INTERIOR OPERATORS AND THEIR APPLICATIONS

Fikreyohans Solomon Assfaw

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University of the Western Cape

Supervisor: Professor David B. Holgate
Co-supervisor: Doctor Ando D. Razafindrakoto

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Connectedness

Compactness

ABSTRACT

Categorical closure operators were introduced by Dikranjan and Giuli in [DG87] and then developed by these authors and Tholen in [DGT89]. These operators have played an important role in the development of Categorical Topology by introducing topological concepts, such as connectedness, separatedness and compactness, in an arbitrary category and they provide a unified approach to various mathematical notions. Motivated by the theory of these operators, the categorical notion of interior operators was introduced by Vorster in [Vor00]. While there is a notational symmetry between categorical closure and interior operators, a detailed analysis shows that the two operators are not categorically dual to each other, that is: it is not true in general that whatever one does with respect to closure operators may be done relative to interior operators. Indeed, the continuity condition of categorical closure operators can be expressed in terms of images or equivalently, preimages, in the same way as the usual topological closure describes continuity in terms of images or preimages along continuous maps. However, unlike the case of categorical closure operators, the continuity condition of categorical interior operators can not be described in terms of images. Consequently, the general theory of categorical interior operators is not equivalent to the one of closure operators. Moreover, the categorical dual closure operator introduced in [DT15] does not lead to interior operators. As a consequence, the study of categorical interior operators in their own right is interesting.

Most studies of categorical interior operators have been largely restricted to point set topology; see [CR10, CM13]. A deeper categorical insight into interior operators and their applications beyond the category of topological spaces is still lacking. In this thesis, we conduct a systematic study of categorical interior operators on category $\mathbb{C}$ supplied with an $(\mathcal{E},\mathcal{M})$-factorization structure for morphisms such that $\mathcal{M}$ is a fixed class of monomorphisms. We study the notions of closed, open, initial and final morphisms with respect to an interior operator on $\mathbb{C}$. These morphisms defined via interior operators are shown to have the cancellation, composition and pullback stability properties. Moreover, they are (partially) characterized in terms of open subobjects, for an idempotent interior operator. In particular, open $\mathcal{M}$-morphisms with respect to any interior operator are shown to be the morphisms whose image commutes with the interior. Some properties of the notion of codenseness with respect to an interior operator are also introduced. Indeed, the codenseness is preserved by both images under $\mathcal{M}$-morphisms and dual images under $\mathcal{E}$-morphisms. We also introduce and study a notion of quasi open morphisms with respect to a given interior operator i. More specifically, it is shown that these morphisms are precisely the morphisms which reflect i-codensity. We then introduce a notion of hereditary interior operators on $\mathbb{C}$ using the right adjoint of the preimage of a given morphism and discuss their properties. We show that these operators behave as well as hereditary closure operators. Notably, we obtain a characterization of heredity of a given interior operator i in terms of "initial embeddings" with respect to i. Moreover, we study the relationship between our hereditary and Castellini's (strongly) hereditary interior operators. Furthermore, inspired by the works of [Cle01], we use a concept of interior operators and a relative notion of constant morphisms to investigate the general notions of connectedness and disconnectedness on $\mathbb{C}$. We show that these notions act in the same manner as the notion of connectedness and disconnectedness with respect to closure operators studied in [Cle01] and extend most of the results of [CR10] to a more general categorical setting. The thesis is concluded with interior theoretic approaches of the notion of compactness on $\mathbb{C}$ in which each preimage preserves arbitrary joins. It is shown that under appropriate hypotheses, most classical results about topological compactness can be generalized to these settings.

01 August 2019

DECLARATION

I declare that *Interior Operators and Their Applications* is my own work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Fikreyohans Solomon Assfaw 01 August 2019

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Contents

Introduction

Closure operators were first introduced in Analysis by [Moo09, Rie09] and since then they have been defined, studied and intensively used in Logic [Her22, Tar29], Algebra [Bir37, Pie72], Topology [Kur22, Čec37] and Lattice theory [Bir40]. A categorical closure operator on an arbitrary category is a family of functions which are expansive, order preserving and compatible with taking images or equivalently, preimages, in the same way as the usual topological closure is compatible with continuous maps. The formal theory of categorical closure operators was introduced by Dikranjan and Giuli in [DG87] and then developed by these authors and Tholen in [DGT89]. The theory was largely inspired by Salbany's paper [Sal76], where regular closure operators on the category of topological spaces and continuous maps were introduced. This categorical notion generalizes both the lattice theoretic closure operations and universal closure operations of Topos- and Sheaf theory. In fact, it has unified various important concepts and has led to interesting examples and applications in diverse areas of Mathematics. Since their introduction, categorical closure operators have played a crucial role in the development of Categorical Topology. They have been employed to characterize epimorphisms and investigate cowellpoweredness in certain categories; see for example, [DG84, DG85, Dik92]. These operators led to an extensive programme of research introducing and studying classical topological notions such as separation, compactness, regularity and connectedness in abstract categories and simultaneously expanding categorical insights into general topology; see in particular the monographs [DT95, Cas03] and the references therein.

The subsequent introduction of categorical interior operators in [Vor00] has only recently received attention and a few papers are published on the subject; see [CR10, Cas11, LTOC11, CM13, Cas15, RH14, Cas16]. An interior operator i on $\mathbb{C}$ is a family of functions which are contractive, order preserving and only compatible with taking preimages. While there is a notational symmetry between categorical closure and interior operators, the two operators are not "dual" to each other. Categorical interior operators are not compatible with taking images unlike closure operators. As a consequence, interior operators cannot be seen as endofunctors on a suitable class $\mathcal{M}$ of embeddings, hence the preservation property of interior operators fails (see Remark 2.1.2(a)). Contrary to this a similar property which is called functorial property holds true for closure operators. This property plays a significant role in the development of closure operators and enables each closure operator to give rise to an endofunctor of the arrow category $\mathcal{M}$; see [Cas03]. Furthermore, the dual closure operator introduced in [DT15] is a categorical dual to closure operator and does not lead to interior operators. Therefore, a categorical understanding of interior operators in their own right makes sense.

The majority of the studies of categorical interior operators have been largely restricted to point set topology. In fact, in [CR10] and [CM13] interior operators are used to study notions of connectedness and separation on the category of topological spaces and continuous maps. A deeper categorical insight into interior operators and their applications beyond the category of topological spaces is still lacking. This thesis takes a further step in the study of categorical interior operators. Indeed, as the title of the thesis suggests our aim is to conduct a systematic study of categorical interior operators and their applications in a more general categorical setting. To this purpose, we consider an $\mathcal{M}$-complete category $\mathbb{C}$ supplied with an $(\mathcal{E}, \mathcal{M})$-factorization structure for morphisms such that $\mathcal{M}$ is a fixed class of monomorphisms. We further assume that the preimage $f^*(-)$ preserves arbitrary joins for every morphism f in $\mathbb{C}$ and consider an interior operator i on $\mathbb{C}$ with respect to $\mathcal{M}$. In fact, the interior operator i provides enough structure for the abstract category $\mathbb{C}$ to be able to regard its objects as spaces and establish a general theory of some topological notions. We first start with further studies of interior operators on $\mathbb{C}$ with respect to $\mathcal{M}$; we discuss some of their properties, study their relations with the other operators such as closure, dual closure, neighbourhood operators and topogenous orders and present some examples. We then investigate the notions of closed, open, initial and final morphisms with respect to the interior operator i on the category $\mathbb{C}$. We also introduce and study a notion of hereditary interior operators on the abstract category $\mathbb{C}$ using the right adjoint of the preimage of a given morphism that improves the Castellini notion of hereditary interior operators presented in [Cas11]. We demonstrate that these operators behave as well as hereditary closure operators. Moreover, inspired by the works of [Cle01], we explore the general notions of connectedness and disconnectedness in the abstract category $\mathbb{C}$ by using a concept of interior operators and a relative notion of constant morphisms. We show that this notion behaves like the notion of connectedness via closure operators given in [Cle01] and extend most of the results of [CR10] to a wider categorical setting. The thesis is concluded with the interior theoretic approaches to the notions of compactness. Our consideration generalizes the usual approach to topological spaces with respect to the classical interior operator.

We now provide a survey of each chapter:

Chapter 1 establishes the categorical framework for subsequent chapters. A brief overview of some categorical concepts such as Galois connections, factorization structures, subobjects, images and preimages is given. We conclude the chapter by introducing a notion of dual images.

Chapter 2 deals with a further study of categorical interior operators. We present useful characterizations of the continuity condition of an interior operator. These characterizations are essential to explicitly define the notion of closed, open, initial and final morphisms with respect to an interior operator. Further properties of the notions of openness and codenseness with respect to an interior operator are given. Indeed, the codenseness is preserved by both images under $\mathcal{M}$-morphisms and dual images under $\mathcal{E}$-morphisms. We also show that the conglomerate of all interior operators on $\mathbb{C}$ with respect to $\mathcal{M}$ together with composition is a monoid which is compatible with its lattice structure and then present some properties of openness and codenseness relative to composites of interior operators. Moreover, we give a (partial) characterization of $\mathcal{M}$ ($\mathcal{E}$)-morphism in terms of a given interior operator. We conclude the chapter by investigating the relationships between interior operators with closure, dual closure, neighbourhood operators and topogenous orders and providing some examples of interior operators. We show that interior operators are special types of neighbourhood operators (or topogeneous orders) and under some natural conditions one can nicely move from a closure operator to an interior operator and vice versa.

Chapter 3 is devoted to the study of classes of morphisms with respect to an interior operator on an arbitrary category in which the preimage functor for any given morphism preserves arbitrary joins. By using the equivalent descriptions of the continuity condition of an interior operator i, we first explicitly define closed, open, initial and final morphisms relative to i. These morphisms defined via interior operators play a vital role throughout the investigation of the topological notions like connectedness, separatedness and compactness. They are shown to have their respective cancellation, composition and pullback stability properties. We give a (partial) characterization of each of these morphisms in terms of i-open subobjects, for an idempotent interior operator i. We also show that open $\mathcal{M}$-morphisms with respect to an interior operator i are the morphisms whose image commutes with the interior i. We then introduce and study a notion of quasi open morphisms with respect to i. In particular, it is shown that these morphisms are precisely the morphisms which reflect i-codensity and they generalize the i-open morphisms that are studied in [Cas15]. We also explore a notion of quotient maps by using interior operators. Indeed, their fundamental properties are presented. We show that the class of these maps ascends along both open and closed monomorphisms with respect to an interior operator. We conclude the chapter by investigating classes of morphisms with respect to dual closure operators and show that these classes of morphisms behave nearly like the classes of morphisms with respect to an interior operator.

Chapter 4 introduces a general notion of hereditary interior operators in terms of "dual images" on an arbitrary category in which the preimage functor for any given morphism preserves arbitrary joins. In [Cas11], the study of hereditary interior operators was proposed by a direct adaptation from the hereditary behaviour of the classical interior operator in general topology. However, these operators do not lead themselves to a natural and general construction in an abstract category. Furthermore, they do not act in the same manner as hereditary closure operators in the sense of [DG87]; see in particular, [Cas11, Examples 3.8. (b) and (c)], [Cas15, Corollary 2] and they can not be characterized as in Proposition 4.1.16. To this end, we introduce and study the notion of hereditary interior operators using the right adjoint of the preimage of a given morphism by assuming that each preimage commutes with the join of subobjects, as in [LTOC11]. In particular, we show that hereditary interior operators are the counterpart of the notion of hereditary closure operators. We obtain a characterization of heredity of a given interior operator i in terms of "initial embeddings" with respect to i. Moreover, we study the relationship between our hereditary and Castellini's (strongly) hereditary interior operators. The notion of dense morphisms with respect to an interior operator is also introduced and studied. We prove that the class of dense morphisms with respect to a hereditary interior operator is left cancellable with respect to the class $\mathcal{M}$ and the class of dense morphisms with respect to an idempotent interior operator is stable under composition. The chapter is concluded with the investigation of maximal interior operators.

Chapter 5 presents general interior theoretic approaches to connectedness in an arbitrary category $\mathbb{C}$ in which the preimage functor for any given morphism preserves arbitrary joins. We develop two possible notions of connectedness by using a concept of categorical interior operators in a more general categorical setting. The first section investigates the notion of connectedness and disconnectedness with respect to a given interior operator via a relative notion of constant morphisms. Indeed, by introducing the concept of coarse and fine objects with respect to a given interior operator i and a relative notion of constant morphisms we investigate the notions of connectedness and disconnectedness with respect to i on abstract categories in a fashion similar to [Cle01]. We show that our notion generalizes the notions of connectedness and disconnectedness with respect to a given interior operator on the category of topological spaces and continuous maps presented in [CR10]. Furthermore, we construct two Galois connections between the conglomerate of all interior operators on $\mathbb{C}$ with respect to $\mathcal{M}$ with the reverse order, the conglomerate of all full subcategories of $\mathbb{C}$ with inclusion order and the dual of the conglomerate of all full subcategories of $\mathbb{C}$ and prove that the Herrlich-Preuß-Arhangel'skii-Wiegandt (HPAW) "(left-constant, right-constant)" correspondence is the composition of the two Galois connections, under mild conditions on $\mathbb{C}$. In the second section we study connectedness with respect to a given interior operator based on using pseudocomplements in subobject semilattices. We use quotient maps, dense, open and final morphisms with respect to an interior operator to investigate this notion of connectedness. In fact, we show that under some natural conditions the connectedness defined is preserved by preimages of subobjects under quotient maps with respect to an interior operator.

Chapter 6 investigates categorical notions of compactness via interior operators in an arbitrary category $\mathbb{C}$ in which the preimage functor for any given morphism preserves arbitrary joins. We use interior operators to provide two possible categorical approaches of studying compactness such that both ways yield a number of results of the classical theory of compactness in topology as special cases. In the first section, following [Tho99, CGT04], we use closed morphisms with respect to a given interior operator i to study firstly stably i-closed morphisms, thence a notion of compactness relative to i on an arbitrary category $\mathbb{C}$. We establish properties similar to those of compactness with respect to a given closure operator studied in [CGT96]. Moreover, we present a relative notion of Hausdorff separation with respect to i on $\mathbb{C}$ and provide a property which links the two notions. The second section invesigates an internal notion of compact objects with respect to an interior operator following the Borel-Lebesgue definition of compact topological spaces. We define compactness of objects of the category $\mathbb{C}$ in a natural way and study some of its properties.

Some of the results presented in this thesis have been discussed in [AH19a, AH19b].

Prerequisites for reading this thesis include a basic knowledge of Topology, Algebra, Category Theory and Order and Lattice theory. For further references, we suggest the reader consult [Eng89, DF04, AHS90, DP02].

1. Preliminaries

Throughout the thesis, we consider a fixed finitely complete category $\mathbb{C}$ (in particular, $\mathbb{C}$ has a terminal object and admits finite products). Unless stated otherwise, all the objects and morphisms to be considered are assumed to belong to the category $\mathbb{C}$; $X \in \mathbb{C}$ and f in $\mathbb{C}$ will be used to denote X is an object of $\mathbb{C}$ and f is a morphism of $\mathbb{C}$, respectively. We use general categorical terminologies from [AHS90], while for categorical closure operators we refer to [DT95] or [Cas03] or [CGT04]. In this chapter, we discuss the concept of Galois connections, factorization structures, subobjects, images, preimages and dual images, which are required throughout the thesis. We start with the notion of Galois connections.

1.1 Galois connections

A Galois connection is a particular correspondence between preoreded sets or classes. Galois connections are generalizations of the correspondence between subgroups and subfields investigated in Galois theory. They enable us to move back and forth between two different structures. They also enable many proofs to be short, elegant and transparent and are effective tools for research. We recall that a reflexive and transitive relation is a preorder. As a result we have the following definition which is given in [DT95].

Definition 1.1.1. Given preordered classes P and Q, a Galois connection between P and Q is a pair of mappings $f : P \to Q$ and $g : Q \to P$ such that $f(x) \leq y \Leftrightarrow x \leq g(y)$ for all $x \in P,\ y \in Q$.

Remark 1.1.2. The maps f and g in Definition 1.1.1 are order preserving. Indeed, for $x, x' \in P$ such that $x \leq x'$ we have that $x' \leq g(f(x'))$, since $f(x') \leq f(x')$. Consequently, $x \leq g(f(x'))$ and hence $f(x) \leq f(x')$. Similarly, we can show that g is an order preserving map.

Since a preordered class can be viewed as a category, a Galois connection between two preordered classes is a special case of a pair of adjoint functors between two categories. In this context for the maps f and g which are given in Definition 1.1.1, we say that f is left adjoint of g or g is right adjoint of f and write $f \dashv g$. We also use the notation $P \underset{g}{\overset{f}{\rightleftarrows}} Q$ or (f, g) to denote a Galois connection between P and Q. Furthermore, the corresponding fixed points of the Galois connection $P \underset{g}{\overset{f}{\rightleftarrows}} Q$ are $p \in P$ and $q \in Q$ such that $f(p) = q$ and $p = g(q)$. More precisely, p and q are the left and right fixed points, respectively.

