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著者（英）	Kazuo Tanaka, Hiroshi Ohtake, Toshiaki Seo, Motoyasu Tanaka, Hua O. Wang
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Polynomial Fuzzy Observer Designs:A Sum of Squares Approach

Kazuo Tanaka, *Senior Member, IEEE,* Hiroshi Ohtake, *Member, IEEE,* Toshiaki Seo, Motoyasu Tanaka, and Hua O. Wang, *Senior Member, IEEE*

***Abstract*—This paper presents a sum of squares (SOS, for brevity) approach to polynomial fuzzy observer designs for three classes of polynomial fuzzy systems. The proposed SOS-based framework provides a number of innovations and improvements over the existing LMI-based approaches to Takagi-Sugeno (T-S) fuzzy controller and observer designs. First, we briefly summarize previous results with respect to a polynomial fuzzy system that is more general representation of the well-known T-S fuzzy system. Next, we propose polynomial fuzzy observers to estimate states in three classes of polynomial fuzzy systems and derive SOS conditions to design polynomial fuzzy controllers and observers. A remarkable feature of the SOS design conditions for the first two classes (Classes I and II) is that they realize the so-called separation principle, that is, that a polynomial fuzzy controller and observer for each class can be separately designed without lack of guaranteeing the stability of the overall control system in addition to converging state estimation error (via the observer) to zero. Although, for the last class (Class III), the separation principle does not hold, we propose an algorithm to design a polynomial fuzzy controller and observer satisfying the stability of the overall control system in addition to converging state estimation error (via the observer) to zero. All the design conditions in the proposed approach can be represented in terms of SOS and is symbolically and numerically solved via the recent developed SOSTOOLS and a semidefinite program (SDP) solver, respectively. To illustrate the validity and applicability of the proposed approach, three design examples are provided. The examples demonstrate advantages of the SOS-based approaches for the existing LMI approaches to T-S fuzzy observer designs.**

***Index Terms*—polynomial fuzzy system, polynomial fuzzy observer, separation principle, stability, sum of squares.**

I. Introduction

THE Takagi-Sugeno (T-S) fuzzy model-based control methodology [1], [2] has received a great deal of attention after LMI-based designs have been discussed in [3]-[4]. The fuzzy model-based control methodology provides a natural, simple and effective design approach to complement other nonlinear control techniques (e.g., [5]) that require special and rather involved knowledge.

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Kazuo Tanaka, Toshiaki Seo and Motoyasu Tanaka are with the Department of Mechanical Engineering and Intelligent Systems, The University of Electro-Communications, Chofu, Tokyo 182-8585 Japan (email: ktanaka@mce.uec.ac.jp; toshiseo@rc.mce.uec.ac.jp; mtanaka@rc.mce.uec.ac.jp).

Hiroshi Ohtake is with the Department of Mechanical Information Science and Technology, Kyushu Institute of Technology, Iizuka, Fukuoka 820-8502 Japan (email: hohtake@mse.kyutech.ac.jp).

Hua O. Wang is with the Department of Mechanical Engineering, Boston University, Boston, MA 02215 USA (email: wangh@bu.edu).

Recently, the authors have first presented a sum of squares (SOS, for brevity) approach [6]-[11] to polynomial fuzzy control system designs. This is a completely different approach from the existing LMI approaches [2], [12]-[27]. Our SOS approach [6]-[11] provided more extensive results for the existing LMI approaches to T-S fuzzy model and control. However, to the best of our knowledge, there exists no literature on SOS-based observer designs for polynomial fuzzy systems.

This paper presents SOS-based observer designs to estimate the states of polynomial fuzzy systems. The proposed SOS-based framework for polynomial fuzzy systems provides a number of innovations and improvements over the existing LMI approaches to T-S fuzzy observer-based control, e.g., [2], [12], [13]. First, it is known that nonlinear systems with polynomial terms can not be generally converted to globally exact T-S fuzzy models. Only local or semi-global T-S fuzzy models can be constructed for such nonlinear systems [2]. Thus, resulting control design conditions guarantee global stabilization and global state-estimation convergence only for local or semi-global models, but not always guarantee global stabilization and global state-estimation convergence for original nonlinear systems. On the other hand, it is possible to convert even nonlinear systems with polynomial terms to globally exact polynomial fuzzy models. Hence all the conditions derived here guarantee global stabilization and global state-estimation convergence for original nonlinear systems that are perfectly equivalent to polynomial fuzzy models. Secondly, even if local or semi-global T-S fuzzy models are permitted to use in practical sense, variables in polynomial terms appear in premise (part) variables of T-S fuzzy models. In polynomial fuzzy models, variables in polynomial terms do not appear in their premise parts and remain in system polynomial matrices $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ in consequence parts of polynomial fuzzy models. The difference is quite large from fuzzy observer design points of view. In general, fuzzy observer designs are not permitted to have premise variables depending on the states to be estimated. Therefore, T-S fuzzy observer designs can not be generally applied to nonlinear systems with polynomial terms. Conversely, the polynomial fuzzy observer designs proposed in this paper can be applied to even such systems. We will see these facts in the design examples later.

This paper presents three types of SOS-based observer designs according to three classes of polynomial fuzzy systems. First, we present an observer-based design for the polynomial fuzzy systems with the polynomial matrices $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ being independent of the states $\boldsymbol{x}$ to be estimated (shortly name it as Class I). Secondly, we discuss an observer-based design for a wider class of polynomial fuzzy systems with the polynomial matrices $\boldsymbol{A}_i$ that are permitted to be dependent of the states $\boldsymbol{x}$ to be estimated (shortly name it as Class II). It should be emphasized that this paper realizes the so-called separation design for both of the classes. This paper also presents a polynomial fuzzy observer design for a more complicated class of polynomial fuzzy systems, i.e., the polynomial fuzzy systems with the polynomial matrices $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ that are permitted to be dependent of the states $\boldsymbol{x}$ to be estimated (shortly name it as Class III). All the design conditions discussed here are represented in terms of SOS.

It is well known that stability conditions for the T-S fuzzy system reduce to LMIs, e.g., [2]. Hence, the stability conditions can be solved numerically and efficiently by interior point algorithms, e.g., by LMI solvers. On the other hand, some kinds of control design conditions [6]-[11] for polynomial fuzzy systems reduce to SOS problems. Clearly, the problems are never directly solved by LMI solvers and can be solved via the SOSTOOLS [28] and an SDP solver. Thus, SOS can be regarded as an extensive representation of LMIs. The computational method used in this paper relies on the SOS decomposition of multivariate polynomials. A multivariate polynomial $f(\boldsymbol{x}(t))$ (where $\boldsymbol{x}(t) \in R^n$) is an SOS if there exist polynomials $f_1(\boldsymbol{x}(t)), \cdots, f_k(\boldsymbol{x}(t))$ such that $f(\boldsymbol{x}(t)) = \sum_{i=1}^{k} f_i^2(\boldsymbol{x}(t))$. It is clear that $f(\boldsymbol{x}(t))$ being an SOS naturally implies $f(\boldsymbol{x}(t)) \geq 0$ for all $\boldsymbol{x}(t) \in R^n$. For more details of SOS, see [28].

The rest of the paper is organized as follows. Section II recalls a polynomial fuzzy system defined in [6]-[11]. Sections III, IV and V discuss SOS-based polynomial fuzzy controller and observer designs for Classes I, II and III, respectively. In addition, each section entails a design example to demonstrate the viability of our SOS design approach.

In this paper, to save the space, we employ the following short notations with respect to matrix representation.

$\mathcal{L}\{\boldsymbol{M}\} = \boldsymbol{M}^T + \boldsymbol{M}$,

$\boldsymbol{E}_1 = diag[\epsilon_{11}\ \epsilon_{12}\ \cdots\ \epsilon_{1s}]$,

$\boldsymbol{E}_{2i}(\boldsymbol{x}) = diag[\epsilon_{2i1}(\boldsymbol{x})\ \epsilon_{2i2}(\boldsymbol{x})\ \cdots\ \epsilon_{2is}(\boldsymbol{x})]$,

where $\boldsymbol{M}$ is an arbitrary square matrix. ϵ_{1k} $(k = 1, 2, \cdots, s)$ are positive values and $\epsilon_{2ik}(\boldsymbol{x})$ $(i = 1, 2, \cdots, r,\ k = 1, 2, \cdots, s)$ are nonnegative polynomials such that $\epsilon_{2ik}(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \neq \boldsymbol{0}$. ϵ_{1k} and $\epsilon_{2ik}(\boldsymbol{x})$ ($\boldsymbol{E}_1$ and $\boldsymbol{E}_{2i}(\boldsymbol{x})$) will be used as slack variables (matrices) to keep positivity of SOS conditions derived in this paper. s is the matrix size of $\boldsymbol{E}_1$ and $\boldsymbol{E}_{2i}(\boldsymbol{x})$ that are assumed to have appropriate dimensions. r is the number of fuzzy model rules.

II. Takagi-Sugeno fuzzy model and polynomial fuzzy model

In this section, we recall the Takagi-Sugeno fuzzy model. The Takagi-Sugeno fuzzy model is described by fuzzy IF-THEN rules which represent local linear input-output relations of a nonlinear system. The main feature of this model is to express the local dynamics of each fuzzy implication (rule) by a linear system model. The overall fuzzy model of the system is achieved by fuzzy blending of the linear system models.

Consider the following nonlinear system:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)), \tag{1}$$

where f is a smooth nonlinear function such that $f(\boldsymbol{0}, \boldsymbol{0}) = \boldsymbol{0}$. $\boldsymbol{x}(t) = [x_1(t)\ x_2(t)\ \cdots\ x_n(t)]^T$ is the state vector and $\boldsymbol{u}(t) = [u_1(t)\ u_2(t)\ \cdots\ u_m(t)]^T$ is the input vector. Based on the sector nonlinearity concept [2], we can exactly represent (1) with the following Takagi-Sugeno fuzzy model (globally or at least semi-globally).

Model Rule i:

$$\begin{aligned} &If\ z_1(t)\ is\ M_{i1}\ and \cdots\ and\ \ z_p(t)\ is\ M_{ip} \\ &then\ \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t) \quad i = 1, 2, \cdots, r, \end{aligned} \tag{2}$$

where $z_j(t)$ $(j = 1, 2, \cdots, p)$ is the premise variable. The membership function associated with the ith $Model\ Rule$ and jth premise variable component is denoted by M_{ij}. r denotes the number of $Model\ Rules$. Note that $z_j(t)$ is assumed to be independent of the states $\boldsymbol{x}$ to be estimated. In other words, each $z_j(t)$ is a measurable time-varying quantity that may be states, measurable external variables and/or time. The defuzzification process of the model (2) can be represented as

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \frac{\sum_{i=1}^{r} w_i(\boldsymbol{z}(t))\{\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t)\}}{\sum_{i=1}^{r} w_i(\boldsymbol{z}(t))} \\ &= \sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\{\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t)\}, \end{aligned} \tag{3}$$

where

$$\boldsymbol{z}(t) = [z_1(t) \cdots z_p(t)]$$

and

$$w_i(\boldsymbol{z}(t)) = \prod_{j=1}^{p} M_{ij}(z_j(t)).$$

It should be noted from the properties of membership functions that the following relations hold.

$$\sum_{i=1}^{r} w_i(\boldsymbol{z}(t)) > 0, \quad w_i(\boldsymbol{z}(t)) \geq 0 \quad i = 1, 2, \cdots, r$$

Hence,

$$h_i(\boldsymbol{z}(t)) = \frac{w_i(\boldsymbol{z}(t))}{\sum_{i=1}^{r} w_i(\boldsymbol{z}(t))} \geq 0, \quad \sum_{i=1}^{r} h_i(\boldsymbol{z}(t)) = 1.$$

In [6] and [9], we proposed a new type of fuzzy model with polynomial model consequence, i.e., fuzzy model whose consequent parts are represented by polynomials. Using the sector nonlinearity concept [2], we exactly represent (1) with the following polynomial fuzzy model (4). The main difference between the T-S fuzzy model [29] and the polynomial fuzzy model is consequent part representation. The fuzzy model (4) has a polynomial model consequence.

Model Rule i:

$$\begin{aligned} &If\ z_1(t)\ is\ M_{i1}\ and \cdots\ and\ \ z_p(t)\ is\ M_{ip} \\ &then\ \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i(\boldsymbol{x}(t))\boldsymbol{x}(t) + \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{u}(t), \end{aligned} \tag{4}$$

where $i = 1, 2, \cdots, r$. r denotes the number of $Model\ Rules$. $\boldsymbol{A}_i(\boldsymbol{x}(t)) \in \boldsymbol{R}^{n\times n}$ and $\boldsymbol{B}_i(\boldsymbol{x}(t)) \in \boldsymbol{R}^{n\times m}$ are polynomial matrices in $\boldsymbol{x}(t)$. Therefore, $\boldsymbol{A}_i(\boldsymbol{x}(t))\boldsymbol{x}(t) + \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{u}(t)$ is a polynomial vector. Thus, the polynomial fuzzy model (4) has a polynomial in each consequent part.

