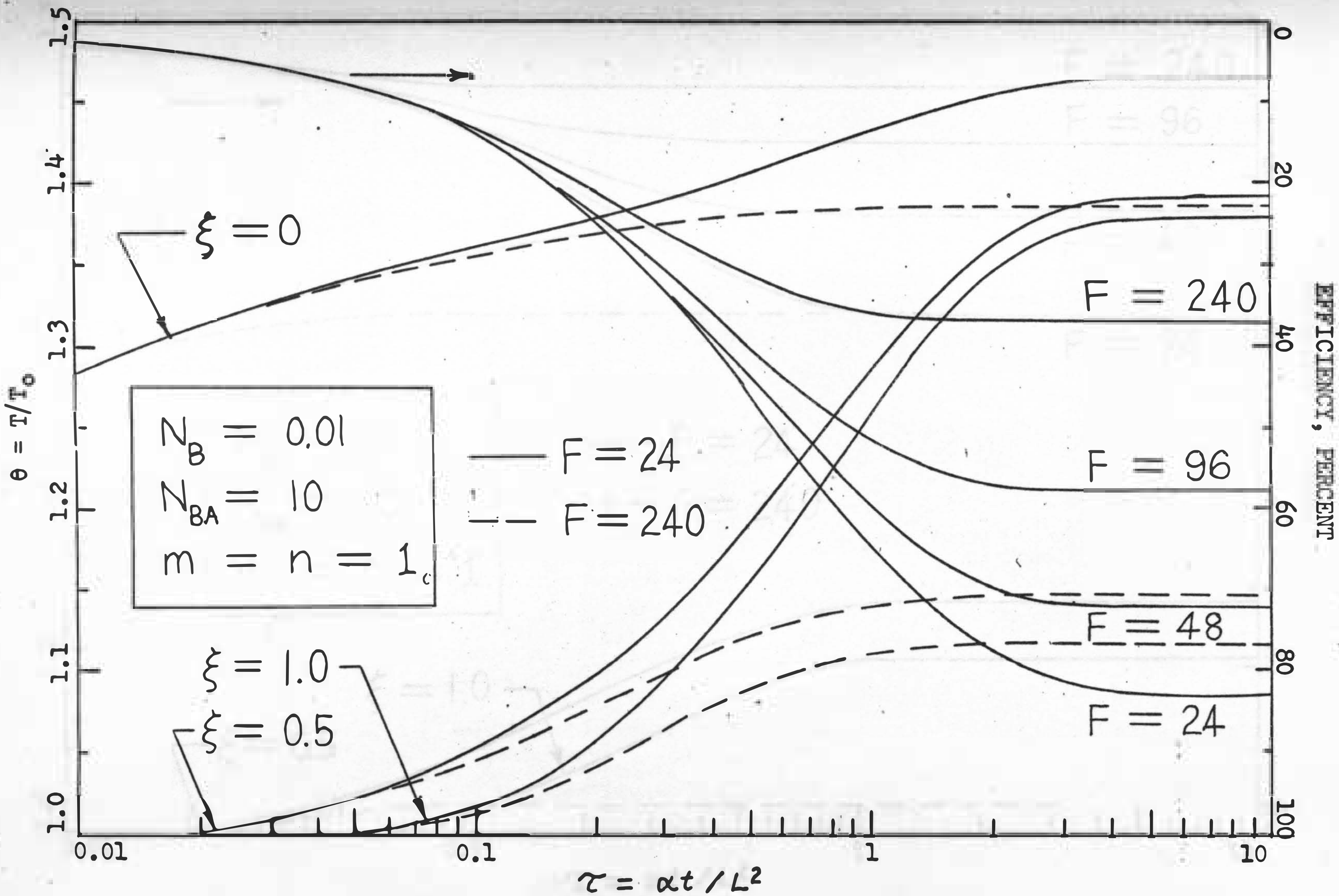


Fig. 9 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.1$, $N_r = 0.0001$

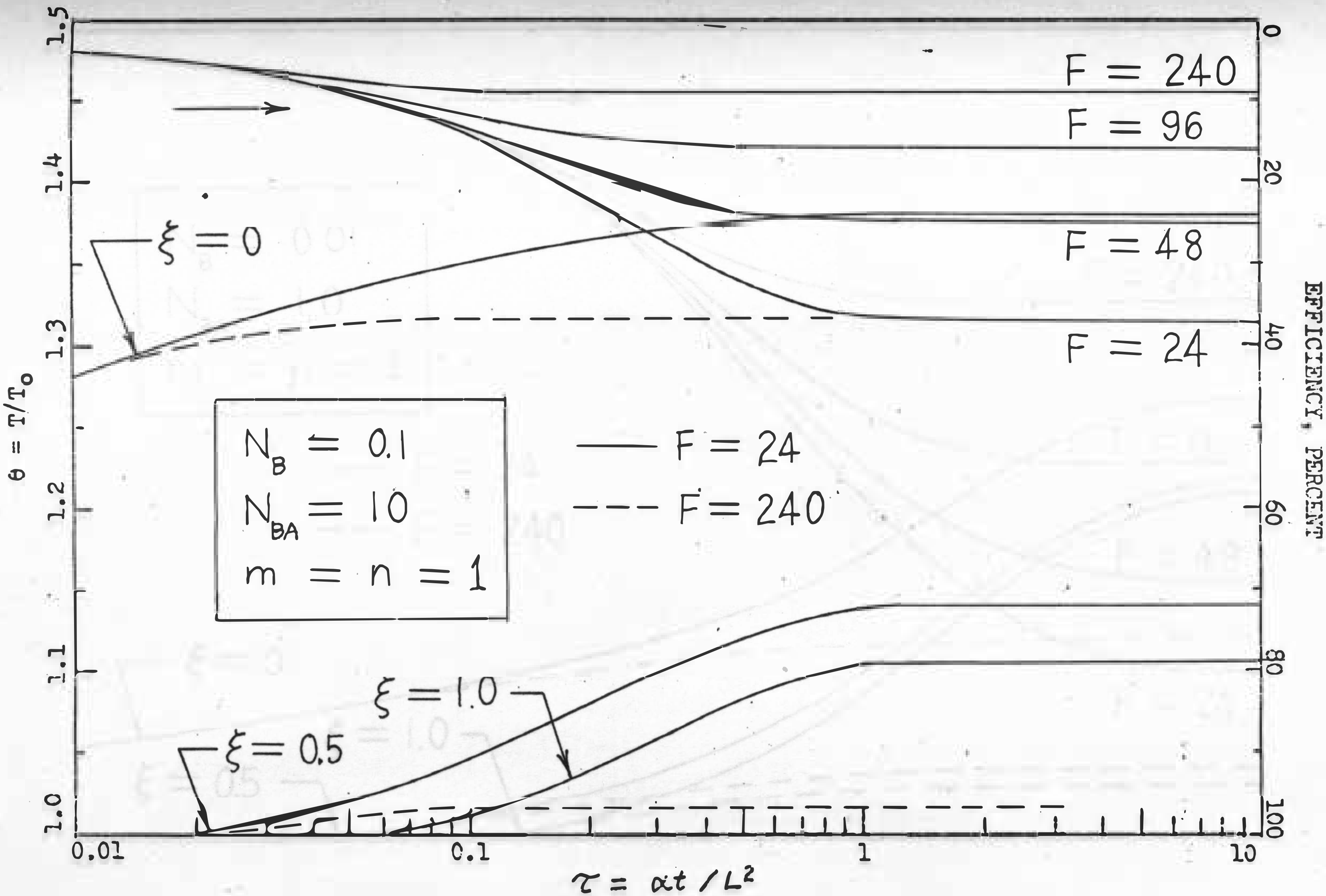


Fig. 10 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.1$, $N_r = 0.0001$

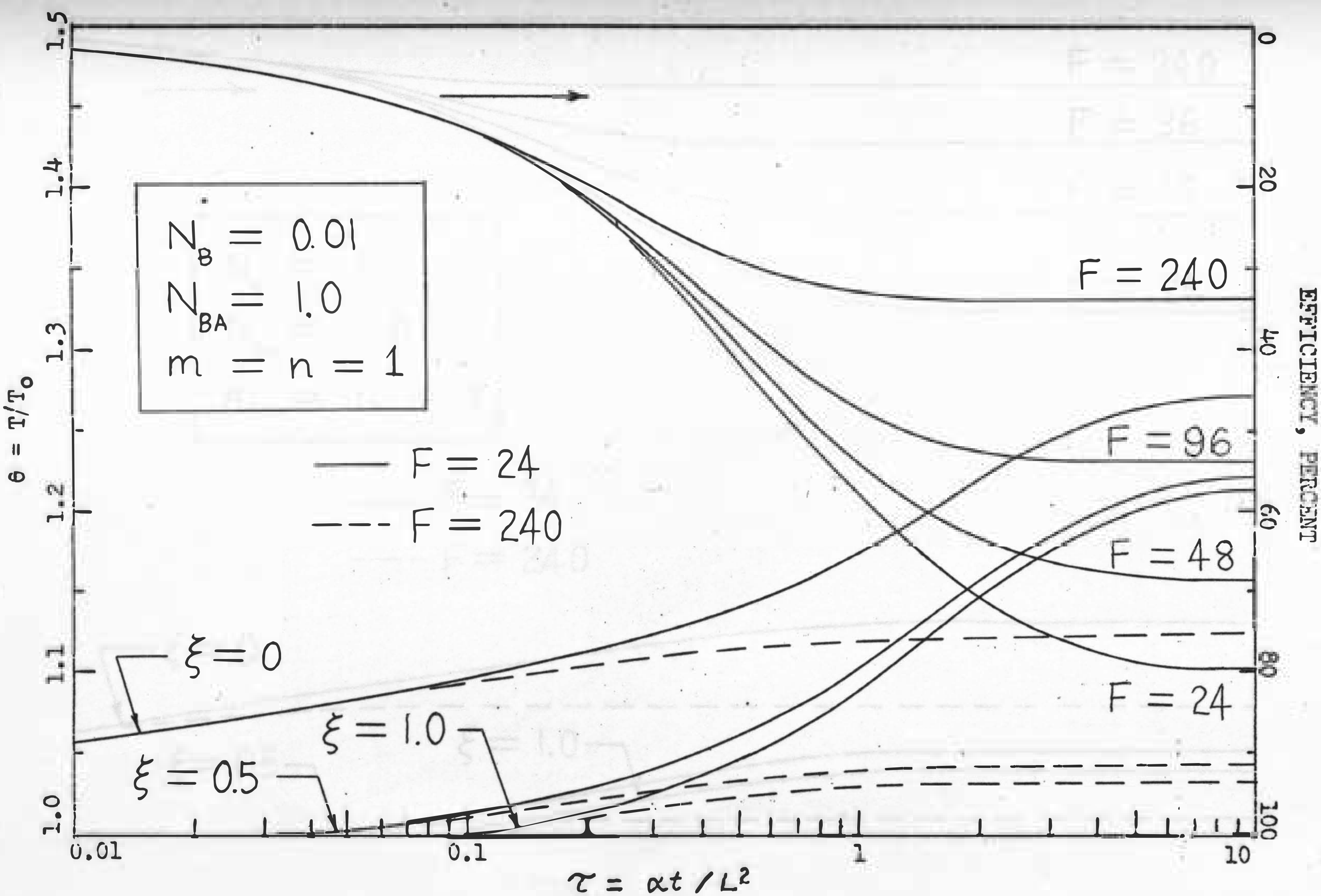


Fig. 11 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.1$, $N_r = 0.0005$

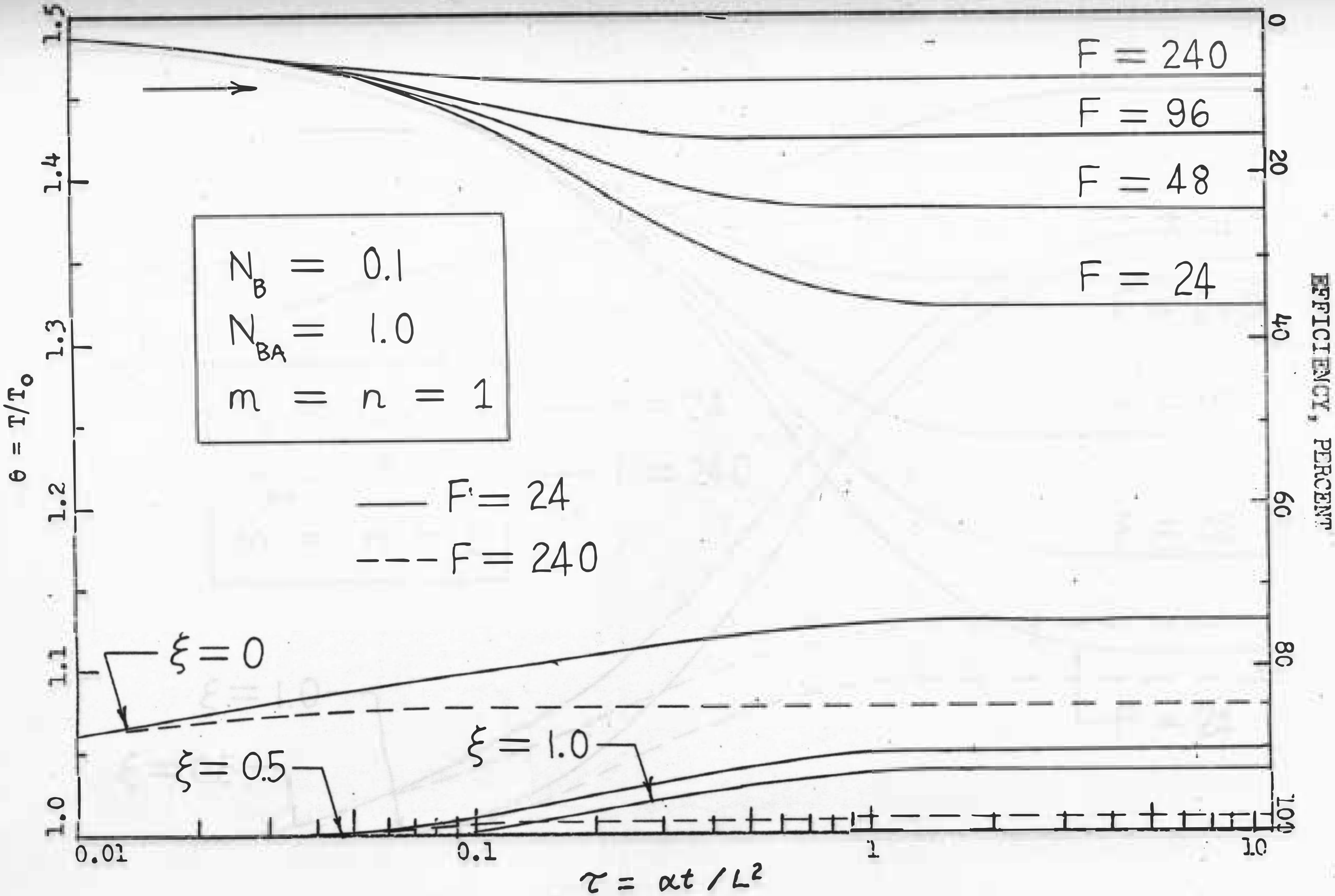


Fig. 12 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.1$, $N_r = 0.0005$

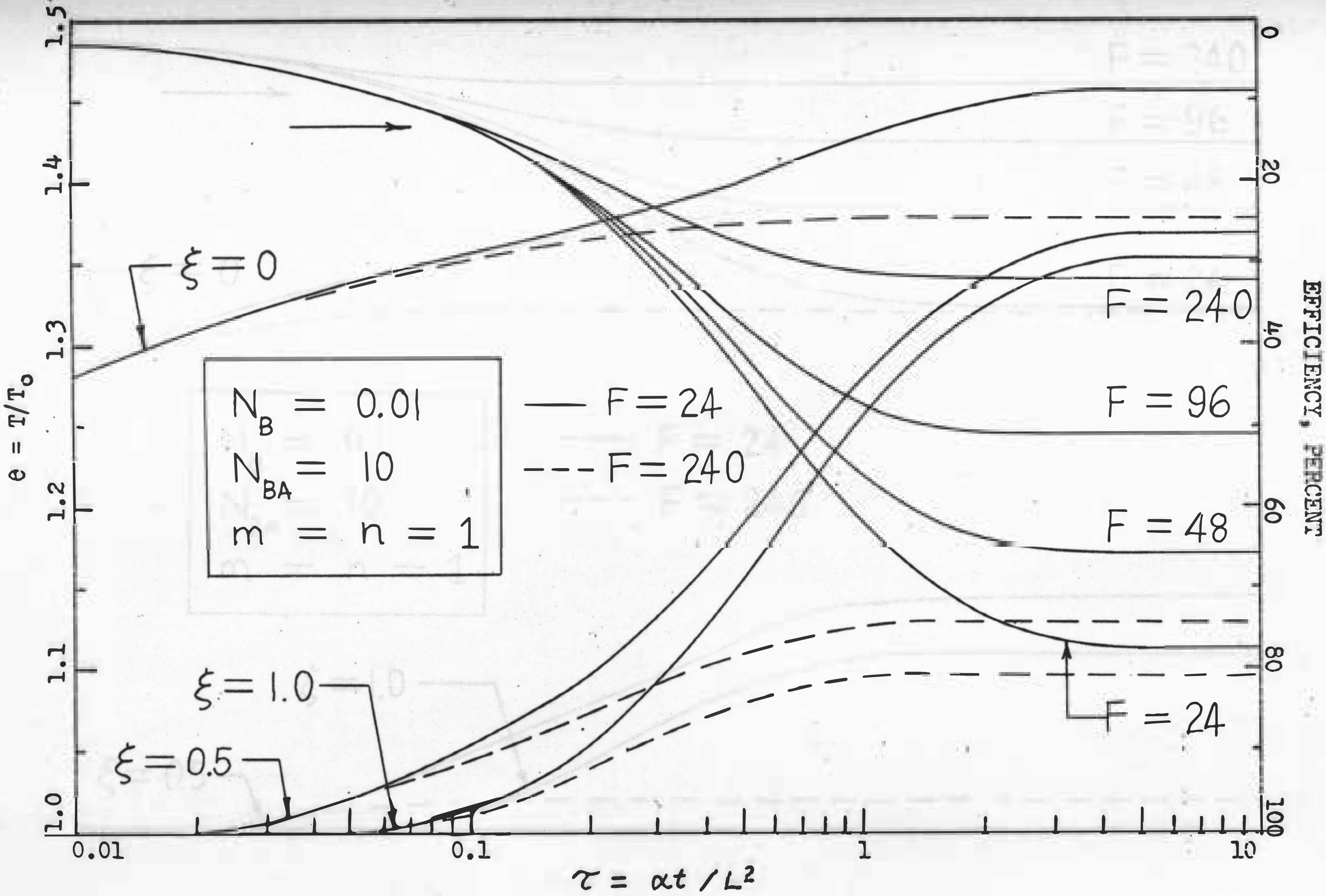


Fig. 13 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.1$, $N_r = 0.0005$

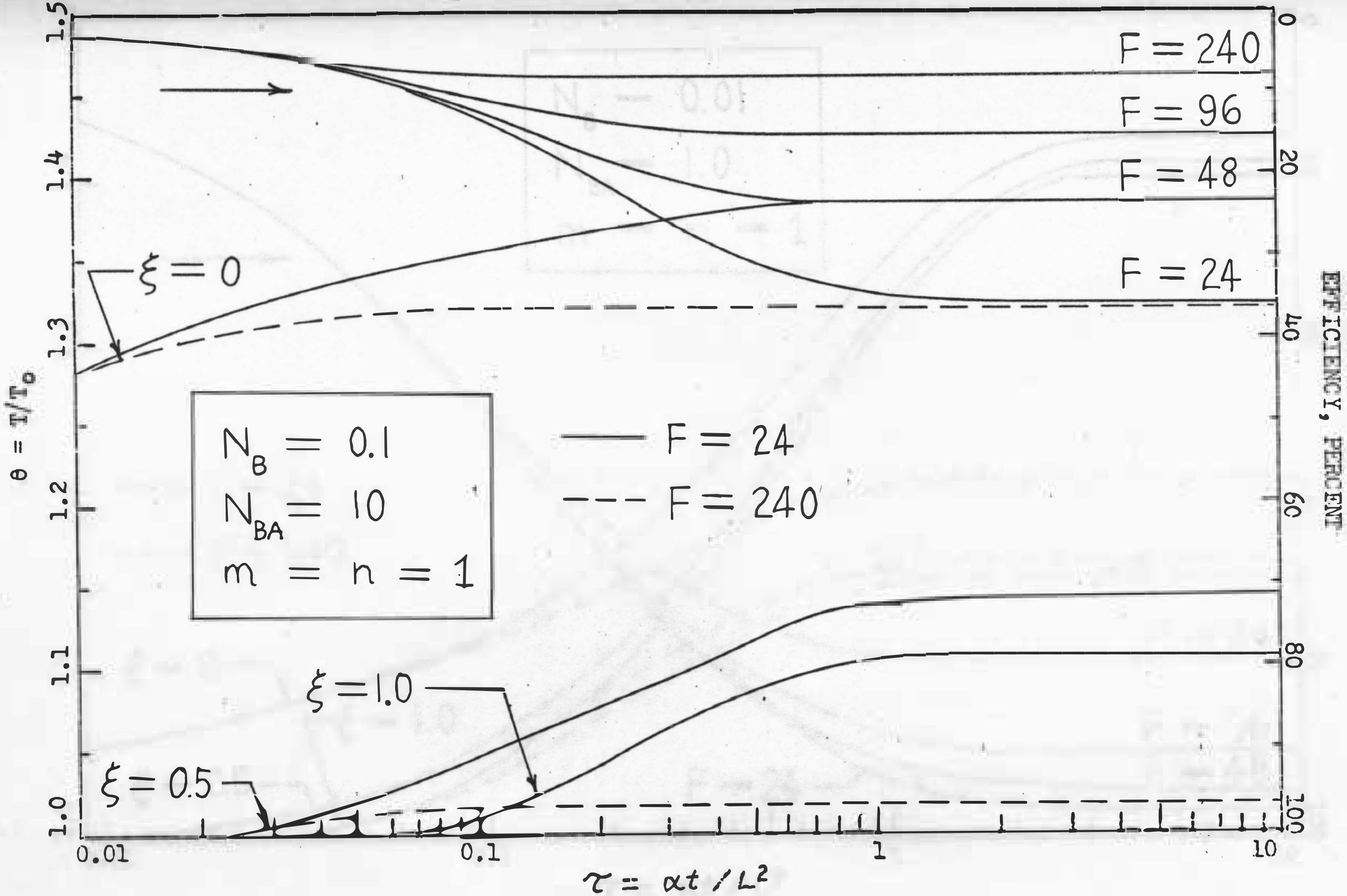


Fig. 14 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.1$, $N_r = 0.0005$

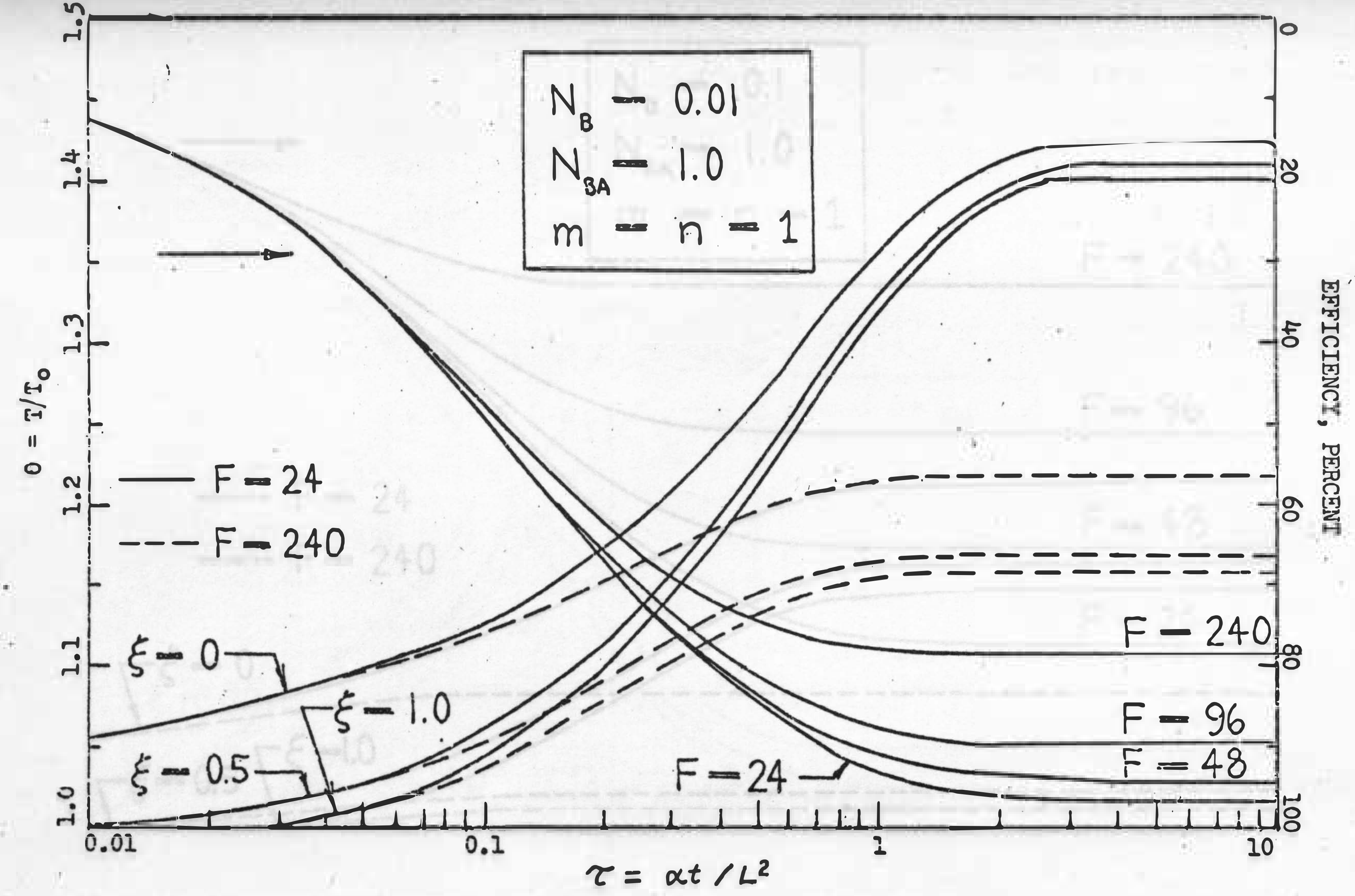


Fig. 15 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0001$

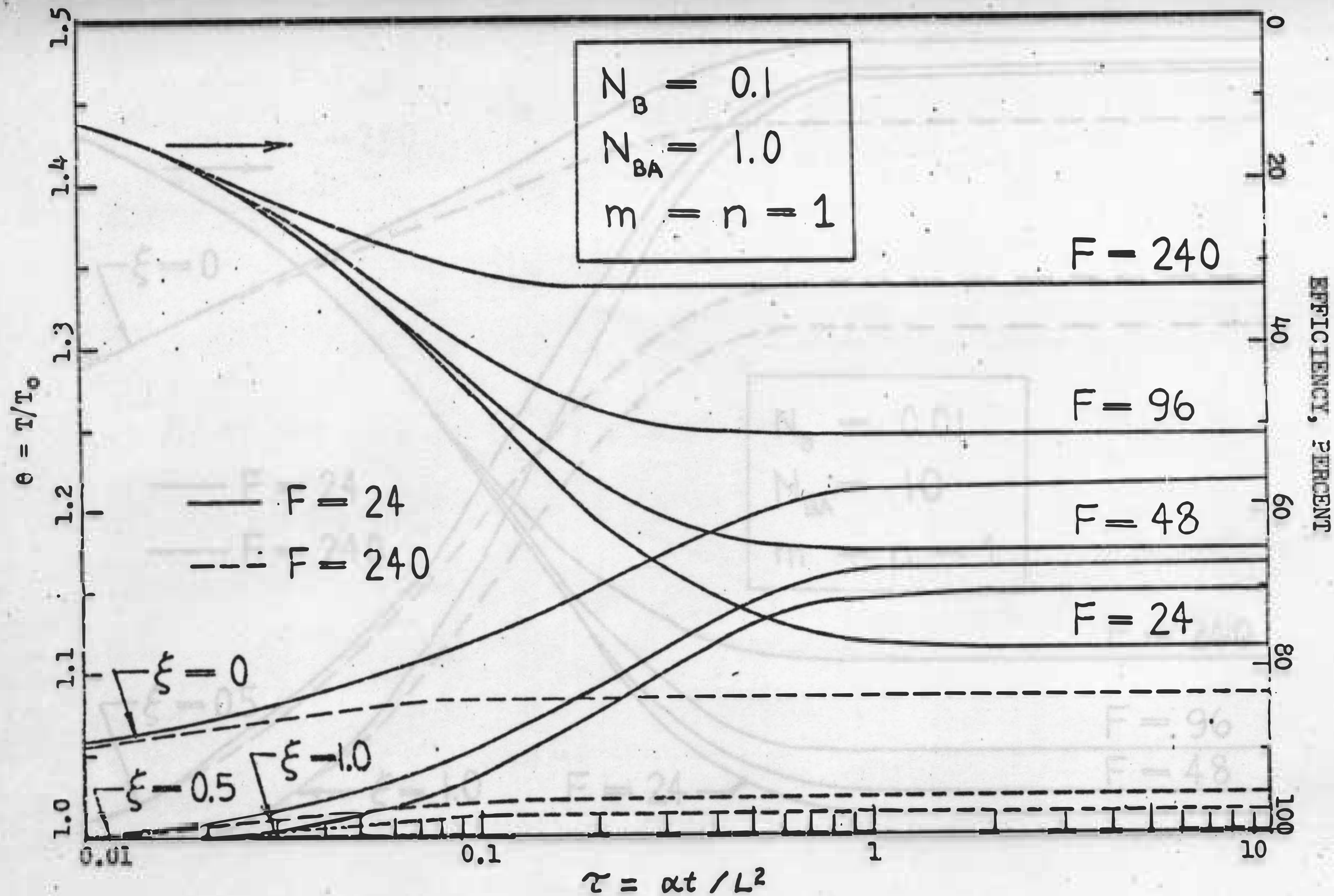


Fig. 16 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0001$

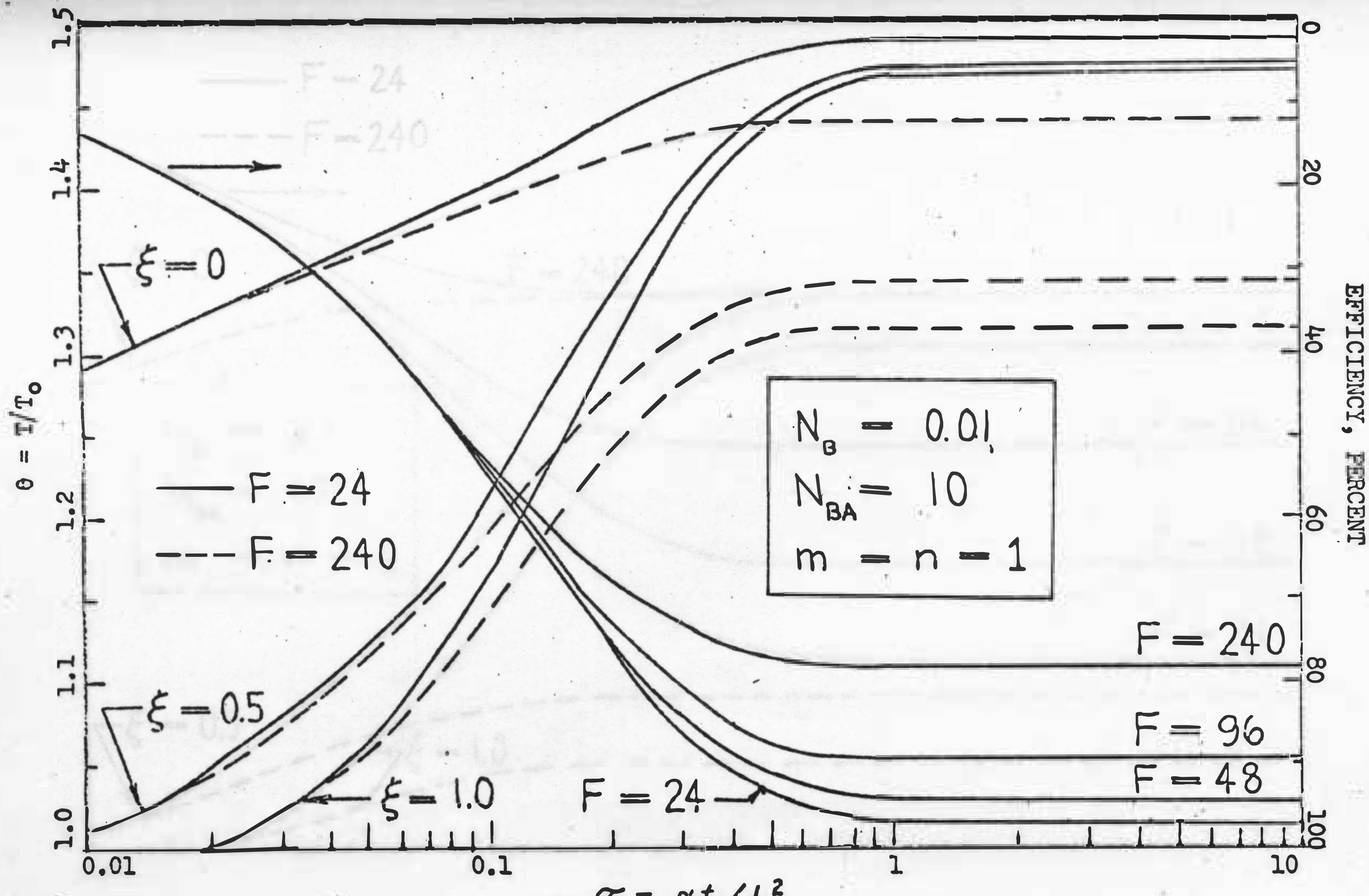


Fig. 17 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0001$

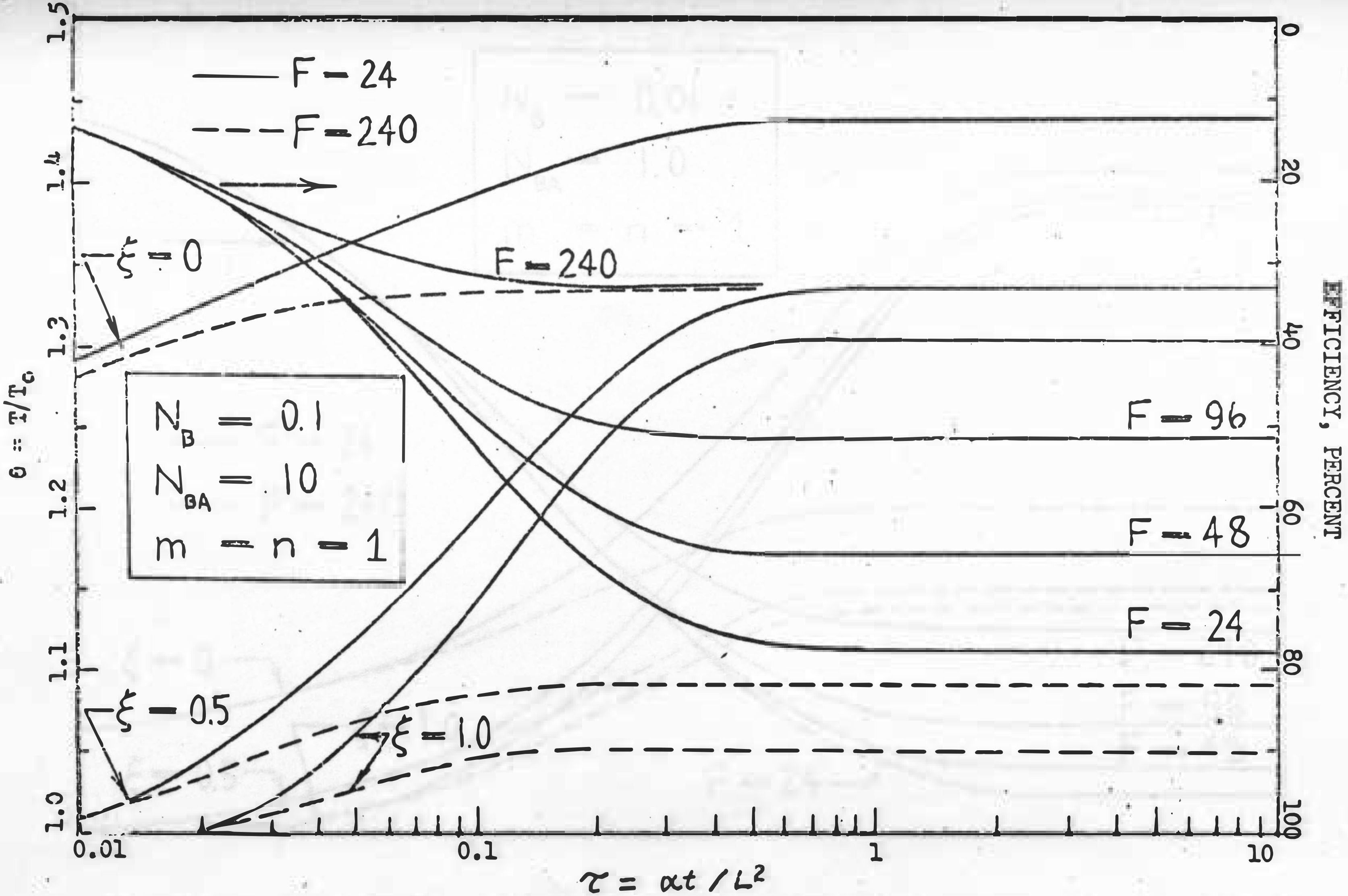


Fig. 18 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0001$

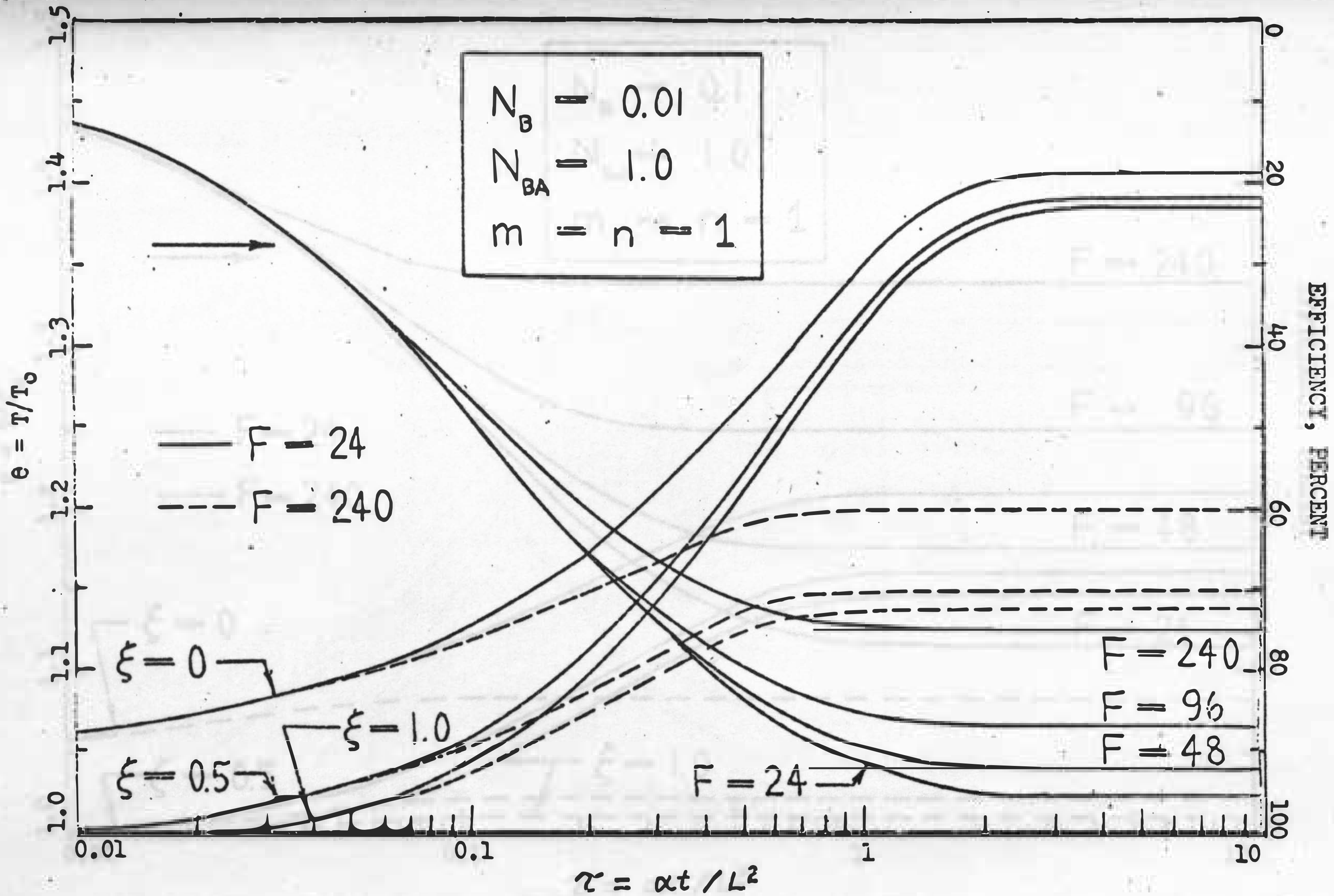


Fig. 19 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0005$