Remark 1.1.3. [Cas03]

- (a) Adjoints determine each other uniquely, up to the equivalence relation given by $p \cong q \Leftrightarrow p \leq q$ and $q \leq p$. Indeed, if (f, g) and (f, g') are Galois connections between P and Q then for any $q \in Q$ one has $g(q) \leq g(q) \Leftrightarrow f(g(q)) \leq q \Leftrightarrow g(q) \leq g'(q)$. Moreover, $g'(q) \leq g'(q) \Leftrightarrow f(g'(q)) \leq q \Leftrightarrow g'(q) \leq g(q)$. Therefore, we deduce that $g(q) \cong g'(q)$. Dual reasoning yields the uniqueness of the left adjoint.
- (b) The composition of Galois connections is a Galois connection.

Note that for a Galois connection $P \underset{g}{\overset{f}{\rightleftarrows}} Q$ of preodred classes P and Q with the least elements 0_P in P and 0_Q in Q and the largest elements 1_P in P and and 1_Q in Q, one has $f(0_P) \cong 0_Q$ and $g(1_Q) \cong 1_P$ since $0_P \leq g(0_Q)$ and $f(1_P) \leq 1_Q$. Therefore, the left adjoint is bottom-preserving and the right adjoint is top-preserving. The following lemma gives characterizations of Galois connections.

Lemma 1.1.4. [DT95] Let $f : P \to Q$ and $g : Q \to P$ be an arbitrary pair of maps of preodered classes. Then the following are equivalent:

(a) $f \dashv g$;

(b) f and g are monotone, and $p \leq (g \circ f)(p)$ and $(f \circ g)(q) \leq q$ for all $p \in P$ and $q \in Q$;

(c) f is monotone and $g(q) \cong \max\{p \in P : f(p) \leq q\}$ for all $q \in Q$;

(d) g is monotone and $f(p) \cong \min\{q \in Q : p \leq g(q)\}$ for all $p \in P$.

Proof. $(a) \Rightarrow (b)$: By Remark 1.1.2, both f and g are monotone. Furthermore, since $f(p) \leq f(p)$, one has $p \leq (g \circ f)(p)$, since $g(q) \leq g(q)$, one has $(f \circ g)(q) \leq q$ for all $p \in P$ and $q \in Q$.

$(b) \Rightarrow (c)$: Since $(f \circ g)(q) \leq q$ for all $q \in Q$, one has $g(q) \in \{p \in P : f(p) \leq q\}$, hence $g(q) \leq \max\{p \in P : f(p) \leq q\}$. Furthermore, for all $p \in P$ such that $f(p) \leq q$, one has $p \leq (g \circ f)(p) \leq g(q)$, hence $\max\{p \in P : f(p) \leq q\} \leq g(q)$. Therefore, $g(q) \cong \max\{p \in P : f(p) \leq q\}$.

$(c) \Rightarrow (d)$: Let $q \leq q^{'}$ in Q. Then $\{p \in P : f(p) \leq q\} \subseteq \{p \in P : f(p) \leq q^{'}\}$, hence $g(q) \cong \max\{p \in P : f(p) \leq q\} \leq \max\{p \in P : f(p) \leq q^{'}\} \cong g(q^{'})$. Thus g is monotone. Furthermore, for all $q \in Q$ such that $p \leq g(q)$, one has $f(p) \leq f(g(q)) \leq q$ since f is monotone and $g(q) \in \{p \in P : f(p) \leq q\} \Leftrightarrow (f \circ g)(q) \leq q$, hence $f(p) \leq \min\{q \in Q : p \leq g(q)\}$. One also has $\min\{q \in Q : p \leq g(q)\} \leq f(p)$ since $f(p) \in \{q \in Q : p \leq g(q)\}$ as $p \leq (g \circ f)(p) \cong \max\{x \in P : f(x) \leq f(p)\}$. Therefore, $f(p) \cong \min\{q \in Q : p \leq g(q)\}$.

$(d) \Rightarrow (a)$: Suppose $f(x) \leq y$ for $x \in P, y \in Q$. Then since g is monotone and $f(x) \in \{q \in Q : x \leq g(q)\}$, one has $x \leq (g \circ f)(x) \leq g(y)$. Furthermore, for $x \in P, y \in Q$ such that $x \leq g(y)$, one has $f(x) \leq (f \circ g)(y) \leq y$ since f is monotone and $f(g(y)) \cong \min\{q \in Q : g(y) \leq g(q)\}$. Hence, $f(x) \leq y \Leftrightarrow x \leq g(y)$ for all $x \in P,\ y \in Q$.

□

As an immediate consequence of Lemma 1.1.4 (d) one has $(f \circ g \circ f)(p) \cong f(p)$ and $(g \circ f \circ g)(q) \cong g(q)$ for all $p \in P, q \in Q$. As a result, f and g give a bijective correspondence between $f(P)$ and $g(Q)$. It turns out that every Galois connection gives rise to an isomorphism of certain sub-preorders. We are now ready to show that left adjoints preserve arbitrary joins and right adjoints preserve arbitrary meets.

Proposition 1.1.5. [DT95] Let $f \dashv g$. Then $f(\bigvee_{i \in I} x_i) \cong \bigvee_{i \in I} f(x_i)$ and $g(\bigwedge_{i \in I} y_i) \cong \bigwedge_{i \in I} g(y_i)$.

Proof. Since g is monotone and $\bigwedge_{i \in I} y_i \leq y_i$ for all $i \in I$, one has $g(\bigwedge_{i \in I} y_i) \leq g(y_i)$ for all $i \in I$, hence $g(\bigwedge_{i \in I} y_i) \leq \bigwedge_{i \in I} g(y_i)$. Furthermore, since $f \dashv g$ and $\bigwedge_{i \in I} g(y_i) \leq g(y_i)$ for all $i \in I$, one has $f(\bigwedge_{i \in I} g(y_i)) \leq y_i$ for all $i \in I$, hence $f(\bigwedge_{i \in I} g(y_i)) \leq \bigwedge_{i \in I} y_i$. Consequently, $\bigwedge_{i \in I} g(y_i) \leq g(\bigwedge_{i \in I} y_i)$. Therefore, g preserves meets. Dually, f preserves joins. □

The following result shows the conditions under which the converse of the above proposition is true.

Theorem 1.1.6. *[DT95] Let P and Q be preordered classes.*

(a) *If arbitrary joins exist in P. Then any map $f : P \to Q$ that preserves arbitrary joins has a right adjoint.*

(b) *If arbitrary meets exist in Q. Then any map $g : Q \to P$ that preserves arbitrary meets has a left adjoint.*

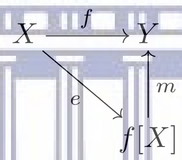
Proof. (a) Since arbitrary joins exist in P define g by $g(q) \cong \bigvee\{p \in P : f(p) \leq q\}$. Then one obtains $f(g(q)) \cong \bigvee\{f(p) \in Q : f(p) \leq q\} \leq q$ and $p' \leq g(f(p')) \cong \bigvee\{p \in P : f(p) \leq f(p')\}$. Furthermore, since f preserves suprema, for $p \leq p'$ in P one has $p' = p \vee p'$, hence $f(p') = f(p \vee p') \cong f(p) \vee f(p')$. Thus $f(p) \leq f(p')$, that is: f is monotone. For $q \leq q'$ in Q one also has $\{p \in P : f(p) \leq q\} \subseteq \{p \in P : f(p) \leq q'\}$, hence $g(q) \cong \bigvee\{p \in P : f(p) \leq q\} \leq \bigvee\{p \in P : f(p) \leq q'\} \cong g(q')$, that is: g is monotone. Therefore, by Lemma 1.1.4, $f \dashv g$.

(b) It follows by dualizing (a).

□

1.2 Factorization structures

In the category of **Sets** every function $f : X \to Y$ can be expressed as the composite of a surjective function e followed by an injective function m



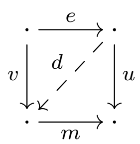
where $f[X]$ is the image of f, $e(x) = f(x)$ for all $x \in X$ and m is an inclusion map. Factorization structures are a generalization of this situation in category theory and allow us to define "image, preimage, dual image of a subobject ". In the sequel we use Mor($\mathbb{C}$), Sect($\mathbb{C}$), Mono($\mathbb{C}$), Retr($\mathbb{C}$), Epi($\mathbb{C}$) and Iso($\mathbb{C}$) to denote the classes of all morphisms, sections, monomorphisms, retractions, epimorphisms and isomorphisms in $\mathbb{C}$.

Definition 1.2.1. [AHS90] A factorization structure for morphisms in $\mathbb{C}$ is a pair $(\mathcal{E}, \mathcal{M})$ of any morphism classes in $\mathbb{C}$ such that

(a) $\mathcal{E}$ and $\mathcal{M}$ are closed under composition with isomorphisms from the left and right, respectively; that is: Iso($\mathbb{C}$) $\circ\, \mathcal{E} \subseteq \mathcal{E}$, $\mathcal{M} \circ$ Iso($\mathbb{C}$) $\subseteq \mathcal{M}$,

(b) $\mathbb{C}$ has $(\mathcal{E}, \mathcal{M})$-factorizations of morphisms; that is: f factors into an $\mathcal{E}$-morphism e followed by an $\mathcal{M}$-morphism m ($f = m \circ e \Leftrightarrow$ Mor($\mathbb{C}$) $= \mathcal{M} \circ \mathcal{E}$), and

(c) $\mathbb{C}$ has the unique $(\mathcal{E}, \mathcal{M})$-diagonalization property; that is: for each commutative solid-arrow square



with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there exists a unique diagonal d such that $d \circ e = v$ and $m \circ d = u$.

If $(\mathcal{E}, \mathcal{M})$ is a factorization structure for morphisms in $\mathbb{C}$ then $\mathbb{C}$ is called $(\mathcal{E}, \mathcal{M})$-structured.

Remark 1.2.2. As a consequence of the above Definition one has the following facts.

(a) The notion of factorization structure is self-dual in the sense that $\mathbb{C}$ is $(\mathcal{E}, \mathcal{M})$-structured if and only if $\mathbb{C}^{op}$ is $(\mathcal{M}, \mathcal{E})$-structured.

(b) $(\mathcal{E}, \mathcal{M})$-factorizations are unique up to isomorphism (essentially unique), that is: if $f \in C$ has $(\mathcal{E}, \mathcal{M})$-factorizations given by $f = m \circ e = m^{'} \circ e^{'}$ then there exists a unique isomorphism d such that $m = m^{'} \circ d$ and $e^{'} = d \circ e$; in particular, $m \cong m^{'}$. Indeed, this follows from the unique $(\mathcal{E}, \mathcal{M})$-diagonalization property.

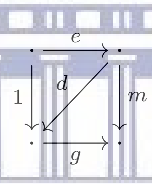
Examples 1.2.3. [AHS90]

(a) Both $(\text{Iso}(\mathbb{C}), \text{Mor}(\mathbb{C}))$ and $(\text{Mor}(\mathbb{C}), \text{Iso}(\mathbb{C}))$ are trivial factorization structures for morphisms in $\mathbb{C}$.

(b) $(\text{regular epimorphism}, \text{monomorphism})$ is a factorization structure for morphisms for categories **Set**, **Vect**, **Grp**, **Mon**.

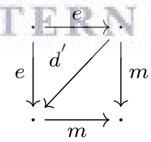
(c) $(\text{surjection}, \text{embedding})$ and $(\text{quotient}, \text{injection})$ are factorization structures for morphisms in **Top**, but $(\text{surjection}, \text{injection})$ is not.

Lemma 1.2.4. [AHS90] Let $\mathbb{C}$ be $(\mathcal{E}, \mathcal{M})$-structured with $e \in \mathcal{E}$ and $m \in \mathcal{M}$ such that the diagram



commutes. Then e is an isomorphism and $g \in \mathbb{M}$.

Proof. Since the diagram



commutes for $d^{'} = e \circ d$ and $d^{'} = 1$ then the unique diagonalization property yields $e \circ d = 1$. Therefore, $e \in \text{Iso}(\mathbb{C})$. Consequently, by Definition 1.2.1 (a), one has $g \in \mathcal{E} \cap \mathcal{M}$. ☐

Before we move to the next proposition let us recall the following from [Cas03].

Definition 1.2.5. A multiple pullback of a sink $(r_i : R_i \to X)_{i \in I}$ is a pair $(r, \mathbb{S})$ consisting of a morphism $r : R \to X$ and a source $\mathbb{S} = (R \xrightarrow{j_i} R_i)_{i \in I}$ in $\mathbb{C}$ such that $r = r_i \circ j_i$ for all $i \in I$ and for each pair $(r^{'}, \mathbb{S}^{'})$ with $r^{'} : R^{'} \to X$ a morphism and $\mathbb{S}^{'} = (R^{'} \xrightarrow{j_i^{'}} R_i)_{i \in I}$ a source in $\mathbb{C}$ for which $r^{'} = r_i \circ j_i^{'}$ for all $i \in I$, there exists a unique morphism $u : R^{'} \to R$ with $r^{'} = r \circ u$ and $j_i^{'} = j_i \circ u$ for all $i \in I$.

We have the following properties of the class $\mathcal{M}$ and $\mathcal{E}$.

Proposition 1.2.6. [AHS90] Let $\mathbb{C}$ be equipped with $(\mathcal{E},\mathcal{M})$-factorization structure for morphisms.

(a) $\mathcal{E}\cap\mathcal{M} = \mathsf{Iso}(\mathbb{C})$.

(b) Both $\mathcal{M}$ and $\mathcal{E}$ are closed under composition.

(c) $\mathcal{M}$ is weakly left-cancellable $(g\circ f, g\in\mathcal{M}\Rightarrow f\in\mathcal{M})$, left-cancellable with respect to Mon$(\mathbb{C})$ $(g\circ f\in\mathcal{M}$ and $g\in \text{Mon}(\mathbb{C})\Rightarrow f\in\mathcal{M})$ and right-cancellable with respect to Retr$(\mathbb{C})$ $(g\circ f\in\mathcal{M}$ and $f\in\text{Retr}(\mathbb{C})\Rightarrow g\in\mathcal{M})$.

(d) $\mathcal{M}$ is stable under pullback, that is: for any pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{q} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[f]{} & W \end{array}$$

$f\in\mathcal{M}\Rightarrow q\in\mathcal{M}$.

(e) $\mathcal{M}$ is stable under multiple pullback, that is: for any multiple pullback diagram

$$\begin{array}{ccc} R & & \\ {\scriptstyle j_i}\downarrow & \searrow^{r} & \\ R_i & \xrightarrow[r_i]{} & X \end{array}$$

one has $r\in\mathcal{M}$ whenever $r_i\in\mathcal{M}$ for all $i\in I$.

(f) $\mathcal{M}$ is closed under products, that is: if $r_i\in\mathcal{M}$ for each $i\in I$ then $\prod\limits_{i\in I} r_i\in\mathcal{M}$ (if the products exist).

Proof. The proof can be found in [AHS90]. □

The dual properties to (c) to (f) are possessed by the class $\mathcal{E}$.

The following theorem whose dual result is given in [AHS90] deals with the existence of factorization structures.

Theorem 1.2.7. *Let $\mathcal{M}$ be a class of morphisms in $\mathbb{C}$ such that it*

(a) *contains isomorphisms and is contained in Mono$(\mathbb{C})$,*

(b) *is closed under composition,*

(c) *is stable under pullback, and*

(d) *is stable under multiple pullback (= intersection).*

Then there is a uniquely determined class $\mathbb{E}$ of sinks in $\mathbb{C}$ for which $(\mathbb{E},\mathcal{M})$ is a factorization system for sinks in $\mathbb{C}$. Consequently, there is a uniquely determined class $\mathcal{E}$ of $\mathbb{C}$-morphisms that (considered as 1-sinks) belong to $\mathbb{E}$ such that $\mathbb{C}$ is $(\mathcal{E},\mathcal{M})$-structured.

Following [DT95], for a fixed class $\mathcal{M}$ of $\mathbb{C}$-monomorphisms we say that $\mathbb{C}$ has $\mathcal{M}$-pullbacks if $\mathcal{M}$ is stable under pullback, that is: pullbacks of $\mathcal{M}$-morphisms along $\mathbb{C}$-morphisms exist and belong to $\mathcal{M}$. We also say that $\mathbb{C}$ has $\mathcal{M}$-intersections if $\mathcal{M}$ is stable under multiple pullback. Furthermore, $\mathbb{C}$ is called $\mathcal{M}$-complete if $\mathbb{C}$ has both $\mathcal{M}$-pullbacks and $\mathcal{M}$-intersections.

Remark 1.2.8. In order to develop the theory of interior operators in the following chapters of the thesis we need a fixed class $\mathcal{M}$ of $\mathbb{C}$-monomorphisms, which is closed under composition and contains all $\mathbb{C}$-isomorphisms and the assumption that $\mathbb{C}$ is $\mathcal{M}$-complete. Consequently, by Theorem 1.2.7, there is a uniquely determined class $\mathcal{E}$ of morphisms in $\mathbb{C}$ such that $\mathbb{C}$ is $(\mathcal{E}, \mathcal{M})$-structured. Indeed, hereafter throughout the thesis, unless stated otherwise, we work with an $\mathcal{M}$-complete category $\mathbb{C}$ equipped with $(\mathcal{E}, \mathcal{M})$-factorization structure for morphisms such that $\mathcal{M}$ is a fixed class of monomorphisms. This in turn implies the features of $\mathcal{M}$ and $\mathcal{E}$ listed in Proposition 1.2.6.