The defuzzification process of the model (4) can be represented as

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\{\boldsymbol{A}_i(\boldsymbol{x}(t))\boldsymbol{x}(t) + \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{u}(t)\}. \qquad (5)$$

Thus, the overall fuzzy model is achieved by fuzzy blending of the polynomial system models.

Remark 1. *The polynomial fuzzy model is an extension of the T-S fuzzy model. Hence the SOS conditions derived in this paper may be regarded as an extension of the previous LMI conditions for the T-S fuzzy model. However, it will be seen through the design examples in this paper that the polynomial fuzzy models are exact global models for the original nonlinear systems although the T-S fuzzy models are not global models for the original nonlinear systems. In addition, the previous T-S fuzzy observer technique dose not work completely for both of Classes II and III due to a premise variable restriction. For more details, we will mention again in the design examples later.*

As will be mentioned later, it is in general difficult to separately design a polynomial controller and a polynomial observer for (5) since $\boldsymbol{A}_i(\boldsymbol{x}(t))$ and $\boldsymbol{B}_i(\boldsymbol{x}(t))$ are dependent of the states $\boldsymbol{x}(t)$ to be estimated. Hence, as a first step, we introduce the following representation of polynomial fuzzy systems.

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\{\boldsymbol{A}_i(\boldsymbol{\rho}_A(t))\boldsymbol{x}(t) + \boldsymbol{B}_i(\boldsymbol{\rho}_B(t))\boldsymbol{u}(t)\}, \qquad (6)$$

where (6) reduces to (5) when $\boldsymbol{\rho}_A(t) = \boldsymbol{\rho}_B(t) = \boldsymbol{x}(t)$. In this paper, we discuss three types of polynomial observer-based control according to three classes of polynomial fuzzy systems:

Class I: $\boldsymbol{\rho}_A(t) = \boldsymbol{\zeta}(t)$ and $\boldsymbol{\rho}_B(t) = \boldsymbol{\zeta}(t)$.
Class II: $\boldsymbol{\rho}_A(t) = \boldsymbol{x}(t)$ and $\boldsymbol{\rho}_B(t) = \boldsymbol{\zeta}(t)$.
Class III: $\boldsymbol{\rho}_A(t) = \boldsymbol{\rho}_B(t) = \boldsymbol{x}(t)$.

$\boldsymbol{\zeta}(t)$ is a measurable time-varying vector that may be measurable external variables, outputs and/or time. In other words, $\boldsymbol{\zeta}(t)$ is assumed to be independent of the states $\boldsymbol{x}(t)$ to be estimated. As we can see, Class III is the most complicated class.

From now, to lighten the notation, we will drop the notation with respect to time t. For instance, we will employ $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ instead of $\boldsymbol{x}(t)$ and $\hat{\boldsymbol{x}}(t)$, respectively, where $\hat{\boldsymbol{x}}(t)$ denotes the state estimated by a polynomial fuzzy observer as will be discussed later. Thus, we drop the notation with respect to time t, but it should be kept in mind that $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ means $\boldsymbol{x}(t)$ and $\hat{\boldsymbol{x}}(t)$, respectively.

Next, we define the outputs for the polynomial fuzzy model as

$$\boldsymbol{y} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{C}_i\boldsymbol{x}, \qquad (7)$$

where $\boldsymbol{y} \in \mathbb{R}^q$ is the output.

III. POLYNOMIAL CONTROLLER AND OBSERVER DESIGN (CLASS I)

Consider the following polynomial fuzzy system. The system matrices $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ depend on the vector $\boldsymbol{\zeta}$,

$$\begin{cases} \dot{\boldsymbol{x}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\{\boldsymbol{A}_i(\boldsymbol{\zeta})\boldsymbol{x} + \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{u}\} \\ \boldsymbol{y} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{C}_i\boldsymbol{x}, \end{cases} \qquad (8)$$

where $\boldsymbol{y} \in \mathbb{R}^q$ denotes the output.

We design a polynomial fuzzy observer to estimate the states of (8).

$$\begin{cases} \dot{\hat{\boldsymbol{x}}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\{\boldsymbol{A}_i(\boldsymbol{\zeta})\hat{\boldsymbol{x}} + \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{u} + \boldsymbol{L}_i(\boldsymbol{\zeta})(\boldsymbol{y} - \hat{\boldsymbol{y}})\} \\ \hat{\boldsymbol{y}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{C}_i\hat{\boldsymbol{x}}, \end{cases} \qquad (9)$$

where $\hat{\boldsymbol{x}} \in \mathbb{R}^n$ is the sate vector estimated by the fuzzy observer and $\hat{\boldsymbol{y}} \in \mathbb{R}^q$ is estimated output calculated from $\hat{\boldsymbol{y}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{C}_i\hat{\boldsymbol{x}}$.

To stabilize the system (8) and (9), we design a polynomial fuzzy controller with the state-feedback estimated by the polynomial fuzzy observer.

$$\boldsymbol{u} = -\sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{F}_i(\boldsymbol{\zeta})\hat{\boldsymbol{x}} \qquad (10)$$

Theorem 1 provides SOS conditions to separately design the polynomial fuzzy controller (10) and the polynomial fuzzy observer (9).

Theorem 1. *If there exist positive definite matrices $\boldsymbol{X}_1 \in \mathbb{R}^{n\times n}$, $\boldsymbol{X}_2 \in \mathbb{R}^{n\times n}$ and polynomial matrices $\boldsymbol{M}_i(\boldsymbol{\zeta}) \in \mathbb{R}^{p\times n}$, $\boldsymbol{N}_i(\boldsymbol{\zeta}) \in \mathbb{R}^{n\times q}$ such that (11)∼(16) are satisfied, the polynomial fuzzy controller (10) stabilizes the system (8) and the estimation error via the polynomial observer (9) tends to zero.*

$$\boldsymbol{v}_1^T\left(\boldsymbol{X}_1 - \boldsymbol{E}_1\right)\boldsymbol{v}_1 \quad is \quad SOS \qquad (11)$$

$$\boldsymbol{v}_2^T\left(\boldsymbol{X}_2 - \boldsymbol{E}_2\right)\boldsymbol{v}_2 \quad is \quad SOS \qquad (12)$$

$$-\boldsymbol{v}_3^T\left(\mathcal{L}\{\boldsymbol{A}_i(\boldsymbol{\zeta})\boldsymbol{X}_1 - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{M}_i(\boldsymbol{\zeta})\} + \boldsymbol{E}_{3i}(\boldsymbol{\zeta})\right)\boldsymbol{v}_3 \quad is \quad SOS \qquad (13)$$

$$-\boldsymbol{v}_4^T\left(\mathcal{L}\{\boldsymbol{X}_2\boldsymbol{A}_i(\boldsymbol{\zeta}) - \boldsymbol{N}_i(\boldsymbol{\zeta})\boldsymbol{C}_i\} + \boldsymbol{E}_{4i}(\boldsymbol{\zeta})\right)\boldsymbol{v}_4 \quad is \quad SOS \qquad (14)$$

$$-\boldsymbol{v}_5^T\Big(\mathcal{L}\{\boldsymbol{A}_i(\boldsymbol{\zeta})\boldsymbol{X}_1-\boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{M}_j(\boldsymbol{\zeta})\}+\mathcal{L}\{\boldsymbol{A}_j(\boldsymbol{\zeta})\boldsymbol{X}_1-\boldsymbol{B}_j(\boldsymbol{\zeta})\boldsymbol{M}_i(\boldsymbol{\zeta})\}\Big)\boldsymbol{v}_5 \quad is\ \ SOS \tag{15}$$

$$-\boldsymbol{v}_6^T\Big(\mathcal{L}\{\boldsymbol{X}_2\boldsymbol{A}_i(\boldsymbol{\zeta})-\boldsymbol{N}_i(\boldsymbol{\zeta})\boldsymbol{C}_j\}+\mathcal{L}\{\boldsymbol{X}_2\boldsymbol{A}_j(\boldsymbol{\zeta})-\boldsymbol{N}_j(\boldsymbol{\zeta})\boldsymbol{C}_i\}\Big)\boldsymbol{v}_6 \quad is\ \ SOS \tag{16}$$

where $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, $\boldsymbol{v}_4$ $\boldsymbol{v}_5$ and $\boldsymbol{v}_6 \in \mathbb{R}^n$ denote vectors that are independent of $\boldsymbol{x}$, $\hat{\boldsymbol{x}}$ and $\boldsymbol{\zeta}$. From the solutions $\boldsymbol{X}_1$ and $\boldsymbol{M}_i(\boldsymbol{\zeta})$, we obtain polynomial feedback gains $\boldsymbol{F}_i(\boldsymbol{\zeta})$ as $\boldsymbol{F}_i(\boldsymbol{\zeta}) = \boldsymbol{M}_i(\boldsymbol{\zeta})\boldsymbol{X}_1^{-1}$. From the solutions $\boldsymbol{X}_2$ and $\boldsymbol{N}_i(\boldsymbol{\zeta})$, we obtain polynomial observer gains $\boldsymbol{L}_i(\boldsymbol{\zeta})$ as $\boldsymbol{L}_i(\boldsymbol{\zeta}) = \boldsymbol{X}_2^{-1}\boldsymbol{N}_i(\boldsymbol{\zeta})$ as well.

Proof: We define the estimation error vector $\boldsymbol{e}$ as $\boldsymbol{e} = \boldsymbol{x} - \hat{\boldsymbol{x}}$. Then, the error dynamics can be described as

$$\dot{\boldsymbol{e}} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\{\boldsymbol{A}_i(\boldsymbol{\zeta}) - \boldsymbol{L}_i(\boldsymbol{\zeta})\boldsymbol{C}_j\}\boldsymbol{e}.$$

Next, using the augmented vector $\boldsymbol{x}_v = \begin{bmatrix}\hat{\boldsymbol{x}}^T & \boldsymbol{e}^T\end{bmatrix}^T$, the augmented system consisting of the system, the polynomial fuzzy controller and observer can be represented as

$$\begin{aligned}\dot{\boldsymbol{x}}_v &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\boldsymbol{G}_{ij}(\boldsymbol{\zeta})\boldsymbol{x}_v \\ &= \sum_{i=1}^{r} h_i^2(\boldsymbol{z})\boldsymbol{G}_{ii}(\boldsymbol{\zeta})\boldsymbol{x}_v \\ &\quad + \sum_{i=1}^{r}\sum_{i<j}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\left(\boldsymbol{G}_{ij}(\boldsymbol{\zeta}) + \boldsymbol{G}_{ji}(\boldsymbol{\zeta})\right)\boldsymbol{x}_v,\end{aligned} \tag{17}$$

where

$$\begin{aligned}\boldsymbol{G}_{ij}(\boldsymbol{\zeta}) &= \begin{bmatrix}\boldsymbol{G}_{11_{ij}}(\boldsymbol{\zeta}) & \boldsymbol{G}_{12_{ij}}(\boldsymbol{\zeta}) \\ \boldsymbol{0} & \boldsymbol{G}_{22_{ij}}(\boldsymbol{\zeta})\end{bmatrix}, \\ \boldsymbol{G}_{11_{ij}}(\boldsymbol{\zeta}) &= \boldsymbol{A}_i(\boldsymbol{\zeta}) - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{F}_j(\boldsymbol{\zeta}), \\ \boldsymbol{G}_{12_{ij}}(\boldsymbol{\zeta}) &= \boldsymbol{L}_i(\boldsymbol{\zeta})\boldsymbol{C}_j, \\ \boldsymbol{G}_{22_{ij}}(\boldsymbol{\zeta}) &= \boldsymbol{A}_i(\boldsymbol{\zeta}) - \boldsymbol{L}_i(\boldsymbol{\zeta})\boldsymbol{C}_j.\end{aligned}$$

Next, consider a candidate Lyapunov function

$$V(\boldsymbol{x}_v) = \boldsymbol{x}_v^T\tilde{\boldsymbol{X}}\boldsymbol{x}_v, \tag{18}$$

where

$$\tilde{\boldsymbol{X}} = \begin{bmatrix}\alpha\boldsymbol{X}_1^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{X}_2\end{bmatrix}. \tag{19}$$

α is a positive value, $\boldsymbol{X}_1^{-1} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{X}_2 \in \mathbb{R}^{n\times n}$ are positive definite matrices. Note that $V(\boldsymbol{x}_v) > 0$ at $\boldsymbol{x}_v \neq \boldsymbol{0}$. It is clear from Lyapunov theory that the overall control system (17) is stable if it is proved that $\dot{V}(\boldsymbol{x}_v) < 0$ at $\boldsymbol{x}_v \neq \boldsymbol{0}$.

