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Abstract

For each of the two and three-dimensional indecomposable Lie algebras the geodesic equations of the associated canonical Lie group connection are given. In each case a basis for the associated Lie algebra of symmetries is constructed and the corresponding Lie brackets are written down.

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1 Introduction

Any Lie group comes equipped with a natural linear torsion-free connection and consequently a canonical system of geodesic equations. This connection was introduced as long ago as 1926 [2]: it is in fact E. Cartan's "0"-connection [4]. More recently, it has appeared in the context of the inverse problem of Lagrangian mechanics [6,3,9]. In this paper we embark on a study of the Lie symmetry properties of the canonical geodesic system and it is confined to the case of indecomposable Lie groups in dimensions two and three.

In the main body of this paper we provide systematically the Lie symmetry algebras for each of the three-dimensional Lie algebras. For each such algebra we provide a group matrix S, the left and right-invariant vector fields and one-forms and the associated system of geodesics. We list in each case a basis for the Lie symmetries and the corresponding Lie brackets. Of course constructing these symmetry algebras is a labor intensive albeit routine process [7] that is aided considerably by symbolic programs such as MAPLE, which is what was used in this paper. Nonetheless we forgo completely all the elementary calculations and simply present the results.

The notation for the Lie groups in dimension three and their associated Lie algebras is taken from [8]. However, in the interests of efficiency, we prefer to consolidate cases $3.3, 3.4$ and 3.5 into a single case and likewise for cases 3.6 and 3.7. It is also useful to consult Jacobson's approach at the end of the first chapter of his book [5]. In the future we hope to investigate Lie symmetry properties of canonical geodesic systems in dimension four and higher. We will use $\mathbb{R}^m \rtimes \mathbb{R}^n$ to denote a semi-direct product of abelian Lie algebras in which $\mathbb{R}^m$ is a subalgebra and $\mathbb{R}^n$ an ideal.

2 The canonical Lie group connection

On right invariant vector fields X and Y the canonical symmetric connection ∇ on a Lie group G is defined by

$$\nabla_X Y = \frac{1}{2}\ [X, Y] \tag{1}$$

and then extended to arbitrary vector fields using linearity and the Leibnitz rule. One could just as well use left-invariant vector fields to define ∇ but one must check that the definition is well defined. Some properties of the canonical connection have been derived in [3,9]. Here we shall be content to review them as follows:

- The connection has torsion zero
- The curvature is given by $R(X,Y)Z = \frac{1}{4}[[X,Y],Z]$
- The curvature tensor R is covariantly constant
- The curvature tensor R is zero if and only if the Lie algebra is two-step nilpotent
- The Ricci tensor is symmetric and in fact a multiple of the Killing form
- The Ricci tensor is bi-invariant

- Any left or right invariant vector field is a symmetry of the connection
- Any left or right invariant one-form defines a linear first integral of the geodesics
- Geodesic curves are translates of one-parameter subgroups
- Any vector field in the center of the Lie algebra is bi-invariant

3 Free particle systems

In this Section we shall review the Lie symmetries of a free particle system, being the most extreme case of a flat connection. However, this case of course transcends issues of any Lie group structure. The same results have been rediscovered many times but we shall refer to [1] as one source. The geodesic equations will be written as

$$\ddot{x}^i = 0 \tag{2}$$

where (x^i) are a system of local coordinates on some manifold M. It will be helpful to define the dilation vector field Δ on M by

$$\Delta = tD_t + x^i D_i \tag{3}$$

where D_i denotes the partial derivative operator with respect to x^i and there is a sum over i from 1 to n, the latter being the dimension of M. Then the following vector fields comprise a basis for the space of Lie symmetries of eq(2):

$$D_t, D_i, tD_t, x^i D_t, tD_i, x^i D_j, t\Delta, x^i\Delta. \tag{4}$$

Adding up we obtain a space of dimension $n^2 + 4n + 3 = (n+2)^2 - 1$ and indeed we obtain a representation of the simple Lie algebra $\mathfrak{sl}(n+2, \mathbb{R})$.