1.3 Subobjects, Images, Preimages

A subobject of an object in a category is a concept analogous to the concept of a substructure of a mathematical structure. It generalizes concepts such as subsets from set theory, subgroups from group theory, subspaces from topology. Let us now formally define subobject of an object in $\mathbb{C}$.

Definition 1.3.1. [DT95] For a given $X \in \mathbb{C}$, subobjects of X is a class given by

$$\text{sub}X := \{r \in \mathcal{M} : r : R \to X\}.$$

Objects of sub$X$ are known as $\mathcal{M}$-subobjects of X. sub$X$ is a preordered class with order $r \leq s \Leftrightarrow r = s \circ t$ for some morphism t (which we shall denote by r_s);

$$\begin{array}{ccc} R & \xrightarrow{t=r_s} & S \\ & {\scriptstyle r}\searrow & \downarrow{\scriptstyle s} \\ & & X \end{array}$$

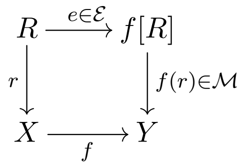
In fact, since s is monic, r_s is unique; since $\mathcal{M}$ is weakly left-cancellable, $r_s \in \mathcal{M}$. Furthermore, if $r \leq s$ and $s \leq r$ then r_s is an isomorphism between the domains R of r and S of s, hence r and s are isomorphic and we write $r \cong s$. We do not distinguish between isomorphic subobjects. In fact, the preorder relation " $\leq$ " induces an equivalence relation " $\cong$ " between $\mathcal{M}$-subobjects of X, which is given by $r \cong s \Leftrightarrow r \leq s$ and $s \leq r$. Thus sub$X$ modulo $\cong$ is a partially ordered class and we use the usual lattice theoretic terminology and notations such as $\wedge, \vee, \bigwedge, \bigvee$, etc in sub$X$. As a consequence of the above assumptions and terminologies one has, for each $X \in \mathbb{C}$, sub$X$ is a complete lattice with $0_X : O_X \to X$ and $1_X : X \to X$ as the least and greatest member of the lattice, respectively.

Definition 1.3.2. [Cas03] A subobject $r : R \to X$ is called an $\mathcal{M}$-intersection of a family $(r_i : R_i \to X)_{i \in I}$ in sub$X$ provided that r is the meet of $(r_i)_{i \in I}$, that is: $r \leq r_i$ for all $i \in I$ and any morphism that factors through each r_i must also factor through r.

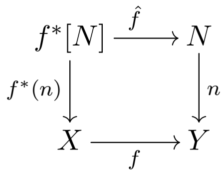
As the consequence of the definition, $r \cong \bigwedge\limits_{i \in I} r_i : \bigwedge\limits_{i \in I} R_i \to X$. One can also notice that intersections are unique up to isomorphism and categorically characterized as a multiple pullback.

Definition 1.3.3 (image, preimage)**.** [CGT04] For $f : X \to Y$ in $\mathbb{C}$, $r \in \text{sub}X$ and $n \in \text{sub}Y$.

(a) The image $f(r)$ of r under f is the $\mathcal{M}$-component of the $(\mathcal{E}, \mathcal{M})$-factorization of $f \circ r$, which is described by the commutative diagram below.



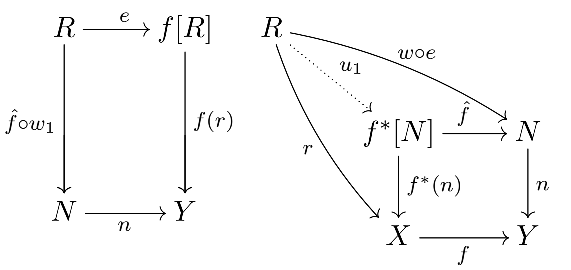
(b) The preimage $f^*(n)$ of n under f is the pullback of n along f, which is shown by the commutative diagram below. The pullback $\hat{f}$ of f is also called a restriction of f.



The fact that $(\mathcal{E}, \mathcal{M})$-factorizations and pullbacks are unique up to isomorphism implies both image and preimage are uniquely defined, up to isomorphism. As a consequence of the definition one has a pair of adjoint functors given as follows:

Proposition 1.3.4. [CGT04] Every morphism $f : X \to Y$ in $\mathbb{C}$ induces an image-preimage adjunction: $\text{sub}X \underset{f^*(-)}{\overset{f(-)}{\underset{\perp}{\rightleftarrows}}} \text{sub}Y$, that is: $f(r) \leq n \Leftrightarrow r \leq f^*(n)$ for all $r \in \text{sub}X$ and $n \in \text{sub}Y$.

Proof. Let $r \in \text{sub}X$ and $n \in \text{sub}Y$ and $(e, f(r))$ be the $(\mathcal{E}, \mathcal{M})$-factorization of $f \circ r$.
$(\Rightarrow)$ Assume $f(r) \leq n$. Then $\exists!\ w : f[R] \to N$ such that $f(r) = n \circ w$. Consequently, $f \circ r = f(r) \circ e = n \circ w \circ e$. Hence the solid arrow right diagram below



commutes. Thus, by the pullback property $\exists!\ u_1 : R \to f^*[N]$ such that $r = f^*(n) \circ u_1$ and $w \circ e = p \circ u_1$. Hence $r \leq f^*(n)$.
$(\Leftarrow)$ Suppose $r \leq f^*(n)$. Then $\exists!\ w_1 : R \to f^*[N]$ such that $r = f^*(n) \circ w_1$. As a result one has $f(r) \circ e = f \circ r = f \circ f^*(n) \circ w_1 = n \circ \hat{f} \circ w_1$ and hence the left diagram above commutes. Therefore, $\exists!\ d : f[R] \to N$ such that $f(r) = n \circ d$ and $\hat{f} \circ w_1 = d \circ e$. Thus $f(r) \leq n$. □

The following statements follow from Section 1.1 and Proposition 1.3.4.

Remark 1.3.5. Let $f : X \to Y$ be any morphism in $\mathbb{C}$, $r, s \in \text{sub}X$ and $n, k \in \text{sub}Y$. Then since sub$X$ and sub$Y$ are preordered classes one has

(a) $r \leq s \Rightarrow f(r) \leq f(s)$;

(b) $n \leq k \Rightarrow f^*(n) \leq f^*(k)$;

(c) $r \leq f^*(f(r))$ and $f(f^*(n)) \leq n$;

(d) $f(\bigvee_{i\in I} r_i) = \bigvee_{i\in I} f(r_i)$, where $r_i \in \text{sub}X$ for all $i \in I$;

(e) $f^*(\bigwedge_{i\in I} n_i) = \bigwedge_{i\in I} f^*(n_i)$, where $n_i \in \text{sub}Y$ for all $i \in I$;

(f) $f(0_X) \cong 0_Y$ and $f^*(1_Y) \cong 1_X$.

Next we see that the image of a subobject under a morphism in $\mathcal{M}$ can be described in terms of composition.

Remark 1.3.6. [DT95] Let $f : X \to Y$ be any morphism in $\mathbb{C}$. Then, the following statements hold.

(a) $f : X \to Y \in \mathcal{M} \Rightarrow f(r) \cong f \circ r$ for all $r \in \text{sub}X$. Indeed, if $f \in \mathcal{M}$ then for any $r \in \text{sub}X$, one has both $(1, f \circ r)$ and $(e, f(r))$ for some $e \in \mathcal{E}$ are $(\mathcal{E}, \mathcal{M})$-factorizations of $f \circ r$. Therefore, by Remark 1.2.2, one has $f(r) \cong f \circ r$.

(b) $f : X \to Y \in \mathcal{E} \Leftrightarrow f(1_X) \cong 1_Y$. Indeed, if $f \in \mathcal{E}$ then both $(f, 1_Y)$ and $(e, f(1_X))$ for some $e \in \mathcal{E}$ are $(\mathcal{E}, \mathcal{M})$-factorizations of $f \circ 1_X = f$. Therefore, by Remark 1.2.2, one has $f(1_X) \cong 1_Y$. Conversely, if $f(1_X) \cong 1_Y$ then $f = f \circ 1_X = f(1_X) \circ e \cong 1_Y \circ e = e \in \mathcal{E}$. Therefore, $f \in \mathcal{E} \Leftrightarrow f(1_X) \cong 1_Y$.

Remark 1.3.7. For each object $X \in \mathbb{C}$, sub$X$ has binary meets. Indeed, the meet of $s : S \to X$ and $t : T \to X$ in sub$X$ is given by $s \wedge t \cong s \circ s^*(t) \cong t \circ t^*(s)$, which is the diagonal of the pullback diagram

$$\begin{array}{ccc} S \wedge T & \longrightarrow & T \\ {\scriptstyle s^*(t)}\downarrow & & \downarrow{\scriptstyle t} \\ S & \underset{s}{\longrightarrow} & X \end{array}$$

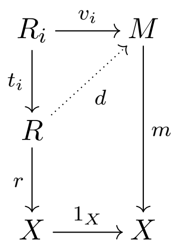
Note that since $\mathcal{M}$ is stable under pullback, one has $s^*(t) \in \mathcal{M}$. Consequently, since $\mathcal{M}$ is closed under composition, one has $s \wedge t \cong s \circ s^*(t) \in \mathcal{M}$.

More generally, as a consequence of our assumption on $\mathcal{M}$, for each $X \in \mathbb{C}$, sub$X$ is a complete lattice, infima are formed via intersections and suprema are formed via $(\mathcal{E}, \mathcal{M})$-factorizations. Indeed, the join of $(r_i)_{i\in I}$ in sub$X$ can be seen as the meet of all upper bounds of $(r_i)_{i\in I}$. In fact, the categorical property of join is given as follows.

Definition 1.3.8. [DT95] A subobject $r : R \to X$ is called an $\mathcal{M}$-union of a family $(r_i : R_i \to X)_{i\in I}$ in sub$X$ provided that $r_i \leq r$ for all $i \in I$ and any morphism m in sub$X$ such that $r_i \leq m$ for all $i \in I$ must satisfy the relation $r \leq m$, that is:

(a) there are morphisms $t_i : R_i \to R,\ i \in I$ with the property $r_i = r \circ t_i$ for all $i \in I$, and

(b) for any commutative solid diagram



in $\mathbb{C}$ with $m \in \mathcal{M}$ $\exists!$ $d : R \to M$ with $r = m \circ d$ and $v_i = d \circ t_i$ for all $i \in I$.

Consequently, we write $r \cong \bigvee_{i \in I} r_i : \bigvee_{i \in I} R_i \to X$. The following are formulas for images and preimages of composites.

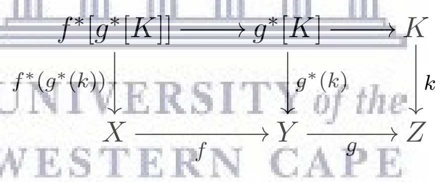
Proposition 1.3.9. [DT95] For morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathbb{C}$, one has

(a) $(g \circ f)(r) \cong g(f(r))$ for all $r \in \text{sub}X$;

(b) $(g \circ f)^*(k) \cong f^*(g^*(k))$ for all $k \in \text{sub}Z$.

Proof. (a) Let $r \in \text{sub}X$. Then by Definition 1.3.3 (a), one has $f \circ r = f(r) \circ e$, $g \circ f(r) = g(f(r)) \circ e^{'}$ and $g \circ f \circ r = (g \circ f)(r) \circ e^{''}$ for some $e, e^{'}, e^{''} \in \mathcal{E}$. Consequently, $(g \circ f)(r) \circ e^{''} = g \circ f \circ r = g \circ f(r) \circ e = g(f(r)) \circ e^{'} \circ e$. Hence, both $(e^{''}, (g \circ f)(r))$ and $(e^{'} \circ e, g(f(r)))$ are $(\mathcal{E}, \mathcal{M})$-factorizations of $g \circ f \circ r$. Note that $\mathcal{E}$ is closed under composition. Therefore, $(g \circ f)(r)) \cong g(f(r))$ since by Remark 1.2.2 $(\mathcal{E}, \mathcal{M})$-factorizations are unique up to isomorphism.

(b) Let $k \in \text{sub}Z$. Then by Definition 1.3.3 (b), one has $(g \circ f)^*(k)$ as the pullback of k along $g \circ f$ and the two pullback squares



Consequently, the outer rectangle is a pullback since the composition of pullbacks is a pullback. Hence, $f^*(g^*(k))$ is also the pullback of k along $g \circ f$. Therefore, $(g \circ f)^*(k) \cong f^*(g^*(k))$ since pullbacks are unique up to isomorphism.

□

In the sequel we use $\mathcal{E}^{'}$ and $\mathcal{E}^*$ to denote the class of morphisms in $\mathcal{E}$ that are stable under pullback along $\mathcal{M}$-morphisms and the largest pullback-stable subclass of $\mathcal{E}$, respectively. Hence, if $\mathcal{E}$ is stable under pullback then $\mathcal{E}$ is stable under pullback along $\mathcal{M}$-morphisms. Therefore, $f \in \mathcal{E} = \mathcal{E}^* \Rightarrow f \in \mathcal{E}^{'}$, that is: $\mathcal{E} = \mathcal{E}^* \subseteq \mathcal{E}^{'}$. The following result shows the image-preimage functors are partially inverse to each other under particular conditions.

Proposition 1.3.10. [GT00] For any $f : X \to Y$ in $\mathbb{C}$, one has:

(a) $f^*(f(r)) \cong r$ for all $r \in \text{sub}X$ provided that $f \in \mathcal{M}$ (or f is monic and $\mathcal{E}$ is stable under pullback along monomorphisms);

(b) $f(f^*(k)) \cong k$ for all $k \in \text{sub}Y$ provided that $f \in \mathcal{E}^{'}$.

Proof. (a) Suppose $f \in \mathcal{M}$. Then $f \in \text{Mono}(\mathbb{C})$, hence the diagram

$$\begin{array}{ccc} R & \xrightarrow{1_R} & R \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle f\circ r} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a pullback. In particular, r is the pullback of $f \circ r$ along f, that is: $r \cong f^*(f \circ r)$. Consequently, by Remark 1.3.6 (a), $r \cong f^*(f \circ r) \cong f^*(f(r))$. For the case f is monic and $\mathcal{E}$ is stable under pullback along monomorphisms, see [GT00].

(b) Suppose $f \in \mathcal{E}'$ and $k \in \text{sub}Y$. Then

$$\begin{array}{ccc} f^*[K] & \xrightarrow{\hat{f}} & K \\ {\scriptstyle f^*(k)}\downarrow & & \downarrow{\scriptstyle k} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a pullback diagram with $\hat{f} \in \mathcal{E}$. Hence $(\hat{f},\ k)$ is the $(\mathcal{E},\ \mathcal{M})$-factorization of $f \circ f^*(k)$. Consequently, $k \cong f(f^*(k))$ since the image of $f^*(k)$ under f is the $\mathcal{M}$ part of the $(\mathcal{E},\ \mathcal{M})$-factorization of $f \circ f^*(k)$ (see Definition 1.3.3).

□

Corollary 1.3.11. Let $f : X \to Y$ be in $\mathbb{C}$.

(a) $f \in \mathcal{M} \Rightarrow f^*(0_Y) \cong 0_X$.

(b) $f \in \mathcal{E}' \Rightarrow f(1_X) \cong 1_Y$. Furthermore, if $\mathcal{E} \subseteq \mathcal{E}'$ then the converse is also true.

Proof. (a) Suppose $f \in \mathcal{M}$ then Remark 1.3.5 and Proposition 1.3.10 (a) yield $f^*(0_Y) \cong f^*(f(0_X)) \cong 0_X$.

(b) Suppose $f \in \mathcal{E}'$ then by Remark 1.3.5 and Proposition 1.3.10 (b), one obtains $f(1_X) \cong f(f^*(1_Y)) \cong 1_Y$. Conversely, if $f(1_X) \cong 1_Y$ then by Remark 1.3.6(b) one has $f \in \mathcal{E}$, hence $f \in \mathcal{E}'$ since $\mathcal{E} \subseteq \mathcal{E}'$.

□

1.4 Dual images

This section is devoted to the notion of dual images. In order to deal with this concept we need to make the following further assumption. We assume that the preimage $f^*(-) : \text{sub}Y \to \text{sub}X$ preserves arbitrary joins for every morphism $f : X \to Y$ in $\mathbb{C}$, as in [LTOC11], throughout the section. Consequently, by Theorem 1.1.6 (a), f^* has a right adjoint f_*, which is given by $f_*(r) = \bigvee\{u \in \text{sub}Y : f^*(u) \leq r\}$. Hence, one has $f^*(k) \leq r$ if and only if $k \leq f_*(r)$ for all $r \in \text{sub}X$ and $k \in \text{sub}Y$.

Definition 1.4.1. Let $f : X \to Y$ be a morphism in $\mathbb{C}$ and $r \in \text{sub}X$. The dual image $f_*(r)$ of r is the greatest subobject $t : T \to Y$ such that the pullback $f^*(t) : f^*(T) \to X$ factors throuh r.