The time derivative of $V(\boldsymbol{x}_v)$ along the trajectory of the system is obtained as

$$\begin{aligned}\dot{V}(\boldsymbol{x}_v) &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\boldsymbol{x}_v^T\mathcal{L}\{\tilde{\boldsymbol{X}}\boldsymbol{G}_{ij}(\boldsymbol{\zeta})\}\boldsymbol{x}_v \\ &= \sum_{i=1}^{r} h_i^2(\boldsymbol{z})\boldsymbol{x}_v^T\mathcal{L}\{\tilde{\boldsymbol{X}}\boldsymbol{G}_{ii}(\boldsymbol{\zeta})\}\boldsymbol{x}_v \\ &\quad + \sum_{i=1}^{r}\sum_{i<j}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\times \\ &\qquad \boldsymbol{x}_v^T\mathcal{L}\{\tilde{\boldsymbol{X}}\left(\boldsymbol{G}_{ij}(\boldsymbol{\zeta}) + \boldsymbol{G}_{ji}(\boldsymbol{\zeta})\right)\}\boldsymbol{x}_v.\end{aligned}$$

If the following conditions are satisfied, $\dot{V}(\boldsymbol{x}_v) < 0$ at $\boldsymbol{x}_v \neq \boldsymbol{0}$.

$$\mathcal{L}\{\tilde{\boldsymbol{X}}\boldsymbol{G}_{ii}(\boldsymbol{\zeta})\} < \boldsymbol{0} \tag{20}$$

$$\mathcal{L}\{\tilde{\boldsymbol{X}}\left(\boldsymbol{G}_{ij}(\boldsymbol{\zeta}) + \boldsymbol{G}_{ji}(\boldsymbol{\zeta})\right)\} \leq \boldsymbol{0} \quad i < j \leq r \tag{21}$$

(20) can be rewritten as

$$\mathcal{L}\{\tilde{\boldsymbol{X}}\boldsymbol{G}_{ii}(\boldsymbol{\zeta})\} = \begin{bmatrix}\alpha\boldsymbol{\Omega}_{11_{ii}}(\boldsymbol{\zeta}) & \alpha\boldsymbol{\Omega}_{12_{ii}}(\boldsymbol{\zeta}) \\ \alpha\boldsymbol{\Omega}_{12_{ii}}^T(\boldsymbol{\zeta}) & \boldsymbol{\Omega}_{22_{ii}}(\boldsymbol{\zeta})\end{bmatrix} < \boldsymbol{0}, \tag{22}$$

where

$$\begin{aligned}\boldsymbol{\Omega}_{11_{ii}}(\boldsymbol{\zeta}) &= \mathcal{L}\{\boldsymbol{X}_1^{-1}\boldsymbol{G}_{11_{ii}}(\boldsymbol{\zeta})\}, \\ \boldsymbol{\Omega}_{12_{ii}}(\boldsymbol{\zeta}) &= \boldsymbol{X}_1^{-1}\boldsymbol{G}_{12_{ii}}(\boldsymbol{\zeta}), \\ \boldsymbol{\Omega}_{22_{ii}}(\boldsymbol{\zeta}) &= \mathcal{L}\{\boldsymbol{X}_2\boldsymbol{G}_{22_{ii}}(\boldsymbol{\zeta})\}.\end{aligned}$$

From Schur complement, (22) can be converted into

$$\boldsymbol{\Omega}_{22_{ii}}(\boldsymbol{\zeta}) < \boldsymbol{0}, \tag{23}$$

$$\boldsymbol{\Omega}_{11_{ii}}(\boldsymbol{\zeta}) - \alpha\boldsymbol{\Omega}_{12_{ii}}(\boldsymbol{\zeta})(\boldsymbol{\Omega}_{22_{ii}}(\boldsymbol{\zeta}))^{-1}\boldsymbol{\Omega}_{12_{ii}}^T(\boldsymbol{\zeta}) < \boldsymbol{0}. \tag{24}$$

From (23) and (24), we have

$$\boldsymbol{\Omega}_{11_{ii}}(\boldsymbol{\zeta}) < \alpha\boldsymbol{\Omega}_{12_{ii}}(\boldsymbol{\zeta})(\boldsymbol{\Omega}_{22_{ii}}(\boldsymbol{\zeta}))^{-1}\boldsymbol{\Omega}_{12_{ii}}^T(\boldsymbol{\zeta}) \leq \boldsymbol{0}.$$

Hence, if (25) and (26) hold, then (20) is satisfied.

$$\mathcal{L}\{\boldsymbol{X}_1^{-1}(\boldsymbol{A}_i(\boldsymbol{\zeta}) - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{F}_i(\boldsymbol{\zeta}))\} < \boldsymbol{0} \tag{25}$$

$$\mathcal{L}\{\boldsymbol{X}_2(\boldsymbol{A}_i(\boldsymbol{\zeta}) - \boldsymbol{L}_i(\boldsymbol{\zeta})\boldsymbol{C}_i)\} < \boldsymbol{0} \tag{26}$$

Multiplying both side of (25) by $\boldsymbol{X}_1$ and defining a new variable $\boldsymbol{M}_i(\boldsymbol{\zeta}) = \boldsymbol{F}_i(\boldsymbol{\zeta})\boldsymbol{X}_1$, we obtain the following conditions.

$$\mathcal{L}\{\boldsymbol{A}_i(\boldsymbol{\zeta})\boldsymbol{X}_1 - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{M}_i(\boldsymbol{\zeta})\} < \boldsymbol{0} \tag{27}$$

Defining another new variable $\boldsymbol{N}_i(\boldsymbol{\zeta}) = \boldsymbol{X}_2\boldsymbol{L}_i(\boldsymbol{\zeta})$, (26) can be described as

$$\mathcal{L}\{\boldsymbol{X}_2\boldsymbol{A}_i(\boldsymbol{\zeta}) - \boldsymbol{N}_i(\boldsymbol{\zeta})\boldsymbol{C}_i\} < \boldsymbol{0}. \tag{28}$$

In the same way as above, (21) can be also represented as

$$\begin{aligned}\mathcal{L}\{&\boldsymbol{A}_i(\boldsymbol{\zeta})\boldsymbol{X}_1 - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{M}_j(\boldsymbol{\zeta}) \\ &+ \boldsymbol{A}_j(\boldsymbol{\zeta})\boldsymbol{X}_1 - \boldsymbol{B}_j(\boldsymbol{\zeta})\boldsymbol{M}_i(\boldsymbol{\zeta})\} \leq \boldsymbol{0},\end{aligned} \tag{29}$$

$$\begin{aligned}\mathcal{L}\{&\boldsymbol{X}_2\boldsymbol{A}_i(\boldsymbol{\zeta}) - \boldsymbol{N}_i(\boldsymbol{\zeta})\boldsymbol{C}_j \\ &+ \boldsymbol{X}_2\boldsymbol{A}_j(\boldsymbol{\zeta}) - \boldsymbol{N}_j(\boldsymbol{\zeta})\boldsymbol{C}_i\} \leq \boldsymbol{0},\end{aligned} \tag{30}$$

for $i < j \leq r$. It is clear from the inequality conditions (27)-(30) that $\dot{V}(\boldsymbol{x}_v) < 0$ at $\boldsymbol{x}_v \neq \boldsymbol{0}$ if the SOS conditions (11)-(16) hold.

■

Remark 2. *The conditions (11), (13) and (15) are for SOS conditions of polynomial fuzzy controller design. The conditions (12), (14) and (16) are for SOS conditions of polynomial fuzzy observer design. Thus, Theorem 1 provides SOS design conditions to separately design polynomial fuzzy controllers and observers.*

Remark 3. *If $\boldsymbol{A}_i(\boldsymbol{\zeta})$, $\boldsymbol{B}_i(\boldsymbol{\zeta})$, $\boldsymbol{L}_i(\boldsymbol{\zeta})$ and $\boldsymbol{F}_i(\boldsymbol{\zeta})$ reduce to constant matrices in (8), (9) and (10), they reduce to the ordinary T-S fuzzy model, the T-S fuzzy controller and observer, respectively. In addition, Theorem 1 reduces to the existing LMI design conditions, e.g., [13], for the T-S fuzzy controller and observer. Hence, Theorem 1 provides more general results.*

Remark 4. *Currently, sum of squares programs (SOSPs) are solved by reformulating them as semidefinite programs (SDPs), which in turn are solved efficiently, e.g., using interior point methods. Several commercial as well as non-commercial software packages are available for solving SDPs. While the conversion from SOSPs to SDPs can be manually performed for small size instances or tailored for specific problem classes, such a conversion can be quite cumbersome to perform in general. It is therefore desirable to have a computational aid that automatically performs this conversion for general SOSPs. This is exactly where SOSTOOLS comes to play. SOSTOOLS automates the conversion from SOSP to SDP, calls the SDP solver, and converts the SDP solution back to the solution of the original SOSP. At present, it uses other free MATLAB add-ons such as SeDuMi [30] or SDPT3 [31] as the SDP solver. It should be noted that we can numerically find the SOS variables (matrices) $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\boldsymbol{M}_i(\boldsymbol{\zeta})$ and $\boldsymbol{N}_i(\boldsymbol{\zeta})$ satisfying the SOS conditions in Theorem 1 via SeDuMi in addition to SOSTOOLS. Because Theorem 1 provides the SOS conditions that are convex with respect to the SOS variables (matrices) $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\boldsymbol{M}_i(\boldsymbol{\zeta})$ and $\boldsymbol{N}_i(\boldsymbol{\zeta})$. If non-convex terms exist in SOS conditions, they can not be numerically solved in general even via SOSTOOLS and SeDuMi. All the SOS conditions derived in this paper are convex with respect to SOS variables. Thus, our SOS-based designs proposed in this paper become numerically feasibility problems. For more details of how to solve the SDPs using SeDuMi, see [28] and [30].*

Remark 5. *To obtain more reliable solutions for SOS conditions, we perform the following double checking throughout this paper. We first carefully check whether the command 'sossolve' find a solution without any error messages, i.e., pinf=0, dinf=0 and numerr=0, or not. If any error messages exist, we judge 'infeasible'. After getting the feasible solutions using the command 'sossolve', the 'findsos' command is employed to check the feasibility of SOS conditions by substituting solutions into SOS conditions. We also carefully check whether the command 'findsos' provides a feasibility solution or not. If the command 'findsos' returns an infeasible result, we also judge 'infeasible'. This double checking is important to have reliable solutions in the use of SOSTOOLS [28] and SeDuMi [30].*

Remark 6. *The conditions $\epsilon_{1k} > 0$, $\epsilon_{2k} > 0$, $\epsilon_{3ik}(\boldsymbol{\zeta}) > 0$ and $\epsilon_{4ik}(\boldsymbol{\zeta}) > 0$ for $\boldsymbol{\zeta} \neq 0$ can be accommodated by sum of squares optimization in a similar way as in [32].*

A. Design Example I

Consider the following nonlinear system.

$$\left\{ \begin{array}{l} \dot{x}_1 = 0.1x_1^3 - x_2 + u \\ \dot{x}_2 = \sin x_1 - x_1^2 x_2 \end{array} \right. \tag{31}$$

This system has polynomial terms $0.1x_1^3$ and $x_1^2 x_2$. To obtain a T-S fuzzy model using the well-known sector nonlinearity [2], we need to assume the range of x_1, i.e., $x_1 \in [-d \ d]$, where d is a positive value. For outside the range, i.e., $x_1 < -d$ or $x_1 > d$, the T-S fuzzy model dynamics never agree with the original system dynamics. Thus, the T-S fuzzy model constructed for (31) is a local model. This means that the T-S fuzzy model stabilization and state-estimation convergence are not guaranteed for outside the range. Conversely, the polynomial fuzzy model constructed in this example can exactly and globally represent the dynamics of the original system.

Assume that x_1 is measurable and $y = x_1$. Fig.1 shows the behavior of this system without input. It can be seen that the system is unstable.

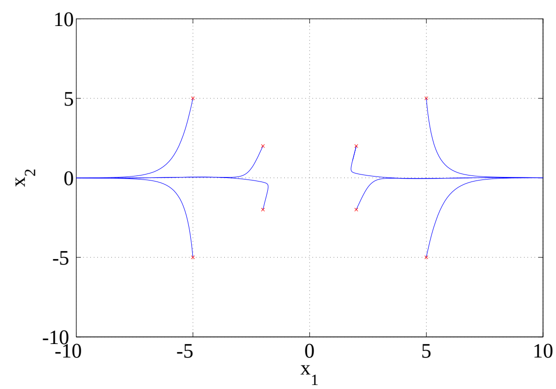

Fig. 1. System behavior without input.

