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Abstract

This paper is concerned with finding minimal dimension linear representations for six-dimensional real, indecomposable nilpotent Lie algebras. It is known that all such Lie algebras can be represented in $\mathfrak{gl}(6,\mathbb{R})$. After discussing the classification of the 24 such Lie algebras, it is shown that only one algebra can be represented in $\mathfrak{gl}(4,\mathbb{R})$. A Theorem is then presented that shows that 13 of the algebras can be represented in $\mathfrak{gl}(5,\mathbb{R})$. The special case of filiform Lie algebras is considered, of which there are five, and it is shown that each of them can be represented in $\mathfrak{gl}(6,\mathbb{R})$ and not $\mathfrak{gl}(5,\mathbb{R})$. Of the remaining five algebras, four of them can be represented minimally in $\mathfrak{gl}(5,\mathbb{R})$. That leaves one difficult case that is treated in detail in an Appendix.

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1 introduction

This paper forms part of a series whose goal is to find linear representations of minimal dimension for all the Lie algebras of low dimensions. In a previous work we have succeeded in finding linear representations for all indecomposable Lie algebras of dimension five and less [5] and very recently we have been able to carry out the same program for decomposable algebras of dimension five and less [7]. We have also extended our work to Lie algebras that admit a non-trivial Levi decomposition up to and including dimension eight [6]. Several other authors have investigated the problem of finding minimal dimensional representations of Lie algebras, see [2, 9] for example.

Of course it is an interesting and challenging mathematical problem to find such minimal dimension representations but there are also compelling practical reasons. Besides the value of having explicit representations of low-dimensional Lie algebras, they also add to the growing body of results that seek to provide alternatives to Ado's Theorem for the construction of representations, see [3]. Although Ado's theorem guarantees the existence of a matrix representation, it is of no practical utility in constructing them and certainly not helpful in finding representations of minimal dimension. Calculations involving symbolic programs such as Maple and Mathematica use up lots of memory when storing matrices; accordingly, calculations are likely to be faster if one can represent matrix Lie algebras using matrices of a small size. On the downside, it is true that choosing a smaller size representation may entail using more complicated entries in the representing matrices; one can see this phenomenon even more clearly if one constructs a matrix Lie group that gives rise to a matrix representation of the algebra and an attendant set of invariant vector fields or one-forms. One cannot have simultaneous simplicity in all aspects of a Lie algebra or Lie group representation and so various choices among such representations have to be made according to the kind of application one has in mind.

In this paper we are concerned with finding linear representations for real indecomposable nilpotent Lie algebras in dimension six that are of minimal dimension. It is convenient to define an invariant $\mu(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$ to be the dimension of a minimal dimensional representation of $\mathfrak{g}$. It is known from previous work [4] that for all the indecomposable six-dimensional nilpotent Lie algebras $\mu \leq 6$. We adopt the numbering given in [10] and [12]. Refer also to Section 2 below for a refinement in this classification.

In this article we give a Lie *group* corresponding to each of the 24 six-dimensional nilpotent Lie algebras that is a subgroup of $GL(4,\mathbb{R})$, $GL(5,\mathbb{R})$ or $GL(6,\mathbb{R})$, respectively. The representation for the Lie algebra is then easily obtained by differentiating and evaluating at the identity. In order to ensure that we have a bona fide group representation we provide also a list of right-invariant vector fields in each case.

An outline of this paper is as follows. In Section 2 we give a brief description of the indecomposable six-dimensional nilpotent Lie algebras. Since we know that for the indecomposable six-dimensional nilpotent Lie algebras $\mu \leq 6$, in

Section 3 we take up the issue of characterizing those algebras for which $\mu = 4$. In fact there is just one, $6.14\, a = -1$, which is isomorphic to the space of 4×4 strictly upper triangular matrices. Thus, for the remaining 23 algebras, we know that $\mu = 5$ or 6. In Section 4 we present a result about representations that, where applicable, implies that $\mu = 5$. In fact it is applicable to 13 of the 24 algebras. In Section 5 we discuss an important class of nilpotent Lie algebras known as filiform Lie algebras. We show quite generally, that for an n-dimensional filiform, $\mu \geq n$. Among the indecomposable six-dimensional nilpotent Lie algebras, just five are filiform and it happens that we know representations in $\mathfrak{gl}(6, \mathbb{R})$ for each of these five algebras and hence the value of μ is settled in those cases. Remarkably, that leaves just one "difficult" case, algebra 6.10, to discuss ; what makes it difficult, is that Theorem 4.1 is not applicable and we cannot find a representation in $\mathfrak{gl}(5, \mathbb{R})$ although we have one in $\mathfrak{gl}(6, \mathbb{R})$; therefore to claim that $\mu = 6$ we have to *prove* that there is no representation in $\mathfrak{gl}(5, \mathbb{R})$. It is much to be emphasized that, although the conditions that arise in finding representations are polynomial, they are far too complicated to solve in a simple algorithmic manner; rather, the conditions have to be solved interactively according to their significance for the algebra concerned. One can see the kind of complications that are involved in the algebra 6.10 in the Appendix.

For the convenience of the reader we have listed also minimal dimension representations for the indecomposable nilpotent Lie algebras of dimension three, four and five. The representations are listed in Section 6. Most of the calculations were done with MAPLE.

2 Classifying nilpotent Lie algebras in dimension six

The six-dimensional nilpotent Lie algebras have been studied and classified independently by several different groups of authors. We summarize and synthesize these various accounts. In 1958 Morozov obtained a classification of the six-dimensional nilpotent Lie algebras over $\mathbb{R}$ [10]. Morozov discerned 22 classes of algebra of which four depended on a single parameter that he denoted by a in each case. In [10] these four cases were numbered $6.5, 6.10, 6.14$ and 6.18, respectively. This classification is reproduced verbatim in [12] and is used below in Section 5. Later on Winternitz et al. refined Morozov's classification in, as far as we are aware, unpublished notes. In the later version they realized that in all four cases where the parameter a entered, the parameter could be removed and reduced to ± 1. Furthermore it turned out that the case $6.5(a = 1)$ was decomposable and that $6.10(a = 1)$ was equivalent to case 6.8. In Section 6 we give this isomorphism. Thus altogether there are 24 indecomposable nilpotent Lie algebras in dimension six, none of which contains parameters, up to isomorphism. Precisely the same number is found in [1], [11] and [14], albeit using different bases.

3 Algebras for which $\mu = 4$

3.1 The centralizer of a central element

Suppose that we have a six-dimensional nilpotent indecomposable Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(4,\mathbb{R})$. Since $\mathfrak{g}$ is nilpotent it must have a non-trivial center that we may assume contains a strictly upper triangular matrix and we denote it by C. The only possibilities are that C has rank three, two or one. It is impossible for C to have rank three, for in that case the centralizer of C is four-dimensional abelian.

Next suppose that C has rank two; then we may assume that $C = \begin{bmatrix} 0 & I_2 \\ 0 & 0 \end{bmatrix}$ where I_2 denotes the 2×2 identity matrix and each of the zeroes are also 2×2. The centralizer of C is of the form $\begin{bmatrix} A & B \\ 0 & A \end{bmatrix}$ where A, B are arbitrary 2×2 matrices and comprises an eight-dimensional subalgebra. Given two such matrices $\begin{bmatrix} A_1 & B_1 \\ 0 & A_1 \end{bmatrix}$ and $\begin{bmatrix} A_2 & B_2 \\ 0 & A_2 \end{bmatrix}$ their commutator is $\begin{bmatrix} [A_1,A_2] & [A_1,B_2]+[B_1,A_2] \\ 0 & [A_1,A_2] \end{bmatrix}$ and hence the central element C would not be in the derived algebra. In conclusion, C must have rank one and as such we may assume that $C = \begin{bmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$.

3.2 Trace argument

Next, we shall make the following remark about taking traces. In a matrix representation of a Lie algebra we can always modify matrices not contained in the derived algebra by adding multiples of the identity and in particular, we may always assume that such a matrix has trace zero. We shall refer to this remark as the "trace argument" and we shall make use of it in the Appendix when investigating algebra 6.10.