Remark 1.4.2. [RH14] The assumption that each preimage commutes with joins is not restrictive. Such a condition naturally arises in topological categories $\mathbb{C}$ over **Set**. Indeed, in a such category $\mathbb{C}$, for any map $f : X \to Y$, the preimage $f^*(-) : P(Y) \to P(X)$, which is given by $f^*(N) = \{r \in R : \ f(r) \in N\}$ for all $N \subseteq Y$, where $P(X)$ is the power set of X, commutes with joins. Consequently, $f^*(-)$ admits a right adjoint (or dual image) $f_*(-) : P(X) \to P(Y)$, which is given by $f_*(R) = \bigcup\{N \in P(Y) : f^*(N) \subseteq R\} = \{y \in Y : f^*(y) \subseteq R\} = Y \setminus f(X \setminus R)$ for any $R \subseteq X$. Note that $f_*(R)$ is the largest subset of Y whose preimage by f is contained in R. In fact, f^{-1} also admits left adjoint $f : P(X) \to P(Y)$ given by $f(R) = \{f(r) : r \in R\}$ for all $R \subseteq Y$.

As the consequence of the above assumption on preimages and Section 1.1 one has:

Remark 1.4.3. Let $f : X \to Y$ be any morphism in $\mathbb{C}$.

(a) f induces a preimage-dual image Galois connection: $\text{sub}Y \underset{f_*(-)}{\overset{f^*(-)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \text{sub}X$.

(b) $n \leq f_*(f^*(n))$ for all $n \in \text{sub}Y$ and $f^*(f_*(r)) \leq r$ for all $r \in \text{sub}X$.

(c) $f^*(0_Y) \cong 0_X$ and $f_*(1_X) \cong 1_Y$.

(d) f_* preserves meets. On the other hand, $\bigvee_{k\in K} f_*(r_k) \leq f_*\left(\bigvee_{k\in K} r_k\right)$ for any family $(r_k)_{k\in K}$ in $\text{sub}X$.

(e) $f_*(r) \vee n \leq f_*(r \vee f^*(n))$ for all $r \in \text{sub}X$ and $n \in \text{sub}Y$.

One says that a morphism $f : X \to Y$ reflects the least subobject 0_Y if $f^*(0_Y) \cong 0_X$, or equivalently, $f(m) \cong 0_Y \Leftrightarrow m \cong 0_X$ (see [HŠ11]). Consequently, by Remark 1.4.3 (c), any morphism in $\mathbb{C}$ reflects the least subobject. However, by Corollary 1.3.11, each subobject morphism reflects the least subobject without assuming that joins commute with preimage. Moreover, the above assumption on preimages produces the following.

Proposition 1.4.4. Let $f : X \to Y$ in $\mathbb{C}$, $r \in \text{sub}X$ and $n \in \text{sub}Y$.

(a) If $f \in \mathcal{M}$ (or f is monic and $\mathcal{E}$ is stable under pullback along monomorphisms) then $f(r) \leq f_*(r)$ and $f^*(f_*(r)) \cong r$.

(b) If $f \in \mathcal{E}'$ then $f_*(f^*(n)) \cong n$ and $f_*(r) \leq f(r)$.

(c) $f \in \mathcal{E}' \Rightarrow f_*(0_X) \cong 0_Y$.

Proof. (a) Let $f \in \mathcal{M}$ (or f is monic and $\mathcal{E}$ is stable under pullback along monomorphisms). Then by Proposition 1.3.10 (a), one has $f^*(f(r)) \leq r$, hence $f^* \dashv f_*$ gives $f(r) \leq f_*(r)$. Consequently, $r \cong f^*(f(r)) \leq f^*(f_*(r))$. Therefore, $f^*(f_*(r)) \cong r$ since by Remark 1.4.3 (b), $f^*(f_*(r)) \leq r$.

(b) Let $f \in \mathcal{E}'$. Then for any $k \in \text{sub}Y$ such that $f^*(k) \leq f^*(n)$, one has $f(f^*(k)) \leq f(f^*(n))$. Consequently, by by Proposition 1.3.10 (b), one has $k \cong f(f^*(k)) \leq f(f^*(n)) \cong n$. Hence, $f_*(f^*(n)) = \bigvee\{k \in \text{sub}Y : f^*(k) \leq f^*(n)\} \leq n$. Therefore, this together with Remark 1.4.3 (b) imply $f_*(f^*(n)) \cong n$. Moreover, by Remark 1.3.5 (c), $r \leq f^*(f(r))$. Hence, $f_*(r) \leq f_*(f^*(f(r))) \cong f(r)$.

(c) Suppose $f \in \mathcal{E}'$ then for any $n \in \text{sub}Y$ such that $f^*(n) \leq 0_X$, one has $n \cong f(f^*(n)) \leq f(0_X) = 0_Y$. Consequently, $f_*(0_X) = \bigvee\{n \in \text{sub}Y : f^*(n) \leq 0_X\} \cong 0_Y$. Of course, the assertion in (c) follows from (b) by setting $r = 0_X$.

□

The next corollary gives some characterizations of morphisms in the class $\mathcal{E}'$.

Corollary 1.4.5. The following three conditions are equivalent for a morphism $f : X \to Y$ in $\mathbb{C}$:

(a) $f \in \mathcal{E}'$;

(b) $f_*(r) \leq f(r)$ for all $r \in \text{sub}X$;

(c) $f_*(r) \cong f(f^*(f_*(r)))$ for all $r \in \text{sub}X$ provided that $\mathcal{E} \subseteq \mathcal{E}'$.

Proof. $(a) \Rightarrow (b)$: follows from Proposition 1.4.4(b).

$(b) \Rightarrow (c)$: Suppose $f_*(m) \leq f(m)$ for all $m \in \text{sub}X$ and $r \in \text{sub}X$. Then, in particular for $m = f^*(f_*(r))$, one has $f_*(r) \leq f_*(f^*(f_*(r))) \leq f(f^*(f_*(r)))$. Consequently, $f_*(r) \cong f(f^*(f_*(r)))$ since one always has $f(f^*(f_*(r))) \leq f_*(r)$.

$(c) \Rightarrow (a)$: Suppose $f_*(r) \cong f(f^*(f_*(r)))$ for all $r \in \text{sub}X$. Then, in particular for $r = 1_X$, $1_Y \cong f_*(1_X) \cong f(f^*(f_*(1_X))) \cong f(f^*(1_Y)) \cong f(1_X)$ and hence by Corollary 1.3.11(b), $f \in \mathcal{E}'$.

□

Remark 1.4.6. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathbb{C}$. Then, from the general theory of Galois connections one has that $(g \circ f)_* \cong g_* \circ f_*$ is the right adjoint of $(g \circ f)^* \cong f^* \circ g^*$.

Recall that a complement of $m \in \text{sub}X$ is an $\overline{m} \in \text{sub}X$ such that $m \wedge \overline{m} \cong 0_X$ and $m \vee \overline{m} \cong 1_X$. If such an $\overline{m}$ exists then m is said to be complemented in sub$X$. In the sequel we use $\overline{m}$ to denote the complement of $m \in \text{sub}X$. In categories with complemented subobjects one can express the dual image in terms of image which is given as follows.

Lemma 1.4.7. Let $f : X \to Y$ be any $\mathbb{C}$-morphism, sub$X$ be a Boolean algebra for each $X \in \mathbb{C}$.

(a) f^* preserves complements, that is: $f^*(\overline{n}) \cong \overline{f^*(n)}$ for all $n \in \text{sub}Y$.

(b) $f_*(m) = \bigvee\{n \in \text{sub}Y |\ f^*(n) \leq m\} \cong \overline{f(\overline{m})}$ for all $m \in \text{sub}X$.

Proof. (a) Let $n \in \text{sub}Y$. Then by the assumption on $f^*(-)$ and Remarks 1.4.3 (c) and 1.3.5 (c), one has $f^*(n) \wedge f^*(\overline{n}) \cong f^*(n \wedge \overline{n}) \cong f^*(0_Y) \cong 0_X$ and $f^*(n) \vee f^*(\overline{n}) \cong f^*(n \vee \overline{n}) \cong f^*(1_Y) \cong 1_X$.

(b) Let $m \in \text{sub}X$. Then by (a), one has $f^*(\overline{f(\overline{m})}) \cong \overline{f^*(f(\overline{m}))} \leq \overline{\overline{m}} = m$. Hence, $\overline{f(\overline{m})} \in \{n \in \text{sub}Y |\ f^*(n) \leq m\}$. Consequently, $\overline{f(\overline{m})} \leq \bigvee\{n \in \text{sub}Y |\ f^*(n) \leq m\} = f_*(m)$. Moreover, for $n \in \text{sub}Y$ such that $f^*(n) \leq m$, one has by (a), $\overline{m} \leq \overline{f^*(n)} \cong f^*(\overline{n})$. This in turn together with Remark 1.3.5 (c) implies $f(\overline{m}) \leq f(f^*(\overline{n})) \leq \overline{n}$. Consequently, $n = \overline{\overline{n}} \leq \overline{f(\overline{m})}$ for all $n \in \text{sub}Y$. Therefore, $f_*(m) = \bigvee\{n \in \text{sub}Y |\ f^*(n) \leq m\} \leq \overline{f(\overline{m})}$.

□

Remark 1.4.8. For each $X \in \mathbb{C}$, sub$X$ has a structure of a frame. Indeed, as is mentioned before each sub$X$ is a complete lattice. Moreover, one has $m \wedge \bigvee_{i \in I} r_i \cong m \circ m^*\left(\bigvee_{i \in I} r_i\right) \cong m \circ \bigvee_{i \in I} m^*(r_i) \cong m\left(\bigvee_{i \in I} m^*(r_i)\right) \cong \bigvee_{i \in I} m(m^*(r_i)) \cong \bigvee_{i \in I} m \circ m^*(r_i) \cong \bigvee_{i \in I} (m \wedge r_i)$ for all $m, r_i \in \text{sub}X$, $i \in I$ by Proposition 1.1.5 and Remarks 1.3.6(a), 1.3.7, hence meets distribute over arbitary joins in each sub$X$.

2. Interior Operators

In general topology, the complement of the interior is the closure of the complement and the complement of the closure is the interior of the complement. More generally, closure and interior operators characterize each other in a category with categorical transformation operator (see [Vor00]). In fact, there is a bijective correspondence between closure and interior operators on a category having a categorical transformation operator. Consequently, most of the theory of interior operators can be derived from that of closure operators and vice versa. But in general, it is not true that whatever one does with respect to closure operators may be done relative to interior operators and vice versa. Indeed, it is shown in [Vor00] that the category of groups does not have a categorical transformation, hence the two notions are not necessarily equivalent. Moreover, in any category for which all subobjects are normal, in particular, in all abelian categories (such as the category of modules over a ring, or the category of all abelian groups), while there is an abundance of closure operators there is a unique interior operator, which is the discrete one (see [DT15]). As a consequence the study of a categorical notion of interior operators for its own sake is interesting enough. In this chapter, we further study categorical interior operators which are introduced by S.J.R. Vorster in [Vor00]. We give important equivalent characterizations of the continuity condition of a given interior operator i and discuss some properties of the notions of openness and codenseness relative to i. By investigating the interaction of interior (or closure) operators with neighbourhood operators and topogenous orders, we also provide a nice way of moving from a given closure operator to an interior operator and vice versa. We close the chapter by looking at the relationship between interior and dual closure operators. As already mentioned in Remark 1.2.8, throughout this chapter (with the exception of Section 2.4) we work with an $\mathcal{M}$-complete category $\mathbb{C}$ equipped with an $(\mathcal{E},\mathcal{M})$-factorization structure for morphisms such that $\mathcal{M}$ is a fixed class of monomorphisms. Consequently, for each morphism f in $\mathbb{C}$, the image $f(-)$ is a left adjoint to the preimage $f^*(-)$ (see Proposition 1.3.4).

2.1 Basic properties of interior operators

The following definition of an interior operator in an arbitrary category was introduced by S.J.R. Vorster [Vor00].

Definition 2.1.1. An interior operator i on $\mathbb{C}$ with respect to $\mathcal{M}$ is a family

$$i = (i_X : \mathsf{sub}X \to \mathsf{sub}X)_{X\in\mathbb{C}}$$

of functions which are

(I_1) contractive: $i_X(r) \le r$,

(I_2) monotone: if $r \le s$ then $i_X(r) \le i_X(s)$, and which satisfy

(I_3) the continuity condition: $f^*(i_Y(k)) \le i_X(f^*(k))$,

for all $f : X \to Y$ in $\mathbb{C}$ and $r, s \in \mathsf{sub}X$ and $k \in \mathsf{sub}Y$.

From now on throughout the thesis, unless stated otherwise, we use i to denote an interior operator i on

$\mathbb{C}$ with respect to $\mathcal{M}$. As a consequence of the contraction property one has a canonical factorization

$$\begin{array}{ccc} i_X[R] & \xrightarrow{j_r} & R \\ & {\scriptstyle i_X(r)} \searrow & \downarrow {\scriptstyle r} \\ & & X \end{array}$$

with $i_X(r) \in \mathcal{M}$ and $j_r \in \mathcal{M}$ for every interior operator i on $\mathbb{C}$ and for every $\mathcal{M}$-subobject $r : R \to X$. The prototypical example of an interior operator is the Kuratowski interior operator k^{in} in the category **Top**, which assigns the usual topological interior R° to each subspace R of a topological space X, that is: $k^{\text{in}}_X(R) = \bigcup\{O \text{ open in } X : O \subseteq R\}$. The operator given by
$k^{*\text{in}}_X(R) = \bigcup\{C \text{ closed in } X : C \subseteq R\} = \{x \in R : k_X(\{x\}) \subseteq R\}$, where $k_X(\{x\})$ is the Kuratowski closure of $\{x\}$ in the topology of X is also an interior operator in **Top**. $k^{*\text{in}}$ is called the inverse Kuratowski interior operator. Note that the interior operator k^{in} ($k^{*\text{in}}$, resp.) are induced from the Kuratowski k (inverse Kuratowski k^*) closure operator given in [DT95] via set theoretic complementation. We will include additional examples later towards the end of this chapter.

Remark 2.1.2. Let i be an interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$, $f : X \to Y$ be any morphism in $\mathbb{C}, r \in \text{sub}X$ and $n \in \text{sub}Y$.

(a) Recall from [Cas03] that for a given categorical closure operator c, the continuity condition of c: $f(c_X(r)) \leq c_Y(f(r))$ yields the functorial property of closure operators (also called the diagonalization lemma in [DT95]). This property has played a crucial role in the development of the theory of closure operators and is essential in proving that any closure operator gives rise to an endofunctor of the arrow category $\mathcal{M}$, whose objects are the $\mathcal{M}$-morphisms. Unlike the categorical closure operator case, the continuity condition of i can not be described in terms of direct images, that is: the inequality $f(i_X(r)) \leq i_Y(f(r))$ is not true in general. Indeed, in the category of **Top**, for the set $\Re$ of real numbers with the usual Euclidean topology with $f(x) = x^2 + 1, R = [-2, 2]$ and i be the interior operator induced by the topology then $f(i(R)) = f((-2, 2)) = [1, 5)$ and $i(f(R)) = i(f[-2, 2]) = i[1, 5] = (1, 5)$. Thus $f(i(R)) \nsubseteq i(f(R))$. Consequently, the functorial property (also called the preservation property in [Cas15, Cas16]) does not hold for interior operators. Note that the preservation property of i holds true if and only if f is an open morphism with respect to i (see Remark 3.1.18(a)). Moreover, the inequality $i_Y(f(r)) \leq f(i_X(r))$ is not also true in general. In fact, if we consider $A = \{0, 1\}$, the two element indiscrete topological space, $B = \{1\}$, the singleton topological space, $r : R = \{0\} \to A$ as inclusion map and $g : A \to B$ as the only possible function then $i(g(R)) = B, g(i(R)) = \emptyset$ and hence $i(g(R)) \nsubseteq g(i(R))$; see [Cas11].

(b) The continuity condition of i: $f^*(i_Y(n)) \leq i_X(f^*(n))$ is equivalent to $[f^*(n) \leq r \Rightarrow f^*(i_Y(n)) \leq i_X(r)]$. Furthermore, the latter is equivalent to $[n \leq f_*(r) \Rightarrow i_Y(n) \leq f_*(i_X(r))]$ provided that each preimage preserves arbitrary joins in the category $\mathbb{C}$.

(c) If $\mathbb{C}$-morphisms reflect 0 then $t^{\text{in}} = (t^{\text{in}}_X)_{X \in \mathbb{C}}$ with $t^{\text{in}}_X(r) \cong 0_X$ for all $r \in \text{sub}X$ is an interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$. We call t^{in} the trivial interior operator.

We recall the following definition from [Vor00, Cas11, Cas15] which will be used in the sequel.

Definition 2.1.3. Given an interior operator i on $\mathbb{C}$ with respect to $\mathcal{M}$, we call

(a) an $\mathcal{M}$-subobject $r : R \to X$ i-open (in X) if $j_r : i_X[R] \to R$ is an isomorphism, that is: $i_X(r) \cong r$;

(b) i idempotent if for each $X \in \mathbb{C}$ and $r \in \text{sub}X$, $i_X(r)$ is i-open in X, that is: $i_X(i_X(r)) \cong i_X(r)$;

(c) i standard if for all $X \in \mathbb{C}$, 1_X is i-open subobject of X, that is: $i_X(1_X) \cong 1_X$ and

(d) i additive if $i_X(r \wedge s) \cong i_X(r) \wedge i_X(s)$ for all $r, s \in \text{sub}X$ and $X \in \mathbb{C}$.

In the sequel, we use O^i to denote the class of i-open $\mathcal{M}$-subobjects, respectively.