1) Existing LMI design approach based on Takagi-Sugeno fuzzy systems: The existing LMI design approach for Takagi-Sugeno fuzzy models can be applied only to Class I. First we construct the Takagi-Sugeno fuzzy model (32) for the nonlinear dynamics using the sector nonlinearity idea [2].

$$\begin{cases} \dot{\boldsymbol{x}} = \displaystyle\sum_{i=1}^{r} h_i(\boldsymbol{z})\{\boldsymbol{A}_i\boldsymbol{x} + \boldsymbol{B}_i\boldsymbol{u}\}, \\ \boldsymbol{y} = \displaystyle\sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{C}_i\boldsymbol{x}, \end{cases} \tag{32}$$

where

$$\boldsymbol{A}_1 = \begin{bmatrix} 0.1d^2 & -1 \\ 1 & -d^2 \end{bmatrix}, \boldsymbol{A}_2 = \begin{bmatrix} 0.1d^2 & -1 \\ -0.217 & -d^2 \end{bmatrix},$$
$$\boldsymbol{A}_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \boldsymbol{A}_4 = \begin{bmatrix} 0 & -1 \\ -0.217 & 0 \end{bmatrix},$$
$$\boldsymbol{B}_1 = \boldsymbol{B}_2 = \boldsymbol{B}_3 = \boldsymbol{B}_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\boldsymbol{C}_1 = \boldsymbol{C}_2 = \boldsymbol{C}_3 = \boldsymbol{C}_4 = \begin{bmatrix} 1 & 0 \end{bmatrix},$$
$$h_1(\boldsymbol{z}) = \frac{x_1^2}{d^2}\frac{sinx_1 + 0.217x_1}{1.217x_1},$$
$$h_2(\boldsymbol{z}) = \frac{x_1^2}{d^2}\frac{x_1 - sinx_1}{1.217x_1},$$
$$h_3(\boldsymbol{z}) = \frac{d^2 - x_1^2}{d^2}\frac{sinx_1 + 0.217x_1}{1.217x_1},$$
$$h_4(\boldsymbol{z}) = \frac{d^2 - x_1^2}{d^2}\frac{x_1 - sinx_1}{1.217x_1}.$$

As mentioned just before, to obtain the Takagi-Sugeno fuzzy model, we need to assume the modeling range of x_1, i.e., $-d < x_1 < d$, where $d > 0$, since the original nonlinear system has polynomial terms. This means that the constructed fuzzy model is a semi-global model even if we select a larger value of d. We can see in Section III-A2 that the polynomial fuzzy model becomes a global model that is equivalent to the nonlinear dynamics of (31) for any x_1. This is an advantage point using the polynomial fuzzy model and our SOS based designs. In addition, it should be noted that the existing LMI design approach for Takagi-Sugeno fuzzy models can not be applied to more complicated classes, i.e., Classes II and III.

The LMI design conditions [2], [13] based on Takagi-Sugeno fuzzy systems are derived as

$$\boldsymbol{P}_1, \boldsymbol{P}_2 > \boldsymbol{0} \tag{33}$$
$$\boldsymbol{P}_1\boldsymbol{A}_i^T - \boldsymbol{M}_{1i}^T\boldsymbol{B}_i^T + \boldsymbol{A}_i\boldsymbol{P}_1 - \boldsymbol{B}_i\boldsymbol{M}_{1i} < \boldsymbol{0} \tag{34}$$
$$\boldsymbol{A}_i^T\boldsymbol{P}_2 - \boldsymbol{C}_i^T\boldsymbol{N}_{2i}^T + \boldsymbol{P}_2\boldsymbol{A}_i - \boldsymbol{N}_{2i}\boldsymbol{C}_i < \boldsymbol{0} \tag{35}$$
$$\begin{aligned}&\boldsymbol{P}_1\boldsymbol{A}_i^T - \boldsymbol{M}_{1j}^T\boldsymbol{B}_i^T + \boldsymbol{A}_i\boldsymbol{P}_1 - \boldsymbol{B}_i\boldsymbol{M}_{1j} \\ &\quad + \boldsymbol{P}_1\boldsymbol{A}_j^T - \boldsymbol{M}_{1i}^T\boldsymbol{B}_j^T + \boldsymbol{A}_j\boldsymbol{P}_1 - \boldsymbol{B}_j\boldsymbol{M}_{1i} < \boldsymbol{0}, \quad i < j\end{aligned} \tag{36}$$
$$\begin{aligned}&\boldsymbol{A}_i^T\boldsymbol{P}_2 - \boldsymbol{C}_j^T\boldsymbol{N}_{2i}^T + \boldsymbol{P}_2\boldsymbol{A}_i - \boldsymbol{N}_{2i}\boldsymbol{C}_j \\ &\quad + \boldsymbol{A}_j^T\boldsymbol{P}_2 - \boldsymbol{C}_i^T\boldsymbol{N}_{2j}^T + \boldsymbol{P}_2\boldsymbol{A}_j - \boldsymbol{N}_{2j}\boldsymbol{C}_i < \boldsymbol{0}, \quad i < j\end{aligned} \tag{37}$$

For all the ranges from a smaller d ($d = 10^{-3}$) to a larger d ($d = 10^9$), the LMI conditions (33)-(37) are infeasible. This means that the Takagi-Sugeno fuzzy controller and observer for the nonlinear system can not be designed using the existing approach. Conversely, we will see in Section III-A2 that the SOS design approach based on the polynomial fuzzy systems realizes that the polynomial fuzzy controller stabilizes the system and the estimation error via the polynomial fuzzy observer tends to zero.

2) SOS design approach based on polynomial fuzzy systems: The dynamics of the nonlinear system (31) can be exactly represented as the polynomial fuzzy system (8), where $r = 2$, $\boldsymbol{z} = \boldsymbol{\zeta} = y$,

$$\boldsymbol{A}_1(\boldsymbol{\zeta}) = \begin{bmatrix} 0.1y^2 & -1 \\ 1 & -y^2 \end{bmatrix}, \quad \boldsymbol{A}_2(\boldsymbol{\zeta}) = \begin{bmatrix} 0.1y^2 & -1 \\ -0.2172 & -y^2 \end{bmatrix}$$
$$\boldsymbol{B}_1(\boldsymbol{\zeta}) = \boldsymbol{B}_2(\boldsymbol{\zeta}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{C}_1 = \boldsymbol{C}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix},$$
$$h_1(\boldsymbol{z}) = \frac{\sin y + 0.2172y}{1.2172y}, \quad h_2(\boldsymbol{z}) = \frac{y - \sin y}{1.2172y}.$$

By solving the SOS conditions in Theorem 1, we have $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\boldsymbol{M}_i(\boldsymbol{\zeta})$ and $\boldsymbol{N}_i(\boldsymbol{\zeta})$, where the orders of $\boldsymbol{M}_i(\boldsymbol{\zeta})$ and $\boldsymbol{N}_i(\boldsymbol{\zeta})$ are two. e^{-10} and e^{-2} mean 10^{-10} and 10^{-2}, respectively.

$$\boldsymbol{X}_1 = \begin{bmatrix} 0.61825 & -0.5326e^{-10} \\ -0.5326e^{-10} & 0.42137 \end{bmatrix}$$
$$\boldsymbol{X}_2 = \begin{bmatrix} 0.68214 & 0.27426 \\ 0.27426 & 0.46738 \end{bmatrix}$$
$$\boldsymbol{M}_1(\boldsymbol{\zeta}) = \begin{bmatrix} 0.14778 + 0.41613y^2 \\ \quad 0.19687 - 0.53405e^{-2}y^2 \end{bmatrix}$$
$$\boldsymbol{M}_2(\boldsymbol{\zeta}) = \begin{bmatrix} 0.44549 + 0.41613y^2 \\ \quad -0.55566 - 0.53404e^{-2}y^2 \end{bmatrix}$$
$$\boldsymbol{N}_1(\boldsymbol{\zeta}) = \begin{bmatrix} 0.61756 + 0.42283y^2 \\ -0.20621 - 0.21828y^2 \end{bmatrix}$$
$$\boldsymbol{N}_2(\boldsymbol{\zeta}) = \begin{bmatrix} 0.30425 + 0.42283y^2 \\ -0.72299 - 0.21828y^2 \end{bmatrix}$$

From the solutions $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\boldsymbol{M}_i(\boldsymbol{\zeta})$ and $\boldsymbol{N}_i(\boldsymbol{\zeta})$, the polynomial feedback gains $\boldsymbol{F}_i(\boldsymbol{\zeta})$ and observer gains $\boldsymbol{L}_i(\boldsymbol{\zeta})$ are given as

$$\boldsymbol{F}_1(\boldsymbol{\zeta}) = \begin{bmatrix} 0.23903 + 0.67308y^2 \\ \quad 0.46721 - 0.12674e^{-1}y^2 \end{bmatrix},$$
$$\boldsymbol{F}_2(\boldsymbol{\zeta}) = \begin{bmatrix} 0.72057 + 0.67308y^2 \\ \quad -1.31870 - 0.12674e^{-1}y^2 \end{bmatrix},$$
$$\boldsymbol{L}_1(\boldsymbol{\zeta}) = \begin{bmatrix} 1.41704 + 1.05701y^2 \\ -1.27273 - 1.08729y^2 \end{bmatrix},$$
$$\boldsymbol{L}_2(\boldsymbol{\zeta}) = \begin{bmatrix} 1.39773 + 1.05701y^2 \\ -2.36709 - 1.08729y^2 \end{bmatrix}.$$

Fig. 2 shows the control and estimation result by the designed polynomial fuzzy controller and observer with their gains $\boldsymbol{F}_i(\boldsymbol{\zeta})$ and $\boldsymbol{L}_i(\boldsymbol{\zeta})$, where the initial states are $\boldsymbol{x}(0) = [5\ 5]$ and $\hat{\boldsymbol{x}}(0) = [-5\ \ -5]$. Fig.3 shows phase plots of control results for the same initial states as in Fig 1. It can be seen from these figures that the polynomial fuzzy controller stabilizes the system and the estimation error via the polynomial observer tends to zero.

IV. POLYNOMIAL CONTROLLER AND OBSERVER DESIGN (CLASS II)

In Section III, we discussed an observer design for the polynomial fuzzy system (8) with $\boldsymbol{A}_i(\zeta)$ and $\boldsymbol{B}_i(\zeta)$ matrices. This section presents a more complicated class, i.e., $\boldsymbol{A}_i$ depends on the state $\boldsymbol{x}$ instead of the vector $\boldsymbol{\zeta}$. Although the separation design for Class II is difficult, we derive SOS conditions to achieve it in this section. The reason will be

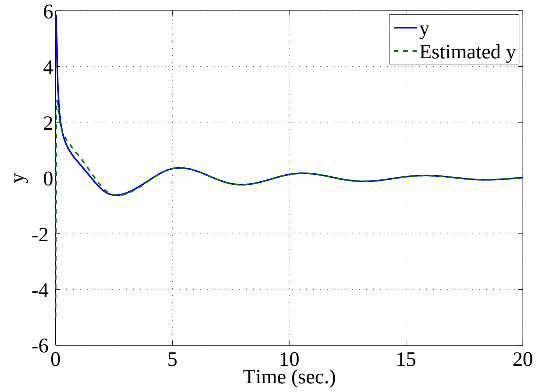

Fig. 2. Control and estimation result.

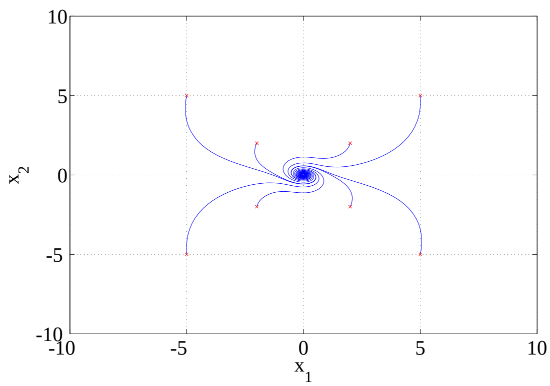

Fig. 3. Control trajectory for same initial states as in Fig 1.

mentioned in Remark 7. Consider the following polynomial fuzzy system.

$$\begin{cases} \dot{\boldsymbol{x}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\{\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{u}\} \\ \boldsymbol{y} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{C}_i\boldsymbol{x} \end{cases} \tag{38}$$

We design a polynomial fuzzy observer to estimate the states of (38).

$$\begin{cases} \dot{\hat{\boldsymbol{x}}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\{\boldsymbol{A}_i(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}} + \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{u} + \boldsymbol{L}_i(\hat{\boldsymbol{x}})(\boldsymbol{y} - \hat{\boldsymbol{y}})\} \\ \hat{\boldsymbol{y}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{C}_i\hat{\boldsymbol{x}} \end{cases} \tag{39}$$

To stabilize the system, we design a polynomial fuzzy controller with the state-feedback estimated by the polynomial observer.

$$\boldsymbol{u} = -\sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{F}_i(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}} \tag{40}$$

The difference between (40) and (10) is that (40) has the polynomial feedback gains in $\hat{\boldsymbol{x}}$ instead of those in $\boldsymbol{\zeta}$ in (10). Theorem 2 provides SOS conditions to separately design the polynomial fuzzy controller (40) and the polynomial fuzzy observer (39).