3.3 Reducing the dimension of the subalgebra

Coming back to the rank one matrix C, its centralizer is given by $\begin{bmatrix} a&b&c&d \\ 0&e&f&g \\ 0&h&i&j \\ 0&0&0&a \end{bmatrix}$. The 2×2 central block must be a nilpotent subalgebra of $\mathfrak{gl}(2,\mathbb{R})$ and so it may be assumed to be of the form $\begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} e & 0 \\ 0 & i \end{bmatrix}$ or $\begin{bmatrix} e & f \\ -f & e \end{bmatrix}$ and the nilpotent algebra that we seek must be a subalgebra of one of these seven or eight dimensional subalgebras. Furthermore we may assume in these last two cases that $a = 0$ by taking traces. Now we have three solvable seven-dimensional algebras; the first of them has a six-dimensional nilradical that is isomorphic to $6.14\, a = -1$

and the others have five-dimensional nilradical that is isomorphic to the five-dimensional Heisenberg algebra. We consider each of them in turn.

In the first case we have the space of matrices $\begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ 0 & 0 & s_5 & s_6 \\ 0 & 0 & 0 & s_7 \\ 0 & 0 & 0 & s_1 \end{bmatrix}$ that we take in the order $s_1, s_2, s_3, s_4, s_5, s_6, s_7$ and obtain the following Lie brackets

$$\begin{aligned} &[e_1, e_2] = e_2, [e_1, e_3] = e_3, [e_1, e_6] = -e_6, [e_1, e_7] = -e_7, \\ &[e_2, e_5] = e_3, [e_2, e_6] = e_4, [e_3, e_7] = e_4, [e_5, e_7] = e_6. \end{aligned} \tag{3.1}$$

We enquire whether it is possible to have a six-dimensional nilpotent subalgebra that involves e_1. First of all we dualize the structure equations so as to obtain

$$\begin{aligned} &d\theta^1 = 0, d\theta^2 = -\theta^1\theta^2, d\theta^3 = -\theta^1\theta^3 - \theta^2\theta^5, \\ &d\theta^4 = -\theta^2\theta^6 - \theta^3\theta^7, d\theta^5 = 0, d\theta^6 = \theta^1\theta^6 - \theta^5\theta^7, \\ &d\theta^7 = \theta^1\theta^7. \end{aligned} \tag{3.2}$$

where the $\theta^i, i = 1..7$ comprise the basis dual to $e_i, i = 1..7$. We suppose that

$$\theta^1 = b\theta^2 + c\theta^3 + a\theta^4 + e\theta^5 + f\theta^6 + g\theta^7 \tag{3.3}$$

is a dependence relation among the one-forms which expresses the existence of a six-dimensional subalgebra that is *not* obtained by setting just $\theta_1 = 0$. Now we substitute 3.3 into 3.2 and $d\theta^1 = 0$ gives

$$\begin{aligned} &ab\theta^2\theta^4 + (be-1)\theta^2\theta^5 + (b(f+1)-a)\theta^2\theta^6 + 2bg\theta^2\theta^7 + ac\theta^3\theta^4 + ce\theta^3\theta^5 \\ &+c(f+1)\theta^3\theta^6 + (2cg-1)\theta^3\theta^7 + a\theta^4\theta^6 + ag\theta^4\theta^7 + e\theta^5\theta^6 + eg\theta^5\theta^7 \\ &+(f-1)g\theta^6\theta^7 = 0. \end{aligned} \tag{3.4}$$

Condition 3.4 is impossible to satisfy; for example from the $\theta^2\theta^5$ term we have $e = \frac{1}{b}$ and from the $\theta^3\theta^7$ term we have $g = \frac{1}{2c}$ in which case the $\theta^5\theta^7$ term is not zero. Thus the only six-dimensional subalgebra is the obvious one obtained by putting $e_1 = 0$ and a fortiori no such nilpotent subalgebra.

Now we consider the second case where the matrix Lie algebra is given by $\begin{bmatrix} 0 & s_3 & s_4 & s_5 \\ 0 & s_1 & 0 & s_6 \\ 0 & 0 & s_2 & s_7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with the following Lie brackets

$$[e_1, e_3] = -e_3, [e_1, e_6] = e_6, [e_2, e_4] = -e_4, [e_2, e_7] = e_7, [e_3, e_6] = e_5, [e_4, e_7] = e_5. \tag{3.5}$$

The dual structure equations are given by

$$d\theta^1 = 0, d\theta^2 = -\theta^1\theta^2, d\theta^3 = -\theta^1\theta^3 - \theta^2\theta^5, d\theta^4 = -\theta^2\theta^6 - \theta^3\theta^7, d\theta^5 = 0,$$
$$d\theta^6 = \theta^1\theta^6 - \theta^5\theta^7, d\theta^7 = \theta^1\theta^7. \tag{3.6}$$

Now we assume that there is a dependence relation of the form

$$a\theta^1 + b\theta^2 + c\theta^3 + h\theta^4 + e\theta^5 + f\theta^6 + g\theta^7 = 0. \tag{3.7}$$

Then taking the exterior derivative of 3.7 and substituting the exterior derivatives from 3.6 gives

$$-bc\theta^2\theta^3 + ah\theta^2\theta^4 + bf\theta^2\theta^6 - ag\theta^2\theta^7 + ch\theta^3\theta^4 + ce\theta^3\theta^5 + (2cf - ae)\theta^3\theta^6$$
$$+cg\theta^3\theta^7 + fh\theta^4\theta^6 - ae\theta^4\theta^7 + ef\theta^5\theta^6 - fg\theta^6\theta^7 = 0. \tag{3.8}$$

Condition 3.8 may be solved in four different ways: $c = e = f = g = h = 0$, $b = e = f = g = h = 0$, $b = c = e = g = h = 0$, $a = c = f = 0$ and the corresponding one-form condition is given by $a\theta^1 + b\theta^3 = 0$, $a\theta^1 + c\theta^3 = 0$, $a\theta^1 + f\theta^6 = 0$, $b\theta^2 + e\theta^5 + g\theta^6 + h\theta^7 = 0$, respectively. However these conditions are imposed, there always remains a condition of the form $d\theta^i = c\theta^i\theta^j + ...$ where $c \neq 0$, which implies that the six-dimensional subalgebra that we are seeking is not nilpotent. The argument for the third matrix subalgebra is similar and indeed the second and third cases are complex equivalent. What we have just argued is that up to isomorphism, the only six-dimensional nilpotent subalgebra of $\mathfrak{gl}(4,\mathbb{R})$ is the space of strictly upper triangular matrices.

Theorem 3.1 *The only six-dimensional nilpotent subalgebra of $\mathfrak{gl}(4,\mathbb{R})$ is isomorphic to the subspace of strictly upper triangular matrices. As an abstract Lie algebra it is $6.14\, a = -1$.*

Corollary 3.1 *The most general representation of $6.14\, a = -1$ in $\mathfrak{gl}(4,\mathbb{R})$ is given by, up to change of basis,*

$$\begin{bmatrix} \lambda a+d\mu+f\nu & a & b & c \\ 0 & \lambda a+d\mu+f\nu & d & e \\ 0 & 0 & \lambda a+d\mu+f\nu & f \\ 0 & 0 & 0 & \lambda a+d\mu+f\nu \end{bmatrix}$$

where $\lambda, \mu, \nu \in \mathbb{R}$ are arbitrary.

If we order the basis according to the parameters a, b, c, d, e, f we find the following non-zero brackets for $6.14\, a = -1$

$$[e_1, e_4] = e_2, [e_1, e_5] = e_3, [e_2, e_6] = e_3, [e_4, e_6] = e_5. \tag{3.9}$$

If we make the change of basis

$$\overline{e_1} = e_1 - e_6, \overline{e_2} = e_1 + e_6, \overline{e_3} = e_4, \overline{e_4} = e_2 + e_5, \overline{e_5} = e_2 - e_5, \overline{e_6} = 2e_3 \tag{3.10}$$

we obtain

$$[\overline{e_1}, \overline{e_3}] = \overline{e_4}, [\overline{e_1}, \overline{e_4}] = \overline{e_6}, [\overline{e_2}, \overline{e_3}] = \overline{e_5}, [\overline{e_2}, \overline{e_5}] = -\overline{e_6} \tag{3.11}$$

which is the form given in [12].

4 A representation theoretic result

In this Section we present a result which has been given before in the context of solvable Lie algebras.