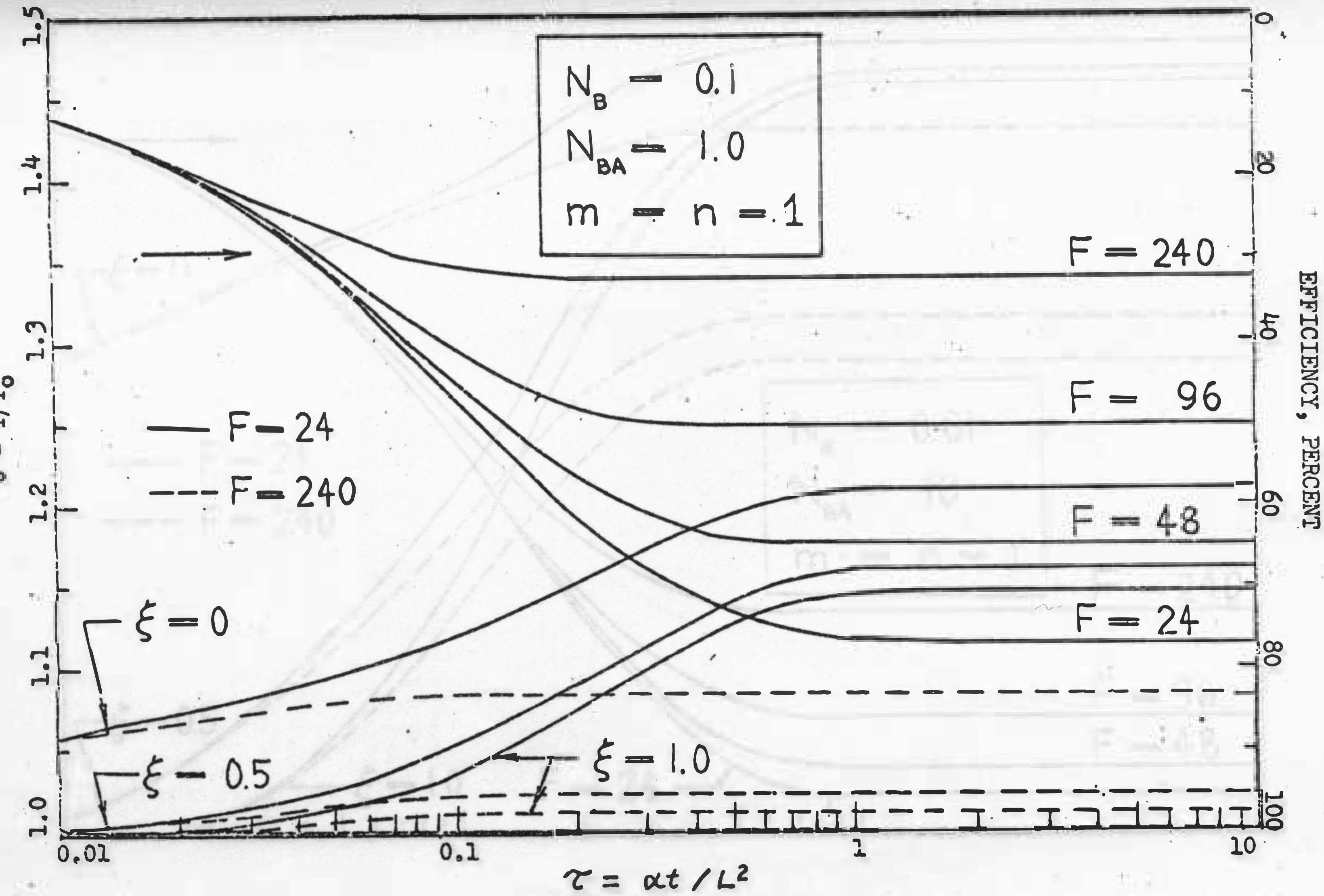


Fig. 20 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0005$

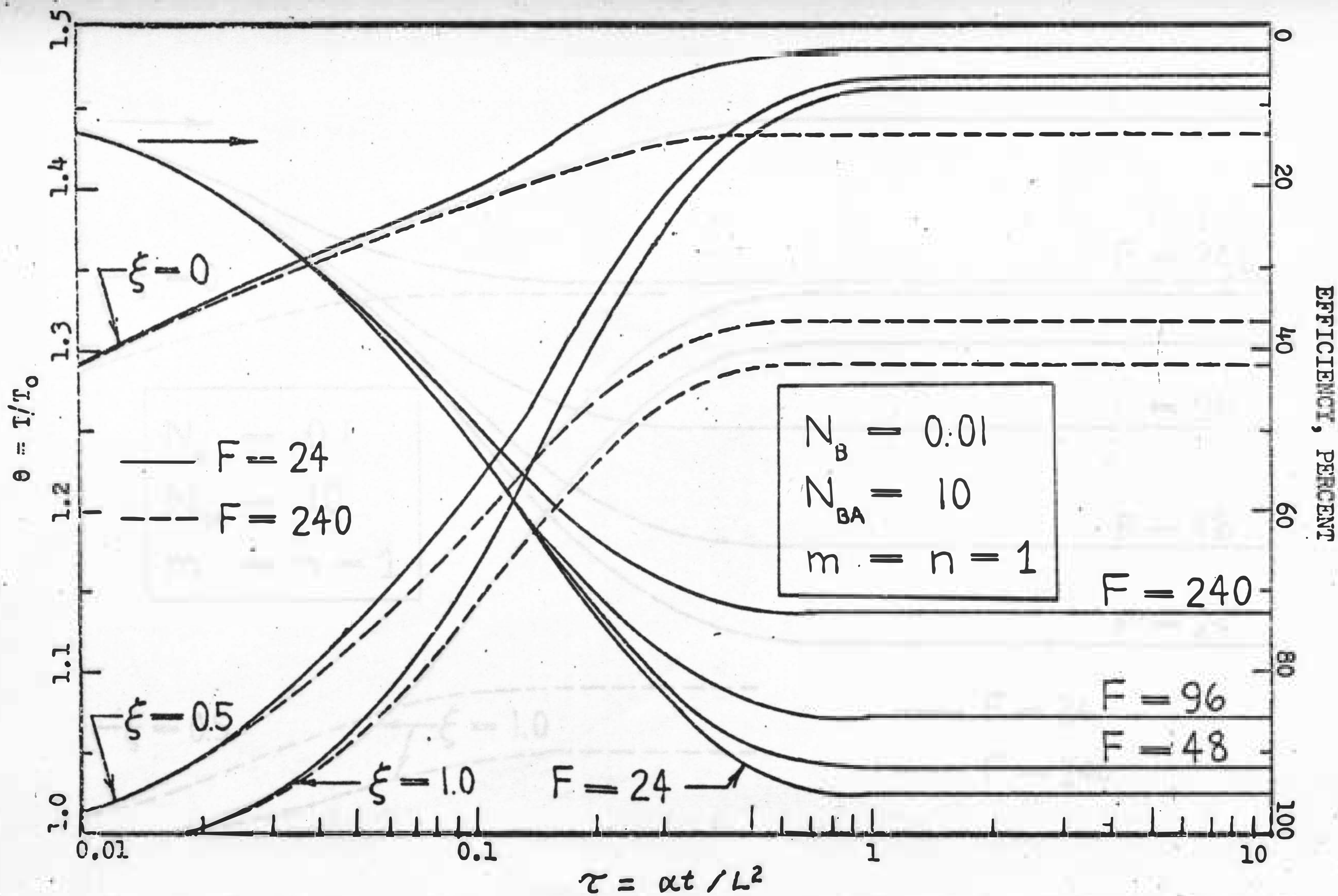


Fig. 21 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0005$

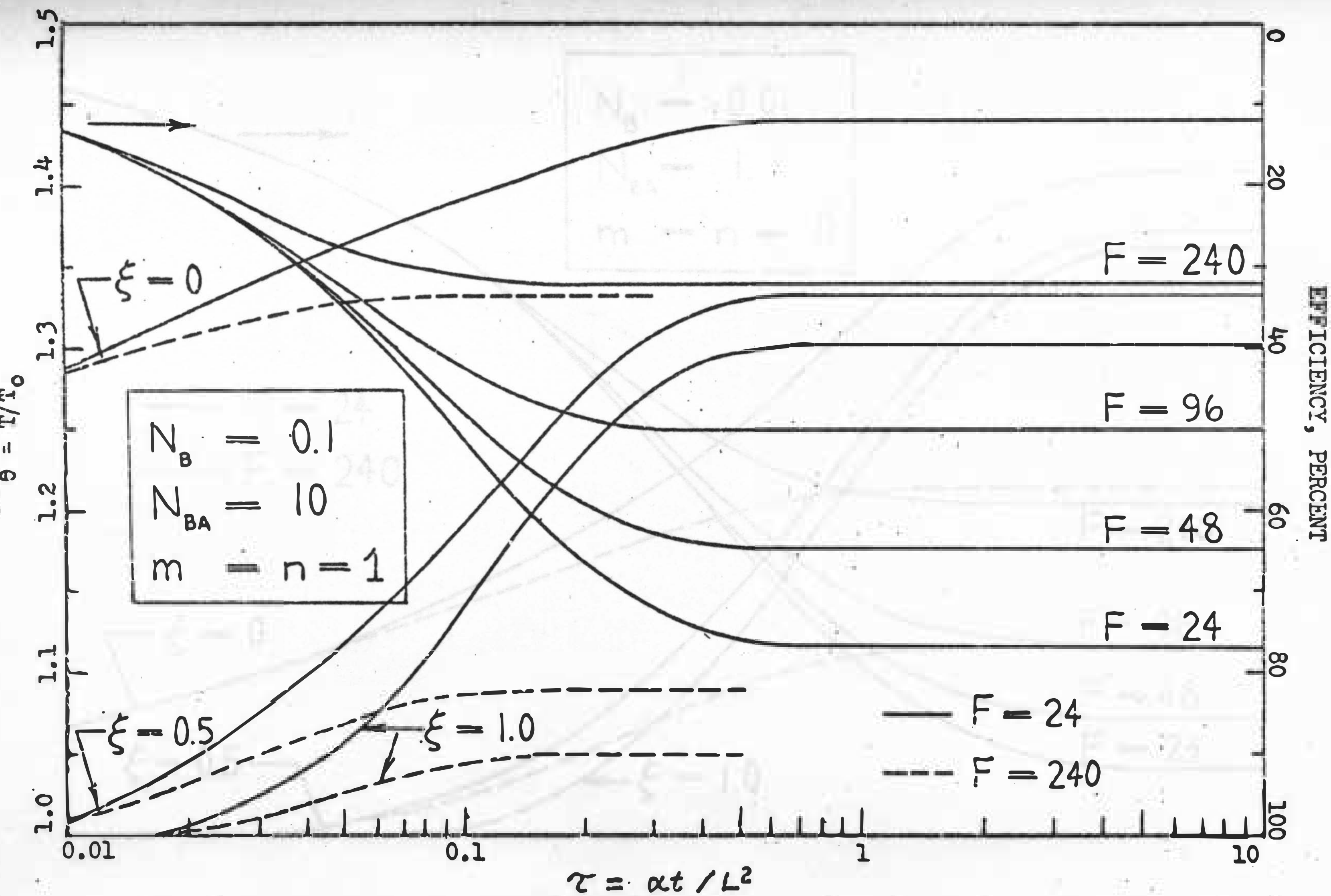


Fig. 22 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0005$

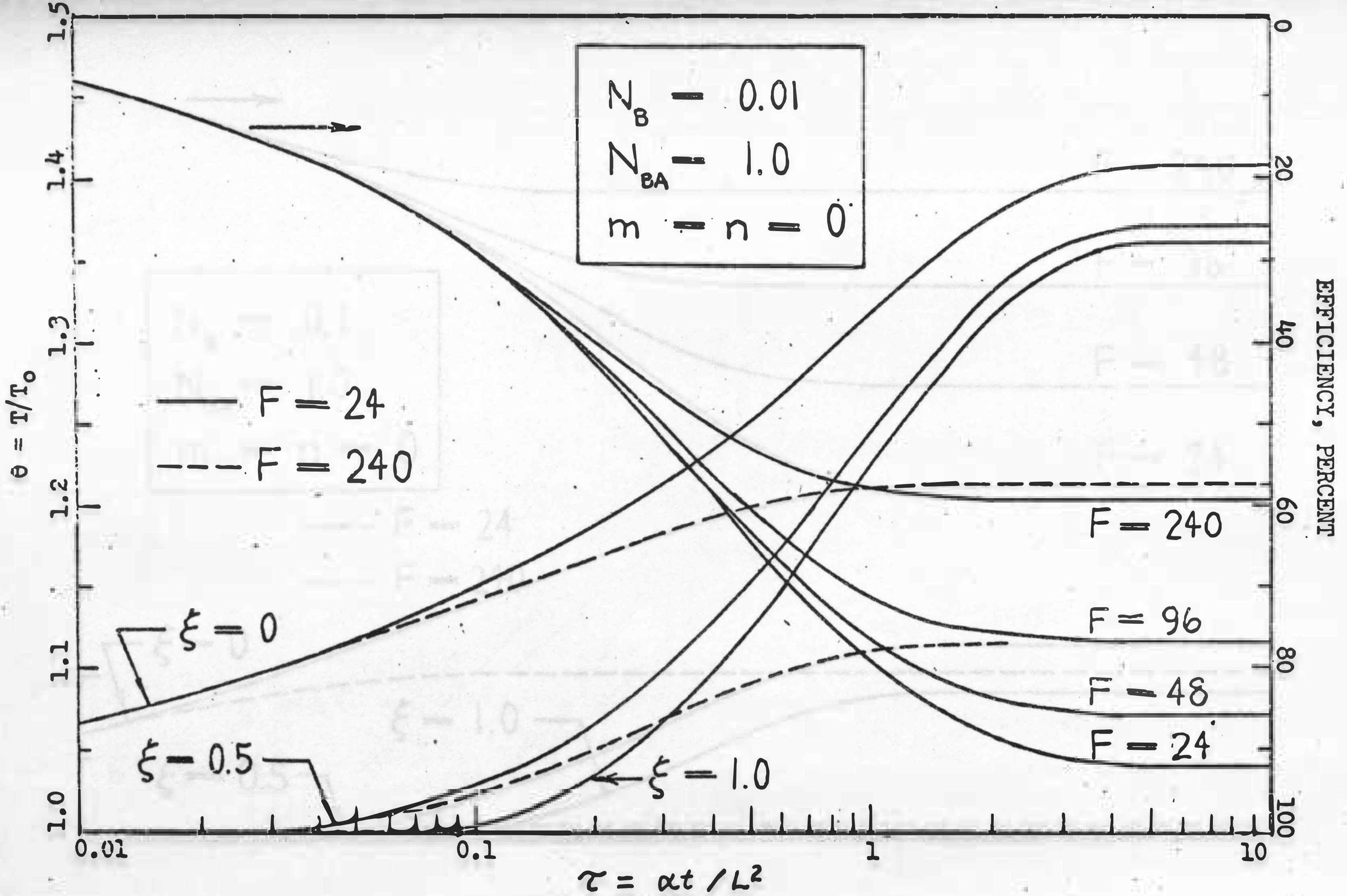


Fig. 23 Transient temperatures and efficiency for a constant-area straight fin with combined convection and radiation, for $\omega = 0$, $N_r = 0.0001$

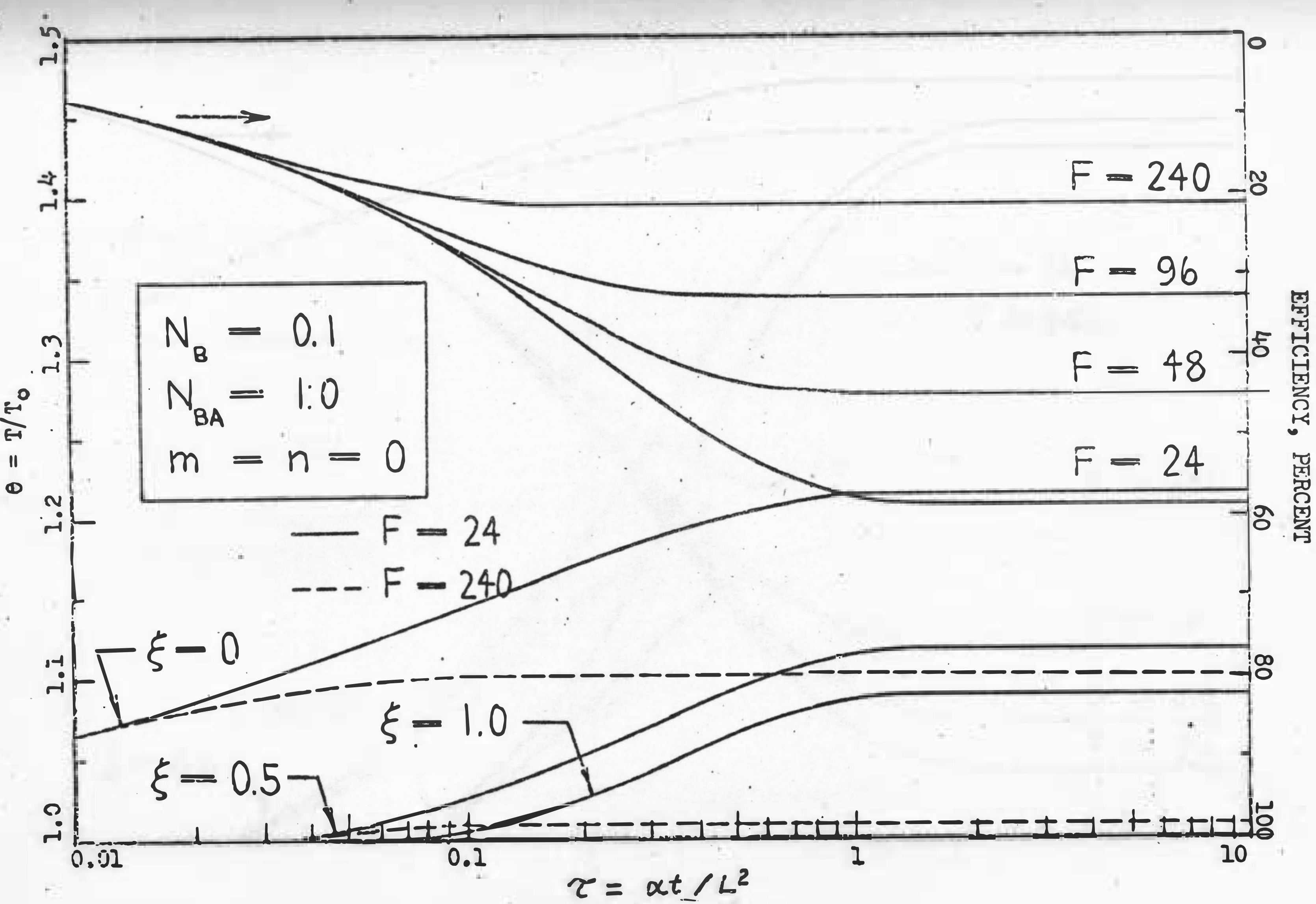


Fig. 24 Transient temperatures and efficiency for a constant-area straight fin with combined convection and radiation, for $\omega = 0$, $N_r = 0.0001$

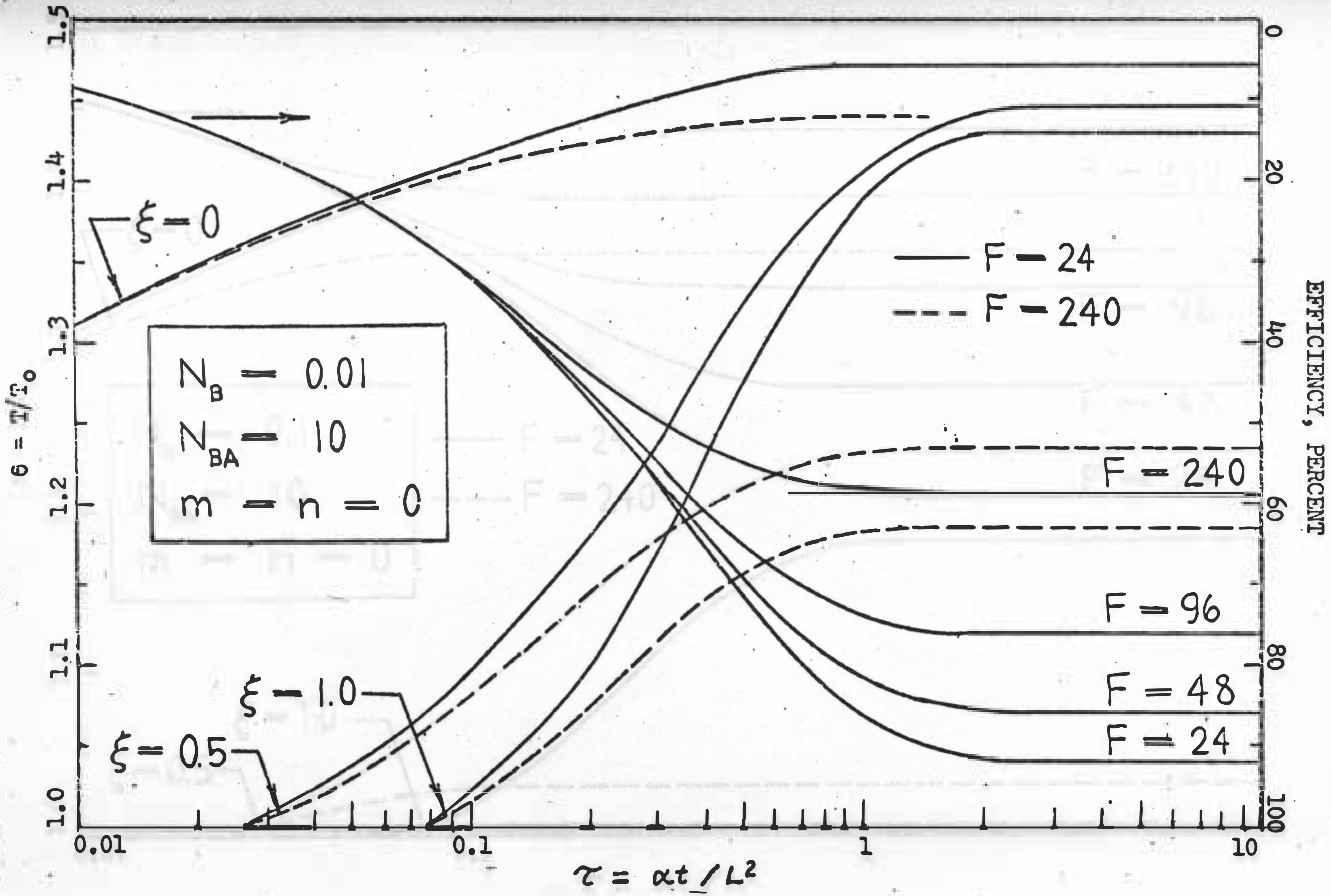


Fig. 25 Transient temperatures and efficiency for a constant-area straight fin with combined convection and radiation, for $\omega = 0$, $N_r = 0.0001$

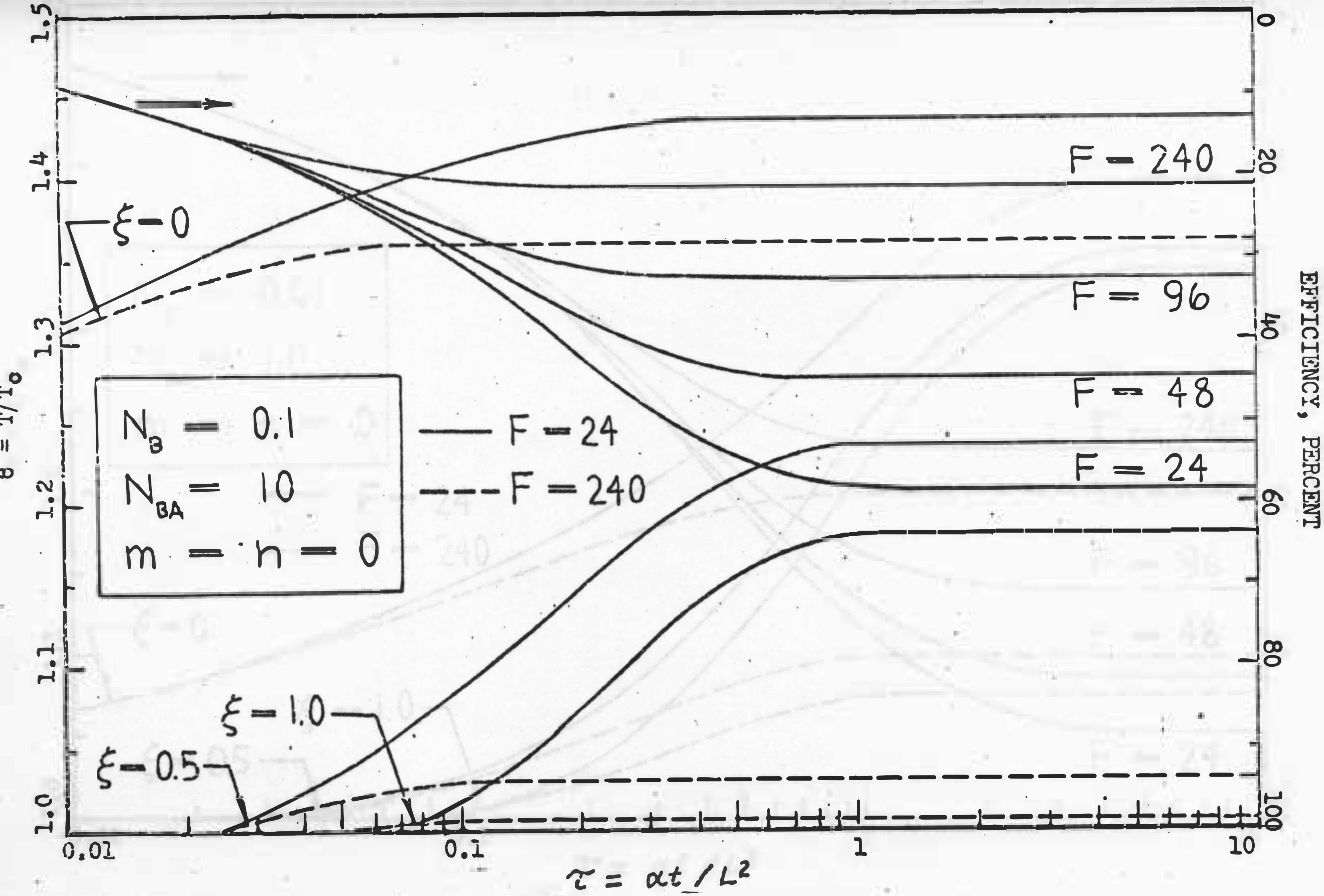


Fig. 26 Transient temperatures and efficiency for a constant-area straight fin with combined convection and radiation, for $\omega = 0$, $N_r = 0.0001$

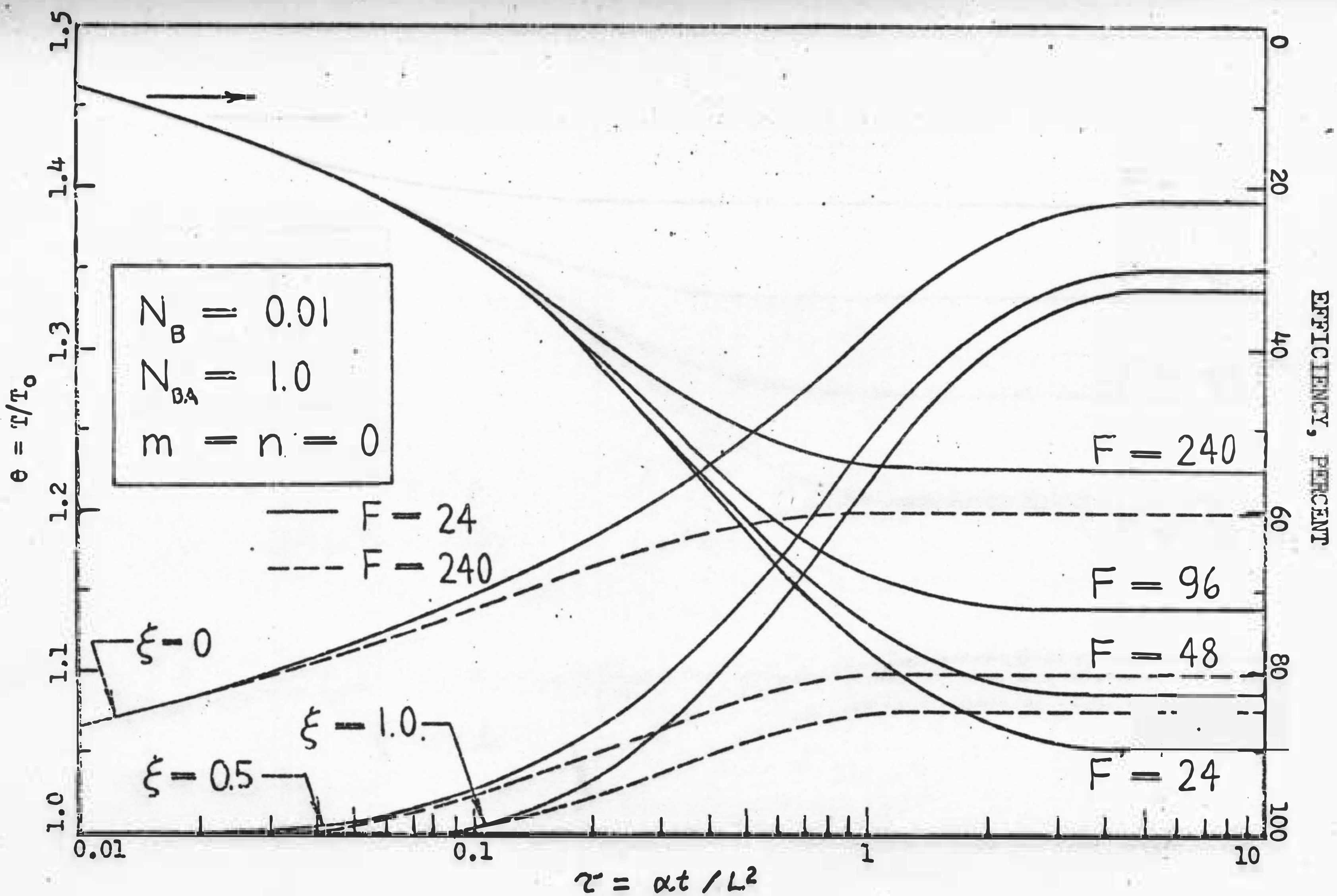


Fig. 27 Transient temperatures and efficiency for a constant-area straight fin with combined convection and radiation, for $\omega = 0$, $N_r = 0.0005$