Theorem 2. *If there exist positive definite matrices $\boldsymbol{X}_1 \in \mathbb{R}^{n\times n}$, $\boldsymbol{X}_2 \in \mathbb{R}^{n\times n}$ and polynomial matrices $\boldsymbol{M}_i(\hat{\boldsymbol{x}}) \in \mathbb{R}^{p\times n}$, $\boldsymbol{N}_i(\hat{\boldsymbol{x}}) \in \mathbb{R}^{n\times q}$ satisfying (41)∼(46), the polynomial fuzzy controller (40) stabilizes the system (38) and the estimation error via the polynomial fuzzy observer (39) tends to zero.*

$$\boldsymbol{v}_1^T(\boldsymbol{X}_1 - \boldsymbol{E}_1)\boldsymbol{v}_1 \quad is \quad SOS \tag{41}$$

$$\boldsymbol{v}_2^T(\boldsymbol{X}_2 - \boldsymbol{E}_2)\boldsymbol{v}_2 \quad is \quad SOS \tag{42}$$

$$-\boldsymbol{v}_3^T\Big(\mathcal{L}\{\boldsymbol{A}_i(\hat{\boldsymbol{x}})\boldsymbol{X}_1 - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{M}_i(\hat{\boldsymbol{x}})\} + \boldsymbol{E}_{3i}(\boldsymbol{\zeta},\hat{\boldsymbol{x}})\Big)\boldsymbol{v}_3 \quad is \quad SOS \tag{43}$$

$$-\boldsymbol{v}_4^T\Big(\mathcal{L}\{\boldsymbol{X}_2\bar{\boldsymbol{A}}_i(\boldsymbol{x},\hat{\boldsymbol{x}}) - \boldsymbol{N}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_i\} + \boldsymbol{E}_{4i}(\boldsymbol{x},\hat{\boldsymbol{x}})\Big)\boldsymbol{v}_4 \quad is \quad SOS \tag{44}$$

$$-\boldsymbol{v}_5^T\Big(\mathcal{L}\{\boldsymbol{A}_i(\hat{\boldsymbol{x}})\boldsymbol{X}_1 - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{M}_j(\hat{\boldsymbol{x}})\} + \mathcal{L}\{\boldsymbol{A}_j(\hat{\boldsymbol{x}})\boldsymbol{X}_1 - \boldsymbol{B}_j(\boldsymbol{\zeta})\boldsymbol{M}_i(\hat{\boldsymbol{x}})\}\Big)\boldsymbol{v}_5 \quad is \quad SOS \quad i<j\le r \tag{45}$$

$$-\boldsymbol{v}_6^T\Big(\mathcal{L}\{\boldsymbol{X}_2\bar{\boldsymbol{A}}_i(\boldsymbol{x},\hat{\boldsymbol{x}}) - \boldsymbol{N}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j\} + \mathcal{L}\{\boldsymbol{X}_2\bar{\boldsymbol{A}}_j(\boldsymbol{x},\hat{\boldsymbol{x}}) - \boldsymbol{N}_j(\hat{\boldsymbol{x}})\boldsymbol{C}_i\}\Big)\boldsymbol{v}_6 \quad is \quad SOS \quad i<j\le r \tag{46}$$

where $\bar{\boldsymbol{A}}_i(\boldsymbol{x},\hat{\boldsymbol{x}})\boldsymbol{e} = \boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} - \boldsymbol{A}_i(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}}$. $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, $\boldsymbol{v}_4$, $\boldsymbol{v}_5$, $\boldsymbol{v}_6$ $\in \mathbb{R}^n$ denote vectors that are independent of $\boldsymbol{x}$, $\hat{\boldsymbol{x}}$ and $\boldsymbol{\zeta}$. From the solutions $\boldsymbol{X}_1$ and $\boldsymbol{M}_i(\hat{\boldsymbol{x}})$, we obtain polynomial feedback gains $\boldsymbol{F}_i(\hat{\boldsymbol{x}})$ as $\boldsymbol{F}_i(\hat{\boldsymbol{x}}) = \boldsymbol{M}_i(\hat{\boldsymbol{x}})\boldsymbol{X}_1^{-1}$. From the solutions $\boldsymbol{X}_2$ and $\boldsymbol{N}_i(\hat{\boldsymbol{x}})$, we obtain polynomial observer gains $\boldsymbol{L}_i(\hat{\boldsymbol{x}})$ as $\boldsymbol{L}_i(\hat{\boldsymbol{x}}) = \boldsymbol{X}_2^{-1}\boldsymbol{N}_i(\hat{\boldsymbol{x}})$ as well.

Proof: Consider the estimation error, $\boldsymbol{e} = \boldsymbol{x} - \hat{\boldsymbol{x}}$, by the observer. Then, the error system with respect to $\boldsymbol{e}$ can be represented as

$$\begin{aligned}\dot{\boldsymbol{e}} &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\{\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} - \boldsymbol{A}_i(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}} - \boldsymbol{L}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j\boldsymbol{e}\} \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\{\bar{\boldsymbol{A}}_i(\boldsymbol{x},\hat{\boldsymbol{x}}) - \boldsymbol{L}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j\}\boldsymbol{e},\end{aligned}$$

where $\bar{\boldsymbol{A}}(\boldsymbol{x},\hat{\boldsymbol{x}})\boldsymbol{e} = \boldsymbol{A}(\boldsymbol{x})\boldsymbol{x} - \boldsymbol{A}(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}}$. The augmented system with the augmented vector $\boldsymbol{x}_v = \begin{bmatrix}\hat{\boldsymbol{x}}^T & \boldsymbol{e}^T\end{bmatrix}^T$ is given as

$$\begin{aligned}\dot{\boldsymbol{x}}_v &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z}) \\ &\quad\times \begin{bmatrix}\boldsymbol{A}_i(\hat{\boldsymbol{x}}) - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{F}_j(\hat{\boldsymbol{x}}) & \boldsymbol{L}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j \\ \boldsymbol{0} & \bar{\boldsymbol{A}}_i(\boldsymbol{x},\hat{\boldsymbol{x}}) - \boldsymbol{L}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j\end{bmatrix}\boldsymbol{x}_v \\ &= \sum_{i=1}^{r} h_i^2(\boldsymbol{z})\boldsymbol{G}_{ii}(\boldsymbol{x},\boldsymbol{\zeta},\hat{\boldsymbol{x}})\boldsymbol{x}_v \\ &\quad + \sum_{i=1}^{r}\sum_{i<j}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\left(\boldsymbol{G}_{ij}(\boldsymbol{x},\boldsymbol{\zeta},\hat{\boldsymbol{x}}) + \boldsymbol{G}_{ji}(\boldsymbol{x},\boldsymbol{\zeta},\hat{\boldsymbol{x}})\right)\boldsymbol{x}_v\end{aligned} \tag{47}$$

where

$$\begin{aligned}
&\boldsymbol{G}_{ij}(\boldsymbol{x},\boldsymbol{\zeta},\hat{\boldsymbol{x}}) = \begin{bmatrix} \boldsymbol{G}_{11_{ij}}(\boldsymbol{\zeta},\hat{\boldsymbol{x}}) & \boldsymbol{G}_{12_{ij}}(\hat{\boldsymbol{x}}) \\ \boldsymbol{0} & \boldsymbol{G}_{22_{ij}}(\boldsymbol{x},\hat{\boldsymbol{x}}) \end{bmatrix},\\
&\boldsymbol{G}_{11_{ij}}(\boldsymbol{\zeta},\hat{\boldsymbol{x}}) = \boldsymbol{A}_i(\hat{\boldsymbol{x}}) - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{F}_j(\hat{\boldsymbol{x}}),\\
&\boldsymbol{G}_{12_{ij}}(\hat{\boldsymbol{x}}) = \boldsymbol{L}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j,\\
&\boldsymbol{G}_{22_{ij}}(\boldsymbol{x},\hat{\boldsymbol{x}}) = \bar{\boldsymbol{A}}_i(\boldsymbol{x},\hat{\boldsymbol{x}}) - \boldsymbol{L}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j.
\end{aligned}$$

Now, consider a candidate of Lyapunov function.

$$V(\boldsymbol{x}_v) = \boldsymbol{x}_v^T \tilde{\boldsymbol{X}} \boldsymbol{x}_v, \tag{48}$$

where

$$\tilde{\boldsymbol{X}} = \begin{bmatrix} \alpha \boldsymbol{X}_1^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{X}_2 \end{bmatrix}, \tag{49}$$

α is a positive value, $\boldsymbol{X}_1^{-1} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{X}_2 \in \mathbb{R}^{n\times n}$ are positive definite matrices. Note that $V(\boldsymbol{x}_v) > 0$ at $\boldsymbol{x}_v \neq \boldsymbol{0}$. It is clear from Lyapunov theory that the overall control system (47) is stable if it is proved that $\dot{V}(\boldsymbol{x}_v) < 0$ at $\boldsymbol{x}_v \neq \boldsymbol{0}$.

The time derivative of $V(\boldsymbol{x}_v)$ along the trajectory of the system is obtained as

$$\begin{aligned}
\dot{V}(\boldsymbol{x}_v) =& \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\boldsymbol{x}_v^T \mathcal{L}\{\tilde{\boldsymbol{X}}\boldsymbol{G}_{ij}(\boldsymbol{x},\boldsymbol{\zeta},\hat{\boldsymbol{x}})\}\boldsymbol{x}_v\\
=& \sum_{i=1}^{r} h_i^2(\boldsymbol{z})\boldsymbol{x}_v^T \mathcal{L}\{\tilde{\boldsymbol{X}}\boldsymbol{G}_{ii}(\boldsymbol{x},\boldsymbol{\zeta},\hat{\boldsymbol{x}})\}\boldsymbol{x}_v\\
&+ \sum_{i=1}^{r}\sum_{i<j}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\times\\
&\boldsymbol{x}_v^T \mathcal{L}\{\tilde{\boldsymbol{X}}\left(\boldsymbol{G}_{ij}(\boldsymbol{x},\boldsymbol{\zeta},\hat{\boldsymbol{x}}) + \boldsymbol{G}_{ji}(\boldsymbol{x},\boldsymbol{\zeta},\hat{\boldsymbol{x}})\right)\}\boldsymbol{x}_v.
\end{aligned}$$

If the following conditions are satisfied, $\dot{V}(\boldsymbol{x}_v) < 0$ at $\boldsymbol{x}_v \neq \boldsymbol{0}$.

$$\mathcal{L}\{\tilde{\boldsymbol{X}}\boldsymbol{G}_{ii}(\boldsymbol{x},\boldsymbol{\zeta},\hat{\boldsymbol{x}})\} < \boldsymbol{0} \tag{50}$$

$$\mathcal{L}\{\tilde{\boldsymbol{X}}\left(\boldsymbol{G}_{ij}(\boldsymbol{x},\boldsymbol{\zeta},\hat{\boldsymbol{x}}) + \boldsymbol{G}_{ji}(\boldsymbol{x},\boldsymbol{\zeta},\hat{\boldsymbol{x}})\right)\} \leq \boldsymbol{0} \quad i < j \leq r \tag{51}$$

As well as in Theorem 1, (50) can be separately rewritten as

$$\mathcal{L}\{\boldsymbol{X}_1^{-1}(\boldsymbol{A}_i(\hat{\boldsymbol{x}}) - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{F}_i(\hat{\boldsymbol{x}}))\} < \boldsymbol{0}, \tag{52}$$

$$\mathcal{L}\{\boldsymbol{X}_2(\bar{\boldsymbol{A}}_i(\boldsymbol{x},\hat{\boldsymbol{x}}) - \boldsymbol{L}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_i)\} < \boldsymbol{0}. \tag{53}$$

Multiplying both side of (52) by $\boldsymbol{X}_1$ and defining a new variable $\boldsymbol{M}_i(\hat{\boldsymbol{x}}) = \boldsymbol{F}_i(\hat{\boldsymbol{x}})\boldsymbol{X}_1$, we obtain the following conditions.

$$\mathcal{L}\{\boldsymbol{A}_i(\hat{\boldsymbol{x}})\boldsymbol{X}_1 - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{M}_i(\hat{\boldsymbol{x}})\} < \boldsymbol{0} \tag{54}$$

Defining another new variable $\boldsymbol{N}_i(\hat{\boldsymbol{x}}) = \boldsymbol{X}_2\boldsymbol{L}_i(\hat{\boldsymbol{x}})$, the inequality (53) can be described as

$$\mathcal{L}\{\boldsymbol{X}_2\bar{\boldsymbol{A}}_i(\boldsymbol{x},\hat{\boldsymbol{x}}) - \boldsymbol{N}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_i\} < \boldsymbol{0}. \tag{55}$$

In the same way as above, (51) can be also represented as

$$\begin{aligned}
&\mathcal{L}\{\boldsymbol{A}_i(\hat{\boldsymbol{x}})\boldsymbol{X}_1 - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{M}_j(\hat{\boldsymbol{x}})\}\\
&+ \mathcal{L}\{\boldsymbol{A}_j(\hat{\boldsymbol{x}})\boldsymbol{X}_1 - \boldsymbol{B}_j(\boldsymbol{\zeta})\boldsymbol{M}_i(\hat{\boldsymbol{x}})\} \leq \boldsymbol{0}, \qquad (56)\\
&\mathcal{L}\{\boldsymbol{X}_2\bar{\boldsymbol{A}}_i(\boldsymbol{x},\hat{\boldsymbol{x}}) - \boldsymbol{N}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j\}\\
&+ \mathcal{L}\{\boldsymbol{X}_2\bar{\boldsymbol{A}}_j(\boldsymbol{x},\hat{\boldsymbol{x}}) - \boldsymbol{N}_j(\hat{\boldsymbol{x}})\boldsymbol{C}_i\} \leq \boldsymbol{0} \qquad (57)
\end{aligned}$$

for $i < j \leq r$. It is clear from the inequality conditions (54)-(57) that $\dot{V}(\boldsymbol{x}_v) < 0$ at $\boldsymbol{x}_v \neq \boldsymbol{0}$ if the SOS conditions (41)∼(46) hold.