Theorem 4.1 **([13])** *Suppose that the n-dimensional Lie algebra $\mathfrak{g}$ has a basis $\{e_1, e_2, ..., e_n\}$ and only the following non-zero brackets: $[e_a, e_i] = C^j_{ai} e_j$, where $1 \leq i, j \leq r$ and $r+1 \leq a, b, c \leq n$. Then $\mathfrak{g}$ has a faithful representation as a subalgebra of $\mathfrak{gl}(r+1, \mathbb{R})$.*

In the proof of the Theorem 4.1 each e_a is represented by the matrix C^j_{ai} (for fixed a) augmented by $(r+1)$th rows and columns of zeroes. For the ideal spanned by the e_i's, map e_i to an $(r+1) \times (r+1)$ matrix whose only non-zero entry is 1 in the $(r+1, i)$th position. Said differently, the Theorem is applicable to an algebra that has an abelian ideal and complementary abelian subalgebra. In the context of the six-dimensional nilpotent indecomposable Lie subalgebras, Theorem 4.1 is applicable to algebras 6.1 and 6.2 where $r = 5$ and $6.3, 6.4, 6.5, 6.7, 6.12, 6.16$ and 6.17 where $r = 4$. The representations appearing in Section 6 are not necessarily constructed by using Theorem 4.1 although they are all minimal.

5 The standard filiform algebra

In this Section we shall consider a very special class of nilpotent Lie algebras

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_4, \quad [e_1, e_4] = e_5, ..., \quad [e_1, e_{n-1}] = e_n. \tag{5.1}$$

There is one such algebra in each dimension n beginning with $n = 3$. In [12] these algebras appear as $3.1, 4.1, 5.2, 6.2$ for $n = 3, 4, 5, 6$. Of course the main concern in this paper is the case $n = 6$. More generally a nilpotent Lie algebra is said to be *filiform* if the lower central series is of maximal length [16], which means that its dimensions are $n, n-2, n-3, n-4, ...1, 0$. For $n = 6$ the filiforms are $6.2, 6.19, 6.20, 6.21, 6.22$. We shall refer to conditions 5.1 as the *standard filiform Lie algebra*. All filiforms are infinitesimal deformations of the standard one in the sense that in a suitable basis they contain all the brackets in 5.1 as well as possibly others: see [16] and [8].

Theorem 5.1 *For the standard filiform Lie algebra of dimension n we have $\mu \geq n$.*

Proof 5.1 *First of all we begin by treating 5.1 as a complex Lie algebra. We shall argue that if we cannot find a representation with $\mu < n$ for the complex Lie algebra, then there can be no representation with $\mu < n$ for the real Lie algebra.*

Suppose then that we have a representation in $\mathfrak{gl}(r, \mathbb{R})$ for the complex Lie algebra. Since we are working over $\mathbb{C}$, it may be assumed in view of Lie's Theorem that the matrices in the representation are simultaneously upper triangular. Indeed, it is sufficient to suppose that the matrices representing e_1 and e_2, that we shall denote by E_1 and E_2, respectively, and likewise for all the remaining basis vectors, are simultaneously upper triangular.

Next, observe that $[e_1, e_n] = 0$ and hence $[E_1, E_n] = 0$. Furthermore we have that $E_4 = [E_1, [E_1, E_2]]$ and more generally $E_k = ad(E_1)^{k-2}E_2$ for $k \geq 3$. It follows that E_k is strictly upper triangular for $3 \leq k \leq n$.

Now we shall have to be more specific, so we shall label the entries of E_1 and E_2 as a_{ij} and b_{ij}, respectively. Then we find that

$$E_3 = \begin{bmatrix} 0 & b_{12}(a_{11}-a_{22})-a_{12}(b_{11}-b_{22}) & * & * & * & \dots & * \\ 0 & 0 & b_{23}(a_{22}-a_{33})-a_{23}(b_{22}-b_{33}) & * & * & \dots & * \\ 0 & 0 & 0 & E_{3,3,4} & * & \dots & * \\ 0 & 0 & 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & * & \ddots & E_{3,n-1,n} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{5.2}$$

where

$$E_{3,3,4} = b_{34}(a_{33} - a_{44}) - a_{34}(b_{33} - b_{44}),$$

and

$$E_{3,n-1,n} = b_{n,n-1}(a_{n-1,n-1} - a_{n,n}) - a_{n-1,n}(b_{n-1,n-1} - b_{n,n}).$$

Moreover we find that

$$E_4 = \begin{bmatrix} 0 & (a_{11}-a_{22})(b_{12}(a_{11}-a_{22})-a_{12}(b_{11}-b_{22})) & * & * & \dots \\ 0 & 0 & (a_{22}-a_{33})(b_{23}(a_{22}-a_{33})-a_{23}(b_{22}-b_{33})) & * & \dots \\ 0 & 0 & 0 & E_{4,3,4} & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.3}$$

where $E_{4,3,4} = (a_{33} - a_{44})(b_{34}(a_{33} - a_{44}) - a_{34}(b_{33} - b_{44}))$ and

$$[E_2, E_3] = \begin{bmatrix} 0 & (b_{11}-b_{22})(b_{12}(a_{11}-a_{22})-a_{12}(b_{11}-b_{22})) & * & * & \dots \\ 0 & 0 & E_{2,3,2,3} & * & \dots \\ 0 & 0 & 0 & E_{2,3,3,4} & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.4}$$

where

$$E_{2,3,2,3} = (b_{22} - b_{33})(b_{23}(a_{22} - a_{33}) - a_{23}(b_{22} - b_{33})),$$

$$E_{2,3,3,4} = (b_{33} - b_{44})(b_{34}(a_{33} - a_{44}) - a_{34}(b_{33} - b_{44})),$$

and

$$E_n = \begin{bmatrix} 0 & (a_{11}-a_{22})^{n-3}(b_{12}(a_{11}-a_{22})-a_{12}(b_{11}-b_{22})) & * & * & \cdots \\ 0 & 0 & E_{n,2,3} & * & \cdots \\ 0 & 0 & 0 & E_{n,3,4} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.5}$$

where

$$E_{n,2,3} = (a_{22} - a_{33})^{n-2}(b_{23}(a_{22} - a_{33}) - a_{23}(b_{22} - b_{33}))$$

$$E_{n,3,4} = (a_{33} - a_{44})^{n-3}(b_{34}(a_{33} - a_{44}) - a_{34}(b_{33} - b_{44}))$$

and

$$[E_1, E_n] = \begin{bmatrix} 0 & (a_{11}-a_{22})^{n-2}(b_{12}(a_{11}-a_{22})-a_{12}(b_{11}-b_{22})) & * & * & \cdots \\ 0 & 0 & E_{1,n,2,3} & * & \cdots \\ 0 & 0 & 0 & E_{1,n,3,4} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.6}$$

where

$$E_{1,n,2,3} = (a_{22} - a_{33})^{n-2}(b_{23}(a_{22} - a_{33}) - a_{23}(b_{22} - b_{33}))$$

$$E_{1,n,3,4} = (a_{33} - a_{44})^{n-2}(b_{34}(a_{33} - a_{44}) - a_{34}(b_{33} - b_{44}))$$

Now since $[E_1, E_n] = 0$ *it follows that for* $1 \le k \le n-1$

$$(a_{k,k} - a_{k+1,k+1})(b_{k,k+1}(a_{k,k} - a_{k+1,k+1}) - a_{k,k+1}(b_{kk} - b_{k+1,k+1})) = 0 \tag{5.7}$$

and from $[E_2, E_3] = 0$ *we find that for* $1 \le k \le n-1$

$$(b_{k,k} - b_{k+1,k+1})(b_{k,k+1}(a_{k,k} - a_{k+1,k+1}) - a_{k,k+1}(b_{kk} - b_{k+1,k+1})) = 0. \tag{5.8}$$

From 5.7 and 5.8 we deduce that

$$b_{k,k+1}(a_{k,k} - a_{k+1,k+1}) - a_{k,k+1}(b_{kk} - b_{k+1,k+1}) = 0 \tag{5.9}$$

for $1 \le k \le n-1$*. Hence in* E_3 *the second upper diagonal, that is, the one above the main diagonal is zero. Now we calculate again with the new form of* E_3*. Then we find*

$$E_4 = \begin{bmatrix} 0 & 0 & (a_{11}-a_{33})E_{3,1,2} & * & * & \cdots \\ 0 & 0 & 0 & (a_{22}-a_{44})E_{3,2,3} & * & \cdots \\ 0 & 0 & 0 & 0 & (a_{33}-a_{55})E_{3,3,4} & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & 0 & 0 & \end{bmatrix} \tag{5.10}$$

and

$$E_n = \begin{bmatrix} 0 & 0 & (a_{11}-a_{33})^{n-3}E_{3,1,2} & * & * & \cdots \\ 0 & 0 & 0 & (a_{22}-a_{44})^{n-3}E_{3,2,3} & * & \cdots \\ 0 & 0 & 0 & 0 & (a_{33}-a_{55})^{n-3}E_{3,3,4} & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & 0 & 0 & \end{bmatrix} \tag{5.11}$$

and

$$[E_1, E_n] = \begin{bmatrix} 0 & 0 & (a_{11}-a_{33})^{n-2}E_{3,1,2} & * & * & \cdots \\ 0 & 0 & 0 & (a_{22}-a_{44})^{n-2}E_{3,2,3} & * & \cdots \\ 0 & 0 & 0 & 0 & (a_{33}-a_{55})^{n-2}E_{3,3,4} & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & 0 & 0 & \end{bmatrix}. \tag{5.12}$$

It follows that in E_4 the third upper diagonal is zero. Continuing in the same manner we deduce that the fourth upper diagonal of E_5 is zero and eventually that the first $n-1$ upper diagonals in E_n are zero. Since the representation is in $\mathfrak{gl}(r,\mathbb{R})$, if $r < n$ we would have $E_n = 0$. Hence there can be no no such representation.