4 2-dimensional Lie group, geodesics and symmetry algebra

2.1: $[e_1, e_2] = e_1$:

$$S = \begin{bmatrix} e^y & x \\ 0 & 1 \end{bmatrix}.$$

Left-invariant vector fields $e^y D_x, -D_y$
Left-invariant one forms $e^{-y}dx, dy$
Right-invariant vector fields $D_x, D_y + xD_x$
Right-invariant one forms $dx - xdy, dy$.

Geodesics:

$$\ddot{x} = \dot{x}\dot{y}, \;\; \ddot{y} = 0. \tag{5}$$

Symmetry algebra basis and brackets:

$$e_1 = D_t, e_2 = D_x, e_3 = yD_t, e_4 = e^y D_x, e_5 = tD_t, e_6 = xD_x, e_7 = -D_y. \tag{6}$$

$$[e_1, e_5] = e_1, [e_2, e_6] = e_6, [e_3, e_5] = e_3, [e_3, e_7] = e_1, [e_4, e_6] = e_4, [e_4, e_7] = e_4.$$

The corresponding symmetry algebra is seven-dimensional indecomposable: the nilradical is abelian spanned by e_1, e_2, e_3, e_4 and there is an abelian complement spanned by e_5, e_6, e_7 so the symmetry algebra is $\mathbb{R}^3 \rtimes \mathbb{R}^4$.

5 3-dimensional Lie groups, geodesics and symmetry algebras

3.1: $[e_2, e_3] = e_1$:

$$S = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Left-invariant vector fields $D_z, D_x, D_y + xD_z$
Left-invariant one forms $dz - xdy, dy, dx$
Right-invariant vector fields $D_z, D_y, D_x + yD_z$
Right-invariant one forms $dz - ydx, dx, dy$.

Geodesics:

$$\ddot{z} = \dot{x}\dot{y}, \;\; \ddot{x} = 0, \;\; \ddot{y} = 0. \tag{7}$$

By making the change of variable $\overline{z} = z - \frac{xy}{2}$, the geodesic equations can be changed to the "free particle" system

$$\ddot{\overline{z}} = 0, \;\; \ddot{x} = 0, \;\; \ddot{y} = 0. \tag{8}$$

Hence the symmetry Lie Algebra is $sl(5, \mathbb{R})$ as was explained in Section 3. We shall forgo the irksome task of transforming the generators of the free particle into generators for the Heisenberg group under the transformation induced by the variable $\overline{z}$ introduced above.

3.2: $[e_1, e_3] = e_1, [e_2, e_3] = e_1 + e_2$:

$$S = \begin{bmatrix} e^z & ze^z & x \\ 0 & e^z & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Left-invariant vector fields $e^z D_x, e^z(D_y + zD_x), -D_z$
Left-invariant one forms $e^{-z}(dx - zdy), e^{-z}dy, dz$
Right-invariant vector fields $D_x, D_y, D_z + (x+y)D_x + yD_y$
Right-invariant one forms $dx - (x+y)dz, dy - ydz, dz$.

Geodesics:

$$\ddot{x} = (\dot{x} + \dot{y})\dot{z},\ \ddot{y} = \dot{y}\dot{z},\ \ddot{z} = 0. \tag{9}$$

Symmetry algebra basis and brackets:

$$\begin{array}{l} e_1 = D_z,\ e_2 = -(xD_x + yD_y),\ e_3 = -tD_t,\ e_4 = -yD_x,\ e_5 = D_y, \\ e_6 = e^z(D_y + zD_x),\ e_7 = D_x,\ e_8 = e^z D_x,\ e_9 = D_t,\ e_{10} = zD_t. \end{array} \tag{10}$$

$[e_1, e_6] = e_6 + e_8, [e_1, e_8] = e_8, [e_1, e_{10}] = e_9, [e_2, e_5] = e_5, [e_2, e_6] = e_6, [e_2, e_7] = e_7, [e_2, e_8] = e_8, [e_3, e_9] = e_9, [e_3, e_{10}] = e_{10}, [e_4, e_5] = e_7, [e_4, e_6] = e_8$.

It is a 10-dimensional indecomposable, solvable Lie algebra with a 7-dimensional nilradical. In fact the nilradical is $\mathbb{R}^2 \oplus A_{5.1}$ where $\mathbb{R}^2$ and $A_{5.1}$ are spanned by e_9, e_{10} and e_4, e_5, e_6, e_7, e_8 and the 3-dimensional abelian complement is spanned by e_1, e_2, e_3, respectively. The algebra as a whole is isomorphic to $\mathbb{R}^3 \rtimes (\mathbb{R}^2 \oplus A_{5.1})$.