■

Remark 7. *As we can see, Theorems 1 and 2 show that the so-called separation principle is realized, i.e., that the fuzzy polynomial controller and observer can be separately designed without lack of guaranteeing the stability of the overall control system in addition to converging state estimation error (via the observer) to zero. This is a very important point in our fuzzy polynomial controller and observer design. In particular, in Theorem 2, a key feature of realizing the separation design is that, by introducing the transformation* $\bar{\boldsymbol{A}}(\boldsymbol{x},\hat{\boldsymbol{x}})\boldsymbol{e} = \boldsymbol{A}(\boldsymbol{x})\boldsymbol{x} - \boldsymbol{A}(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}}$*, the (2,1) element in* $\boldsymbol{G}_{ij}(\boldsymbol{x},\boldsymbol{\zeta},\hat{\boldsymbol{x}})$ *becomes zero element (matrix). This transformation idea leads to the successful separation design.*

A. Design Example II

Consider the following nonlinear system, where x_1 is measurable and $y = x_1$.

$$\begin{cases} \dot{x}_1 = \sin x_1 - 0.3x_2 + (x_1^2+1)u \\ \dot{x}_2 = -1.5x_1 - 2x_2 - x_2^3 \end{cases} \tag{58}$$

This system has polynomial terms $(x_1^2+1)u$ and x_2^3. To obtain a T-S fuzzy model, we need to assume the ranges of x_1 and x_2. Thus, as well as in Example I, the T-S fuzzy model is a local model. This means that the T-S fuzzy model stabilization and state-estimation convergence are not guaranteed for outside the ranges. The polynomial fuzzy model constructed in this example can exactly and globally represent the dynamics of the original system. Even if a local or semi-global T-S fuzzy model is permitted to use in practical sense, the premise variable vector $\boldsymbol{z}$ contain x_2 to be estimated. Hence, the previous LMI conditions mentioned in Section III-A1 can not be applied to the nonlinear system. On the other hand, the premise variable vector $\boldsymbol{z}$ in polynomial fuzzy model does not contain x_2 and x_2 appears in polynomial system matrices $\boldsymbol{A}_i$ in consequent parts of polynomial fuzzy models. Since the Class II design permits to have unmeasurable states in $\boldsymbol{A}_i$ matrices, it is possible to design a polynomial fuzzy observer in this example.

The dynamics of the nonlinear system can be exactly represented as the polynomial fuzzy system (38), where $r = 2$, $\boldsymbol{z} = \boldsymbol{\zeta} = y$,

$$\begin{aligned}
&\boldsymbol{A}_1(\boldsymbol{x}) = \begin{bmatrix} 1 & -0.3x_2 \\ -1.5 & -2-x_2^2 \end{bmatrix},\\
&\boldsymbol{A}_2(\boldsymbol{x}) = \begin{bmatrix} -0.2172 & -0.3x_2 \\ -1.5 & -2-x_2^2 \end{bmatrix},\\
&\boldsymbol{B}_1(\boldsymbol{\zeta}) = \boldsymbol{B}_2(\boldsymbol{\zeta}) = \begin{bmatrix} y^2+1 \\ 0 \end{bmatrix},\ \boldsymbol{C}_1 = \boldsymbol{C}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix},\\
&h_1(\boldsymbol{z}) = \frac{\sin y + 0.2172y}{1.2172y},\ h_2(\boldsymbol{z}) = \frac{y - \sin y}{1.2172y}.
\end{aligned}$$

In this example, note that

$$\begin{aligned}
&\bar{\boldsymbol{A}}_1(\boldsymbol{x},\hat{\boldsymbol{x}})\boldsymbol{e} = \boldsymbol{A}_1(\boldsymbol{x})\boldsymbol{x} - \boldsymbol{A}_1(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}}\\
&= \begin{bmatrix} 1 & -0.3(x_2+\hat{x}_2) \\ -1.5 & -2-x_2^2-x_2\hat{x}_2-\hat{x}_2^2 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \qquad (59)
\end{aligned}$$

$$\bar{\boldsymbol{A}}_2(\boldsymbol{x},\hat{\boldsymbol{x}})\boldsymbol{e} = \boldsymbol{A}_2(\boldsymbol{x})\boldsymbol{x} - \boldsymbol{A}_2(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}}$$
$$= \begin{bmatrix} -0.2172 & -0.3(x_2+\hat{x}_2) \\ -1.5 & -2-x_2^2-x_2\hat{x}_2-\hat{x}_2^2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \quad (60)$$

By solving the SOS conditions in Theorem 2, we obtain the following polynomial feedback and observer gains, where the orders of $\boldsymbol{M}_i(\hat{\boldsymbol{x}})$ and $\boldsymbol{N}_i(\hat{\boldsymbol{x}})$ are two.

$$\boldsymbol{F}_1(\hat{\boldsymbol{x}}) = [\ 2.17028 + 0.31476e^{-17}\hat{x}_2^2 \quad 0.35016e^{-5} - 0.37934e^{-11}\hat{x}_2^2\ ]$$

$$\boldsymbol{F}_2(\hat{\boldsymbol{x}}) = [\ 1.38495 + 0.31482e^{-17}\hat{x}_2^2 \quad 0.34413e^{-5} - 0.37942e^{-11}\hat{x}_2^2\ ]$$

$$\boldsymbol{L}_1(\boldsymbol{y},\hat{\boldsymbol{x}}) = \begin{bmatrix} 1.75626 + 0.650097e^{-11}\hat{x}_2^2 \\ -1.46221 - 0.52724e^{-5}\hat{x}_2^2 \end{bmatrix}$$

$$\boldsymbol{L}_2(\boldsymbol{y},\hat{\boldsymbol{x}}) = \begin{bmatrix} 0.64328 + 0.65012e^{-11}\hat{x}_2^2 \\ -1.41280 - 0.52725e^{-5}\hat{x}_2^2 \end{bmatrix}$$

Fig. 4 shows the control and estimation result by the designed polynomial fuzzy controller and observer, where the initial states are $\boldsymbol{x}(0) = [1\ 1]$ and $\hat{\boldsymbol{x}}(0) = [0\ 0]$. It can be seen that the designed controller stabilizes the nonlinear system and the estimation error via the polynomial fuzzy observer tends to zero.

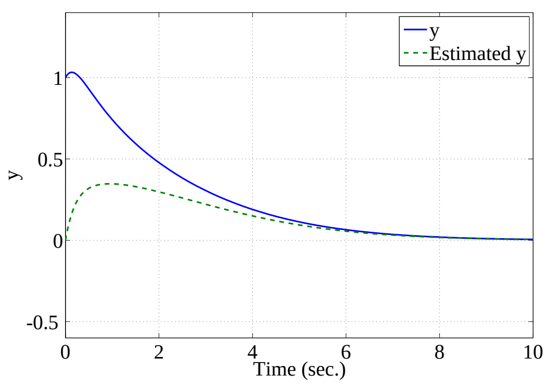

Fig. 4. Control and estimation result.

Remark 8. *Since $\boldsymbol{A}_1(\boldsymbol{x})$ and $\boldsymbol{A}_2(\boldsymbol{x})$ have unmeasurable x_2 in this design example, the Class I SOS-based observer design (Theorem 1) can not be applied to this design example. The previous LMI conditions mentioned in Section III-A1 can not be also applied to the nonlinear system. On the other hand, since the Class II design (Theorem 2) permits to have unmeasurable states in $\boldsymbol{A}_i$ matrices, it is possible to design a polynomial fuzzy observer in this example.*

V. Polynomial Controller and Observer Design (Class III)

In this section, we consider a more complicated class, i.e., $\boldsymbol{A}_i(\boldsymbol{x})$ and $\boldsymbol{B}_i(\boldsymbol{x})$ case. Class III design deals with the polynomial fuzzy system (61) and (7).

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\{\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{u}\} \quad (61)$$

For the system (61) and (7), we design the following polynomial fuzzy observer.

$$\dot{\hat{\boldsymbol{x}}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\{\boldsymbol{A}_i(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}} + \boldsymbol{B}_i(\hat{\boldsymbol{x}})\boldsymbol{u} + \boldsymbol{L}_i(\hat{\boldsymbol{x}})(\boldsymbol{y} - \hat{\boldsymbol{y}}) \quad (62)$$

$$\hat{\boldsymbol{y}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{C}_i\hat{\boldsymbol{x}}, \quad (63)$$

where $\boldsymbol{L}_i(\hat{\boldsymbol{x}})$ for all i are the polynomial observer gain matrices in $\hat{\boldsymbol{x}}$.

It is known that it is extremely difficult to separately design a polynomial fuzzy controller and observer in Class III. In fact, to the best of our knowledge, there exist no literatures on achieving the separation design in this class of polynomial fuzzy systems. To overcome the difficulty, we propose a practical algorithm to design a polynomial fuzzy controller and observer satisfying the stability of the overall augmented system in addition to converging state estimation error (via the observer) to zero.

The algorithm mainly consists of three steps.

Step 1 By assuming that all the states are measurable, we design the following controller.

$$\boldsymbol{u} = -\sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{F}_i(\boldsymbol{x})\boldsymbol{x} \quad (64)$$

The SOS conditions (see Theorem 3 below) derived in [7], [9] are applied to determine the polynomial feedback gains $\boldsymbol{F}_i(\boldsymbol{x})$.

Step 2 We replace the controller designed in Step 1 with

$$\boldsymbol{u} = -\sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{F}_i(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}}, \quad (65)$$

where $\boldsymbol{x}$ is replaced with $\hat{\boldsymbol{x}}$.

Step 3 Note that $\boldsymbol{F}_i(\hat{\boldsymbol{x}})$ and $\boldsymbol{X}_1$ (see Theorem 3 below) obtained in Step 2 are known polynomial matrices in $\hat{\boldsymbol{x}}$ and a positive definite matrix, respectively. We determine the polynomial observer gains $\boldsymbol{L}_i(\hat{\boldsymbol{x}})$ by solving new SOS design conditions (see Theorem 4 below).

We present the previous SOS conditions [7], [9] (Theorem 3 below) to determine the polynomial feedback gains $\boldsymbol{F}_i(\boldsymbol{x})$ and new SOS design conditions (Theorem 4 below) to determine the polynomial observer gains that are newly derived in this paper.

Theorem 3. *[7], [9] The system (61) and (7) can be stabilized by the controller (64) if there exist a positive definite matrix $\boldsymbol{X}_1 \in \mathbb{R}^{n\times n}$ and polynomial matrices $\boldsymbol{M}_i(\boldsymbol{x}) \in \mathbb{R}^{p\times n}$*

satisfying the following SOS conditions.

$$\boldsymbol{v}_1^T(\boldsymbol{X}_1 - \boldsymbol{E}_1^{reg})\boldsymbol{v}_1 \quad is \quad SOS \tag{66}$$

$$-\boldsymbol{v}_2^T\Big(\mathcal{L}\{\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{X}_1 - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{M}_i(\boldsymbol{x})\} + \boldsymbol{E}_{2i}^{reg}(\boldsymbol{x})\Big)\boldsymbol{v}_2 \quad is \quad SOS \tag{67}$$

$$-\boldsymbol{v}_3^T\Big(\mathcal{L}\{\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{X}_1 - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{M}_j(\boldsymbol{x})\} + \mathcal{L}\{\boldsymbol{A}_j(\boldsymbol{x})\boldsymbol{X}_1 - \boldsymbol{B}_j(\boldsymbol{x})\boldsymbol{M}_i(\boldsymbol{x})\}\Big)\boldsymbol{v}_3 \quad is \quad SOS \quad i < j \le r \tag{68}$$

where $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3 \in \mathbb{R}^n$ denote vectors that are independent of $\boldsymbol{x}$. From the solutions $\boldsymbol{X}_1$ and $\boldsymbol{M}_i(\boldsymbol{x})$, the feedback gain can be obtained as $\boldsymbol{F}_i(\boldsymbol{x}) = \boldsymbol{M}_i(\boldsymbol{x})\boldsymbol{X}_1^{-1}$.