Remark 2.1.4. Let i be an interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$.

(a) The contraction property produces $i_X(0_X) \cong 0_X$, that is: 0_X is i-open in X for all $X \in \mathbb{C}$.

(b) The continuity condition of i implies the preimage of an i-open $\mathcal{M}$-subobject is an i-open $\mathcal{M}$-subobject, that is: O^i is stable under $\mathcal{M}$-pullback. In particular, $f^*(0_Y)$ is i-open in X since by (a) one has 0_Y is i-open in Y. Furthermore, O^i satisfies the left cancellation condition: $s \circ t \in O^i, s \in \mathcal{M} \Rightarrow t \in O^i$. In particular, the first factor of an i-open $\mathcal{M}$-subobject is i-open.

(c) The monotonicity condition of i yields arbitrary joins of i-open $\mathcal{M}$-subobjects are i-open $\mathcal{M}$-subobjects, that is: if $r_k \in O^i_X$ for all $k \in K$ then $\bigvee_{k \in K} r_k \in O^i_X$, where O^i_X denotes the class of i-open $\mathcal{M}$-subobjects of $X \in \mathbb{C}$.

(d) Let i be an idempotent interior operator. Then i is additive if and only if O^i is closed under binary meets. Moreover, i induces a Galois connection $O^i_X \underset{i}{\overset{\hookrightarrow}{\underset{\longleftarrow}{\bot}}} \text{sub}X$, that is: the class O^i_X of i-open subobjects of $X \in \mathbb{C}$ is a coreflective subcategory of sub$X$.

One easily sees that the trivial interior operator t^{in} is not standard unless $O_X \cong X$ for all $X \in \mathbb{C}$. On the other hand, the following result shows that there is a smallest standard interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$.

Proposition 2.1.5. Suppose each preimage preserves arbitrary joins in the category $\mathbb{C}$. The operator $s = (s_X : \text{ sub}X \to \text{sub}X)_{X \in \mathbb{C}}$ defined by $s_X(r) = \bigvee\{e^*(1_E) : X \overset{e}{\to} E \in \mathcal{E}, e_*(r) \cong 1_E\}$ for all $r \in$ sub$X$ is the smallest standard interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$.

Proof. (a) Let $e : X \to E \in \mathcal{E}$ such that $e_*(r) \cong 1_E$ for all $r \in \text{sub}X$. Then $e^*(1_E) \cong e^*(e_*(r)) \leq r$ for all $r \in \text{sub}X$. Thus $s_X(r) = \bigvee\{e^*(1_E) : X \overset{e}{\to} E \in \mathcal{E}, e_*(r) \cong 1_E\} \leq r$.

(b) Let $m \leq r$ in sub$X$. Then for all $e : X \to E \in \mathcal{E}$ such that $e_*(m) \cong 1_E$, one has $1_E \cong e_*(m) \leq e_*(r)$, hence $s_X(m) = \bigvee\{e^*(1_E) : X \overset{e}{\to} E \in \mathcal{E}, e_*(m) \cong 1_E\} \leq \bigvee\{e^*(1_E) : X \overset{e}{\to} E \in \mathcal{E}, e_*(r) \cong 1_E\} = s_X(r)$.

(c) Let $f : X \to Y$ be a morphism in $\mathbb{C}$, $r \in \text{sub}X$ and $n \in \text{sub}Y$ such that $n \leq f_*(r)$. Then we need to show that $s_Y(n) \leq f_*(s_X(r))$. To this end, let $g : Y \to Z \in \mathcal{E}$ with $g_*(n) \cong 1_Z$ and (e, m) with $e : X \to E \in \mathcal{E}$ and $m : E \to Z \in \mathcal{M}$ be the $(\mathcal{E}, \mathcal{M})$-factorization of $g \circ f$. Then $1_Z \cong g_*(n) \leq g_*(f_*(r))$. Consequently, $e_*(r) \cong m^*(m_*(e_*(r))) \cong m^*(g_*(f_*(r))) \cong m^*(1_Z) \cong 1_E$. Moreover, $1_Z \cong m_*(1_E) \leq m_*(e_*(e^*(1_E))) \cong g_*(f_*(e^*(1_E)))$, hence $g^*(1_Z) \leq f_*(e^*(1_E))$. Thus, for all $g : Y \to Z \in \mathcal{E}$ with $g_*(n) \cong 1_Z$ there exists $e : X \to E \in \mathcal{E}$ with $e_*(r) \cong 1_E$ such that $g^*(1_Z) \leq f_*(e^*(1_E))$. As a consequence,

$$\begin{aligned} s_Y(n) &= \bigvee\{g^*(1_Z) : Y \xrightarrow{g} Z \in \mathcal{E}, g_*(n) \cong 1_Z\} \\ &\leq \bigvee\{f_*(e^*(1_E)) : X \xrightarrow{e} E \in \mathcal{E}, e_*(r) \cong 1_E\} \\ &\leq f_*(\bigvee\{e^*(1_E) : X \xrightarrow{e} E \in \mathcal{E}, e_*(r) \cong 1_E\}) \\ &\cong f_*(s_X(r)). \end{aligned}$$

Therefore, by Remark 2.1.2(b), f is s-continuous. Hence, by (a), (b) and (c), s is an interior operator.

(d) Since $1_X \in \mathcal{E}$ and $(1_X)_*(1_X) \cong 1_X$, one has $1_X \cong 1_X^*(1_X) \leq \bigvee\{e^*(1_E) : X \xrightarrow{e} E \in \mathcal{E}, e_*(1_X) \cong 1_E\} = s_X(1_X)$, hence $s = (s_X)_{X \in \mathbb{C}}$ is standard.

(e) Let i be any other standard interior operator. Then for all $e : X \to E \in \mathcal{E}$ with $e_*(r) \cong 1_E$ for all $r \in$ sub$X$, one has $e^*(1_E) \cong e^*(i_E(1_E)) \leq i_X(e^*(1_E)) \cong i_X(e^*(e_*(r))) \leq i_X(r)$. Consequently, $s_X(r) = \bigvee\{e^*(1_E) : X \xrightarrow{e} E \in \mathcal{E}, e_*(r) \cong 1_E\} \leq i_X(r)$. Therefore, s is the least standard interior operator.

□

Remark 2.1.6. (a) In **Top** with $\mathcal{M}$ the class of embeddings, the indiscrete interior operator given by $i_X(X) = X$ and $i_X(R) = \emptyset$ for all $R \subset X \in$ **Top** is the smallest standard interior operator. We also note that this operator is the smallest standard interior operator on **Set** with respect to $\mathcal{M}$ = the class of injective maps.

(b) Let each preimage preserves arbitrary joins in the category $\mathbb{C}$. Then $\mathbb{C}$ has at least three interior operators, namely the trivial t^{in}, the indiscrete s and the discrete d^{in} interior operators such that $t^{\text{in}} \leq s \leq d^{\text{in}}$. We call an interior operator proper p^{in} if it is not isomorphic to any of these.

A useful characterization of the continuity condition of an interior operator i is given in the following proposition.

Proposition 2.1.7. Let $f : X \to Y$ be any morphism in $\mathbb{C}$, $m \in$ sub$X$ and $n \in$ sub$Y$. If preimages commute with the joins in the category $\mathbb{C}$ then for a given interior operator i on $\mathbb{C}$ with respect to $\mathcal{M}$, the following statements are equivalent:

(a) f is i-continuous, that is: $f^*(i_Y(n)) \leq i_X(f^*(n))$;

(b) $i_Y(f_*(m)) \leq f_*(i_X(m))$;

(c) $f^*(i_Y(f_*(m)) \leq i_X(m)$;

(d) $i_Y(n) \leq f_*(i_X(f^*(n)))$.

Proof. Let $m \in$ sub$X$ and $n \in$ sub$Y$.

(a) $\Rightarrow$ (b): Suppose $f^*(i_Y(n)) \leq i_X(f^*(n))$ then the adjointness of preimage and dual image (f^*, f_*) yields $f^*(i_Y(f_*(m))) \leq i_X(f^*(f_*(m))) \leq i_X(m)$, hence $i_Y(f_*(m)) \leq f_*(i_X(m))$.

(b) $\Rightarrow$ (c): This follows from the fact that $f^* \dashv f_*$.

$(c) \Rightarrow (d)$: Putting $m := f^*(n)$ in c, one obtains $f^*(i_Y(f_*(f^*(n)))) \leq i_X(f^*(n))$. Therefore, $i_Y(n) \leq i_Y(f_*(f^*(n))) \leq f_*(i_X(f^*(n)))$ by adjointness.

$(d) \Rightarrow (a)$: This is obtained from adjointness of (f^*, f_*).

□

We obtain the following property of $\mathcal{M}$ $(\mathcal{E}')$-morphisms from Propositions 1.4.4 and 2.1.7.

Corollary 2.1.8. Let preimages commute with arbitrary joins in the category $\mathbb{C}$, $f : X \to Y$ be a morphism in $\mathbb{C}$ and i be an interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$.

(a) $f \in \mathcal{M} \Rightarrow i_Y(f(m)) \leq f_*(i_X(m)) \leq f_*(m)$ for all $m \in$ sub$X$.

(b) $f \in \mathcal{E}' \Rightarrow i_Y(f_*(m)) \leq f(i_X(m)) \leq f(m)$ (or $f^*(i_Y(f_*(m))) \leq i_X(m) \leq m$) for all $m \in$ sub$X$. In particular, one has $i_Y(f_*(0_X)) \cong 0_Y$ for all $f \in \mathcal{E}$. Furthermore, if i is standard and $\mathcal{E} \subseteq \mathcal{E}'$ then the converse is true.

Proof. (a) Suppose $f \in \mathcal{M}$. Then by Propositions 1.4.4(a) and 2.1.7 and the contraction property of i one has $i_Y(f(m)) \leq i_Y(f_*(m)) \leq f_*(i_X(m)) \leq f_*(m)$ for all $m \in$ sub$X$.

(b) Suppose $f \in \mathcal{E}$. Then by Propositions 1.4.4(b) and 2.1.7 and the contraction property of i one has $i_Y(f_*(m)) \leq f_*(i_X(m)) \leq f(i_X(m)) \leq f(m)$ for all $m \in$ sub$X$. The converse follows by setting $m = 1_X$. Conversely, assume that $i_Y(f_*(m)) \leq f(i_X(m))$ for all $m \in$sub$X$. Since i is standard then for $m = 1_X$ one has $1_Y \cong i_Y(1_Y) \cong i_Y(f_*(1_X)) \leq f(i_X(1_X)) \cong f(1_X)$, hence $f(1_X) \cong 1_Y$. Consequently, by Remark 1.3.6(b) $f \in \mathcal{E}$. Therefore, $f \in \mathcal{E}'$ since $\mathcal{E} \subseteq \mathcal{E}'$.

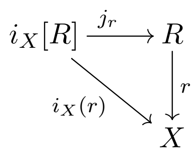
□

Next we introduce some basic properties of the notion of codenseness with respect to an interior operator i on $\mathbb{C}$ with respect to $\mathcal{M}$. We begin with the following:

Remark 2.1.9. Recall from [Eng89] that a subset R of a topological space X is called codense in X if the complement $X \setminus R$ of R in X is dense in X, that is: if $k_X(X \setminus R) = X$, which is equivalent to $k^{\text{in}}_X(R) = \emptyset$, where k and k^{in} are the Kuratowski closure and interior operators, respectively.

The above description of codenseness in terms of the Kuratowski interior operator motivates the following:

Definition 2.1.10. Given an interior operator i, we say that an $\mathcal{M}$-subobject $r : R \to X$ is i-codense (also called i-isolated in [CR10, Cas11]) in X if its i-interior is isomorphic to 0_X, that is: if $i_X(r) \cong 0_X$.



A notion of i-codense subobjects was used in [CR10] to define indiscrete objects with respect to an interior operator i in the category **Top** of topological spaces. We use C^i to denote the class of i-codense $\mathcal{M}$-subobjects.

Remark 2.1.11. (a) An $\mathcal{M}$-subobject $r : R \to X$ is i-codense in X if and only if $j_r : i_X[R] \to R$ is isomorphic to 0_R. Indeed, $i_X(r) \cong 0_X \Leftrightarrow r \circ j_r \cong 0_X \cong r(0_R) \cong r \circ 0_R \Leftrightarrow j_r \cong 0_R$ since r is monic.

(b) An i-codense and i-open subobject is a least subobject. Indeed, let m be an i-open and i-codense subobject of X then $m \cong i_X(m) \cong 0_X$. Note that since $i_X(0_X) \cong 0_X$, 0_X is both i-open and i-codense subobject of X.

(c) The monotonicity condition of i also yields arbitrary meets of i-codense $\mathcal{M}$-subobjects are i-codense $\mathcal{M}$-subobjects, that is: if $r_k \in C^i_X$ for all $k \in K$ then $\bigwedge_{k\in K} r_k \in C^i_X$, where C^i_X denotes the class of i-codense $\mathcal{M}$-subobjects of $X \in \mathbb{C}$ (see [Cas11]).

(d) Let sub$X$ be a Boolean algebra for every $\mathbb{C}$-object X and for every $\mathbb{C}$-morphism f let f^* preserve complements. Let c be a closure operator and i^c be the induced interior operator from c given by $i^c_X(m) = \overline{c_X(\overline{m})}$ for all $m \in$ sub$X$, where $\overline{m}$ denotes the complement of m. Then an $\mathcal{M}$-subobject $r : R \to X$ is i^c-codense in X if and only if $\overline{r}$ is c-dense in X; see for example [DT95].

Examples 2.1.12. (a) In the category **Top**, Q and $\Re \setminus Q$ are codense with respect to the interior operator induced by the Euclidean topology. In fact, for the Kuratowski interior operator k^{in} of **Top**, k^{in}-codense for a subspace inclusion $R \hookrightarrow X$ means codense in the usual topological sense.

(b) Consider the normal interior operator n given by $n_G(H) = \bigvee\{N \trianglelefteq G : N \le H\}$ on the category **Grp**. Then every subgroup $H \neq G$ of a simple group G is n-codense in G. On the other hand, a Dedekind group G has only the trivial subgroup $\{e_G\}$ which is n-codense in G.

(c) For any category $\mathbb{C}$, the least subobject 0_X of $X \in \mathbb{C}$ is the only subobject which is codense in X with respect to the discrete interior operator on $\mathbb{C}$.

The following partial characterization of $\mathcal{M}$-morphisms will be used in our next proof.

Lemma 2.1.13. Let i be an interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$ and let $f : X \to Y$ be a morphism in $\mathcal{M}$. Then $i_Y(f(m)) \le f(i_X(m))$ for all $m \in$ sub$X$.

Proof. Since $f(m) \cong f \circ m \le f$, one has $i_Y(f(m)) \le i_Y(f) \le f$. Consequently, $i_Y(f(m)) \cong f \wedge i_Y(f(m)) \cong f \circ f^*(i_Y(f(m))) \le f \circ i_X(f^*(f(m))) \cong f \circ i_X(m) \cong f(i_X(m))$. □

For any given interior operator i, i-codenseness is preserved by images under $\mathcal{M}$-morphisms:

Proposition 2.1.14. Let $f : X \to Y$ be a morphism in $\mathbb{C}$, $r \in$ sub$X$ and $n \in$ sub$Y$.

(a) If r is i-codense in X and $f \in \mathcal{M}$, then $f(r)$ is i-codense in Y. That is: $\mathcal{M}$-morphisms map i-codense $\mathcal{M}$-subobjects to i-codense $\mathcal{M}$-subobjects.

(b) If $f^*(n)$ is i-codense in X and $f \in \mathcal{E}'$, then n is i-codense in Y.

Proof. (a) Indeed, since $f \in \mathcal{M}$, one has $i_Y(f(r)) \le f(i_X(r))$ by Lemma 2.1.13, hence $i_Y(f(r)) \le f(i_X(r)) \cong f(0_X) \cong 0_Y$.

(b) If $i_X(f^*(n)) \cong 0_X$ and $f \in \mathcal{E}'$, then $i_Y(n) \cong f(f^*(i_Y(n))) \le f(i_X(f^*(n))) \cong f(0_X) \cong 0_Y$.

□

Remark 2.1.15. Let i be an interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$.

(a) The class of i-codense subobjects is stable under composition with the class $\mathcal{M}$ from the left. Indeed, let $t : T \to S$ be an i-codense $\mathcal{M}$-subobject of S and $s : S \to X$ be an $\mathcal{M}$-subobject of X. Since $s \circ t \cong s(t)$, by Proposition 2.1.14(a), one has $s \circ t$ is an i-codense $\mathcal{M}$-subobject of X.

(b) The class of i-codense subobjects is stable under composition with the class $\mathcal{M}$ from the right. To this end, let $s : S \to X$ be an i-codense subobject of X and $t : T \to S$ be a subobject of S. Then $s \circ t \leq s$, hence $i_X(s \circ t) \leq i_X(s) \cong 0_X$. Therefore, $i_X(s \circ t) \cong 0_X$.

(c) Let r and s be subobjects of X such that $r \leq s$. If s is i-codense in X then so is r. Indeed, this is clearly equivalent to (b).