According to Vergne [16] every filiform algebra is a deformation of 5.1 in the sense that in an appropriate basis it contains all the brackets of 5.1 together with extra ones coming from a certain two cycle. If we think about the conditions that were used in Theorem 5.1 we see that in addition to 5.1 all that was required was that $[e_2, e_3] = 0$. Let us examine whether this condition itself can be removed. We note first of all that if $[e_2, e_3]$ is not zero it cannot contain e_1, e_2 or e_3 otherwise the Lie algebra will not be nilpotent in view of Engel's Theorem. Condition 5.7 is valid just as before. Looking at $[E_2, E_3]$ which is a linear combination of $E_4, E_5, ..., E_n$ condition 5.8 no longer holds directly; however, we now have

$$(b_{k,k}-b_{k+1,k+1}-P(a_{k,k}-a_{k+1,k+1}))(b_{k,k+1}(a_{k,k}-a_{k+1,k+1})-a_{k,k+1}(b_{kk}-b_{k+1,k+1}) = 0 \tag{5.13}$$

where P is some polynomial of degree at most $n-3$. Now in view of condition 5.7 condition 5.13 reduces to condition 5.8 and hence the second upper diagonal in E_3 is zero as before. The remainder of the argument proceeds as before. Hence

Corollary 5.1 *For any filiform Lie algebra of dimension n we have $\mu \geq n$.*

Extending the ideas in [16], Burde reached similar conclusions for filiform algebras [2]. Finally, we remark that even if an algebra is not itself filiform, it may possess some filiform characteristics; for example, it may contain a subalgebra that is isomorphic to a filiform subalgebra of lower dimension. In that case one may be able to apply the "filiform argument" more generally. This line of reasoning is followed in the Appendix for algebra 6.10.

6 Minimal Representations

Below, corresponding to each Lie algebra, we give a Lie *group* whose Lie algebra is isomorphic to the given six-dimensional nilpotent Lie algebra. We have found it more convenient to work with the group, which can be obtained by exponentiating the Lie algebra representation. As such we only have to give a single 6×6 or 5×5 matrix rather than six such matrices. The typical element of the group is denoted by S. The representation for the algebra is then easily obtained by differentiating and evaluating at the identity. In the list below the vector fields do give a faithful representation of the algebra.

<u>dimension 3</u>

<u>3.1</u>: $[e_2, e_3] = e_1$:

$$S = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields $D_z, D_y, D_x + yD_z$.

<u>dimension 4</u>

<u>4.1</u>: $[e_2, e_4] = e_1, [e_3, e_4] = e_2$:

$$S = \begin{bmatrix} 1 & w & \frac{w^2}{2} & x \\ 0 & 1 & w & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_x, D_y, D_z, D_w + yD_x + zD_y$

<u>dimension 5</u>

<u>5.1</u>: $[e_3, e_5] = e_1, [e_4, e_5] = e_2$:

$$S = \begin{bmatrix} 1 & 0 & x & z \\ 0 & 1 & w & y \\ 0 & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_y, D_z, D_q, D_x + qD_z, D_w + qD_y$.

5.2: $[e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_3$: (filiform)

$$S = \begin{bmatrix} 1 & w & \frac{w^2}{2} & \frac{w^3}{6} & q \\ 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_q, D_x, D_y, D_z, D_w + xD_q + yD_x + zD_y$.

5.3: $[e_3, e_4] = e_2$, $[e_3, e_5] = e_1$, $[e_4, e_5] = e_3$:

$$S = \begin{bmatrix} 1 & 0 & -\frac{z}{2} & \frac{y-zw}{2} & q \\ 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_x, -D_q, D_y + \frac{z}{2}D_q, D_z - \frac{y}{2}D_q, D_w + yD_x + zD_y$.

5.4: $[e_2, e_4] = e_1, [e_3, e_5] = e_1$:

$$S = \begin{bmatrix} 1 & x & y & q \\ 0 & 1 & 0 & z \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_q, D_z, D_w, D_x + zD_q, D_y + wD_q$.

5.5: $[e_3, e_4] = e_1, [e_2, e_5] = e_1$ $[e_3, e_5] = e_2$:

$$S = \begin{bmatrix} 1 & q & w + \frac{q^2}{2} & x \\ 0 & 1 & q & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_x, D_y, D_z, D_w + zD_x, D_q + yD_x + zD_y$.

5.6: $[e_3, e_4] = e_1, [e_2, e_5] = e_1$, $[e_3, e_5] = e_2, [e_4, e_5] = e_3$: (filiform)

$$S = \begin{bmatrix} 1 & 2w & w^2 - z & y - zw + \frac{w^3}{3} & q \\ 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $2D_q, -D_x, D_y+zD_q, -D_z+yD_q, -(D_w+2xD_q+yD_x+zD_y)$

<u>dimension 6</u>

<u>6.1</u>: $[e_1,e_2]=e_3, [e_1,e_3]=e_4, [e_1,e_5]=e_6$:

$$S=\begin{bmatrix} 1 & w & \frac{w^2}{2} & x & p \\ 0 & 1 & w & y & q \\ 0 & 0 & 1 & z & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right invariant vector fields: $-qD_p-yD_x-zD_y-D_w, D_z, D_y, D_x, D_q, D_p$.

<u>6.2</u>: $[e_1,e_2]=e_3, [e_1,e_3]=e_4, [e_1,e_4]=e_5, [e_1,e_5]=e_6$:

$$S=\begin{bmatrix} 1 & w & \frac{w^2}{2} & \frac{w^3}{6} & \frac{w^4}{24} & p \\ 0 & 1 & w & \frac{w^2}{2} & \frac{w^3}{6} & q \\ 0 & 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right invariant vector fields: $-qD_p-xD_q-yD_x-zD_y-D_w, D_z, D_y, D_x, D_q, D_p$.

<u>6.3</u>: $[e_1,e_2]=e_6, [e_1,e_3]=e_4, [e_2,e_3]=e_5$:

$$S=\begin{bmatrix} 1 & 0 & z & \frac{z^2}{2} & p \\ 0 & 1 & q & x & y \\ 0 & 0 & 1 & z & w \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Right-invariant vector fields: $D_w, D_z+wD_p, D_q+wD_y+zD_x, D_y, D_x, D_p$.

<u>6.4</u>: $[e_1,e_2]=e_5, [e_1,e_3]=e_6, [e_2,e_4]=e_6$:

$$S=\begin{bmatrix} 1 & x & y & q & p \\ 0 & 1 & 0 & z & y \\ 0 & 0 & 1 & w & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $-D_x - yD_p - zD_q, D_y + wD_q, D_z, -D_w, D_p, D_q$.

<u>6.5</u>: $[e_1, e_3] = e_5, [e_1, e_4] = e_6, [e_2, e_3] = -e_6, [e_2, e_4] = e_5$:

$$S = \begin{bmatrix} 1 & x & y & q & p \\ 0 & 1 & 0 & w & z \\ 0 & 0 & 1 & -z & w \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_w, D_z, D_y - zD_q + wD_p, D_x + zD_p + wD_q, D_p, D_q$.