Theorem 4. *The system (61) and (7) can be stabilized by the polynomial fuzzy controller (65) and the estimation error via the polynomial fuzzy observer (62) and (63) tends to zero if there exist a positive definite matrix $\boldsymbol{X}_2 \in \mathbb{R}^{n\times n}$ and polynomial matrices $\boldsymbol{N}_i(\hat{\boldsymbol{x}}) \in \mathbb{R}^{n\times q}$ satisfying the following SOS conditions, where $\boldsymbol{X}_1$ and $\boldsymbol{F}_j(\hat{\boldsymbol{x}})$ are solutions satisfying the SOS conditions in Theorem 3 and are given (known) matrices in Theorem 4.*

$$\boldsymbol{x}_v^T\left(\begin{bmatrix}\boldsymbol{X}_1^{-1}\boldsymbol{X}_2 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{X}_2\end{bmatrix} - \boldsymbol{E}_1^{obs}\right)\boldsymbol{x}_v \quad is \quad SOS \tag{69}$$

$$-\boldsymbol{x}_v^T\Big(\boldsymbol{\Omega}_{ii}(\boldsymbol{x},\hat{\boldsymbol{x}}) + \boldsymbol{E}_{2i}^{obs}(\boldsymbol{x},\hat{\boldsymbol{x}})\Big)\boldsymbol{x}_v \quad is \quad SOS \tag{70}$$

$$-\boldsymbol{x}_v^T\Big(\boldsymbol{\Omega}_{ij}(\boldsymbol{x},\hat{\boldsymbol{x}}) + \boldsymbol{\Omega}_{ji}(\boldsymbol{x},\hat{\boldsymbol{x}})\Big)\boldsymbol{x}_v \quad is \quad SOS \quad i < j \le r \tag{71}$$

where

$$\begin{aligned}
&\boldsymbol{\Omega}_{ij}(\boldsymbol{x},\hat{\boldsymbol{x}}) = \begin{bmatrix}\boldsymbol{\Omega}_{ij}^{11}(\hat{\boldsymbol{x}}) & \boldsymbol{\Omega}_{ij}^{12}(\hat{\boldsymbol{x}})\\ \boldsymbol{\Omega}_{ij}^{21}(\boldsymbol{x},\hat{\boldsymbol{x}}) & \boldsymbol{\Omega}_{ij}^{22}(\boldsymbol{x},\hat{\boldsymbol{x}})\end{bmatrix},\\
&\boldsymbol{\Omega}_{ij}^{11}(\hat{\boldsymbol{x}}) = \boldsymbol{X}_1^{-1}\boldsymbol{X}_2(\boldsymbol{A}_i(\hat{\boldsymbol{x}}) - \boldsymbol{B}_i(\hat{\boldsymbol{x}})\boldsymbol{F}_j(\hat{\boldsymbol{x}})),\\
&\boldsymbol{\Omega}_{ij}^{12}(\hat{\boldsymbol{x}}) = \boldsymbol{X}_1^{-1}\boldsymbol{N}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j,\\
&\boldsymbol{\Omega}_{ij}^{21}(\boldsymbol{x},\hat{\boldsymbol{x}}) = \boldsymbol{X}_2(\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{A}_i(\hat{\boldsymbol{x}}) - (\boldsymbol{B}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\hat{\boldsymbol{x}}))\boldsymbol{F}_j(\hat{\boldsymbol{x}})),\\
&\boldsymbol{\Omega}_{ij}^{22}(\boldsymbol{x},\hat{\boldsymbol{x}}) = \boldsymbol{X}_2\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{N}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j,
\end{aligned}$$

$\boldsymbol{x}_v = [\hat{\boldsymbol{x}}^T \quad \boldsymbol{e}^T]^T$ and $\boldsymbol{e} = \boldsymbol{x} - \hat{\boldsymbol{x}}$. From the solutions $\boldsymbol{X}_2$ and $\boldsymbol{N}_i(\hat{\boldsymbol{x}})$, we can obtain observer gain matrices as $\boldsymbol{L}_i(\hat{\boldsymbol{x}}) = \boldsymbol{X}_2^{-1}\boldsymbol{N}_i(\hat{\boldsymbol{x}})$.

Proof: Define the estimation error via the observer as $\boldsymbol{e} = \boldsymbol{x} - \hat{\boldsymbol{x}}$. Then, the error dynamics are represented as

$$\begin{aligned}\dot{\boldsymbol{e}} = &\sum_{i=1}^r\sum_{j=1}^r h_i(\boldsymbol{z})h_j(\boldsymbol{z})\times\\ &\{(\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{A}_i(\hat{\boldsymbol{x}}) - (\boldsymbol{B}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\hat{\boldsymbol{x}}))\boldsymbol{F}_j(\hat{\boldsymbol{x}}))\hat{\boldsymbol{x}}\\ &+ (\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{L}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j)\boldsymbol{e}\}.\end{aligned}$$

We obtain the following augmented system:

$$\dot{\boldsymbol{x}}_v = \sum_{i=1}^r\sum_{j=1}^r h_i(\boldsymbol{z})h_j(\boldsymbol{z})\boldsymbol{G}_{ij}(\boldsymbol{x},\hat{\boldsymbol{x}})\boldsymbol{x}_v,$$

where

$$\begin{aligned}
&\boldsymbol{x}_v = \begin{bmatrix}\hat{\boldsymbol{x}}^T & \boldsymbol{e}^T\end{bmatrix}^T,\\
&\boldsymbol{G}_{ij}(\boldsymbol{x},\hat{\boldsymbol{x}}) = \begin{bmatrix}\boldsymbol{G}_{ij}^{11}(\hat{\boldsymbol{x}}) & \boldsymbol{G}_{ij}^{12}(\hat{\boldsymbol{x}})\\ \boldsymbol{G}_{ij}^{21}(\boldsymbol{x},\hat{\boldsymbol{x}}) & \boldsymbol{G}_{ij}^{22}(\boldsymbol{x},\hat{\boldsymbol{x}})\end{bmatrix},\\
&\boldsymbol{G}_{ij}^{11}(\hat{\boldsymbol{x}}) = \boldsymbol{A}_i(\hat{\boldsymbol{x}}) - \boldsymbol{B}_i(\hat{\boldsymbol{x}})\boldsymbol{F}_j(\hat{\boldsymbol{x}}),\\
&\boldsymbol{G}_{ij}^{12}(\hat{\boldsymbol{x}}) = \boldsymbol{L}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j,\\
&\boldsymbol{G}_{ij}^{21}(\boldsymbol{x},\hat{\boldsymbol{x}}) = \boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{A}_i(\hat{\boldsymbol{x}}) - (\boldsymbol{B}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\hat{\boldsymbol{x}}))\boldsymbol{F}_j(\hat{\boldsymbol{x}}),\\
&\boldsymbol{G}_{ij}^{22}(\boldsymbol{x},\hat{\boldsymbol{x}}) = \boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{L}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j.
\end{aligned}$$

Now, consider the following candidate of Lyapunov functions.

$$V(\boldsymbol{x}_v) = \boldsymbol{x}_v^T\tilde{\boldsymbol{X}}\boldsymbol{x}_v, \tag{72}$$

where

$$\tilde{\boldsymbol{X}} = \begin{bmatrix}\boldsymbol{X}_1^{-1}\boldsymbol{X}_2 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{X}_2\end{bmatrix} > \boldsymbol{0}. \tag{73}$$

The time derivative of $V(\boldsymbol{x}_v)$ along the system trajectories is

$$\begin{aligned}\dot{V}(\boldsymbol{x}_v) = &\sum_{i=1}^r\sum_{j=1}^r h_i(\boldsymbol{z})h_j(\boldsymbol{z})\boldsymbol{x}_v^T(\boldsymbol{G}_{ij}^T(\boldsymbol{x},\hat{\boldsymbol{x}})\tilde{\boldsymbol{X}}\\ &+ \tilde{\boldsymbol{X}}\boldsymbol{G}_{ij}(\boldsymbol{x},\hat{\boldsymbol{x}}))\boldsymbol{x}_v.\end{aligned}$$

Since $\boldsymbol{x}_v^T\boldsymbol{H}\boldsymbol{x}_v = \boldsymbol{x}_v^T\boldsymbol{H}^T\boldsymbol{x}_v$ for any square matrix $\boldsymbol{H}$, we have

$$\begin{aligned}\dot{V}(\boldsymbol{x}_v) =&2\sum_{i=1}^r\sum_{j=1}^r h_i(\boldsymbol{z})h_j(\boldsymbol{z})\boldsymbol{x}_v^T\tilde{\boldsymbol{X}}\boldsymbol{G}_{ij}(\boldsymbol{x},\hat{\boldsymbol{x}})\boldsymbol{x}_v\\ =&2\sum_{i=1}^r h_i^2(\boldsymbol{z})\boldsymbol{x}_v^T\tilde{\boldsymbol{X}}\boldsymbol{G}_{ii}(\boldsymbol{x},\hat{\boldsymbol{x}})\boldsymbol{x}_v\\ &+2\sum_{i=1}^r\sum_{i<j}^r h_i(\boldsymbol{z})h_j(\boldsymbol{z})\times\\ &\boldsymbol{x}_v^T\tilde{\boldsymbol{X}}(\boldsymbol{G}_{ij}(\boldsymbol{x},\hat{\boldsymbol{x}}) + \boldsymbol{G}_{ji}(\boldsymbol{x},\hat{\boldsymbol{x}}))\boldsymbol{x}_v.\end{aligned} \tag{74}$$

$\dot{V}(\boldsymbol{x}_v) < 0$ at $\boldsymbol{x}_v \neq \boldsymbol{0}$ if (75) and (76) hold.

$$-\boldsymbol{x}_v^T\tilde{\boldsymbol{X}}\boldsymbol{G}_{ii}(\boldsymbol{x},\hat{\boldsymbol{x}})\boldsymbol{x}_v > \boldsymbol{0}, \tag{75}$$

$$-\boldsymbol{x}_v^T\tilde{\boldsymbol{X}}(\boldsymbol{G}_{ij}(\boldsymbol{x},\hat{\boldsymbol{x}}) + \boldsymbol{G}_{ji}(\boldsymbol{x},\hat{\boldsymbol{x}}))\boldsymbol{x}_v \ge \boldsymbol{0} \quad i < j \le r. \tag{76}$$

By defining as $\boldsymbol{N}_i(\hat{\boldsymbol{x}}) = \boldsymbol{X}_2\boldsymbol{L}_i(\hat{\boldsymbol{x}})$, (75) can be rewritten as

$$\begin{aligned}-\boldsymbol{x}_v^T\tilde{\boldsymbol{X}}\boldsymbol{G}_{ii}(\boldsymbol{x},\hat{\boldsymbol{x}})\boldsymbol{x}_v &= -\boldsymbol{x}_v^T\begin{bmatrix}\boldsymbol{\Omega}_{ii}^{11}(\hat{\boldsymbol{x}}) & \boldsymbol{\Omega}_{ii}^{12}(\hat{\boldsymbol{x}})\\ \boldsymbol{\Omega}_{ii}^{21}(\boldsymbol{x},\hat{\boldsymbol{x}}) & \boldsymbol{\Omega}_{ii}^{22}(\boldsymbol{x},\hat{\boldsymbol{x}})\end{bmatrix}\boldsymbol{x}_v\\ &= -\boldsymbol{x}_v^T\boldsymbol{\Omega}_{ii}(\boldsymbol{x},\hat{\boldsymbol{x}})\boldsymbol{x}_v > \boldsymbol{0},\end{aligned} \tag{77}$$

where

$$\begin{aligned}
&\boldsymbol{\Omega}_{ii}^{11}(\hat{\boldsymbol{x}}) = \boldsymbol{X}_1^{-1}\boldsymbol{X}_2(\boldsymbol{A}_i(\hat{\boldsymbol{x}}) - \boldsymbol{B}_i(\hat{\boldsymbol{x}})\boldsymbol{F}_i(\hat{\boldsymbol{x}})),\\
&\boldsymbol{\Omega}_{ii}^{12}(\hat{\boldsymbol{x}}) = \boldsymbol{X}_1^{-1}\boldsymbol{N}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_i,\\
&\boldsymbol{\Omega}_{ii}^{21}(\boldsymbol{x},\hat{\boldsymbol{x}}) = \boldsymbol{X}_2(\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{A}_i(\hat{\boldsymbol{x}}) - (\boldsymbol{B}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\hat{\boldsymbol{x}}))\boldsymbol{F}_i(\hat{\boldsymbol{x}})),\\
&\boldsymbol{\Omega}_{ii}^{22}(\boldsymbol{x},\hat{\boldsymbol{x}}) = \boldsymbol{X}_2\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{N}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_i.
\end{aligned}$$

Also, (76) can be rewritten as

$$-\boldsymbol{x}_v^T(\boldsymbol{\Omega}_{ij}(\boldsymbol{x},\hat{\boldsymbol{x}}) + \boldsymbol{\Omega}_{ji}(\boldsymbol{x},\hat{\boldsymbol{x}}))\boldsymbol{x}_v \ge \boldsymbol{0}, \quad i < j \le r \tag{78}$$

where

$$\begin{aligned}\Omega_{ij}^{11}(\hat{x}) &= X_1^{-1}X_2(A_i(\hat{x}) - B_i(\hat{x})F_j(\hat{x})),\\ \Omega_{ij}^{12}(\hat{x}) &= X_1^{-1}N_i(\hat{x})C_j,\\ \Omega_{ij}^{21}(x,\hat{x}) &= X_2(A_i(x) - A_i(\hat{x}) \\ &\quad - (B_i(x) - B_i(\hat{x}))F_j(\hat{x})),\\ \Omega_{ij}^{22}(x,\hat{x}) &= X_2A_i(x) - N_i(\hat{x})C_j.\end{aligned}$$

Now, we arrive at the SOSPs (69)-(71). ■

Clearly, the overall control system consisting of (61), (7), (65), (62) and (63) is asymptotically and globally stable and the estimation error tends to zero.