(d) Let preimages commute with arbitrary joins in the category $\mathbb{C}$, $f : X \to Y$ be a morphism in $\mathbb{C}$, $m \in \mathsf{sub}X$ and $n \in \mathsf{sub}Y$. Since $f^*(f_*(m)) \leq m$ and $n \leq f_*(f^*(n))$, it follows from (c) that the statements

(i) if $f^*(n)$ is i-codense in X, then n is i-codense in Y,

(ii) if m is i-codense in X, then $f_*(m)$ is i-codense in Y

are equivalent. Consequently, Proposition 2.1.14(b) yields that i-codenseness is preserved by dual images under $\mathcal{E}'$-morphisms, that is: if $f \in \mathcal{E}'$ then (ii) holds. Of course, Propositions 2.1.7(b) and 1.4.4(c) imply that $\mathcal{E}'$-morphisms map i-codense subobjects to i-codense subobjects.

Some additional properties of the notion of codenseness with respect to i will be discussed later in this section, section 3.2 and section 4.1.

In the remainder of this section we focus on operations on interior operators. We use $\mathsf{INT}(\mathbb{C}, \mathcal{M})$ to denote the conglomerate of all interior operators on $\mathbb{C}$ with respect to $\mathcal{M}$. $\mathsf{INT}(\mathbb{C}, \mathcal{M})$ is preordered as follows. For $i, j \in \mathsf{INT}(\mathbb{C}, \mathcal{M}), i \leq j \Leftrightarrow i_X(m) \leq j_X(m)$ for all $m \in \mathsf{sub}X, X \in \mathbb{C}$, where $\leq$ is the order on subobjects. Consequently, meets and joins of non-empty families of interior operators exist and are formed "pointwise", which is shown in the following result (see [Cas11]).

Proposition 2.1.16. $\mathsf{INT}(\mathbb{C}, \mathcal{M})$ is a large complete lattice.

Proof. Let $(i_l)_{l \in L}$ be a non-empty family of interior operators. Define $\left(\bigwedge_{l \in L} i_l\right)_X (m) := \bigwedge_{l \in L} (i_l)_X (m)$ for all $m \in \mathsf{sub}X$, $X \in \mathbb{C}$. Then one can observe that $\bigwedge_{l \in L} i_l \in \mathsf{INT}(\mathbb{C}, \mathcal{M})$ and is the infimum of the family $(i_l)_{l \in L}$ in $\mathsf{INT}(\mathbb{C}, \mathcal{M})$. Consequently, by the general property of a preordered class, $\bigvee_{l \in L} i_l$ exists in $\mathsf{INT}(\mathbb{C}, \mathcal{M})$. □

The discrete interior operator d^{in} given by $d^{\mathsf{in}}_X(r) \cong r$ for all $r \in \mathsf{sub}X$ is the largest element in $\mathsf{INT}(\mathbb{C}, \mathcal{M})$. In fact, d^{in} is the greatest standard interior operator. If $\mathbb{C}$-morphisms reflect 0 then the trivial interior operator t^{in} is the least element in $\mathsf{INT}(\mathbb{C}, \mathcal{M})$. Furthermore, if preimages commute with the joins in the category $\mathbb{C}$ then the joins of the family $(i_l)_{l \in L}$ are explicitly expressed as: $\left(\bigvee_{l \in L} i_l\right)_X (m) := \bigvee_{l \in L} (i_l)_X (m)$ for all $m \in \mathsf{sub}X$, $X \in \mathbb{C}$. Below we list some properties which are stable under meet or join.

Remark 2.1.17. (a) The property of being standard (additive, resp.) is stable under arbitrary meet, that is: for any family $(i_k)_{k\in K}$ of standard (additive, resp.) interior operators $\bigwedge_{k\in K} i_k$ is also standard (additive, resp.). On the other hand, if $\exists k \in K$ such that i_k is standard interior operator then so is $\bigvee_{k\in K} i_k$.

(b) The property idempotency is stable under arbitrary join (see [Cas11]).

Definition 2.1.18. [Cas11] The composite of two interior operators i and j on $\mathbb{C}$ with respect to $\mathcal{M}$ is given by composing the maps i_X and j_X: $(j \circ i)_X(r) := j_X(i_X(r))$ for all $r \in \text{sub}X$.

One readily checks that the composition $j \circ i$ of two interior operators i and j on $\mathbb{C}$ with respect to $\mathcal{M}$ is in fact an interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$. Consequently, $(\text{INT}(\mathbb{C}, \mathcal{M}), \circ)$ is a monoid which is compatible with its lattice structure. More precisely, one has the following properties:

Lemma 2.1.19. Let $i, j, k \in \text{INT}(\mathbb{C}, \mathcal{M})$.

(a) The composition $\circ$ is associative, that is: $(k \circ j) \circ i = k \circ (j \circ i)$;

(b) The discrete interior operator d is an identity element for the composition $\circ$, that is: $d \circ i = i = i \circ d$;

(c) If preimages commute with the joins in the category $\mathbb{C}$ then the trivial interior operator t is absorbing, that is: $i \circ t \cong t \cong t \circ i$;

(d) If $i \leq j$ then the monotonicity properties $i \circ k \leq j \circ k$ and $k \circ i \leq k \circ j$ hold;

(e) $(\bigwedge_{l\in L} j_l) \circ i \cong \bigwedge_{l\in L} j_l \circ i$ and if preimages commute with the joins in the category $\mathbb{C}$, $(\bigvee_{l\in L} j_l) \circ i \cong \bigvee_{l\in L} j_l \circ i$;

(f) $j \circ (\bigwedge_{l\in L} i_l) \leq \bigwedge_{l\in L} j \circ i_l$ and $j \circ (\bigvee_{l\in L} i_l) \leq \bigvee_{l\in L} j \circ i_l$.

Remark 2.1.20. (a) If both i and j are standard (additive, resp.) interior operators then $j \circ i$ is also standard (additive, resp.). However, the composite of idempotent interior operators need not be idempotent. Indeed, both the inverse Kuratowski interior operator $k^{*\text{in}}$ and the Kuratowski interior operator k^{in} are idempotent interior operators in **Top** but the composition $k^{*\text{in}} \circ k^{\text{in}}$ is not idempotent. To see this, consider $(X = \{1, 2, 3\}, \tau_X = \{\emptyset, \{2\}, \{2, 3\}, \{1, 2\}, X\})$ in **Top**. Then for $R = \{1, 2\}$, one has $k_X^{\text{in}}(R) = R$, hence $k_X^{*\text{in}}(k_X^{\text{in}}(R)) = k_X^{*\text{in}}(R) = \{1\}$. On the other hand, $k_X^{*\text{in}}(k_X^{\text{in}}(k_X^{*\text{in}}(k_X^{\text{in}}(R)))) = k_X^{*\text{in}}(k_X^{\text{in}}(\{1\})) = k_X^{*\text{in}}(\emptyset) = \emptyset$.

(b) Let $i, j, k \in \text{INT}(\mathbb{C}, \mathcal{M})$. If k is additive then one has the distributive law: $k \circ (j \wedge i) \cong (k \circ j) \wedge (k \circ i)$.

The following facts follow immediately from the respective definitions.

Proposition 2.1.21. Let $i, j \in \text{INT}(\mathbb{C}, \mathcal{M})$ such that $i \leq j$ and $r \in \text{sub}X$.

(a) If r is j-codense then it is i-codense, that is: $i \leq j \Rightarrow C^j \subseteq C^i$.

(b) If r is i-open then it is j-open, that is: $i \leq j \Rightarrow O^i \subseteq O^j$.

and for $(i_l)_{l\in L} \subseteq \text{INT}(\mathbb{C}, \mathcal{M})$, as is observed in [Cas11], one has:

(c) r is $\bigwedge_{l\in L} i_l$-open if and only if r is i_l-open for all k;

(d) If preimages commute with the joins in the category $\mathbb{C}$, then r is $\bigvee_{l\in L} i_l$-codense if and only if r is i_l-codense for all l.

The following lemma deals with openness and codenseness for composites.

Lemma 2.1.22. Let $i, j \in \mathsf{INT}(\mathbb{C}, \mathcal{M})$.

(a) $O^i = O^{i\circ i}$.

(b) $C^i \subseteq C^{j\circ i}$.

(c) $i \circ j \leq j \wedge i$. Furthermore, $j \wedge i \leq j \vee i$, provided that preimages commute with arbitrary joins in the category $\mathbb{C}$.

Proof. (a) Since $i \circ i \leq i$ then by Proposition 2.1.21(b) one obtains $O^{i\circ i} \subseteq O^i$. On the other hand, $r \cong i_X(r)$ implies $r \cong i_X(r) \cong i_X(i_X(r))$, hence $O^i \subseteq O^{i\circ i}$.

(b) Since $j \circ i \leq i$ then by Proposition 2.1.21(a) one obtains $C^i \subseteq C^{j\circ i}$.

(c) From Lemma 2.1.19(b) and (f), we obtain $i \circ j = i \circ (j \wedge d) \leq (i \circ j) \wedge (i \circ d) \leq j \wedge i \leq j \vee i$.

□

Given an interior operator i, Lemma 2.1.22(b) gives $I^i \subseteq I^{i\circ i}$. However, $I^{i\circ i}$ is not equal to I^i in general. Indeed, for $(X = \{1, 2, 3\}, \tau_X = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}) \in$ **Top** and the interior operator $\Theta^{\text{in}}_X(R) = \{r \in R : \exists$ a open neighbourhood U_r of r in X such that $k_X(U_r) \subseteq R\}$, where $k_X(U_r)$ is the Kuratowski closure of U_r, defined in [CR10], one has $\theta^{\text{in}}_X(\{1, 3\}) = \{1\}$ and $\theta^{\text{in}}_X\left(\theta^{\text{in}}_X(\{1, 3\})\right) = \theta^{\text{in}}_X(\{1\}) = \emptyset$. Therefore, $\{1, 3\}$ is $\theta^{\text{in}} \circ \theta^{\text{in}}$-codense but not θ^{in}-codense in X.

Theorem 2.1.23. *Let $i, j \in$ INT$(\mathbb{C}, \mathcal{M})$. Then*

(a) $O^{i\circ j} = O^{j\circ i} = O^{i\wedge j} = O^i \cap O^j$.

(b) $C^{i\vee j} = C^i \cap C^j$, *provided that preimages commute with arbitrary joins in the category* $\mathbb{C}$.

Proof. (a) From $i \circ j \leq j \wedge i$ (see Lemma 2.1.22(c)), we obtain $O^{i\circ j} \subseteq O^{j\wedge i}$. On the other hand, let r be $j \wedge i$-open in $X \in \mathbb{C}$. Then $r \cong (j \wedge i)_X(r) = j_X(r) \wedge i_X(r) \leq j_X(r), i_X(r)$, hence $r \cong i_X(r) \cong j_X(r)$. Consequently, $(j \circ j)_X(r) \cong r$. Thus $O^{j\wedge i} \subseteq O^{i\circ j}$.

(b) From $i, j \leq i \vee j$, we obtain $C^{i\vee j} \subseteq C^i, C^j$ (see Proposition 2.1.21(a)), hence $C^{i\vee j} \subseteq C^i \cap C^j$. The other inclusion is obvious.

□

2.2 Interior, neighbourhood operators and topogenous orders

Inspired by the categorical study of convergence, a notion of neighbourhood with respect to a categorical closure operator was introduced in [GŠ05] and then subsequently studied in [Šla08, GŠ09, Šla11]. The formal theory of categorical neighbourhood operators was introduced by Holgate and Šlapal [HŠ11]. Since then, these operators were used to study a categorical notion of convergence, separation and compactness. The following is a list of some of the papers that have contributed to the development of neighbourhood operators: [HŠ11, Raz12, RH14, RH17]. In this section we discusss the relation between interior and neighbourhood operators (or topogenous orders). We show that interior operators are special types of neighbourhood operators (or topogeneous orders). For $X \in \mathbb{C}$, we use P(sub$X$) to denote the class of subclasses of sub$X$.

Definition 2.2.1. [Raz12] A neighbourhood operator on $\mathbb{C}$ with respect to $\mathcal{M}$ is a family

$$\nu = (\nu_X : \text{sub}X \to \text{P}(\text{sub}X))_{X \in \mathbb{C}}$$

of functions which are

(N_1) stack: if $s \leq t$ and $s \in \nu_X(r)$ then $t \in \nu_X(r)$,

(N_2) antimonotone: if $m \leq r$ then $\nu_X(r) \subseteq \nu_X(m)$, and which satisfy

(N_3) the property: if $s \in \nu_X(r)$ then $r \leq s$ and

(N_4) the continuity condition: $f^*(\nu_Y(k)) \subseteq \nu_X(f^*(k))$, that is: if $n \in \nu_Y(k)$ then $f^*(n) \in \nu_X(f^*(k))$,

for all $f : X \to Y$ in $\mathbb{C}$ and $m, r, s, t \in$ sub$X$ and $k, n \in$ sub$Y$.

We note that $f^*(\nu_Y(k)) = \{f^*(n) | \, n \in \nu_Y(k)\}$. Consequently, by applying the antimonotonicity property of ν (see Definition 2.2.1(c)) and the adjointness property (see Remark 1.3.5), one can equivalently describe the continuity condition of ν as follows.

Proposition 2.2.2. [Raz13] Let $f : X \to Y$ be any morphism in $\mathbb{C}$, $m \in$ sub$X$ and $n \in$ sub$Y$. Then for a given neighbourhood operator ν, the following statements are equivalent:

(a) f is ν-continuous, that is: $f^*(\nu_Y(n)) \subseteq \nu_X(f^*(n))$;

(b) $\nu_Y(f(m)) \subseteq f(\nu_X(m))$;

(c) $f^*(\nu_Y(f(m)) \subseteq \nu_X(m)$;

(d) $\nu_Y(n) \subseteq f(\nu_X(f^*(n)))$.

We use NBH$(\mathbb{C}, \mathcal{M})$ to denote the conglomerate of all neighbourhood operators on $\mathbb{C}$ with respect to $\mathcal{M}$. As for INT$(\mathbb{C}, \mathcal{M})$, NBH$(\mathbb{C}, \mathcal{M})$ is ordered pointwise, $\nu \leq \nu'$ if and only if $\nu_X(r) \subseteq \nu'_X(r)$ for all $r \in$ sub$X$ and $X \in \mathbb{C}$. Consequently, NBH$(\mathbb{C}, \mathcal{M})$ is a large complete lattice, that is: arbitrary meets and joins of neighbourhood operators exist in NBH$(\mathbb{C}, \mathcal{M})$. Indeed, the meeet and join of the family $(\nu_k)_{k \in K}$ is given by $\bigcap_{k \in K} (\nu_k)_X(r)$ and $\bigcup_{k \in K} (\nu_k)_X(r)$ for all $r \in$sub$X$ and $X \in \mathbb{C}$, respectively (see [Raz12]). Next we focus on special classes of neighbourhood operators which are stable under join. To this end, let ν be a neighbourhood operator on $\mathbb{C}$ with respect to $\mathcal{M}$ and $G \subseteq$ sub$X$, $X \in \mathbb{C}$. We observe that ν satisfies the property: if $p \in \nu_X(m)$ for all $m \in G$ then $p \in \nu_X(\bigvee G)$ if and only if ν has a right adjoint. This motivates the following definition.

Definition 2.2.3. [RH14] A neighbourhood operator ν on $\mathbb{C}$ with respect to $\mathcal{M}$ is a left adjoint neighbourhood operator if it satisfies the property: if $p \in \nu_X(m)$ for all $m \in G$ then $p \in \nu_X(\bigvee G)$.

The conglomerate of all left adjoint neighbourhood operators on $\mathbb{C}$ with respect to $\mathcal{M}$ is denoted by LNBH$(\mathbb{C}, \mathcal{M})$. Like the NBH$(\mathbb{C}, \mathcal{M})$, LNBH$(\mathbb{C}, \mathcal{M})$ is ordered pointwise and is a complete lattice with set theoretic union and intersection yielding join and meet, respectively. The next proposition states that every interior operator induces a left adjoint neighbourhood operator and each left adjoint neighbourhood operator induces an interior operator.

Proposition 2.2.4. [HŠ11, Raz12, RH14] $\mathsf{LNBH}(\mathbb{C},\mathcal{M})$ is order isomorphic to $\mathsf{INT}(\mathbb{C},\mathcal{M})$.

Proof. For $\nu \in \mathsf{LNBH}(\mathbb{C},\mathcal{M})$ and $i \in \mathsf{INT}(\mathbb{C},\mathcal{M})$ define $i^{\nu}_X(r) := \bigvee \{k \in \mathsf{sub}X : \ r \in \nu_X(k)\}$ and $\nu^{i}_X(r) := \{k \in \mathsf{sub}X : \ r \leq i_X(k)\}$ for all $r \in \mathsf{sub}X$. Then, $r \in \nu_X(k) \Rightarrow k \leq i^{\nu}_X(r)$ and $k \in \nu^{i}_X(r) \Leftrightarrow r \leq i_X(k)$.

Claim-1: $i^{\nu} \in \mathsf{INT}(\mathbb{C},\mathcal{M})$. In order to show this,

(I_1) let $r \in \mathsf{sub}X$. Then for all $k \in \{k \in \mathsf{sub}X : \ r \in \nu_X(k)\}$. By (N_3), one has $k \leq r$, hence $i^{\nu}_X(r) = \bigvee \{k \in \mathsf{sub}X : \ r \in \nu_X(k)\} \leq r$.