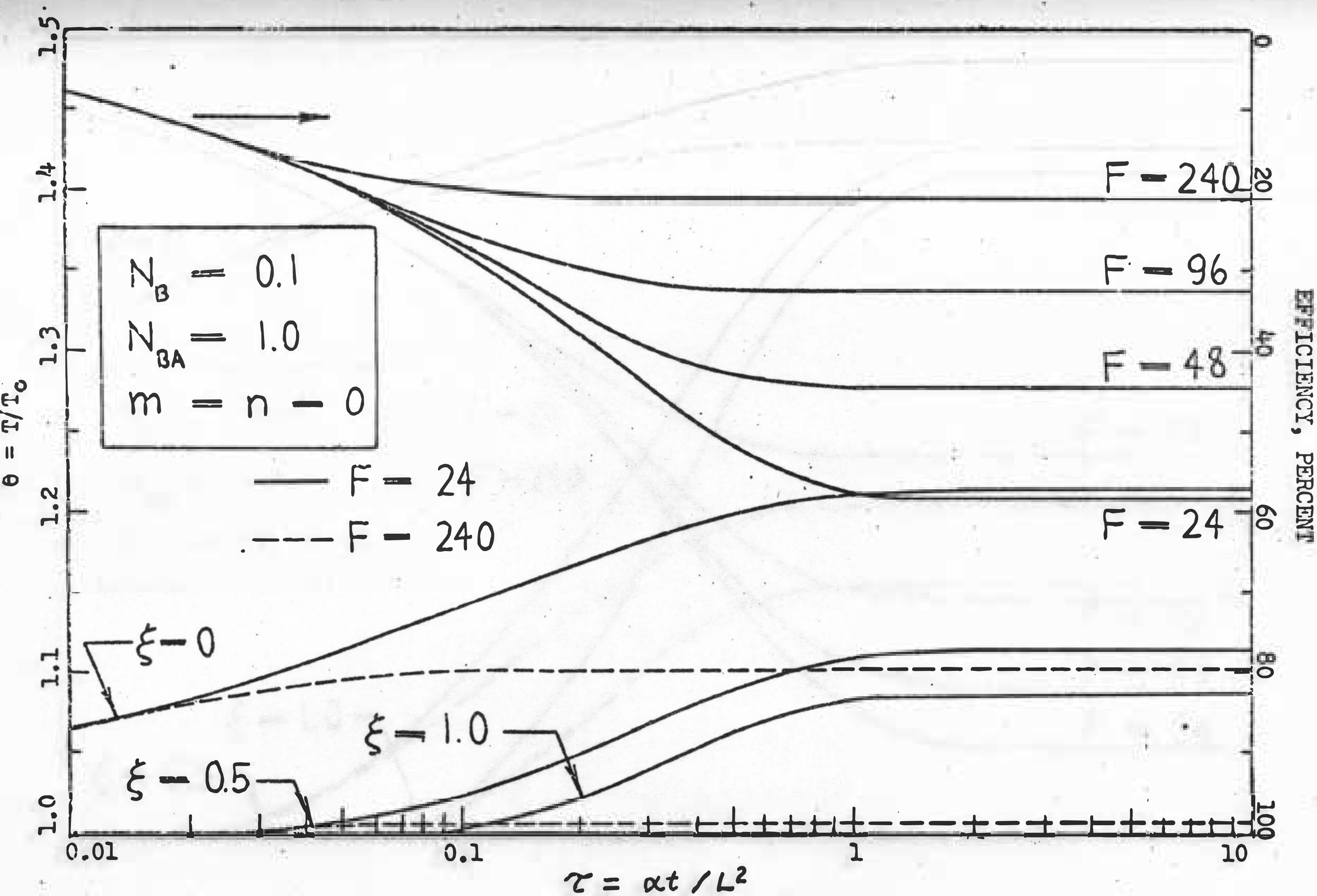


Fig. 28 Transient temperatures and efficiency for a constant-area straight fin with combined convection and radiation, for $\omega = 0$, $N_r = 0.0005$

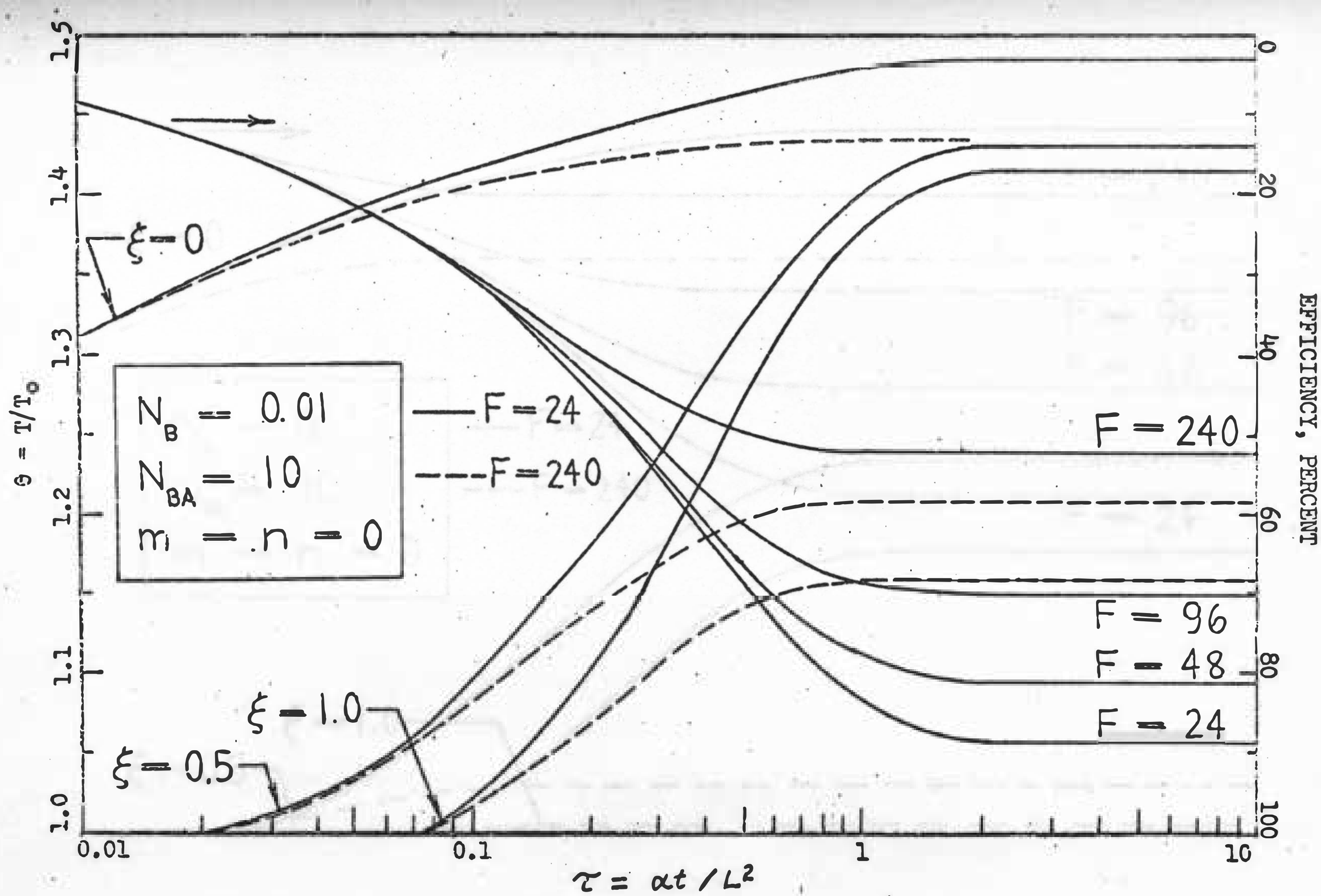


Fig. 29 Transient temperatures and efficiency for a constant-area straight fin with combined convection and radiation, for $\omega = 0$, $N_r = 0.0005$

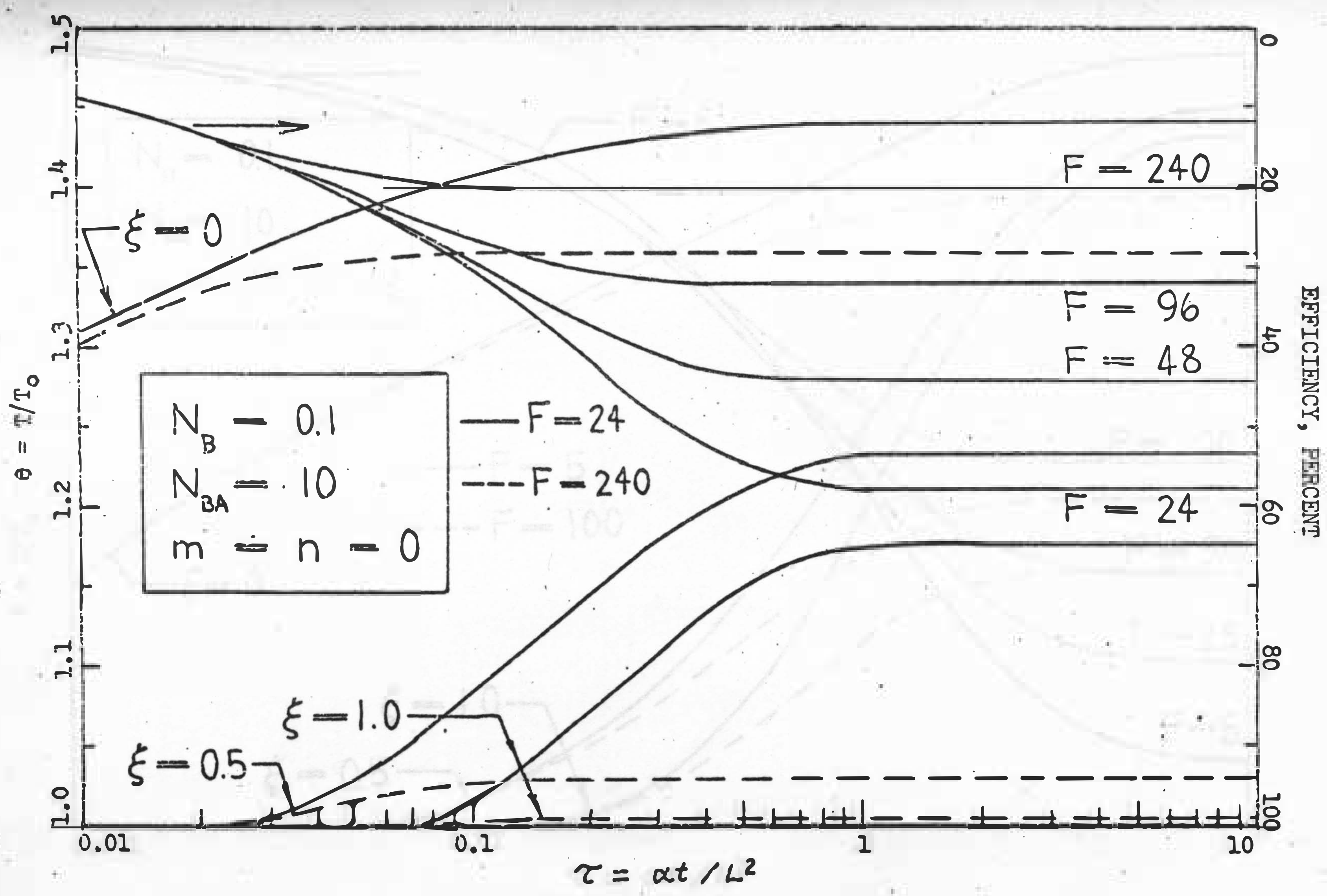


Fig. 30 Transient temperatures and efficiency for a constant-area straight fin with combined convection and radiation for $\omega = 0$, $N_r = 0.0005$

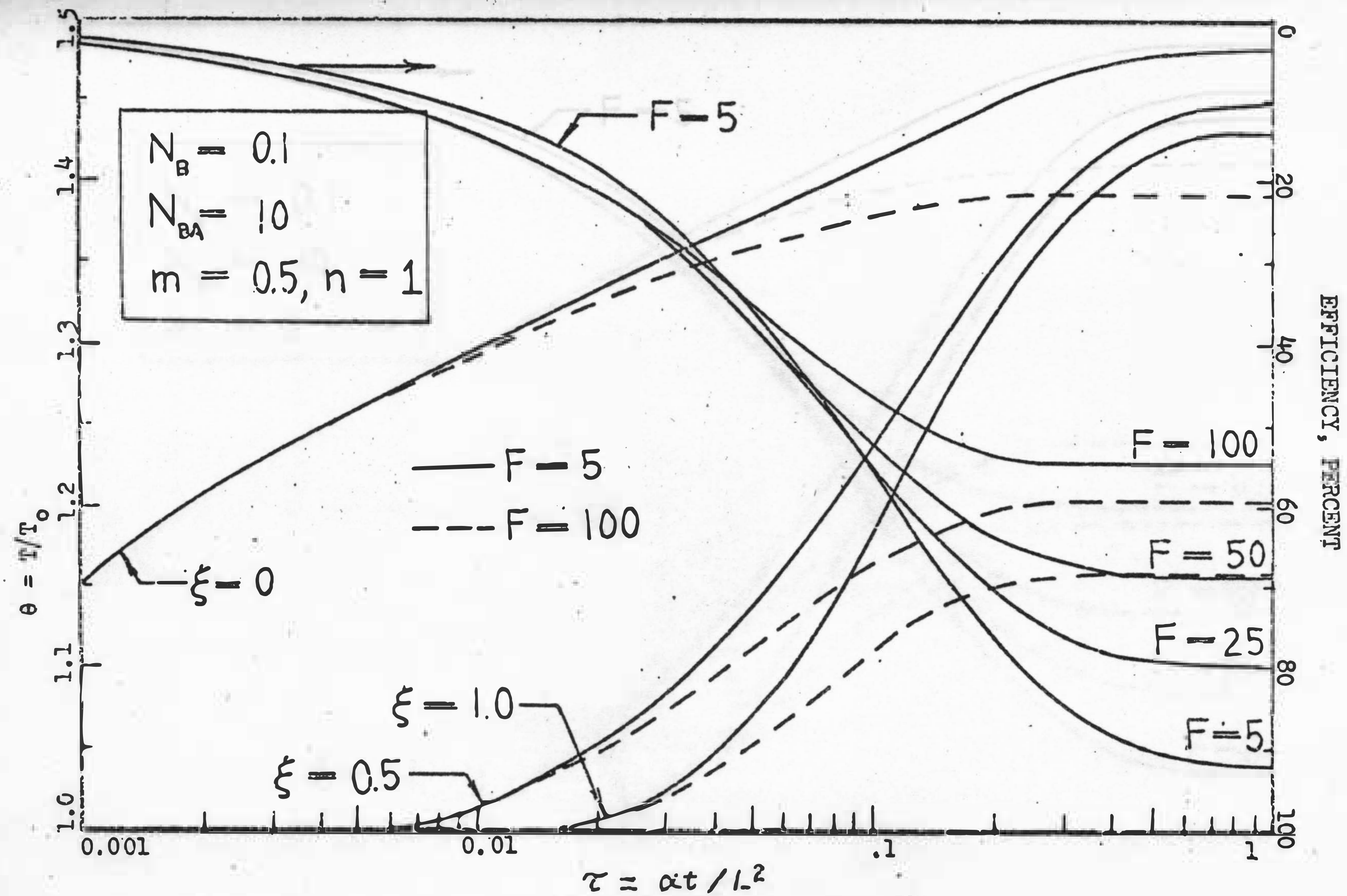


Fig. 31 Transient temperature and efficiency for an annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0005$, $m = .5$, $n = 1$

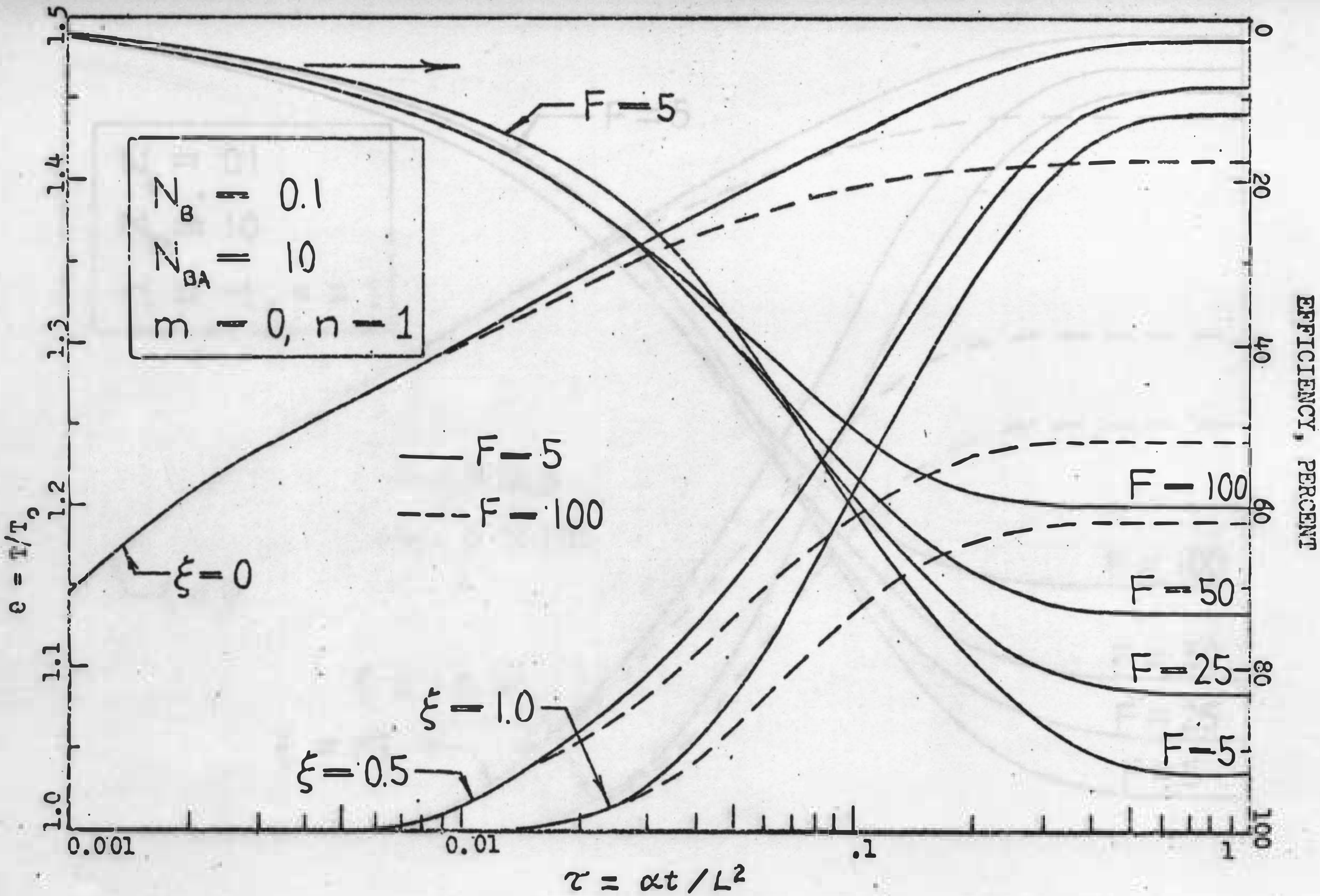


Fig. 32 Transient temperatures and efficiency for a hyperbolic annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0005$

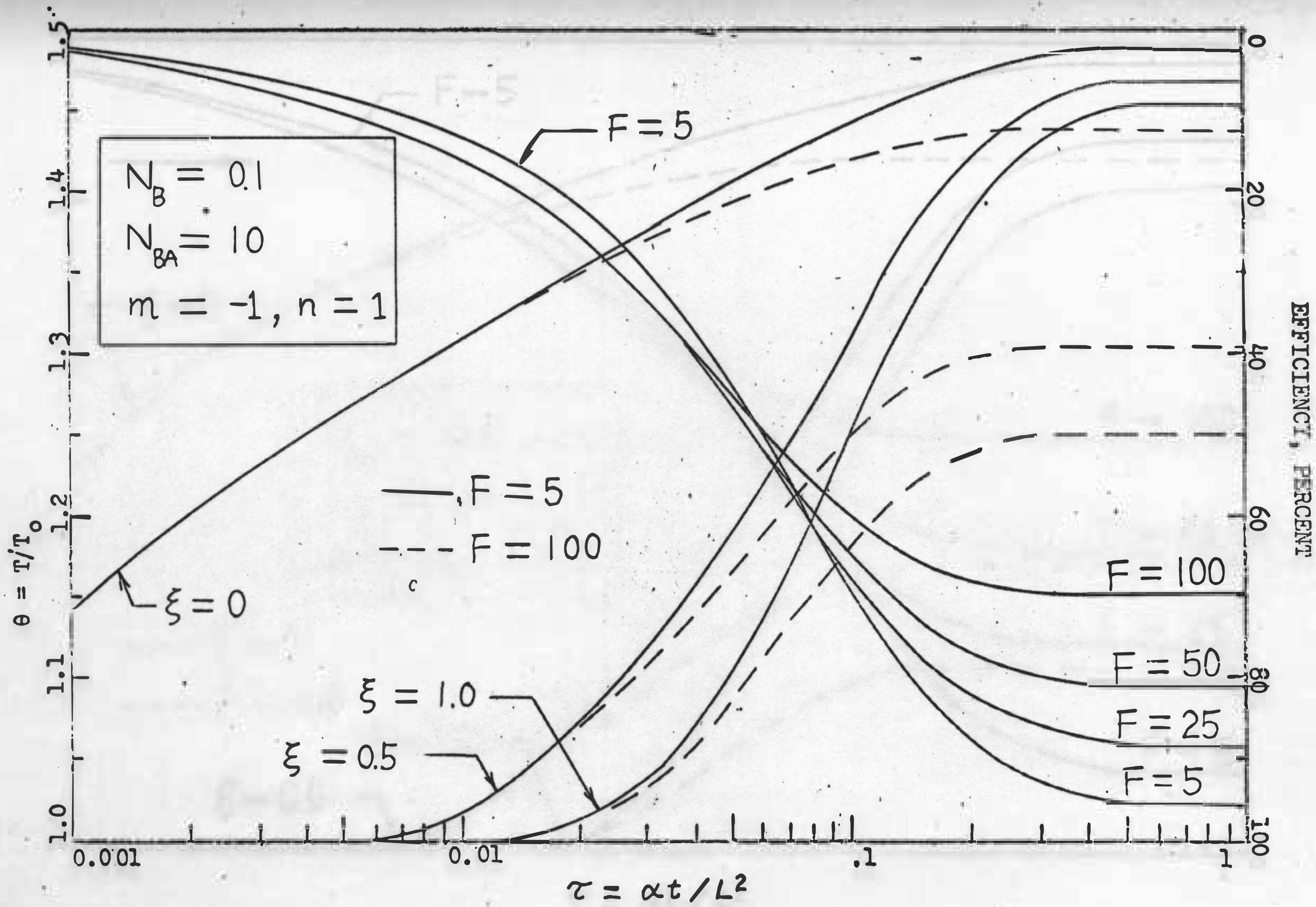


Fig. 33 Transient temperature and efficiency for an annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0005$, $m = -1$, $n = 1$

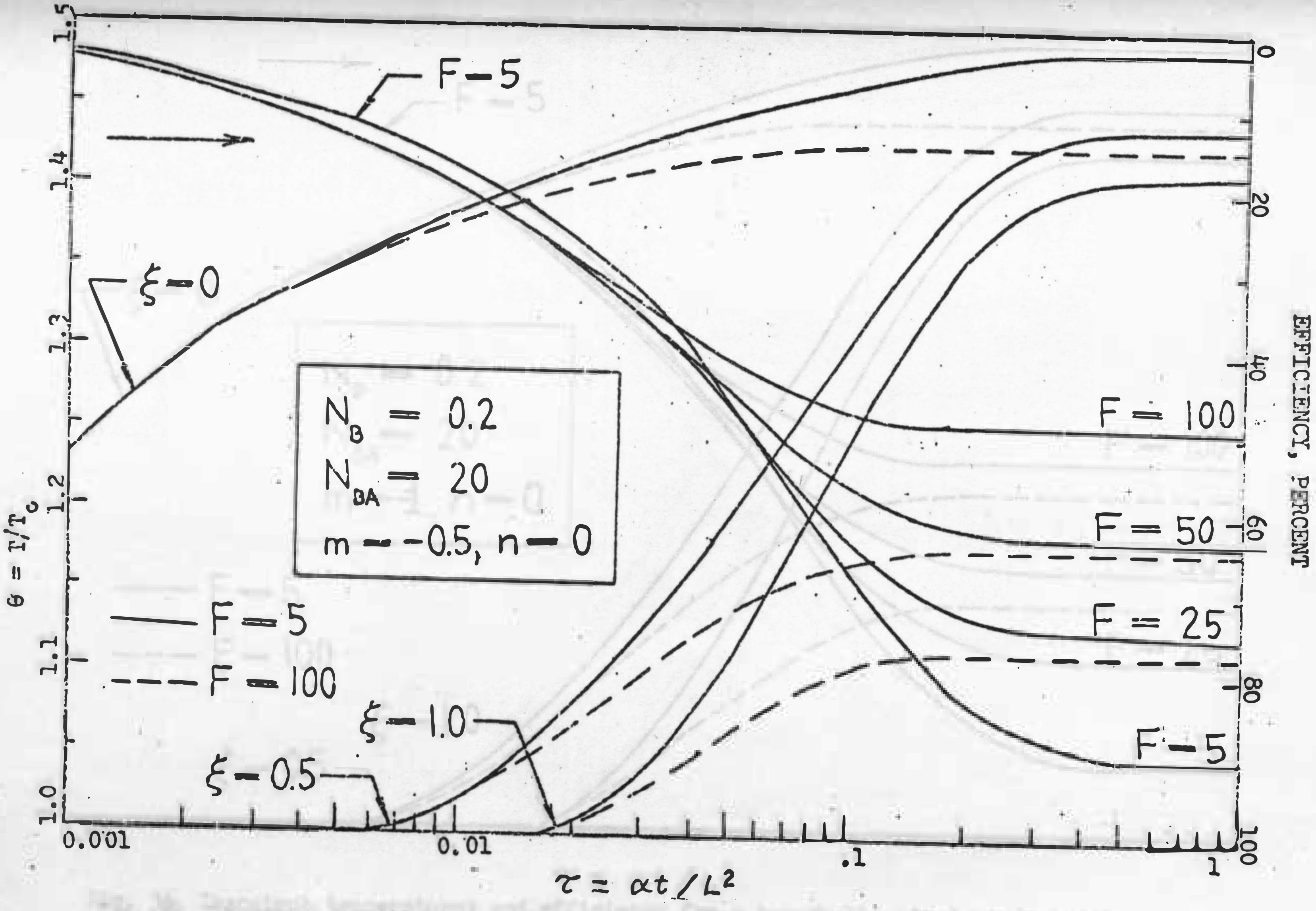


Fig. 34 Transient temperatures and efficiency for a straight fin with combined convection and radiation, for $\omega = 0$, $N_r = 0.001$, $m = -.5$, $n = 0$

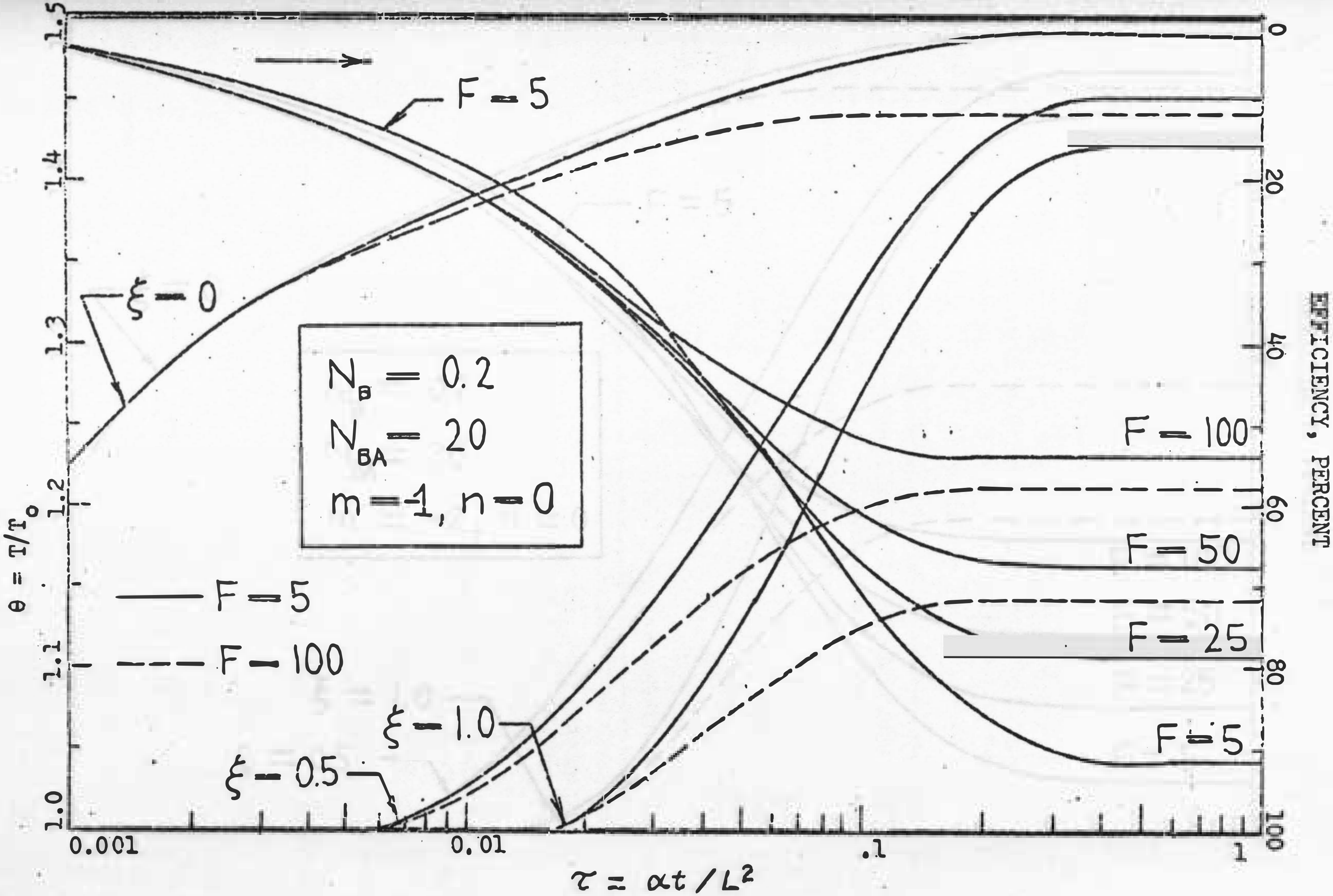


Fig. 35 Transient temperatures and efficiency for a hyperbolic straight fin with combined convection and radiation, for $\omega = 0$, $N_r = 0.001$

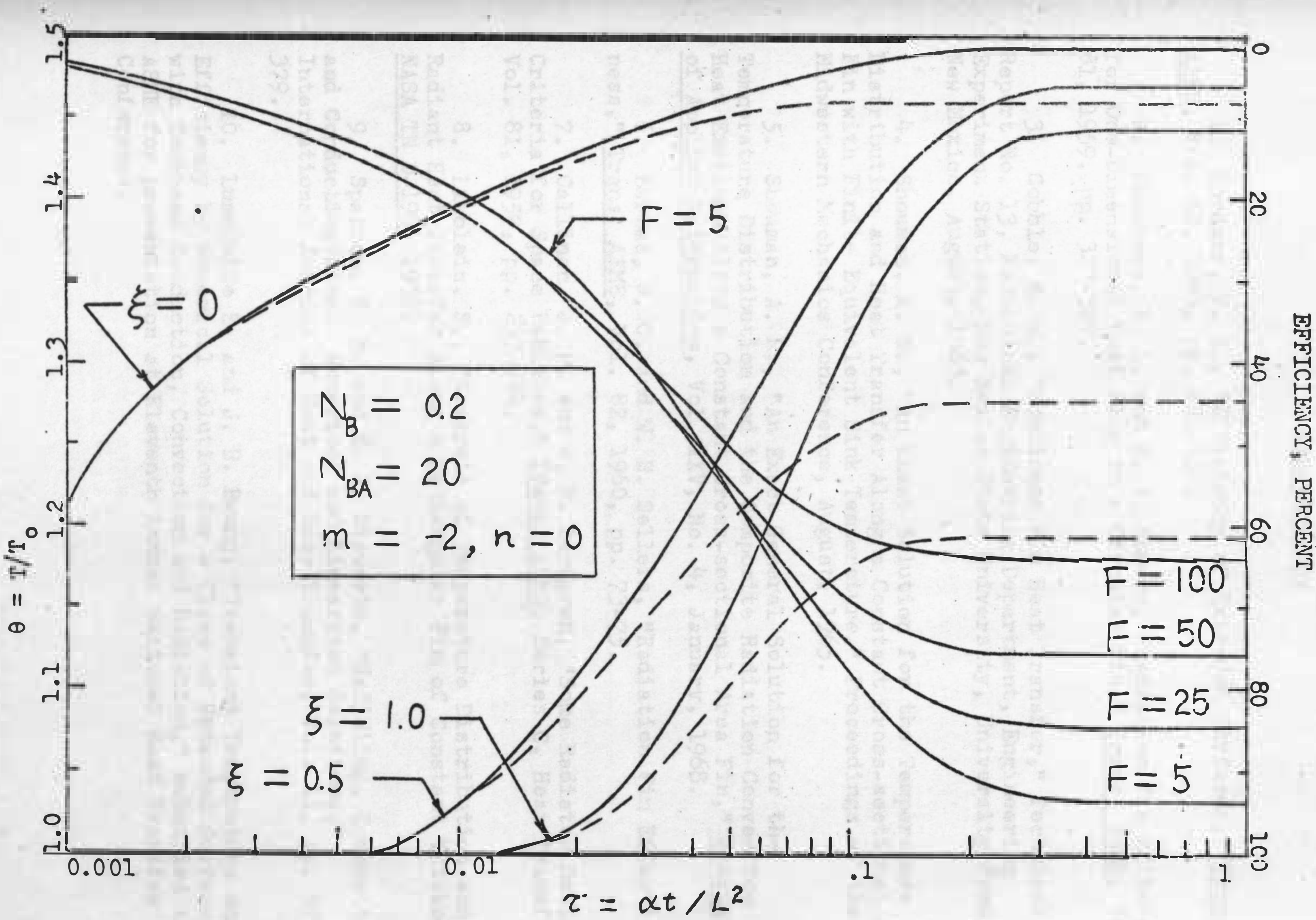


Fig. 36 Transient temperatures and efficiency for a straight fin with combined convection and radiation, for $\omega = 0$, $m = -2$, $n = 0$

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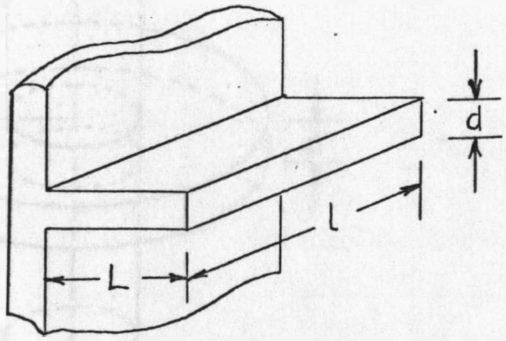
# APPENDIX I