Remark 9. *Note that (73) is different from (19) and (49). (73) is needed to have SOS conditions with respect to variables X_2 and $N_i(\hat{x})$. If we use (19) or (49) instead of (73), the derived conditions have X_2, $N_i(\hat{x})$ and $L_i(\hat{x})$. In this case, due to the constraint $N_i(\hat{x}) = X_2L_i(\hat{x})$, they can not be generally solved by SOSTOOLS and SeDuMi.*

A. Design Example III

Consider the following nonlinear system.

$$\begin{cases}\dot{x}_1 = \sin x_1 - 5x_2 + (x_2^2+5)u\\ \dot{x}_2 = -x_1 - x_2^3\end{cases} \tag{79}$$

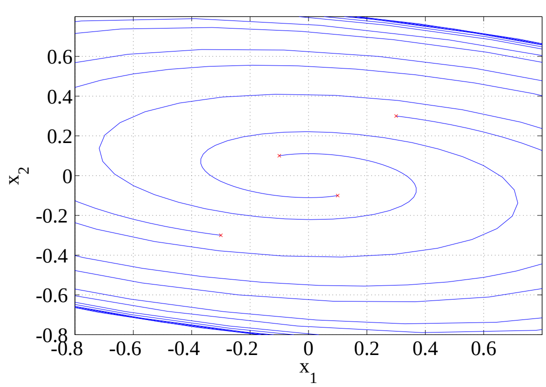

Fig. 5. System behavior without input.

This system has polynomial terms $(x_2^2+5)u$ and x_2^3. As well as in Examples I and II, the polynomial fuzzy model constructed in this example can exactly and globally represent the dynamics of the original system although the T-S fuzzy model for (79) is a local model. In addition, the previous LMI conditions in Section III-A1 can not be applied to the nonlinear system. Conversely, the Class III design can be applied to designing a polynomial fuzzy observer in this example.

Assume that x_1 is measurable and $y = x_1$. Fig. 5 shows the behavior of the nonlinear system without input for several initial states. It is found from the figure that this system is unstable.

The system (79) can be exactly converted into the polynomial fuzzy system (61) and (7) using the sector nonlinearity [2], where $r = 2$, $z = y$,

$$\begin{aligned}&A_1(x) = \begin{bmatrix}1 & 5\\ -1 & -x_2^2\end{bmatrix},\ A_2(x) = \begin{bmatrix}-0.2172 & 5\\ -1 & -x_2^2\end{bmatrix},\\ &B_1(x) = \begin{bmatrix}x_2^2+5\\ 0\end{bmatrix},\ B_2(x) = \begin{bmatrix}x_2^2+5\\ 0\end{bmatrix},\\ &C_1 = C_2 = \begin{bmatrix}1 & 0\end{bmatrix},\\ &h_1(z) = \frac{\sin y + 0.2172y}{1.2172y},\ h_2(z) = \frac{y - \sin y}{1.2172y}.\end{aligned}$$

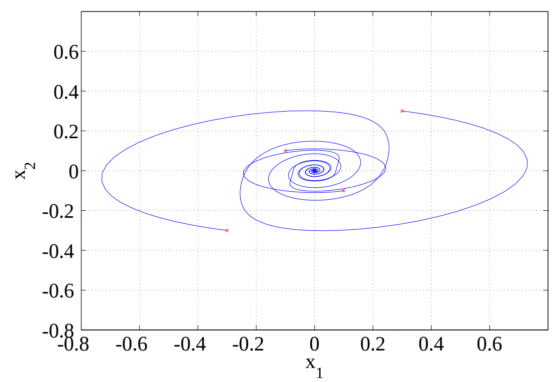

Fig. 6. Control trajectories for same initial states as in Fig. 5.

Fig. 6 shows control result (for the same initial states as Fig. 5) by the polynomial fuzzy controller and observer designed using Theorem 3 and Theorem 4, where the order of $M_i(\hat{x})$ and $N_i(\hat{x})$ are two. Fig. 7 shows the control and estimation result starting from one of the initial states, where $x(0) = [0.3\ 0.3]$ and $\hat{x}(0) = [-0.3\ -0.3]$. The polynomial feedback and observer gains are obtained as follows.

$$\begin{aligned}F_1(\hat{x}) &= [\ 0.29008 + 0.20778\hat{x}_2^2\\ &\qquad 0.63772 - 0.22047e^{-1}\hat{x}_2^2\ ]\\ F_2(\hat{x}) &= [\ 0.46829e^{-1} + 0.22751\hat{x}_2^2\\ &\qquad 0.64532 - 0.24141e^{-1}\hat{x}_2^2\ ]\\ L_1(\hat{x}) &= \begin{bmatrix}2.65691 + 17.71908\hat{x}_2^2\\ 1.08259 + 1.76675\hat{x}_2^2\end{bmatrix}\\ L_2(\hat{x}) &= \begin{bmatrix}3.68595 + 18.01543\hat{x}_2^2\\ 1.52432 + 1.70592\hat{x}_2^2\end{bmatrix}\end{aligned}$$

It can be found from the control results that the designed polynomial fuzzy controller stabilizes the system and the estimation error via the polynomial fuzzy observer tends to zero.

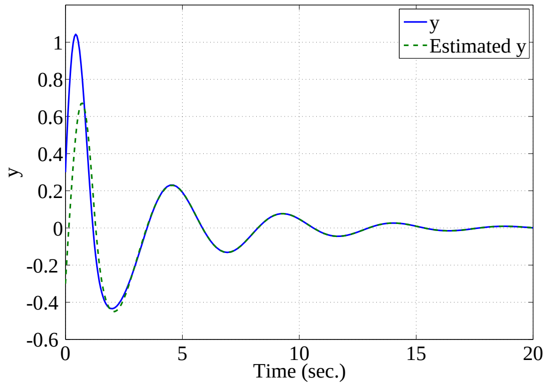

Fig. 7. Control and estimation result.

Remark 10. *Since $\boldsymbol{A}_1(\boldsymbol{x})$, $\boldsymbol{A}_2(\boldsymbol{x})$, $\boldsymbol{B}_1(\boldsymbol{x})$ and $\boldsymbol{B}_2(\boldsymbol{x})$ have unmeasurable x_2 in this design example, the previous SOS-based observer designs (Classes I and II) can not be applied to this design example. Even if the sector nonlinearity concept is applied to construct a T-S fuzzy model for the nonlinear system, the premise variables z contain x_2. Hence, the previous LMI conditions mentioned in Section III-A1 can not be applied to the nonlinear system. On the other hand, since the Class III design permits to have unmeasurable states in both of $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ matrices, it is possible to design a polynomial fuzzy observer in this example.*

VI. Conclusions

This paper has presented a sum of squares (SOS) approach for three classes of polynomial fuzzy controllers and observers. To illustrate the validity and applicability of the proposed approach, three design examples have been provided. The examples have demonstrated advantages of the SOS-based approaches for the existing LMI approaches to T-S fuzzy observer designs.

Our next subjects are to derive SOS observer design conditions to realize the sepration design even for Class III and to apply our observer designs to helicopter control [11].

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Kazuo Tanaka (S’87 - M’91 - SM’09) received the B.S. and M.S. degrees in Electrical Engineering from Hosei University, Tokyo, Japan, in 1985 and 1987, and Ph.D. degree, in Systems Science from Tokyo Institute of Technology, in 1990, respectively.

He is currently a Professor in Department of Mechanical Engineering and Intelligent Systems at The University of Electro-Communications. He was a Visiting Scientist in Computer Science at the University of North Carolina at Chapel Hill in 1992 and 1993. He received the Best Young Researchers Award from the Japan Society for Fuzzy Theory and Systems in 1990, the Outstanding Papers Award at the 1990 Annual NAFIPS Meeting in Toronto, Canada, in 1990, the Outstanding Papers Award at the Joint Hungarian-Japanese Symposium on Fuzzy Systems and Applications in Budapest, Hungary, in 1991, the Best Young Researchers Award from the Japan Society for Mechanical Engineers in 1994, the Outstanding Book Awards from the Japan Society for Fuzzy Theory and Systems in 1995, 1999 IFAC World Congress Best Poster Paper Prize in 1999, 2000 IEEE Transactions on Fuzzy Systems Outstanding Paper Award in 2000, the Best Paper Selection at 2005 American Control Conference in Portland, USA, in 2005. His research interests include intelligent systems and control, nonlinear systems control, robotics, brain-machine interface and their applications.

He is currently serving as an Associate Editor for Automatica and for the IEEE Transactions on Fuzzy Systems, and is on the IEEE Control Systems Society Conference Editorial Board. He is the author of two books and a co-author of 17 books. Recently, he co-authored (with Hua O. Wang) the book Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach (Wiley-Interscience, 2001). He is a senior member of IEEE.

Hiroshi Ohtake (S’02 - M’05) received the B.S. and M.S. degrees in mechanical and control engineering from The University of Electro-Communications, Tokyo, Japan, in 2000 and 2002, respectively.

He is currently an Associate Professor in Department of Mechanical Information Science and Technology, Kyushu Institute of Technology, Fukuoka, Japan. He was a Research Fellow of the Japan Society for the Promotion of Science from 2002 to 2004. He received Outstanding Student Paper Award at the Joint 9th IFSA World Congress and 20th NAFIPS International Conference in Vancouver, Canada, in 2001, the Young Investigators Award from the Japan Society for Fuzzy Theory and Intelligent Informatics in 2003, the Best Presentation Award at the FAN Symposium 2004 in Kochi, Japan, in 2004, 2005 American Control Conference Best Paper Selection, at American Control Conference 2005 in Portland, USA, in 2005. His research interests include nonlinear mechanical systems control and robotics.

Toshiaki Seo received the B.S. degree in Department of Mechanical Engineering and Intelligent System at The University of Electro-Communications, Tokyo, Japan in 2010. He is currently a graduate student in Department of Mechanical Engineering and Intelligent System at The University of Electro-Communications, Tokyo, Japan. His research interests include nonlinear systems control and applications.

Motoyasu Tanaka received the B.S., M.S. and Ph. D. degrees in mechanical engineering and intelligent systems from The University of Electro-Communications, Tokyo, Japan, in 2005, 2007, and 2009, respectively.

He is currently an Assistant Professor in in Department of Mechanical Engineering and Intelligent Systems at The University of Electro-Communications. He was a Research Fellow of the Japan Society for the Promotion of Science from 2007 to 2009. He received IEEE Robotics and Automation Society Japan Chapter Young Award in 2007. His research interests include biologically inspired robotics and dynamic-based nonlinear control.

Hua O. Wang (M’94-SM’01) received the B.S. degree from the University of Science and Technology of China (USTC), Hefei, China, in 1987, the M.S. degree from the University of Kentucky, Lexington, KY, in 1989, and the Ph.D. degree from the University of Maryland, College Park, MD, in 1993, all in Electrical Engineering.

He has been with Boston University where he is currently an Associate Professor of Aerospace and Mechanical Engineering since September 2002. He was with the United Technologies Research Center, East Hartford, CT, from 1993 to 1996, and was a faculty member in the Department of Electrical and Computer Engineering at Duke University, Durham, NC, from 1996 to 2002. Dr. Wang served as the Program Manager (IPA) for Systems and Control with the U.S. Army Research Office (ARO) from August 2000 to August 2002. During 2000 - 2005, he also held the position of Cheung Kong Chair Professor and Director with the Center for Nonlinear and Complex Systems at Huazhong University of Science and Technology, Wuhan, China.

Dr. Wang is a recipient of the 1994 O. Hugo Schuck Best Paper Award of the American Automatic Control Council, the 14th IFAC World Congress Poster Paper Prize, the 2000 IEEE Transactions on Fuzzy Systems Outstanding Paper Award. His research interests include control of nonlinear dynamics, intelligent systems and control, networked control systems, robotics, cooperative control, and applications. He co-authored (with Kazuo Tanaka) the book Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach (Wiley-Interscience, 2001). Dr. Wang has served as an Associate Editor for the IEEE Transactions on Automatic Control and was on the IEEE Control Systems Society Conference Editorial Board. He is an Editor for the Journal of Systems Science and Complexity. He is an appointed member of the 2006 Board of Governors of the IEEE Control Systems Society and a senior member of IEEE.