(I_2) let $r, s \in \mathsf{sub}X$ such that $r \leq s$. Then for all $k \in \mathsf{sub}X$ with $r \in \nu_X(k)$, one has $s \in \nu_X(k)$ by (N_1). Consequently, $\{k \in \mathsf{sub}X : \ r \in \nu_X(k)\} \subseteq \{k \in \mathsf{sub}X : \ s \in \nu_X(k)\}$, hence $i^{\nu}_X(r) = \bigvee \{k \in \mathsf{sub}X : \ r \in \nu_X(k)\} \leq \bigvee \{k \in \mathsf{sub}X : \ s \in \nu_X(k)\} = i^{\nu}_X(s)$, and

(I_3) for any $f : X \to Y \in \mathbb{C}$ and $n \in \mathsf{sub}Y$, one has $i^{\nu}_Y(n) = \bigvee \{p \in \mathsf{sub}Y : \ n \in \nu_Y(p)\}$. Hence $n \in \nu_Y(\bigvee \{p \in \mathsf{sub}Y : \ n \in \nu_Y(p)\}) = \nu_Y(i^{\nu}_Y(n))$ since ν is a left adjoint neighbourhood operator. Consequently, with the continuity condition (N_4) one has $f^*(n) \in \nu_X(f^*(i^{\nu}_Y(n)))$, hence $f^*(i^{\nu}_Y(n)) \in \{k \in \mathsf{sub}X : \ f^*(n) \in \nu_X(k)\}$. Therefore, $f^*(i^{\nu}_Y(n)) \leq \bigvee \{k \in \mathsf{sub}X : \ f^*(n) \in \nu_X(k)\} = i^{\nu}_X(f^*(n))$.

Claim-2: $\nu^{i} \in \mathsf{LNBH}(\mathbb{C},\mathcal{M})$. To this end,

(N_1) let $k \leq m$ in $\mathsf{sub}X$ such that $k \in \nu^{i}_X(r)$. Then $r \leq i_X(k) \leq i_X(m)$. Consequently, $m \in \nu^{i}_X(r)$,

(N_2) let $m \leq r$ in $\mathsf{sub}X$. Let $k \in \nu^{i}_X(r)$. Then $m \leq r \leq i_X(k)$, hence $k \in \nu^{i}_X(m)$. Therefore, $\nu^{i}_X(r) \subseteq \nu^{i}_X(m)$,

(N_3) let $r, s \in \mathsf{sub}X$ such that $s \in \nu^{i}_X(r)$. Then $r \leq i_X(s) \leq s$, and

(N_4) for any $f : X \to Y$ in $\mathbb{C}$ and $n, k \in \mathsf{sub}Y$ such that $n \in \nu^{i}_Y(k)$, one has $k \leq i_Y(n)$. Consequently, with the continuity condition of i one has $f^*(k) \leq f^*(i_Y(n)) \leq i_X(f^*(n))$. Hence $f^*(n) \in \nu^{i}_X(f^*(k))$.

Claim-3: The assignments $\nu \mapsto i^{\nu}$ and $i \mapsto \nu^{i}$ are monotone and inverse to each other. Indeed,

(a) let $i, j \in \mathsf{INT}(\mathbb{C},\mathcal{M})$ such that $i \leq j$. Then for $k \in \nu^{i}_X(r)$, one has $r \leq i_X(k) \leq j_X(k)$, hence $k \in \nu^{j}_X(r)$. Therefore, $i \leq j \Rightarrow \nu^{i} \leq \nu^{j}$, that is: $i \mapsto \nu^{i}$ is a monotone map,

(b) let $\nu, \nu' \in \mathsf{LNBH}(\mathbb{C},\mathcal{M})$ such that $\nu \leq \nu'$. Then $\{k \in \mathsf{sub}X : \ r \in \nu_X(k)\} \subseteq \left\{k \in \mathsf{sub}X : \ r \in \nu'_X(k)\right\}$, hence $i^{\nu}_X(r) = \bigvee \{k \in \mathsf{sub}X : \ r \in \nu_X(k)\} \leq \bigvee \left\{k \in \mathsf{sub}X : \ r \in \nu'_X(k)\right\} = i^{\nu'}_X(r)$. Therefore, $\nu \leq \nu'$ implies $i^{\nu} \leq i^{\nu'}$, that is: the map $\nu \mapsto i^{\nu}$ is monotone,

(c) let $r \in \mathsf{sub}X$. Then $i^{\nu^{i}}_X(r) = \bigvee \left\{k \in \mathsf{sub}X : \ r \in \nu^{i}_X(k)\right\} = \bigvee \{k \in \mathsf{sub}X : \ k \leq i_X(r)\} \cong i_X(r)$, and

(d) let $k \in \nu_X(r)$. Then $r \in \{p \in \mathsf{sub}X : \ k \in \nu_X(p)\}$, hence $r \leq \bigvee \{p \in \mathsf{sub}X : \ k \in \nu_X(p)\} = i^{\nu}_X(k)$. Consequently, $k \in \nu^{i^{\nu}}_X(r) = \{p \in \ \mathsf{sub}X : \ r \leq i^{\nu}_X(p)\}$. Hence $\nu_X(r) \subseteq \nu^{i^{\nu}}_X(r)$. On the other hand, let $k \in \nu^{i^{\nu}}_X(r)$. Then $r \leq i^{\nu}_X(k)$, hence by the animonocity property of ν one has $\nu_X(i^{\nu}_X(k)) \subseteq \nu_X(r)$. Consequently, $k \in \nu_X(r)$ since $k \in \nu_X(i^{\nu}_X(k))$.

□

The previous proposition shows that interior operators are special neighbourhood operators which respect joins. Indeed, $\mathsf{INT}(\mathbb{C},\mathcal{M}) \cong \mathsf{LNBH}(\mathbb{C},\mathcal{M})$ is reflective in $\mathsf{NBH}(\mathbb{C},\mathcal{M})$, which is shown below.

Proposition 2.2.5. [Raz12] $\mathsf{INT}(\mathbb{C},\mathcal{M}) \cong \mathsf{LNBH}(\mathbb{C},\mathcal{M})$ is reflective in $\mathsf{NBH}(\mathbb{C},\mathcal{M})$ and the reflection of a neighbourhood operator ν is $\nu^+ = \bigcap\left\{\nu' \in \mathsf{LNBH}(\mathbb{C},\mathcal{M}) :\ \nu \leq \nu'\right\}$.

Proof. As stated above, an arbitrary intersection of left neighbourhood operators is a left adjoint neighbourhood operator, hence $\nu^+ \in \mathsf{LNBH}(\mathbb{C},\mathcal{M})$. Consequently, one has the adjunction

$$\mathsf{NBH}(\mathbb{C},\mathcal{M}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{LNBH}(\mathbb{C},\mathcal{M}) \cong \mathsf{INT}(\mathbb{C},\mathcal{M})\ .$$

Therefore, the inclusion functor $\nu' \hookrightarrow \nu'$ has a left adjoint. Therefore, $\mathsf{INT}(\mathbb{C},\mathcal{M}) \cong \mathsf{LNBH}(\mathbb{C},\mathcal{M})$ is reflective in $\mathsf{NBH}(\mathbb{C},\mathcal{M})$. □

Remark 2.2.6. If preimages commute with the joins in the category $\mathbb{C}$ then for any neighbourhood operator (not only left adjoint neighbourhood operator) ν, the assignment $i^{\nu}_X(r) := \bigvee\{k \in \mathsf{sub}X :\ r \in \nu_X(k)\}$ is an interior operator. Indeed, for any $f : X \to Y$in $\mathbb{C}$ and $n \in \mathsf{sub}Y$ the continuity condition (N_4) of ν yields

$$\begin{aligned} f^*(i^{\nu}_Y(n)) &= f^*(\bigvee\{p \in \mathsf{sub}Y :\ n \in \nu_Y(p)\}) \\ &\cong \bigvee\{f^*(p) \in \mathsf{sub}X :\ n \in \nu_Y(p)\} \\ &\leq \bigvee\{f^*(p) \in \mathsf{sub}X :\ f^*(n) \in \nu_X(f^*(p))\} \\ &\leq \bigvee\{k \in \mathsf{sub}X :\ f^*(n) \in \nu_X(k)\} \\ &\cong i^{\nu}_X(f^*(n)). \end{aligned}$$

In the remainder of this section we discuss the relation between interior operators and topogenous orders. Topogenous orders on an arbitrary category were introduced in [HIR16] with the assistance of both categorical closure and interior operators, inspired by the works of Császár on syntopogenous structures and spaces presented in [Csá63]. These orders provide a unified categorical framework for closure, interior and neighbourhood operators and are defined as follows.

Definition 2.2.7. [HIR16] A topogenous order on $\mathbb{C}$ with respect to $\mathcal{M}$ is a family

$$\sqsubseteq = (\sqsubseteq_X)_{X\in\mathbb{C}}$$

of relations which satisfy the properties

(T_1) if $r \sqsubseteq_X s$ then $r \leq s$,

(T_2) if $r \leq s \sqsubseteq_X p \leq q$ then $r \sqsubseteq_X q$, and

(T_3) the continuity condition: if $n \sqsubseteq_Y k$ then $f^*(n) \sqsubseteq_X f^*(k)$

for all $f : X \to Y$ in $\mathbb{C}$ and $r, s, p, q \in \mathsf{sub}X$ and $k, n \in \mathsf{sub}Y$.

The conglomerate TORD$(\mathbb{C},\mathcal{M})$ of all topogenous orders on $\mathbb{C}$ with respect to $\mathcal{M}$ is a preordered class and ordered by the relation $\sqsubseteq \subseteq \sqsubseteq'$ if and only if $r \sqsubseteq_X s$ implies $r \sqsubseteq'_X s$ for all $r, s \in$ sub$X$ and $X \in \mathbb{C}$. Consequently, TORD$(\mathbb{C},\mathcal{M})$ is a large complete lattice, that is: arbitrary meets and joins of topogenous orders exist in TORD$(\mathbb{C},\mathcal{M})$. Indeed, the meeet and join of the family $(\sqsubseteq_k)_{k\in K}$ is given by $\bigcap_{k\in K} \sqsubseteq_k$ and $\bigcup_{k\in K} \sqsubseteq_k$, respectively (see [HIR16]). In the next proposition we see that every neighbourhood operator ν induces a topogenous order $\sqsubseteq^{\nu}$ and each topogenous order $\sqsubseteq$ induces neighbourhood operator $\nu^{\sqsubseteq}$.

Proposition 2.2.8. [HIR16] TORD$(\mathbb{C},\mathcal{M}) \cong$ NBH$(\mathbb{C},\mathcal{M})$.

Proof. Let $\nu \in$ NBH$(\mathbb{C},\mathcal{M})$ and $\sqsubseteq \in$ TORD$(\mathbb{C},\mathcal{M})$. Then the order relation $\sqsubseteq^{\nu}$ given by
$r \sqsubseteq^{\nu}_X s \Leftrightarrow s \in \nu_X(r)$ is a topogenous order and the class $\nu^{\sqsubseteq}_X(r) = \{k \in \text{sub}X : r \sqsubseteq_X k\}$ is a neighbourhood operator. Indeed, these are consequences of $(N_3) \Leftrightarrow (T_1)$, (N_1) and $(N_2) \Leftrightarrow (T_2)$ and $(N_4) \Leftrightarrow (T_3)$. Furthermore, it is clear that both $\sqsubseteq \mapsto \nu^{\sqsubseteq}$ and $\nu \mapsto \sqsubseteq^{\nu}$ are monotone and inverse to each other. □

Consequently, topogenous orders are precisely neighbourhood operators.

Definition 2.2.9. [HIR16] Let $X \in \mathbb{C}$, $s \in$ sub$X$ and $\{r_k : k \in K\} \subseteq$ sub$X$. A topogenous order $\sqsubseteq$ with the property:

$$\text{if } r_k \sqsubseteq_X s \text{ for all } k \in K \text{ then } \bigvee_{k\in K} r_k \sqsubseteq_X s$$

is called a topogenous order which respects joins.

The conglomerate of all topogenous orders on $\mathbb{C}$ with respect to $\mathcal{M}$ which respect joins is denoted by $\bigvee$-TORD$(\mathbb{C},\mathcal{M})$. Similar to TORD$(\mathbb{C},\mathcal{M})$, $\bigvee$-TORD$(\mathbb{C},\mathcal{M})$ is stable under arbitrary intersections. The following proposition shows that topogenous orders which respect joins are precisely the interior operators, that is: every topogenous order which respect joins induces an interior operator and each interior operator induces a topogenous order which respect joins.

Proposition 2.2.10. [HIR16] $\bigvee$-TORD$(\mathbb{C},\mathcal{M}) \cong$ INT$(\mathbb{C},\mathcal{M})$.

Proof. Let $\sqsubseteq \in \bigvee$-TORD$(\mathbb{C},\mathcal{M})$ and $i \in$ INT$(\mathbb{C},\mathcal{M})$.

(a) The order relation given by $r \sqsubseteq^i_X s \Leftrightarrow r \leq i_X(s)$ for all $r \in$ sub$X$ is a topogenous order which respects joins. Indeed, this is due to $(I_1) \Rightarrow (T_1)$, $(I_2) \Rightarrow (T_2)$ and $(I_3) \Rightarrow (T_3)$ and $(\forall k \in K)\,\big(r_k \sqsubseteq^i_X s \Rightarrow r_k \leq i_X(s)\big)$, hence $\bigvee_{k\in K} r_k \leq i_X(s)$. Therefore, $\bigvee_{k\in K} r_k \sqsubseteq^i s$.

(b) The operator given by $i^{\sqsubseteq}_X(r) := \bigvee\{k \in \text{sub}X : k \sqsubseteq_X r\}$ is an interior operator. Indeed, $(T_1) \Rightarrow (I_1)$, $(T_2) \Rightarrow (I_2)$ and for any $f : X \to Y$ in $\mathbb{C}$ and $n \in$ sub$Y$, one has $i^{\sqsubseteq}_Y(n) = \bigvee\{p \in \text{sub}Y : p \sqsubseteq_Y n\}$, hence $i^{\sqsubseteq}_Y(n) \sqsubseteq_Y n$ since $\sqsubseteq \in \bigvee$-TORD. Consequently, with (T_3) one obtains $f^*(^{\sqsubseteq}_Y(n)) \sqsubseteq_X f^*(n)$. Therefore, $f^*(^{\sqsubseteq}_Y(n)) \leq i_X(f^*(n)) = \bigvee\{k \in \text{sub}X : k \sqsubseteq_X f^*(n)\}$.

(c) It is clear that both $\sqsubseteq \mapsto i^{\sqsubseteq}$ and $i \mapsto \sqsubseteq^i$ are monotone and inverse to each other.

□

Consequently, one has the following proposition.

Corollary 2.2.11. [HIR16] INT$(\mathbb{C},\mathcal{M}) \cong \bigvee$-TORD$(\mathbb{C},\mathcal{M})$ is reflective in TORD$(\mathbb{C},\mathcal{M})$ and the reflection of a topogenous order $\sqsubseteq$ is $\sqsubseteq^{+}=\bigcap\left\{\sqsubseteq' \in \bigvee\text{-TORD}(\mathbb{C},\mathcal{M}) : \sqsubseteq\subseteq\sqsubseteq'\right\}$.

Analogous to Remark 2.2.6, one obtains the following.

Remark 2.2.12. If preimages commute with the joins in the category $\mathbb{C}$ then any topogenous order (not only topogenous order which respect joins) $\sqsubseteq$, the assignment $i_X^{\sqsubseteq}(r) := \bigvee\{k \in \text{sub}X : \ k \sqsubseteq_X r\}$ is an interior operator. Indeed, for any $f : X \to Y$in $\mathbb{C}$ and $n \in \text{sub}Y$ the continuity condition (T_3) of $\sqsubseteq$ yields

$$\begin{aligned} f^*(i_Y^{\sqsubseteq}(n)) &= f^*(\bigvee\{p \in \text{sub}Y : \ p \sqsubseteq_Y n\}) \\ &\cong \bigvee\{f^*(p) \in \text{sub}X : \ p \sqsubseteq_Y n\} \\ &\leq \bigvee\{f^*(p) \in \text{sub}X : \ f^*(p) \sqsubseteq_X f^*(n)\} \\ &\leq \bigvee\{k \in \text{sub}X : \ k \sqsubseteq_X f^*(n)\} \\ &\cong i_X^{\sqsubseteq}(f^*(n)). \end{aligned}$$

The remark certainly should not surprise us as topogenous orders are essentially the same as neighbourhood operators (see Proposition 2.2.8).

Definition 2.2.13. [HIR16] A topogenous order $\sqsubseteq$ with the property: for all $r \sqsubseteq_X s$ in sub$X$ there exists $p \in \text{sub}X$ such that $r \sqsubseteq_X p \sqsubseteq_X s$ for all $X \in \mathbb{C}$ is called interpolative topogenous order.

The conglomerate INTORD$(\mathbb{C},\mathcal{M})$ of all interpolative topogenous orders on $\mathbb{C}$ with respect to $\mathcal{M}$ is stable under arbitrary unions. Hence, INTORD$(\mathbb{C},\mathcal{M})$ is coreflective in TORD$(\mathbb{C},\mathcal{M})$. Furthermore, we do have the following relation.

Corollary 2.2.14. [Ira16] Idempotent interior operators are precisely interpolative topogenous orders which respect joins.