## VARIOUS FIN GEOMETRIES

1. Straight Rectangular Fin

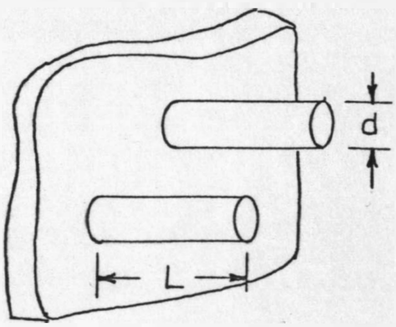



$A = ld$ ;  $B_1 = ld$ ,  $m = 0$

$P = 2l$ ;  $B_2 = 2l$ ,  $n = 0$

$F = 2l / d$

2. Spine of Rectangular Profile

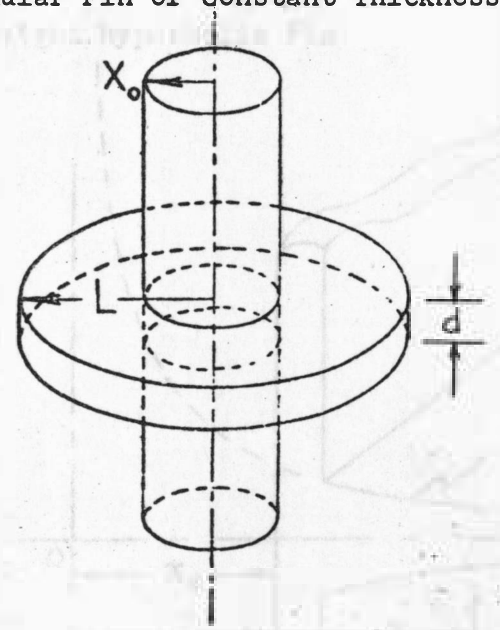



$A = \frac{1}{4}\pi d^2$ ;  $B_1 = \frac{1}{4}\pi d^2$ ,  $m = 0$

$P = \pi d$ ;  $B_2 = \pi d$ ,  $n = 0$

$F = 4L / d$

3. Annular Fin of Constant Thickness

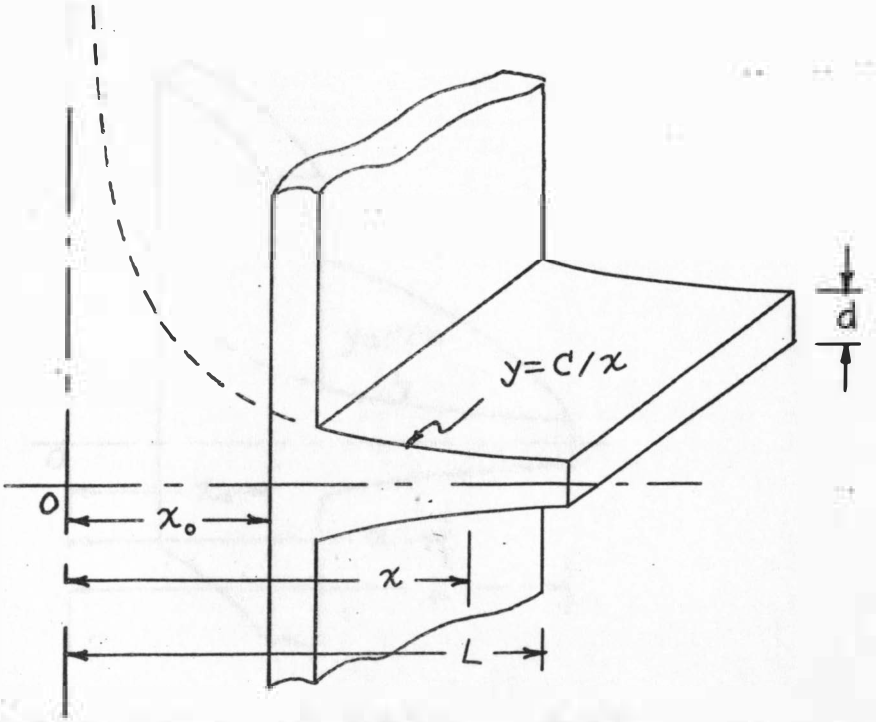



$A = 2\pi x d$ ;   $B_1 = 2\pi d$ ,   $m = 1$

$P = 4\pi x$ ;   $B_2 = 4\pi$ ,   $n = 1$

$F = 2L / d$

4. Straight Hyperbolic Fin



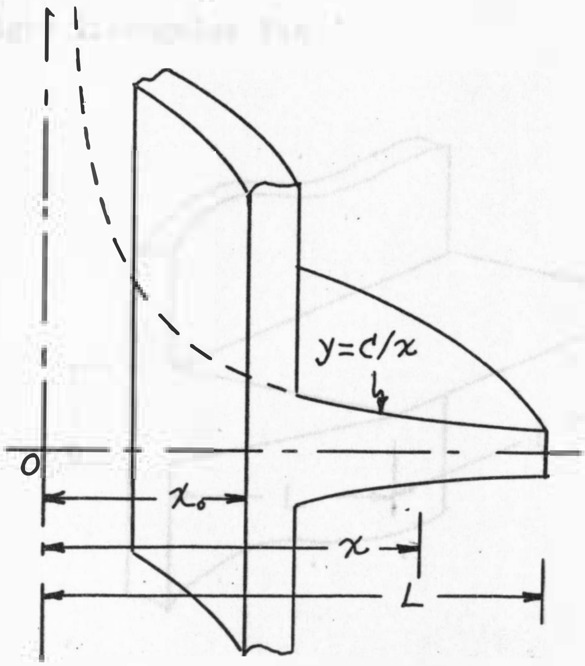

For this type of fin, the origin of the coordinate is at somewhere other than the root. If $x_o/L < 1/2$, then $y' \ll 1$, the following identies will hold:

$A = 2l \cdot (C/x)$ ; $B_1 = 2Cl$ , $m = -1$

$P = 2l$ ; $B_2 = 2l$, $n = 1$

$F = L^2 / C$

5. Annular Hyperbolic Fin

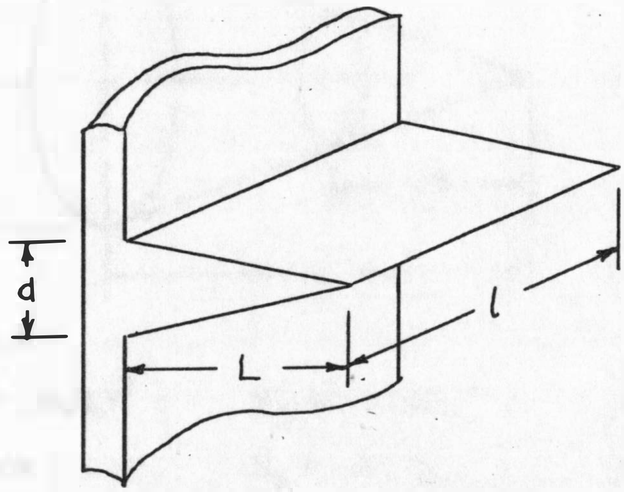



$A = 4\pi x\,(C/x)$ ;  $B_1 = 4\pi C$ ,  $m = 0$

$P = 4\pi x$ ;  $B_2 = 4\pi$ ,  $n = 1$

$F = L^2 / C$

6. Straight Triangular Fin



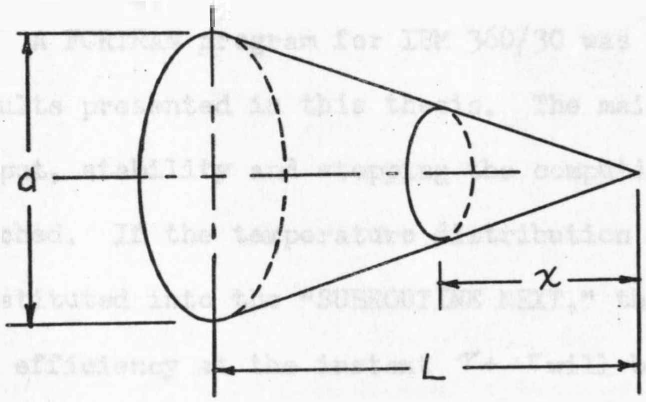

For $l \gg d$

$A = dlx/L$ ; $B_1 = dl/L$ , $m = 1$

$P \simeq 2l$ ; $B_2 = 2l$ , $n = 0$

$F = 2L / d$

7. Spine of Triangular Profile




If $d \ll L$

$A = \frac{1}{4}\pi(dx/L)^2$ ; $B_1 = \frac{1}{4}\pi(d/L)^2$ , $m = 2$

$P = \pi dx / L$ ; $B_2 = d / L$ , $n = 1$

$F = 4L / d$

# APPENDIX II

## PROGRAMMING

A FORTRAN program for IBM 360/30 was prepared to obtain the results presented in this thesis. The main program is for input, output, stability and stopping the computing when steady-state is reached. If the temperature distribution at any instant $\tau$ is substituted into the "SUBROUTINE NEXT," the temperature distribution and efficiency at the instant $\tau + \Delta\tau$ will be obtained. "SUBROUTINE NEWTON" is for carrying out Newton's method for finding roots to Equation (16). In this particular program, the fin was divided into ten parts, thus, $\Delta\xi = .1$.

# Description of Programming Symbols

| Symbol | Description |
| --- | --- |
| B | $N_B$ |
| BA | $N_{BA}$ |
| DT | $\Delta\tau$ |
| DX | $\Delta\xi$ |
| EFF | $\eta$ |
| F | F |
| G | $N_r$ |
| M | m |
| N | n |
| R | $\omega$ |
| S (in next) | $\theta'$ |
| SUMA | Numerator of Equation (25) |
| SUMB | Denominator of Equation (25) |
| T | $\theta$ |
| TM | $\tau$ |
| TYPE | Type of the fin |

```
      REAL M,N
      DIMENSION TYPE(70),T(12),S(12)
      COMMON G,B,R,F,BA,M,N,DT,T,EFF
      T(1)=1.5
      DX=.1
      READ (11,110) (TYPE(I),I=1,70)
      READ (11,101) M,N,G,B,R,F,BA
C     STABILITY
      A=2./(DX*(1.-R))**2
      C=F*(B&4.*G*T(1)**3)
      IF (N-M) 12,11,11
   11 DT=1./(A&C)
      GO TO 13
   12 DT=1./(A&C*R**(N-M))
   13 CONTINUE
      WRITE (12,201) (TYPE(I),I=1,70),B,BA,G,F,M,N,R,DT
C     TEMPERATURE FIELD
      TM=0.
      EFF=0.
      DO 20 I=2,12
   20  T(I)=1.
      WRITE (12,210) TM,(T(I),I=1,12),EFF
      DO 21 J=1,10
      TM=TM&DT
      CALL NEXT
   21 WRITE (12,210) TM,(T(I),I=1,12),EFF
C     S IS OLD TEMP., T IS NEW TEMP., IF THE CORRES-
C     PONDING VALUES OF S AND T ARE EQUAL, I.E.,
C     STEADY-STATE IS REACHED, STOP THE COMPUTING
   22 DO 23 I=2,12
   23 S(I)=T(I)
      DO 24 K=1,10
      TM=TM&DT
   24 CALL NEXT
      WRITE (12,210) TM,(T(I),I=1,12),EFF
      DO 25 J=1,11
      I=13-J
      IF (T(I)-S(I)-.1E-5) 25,25,22
   25 STOP
  101 FORMAT (F15.5)
  110 FORMAT (70A1)
  201 FORMAT (1H1,33H TRANSIENT STUDY OF RADIATION FIN,
     1          21H BY FINITE DIFFERENCE//
     2          16H TYPE OF THE FIN,70A1//
     3          F16.5,2H=B,F18.5,3H=BA,F17.5,2H=G //
     4          F16.5,2H=F,F18.5,2H=M,F18.5,2H=N //
     5          F16.5,2H=R///
     6          10H STABILITY,E15.4,3H=DT///
     7          32H DIMENSIONLESS TEMPERATURE FIELD)
  210 FORMAT (1H ,F7.4,12F9.6,F5.3)
      END
```

```
      SUBROUTINE NEXT
      REAL M,N
      DIMENSION T(12),S(12)
      COMMON G,B,R,F,BA,M,N,DT,T,EFF
      SUMA=0.
      SUMB=0.
      DO 11 I=3,11
      X=.1*(I-2)
      XI=(1.-R)*X&R
      S(I)=T(I)&DT*((5.*M*(T(I&1)-T(I-1))/XI&100.*
     1     (T(I&1)&T(I-1)-2.*T(I))/(1.-R))/(1.-R)
     2     -F*XI**(N-M)*(B*(T(I)-1.)&G*(T(I)**4-1.)))
      SUMA=SUMA&XI**N*(B*(S(I)-1.)&G*(S(I)**4-1.))
      SUMB=SUMB&XI**N
   11 CONTINUE
      CALL NEWTON (S(11),S(12))
      SA=.5+10./F
      SUMA=SUMA&SA*(B*(S(12)-1.)&G*(S(12)**4-1.))
      SUMB=SUMB&SA
      S(2)=(10.*S(3)&BA*(1.-R)*T(1))/(BA*(1.-R)&10.)
      SA=B*(S(2)-1.)&G*(S(2)**4-1.)
      IF (N) 14,15,14
   14 SUMA=SUMA&.5*R**N*SA
      SUMB=(SUMB&.5*R**N)*SA
      GO TO 16
   15 SUMA=SUMA&.5*SA
      SUMB=(SUMB&.5)*SA
   16 IF (G) 18,17,18
   17 IF (B) 18,19,18
   18 EFF=SUMA/SUMB
      GO TO 20
   19 EFF=1.
   20 DO 21 I=2,12
   21 T(I)=S(I)
      RETURN
      END
```

```
      SUBROUTINE NEWTON (T,S)
      COMMON G,B,R
      S=T
    1 Y=G*(1.-R)*S**4&(10.&B*(1.-R))*S
     1  -(10.*T&B*(1.-R)&G*(1.-R))
      DY=4.*G*(1.-R)*S**3&(10.&B*(1.-R))
      DS=Y/DY
      S=S-DS
      IF (ABS(DS)-.0005) 2,1,1
    2 RETURN
      END
```

# HEAT TRANSFER BY NUMERICAL SOLUTION
# FOR A CLASS OF RADIATING FINS

BY

JIA-BO HWANG

A thesis submitted
in partial fulfillment of the requirements for the
degree Master of Science, Major in
Mechanical Engineering, South Dakota
State University

1969

HEAT TRANSFER BY NUMERICAL SOLUTION

FOR A CLASS OF RADIATING FINS

This thesis is approved as a creditable and independent investigation by a candidate for the degree, Master of Science, and is acceptable as meeting the thesis requirements for this degree, but without implying that the conclusions reached by the candidate are necessarily the conclusions of the major department.

Edward Lumsdaine — 17 FEB. 69

Thesis Adviser — Date

John F. Sandfort — 2/17/69

Head, Mechanical Engineering Department — Date

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JBH

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## Nomenclature

| | |
|---|---|
| $A$ | Cross-sectional area of fin |
| $B_1, m$ | Geometric parameters pertaining to area of fin |
| $B_2, n$ | Geometric parameters pertaining to perimeter of fin |
| $C_p$ | Specific heat |
| $H$ | Convection coefficient of the surface and tip of fin |
| $h$ | Convection coefficient of the base |
| $k$ | Conductivity |
| $L$ | Coordinate at the tip of the fin |
| $P$ | Perimeter or the derivative of surface area with respect to x of fin |
| $T$ | Temperature |
| $t$ | Time |
| $x$ | Coordinate at any point of the fin |
| $x_o$ | Coordinate at the base |
| $\epsilon$ | Emissivity |
| $\eta$ | Fin efficiency |
| $\rho$ | Density |
| $\sigma$ | Stefan-Boltzmann constant |

Subscripts:

| | |
|---|---|
| o | Ambient fluid |
| 1,f | Heating fluid |

Dimensionless Parameters:

| | |
|---|---|
| $\xi$ | Coordinate |
| $\theta$ | Temperature |
| $\omega$ | Ratio of the coordinates |
| $\tau$ | Time |
| $N_B$ | Biot number of the fin surface |
| $N_{BA}$ | Biot number of the fin base |
| $N_r$ | Radiation parameter |
| F | Fin parameter |

# CHAPTER I

## INTRODUCTION

It has long been known that the heat transfer from a solid body to an ambient fluid can be increased by increasing the surface area of the solid body. Extended surfaces, or fins, are indispensable for compact heat exchangers. Geometrically, fins may be classified as straight fins, annular fins, and rod fins or spines. In most applications of fins, a fluid is circulating inside the fin-supporting pipe while the outside is exposed to another ambient fluid. The purpose of fin analysis is to find the temperature distribution in the fin and the heat transfer from the fin to the ambient fluid, i.e., the fin efficiency, which is the basis of comparing various fin designs. Conduction is the heat transfer mechanism in the fin, and convection and radiation occur at the surface. The amount of radiation heat transfer, in accordance with Stefan-Boltzmann's law, is proportional to the difference between the fourth power of the temperature of the fin and the ambient fluid. The fourth-power term makes the fin equation non-linear and difficult to solve analytically. Earlier researchers linearized the radiation term by replacing the fourth-power law by an equivalent convection coefficient times the difference of the temperature in order to obtain an analytical solution. Thus, by linearization, Gardner (1)* derived a general equation for the

*Numbers in parentheses refer to the bibliography given at the end of the text.

temperature gradient and fin efficiency for a class of one-dimensional extended surfaces.

Radiating fins have become very important as a result of space exploration. For space vehicles at high altitude, radiation is the dominating heat transfer mechanism. Chamber and Somer (2), in solving an annular radiating fin by numerical methods, showed that the error in fin efficiency introduced by linearizing the radiation term can be as much as 60 percent. Thus, for high-temperature operation of the fin, the non-linear term must be included. Cobble (3) solved the steady-state problem of a one-dimensional constant-area fin with a fixed base temperature and insulated tip analytically by reducing the fourth power through the application of Newton's forward difference. Subsequently, Shouman (4,5) found the exact solution to the same problem. Numerical solutions are also available for steady-state heat transfer of some specific types of fins (6,7,8).

The present thesis deals with the question of transient efficiency of radiating and convecting extended surfaces. The results are compared with data given in some recent publications (3,9) for the case of steady operation of constant-area fins with a constant root temperature and insulated tip. Transient temperature and efficiency curves for various types of extended surfaces are presented for design purposes.

The topic of transient temperatures and efficiencies of convecting and radiating fins was covered, for the first time, in a recent paper by Lumsdaine and Hwang (10).

# CHAPTER II

# ANALYSIS

## 1. Fin Equation

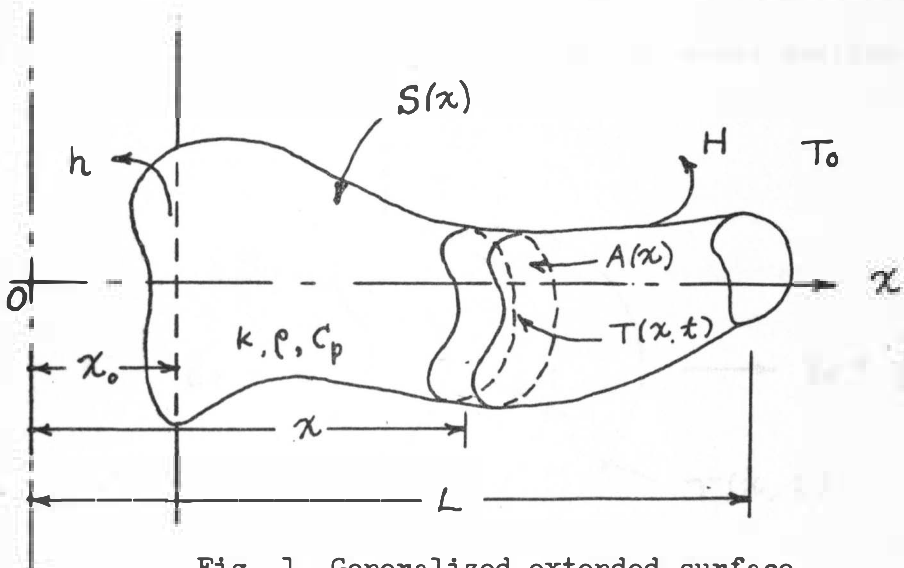


Fig. 1 Generalized extended surface

Consider an extended surface with variable cross section (Fig. 1) with the following assumptions:

1. The fin thickness is so small compared to the width that the problem can be considered as being one-dimensional. The temperature distribution is T(x,t).

2. The fin material is homogeneous and isotropic.

3. There is no heat source in the fin.

4. The thermal conductivity of the fin is constant.

5. The convection coefficient and the emissivity are constant over the fin surface.

In figure 1, the cross-sectional area A(x) normal to x for heat conduction is a function of x and so is the surface area S(x) for convection and radiation to the ambient fluid.

At an element of the fin contained between the cross sections x and $x + \delta x$ (Fig. 2), conduction through the cross section at x is,

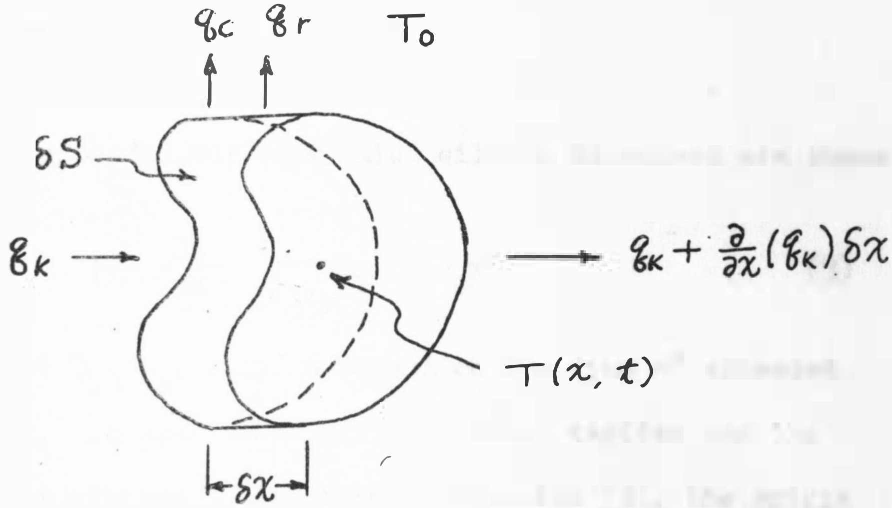


Fig. 2 Heat transfer at a fin element

from Fourier's law, $q_k = -kA\frac{\partial T}{\partial x}$; convection and radiation from the surface $\delta S$ are, from Newton's law and Stefan-Boltzmann's law, $q_c = h\delta S(T-T_o)$ and $q_r = \sigma\epsilon\delta S(T^4-T_o^4)$ respectively. An energy balance on this element gives the fin equation

$$\rho C_p \frac{\partial T}{\partial t}\delta x + q_k + q_r + \frac{\partial q_k}{\partial x} x = 0$$

or

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p} \cdot \frac{1}{A} \frac{\partial}{\partial x}\left(A\frac{\partial T}{\partial x}\right) - \frac{H}{\rho C_p} \frac{P}{A}(T-T_o) - \frac{\sigma\epsilon}{\rho C_p}\frac{P}{A}(T^4-T_o^4) \qquad (1)$$

where $P = dS/dx$ is the perimeter. The boundary conditions are obtained by equating the heat transfer rates at the root and tip respectively, i.e.,

1. $$\frac{\partial T}{\partial x}(x_o,t) = \frac{h}{k}\{T(x_o,t) - T_1(t)\}$$

2. $$\frac{\partial T}{\partial x}(L,t) = -\frac{H}{k}\{T(L,t) - T_o\} - \frac{\sigma\epsilon}{k}\{T^4(L,t) - T_o^4\} \tag{2}$$

and the initial condition is

3. $$T(x,o) = Z(x)$$

The class of extended surfaces which will be discussed are those with

$$A = B_1 x^m \qquad P = \frac{dS}{dx} = \sqrt{1+y'^2} = B_2 x^n \tag{3}$$

where the values of $B_1$, $B_2$, m and n depend on the type of extended surface considered. In some cases when the cross section and the perimeter cannot be written in the form of Equation (3), the origin of the coordinate can be taken at the tip of the fin by simply rearranging the boundary conditions. Some examples of extended surfaces which satisfy Equation (3) are shown in Appendix I.

Upon substitution of Equation (3) into Equation (1), the result is

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{m}{x}\frac{\partial T}{\partial x}\right) - \frac{H}{\rho C_p}\frac{P}{A}(T-T_o) - \frac{\sigma\epsilon}{\rho C_p}\frac{P}{A}(T^4-T_o^4) \tag{4}$$

The introduction of the dimensionless parameters

$$\xi = \frac{x-x_o}{L-x_o} \qquad \omega = \frac{x_o}{L} \qquad \theta = \frac{T}{T_o} \tag{5}$$

and

$$x = (L-x_o) + x_o \qquad \text{or} \qquad \frac{x}{L} = (1-\omega)\xi + \omega$$

into Equation (4) yields

$$T_o \frac{\partial \theta}{\partial t} = \frac{k}{\rho C_p} \left[ \frac{1}{(L - x_o)\xi + x_o} \frac{T_o}{L - x_o} \frac{\partial \theta}{\partial \xi} + \frac{T_o}{(L - x_o)^2} \frac{\partial^2 \theta}{\partial \xi^2} \right]$$

$$- \frac{HT_o}{\rho C_p} \frac{P}{A} (\theta - 1) - \frac{\sigma \epsilon T_o^4}{C_p} \frac{P}{A} (\theta^4 - 1)$$

The multiplication by $\rho C_p L(L-x_o)/kT_o$ on both sides gives

$$\frac{\rho C_p L^2}{k} (1-\omega) \frac{\partial \theta}{\partial t} = \left[ \frac{1}{(1-\omega)\xi + \omega} \frac{\partial \theta}{\partial \xi} + \frac{1}{1-\omega} \frac{\partial^2 \theta}{\partial \xi^2} \right]$$

$$- \frac{PL}{A}(1-\omega) \left[ \frac{HL}{k}(\theta - 1) + \frac{\sigma \epsilon L T_o^3}{k} (\theta^4 - 1) \right] \tag{6}$$

More dimensionless parameters are defined as

$$\tau = \frac{kt}{\rho C_p L^2}, \qquad N_B = \frac{HL}{k} \qquad N_r = \frac{\sigma \epsilon L T_o^3}{k} \tag{7}$$

It can easily be realized that PL/A is dimensionless and

$$\frac{PL}{A} = \frac{B_2 x^n}{B_1 x^m} L = \frac{B_2 L^{n-m+1}}{B_1} \left(\frac{x}{L}\right)^{n-m} = F\left(\frac{x}{L}\right)^{n-m} \tag{8}$$

where $F = (B_2/B_1)L^{n-m+1}$ is dimensionless also. Equation (6) can be rewritten in terms of $\tau$, $N_B$, $N_r$, and F as

$$(1-\omega)\frac{\partial \theta}{\partial t} = \left[ \frac{m}{(1-\omega)\xi + \omega} \frac{\partial \theta}{\partial \xi} + \frac{1}{1-\omega} \frac{\partial^2 \theta}{\partial \xi^2} \right]$$

$$-F \left[ (1-\omega)\xi + \omega \right]^{n-m} (1-\omega) \left[ N_B(\theta - 1) + N_r(\theta^4 - 1) \right] \tag{9}$$

Boundary and initial conditions are now

1. $\frac{\partial \theta}{\partial \xi}(0, \tau) = N_{BA}\,(1-\omega)\,\left[\,\theta(0,\tau) - \theta_1(\tau)\right]\,, \quad N_{BA} = \frac{hL}{k}$

2. $\frac{\partial \theta}{\partial \xi}(1, \tau) = -(1-\omega)\,\left\{ N_B \left[\,\theta(1,\tau) - 1\right] + N_r \left[\,\theta^4(1,\tau) - 1\right]\right\}$ (10)

and $\theta(\xi, 0) = Z(\xi)$, $Z(\xi) = z(x)/T_o$.

Thus, the solution of Equations (9) and (10) is of the form

$$\theta = G(\xi, \tau, \omega, F, m, n, N_B, N_r, N_{BA}) \tag{11}$$

2. Finite Difference Formulation of the Fin Equation

If the fin is divided into q - 2 parts (Fig. 3), with the end points designated by the subscripts 2 and q, then the equivalent formulation of Equations (9) and (10) in finite difference form is

$$(1-\omega)\frac{\theta_i' - \theta_i}{\Delta\tau} = \left[\frac{m}{(1-\omega)\xi_i + \omega}\,\frac{\theta_{i+1} - \theta_{i-1}}{2\cdot\Delta\xi} + \frac{1}{1-\omega}\,\frac{\theta_{i+1} + \theta_{i-1} - 2\theta_i}{(\Delta\xi)^2}\right]$$

$$-F\left[(1-\omega)\xi_i + \omega\right]^{n-m}\,(1-\omega)\left[N_B(\theta_i - 1) + N_r(\theta_i^4 - 1)\right] \tag{12}$$

$$i = 3, 4, \ldots\ (q-1)$$

where $\xi_i = (i-2)\cdot\Delta x$ and

1. $\frac{\theta_3 - \theta_2}{\Delta\xi} = N_{BA}\,(1-\omega)\cdot(\theta_2 - \theta_1)$

2. $\frac{\theta_q - \theta_{q-1}}{\Delta\xi} = -(1-\omega)\left[N_B(\theta_q - 1) + N_r(\theta_q^4 - 1)\right]$ (13)

3. $\theta_i = Z_i$ at $\tau = 0$, with $i = 2, 3, \ldots q$

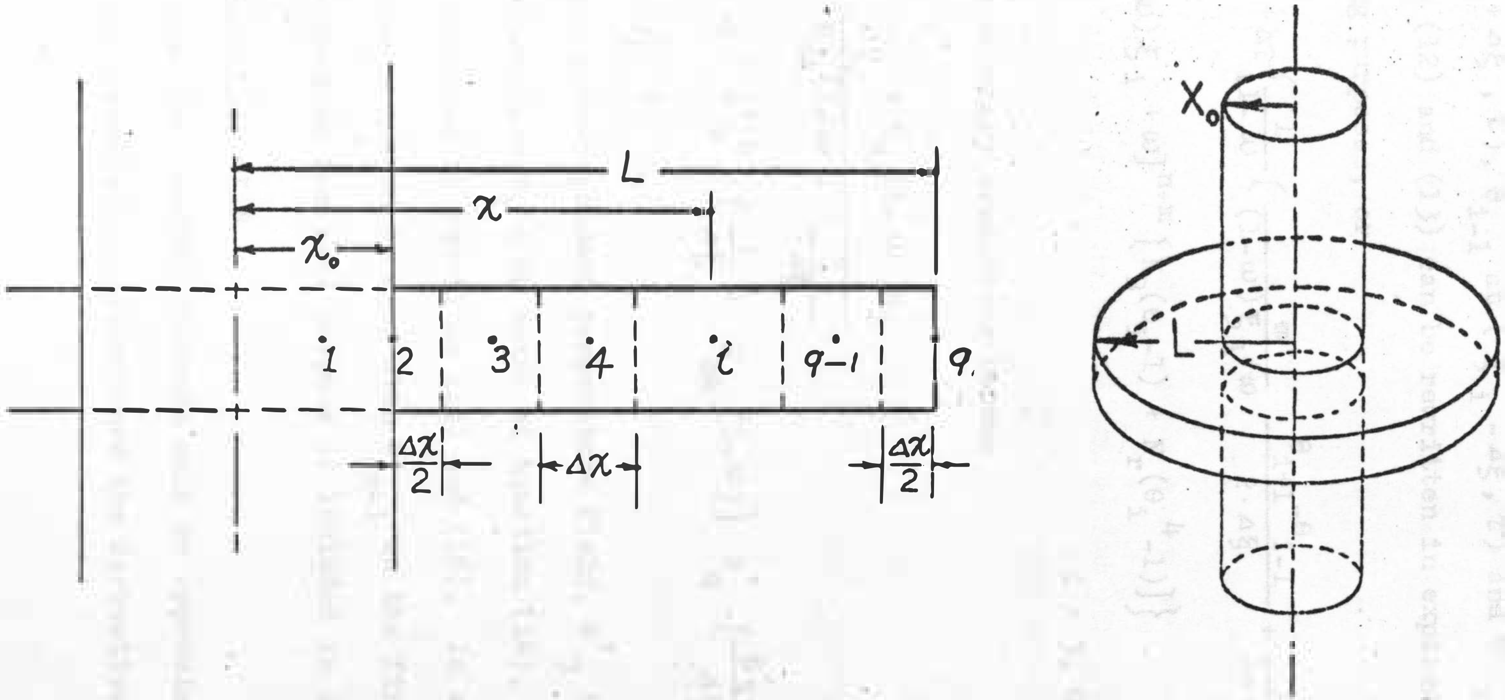


Fig. 3 Coordinate system and subdivision for an annular fin

In the formulation above, $\theta_i$ is the value of $\theta$ at $(\xi_i, \tau)$, $\theta_{i+1}$ at $(\xi_i + \Delta\xi, \tau)$, $\theta_{i-1}$ at $(\xi_i - \Delta\xi, \tau)$ and $\theta'_i$ at $(\xi_i, \tau + \Delta\tau)$. Equations (12) and (13) can be rewritten in explicit form for programing purposes, or

$$\theta_i' = \theta_i + \Delta\tau \left\{ \frac{1}{1-\omega} \left[ \frac{m}{(1-\omega)\xi_i + \omega} \frac{\theta_{i+1} - \theta_{i-1}}{2 \cdot \Delta\xi} + \frac{1}{1-\omega} \frac{\theta_{i+1} + \theta_{i-1} - 2\theta_i}{(\Delta\xi)^2} \right] - F \left[ (1-\omega)\xi_i + \omega \right]^{n-m} \left\{ N_B(\theta_i - 1) + N_r(\theta_i^4 - 1) \right\} \right\} \quad (14)$$

$$i = 3, 4, \ldots (q-1)$$

Also, the boundary conditions become

1. $$\theta_2' = \frac{\frac{\theta_3'}{\Delta\xi} + N_{BA}(1-\omega)\theta_i'}{N_{BA}(1-\omega) + \frac{1}{\Delta\xi}} \quad (15)$$

2. $$N_r(1-\omega)\,\theta_q'^4 + \left[ \frac{1}{\Delta\xi} + N_{BA}(1-\omega) \right] \theta_q' - \left[ \frac{\theta'_{q-1}}{\Delta\xi} + (N_B + N_r)(1-\omega) \right] = 0 \quad (16)$$

To find the time-dependent temperature field, $\theta'_3$ through $\theta'_{q-1}$ were obtained from $\theta_i$'s by means of Equation (14), then $\theta'_2$ and $\theta'_q$ were obtained by Equations (15) and (16). In solving Equation (16), Newton's method is used with $\theta'_{q-1}$ as the first estimate for $\theta'_q$. The program for this purpose is included in Appendix II.

3. Stability Criteria

The numerical method presents only an approximate solution to the original differential equation since the derivatives are replaced by finite differences. A truncation error is introduced by the use of finite subdivision, and the numerical error is due mainly to the accumulation of round-off errors. As the increments are taken smaller and smaller, the numerical results approach the corresponding exact values more closely.

For non-steady numerical problems solved explicitly, the stability condition of the differential equation must be considered. Stability is the condition under which the truncation and numerical errors introduced at one point in time either damp out or increase in amplitude with time.