$3.3(a = 1), 3.4(a = -1), 3.5(a \neq 0, \pm 1)$: $[e_1, e_3] = e_1, [e_2, e_3] = ae_2$:

$$S = \begin{bmatrix} e^z & 0 & x \\ 0 & e^z & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Left-invariant vector fields $e^z D_x, e^{az} D_y, D_z$
Left-invariant one forms $e^{-z}dx, e^{-az}dy, dz$
Right-invariant vector fields $D_x, D_y, D_z + xD_x + ayD_y$
Right-invariant one forms $dx - xdz, dy - aydz, dz$

Geodesics:

$$\ddot{x} = \dot{x}\dot{z},\ \ddot{y} = a\dot{y}\dot{z},\ \ddot{z} = 0. \tag{11}$$

Symmetry algebra basis and brackets 3.3:

$$\begin{aligned}&e_1 = tD_t,\ e_2 = D_z,\ e_3 = xD_x + yD_y,\ e_4 = D_t,\ e_5 = D_x,\ e_6 = D_y\\ &e_7 = zD_t,\ e_8 = e^z D_x,\ e_9 = e^z D_y,\ e_{10} = xD_x - yD_y,\ e_{11} = yD_x,\ e_{12} = xD_y.\end{aligned} \tag{12}$$

$$\begin{aligned}&[e_1, e_4] = -e_4, [e_1, e_7] = -e_7, [e_2, e_7] = e_4, [e_2, e_8] = e_8, [e_2, e_9] = e_9,\\ &[e_3, e_5] = -e_5, [e_3, e_6] = -e_6, [e_3, e_8] = -e_8, [e_3, e_9] = -e_9, [e_5, e_{10}] = e_5,\\ &[e_5, e_{12}] = e_6, [e_6, e_{10}] = -e_6, [e_6, e_{11}] = e_5, [e_8, e_{10}] = e_8, [e_8, e_{12}] = e_9,\\ &[e_9, e_{10}] = -e_9, [e_9, e_{11}] = e_8, [e_{10}, e_{11}] = -2e_{11}, [e_{10}, e_{12}] = 2e_{12},\\ &[e_{11}, e_{12}] = -e_{10}.\end{aligned}$$

The symmetry algebra has a non-trivial Levi decomposition in which e_{10}, e_{11}, e_{12} span the semi-simple part $\mathfrak{sl}(2, \mathbb{R})$; the radical is a semi-direct sum consisting of an abelian nilradical spanned by $e_4, e_5, e_6, e_7, e_8, e_9$ and abelian complement spanned by e_1, e_2, e_3. The algebra as a whole is isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \rtimes (\mathbb{R}^3 \rtimes \mathbb{R}^4)$.

Symmetry algebra basis and brackets 3.4:

$$\begin{aligned}&e_1 = tD_t, e_2 = D_z - yD_y, e_3 = xD_x + yD_y, e_4 = D_t, e_5 = D_x, e_6 = D_y\\ &e_7 = zD_t, e_8 = e^z D_x, e_9 = e^{-z} D_y, e_{10} = xD_x - yD_y, e_{11} = ye^z D_x, e_{12} = xe^{-z} D_y.\end{aligned} \tag{13}$$

$$\begin{aligned}&[e_1, e_4] = -e_4, [e_1, e_7] = -e_7, [e_2, e_6] = e_6, [e_2, e_7] = e_4, [e_2, e_8] = e_8,\\ &[e_3, e_5] = -e_5, [e_3, e_6] = -e_6, [e_3, e_8] = -e_8, [e_3, e_9] = -e_9, [e_5, e_{10}] = e_5,\\ &[e_5, e_{12}] = e_9, [e_6, e_{10}] = -e_6, [e_6, e_{11}] = e_8, [e_8, e_{10}] = e_8, [e_8, e_{12}] = e_6,\\ &[e_9, e_{10}] = -e_9, [e_9, e_{11}] = e_5, [e_{10}, e_{11}] = -2e_{11}, [e_{10}, e_{12}] = 2e_{12}, [e_{11}, e_{12}] = -e_{10}.\end{aligned}$$

The symmetry algebra has a non-trivial Levi decomposition in which e_{10}, e_{11}, e_{12} span the semi-simple part $\mathfrak{sl}(2, \mathbb{R})$; the radical is a semi-direct sum consisting of an abelian nilradical spanned by $e_4, e_5, e_6, e_7, e_8, e_9$ and abelian complement spanned by e_1, e_2, e_3. Again the algebra as a whole is isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \rtimes (\mathbb{R}^3 \rtimes \mathbb{R}^4)$.