<u>6.6</u>: $[e_1, e_2] = e_6, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_5$:

$$S = \begin{bmatrix} 1 & q & w & x & p \\ 0 & 1 & q & y & \frac{q^2}{2} \\ 0 & 0 & 1 & z & q \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Right-invariant vector fields: $D_q + zD_y + yD_x + qD_w + \frac{q^2}{2}D_p, -(D_w + zD_x + qD_p), D_z, -D_y, D_x, -D_p$.

<u>6.7</u>: $[e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_6$:

$$S = \begin{bmatrix} 1 & 0 & 0 & p & q \\ 0 & 1 & z & \frac{z^2}{2} & x \\ 0 & 0 & 1 & z & y \\ 0 & 0 & 0 & 1 & w \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $-(D_w + zD_y + yD_x), -(D_q + zD_p), D_z, D_y, D_x, D_p$.

<u>6.8</u>: $[e_1, e_2] = e_3 + e_5, [e_1, e_3] = e_4, [e_2, e_5] = e_6$:

$$S = \begin{bmatrix} 1 & 0 & -\frac{1}{2}z & p + \frac{1}{2}y - \frac{1}{2}zw & q \\ 0 & 1 & w & \frac{1}{2}w^2 & x \\ 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_z - \frac{y}{2}D_q, -(zD_y + yD_x + D_w), -D_p - zD_q, -D_q, \frac{z}{2}D_q - D_y + D_p, -D_x$.

6.9: $[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_5] = e_6, [e_2, e_3] = e_6$:

$$S = \begin{bmatrix} 1 & 0 & w - y & z & p \\ 0 & 1 & 2\,w - y & wy - x + 2\,z & q \\ 0 & 0 & 1 & w & x \\ 0 & 0 & 0 & 1 & y \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_w + yD_x + wD_z + xD_p + 2xD_q, -D_x + yD_q, D_y - wD_z - xD_p - xD_q, D_p, -D_z - yD_p - 2yD_q, D_q$.

6.10: $[e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_1, e_4] = e_6, [e_2, e_3] = ae_6, [e_2, e_4] = e_5$: As in case **6.5** a can be reduced to $a = \pm 1$. Case $a = 1$ algebra is equivalent to algebra **6.8**. The change of basis that changes **6.8** to **6.10**$(a = 1)$ is given by: $\overline{e_1} = e_1 + e_2$, $\overline{e_2} = e_1 - e_2$, $\overline{e_3} = -2(e_3 + e_5)$, $\overline{e_4} = 2(e_5 - e_3)$, $\overline{e_5} = -2(e_4 + e_6), \overline{e_6} = 2(e_6 - e_4)$.

$a = -1$

$$S = \begin{bmatrix} 1 & 0 & -z & w & y - zw & q \\ 0 & 1 & w & z & \frac{w^2}{2} & x \\ 0 & 0 & 1 & 0 & w & 2y \\ 0 & 0 & 0 & 1 & 0 & p \\ 0 & 0 & 0 & 0 & 1 & 2z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $\sqrt{2}(pD_q + 2yD_x + zD_y + D_w, -2yD_q + pD_x + D_z, \sqrt{2}(-2zD_q - D_y), -4D_p, 4D_x, 4\sqrt{2}D_q$.

6.11: $[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_6$:

$$S = \begin{bmatrix} 1 & y & x - yz + \frac{y^2}{2} & w & p \\ 0 & 1 & y - z & yz - \frac{z^2}{2} - x & q \\ 0 & 0 & 1 & z & x \\ 0 & 0 & 0 & 1 & y \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $\frac{1}{2.3^{\frac{1}{3}}}(2D_y + (2yz - z^2 - 2x)D_w + 2qD_p + 2xD_q)$, $3^{\frac{2}{3}}(D_z + yD_x - xD_q)$, $3^{\frac{1}{3}}(D_x + zD_w + xD_p - yD_q)$, $D_w + yD_p - 2D_q$, $3^{\frac{2}{3}}D_p$, $3(D_w + yD_p +$

D_q).

<u>6.12</u>: $[e_1, e_3] = e_4, \quad [e_1, e_4] = e_6, \quad [e_2, e_5] = e_6$:

$$S = \begin{bmatrix} 1 & w & \frac{w^2}{2} & q & x \\ 0 & 1 & w & 0 & y \\ 0 & 0 & 1 & 0 & z \\ 0 & 0 & 0 & 1 & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $-(D_w + yD_x + zD_y), D_p, D_z, D_y, D_q + pD_x, D_x$.

<u>6.13</u>: $[e_1, e_2] = e_5, \quad [e_1, e_3] = e_4, \quad [e_1, e_4] = e_6, \quad [e_2, e_5] = e_6$:

$$S = \begin{bmatrix} 1 & w & \frac{w^2}{2} & q & x \\ 0 & 1 & w & p & y \\ 0 & 0 & 1 & 0 & z \\ 0 & 0 & 0 & 1 & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields : $(D_w+pD_q+yD_x+zD_y), D_p+pD_y, -D_z, D_y, -D_q-pD_x, -D_x$.

<u>6.14</u>: $[e_1, e_3] = e_4, [e_1, e_4] = e_6, [e_2, e_3] = e_5, [e_2, e_5] = ae_6$: a can be reduced to $a = \pm 1$.

1. $a = 1$

$$S = \begin{bmatrix} 1 & p & q & \frac{1}{2}(p^2+q^2) & x \\ 0 & 1 & 0 & p & y \\ 0 & 0 & 1 & q & z \\ 0 & 0 & 0 & 1 & w \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

   Right-invariant vector fields: $D_p+yD_x+wD_y, D_q+zD_x+wD_z, D_w, -D_y, -D_z, D_x$.

2. $a = -1$

$$S = \begin{bmatrix} 1 & x & y & z \\ 0 & 1 & w & p \\ 0 & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   Right-invariant vector fields: $\frac{1}{2}(D_q + D_x + wD_y + pD_z), \frac{1}{2}(D_x - D_q + wD_y + pD_z), -2D_w - 2qD_p, D_y - D_p + qD_z, D_p + D_y + qD_z, D_z$.

<u>6.15</u>: $[e_1, e_2] = e_3 + e_5, [e_1, e_3] = e_4, [e_1, e_4] = e_6, [e_2, e_5] = e_6$:

$$S = \begin{bmatrix} 1 & 2w & w^2 - z & p + y - zw + \frac{w^3}{3} & q \\ 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields are: $D_w + zD_y + yD_x + 2xD_q, D_z - yD_q, -D_y + zD_q + 2D_p, D_x, -2D_p - 2zD_q, -2D_q$.

<u>6.16</u>: $[e_1, e_3] = e_4, \quad [e_1, e_4] = e_5, \quad [e_1, e_5] = e_6, \quad [e_2, e_3] = e_5, \quad [e_2, e_4] = e_6$:

$$S = \begin{bmatrix} 1 & w & p + \frac{w^2}{2} & pw + \frac{w^3}{6} & q \\ 0 & 1 & w & p + \frac{w^2}{2} & x \\ 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $-(D_w + xD_q + yD_x + zD_y), -(D_p + yD_q + zD_x), D_z, D_y, D_x, D_q$.

<u>6.17</u>: $[e_1, e_2] = e_3, \quad [e_1, e_3] = e_4, \quad [e_1, e_4] = e_6, \quad [e_2, e_5] = e_6$:

$$S = \begin{bmatrix} 1 & w & \frac{w^2}{2} & p + \frac{w^3}{6} & q \\ 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $-(D_w + xD_q + yD_x + zD_y), D_z, D_y, D_x, D_p + zD_q, D_q$.

<u>6.18</u>: $[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_6, [e_2, e_3] = e_5, [e_2, e_5] = ae_6$. As in **6.5**, **6.10** and **6.14** a can be reduced to $a = \pm 1$.

1. $a = 1$

$$S = \begin{bmatrix} 1 & y + 3\,z & \frac{y^2}{2} + 5yz + \frac{15z^2}{2} - 2x & w & p \\ 0 & 1 & y + 5\,z & yz + \frac{5z^2}{2} - x & q \\ 0 & 0 & 1 & z & x \\ 0 & 0 & 0 & 1 & y \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_y + (\frac{5}{2}z^2 + yz - x)D_w + qD_p + xD_q, D_z + yD_x + (3yz + \frac{15}{2}z^2 - 3x)D_w + 3qD_p + 5xD_q, D_x - 2zD_w - 2xD_p - yD_q, D_q, D_p, D_w + yD_p$.