From physical reasoning the higher the $\theta_i$, the higher the $\theta'_i$, thus the derivative of $\theta'_i$ with respect to $\theta_i$ must be positive. From Equation (14),

$$\frac{d\theta'_i}{d\theta_i} = 1 + \Delta\tau\left\{\frac{-2}{(1-\omega)^2}\frac{1}{(\Delta\xi)^2} - F\left[(1-\omega)\xi_i + \omega\right]^{n-m}\left[N_B + 4N_r\theta_i^{\,3}\right]\right\} > 0 \quad (17)$$

Notice the following inequalities,

$$\theta_1 > \theta_i > 1 \quad (18)$$

$$0 < \omega < (1-\omega)\,\xi_i + \omega < 1 \qquad i = 2, 3, \ldots\ldots q$$

and depending upon the values of m and n, one of the following conditions will hold

$$\text{If } n < m, \text{ then} \quad \omega^{n-m} > \left[(1-\omega)\xi_i + \omega\right]^{n-m} > 1$$

$$\text{If } n = m, \text{ then} \quad \left[(1-\omega)\xi_i + \omega\right]^{n-m} = 1 \quad (19)$$

$$\text{If } n > m, \text{ then} \quad \omega^{n-m} < \left[(1-\omega)\xi_i + \omega\right]^{n-m} < 1$$

For $n < m$,

$$1 + \Delta\tau\left\{\frac{-2}{(1-\omega)^2}\ \frac{1}{\Delta\xi} - F\left[(1-\omega)\xi_i + \omega\right]^{n-m}(N_B + 4N_r\theta_i^3)\right\}$$

$$> 1 + \Delta\tau\left\{\frac{-2}{(1-\omega)^2}\ \frac{1}{\Delta\xi} - F\omega^{n-m}(N_B + 4N_r\theta_i^3)\right\} > 0$$

Therefore, the stability condition can be established from the last inequality as

$$\Delta\tau < \frac{1}{\frac{2}{(1-\omega)^2}\ \frac{1}{(\Delta\xi)^2} + F\omega^{n-m}(N_B + 4N_r\theta_1^3)} \tag{20}$$

and similarly, for $n \geq m$

$$\Delta\tau < \frac{1}{\frac{2}{(1-\omega)^2}\ \frac{1}{(\Delta\xi)^2} + F(N_B + 4N_r\theta_1^3)} \tag{21}$$

4. Fin Efficiency

The fin efficiency $\eta$ is defined as the ratio of the actual heat transfer from the extended surface to the heat transfer from the extended surface if the whole surface were at base temperature, or mathematically,

$$\eta = \frac{\int_S \left[H(T-T_o) + \sigma\epsilon(T^4-T_o^4)\right] dS}{\left\{H\left[T(x_o,t) - T_o\right] + \sigma\epsilon\left[T^4(x_o,t) - T_o^4\right]\right\} S} \tag{22}$$

If the surface is divided in the way shown by dotted lines in Figure 3, the finite difference form of Equation (22) is

$$\eta = \Big\{ P_2 \frac{\Delta x}{2} \Big[ H\,(T_2 - T_o) + \sigma\epsilon (T_2^{\,4} - T_o^{\,4}) \Big] + \sum_{i=3}^{q-1} P_i \Delta x \Big[ H(T_i - T_o)$$

$$+ \sigma\epsilon (T_i^{\,4} - T_o^{\,4}) \Big] + (P_q \frac{x}{2} + A_q) \Big[ H\,(T_q - T_o) + \sigma\epsilon (T_q^{\,4} - T_o^{\,4}) \Big] \Big\} \quad (23)$$

$$\Big\{ \Big[ H(T_2 - T_o) + \sigma\epsilon (T_2^{\,4} - T_o^{\,4}) \Big] \Big[ P_2 \frac{\Delta x}{2} + \sum_{i=3}^{q-1} P_i \Delta x + P_q \frac{\Delta x}{2} + A_q \Big] \Big\}^{-1}$$

When both denominator and numerator are divided by $kT_o \Delta x B_2 L^{n-1}$ and with

$$\frac{A_2}{\Delta x B_2 L^n} = \frac{B_1 L^m}{B_2 L^n} \frac{1}{\Delta x} = \frac{B_1}{B_2 L^{n-m+1}} \frac{L}{\Delta x} = \frac{q-2}{F} \quad (24)$$

Equation (23) can be rewritten in terms of dimensionless parameters as

$$\eta = \Big\{ \frac{\omega^n}{2} \Big[ N_B(\theta_2 - 1) + N_r(\theta_2^{\,4} - 1) \Big] + \sum_{i=3}^{q-1} \Big[ (1-\omega)\xi_i + \omega \Big]^n \Big[ N_B(\theta_i - 1)$$

$$+ N_r(\theta_i^{\,4} - 1) \Big] + \Big( \frac{1}{2} + \frac{q-2}{F} \Big) \Big[ N_B(\theta_q - 1) + N_r(\theta_q^{\,4} - 1) \Big] \Big\} \quad (25)$$

$$\Big\{ \Big[ N_B(\theta_2 - 1) + N_r(\theta_q^{\,4} - 1) \Big] \Big[ \frac{\omega^n}{2} + \sum_{i=3}^{q-1} \overline{(1-\omega\xi_i} + \omega)^n + \frac{1}{2} + \frac{q-2}{F} \Big] \Big\}^{-1}$$

Since it is impractical to store the data for the time-dependent temperature, the program for efficiency is included in the program for temperature so that both results can be obtained simultaneously.

# CHAPTER III

## RESULTS AND DISCUSSION

To check the accuracy of the present solution, comparisons were made with two recent publications for steady-state heat transfer in a constant-area straight fin with convection and radiation at the surface. Figure 4 shows the comparison with the analytical and experimental results of Cobble (3) and Figure 5 is a comparison with the work of Sparrow and Nierwith (6). The discrepancy between the present numerical results and the numerical results of reference (6) is not large and certainly within engineering accuracy. Figure 6 gives the steady-state efficiency of the same fin with and without tip heat transfer. Because of the numerous combinations possible, a parametric study will be quite involved but should be done at a future date. It was decided to present curves for a typical range of values of the dimensionless parameters.

Figures 7 to 22 give the transient temperature and efficiency for annular fins and Figures 23 to 30 for straight fins. The effect of changing geometry on transient temperature and efficiency is given in Figures 31 to 36. The values are selected for typical applications in engineering practice.

A constant ambient temperature was assumed in Chapter II. If it were time-dependent, θ would be defined as

$$\theta = T\ (x,t)\ /\ T_o\ (0)$$

$$\theta_o = T_o(t)\ /\ T_o\ (0)$$

and ( $\theta - \theta_o$ ) would take the place of ( $\theta - 1$ ) and ( $\theta^4 - \theta_o^4$) for ( $\theta^4 - 1$ ) in all the formulas.

Since curves on Figure 7 through 36 were run for comparison purposes, the temperatures of the circulating fluid and the ambient fluid were assumed to be constant. In practical applications, the values of time-dependent temperatures should be substituted during calculation.

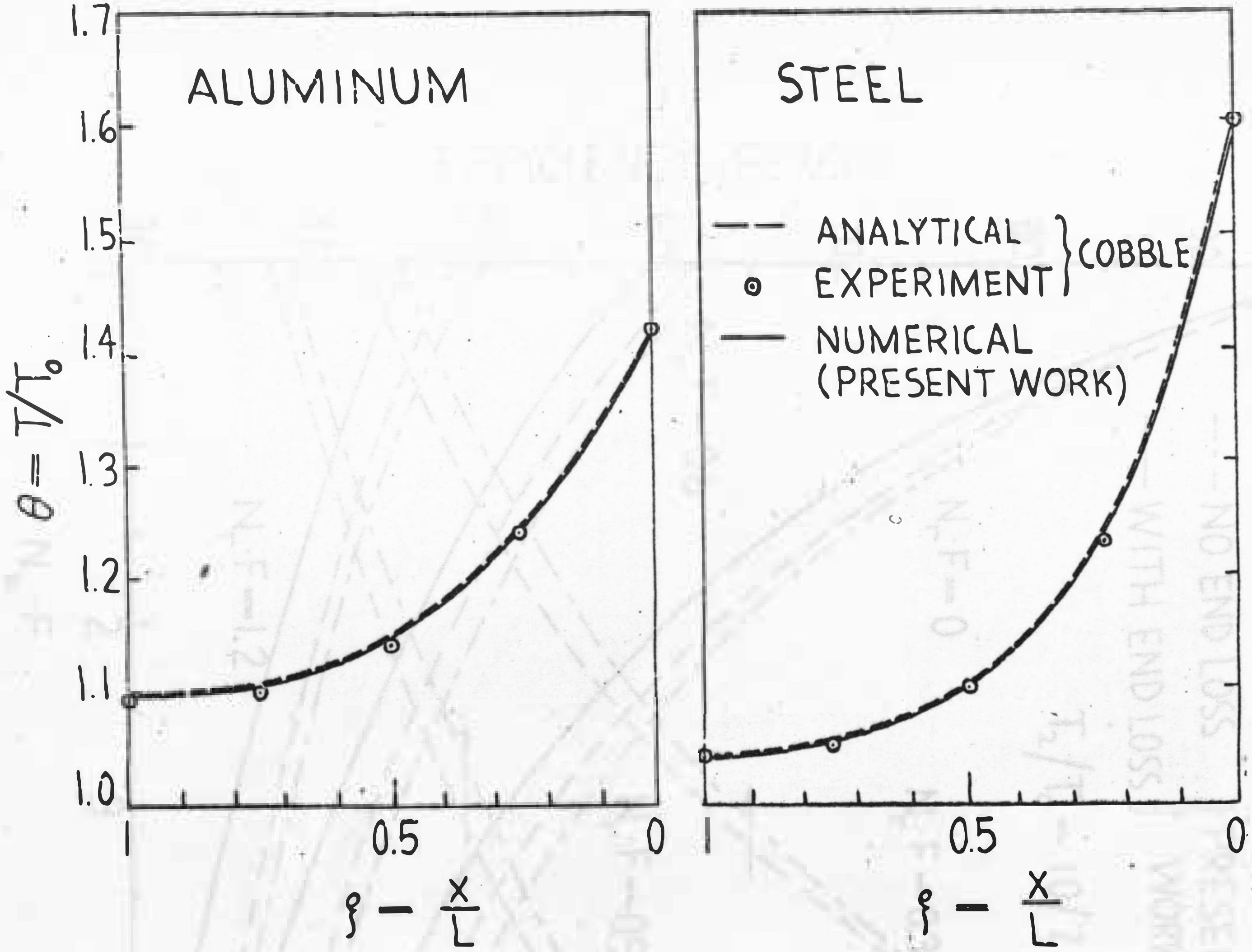


Fig. 4 Steady-state temperature distribution for straight steel and aluminum fins of constant area

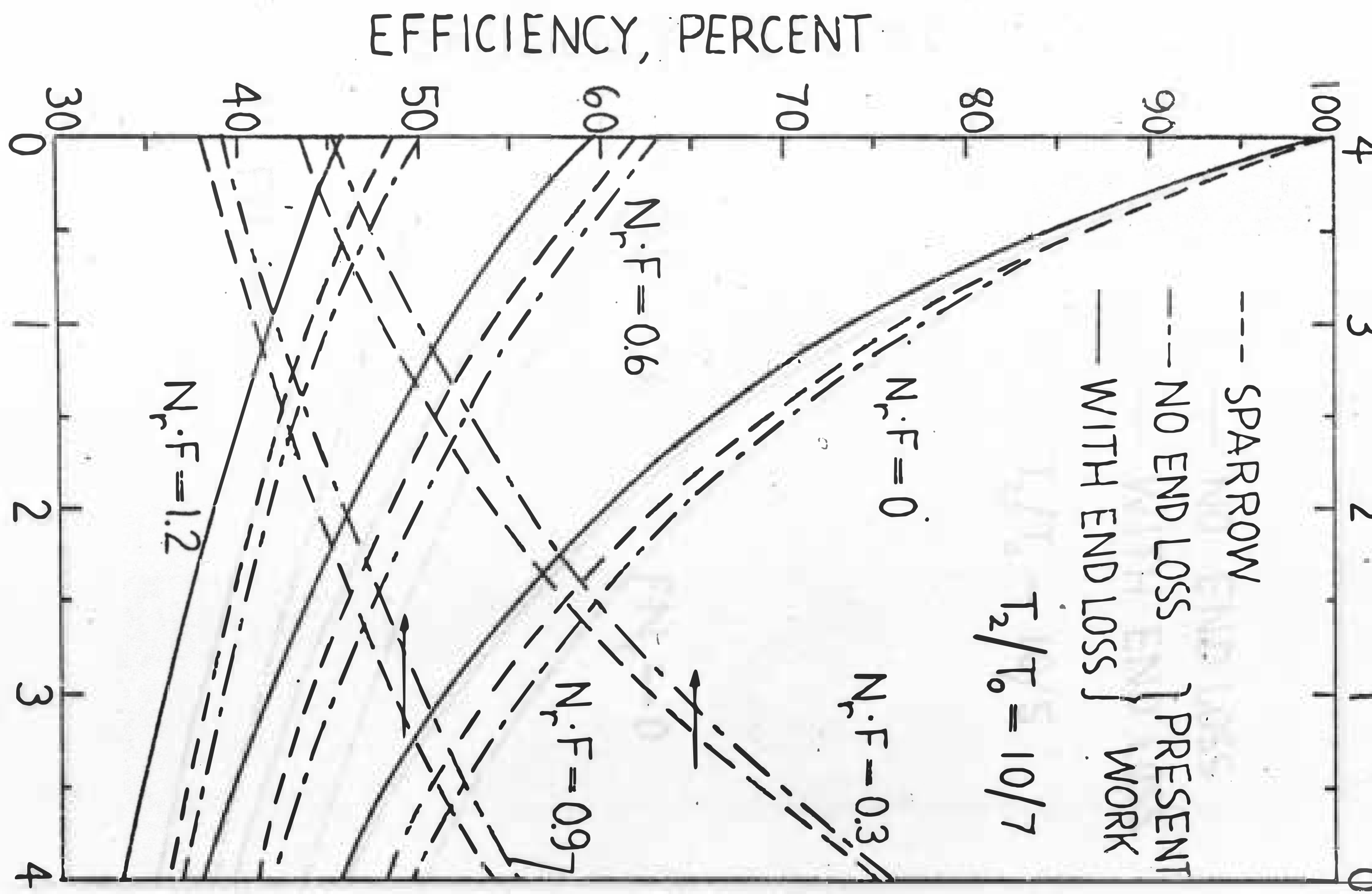


Fig. 5 Steady-state efficiency of convecting and radiating straight constant-area fins, with $\omega = 0$

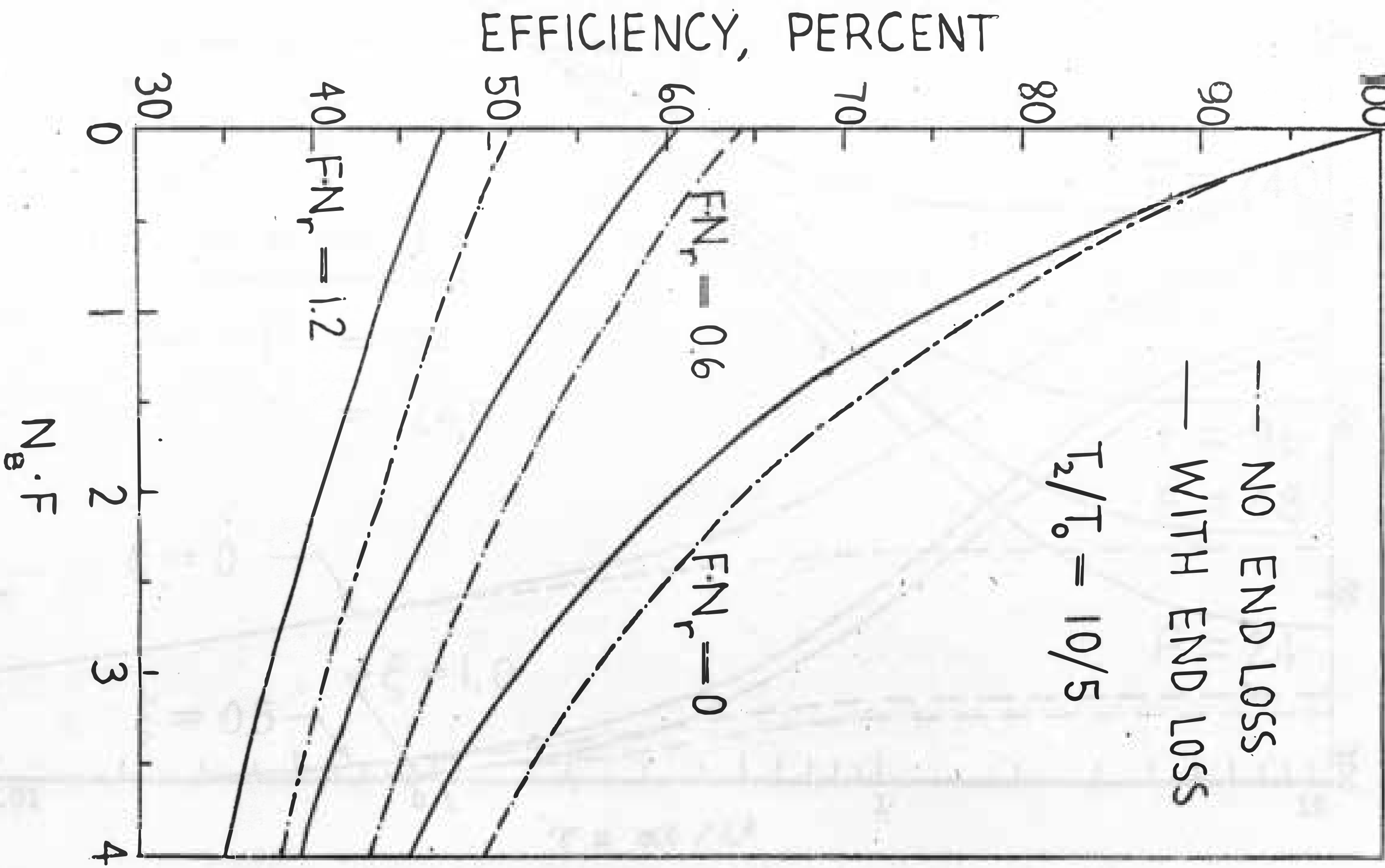


Fig. 6 Steady-state efficiency of convecting and radiating straight constant-area fins, with $\omega = 0$

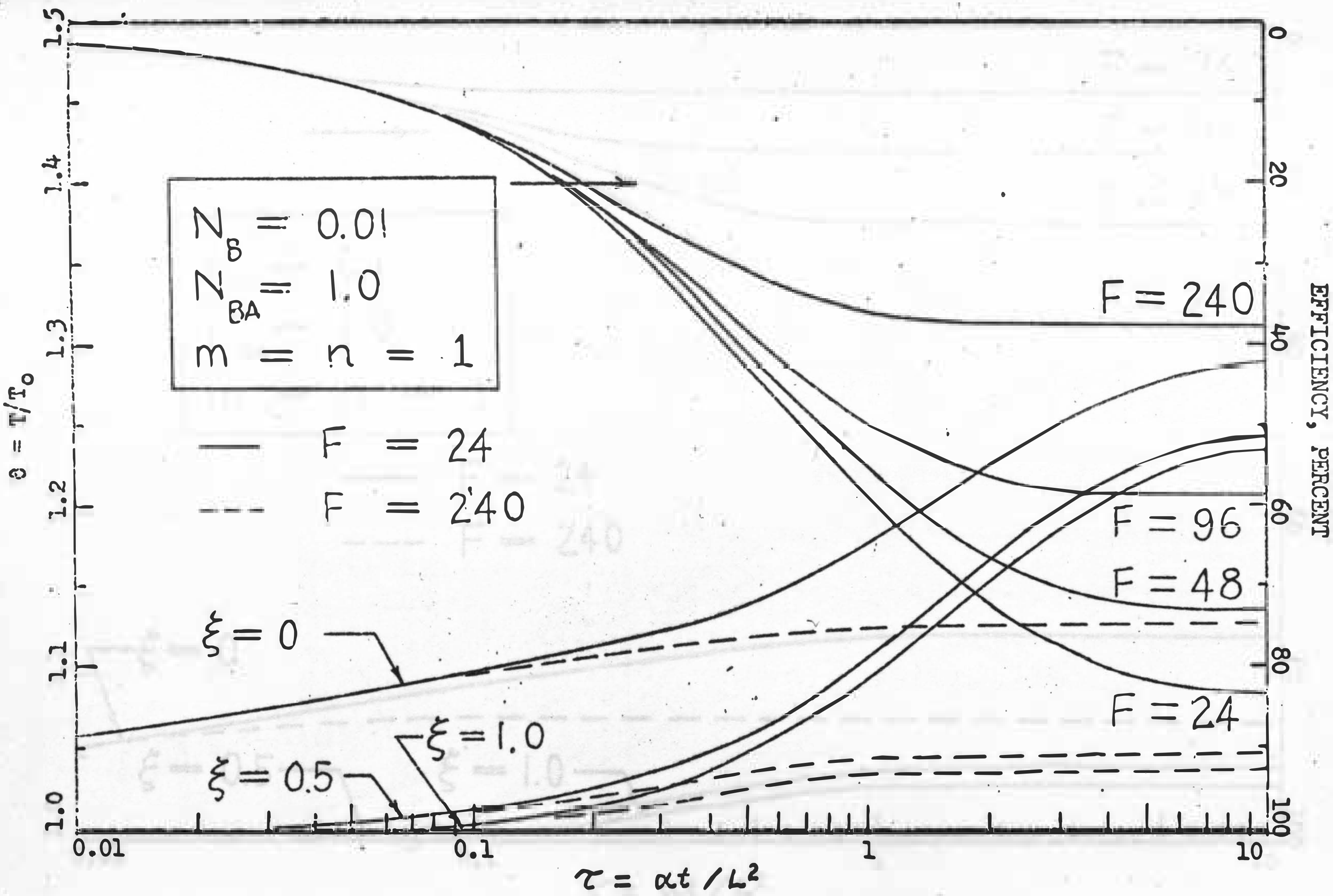


Fig. 7 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.1$, $N_r = 0.0001$

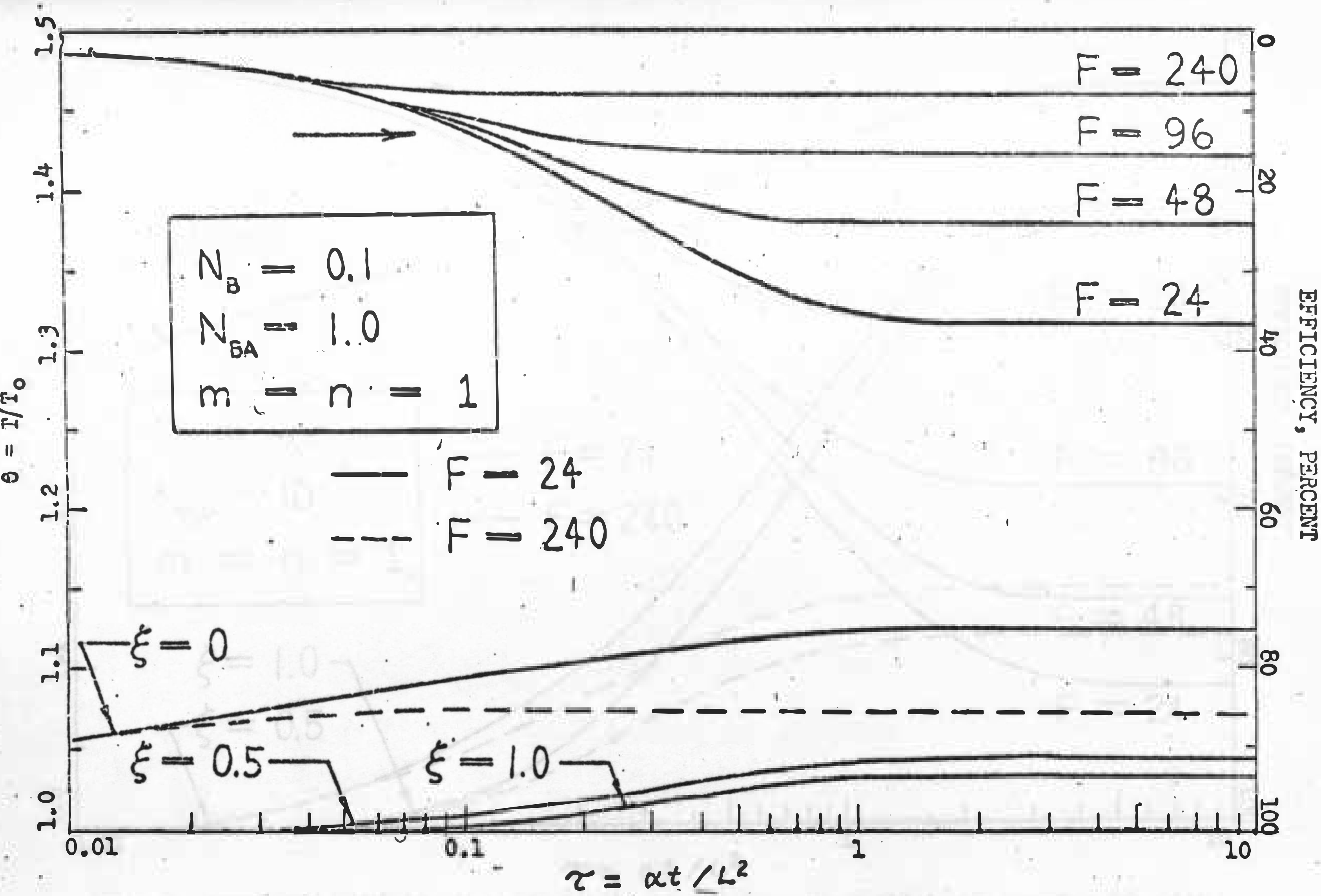


Fig. 8 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.1$, $N_r = 0.0001$

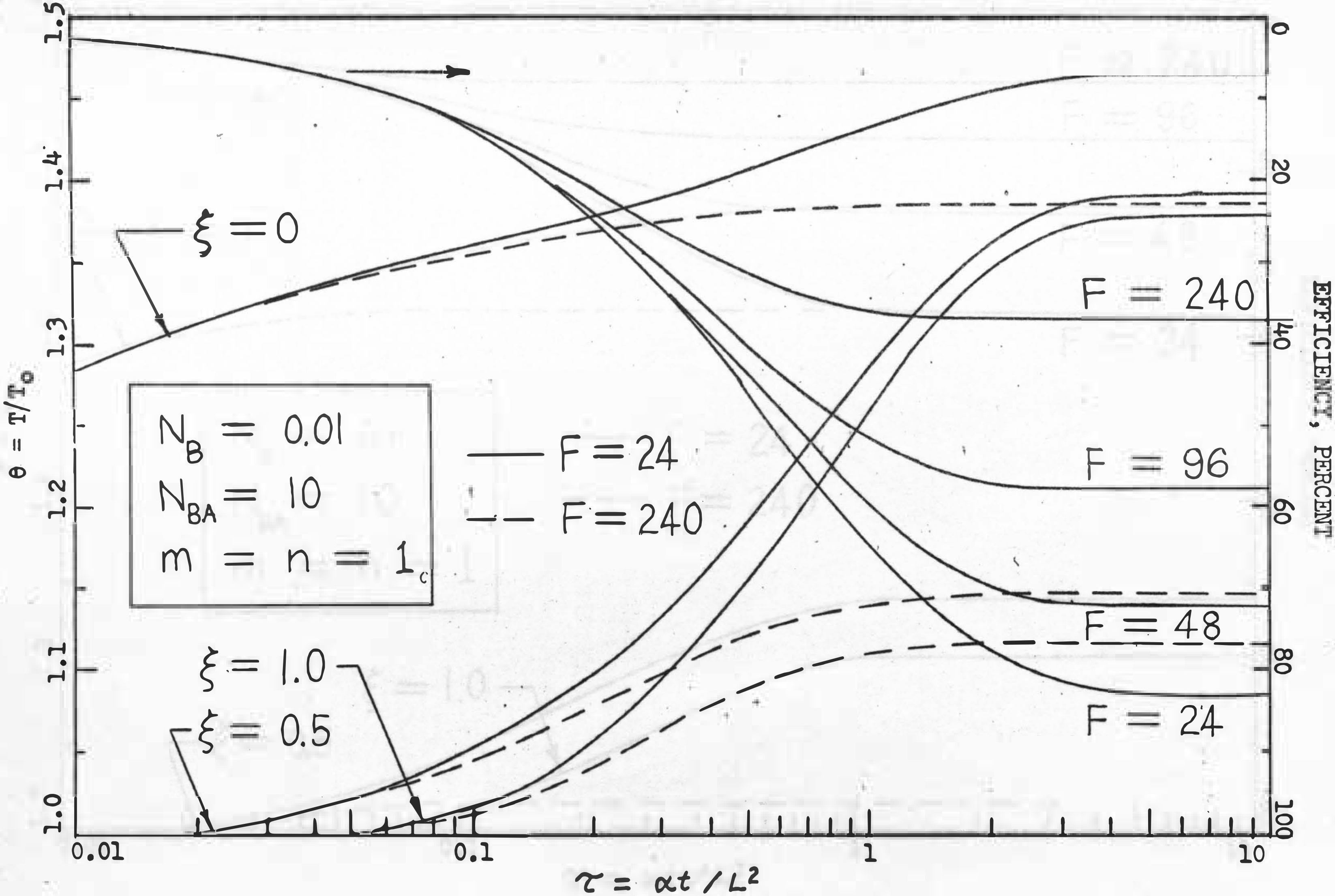


Fig. 9 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega$ = 0.1, $N_r$ = 0.0001