Symmetry algebra basis and brackets 3.5:

$$\begin{array}{lllll} e_1 = D_z, & e_2 = tD_t & e_3 = xD_x, & e_4 = yD_y, & e_5 = D_t, \\ e_6 = D_x, & e_7 = D_y, & e_8 = zD_t, & e_9 = e^z D_x, & e_{10} = e^{az} D_y. \end{array} \tag{14}$$

$[e_1, e_8] = e_5, [e_1, e_9] = e_9, [e_1, e_{10}] = ae_{10}, [e_2, e_5] = -e_5, [e_2, e_8] = -e_8, [e_3, e_6] = -e_6, [e_3, e_9] = -e_9, [e_4, e_7] = -e_7, [e_4, e_{10}] = -e_{10}$.

It is a 10-dimensional indecomposable solvable Lie algebra. It has a 6-dimensional abelian nilradical spanned by $e_5, e_6, e_7, e_8, e_9, e_{10}$ and a 4-dimensional abelian complement spanned by e_1, e_2, e_3, e_4. Hence, the symmetry algebra is isomorphic to $\mathbb{R}^4 \rtimes \mathbb{R}^6$.

$3.6(a = 0), 3.7a(a \neq 0) : [e_1, e_3] = ae_1 - e_2, [e_2, e_3] = e_1 + ae_2$:

$$S = \begin{bmatrix} e^{az}\cos z & e^{az}\sin z & x \\ -e^{az}\sin z & e^{az}\cos z & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Left-invariant vector fields $e^{az}(\cos z\, D_x - \sin z D_y), e^{az}(\sin z\, D_x + \cos z\, D_y), D_z$
Left-invariant one-forms $e^{-az}(\cos z\, dx - \sin z\, dy), e^{-az}(\sin z\, dx + \cos z\, dy), dz$
Right-invariant vector fields $D_x, D_y, D_z + (ax + y)D_x - (x - ay)D_y$
Right-invariant one forms $dx - (ax + y)dz, dy - (ay - x)dz, dz$.

Geodesics:

$$\ddot{x} = (a\dot{x} + \dot{y})\dot{z}, \;\; \ddot{y} = (a\dot{y} - \dot{x})\dot{z}, \;\; \ddot{z} = 0. \tag{15}$$

Symmetry algebra basis and brackets 3.6:

$$\begin{aligned} & e_1 = D_z - \tfrac{1}{2}(-\cos z\, D_x + \sin z\, D_y),\ e_2 = xD_x + yD_x\,, e_3 = tD_t,\ e_4 = D_t \\ & e_5 = D_y, e_6 = zD_t,\ e_7 = D_x,\ e_8 = -yD_x + xD_y,\ e_9 = \sin z\, D_x + \cos z\, D_y, \\ & e_{10} = -\cos z\, D_x + \sin z\, D_y, \\ & e_{11} = (-x\cos z + y\sin z\,)D_x + (y\cos z\, + x\sin z\,)D_y, \\ & e_{12} = (y\cos z\, + x\sin z\,)D_x + (x\cos z\, - y\sin z\,)D_y. \end{aligned} \tag{16}$$