2. $a = -1$

$$S = \begin{bmatrix} 1 & p & x & y & w \\ 0 & 1 & q & pq - x & z \\ 0 & 0 & 1 & p & x \\ 0 & 0 & 0 & 1 & q \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Right-invariant vector fields: $\frac{1}{2}(-D_p + 2\,D_q - (q-2)\,D_x + (2\,p + x - pq)\,D_y + 2\,(x-q)\,D_z + (2\,x - z)\,D_w), \frac{1}{2}(D_p + 2\,D_q + (q+2)\,D_x + (pq + 2\,p - x)\,D_y + 2\,(x-q)\,D_z + (2\,x + z)\,D_w), D_x + (p-2)\,D_y - qD_z + (x - 2\,q)\,D_w, -D\_y - 2\,D_z - (q+2)\,D_w, D_y - 2\,D_z + (q-2)\,D_w, -2\,D_w$.

<u>6.19</u>: $[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_1, e_5] = e_6, [e_2, e_3] = e_6$:

$$S = \begin{bmatrix} 1 & w & \frac{w^2}{2} & z + \frac{w^3}{6} & \frac{w^4}{24} - y + zw & p \\ 0 & 1 & w & \frac{w^2}{2} & \frac{w^3}{6} & q \\ 0 & 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_w + qD_p + xD_q + yD_x + zD_y, \frac{1}{2}(D_z + yD_p), -\frac{1}{2}(D_y - zD_p), \frac{1}{2}D_x, -\frac{1}{2}D_q, \frac{1}{2}D_p$.

<u>6.20</u>: $[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_1, e_5] = e_6, [e_2, e_3] = e_5, [e_2, e_4] = e_6$:

$$S = \begin{bmatrix} 1 & w & z + w^2 & zw - 2y + \frac{w^3}{3} & 3x - 2yw + \frac{w^2 z}{2} + \frac{w^4}{12} & p \\ 0 & 1 & 2w & w^2 - z & y - zw + \frac{w^3}{3} & q \\ 0 & 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_w + qD_p + 2xD_q + yD_x + zD_y, D_z + xD_p - yD_q, -(D_y - 2yD_p + zD_q), D_x + 3zD_p, -2D_q, 2D_p$.

<u>6.21</u>: $[e_1, e_2] = e_3, [e_1, e_5] = e_6, [e_2, e_3] = e_4, [e_2, e_4] = e_5, [e_3, e_4] = e_6$:

$$S = \begin{bmatrix} 1 & z & -y + zw & x - yw + \frac{1}{2}zw^2 & -q + xw + \frac{zw^3}{6} - \frac{yw^2}{2} & p \\ 0 & 1 & w & \frac{w^2}{2} & \frac{w^3}{3} & q \\ 0 & 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_z + qD_p, D_w + xD_q + yD_x + zD_y, D_y - xD_p, -(D_x + yD_p), D_q - zD_p, -2D_p$.

<u>6.22</u>: $[e_1, e_2] = e_3, \quad [e_1, e_3] = e_5, \quad [e_1, e_5] = e_6, \quad [e_2, e_3] = e_4, \quad [e_2, e_4] = e_5, \quad [e_3, e_4] = e_6$:

$$S = \begin{bmatrix} 1 & z & wz-y & x-yw+\frac{1}{2}zw^2-\frac{1}{3}z^2 & xw-q-\frac{1}{2}yw^2+\frac{1}{3}zy+\frac{1}{6}zw^3-\frac{1}{3}z^2w & p \\ 0 & 1 & w & \frac{w^2}{2}-\frac{z}{3} & \frac{2}{3}y+\frac{1}{6}w^3-\frac{1}{3}zw & q \\ 0 & 0 & 1 & w & \frac{w^2}{2}+\frac{z}{3} & x \\ 0 & 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_z + \frac{1}{3}zD_x - \frac{1}{3}yD_q + qD_p, D_w + zD_y + yD_x + xD_q, D_y + \frac{2}{3}zD_q - xD_p, -D_x - yD_p, D_q - zD_p, -2D_p$.

7 Appendix: Algebra 6.10

The first remark is that the definition of Algebra 6.10 is

$$[e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_1, e_4] = e_6, [e_2, e_3] = -e_6, [e_2, e_4] = e_5. \tag{7.1}$$

Now make the change of basis

$$\overline{e_1} = e_1, \overline{e_2} = e_2, \overline{e_3} = e_3, \overline{e_4} = e_5, \overline{e_5} = -e_4, \overline{e_6} = -e_6. \tag{7.2}$$

Then the algebra becomes

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_6, [e_2, e_3] = e_6, [e_2, e_5] = -e_4. \tag{7.3}$$

Hence we can treat both 6.9 and 6.10 by writing the last bracket as $[e_2, e_5] + Ce_4 = 0$ where $C = 0, 1$. We shall use the form given above for algebra 6.10 with $C = 1$ in order to demonstrate that it has no representation in $\mathfrak{gl}(5, \mathbb{R})$.

Now we assume that there is a matrix representation in which E_1, E_2 and E_5 are upper triangular. Then by applying the filiform argument we find that E_4 and E_6 are of the form

$$E_4 = \begin{bmatrix} 0&0&0&a&c \\ 0&0&0&0&b \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix} E_6 = \begin{bmatrix} 0&0&0&\alpha&\gamma \\ 0&0&0&0&\beta \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}. \tag{7.4}$$

We shall analyze the possible normal forms for the subspace spanned by E_4 and E_6.

Lemma 7.1 *Assuming that the pencil of matrices spanned by E_4 and E_6 is two-dimensional, there is a basis for it in one of the following four normal forms:*

$$\begin{bmatrix} 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}; \begin{bmatrix} 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0&0&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}; \begin{bmatrix} 0&0&0&0&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}; \begin{bmatrix} 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}. \tag{7.5}$$

Proof 7.1 *Suppose first of all that $a^2 + \alpha^2 \neq 0$. Then we may reduce to $a = 1, \alpha = 0$. Next, assuming that $\beta \neq 0$ we can reduce to $b = 0$ and $\beta = 1$. Then we conjugate by the matrix $P = \begin{bmatrix} 1&\delta&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&-c \\ 0&0&0&0&1 \end{bmatrix}$ (on the left by P^{-1} and on the right by P) which reduces c and δ to 0.*

Now loop back and assume from the outset that $a = \alpha = 0$. Then we cannot have $b = \beta = 0$ or else the matrices would be proportional. Without loss of generality we may suppose that $b = 1$ and $\beta = 0$, thence to $\gamma = 0$ and $c = 1$.

The last case to take care of is where $a = 1, \alpha = 0$ and $\beta = 0$. Then we assume that $\gamma = 1$ and $c = 0$. Then, assuming that $b \neq 0$, we conjugate by the matrix $P = \begin{bmatrix} 1&0&0&0&0 \\ 0&b&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix}$ (on the left by P^{-1} and on the right by P) which reduces b to 1.

7.0.1 Case 1

Now we start again from one of these normal forms. For the first such case, the intersection of the centralizers of E_4 and E_6 is a subalgebra consisting of matrices of the form $\begin{bmatrix} a&b&c&d&e \\ 0&a&f&g&h \\ 0&0&i&j&k \\ 0&0&0&a&b \\ 0&0&0&0&a \end{bmatrix}$.