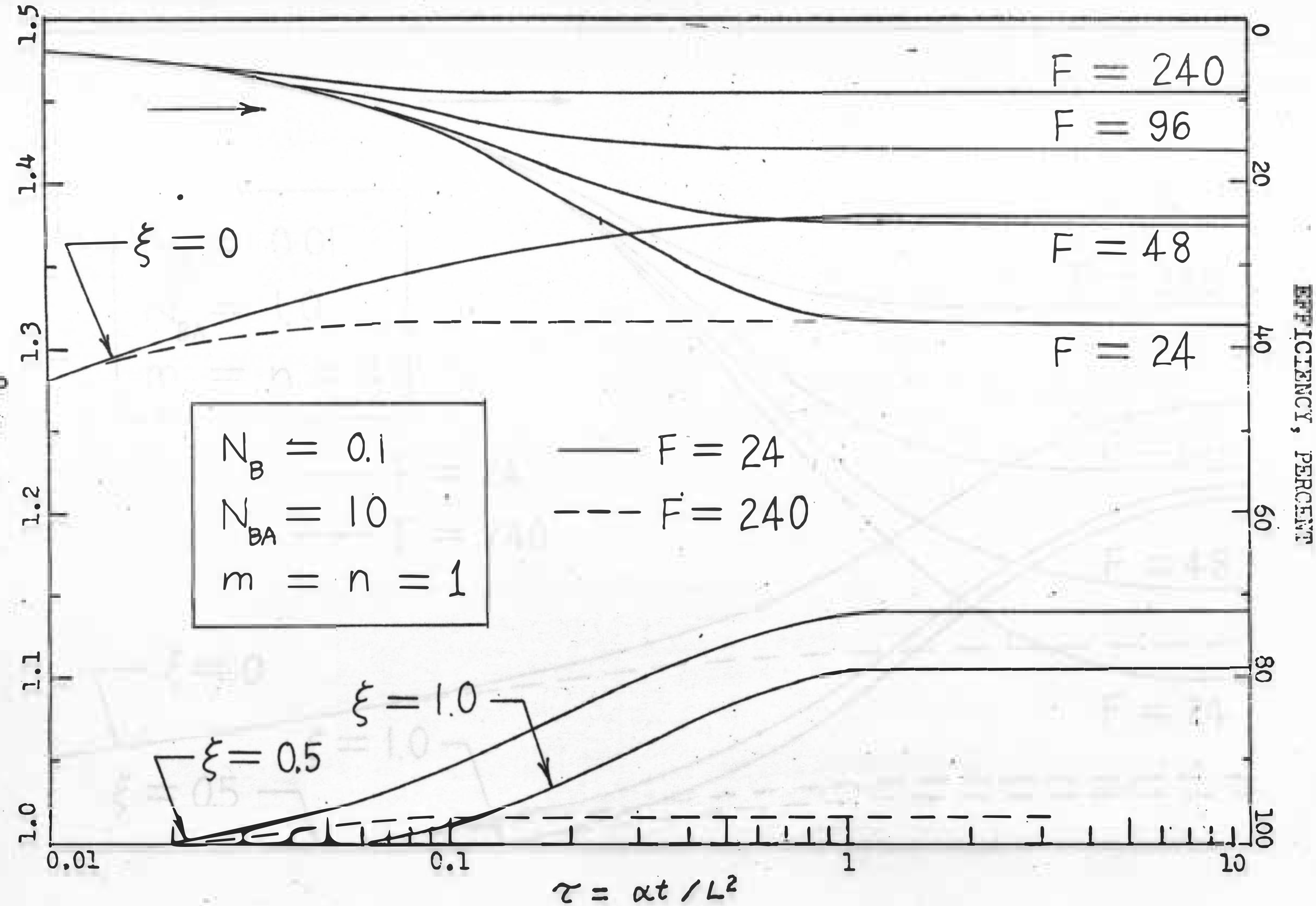


Fig. 10 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.1$, $N_r = 0.0001$

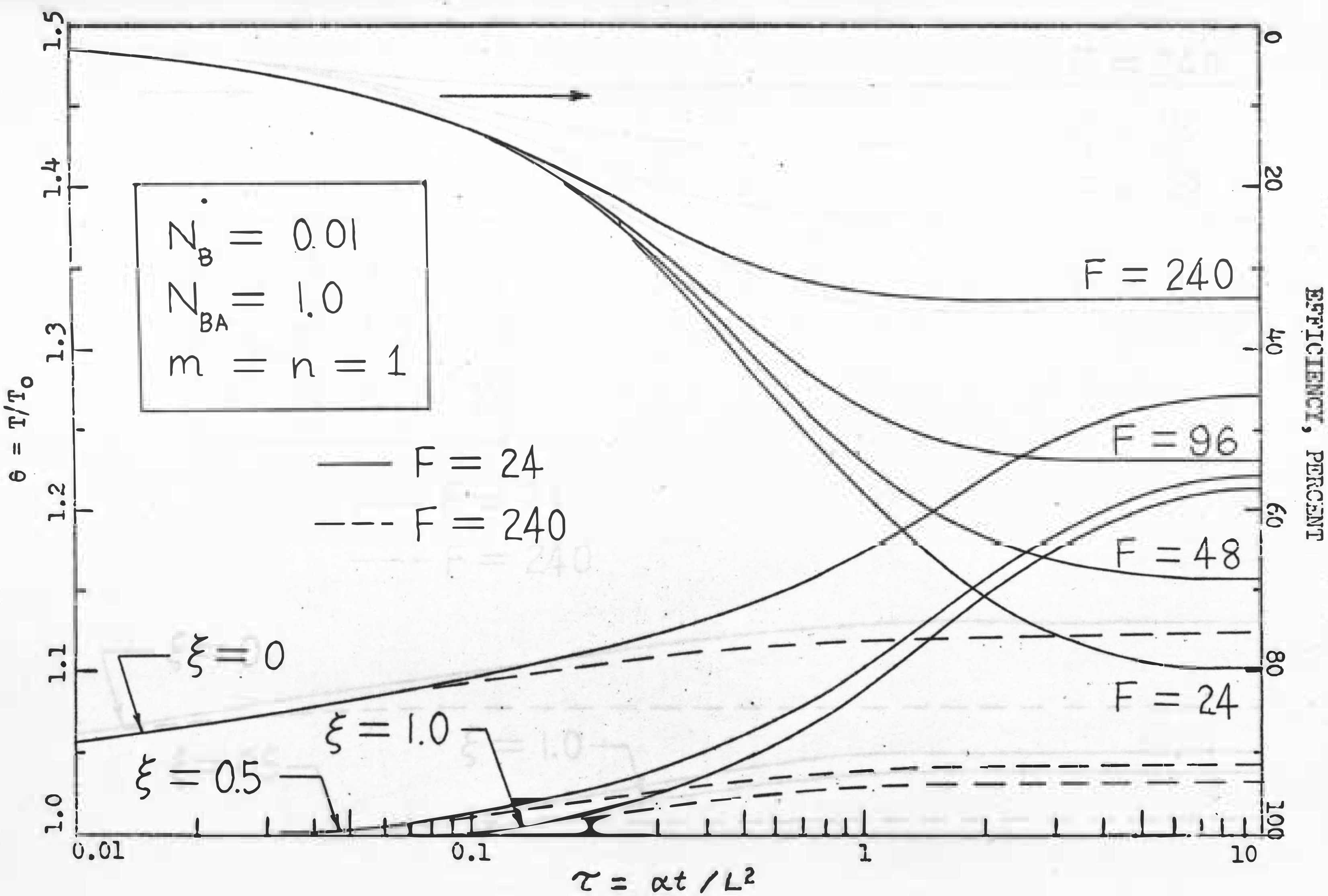


Fig. 11 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.1$, $N_r = 0.0005$

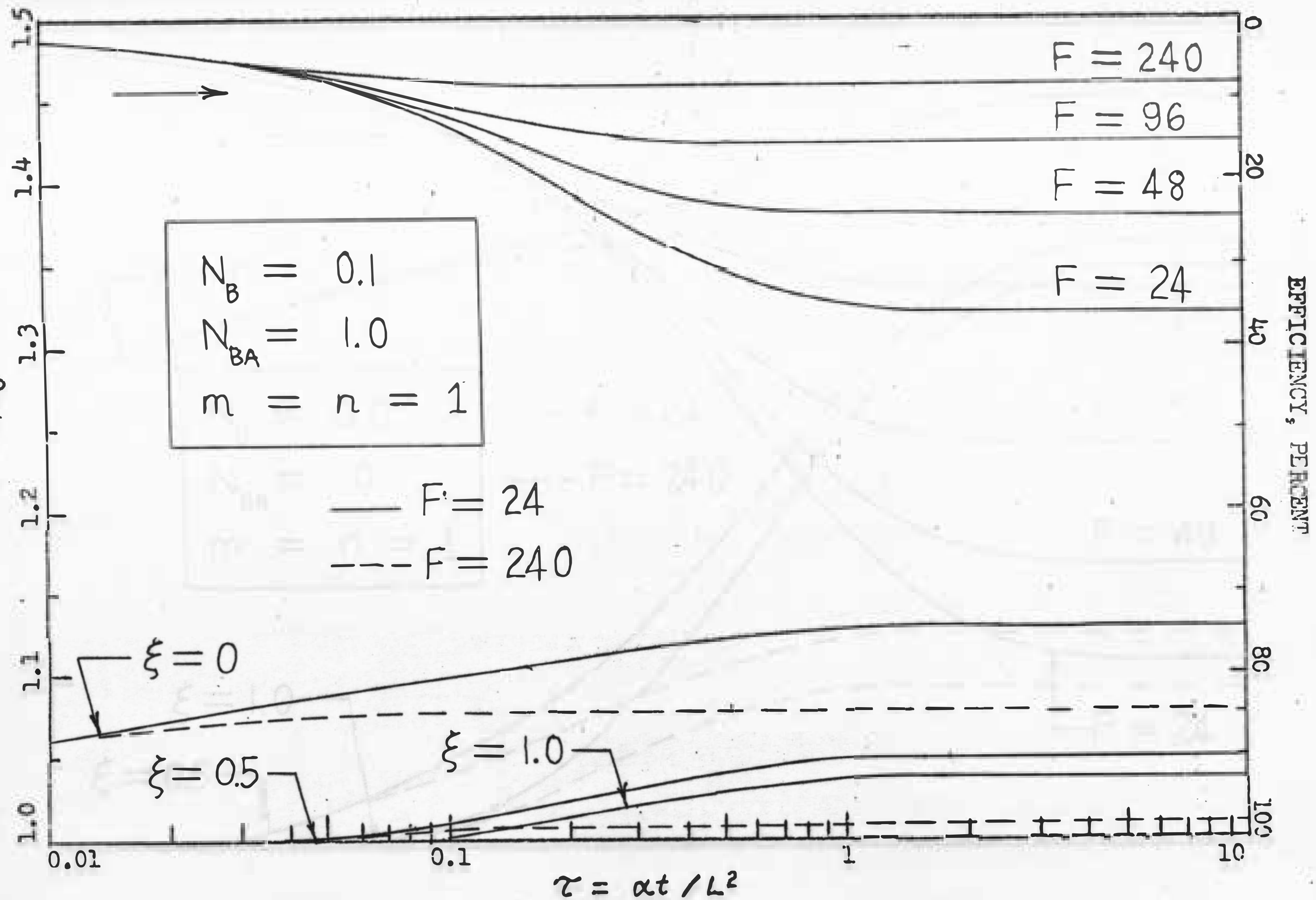


Fig. 12 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.1$, $N_r = 0.0005$

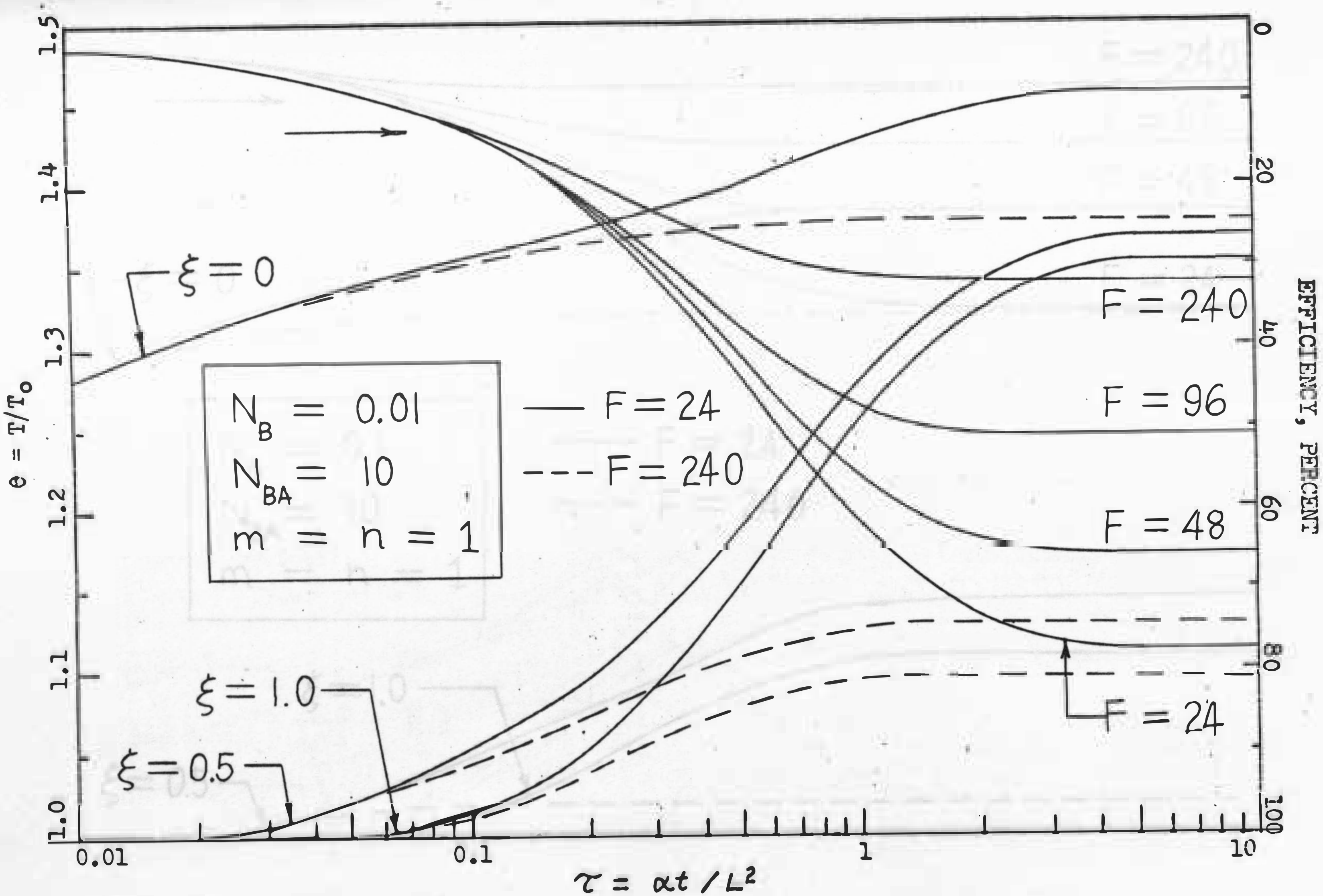


Fig. 13 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.1$, $N_r = 0.0005$

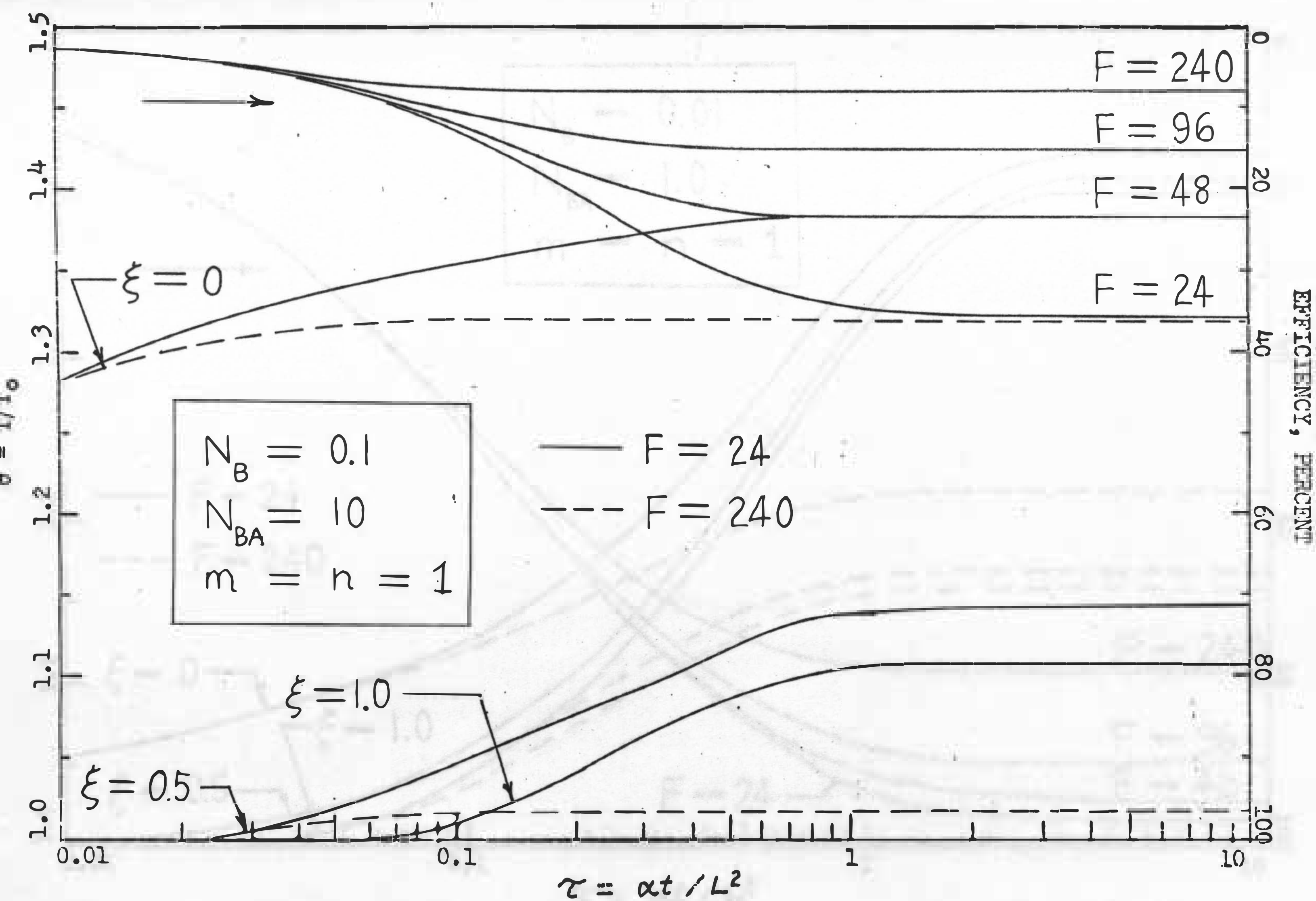


Fig. 14 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.1$, $N_r = 0.0005$

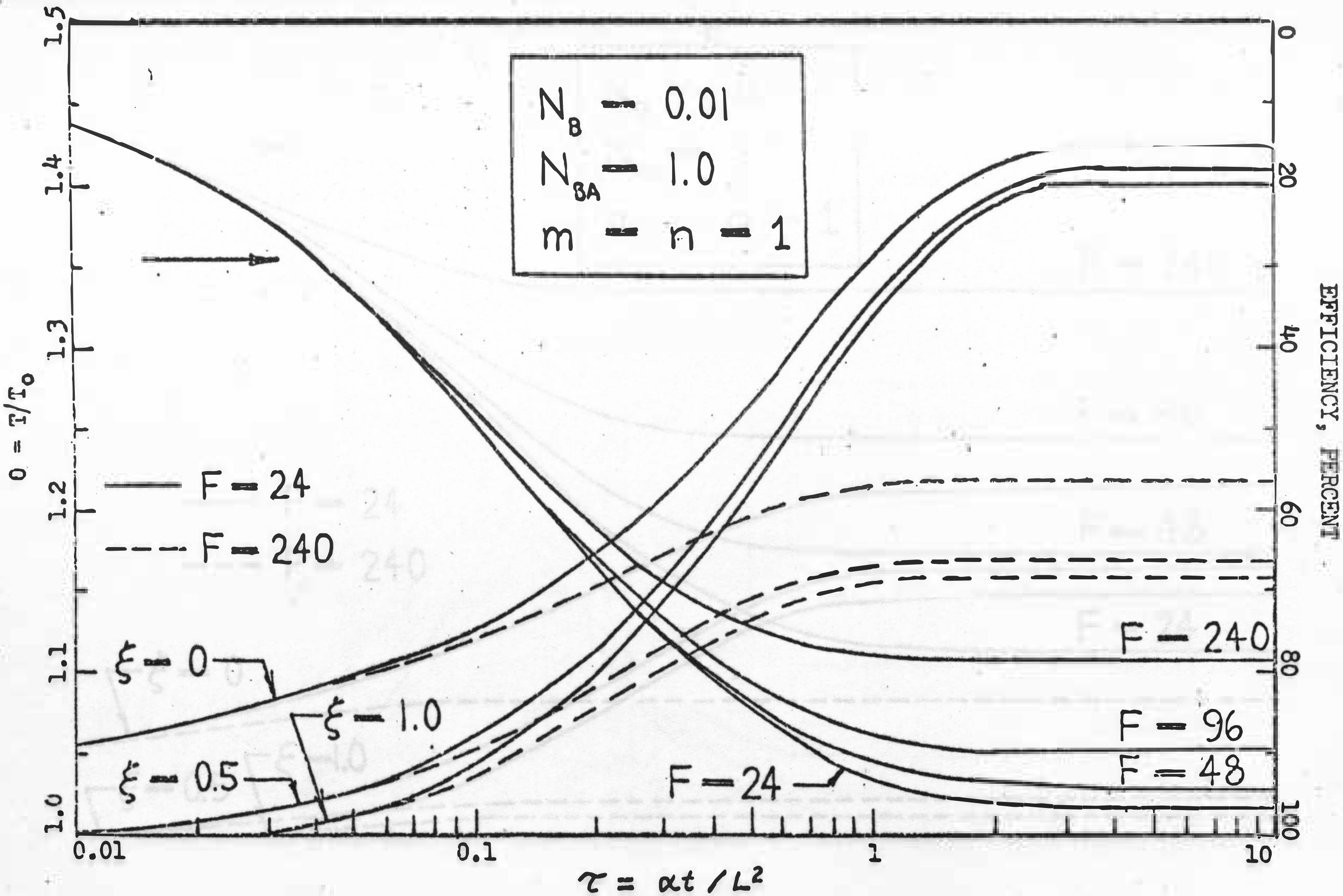


Fig. 15 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0001$

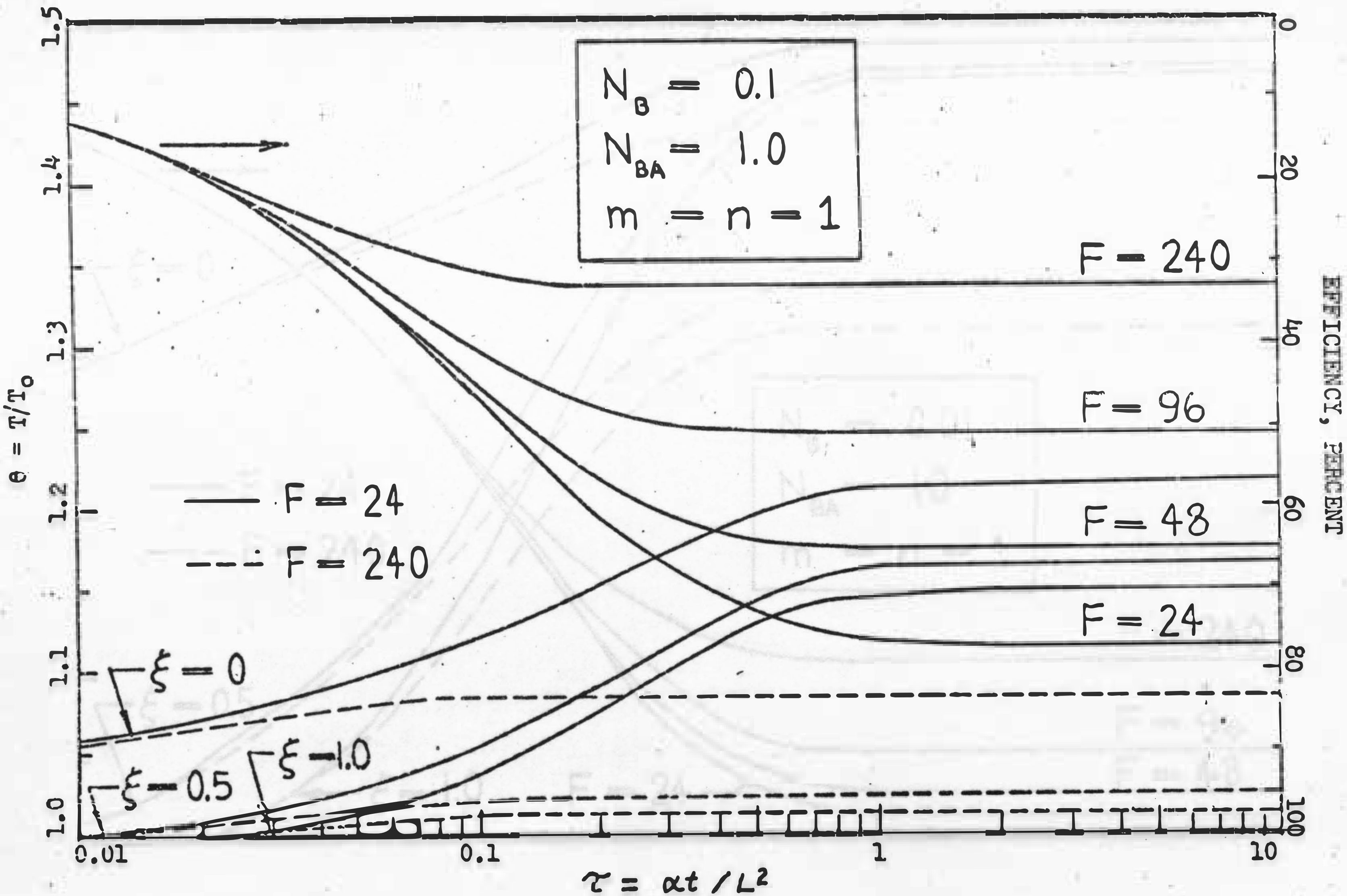


Fig. 16 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0001$

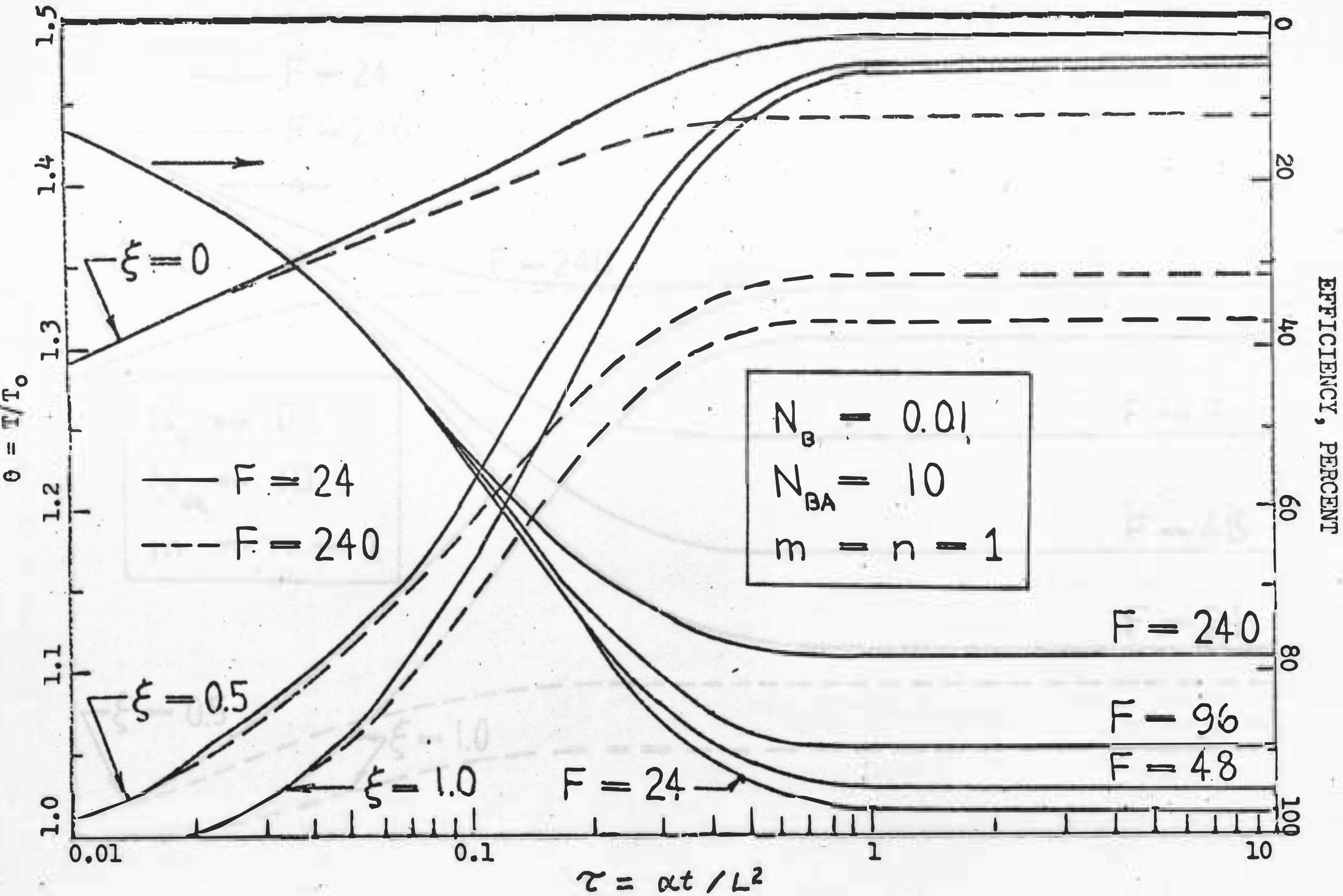


Fig. 17 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0001$

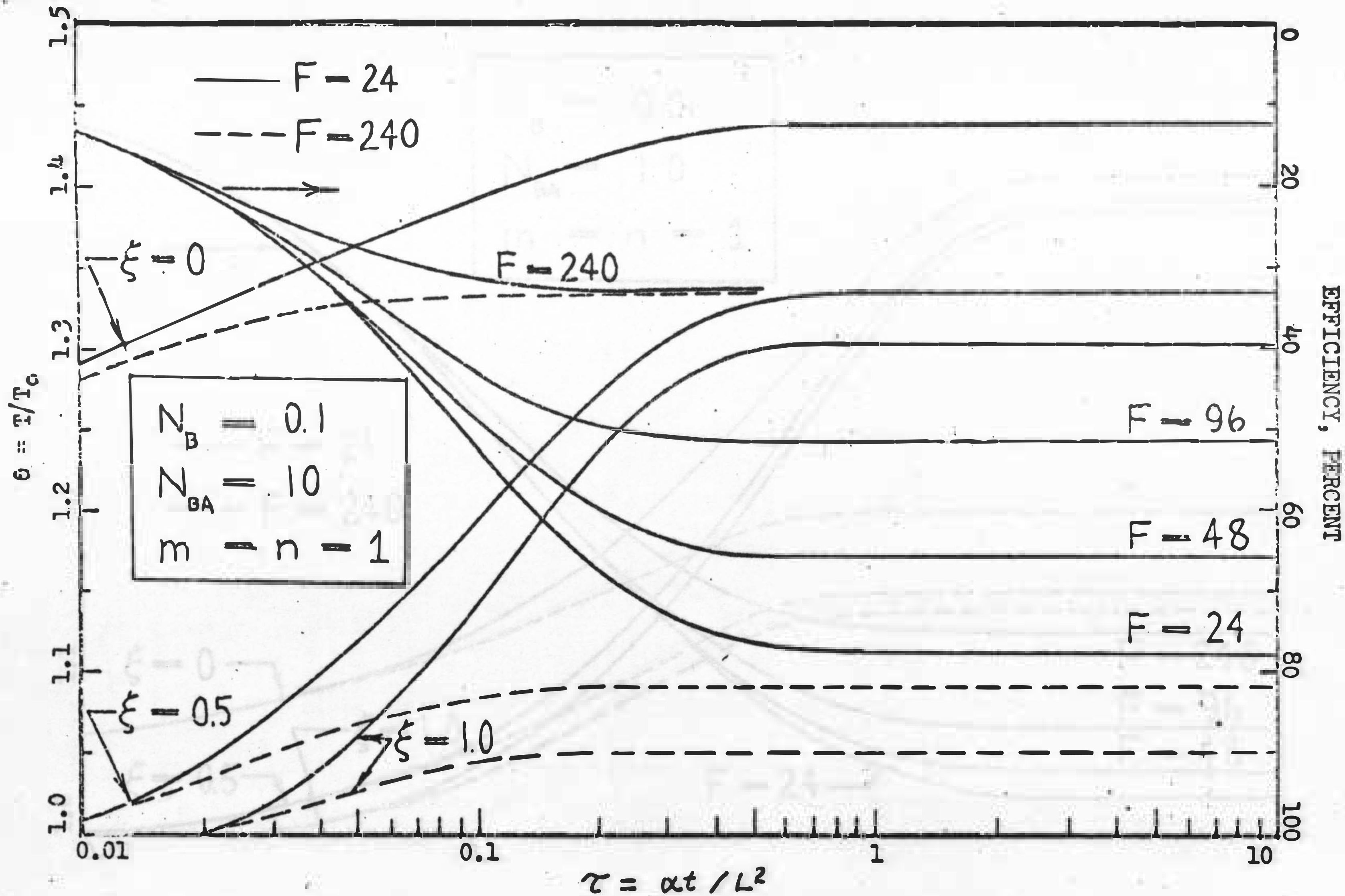


Fig. 18 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0001$

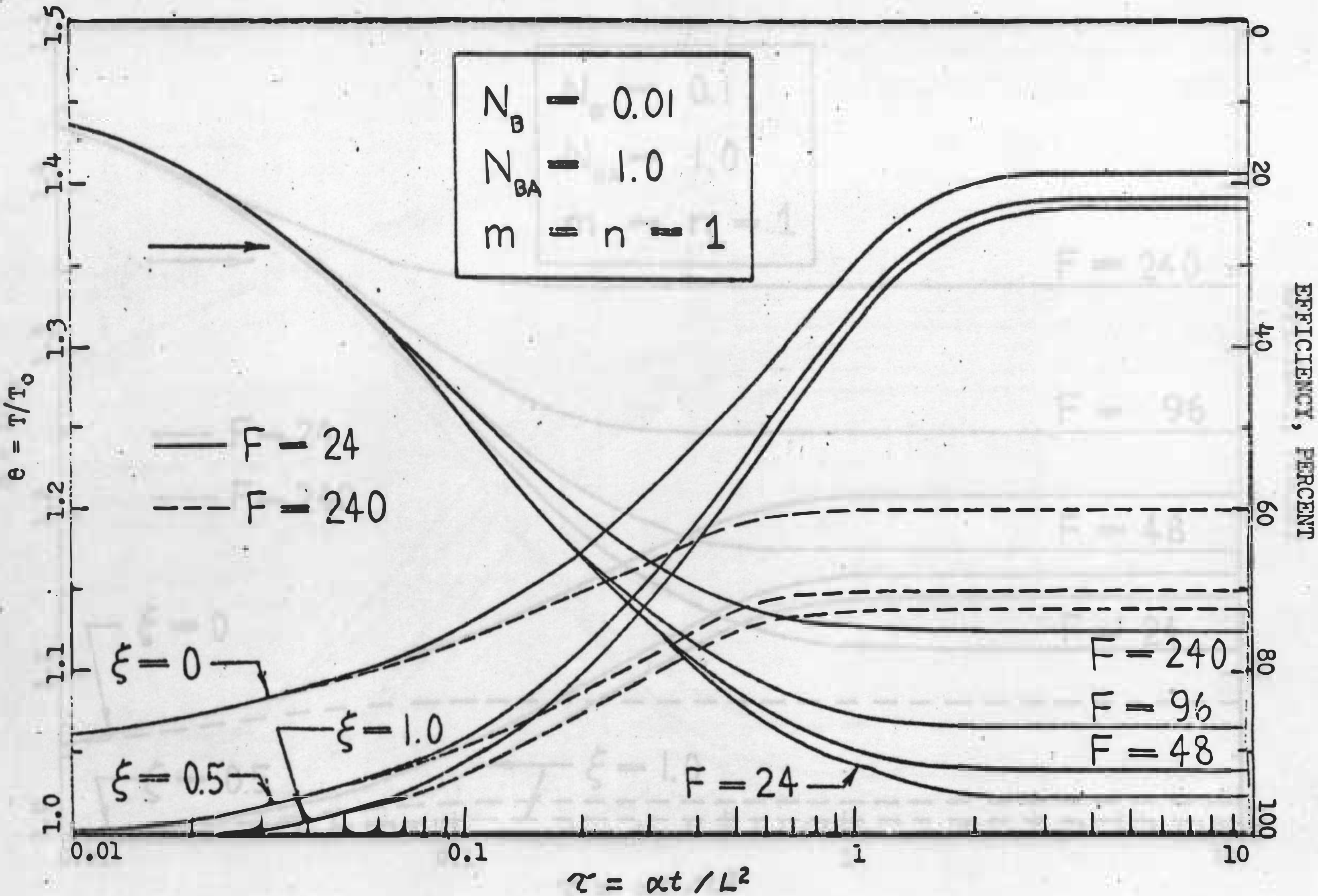


Fig. 19 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0005$

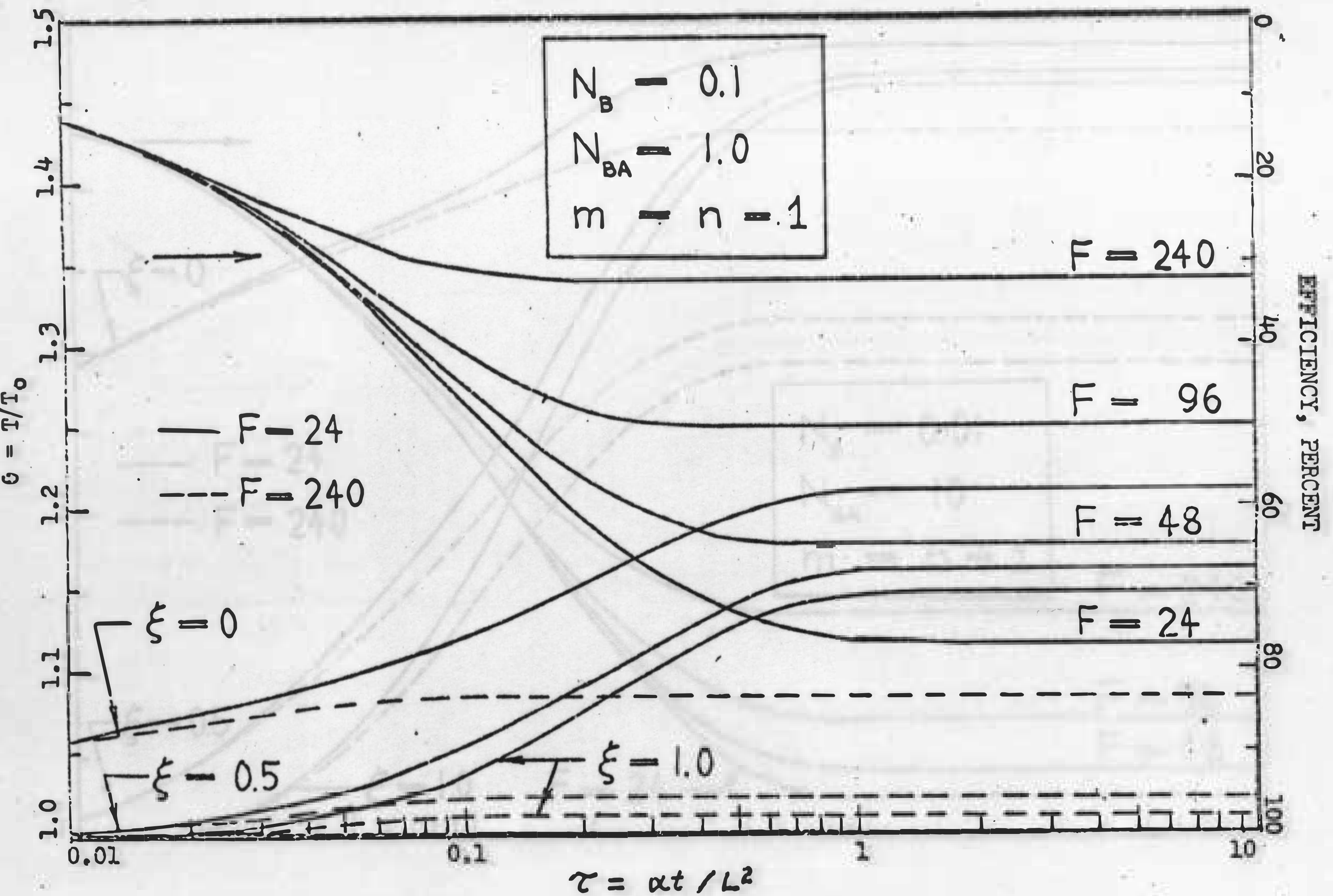


Fig. 20 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0005$

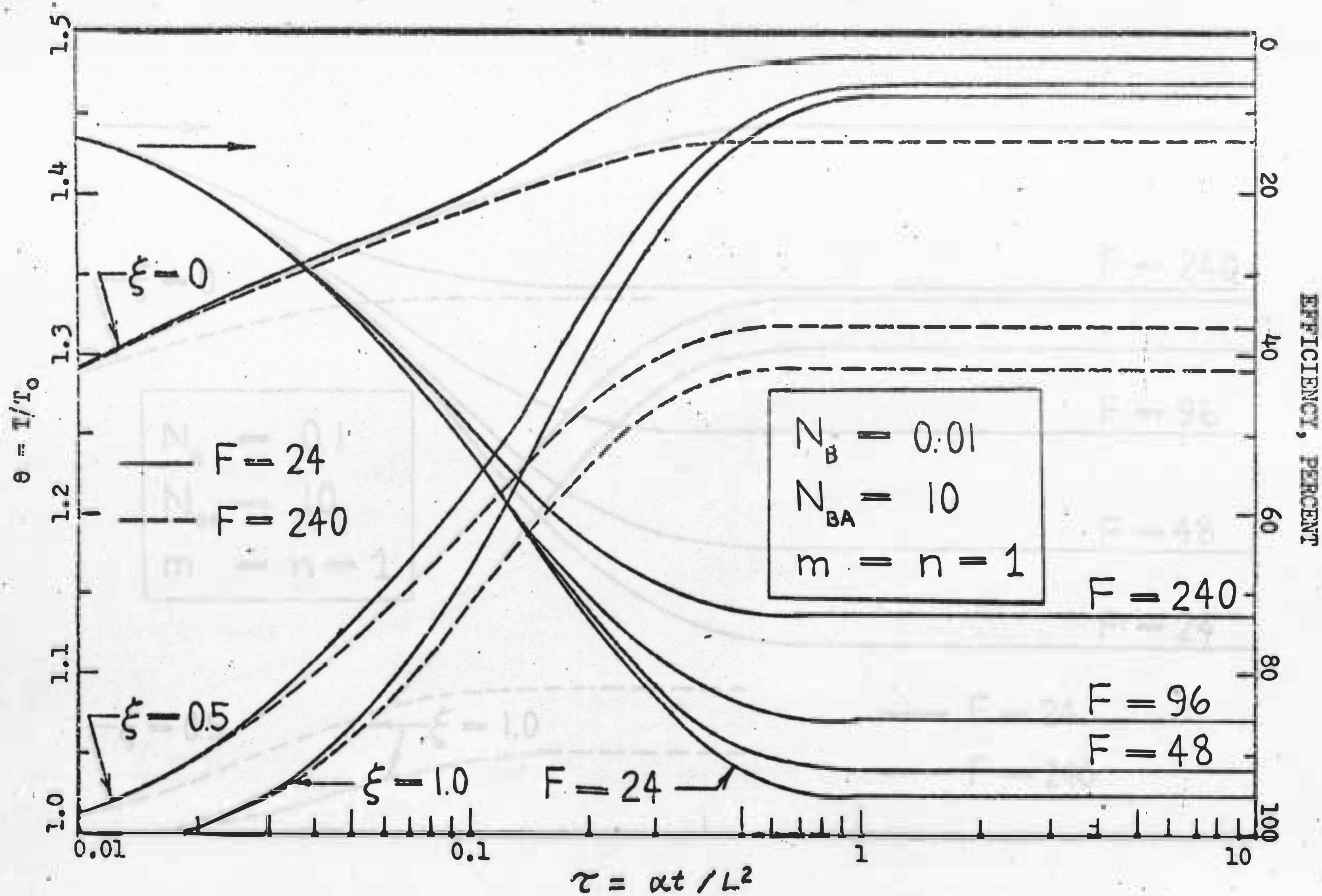


Fig. 21 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0005$

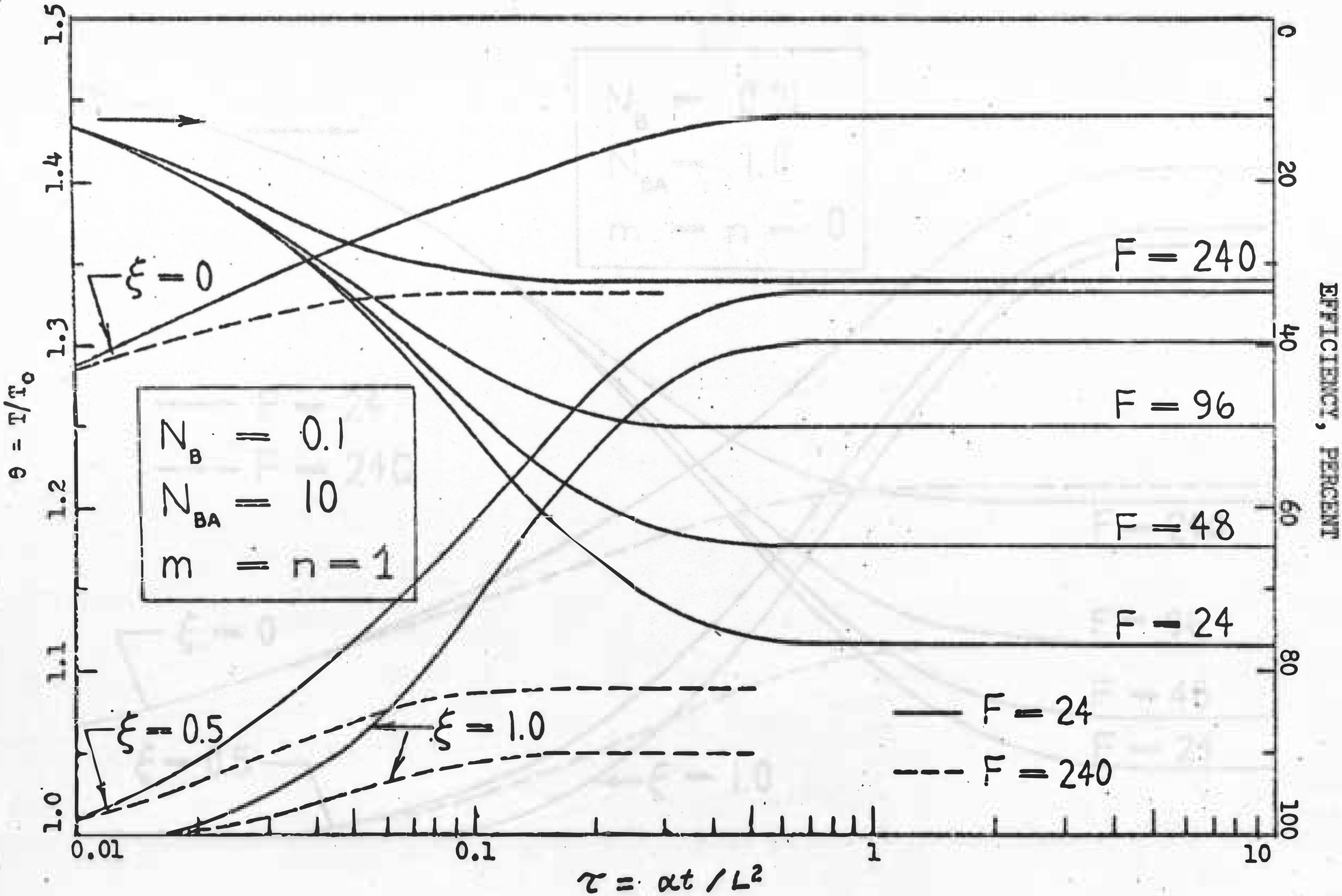


Fig. 22 Transient temperatures and efficiency for a constant-thickness annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0005$