$[e_1,e_5]=-\frac{1}{2}e_7, [e_1,e_6]=e_4, [e_1,e_7]=\frac{1}{2}e_5, [e_1,e_8]=\frac{1}{2}e_9, [e_1,e_9]=-\frac{1}{2}e_8,$
$[e_2,e_5]=-e_5, [e_2,e_7]=-e_7, [e_2,e_8]=-e_8, [e_2,e_9]=-e_9, [e_3,e_4]=-e_4,$
$[e_3,e_6]=-e_6, [e_5,e_{10}]=-e_7, [e_5,e_{11}]=e_9, [e_5,e_{12}]=-e_8, [e_7,e_{10}]=e_5,$
$[e_7,e_{11}]=e_8, [e_7,e_{12}]=e_9, [e_8,e_{10}]=-e_9, [e_8,e_{11}]=e_7, [e_8,e_{12}]=-e_5,$
$[e_9,e_{10}]=e_8, [e_9,e_{11}]=e_5, [e_9,e_{12}]=e_7, [e_{10},e_{11}]=2e_{12}, [e_{10},e_{12}]=-2e_{11},$
$[e_{11},e_{12}]=-2e_{10}.$

The symmetry algebra has a non-trivial Levi decomposition in which e_{10}, e_{11}, e_{12} span the semi-simple part $\mathfrak{sl}(2,\mathbb{R})$; the radical is a semi-direct sum consisting of an abelian nilradical spanned by $e_4, e_5, e_6, e_7, e_8, e_9$ and abelian complement spanned by e_1, e_2, e_3. Hence, the symmetry algebra is isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^3 \rtimes \mathbb{R}^6)$.

Symmetry algebra basis and brackets 3.7:

$$\begin{aligned}&e_1=tD_t,\ e_2=xD_x+yD_y,\ e_3=-yD_x+xD_y,\ e_4=D_z,\ e_5=D_y,\ e_6=zD_t,\\&e_7=D_x, e_8=D_t,\ e_9=e^{az}(\sin z\,D_x+\cos z\,D_y), e_{10}=-e^{az}(\cos z\,D_x-\sin z\,D_y).\end{aligned} \tag{17}$$

$[e_1,e_6]=-e_6, [e_1,e_8]=-e_8, [e_2,e_5]=-e_5, [e_2,e_7]=-e_7, [e_2,e_9]=-e_9, [e_2,e_{10}]=-e_{10}, [e_3,e_5]=e_7, [e_3,e_7]=-e_5, [e_3,e_9]=-e_{10}, [e_3,e_{10}]=e_9, [e_4,e_6]=e_8, [e_4,e_9]=ae_9-e_{10}, [e_4,e_{10}]=ae_{10}+e_9.$

It is a 10-dimensional indecomposable solvable Lie algebra. It has a 6-dimensional abelian nilradical spanned by $e_5, e_6, e_7, e_8, e_9, e_{10}$ and a 4-dimensional abelian complement spanned by e_1, e_2, e_3, e_4. Hence, the symmetry algebra is isomorphic to $\mathbb{R}^4 \rtimes \mathbb{R}^6$.

3.8 (simple $\mathfrak{sl}(2,\mathbb{R})$) $[e_1,e_2]=2e_2, [e_1,e_3]=-2e_3, [e_2,e_3]=e_1$:

$$S=\begin{bmatrix} e^{2z}(1+xy)^2 & 2xe^{2z}(1+xy) & e^{2z}x^2\\ y(1+xy) & 1+2xy & x\\ e^{-2z}y^2 & 2e^{-2z}y & e^{-2z}\end{bmatrix}.$$

Left-invariant vector fields: $-2xD_x+2yD_y+D_z, (1+2xy)D_x-y^2D_y-yD_z, D_y$
Left-invariant one forms: $ydx+(1+2xy)dz, dx+2xdz, -y^2dx+dy-2y(1+xy)dz$
Right-invariant vector fields $D_z, e^{2z}(x^2D_x+D_y-xD_z), e^{-2z}D_z$
Right-invariant one forms $dz+xdy, e^{-2z}dy, e^{2z}(dx-x^2dy)$.

Geodesics:

$$\ddot{x} = 4x^2\dot{y}\dot{z} + 2x\dot{x}\dot{y} - 2\dot{x}\dot{z},\ \ddot{y} = 2\dot{y}\dot{z},\ \ddot{z} = -2x\dot{y}\dot{z} - \dot{x}\dot{y}. \tag{18}$$

Symmetry algebra basis and brackets:

$$\begin{aligned}&e_1 = D_y,\ e_2 = D_z + 2(x^2 D_x + D_y - xD_z),\ e_3 = (-xy - \tfrac{1}{2})D_x + \tfrac{y^2}{2}D_y + \tfrac{y}{2}D_z,\\ &e_4 = D_z, e_5 = e^{-2z}D_x,\ e_6 = e^{2z}(x^2 D_x + D_y - xD_z),\ e_7 = tD_t,\ e_8 = D_t.\end{aligned} \tag{19}$$