Accordingly we put

$$E_1 = \begin{bmatrix} a&b&c&d&e \\ 0&a&g&h&i \\ 0&0&j&k&m \\ 0&0&0&a&b \\ 0&0&0&0&a \end{bmatrix} E_2 = \begin{bmatrix} \alpha&\beta&\gamma&\delta&\epsilon \\ 0&\alpha&\lambda&\sigma&\iota \\ 0&0&\tau&\kappa&\mu \\ 0&0&0&\alpha&\beta \\ 0&0&0&0&\alpha \end{bmatrix} E_5 = \begin{bmatrix} r&s&t&u&v \\ 0&r&x&y&z \\ 0&0&\theta&\xi&\zeta \\ 0&0&0&r&s \\ 0&0&0&0&r \end{bmatrix}. \tag{7.6}$$

and then find that

$$E_3 = \begin{bmatrix} 0 & 0 & b\lambda - \beta\, g + c\tau - j\gamma & b\sigma - \beta\, h + c\kappa - \gamma\, k & -b\delta + b\iota + \beta\, d - \beta\, i + c\mu - \gamma\, m \\ 0 & 0 & g\tau - j\lambda & g\kappa - k\lambda & -b\sigma + \beta\, h + g\mu - \lambda\, m \\ 0 & 0 & 0 & j\kappa - k\tau & -b\kappa + \beta\, k + j\mu - m\tau \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.7}$$

$$E_4 = \begin{bmatrix} 0 & 0 & bg\tau - 2\, bj\lambda + \beta\, gj - cj\tau + \gamma\, j^2 & bg\kappa - 2\, bk\lambda + \beta\, gk + cj\kappa - 2\, ck\tau + \gamma\, jk & * \\ 0 & 0 & -(g\tau - j\lambda)j & gj\kappa - 2\, gk\tau + jk\lambda & E_{4,2,5} \\ 0 & 0 & 0 & j(j\kappa - k\tau) & E_{4,3,5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.8}$$

where

$$E_{4,2,5} = -2\, bg\kappa + bk\lambda + \beta\, gk + gj\mu - 2\, gm\tau + j\lambda\, m$$

$$E_{4,3,5} = -2\, bj\kappa + bk\tau + \beta\, jk + j^2\mu - jm\tau$$

$$E_6 = \begin{bmatrix} 0 & 0 & -b\lambda\,\tau + 2\,\beta\, g\tau - \beta\, j\lambda - c\tau^2 + \gamma\, j\tau & -b\kappa\,\lambda + 2\,\beta\, g\kappa - \beta\, k\lambda - c\kappa\,\tau + 2\,\gamma\, j\kappa - \gamma\, k\tau & * \\ 0 & 0 & -(g\tau - j\lambda)\tau & -g\kappa\,\tau + 2\, j\kappa\,\lambda - k\lambda\,\tau & E_{6,2,5} \\ 0 & 0 & 0 & \tau\,(j\kappa - k\tau) & E_{6,3,5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.9}$$

where

$$E_{6,2,5} = -b\kappa\,\lambda - \beta\, g\kappa + 2\,\beta\, k\lambda - g\mu\,\tau + 2\, j\lambda\,\mu - \lambda\, m\tau$$

$$E_{6,3,5} = -b\kappa\,\tau - \beta\, j\kappa + 2\,\beta\, k\tau + j\mu\,\tau - m\tau^2$$

$$[E_1, E_4] = \begin{bmatrix} 0 & 0 & -j\left(2\, bg\tau - 3\, bj\lambda + \beta\, gj - cj\tau + \gamma\, j^2\right) & bgj\kappa - 3\, bgk\tau + 3\, bjk\lambda - \beta\, gjk + cj^2\kappa - \gamma\, j^2 k & * \\ 0 & 0 & (g\tau - j\lambda)j^2 & j^2(g\kappa - k\lambda) & E_{1,4,2,5} \\ 0 & 0 & 0 & j^2(j\kappa - k\tau) & E_{1,4,3,5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.10}$$

where

$$E_{1,4,2,5} = 3\, bgk\tau - 3\, bgj\kappa - bjk\lambda + \beta\, gjk + gj^2\mu - j^2\lambda\, m$$

$$E_{1,4,3,5} = -j\left(3\, bj\kappa - 2\, bk\tau - \beta\, jk - j^2\mu + jm\tau\right)$$

$$[E_2, E_6] = \begin{bmatrix} 0 & 0 & -\tau\left(-b\lambda\,\tau+3\,\beta\,g\tau-2\,\beta\,j\lambda-c\tau^2+\gamma\,j\tau\right) & b\kappa\,\lambda\,\tau-3\,\beta\,g\kappa\,\tau+3\,\beta\,j\kappa & * \\ 0 & 0 & (g\tau-j\lambda)\tau^2 & \tau^2(g\kappa-k\lambda) & E_{2,6,2,5} \\ 0 & 0 & 0 & \tau^2(j\kappa-k\tau) & E_{2,6,3,5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{7.11}$$

where

$$E_{2,6,2,5} = \beta\,g\kappa\,\tau - b\kappa\,\lambda\,\tau + -3\,\beta\,j\kappa\,\lambda + 3\,\beta\,k\lambda\,\tau + g\mu\,\tau^2 - \lambda\,m\tau^2$$

$$E_{2,6,3,5} = -\tau\,\left(b\kappa\,\tau + 2\,\beta\,j\kappa - 3\,\beta\,k\tau - j\mu\,\tau + m\tau^2\right)$$

We may assume from the trace argument that $a = \alpha = r = 0$. Now we may argue as follows: assume that $j^2 + \tau^2 \neq 0$ and for concreteness suppose, first of all, that $j \neq 0$. Then from the $(1,3)$, $(2,3)$,2,4, $(3,4)$ and $(3,5)$ entries in $[E_1, E_4] = 0$ we see that the corresponding entries in E_3 must be zero. It follows the only possible non-zero entries in E_4, E_6 are the $(1,5)$ entries and hence E_4 and E_6 are linearly dependent. Precisely the same argument applies if $\tau \neq 0$ using E_2, E_6 and $[E_2, E_6] = 0$.

Thus we assume that $j = \tau = 0$. Now, assuming that $\theta \neq 0$ the $(2,3)$ and $(3,4)$ entries in $[E_1, E_5] = E_6$ and $[E_2, E_5] + E_4 = 0$ imply that $g = k = \gamma = \kappa = 0$ and again E_4 and E_6 are rank one and proportional.

To continue assume that $j = \tau = \theta = 0$. If $b = \beta = 0$ then $E_4 = E_6 = 0$ so we suppose that $b^2 + \beta^2 \neq 0$. Then from $[E_1, E_4] = 0$ or $[E_2, E_6] = 0$ we have we have $g\kappa = k\lambda$. To satisfy this condition we put $\kappa = A\lambda$ and $k = Ag$ for some A. Then from the $(1,4)$ and $(2,5)$ entries in $[E_3, E_5] = 0$ we find that $x = \xi = 0$. Next, from the $(3,5)$ entries in $[E_1, E_5] = E_6$ and $[E_2, E_5] + E_4 = 0$ we have $s = 0$ otherwise E_4 and E_6 are proportional. From the from the $(1,5)$ entry in $[E_3, E_5] = 0$ we have $\zeta = -At$. From the $(1,4)$ and $(2,5)$ entries in $[E_1, E_4] = 0$ and $[E_2, E_5] + E_4 = 0$ we find that $by = y\beta = 0$ and hence $y = 0$ since $b^2 + \beta^2 \neq 0$. From the $(1,4)$ entries in $[E_1, E_5] = E_6$ and $[E_2, E_5] + E_4 = 0$ we have, $t(g^2 + \lambda^2)(b\lambda - g\beta) = 0$. Now $b\lambda - g\beta \neq 0$ or else E_4 and E_6 are proportional and also $g = \lambda = 0$ is untenable. Hence we must have $t = 0$. Looking at the $(1,4)$ entries in $[E_1, E_5] = E_6$ and $[E_2, E_5] + E_4 = 0$, we can only have $\lambda = 0$ in order to avoid E_4 and E_6 being proportional. Now $\beta \neq 0$ or else $E_6 = 0$ and then $[E_2, E_5] + E_4 = 0$ gives $z = u$. Finally, we have that E_6 and $[E_1, E_5] = E_6$ differ only by sign and hence $E_6 = 0$.

7.0.2 Case 2

This case in much simpler than case 1 so we summarize it in mainly verbal terms. This time we note that the common centralizer of E_4, E_6 is of the form

$\begin{bmatrix} a & 0 & c & d & e \\ 0 & b & f & g & h \\ 0 & 0 & i & j & k \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & b \end{bmatrix}$. We choose three matrices E_1, E_2, E_5 of this form, and the $(3,3)$-entry of each may be assumed to be zero. Then we calculate all the required brackets. It turns out that E_3 is of the form $\begin{bmatrix} 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Then looking at the brackets $[E_1, E_4] = 0$ and $[E_2, E_6] = 0$ and applying the filiform argument we conclude that E_3 can be reduced to $\begin{bmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Calculating the brackets again it follows that $[E_1, E_4] = 0$ and $[E_2, E_6] = 0$ which imply that $E_4 = E_6 = 0$.