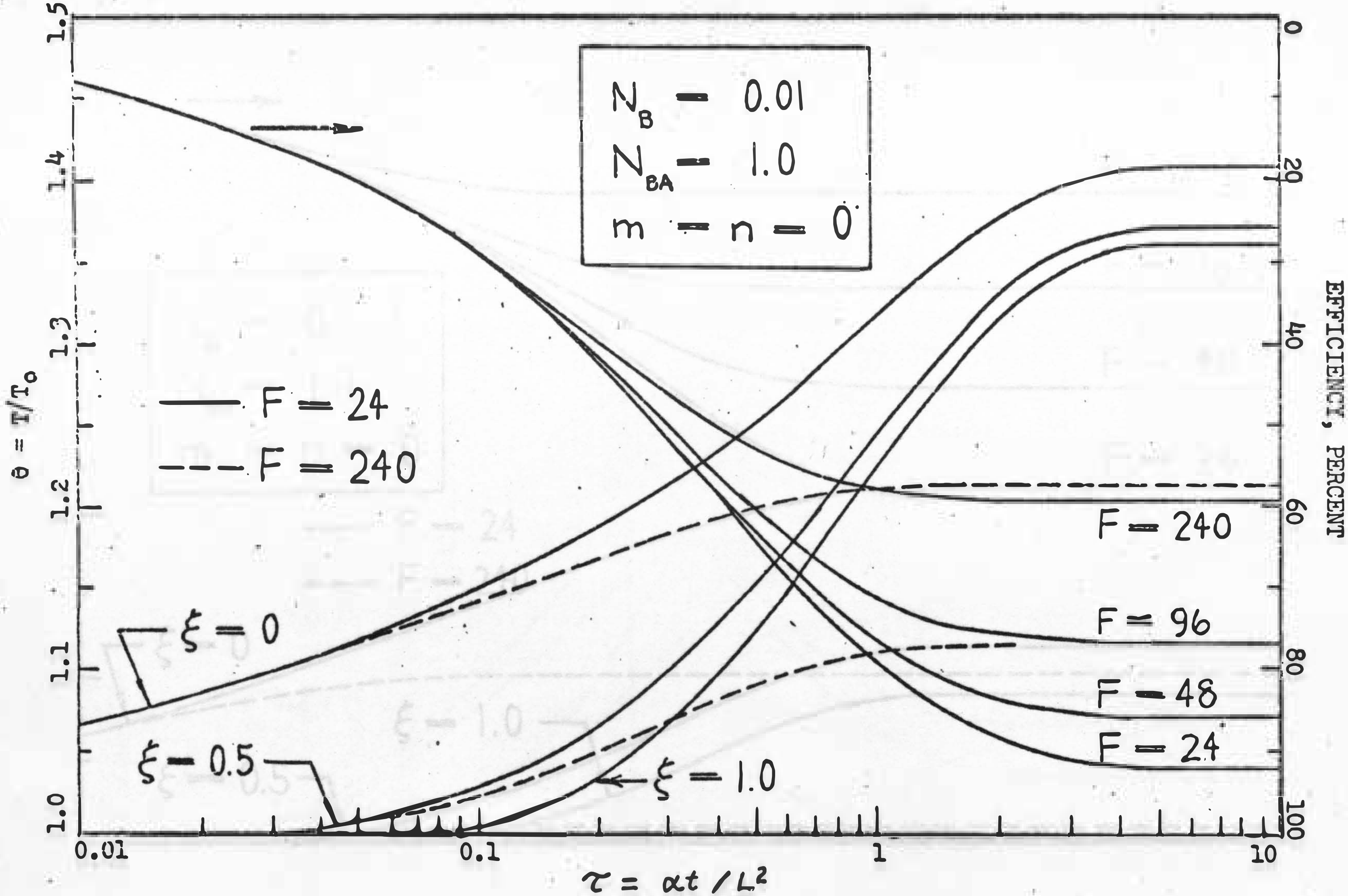


Fig. 23 Transient temperatures and efficiency for a constant-area straight fin with combined convection and radiation, for $\omega = 0$, $N_r = 0.0001$

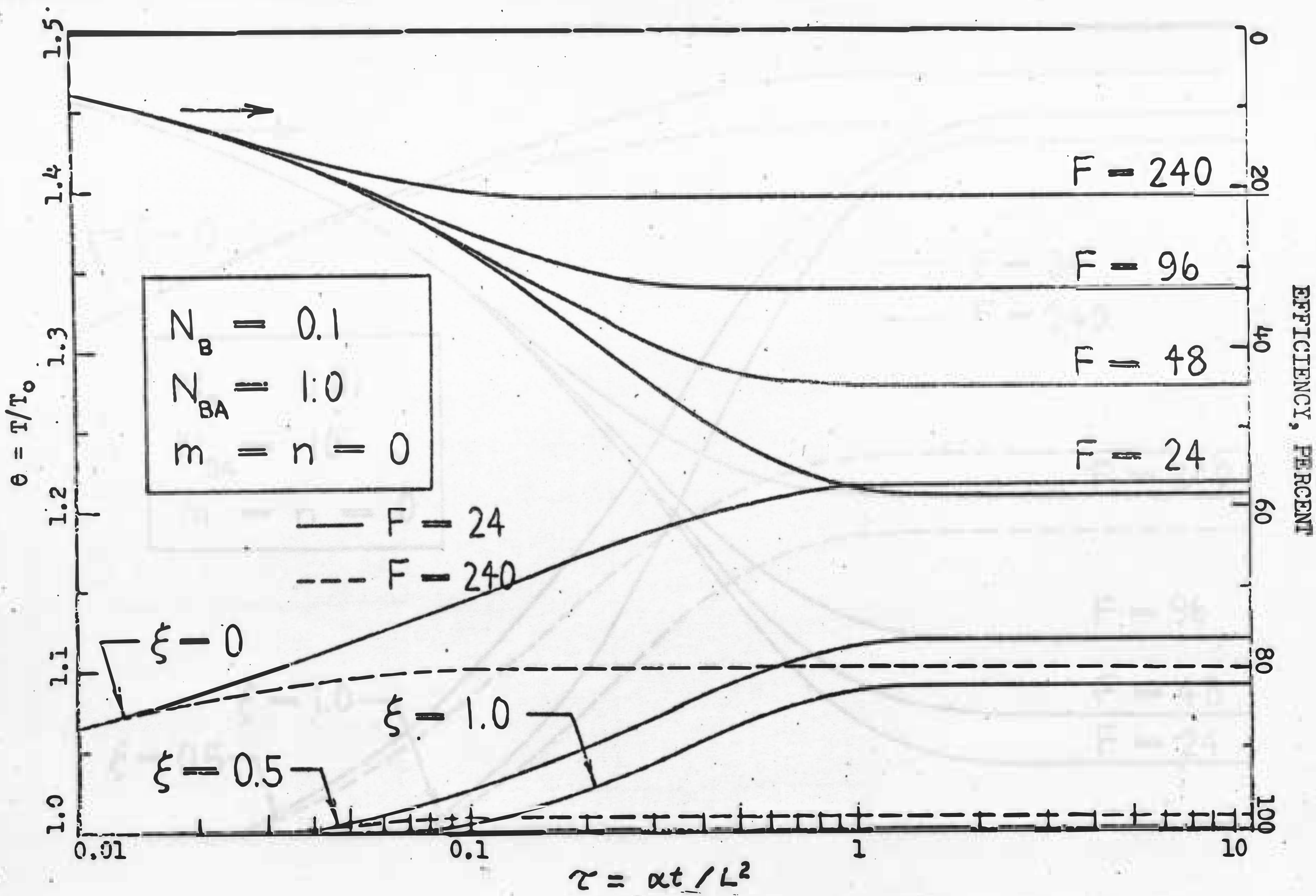


Fig. 24 Transient temperatures and efficiency for a constant-area straight fin with combined convection and radiation, for $\omega = 0$, $N_r = 0.0001$

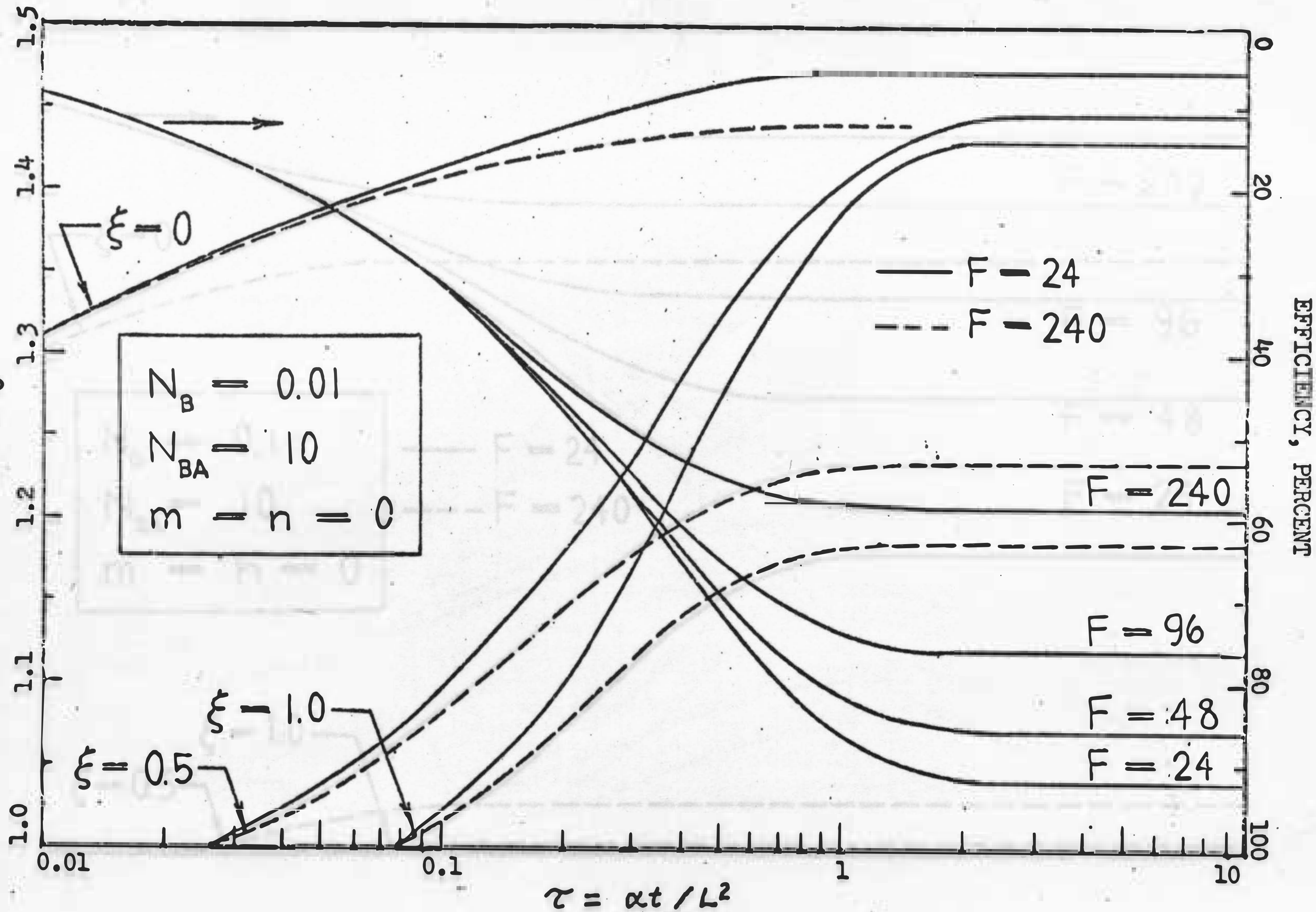


Fig. 25 Transient temperatures and efficiency for a constant-area straight fin with combined convection and radiation, for $\omega = 0$, $N_r = 0.0001$

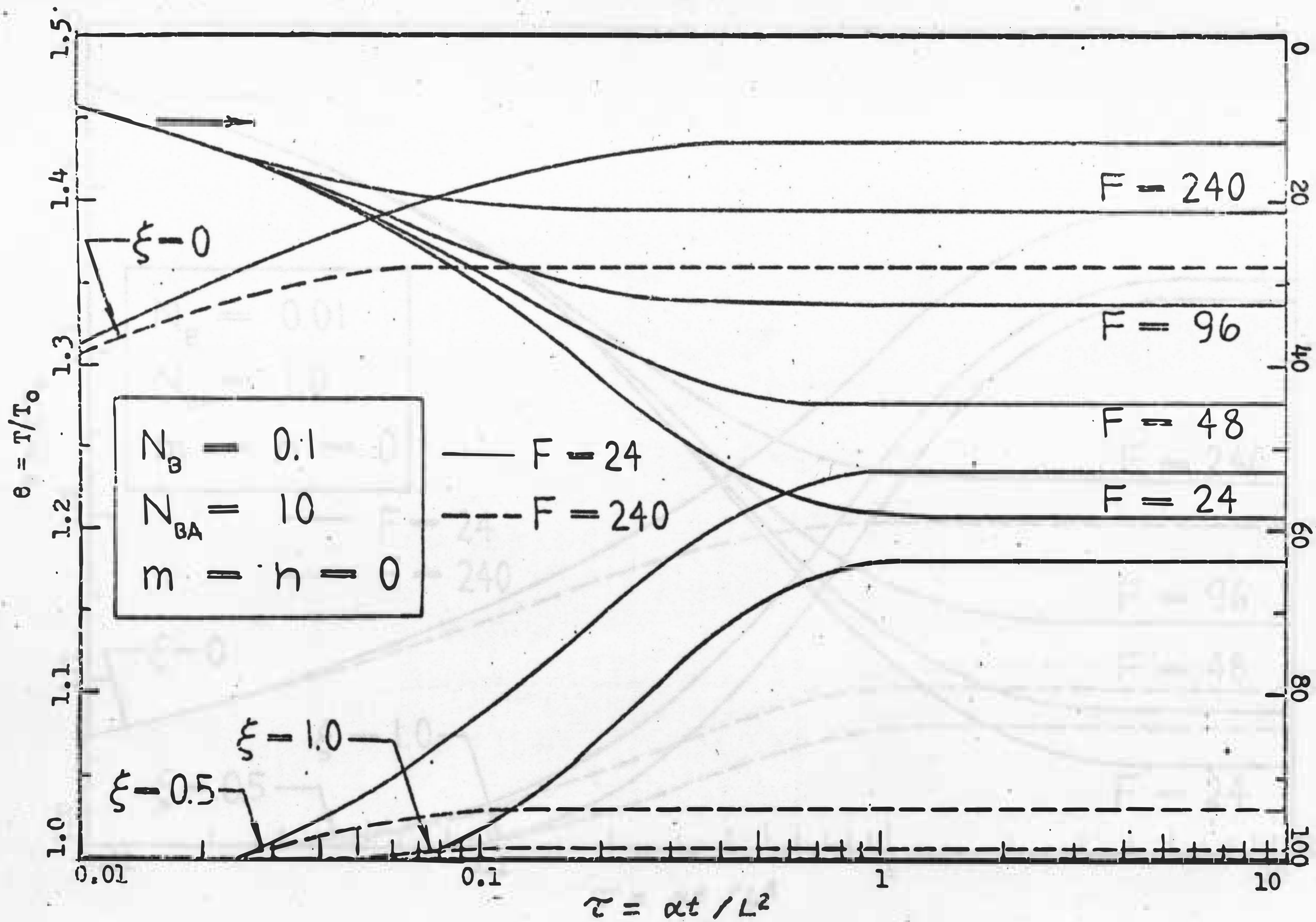


Fig. 26 Transient temperatures and efficiency for a constant-area straight fin with combined convection and radiation, for $\omega = 0$, $N_r = 0.0001$

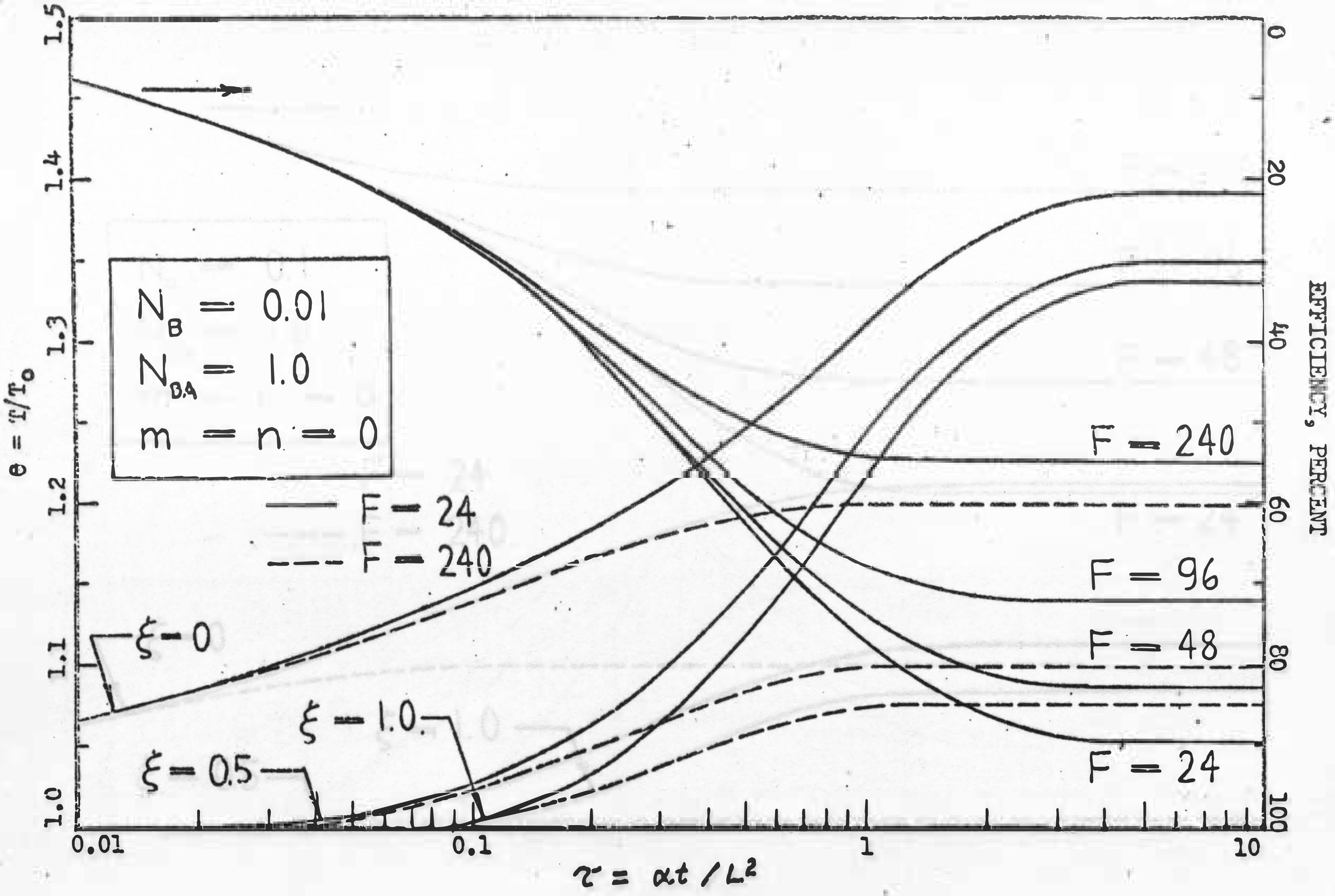


Fig. 27 Transient temperatures and efficiency for a constant-area straight fin with combined convection and radiation, for $\omega = 0$, $N_r = 0.0005$

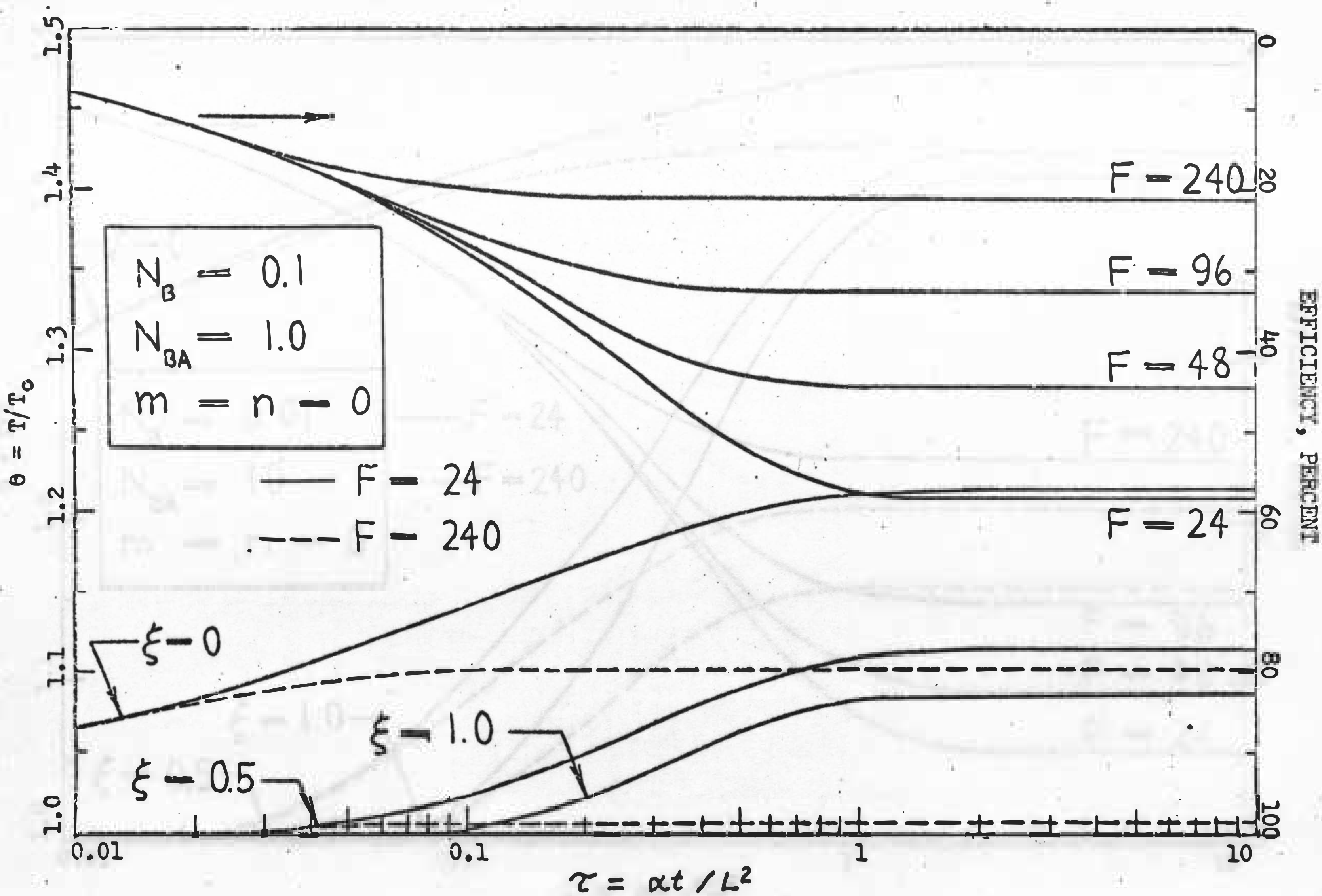


Fig. 28 Transient temperatures and efficiency for a constant-area straight fin with combined convection and radiation, for $\omega = 0$, $N_r = 0.0005$

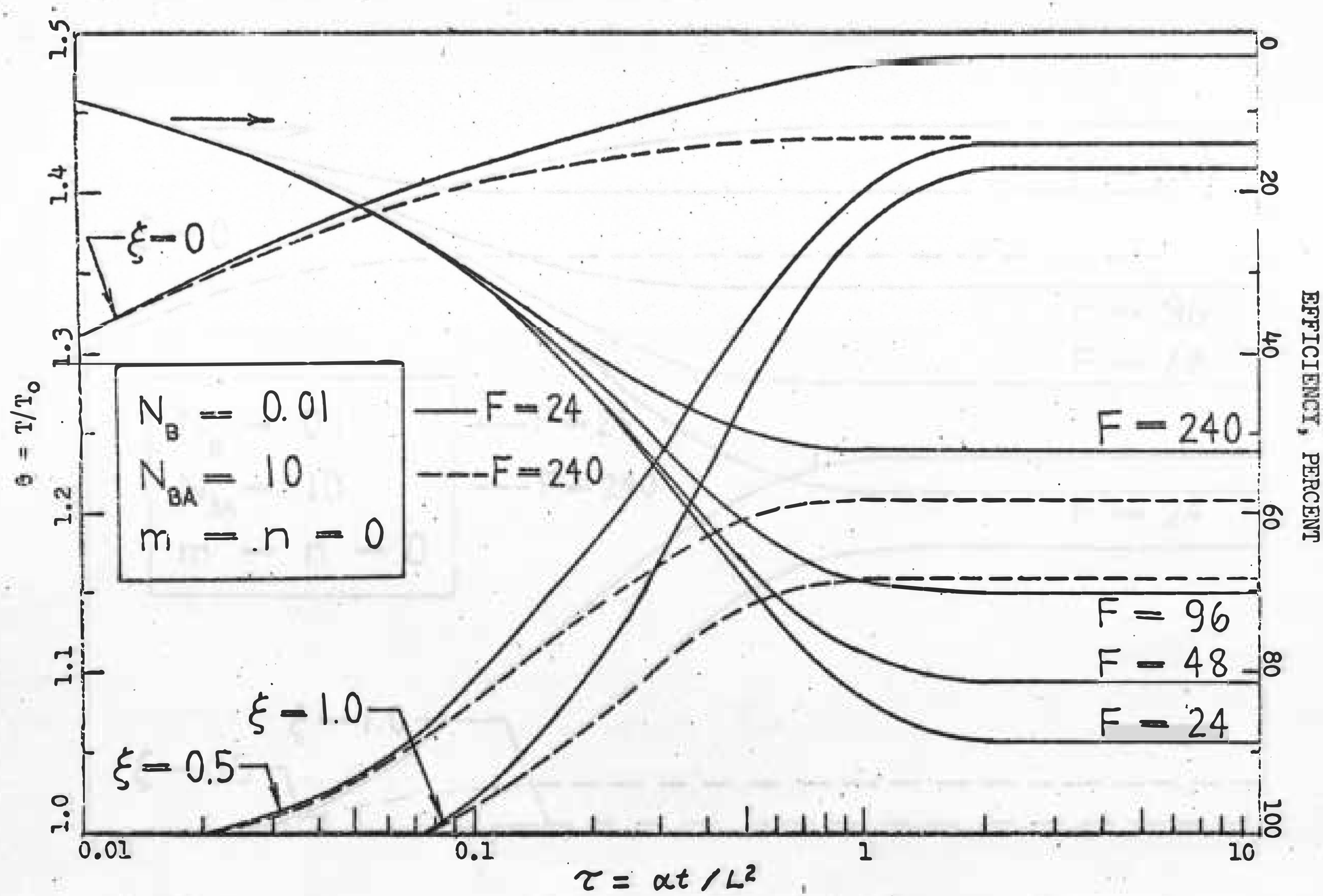

Fig. 29 Transient temperatures and efficiency for a constant-area straight fin with combined convection and radiation, for $\omega = 0$, $N_r = 0.0005$

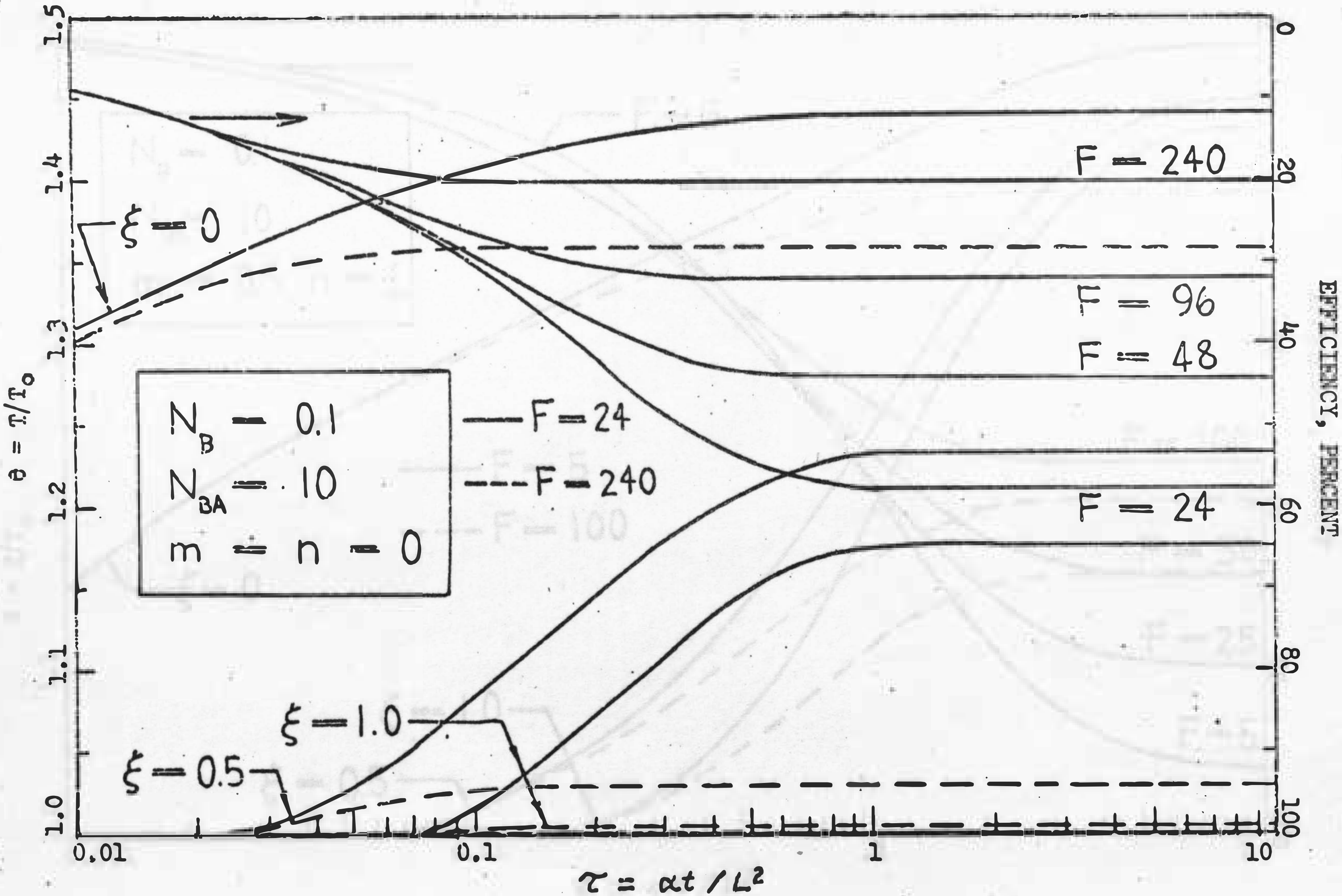


Fig. 30 Transient temperatures and efficiency for a constant-area straight fin with combined convection and radiation for $\omega = 0$, $N_r = 0.0005$