$[e_1, e_2] = 2e_1, [e_1, e_3] = \frac{1}{2}e_2, [e_2, e_3] = 2e_3, [e_4, e_5] = -2e_5, [e_4, e_6] = 2e_6, [e_5, e_6] = -e_4, [e_7, e_8] = -e_8$

The symmetry Lie algebra is an 8-dimensional decomposable algebra. It has two copies of $\mathfrak{sl}(2,\mathbb{R})$ spanned by e_1, e_2, e_3 and e_4, e_5, e_7 and a non-abelian 2-dimensional algebra $A_{2,1}$ spanned by e_7, e_8. Hence, the algebra is isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus A_{2,1}$.

3.9 (simple $\mathfrak{so}(3)$) $[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$:

$$S = \begin{bmatrix} \cos x \cos y\ \cos z - \sin x\ \sin z & \sin x\ \cos y\ \cos z + \cos x \sin z & -\sin y \cos z \\ -\cos x \cos y \sin z - \sin x \cos z & -\sin x \sin z \cos y + \cos x \cos z & \sin y \sin z \\ \cos x \sin y & \sin x \sin y & \cos y \end{bmatrix}.$$

Left-invariant vector fields $D_x, -\frac{\sin x \cos y}{\sin y}D_x + \cos x\, D_y + \frac{\sin x}{\sin y}D_z, -\frac{\cos x\ \cos y}{\sin y}D_z - \sin x\, D_y + \frac{\cos x}{\sin y}D_z$
Left-invariant one forms $dx + \cos y\, dz, \cos x\, dy + \sin x\ \sin y\, dz, -\sin x dy + \cos x\ \sin y\, dz$.
Right-invariant vector fields: $D_z, \frac{\sin z}{\sin y}D_x + \cos z\, D_y - \frac{\cos y\ \sin z}{\sin y}D_z, \frac{\cos z}{\sin y}D_x - \sin z\, D_y - \frac{\cos y\ \cos z}{\sin y}D_z$ Right-invariant one-forms $dz + \cos y\, dx, \sin y \sin z\, dx + \cos z\, dy, \sin y\ \cos z\, dx - \sin z\, dy$.

Geodesics:

$$\ddot{x} = \csc y(\dot{z} - \cos y\dot{x})\dot{y}, \ddot{y} = -\sin y\, \dot{x}\dot{z}, \ddot{z} = \csc y\, (\dot{x} - \cos y\, \dot{z})\dot{y} \tag{20}$$

Symmetry algebra basis and brackets:

$$\begin{aligned}
&e_1 = D_z,\; e_2 = \tfrac{\sin z}{\sin y} D_x + \cos z\, D_y - \tfrac{\cos y\, \sin z}{\sin y} D_z,\\
&e_3 = \tfrac{\cos z}{\sin y} D_x - \sin z\, D_y - \tfrac{\cos y\, \cos z}{\sin y} D_z,\; e_4 = D_x,\\
&e_5 = \tfrac{\cos y \cos x}{\sin y} D_x + \sin x\, D_y - \tfrac{\cos x}{\sin y} D_z,\; e_6 = \tfrac{\cos y \sin x}{\sin y} D_x - \cos x\, D_y - \tfrac{\sin x}{\sin y} D_z,\\
&e_7 = tD_t, \;\; e_8 = D_t.
\end{aligned} \tag{21}$$

$[e_1, e_2] = e_3, [e_1, e_3] = -e_2, [e_2, e_3] = e_1, [e_4, e_5] = -e_6, [e_4, e_6] = e_5, [e_5, e_6] = -e_4, [e_7, e_8] = -e_8$.

The symmetry Lie algebra is an 8-dimensional decomposable algebra. It contains two copies of $\mathfrak{so}(3)$ spanned by e_1, e_2, e_3 and e_4, e_5, e_6 and a non-abelian 2-dimensional algebra $A_{2,1}$ spanned by e_7, e_8. Hence, the algebra is $\mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus A_{2,1}$.

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