7.0.3 Case 3

This time we note that the common centralizer of E_4, E_6 is of the form $\begin{bmatrix} a & 0 & c & d & e \\ 0 & a & f & g & h \\ 0 & 0 & i & j & k \\ 0 & 0 & n & b & m \\ 0 & 0 & 0 & 0 & a \end{bmatrix}$. The space of such matrices forms a decomposable, non-solvable algebra of dimension 13. In order to have a solvable and ultimately nilpotent Lie algebra, we will need to have $n = 0$, since the 2×2 block to which it belongs, comprises a solvable subalgebra. Furthermore, by taking traces, we may assume that $a = 0$. We choose three matrices E_1, E_2, E_5 of this form. Then we calculate all the required brackets. So define:

$$E_1 = \begin{bmatrix} a & 0 & c & d & e \\ 0 & 0 & f & g & h \\ 0 & 0 & i & j & k \\ 0 & 0 & 0 & b & m \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} E_2 = \begin{bmatrix} 0 & 0 & p & \delta & \epsilon \\ 0 & 0 & \lambda & \sigma & \iota \\ 0 & 0 & w & \kappa & \mu \\ 0 & 0 & 0 & \beta & \tau \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} E_5 = \begin{bmatrix} 0 & 0 & t & u & v \\ 0 & 0 & x & y & z \\ 0 & 0 & \theta & \xi & \zeta \\ 0 & 0 & 0 & s & \omega \\ 0 & 0 & 0 & 0 & r \end{bmatrix}. \tag{7.12}$$

Then we find that after applying the filiform argument

$$E_3 = \begin{bmatrix} 0 & 0 & 0 & -b\delta+\beta d+c\kappa-jp & c\mu+d\tau-\delta m-kp \\ 0 & 0 & 0 & -b\sigma+\beta g+f\kappa-j\lambda & f\mu+g\tau-k\lambda-m\sigma \\ 0 & 0 & 0 & 0 & i\mu+j\tau-kw-\kappa m \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.13}$$

$$E_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & b\delta\tau-\beta d\tau-c\kappa\tau+i\mu p+2jp\tau-kpw-\kappa mp \\ 0 & 0 & 0 & 0 & b\sigma\tau-\beta g\tau-f\kappa\tau+i\lambda\mu+2j\lambda\tau-k\lambda w-\kappa\lambda m \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$E_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & b\delta\tau-\beta d\tau-c\kappa\tau+i\mu p+2jp\tau-kpw-\kappa mp \\ 0 & 0 & 0 & 0 & b\sigma\tau-\beta g\tau-f\kappa\tau+i\lambda\mu+2j\lambda\tau-k\lambda w-\kappa\lambda m \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.14}$$

where the following conditions are necessary

$$cw - ip = 0 \tag{7.15}$$
$$fw - i\lambda = 0 \tag{7.16}$$
$$j\beta - b\kappa + i\kappa - jw = 0 \tag{7.17}$$
$$b\tau - m\beta = 0 \tag{7.18}$$
$$b(b\delta - d\beta - c\kappa + jp) = 0 \tag{7.19}$$
$$b(b\sigma - g\beta - f\kappa + j\lambda) = 0 \tag{7.20}$$
$$i(i\mu + j\tau - kw - m\kappa) = 0 \tag{7.21}$$
$$\beta(b\delta - d\beta - c\kappa + jp) = 0 \tag{7.22}$$
$$\beta(b\sigma - g\beta - f\kappa + j\lambda) = 0 \tag{7.23}$$
$$w(i\mu + j\tau - kw - m\kappa) = 0. \tag{7.24}$$

At this point it remains only to satisfy the brackets $[E_1, E_5] = E_6$, $[E_2, E_5] + E_4 = 0$ and $[E_3, E_5] = 0$. Thus

$$[E_1, E_5] - E_6 = \begin{bmatrix} 0 & 0 & c\theta - ti & -ub + c\xi + ds - tj & -b\delta\tau + \beta d\tau + c\kappa\tau - i\mu p - 2jp\tau + kpw + \kappa mp + c\eta + d\omega - tk - um \\ 0 & 0 & f\theta - xi & -yb + f\xi + gs - xj & -b\sigma\tau + \beta g\tau + f\kappa\tau - i\lambda\mu - 2j\lambda\tau + k\lambda w + \kappa\lambda m + f\eta + g\omega - xk - ym \\ 0 & 0 & 0 & -\xi b + i\xi + js - \theta j & i\eta + j\omega - \theta k - \xi m \\ 0 & 0 & 0 & 0 & b\omega - sm \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.25}$$

$$[E_2, E_5] + E_4 = \begin{bmatrix} 0 & 0 & p\theta - tw & -u\beta + \delta s - t\kappa + p\xi & b\delta m - \beta dm + ci\mu + cj\tau - ckw - 2c\kappa m + jmp + \delta\omega + p\eta - t\mu - u\tau \\ 0 & 0 & \lambda\theta - xw & -y\beta - x\kappa + \lambda\xi + \sigma s & bm\sigma - \beta gm + fi\mu + fj\tau - fkw - 2f\kappa m + j\lambda m + \lambda\eta - x\mu + \sigma\omega - y\tau \\ 0 & 0 & 0 & -\xi\beta + \kappa s - \theta\kappa + w\xi & w\eta + \kappa\omega - \theta\mu - \xi\tau \\ 0 & 0 & 0 & 0 & \beta\omega - s\tau \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.26}$$

$$[E_3, E_5] = \begin{bmatrix} 0 & 0 & 0 & -(b\delta - \beta d - c\kappa + jp)s & -b\delta\omega + \beta d\omega + c\kappa\omega - i\mu t - j\omega p - jt\tau + ktw + \kappa mt \\ 0 & 0 & 0 & -(b\sigma - \beta g - f\kappa + j\lambda)s & -b\omega\sigma + \beta g\omega + f\kappa\omega - i\mu x - j\lambda\omega - j\tau x + kwx + \kappa mx \\ 0 & 0 & 0 & 0 & -\theta(i\mu + j\tau - kw - \kappa m) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{7.27}$$

If we calculate the condition for E_4 and E_6 to be linearly independent we see that it contains a factor $i\mu + j\tau - kw - m\kappa$; hence, from 7.21 and 7.24 we see that $i = w = 0$ and from the $(3, 5)$ entry in $[E_3, E_5] = 0$ that $\theta = 0$. From the $(3, 4)$ and $(4, 5)$ entries in $[E_1, E_2] = E_3$ we must have $b = \beta = 0$ and similarly, from the $(3, 4)$ and $(4, 5)$ entries in $[E_1, E_5] = E_6$, $[E_2, E_5] + E_4 = 0$

we have $s = 0$, otherwise $j = m = \kappa = \tau = 0$. From the $(1,5)$ and $(2,5)$ entries in $[E_3, E_2] =$ we must have $t = x = 0$ in order to have $j\tau - m\kappa \neq 0$.

At this point it remains only to satisfy the brackets $[E_1, E_5] = E_6, [E_2, E_5] + E_4 = 0$ which give the conditions:

$$c\kappa\,\tau - 2\,jp\tau + \kappa\,mp + c\eta - um = 0 \tag{7.28}$$
$$f\kappa\,\tau - 2\,j\lambda\,\tau + \kappa\,\lambda\,m + f\eta - ym = 0 \tag{7.29}$$
$$cj\tau - 2cm\kappa + jmp + p\eta - u\tau = 0 \tag{7.30}$$
$$fj\tau - 2\,f\kappa\,m + j\lambda\,m + \lambda\,\eta - y\tau = 0. \tag{7.31}$$

The $(1,5)$ entry of $[E_1, E_5] - E_6\times$ the $(2,5)$ entry of $[E_2, E_5] + E_4$ minus the $(1,5)$ entry of $[E_2, E_5] + E_4\times$ the $(2,5)$ entry of $[E_1, E_5] - E_6$ gives $2(c\lambda - fp)(j\tau - m\kappa)^2$, the non-vanishing of which is required in order to ensure that E_4 and E_6 are linearly independent. However, if, so as to eliminate η, u, y, we form the combination $(f\tau - m\lambda)((\tau 7.28 - m7.30) - ((c\tau - mp))(\tau 7.29 - m7.31))$ we obtain

$$2(m^2 + \tau^2)(j\tau - m\kappa)(c\lambda - fp) = 0$$

and hence the representation that we are seeking does not exist.

7.0.4 Case 4

This case follows from Case 3 by transposing about the anti-diagonal. $\begin{bmatrix} 0&0&0&0&1 \\ 0&0&0&1&0 \\ 0&0&1&0&0 \\ 0&1&0&0&0 \\ 1&0&0&0&0 \end{bmatrix}$.

If there were a representation coming from Case 3 then there would be a not necessarily conjugate representation in Case 4 and vice-versa.

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