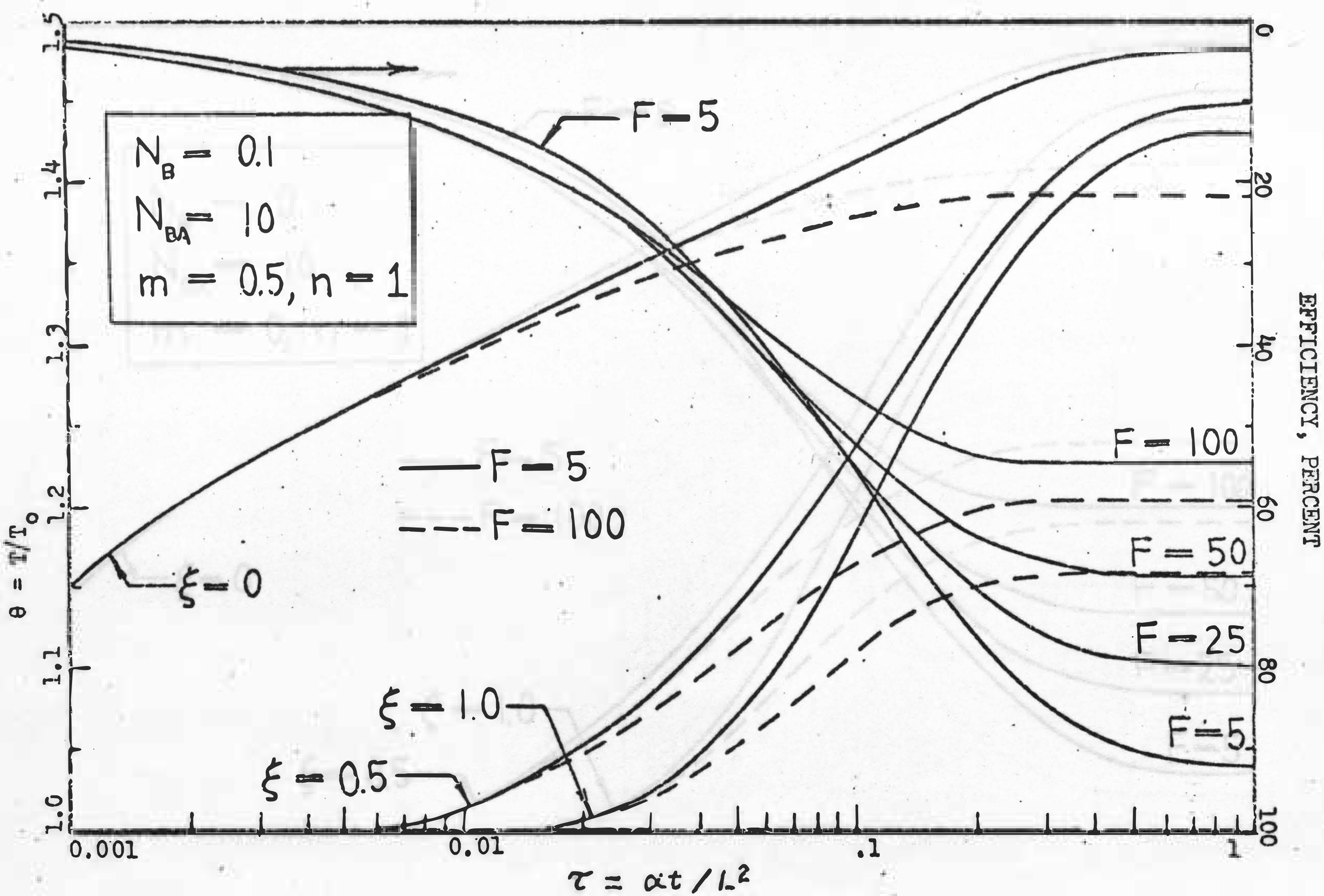


Fig. 31 Transient temperature and efficiency for an annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0005$, $m = .5$, $n = 1$

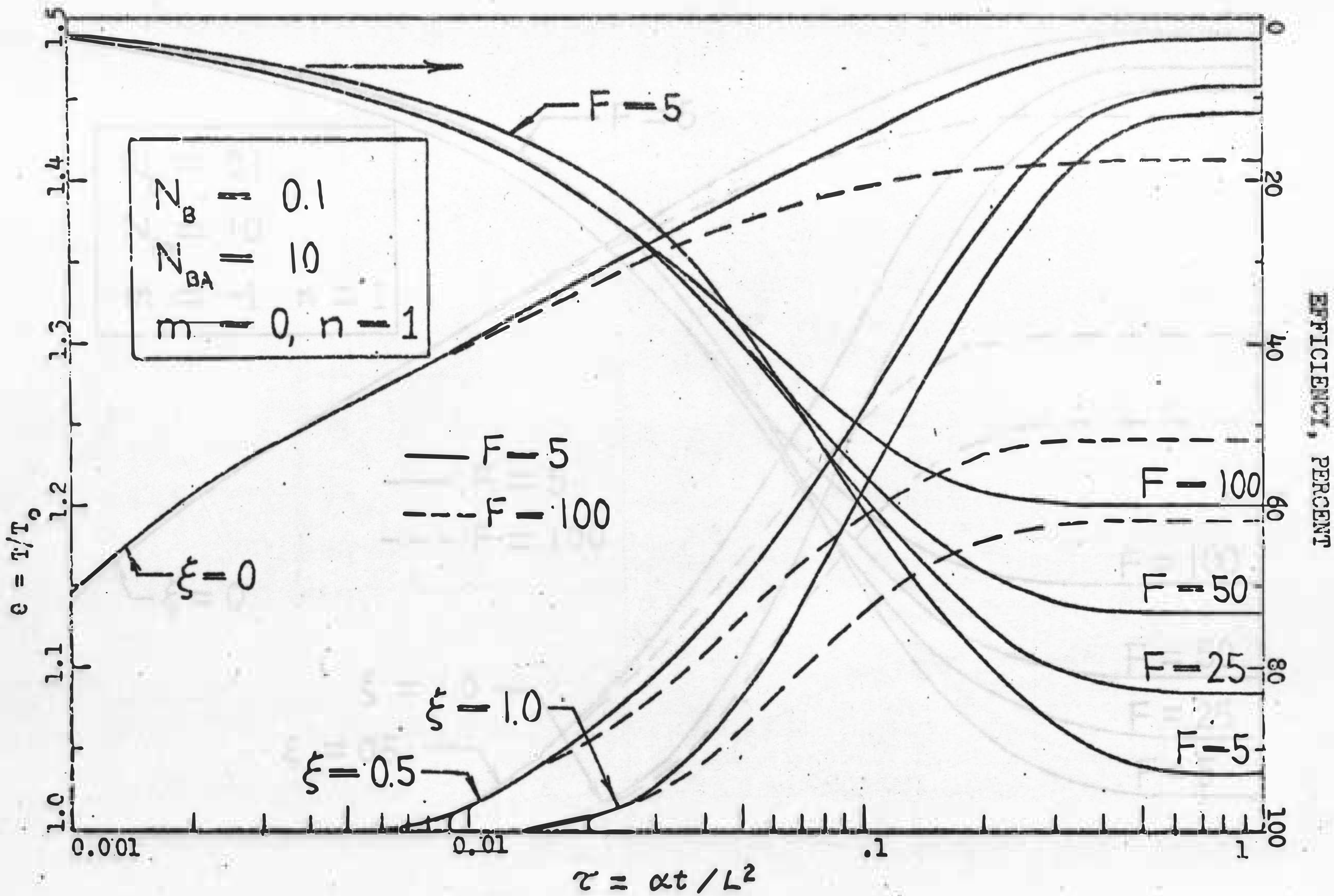


Fig. 32 Transient temperatures and efficiency for a hyperbolic annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0005$

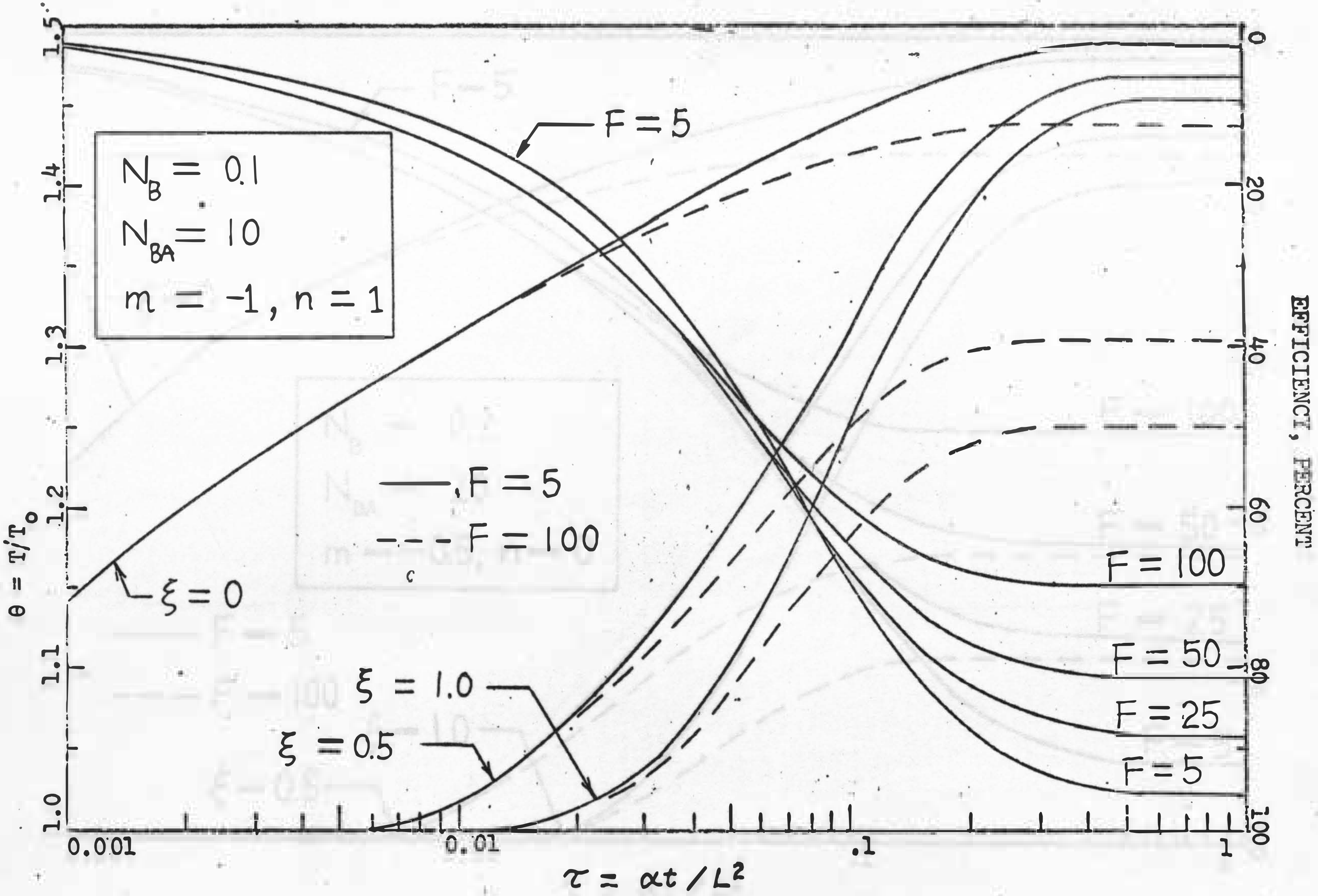


Fig. 33 Transient temperature and efficiency for an annular fin with combined convection and radiation, for $\omega = 0.5$, $N_r = 0.0005$, $m = -1$, $n = 1$

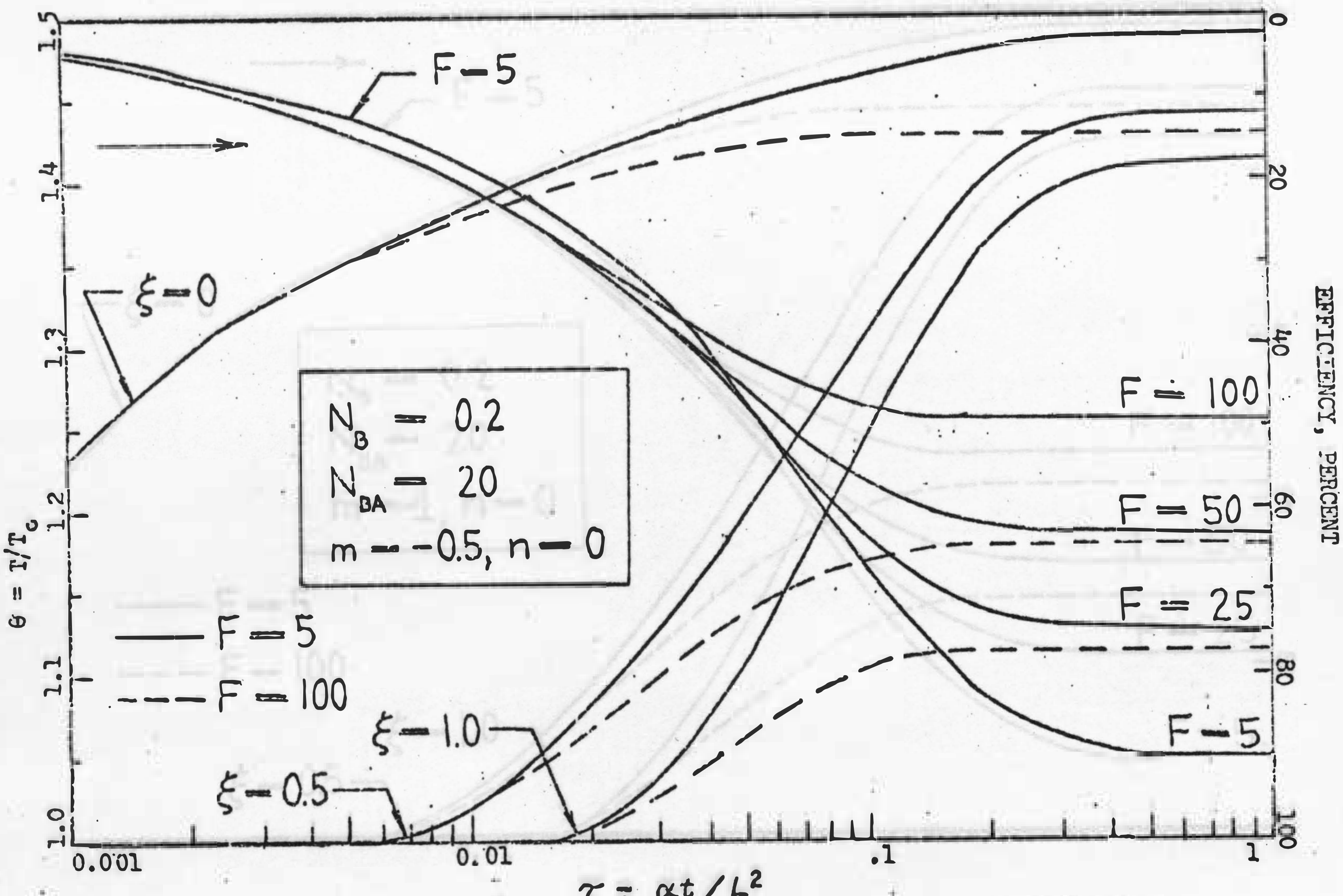


Fig. 34 Transient temperatures and efficiency for a straight fin with combined convection and radiation, for $\omega = 0$, $N_r = 0.001$, $m = -.5$, $n = 0$

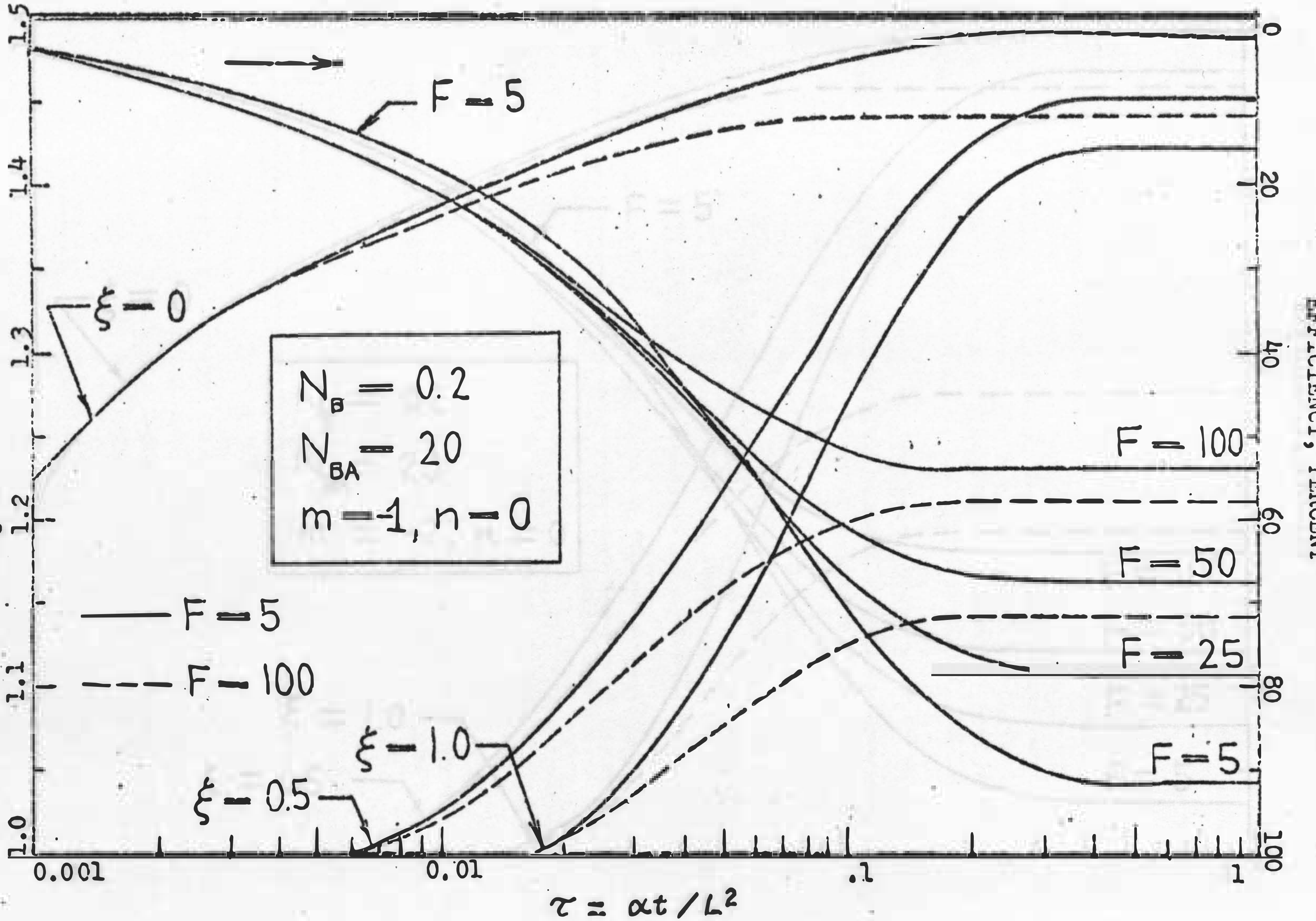


Fig. 35 Transient temperatures and efficiency for a hyperbolic straight fin with combined convection and radiation, for $\omega = 0$, $N_r = 0.001$

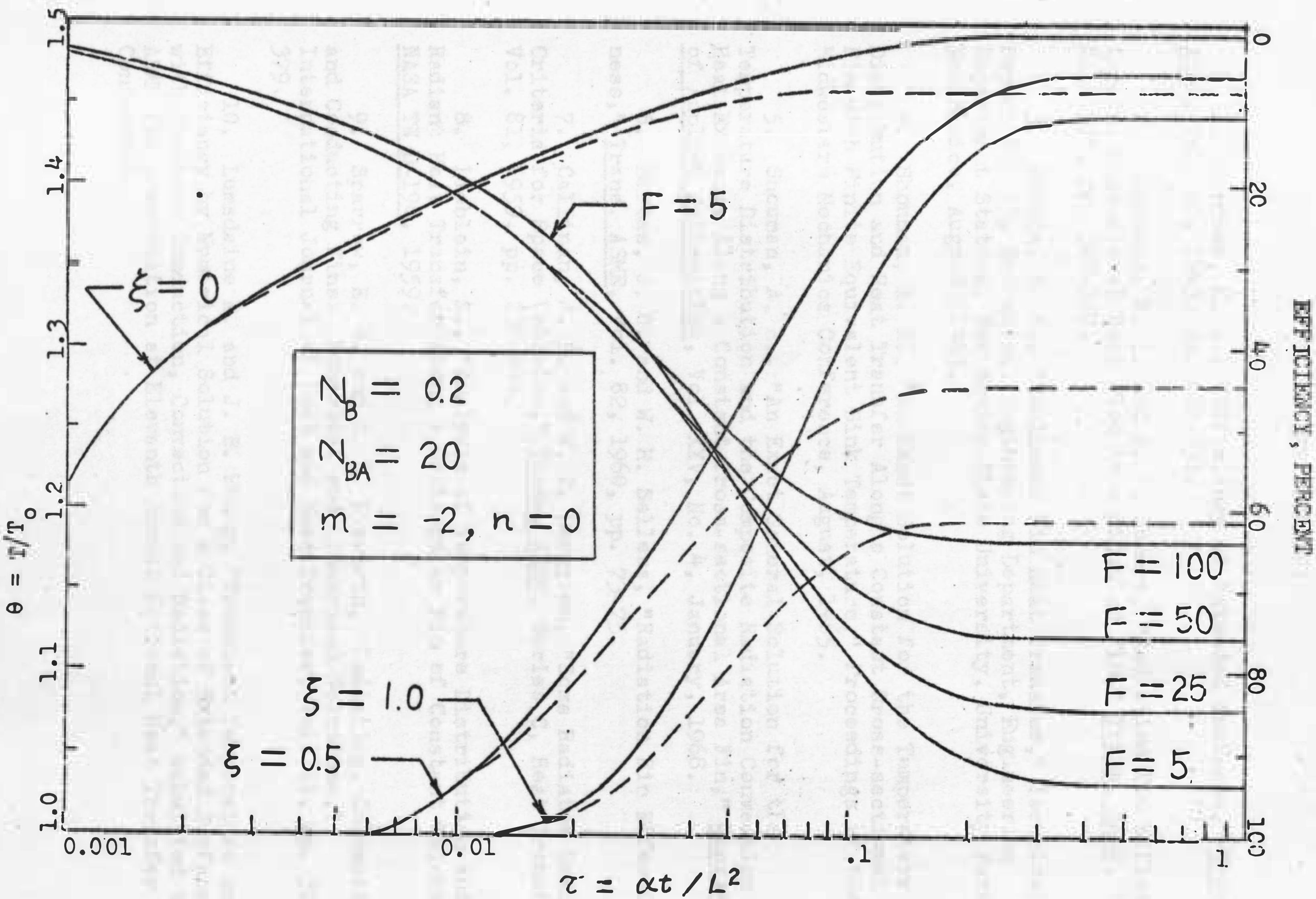


Fig. 36 Transient temperatures and efficiency for a straight fin with combined convection and radiation, for $\omega = 0$, $m = -2$, $n = 0$

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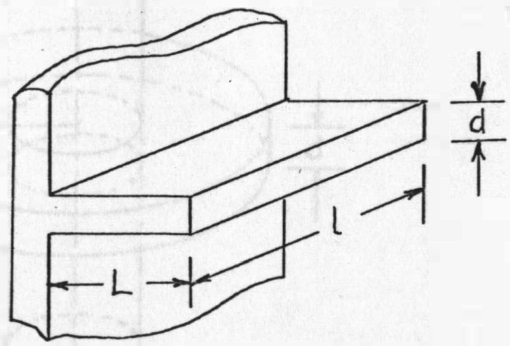
# APPENDIX I

## VARIOUS FIN GEOMETRIES

1. Straight Rectangular Fin

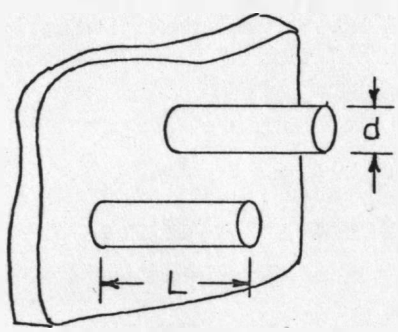



$A = ld$ ; $B_1 = ld$ , $m = 0$

$P = 2l$ ; $B_2 = 2l$ , $n = 0$

$F = 2l / d$

2. Spine of Rectangular Profile

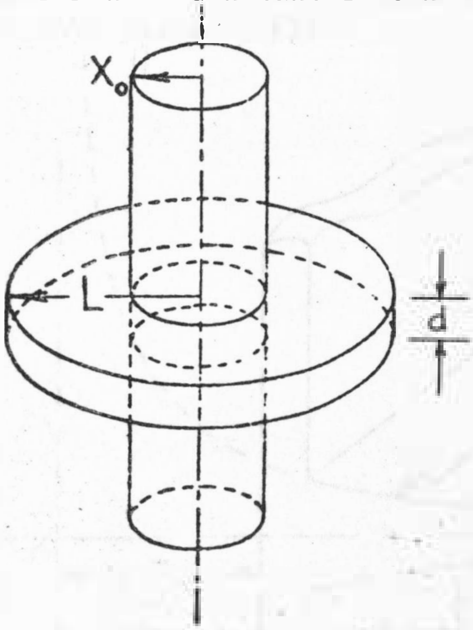



$A = \frac{1}{4}\pi d^2$ ; $B_1 = \frac{1}{4}\pi d^2$ , $m = 0$

$P = \pi d$ ; $B_2 = \pi d$ , $n = 0$

$F = 4L / d$

3. Annular Fin of Constant Thickness

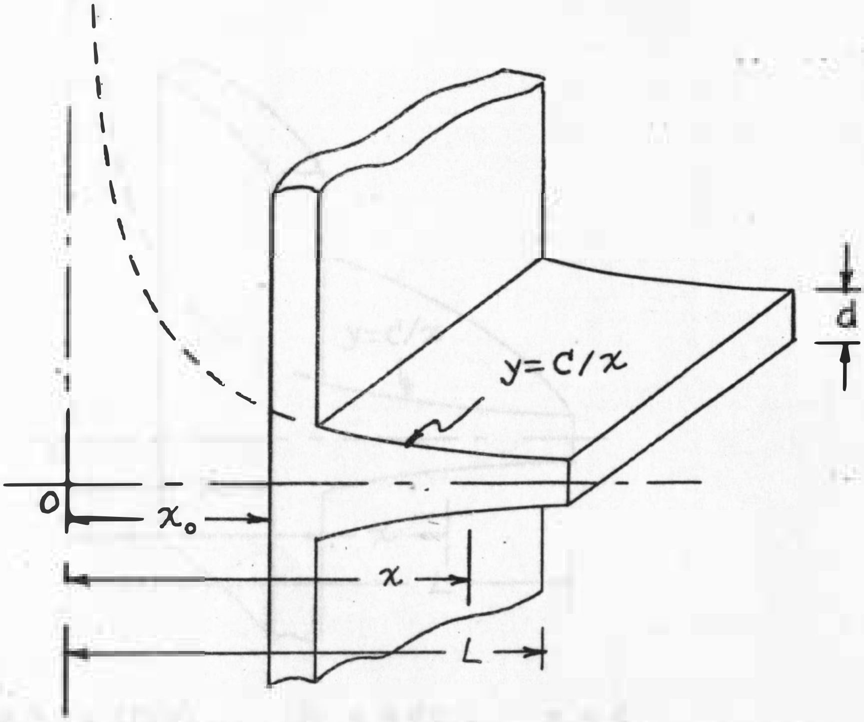



$A = 2\pi xd$ ; $B_1 = 2\pi d$ , $m = 1$

$P = 4\pi x$ ; $B_2 = 4\pi$ , $n = 1$

$F = 2L / d$

4. Straight Hyperbolic Fin



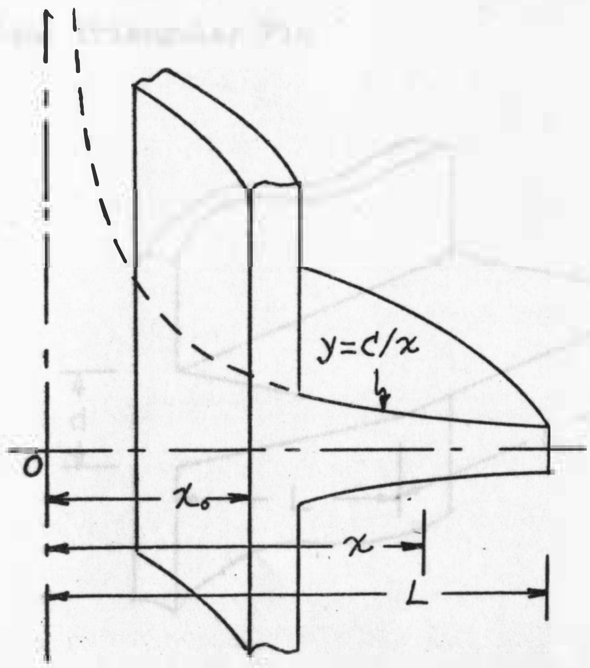

For this type of fin, the origin of the coordinate is at somewhere other than the root. If $x_o/L < 1/2$, then $y' \ll 1$, the following identies will hold:

$A = 2l \cdot (C/x)$ ; $B_1 = 2Cl$ , $m = -1$

$P = 2l$ ; $B_2 = 2l$, $n = 1$

$F = L^2 / C$

5. Annular Hyperbolic Fin

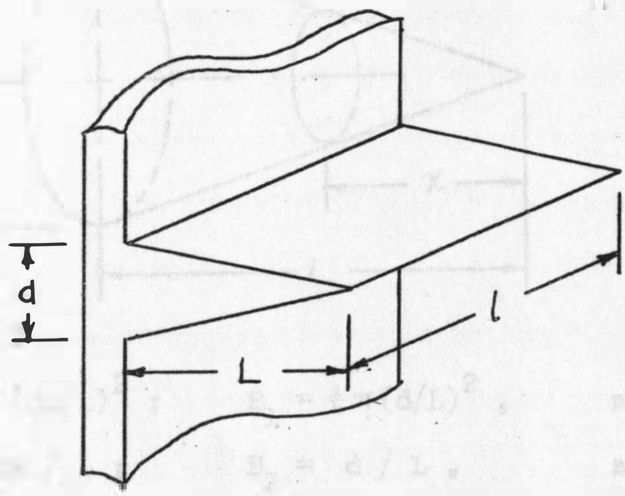



$A = 4\pi x\,(C/x)$ ;   $B_1 = 4\pi C$ ,   $m = 0$

$P = 4\pi x$ ;   $B_2 = 4\pi$ ,   $n = 1$

$F = L^2 / C$

6. Straight Triangular Fin



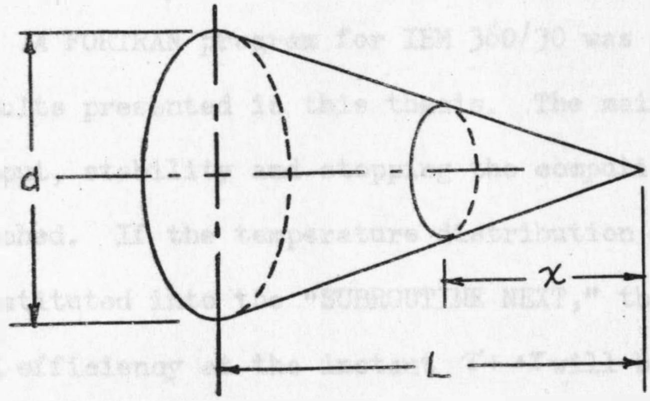

For $l \gg d$

$A = dlx/L$ ; $B_1 = dl/L$ , $m = 1$

$P \simeq 2l$ ; $B_2 = 2l$ , $n = 0$

$F = 2L / d$

7. Spine of Triangular Profile




If $d \ll L$

$A = \frac{1}{4}\pi(dx/L)^2$ ;  $B_1 = \frac{1}{4}\pi(d/L)^2$ ,  $m = 2$

$P = \pi dx / L$ ;  $B_2 = d / L$ ,  $n = 1$

$F = 4L / d$

# APPENDIX II

## PROGRAMMING

A FORTRAN program for IBM 360/30 was prepared to obtain the results presented in this thesis. The main program is for input, output, stability and stopping the computing when steady-state is reached. If the temperature distribution at any instant $\tau$ is substituted into the "SUBROUTINE NEXT," the temperature distribution and efficiency at the instant $\tau + \Delta\tau$ will be obtained. "SUBROUTINE NEWTON" is for carrying out Newton's method for finding roots to Equation (16). In this particular program, the fin was divided into ten parts, thus, $\Delta\xi = .1$.

## Description of Programming Symbols

| Symbol | Description |
|---|---|
| B | $N_B$ |
| BA | $N_{BA}$ |
| DT | $\Delta\tau$ |
| DX | $\Delta\xi$ |
| EFF | $\eta$ |
| F | F |
| G | $N_r$ |
| M | m |
| N | n |
| R | $\omega$ |
| S (in next) | $\theta'$ |
| SUMA | Numerator of Equation (25) |
| SUMB | Denominator of Equation (25) |
| T | $\theta$ |
| TM | $\tau$ |
| TYPE | Type of the fin |

```
      REAL M,N
      DIMENSION TYPE(70),T(12),S(12)
      COMMON G,B,R,F,BA,M,N,DT,T,EFF
      T(1)=1.5
      DX=.1
      READ (11,110) (TYPE(I),I=1,70)
      READ (11,101) M,N,G,B,R,F,BA
C     STABILITY
      A=2./(DX*(1.-R))**2
      C=F*(B&4.*G*T(1)**3)
      IF (N-M) 12,11,11
   11 DT=1./(A&C)
      GO TO 13
   12 DT=1./(A&C*R**(N-M))
   13 CONTINUE
      WRITE (12,201) (TYPE(I),I=1,70),B,BA,G,F,M,N,R,DT
C     TEMPERATURE FIELD
      TM=0.
      EFF=0.
      DO 20 I=2,12
   20  T(I)=1.
      WRITE (12,210) TM,(T(I),I=1,12),EFF
      DO 21 J=1,10
      TM=TM&DT
      CALL NEXT
   21 WRITE (12,210) TM,(T(I),I=1,12),EFF
C     S IS OLD TEMP., T IS NEW TEMP., IF THE CORRES-
C     PONDING VALUES OF S AND T ARE EQUAL, I.E.,
C     STEADY-STATE IS REACHED, STOP THE COMPUTING
   22 DO 23 I=2,12
   23 S(I)=T(I)
      DO 24 K=1,10
      TM=TM&DT
   24 CALL NEXT
      WRITE (12,210) TM,(T(I),I=1,12),EFF
      DO 25 J=1,11
      I=13-J
      IF (T(I)-S(I)-.1E-5) 25,25,22
   25 STOP
  101 FORMAT (F15.5)
  110 FORMAT (70A1)
  201 FORMAT (1H1,33H TRANSIENT STUDY OF RADIATION FIN,
     1          21H BY FINITE DIFFERENCE//
     2          16H TYPE OF THE FIN,70A1//
     3          F16.5,2H=B,F18.5,3H=BA,F17.5,2H=G //
     4          F16.5,2H=F,F18.5,2H=M,F18.5,2H=N //
     5          F16.5,2H=R///
     6          10H STABILITY,E15.4,3H=DT///
     7          32H DIMENSIONLESS TEMPERATURE FIELD)
  210 FORMAT (1H ,F7.4,12F9.6,F5.3)
      END
```

```
      SUBROUTINE NEXT
      REAL M,N
      DIMENSION T(12),S(12)
      COMMON G,B,R,F,BA,M,N,DT,T,EFF
      SUMA=0.
      SUMB=0.
      DO 11 I=3,11
      X=.1*(I-2)
      XI=(1.-R)*X&R
      S(I)=T(I)&DT*((5.*M*(T(I&1)-T(I-1))/XI&100.*
     1    (T(I&1)&T(I-1)-2.*T(I))/(1.-R))/(1.-R)
     2    -F*XI**(N-M)*(B*(T(I)-1.)&G*(T(I)**4-1.)))
      SUMA=SUMA&XI**N*(B*(S(I)-1.)&G*(S(I)**4-1.))
      SUMB=SUMB&XI**N
   11 CONTINUE
      CALL NEWTON (S(11),S(12))
      SA=.5+10./F
      SUMA=SUMA&SA*(B*(S(12)-1.)&G*(S(12)**4-1.))
      SUMB=SUMB&SA
      S(2)=(10.*S(3)&BA*(1.-R)*T(1))/(BA*(1.-R)&10.)
      SA=B*(S(2)-1.)&G*(S(2)**4-1.)
      IF (N) 14,15,14
   14 SUMA=SUMA&.5*R**N*SA
      SUMB=(SUMB&.5*R**N)*SA
      GO TO 16
   15 SUMA=SUMA&.5*SA
      SUMB=(SUMB&.5)*SA
   16 IF (G) 18,17,18
   17 IF (B) 18,19,18
   18 EFF=SUMA/SUMB
      GO TO 20
   19 EFF=1.
   20 DO 21 I=2,12
   21 T(I)=S(I)
      RETURN
      END
```

```
      SUBROUTINE NEWTON (T,S)
      COMMON G,B,R
      S=T
    1 Y=G*(1.-R)*S**4&(10.&B*(1.-R))*S
     1  -(10.*T&B*(1.-R)&G*(1.-R))
      DY=4.*G*(1.-R)*S**3&(10.&B*(1.-R))
      DS=Y/DY
      S=S-DS
      IF (ABS(DS)-.0005) 2,1,1
    2 RETURN
      END
```