Proof. Let i be an idempotent interior operator. Then by Proposition 2.2.10 the relation given by $r \sqsubseteq_X^i s \Leftrightarrow r \leq i_X(s)$ for all $X \in \mathbb{C}$ is a topogenous order in $\bigvee$-TORD$(\mathbb{C},\mathcal{M})$. Moreover, since i is idempotent one has $i_X(s) \cong i_X(i_X(s))$, hence

$$\begin{aligned} r \sqsubseteq_X^i s &\Leftrightarrow r \leq i_X(s) \cong i_X(i_X(s)) \\ &\Leftrightarrow r \leq i_X(i_X(s)) \\ &\Leftrightarrow r \sqsubseteq_X^i i_X(s) \leq i_X(s) \\ &\Leftrightarrow r \sqsubseteq_X^i i_X(s) \sqsubseteq_X^i s. \end{aligned}$$

So, there exists $p = i_X(s)$ in sub$X$ such that $r \sqsubseteq_X^i p \sqsubseteq_X^i s$. Therefore, $\sqsubseteq^i$ is an interpolative topogenous order. On the other hand, if $\sqsubseteq$ is an interoplative topogenous order then by Proposition 2.2.10 the operator given by $i_X^{\sqsubseteq}(r) = \bigvee\{k \in \text{sub}X : \ k \sqsubseteq_X r\}$ is an interior operator. Furthermore, since $\sqsubseteq$ is an interoplative topogenous order, for $k \in \text{sub}X$ such that $k \sqsubseteq_X r$, one has the existence of $p \in \text{sub}X$ such that $k \sqsubseteq_X p \sqsubseteq_X r$, hence $k \leq k \sqsubseteq_X p \leq i_X^{\sqsubseteq}(r)$. Consequently, by (T_2), $k \sqsubseteq_X i_X^{\sqsubseteq}(r)$. Therefore, $\{k \in \text{sub}X : \ k \sqsubseteq_X r\} \subseteq \left\{q \in \text{sub}X : \ q \sqsubseteq_X i_X^{\sqsubseteq}(r)\right\}$. Thus,

$i^{\sqsubseteq}_X(r) = \bigvee\{k \in \mathsf{sub}X :\ k \sqsubseteq_X r\} \leq \bigvee\left\{q \in \mathsf{sub}X :\ q \sqsubseteq_X i^{\sqsubseteq}_X r\right\} = i^{\sqsubseteq}_X(i^{\sqsubseteq}_X(r))$. Hence, $i^{\sqsubseteq}$ is an idempotent interior operator. □

The following diagram summarizes the relation of interior operators with neighbourhood operators and topogenous orders.

$$\mathsf{TORD}(\mathbb{C},\mathcal{M}) \cong \mathsf{NBH}(\mathbb{C},\mathcal{M}) \underset{\perp}{\overset{\longrightarrow}{\hookleftarrow}} \mathsf{LNBH}(\mathbb{C},\mathcal{M}) \cong \bigvee\text{-}\mathsf{TORD}(\mathbb{C},\mathcal{M}) \cong \mathsf{INT}(\mathbb{C},\mathcal{M})\ .$$

2.3 "Duality" between interior and closure operators

Although the associated closure and interior operators provide equivalent descriptions of the topology for a given topological space, categorical closure and interior operators are not "dual" to each other. This is due to the fact that categorical interior operators are only compatible with taking preimages unlike closure operators (see Remark 2.1.2(a)). As a consequence, the preservation property, which is the symmetric counter part of the functorial property of closure operators, does not hold for interior operators, hence results which are analogous to results involving the functorial property of closure operators may not hold (see [CM13, Cas15, Cas16]).

Definition 2.3.1. [DG87] A closure operator c on $\mathbb{C}$ with respect to $\mathcal{M}$ is a family

$$c = (c_X : \mathsf{sub}X \to \mathsf{sub}X)_{X\in\mathbb{C}}$$

of functions which are

(C_1) extensive: $r \leq c_X(r)$,

(C_2) monotone: if $r \leq s$ then $c_X(r) \leq c_X(s)$,

(C_3) and which satisfy the continuity condition: $f(c_X(r)) \leq c_Y(f(r))$,

for all $f : X \to Y$ in $\mathbb{C}$ and $r, s \in$sub$X$.

The image-preimage adjunction allows us to equivalently describe the continuity condition (C_3) as $c_X(f^*(n)) \leq f^*(c_Y(n))$ or $f(c_X(f^*(n))) \leq c_Y(n)$ or $c_X(r) \leq f^*(c_Y(f(r)))$ for all $f : X \to Y$ in $\mathbb{C}$ and $n \in$ sub$Y$ and $r \in$ sub$X$.

Definition 2.3.2. [DG87, DT95] Let c be a closure operator.

(a) An $\mathcal{M}$ subobject $r \in$ sub$X$ is called c-closed if $c_X(r) \cong r$.

(b) An $\mathcal{M}$ subobject $r \in$ sub$X$ is called c-dense if $c_X(r) \cong 1_X$.

(c) A morphism $f : X \to Y$ in $\mathbb{C}$ is called c-dense if $c_Y(f(1_X)) \cong 1_Y$.

(d) c is called idempotent if $c_X(r)$ is c-closed in X, that is: $c_X(c_X(r)) \cong c_X(r)$ for all $r \in$ sub$X$, $X \in \mathbb{C}$.

(e) c is hereditary if $c_X(r_s) \cong s^*(c_X(r)) \cong s^*(c_X(s(r_s)))$ for all $r \leq s$ in sub$X$, $X \in \mathbb{C}$.

We use CLOS$(\mathbb{C},\mathcal{M})$ to denote the conglomerate of all closure operators on $\mathbb{C}$ with respect to $\mathcal{M}$. Like INT$(\mathbb{C},\mathcal{M})$, CLOS$(\mathbb{C},\mathcal{M})$ is preordered by $\leq$. For $c, c' \in$ CLOS$(\mathbb{C},\mathcal{M})$ one defines $c \leq c' \Leftrightarrow c_X(m) \leq c'_X(m)$ for all $m \in$ sub$X, X \in \mathbb{C}$, where $\leq$ is the order on subobjects. Consequently, meets and joins of non-empty families of closure operators exist and are formed "pointwise" (see [DT95]). In order to see the relationship between closure operator and topogenous order and hence interior operator let us first recall the following definition from [HIR16].

Definition 2.3.3. Let $X \in \mathbb{C}$, $s \in \mathsf{sub}X$ and $\{r_k : k \in K\} \subseteq \mathsf{sub}X$. A topogenous order $\sqsubseteq$ with the property: if $s \sqsubseteq_X r_k$ for all $k \in K$ then $s \sqsubseteq_X \bigwedge_{k\in K} r_k$ is called a topogenous order which respects meets.

The conglomerate of all topogenous orders on $\mathbb{C}$ with respect to $\mathcal{M}$ which respect meets is denoted by $\bigwedge$-TORD$(\mathbb{C},\mathcal{M})$. Similar to $\bigvee$-TORD$(\mathbb{C},\mathcal{M})$, $\bigwedge$-TORD$(\mathbb{C},\mathcal{M})$ is stable under arbitrary intersections and hence reflective in TORD$(\mathbb{C},\mathcal{M})$. The following proposition shows that topogenous orders which respect meets are precisely the closure operators.

Proposition 2.3.4. [HIR16] $\bigwedge$-TORD$(\mathbb{C},\mathcal{M}) \cong$ CLOS$(\mathbb{C},\mathcal{M})$.

Proof. We observe that the proof is similar to the proof of Proposition 2.2.10. Let $\sqsubseteq \in \bigwedge$-TORD$(\mathbb{C},\mathcal{M})$ and $c \in$ CLOS$(\mathbb{C},\mathcal{M})$.

(a) The order relation given by $r \sqsubseteq^c_X s \Leftrightarrow c_X(r) \leq s$ for all $r \in \mathsf{sub}X$ is a topogenous order which respects meets. Indeed, this is due to $(C_1) \Rightarrow (T_1)$, $(C_2) \Rightarrow (T_2)$, $(C_3) \Rightarrow (T_3)$ and $(\forall k \in K)\,(s \sqsubseteq^c_X r_k \Rightarrow c_X(s) \leq r_k)$. Hence $c_X(s) \leq \bigwedge_{k\in K} r_k$. Therefore, $s \sqsubseteq^c \bigwedge_{k\in K} r_k$.

(b) The operator given by $c^{\sqsubseteq}_X(r) = \bigwedge\{k \in \mathsf{sub}X :\ r \sqsubseteq_X k\}$ is a closure operator. Indeed, $(T_1) \Rightarrow (C_1)$, $(T_2) \Rightarrow (C_2)$ and for any $f : X \to Y$ in $\mathbb{C}$ and $n \in \mathsf{sub}Y$, one has $c^{\sqsubseteq}_Y(n) = \bigwedge\{p \in \mathsf{sub}Y :\ n \sqsubseteq_Y p\}$. Hence $n \sqsubseteq_Y c^{\sqsubseteq}_Y(n)$ since $\sqsubseteq \in \bigwedge$-TORD. Consequently, with (T_3) one obtains $f^*(n) \sqsubseteq_X f^*(c^{\sqsubseteq}_Y(n))$. Therefore, $c^{\sqsubseteq}_X(f^*(n)) = \bigwedge\{k \in \mathsf{sub}X :\ f^*(n) \sqsubseteq_X k\} \leq f^*(c^{\sqsubseteq}_Y(n))$.

(c) It is clear that both $\sqsubseteq \mapsto c^{\sqsubseteq}$ and $c \mapsto \sqsubseteq^c$ are order reversing and inverse to each other.

□

Remark 2.3.5. As pointed out in [HIR16], because f^* commutes with meets one always obtains the continuity condition (C_3) of a closure operator c, hence $c^{\sqsubseteq}$ is a closure operator for any topogenous order (not only for topogenous orders which respect meets) $\sqsubseteq$.

As a consequence of Proposition 2.3.4, one has the following corollary.

Corollary 2.3.6. [HIR16] CLOS$(\mathbb{C},\mathcal{M}) \cong \bigwedge$-TORD$(\mathbb{C},\mathcal{M})$ is reflective in TORD$(\mathbb{C},\mathcal{M})$ and the reflection of a topogenous order $\sqsubseteq$ is $\sqsubseteq^+ = \bigcap\left\{\sqsubseteq' \in \bigwedge\text{-TORD}(\mathbb{C},\mathcal{M}) : \sqsubseteq \subseteq \sqsubseteq'\right\}$.

Analogous to Corollary 2.2.14 one has the following.

Corollary 2.3.7. [Ira16] Idempotent closure operators are precisely interpolative topogenous orders which respect meets.

The following diagram summarizes the relation of closure operators with neighbourhood operators and topogenous orders.

$$\mathsf{NBH}(\mathbb{C},\mathcal{M}) \cong \mathsf{TORD}(\mathbb{C},\mathcal{M}) \underset{\perp}{\rightleftarrows} \bigwedge\text{-}\mathsf{TORD}(\mathbb{C},\mathcal{M}) \cong \mathsf{CLOS}(\mathbb{C},\mathcal{M})\,.$$

If preimages commute with joins in the category $\mathbb{C}$ then combinining Propositions 2.2.10 and 2.3.4 and Remarks 2.2.12 and 2.3.5 one has a natural correspondence between closure and interior operators, that is: every closure operator gives rise to an interior oparator and vice versa. More precisely, one has the following theorem.

Theorem 2.3.8. *Let* $c \in$ *CLOS*$(\mathbb{C}, \mathcal{M})$ *and* $i \in$ *INT*$(\mathbb{C}, \mathcal{M})$.

(a) *The family* $(c_X)_{X \in \mathbb{C}}$, *where* $c^i_X(r) := \bigwedge\{k \in \text{ sub}X : \; r \leq i_X(k)\}$ *for all* $r \in \text{sub}X$, *is a closure operator.*

(b) *If each preimage commutes with the joins in the category* $\mathbb{C}$ *then the family* $(i_X)_{X\in\mathbb{C}}$, *where* $i^c_X(r) := \bigvee\{k \in \text{ sub}X : \; c_X(k) \leq r\}$ *is an interior operator.*

(c) *The maps* $i \mapsto c^i$ *and* $c \mapsto i^c$ *are order reversing between INT*$(\mathbb{C}, \mathcal{M})$ *and CLOS*$(\mathbb{C}, \mathcal{M})$.

Proof. This follows by composing the maps

$$\mathsf{CLOS} \cong \bigwedge\text{-}\mathsf{TORD} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathsf{TORD} \cong \mathsf{NBH} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{LNBH} \cong \bigvee\text{-}\mathsf{TORD} \cong \mathsf{INT}.$$

□

Theorem 2.3.8 deals with general method of constructing interior operators from closure operators and vice versa.

Remark 2.3.9. The correspondence in Theorem 2.3.8 states that for any interior operator i, the composition $i \mapsto \sqsubseteq^i \mapsto c^{\sqsubseteq^i}$ is a closure operator and if each preimage commutes with the joins in the category $\mathbb{C}$, then for each closure operator c, the composition $c \mapsto \sqsubseteq^c \mapsto i^{\sqsubseteq^c}$ is an interior operator.

As a consequence of Corollaries 2.2.14 and 2.3.7 and Theorem 2.3.8 one has the following.

Corollary 2.3.10. If preimages commute with the joins in the category $\mathbb{C}$ then each idempotent interior operator induces an idempotent closure operator and vice versa.

The maps $i \mapsto c^i$ and $c \mapsto i^c$ in Theorem 2.3.8 are neither Galois connections nor inverse to each other but they are a natural way of moving between interior and closure operators. Consequently, the maps yield a certain "duality" between INT$(\mathbb{C}, \mathcal{M})$ and CLOS$(\mathbb{C}, \mathcal{M})$. They become inverse to each other if for each X, sub$X$ is a Boolean algebra. Indeed, we obtain the following consequence of Lemma 1.4.7(a) and Proposition 1 of [HŠ11].

Proposition 2.3.11. If sub$X$ is a Boolean algebra for every $\mathbb{C}$-object X and for every $\mathbb{C}$-morphism f, $f^*(-)$ preserves complements. Then interior operators i on $\mathbb{C}$ with respect to $\mathcal{M}$ are in bijective correspondence with closure operators c on $\mathbb{C}$ with respect to $\mathcal{M}$, via $c^i_X(m) = \overline{i_X(\overline{m})}$, $i^c_X(m) = \overline{c_X(\overline{m})}$, for all $m \in \text{sub}X$, $X \in \mathbb{C}$.

Remark 2.3.12. Recall from Lemma 1.4.7(a) that if sub$X$ is a Boolean algebra for every $\mathbb{C}$-object X and the preimage $f^*(-)$ preserves arbitrary joins for every morphism f in $\mathbb{C}$ then each f^* preserves complements, hence by the above Proposition there is a bijective correspondence between interior and closure operators on $\mathbb{C}$ with respect to $\mathcal{M}$, via $c^i_X(m) = \overline{i_X(\overline{m})}$, $i^c_X(m) = \overline{c_X(\overline{m})}$, for all $m \in \text{sub}X$, $X \in \mathbb{C}$.

Let us also note that the following is a generalization of Proposition 2.3.11 proved in [Vor00].

Proposition 2.3.13. If the category $\mathbb{C}$ admits a transformation operator there is a bijective correspondence between interior and closure operators on $\mathbb{C}$ with respect to $\mathcal{M}$.

In [HŠ18], by using the notion of relative closed and open they estabilished possible correspondences between closure and interior operators. But unless the subobject lattices are Boolean algebras the correspondences do not yield a satisfactory way of moving from one operator to the other. The situation is discussed in the remainder of this section. Given an interior operator i, based on the topological notion we define closed subobjects with respect to i as follows.

Definition 2.3.14. [HŠ18] Let $i \in \mathsf{INT}(\mathbb{C}, \mathcal{M})$, $X \in \mathbb{C}$. An $\mathcal{M}$-subobject $r : R \to X$ is called:

1. A^i-closed if for all $s \in \mathsf{sub}X$, $i_X(r \vee s) \leq r \vee i_X(s)$;
2. B^i-closed if for all $s \in \mathsf{sub}X$, $r \vee s \cong 1_X \Rightarrow r \vee i_X(s) \cong 1_X$;
3. C^i-closed if r is pseudocomplemented and $r^c \cong i_X(r^c)$, where the pseudocomplement of r is an $\mathcal{M}$-subobject $r^c : R^c \to X$ such that for any $m \in \mathsf{sub}X$, $m \leq r^c \Leftrightarrow m \wedge r \cong 0_X$.

One readily sees that when the subobject lattices are Boolean algebras the above three definitions coincide. One can also derive some relations in between them under some conditions (see [HŠ18]). Furthermore, the three definitions induce three types of closure operators as discussed below.
To this end, we construct the smallest pullback stable class $\mathcal{F}^*$ containing a given a class $\mathcal{F} \subseteq \mathcal{M}$. Indeed, $\mathcal{F}^*$ is given by $\mathcal{F}^* = \{f^*(n) : n \in \mathcal{F},\ f \text{ in } \mathbb{C}\}$ (see [CH03a, HŠ18]).

Proposition 2.3.15. [CH03a, HŠ18] The family $(c^{\mathcal{F}}_X)_{X\in\mathbb{C}}$, where $c^{\mathcal{F}}_X(r) := \bigwedge\{r' \in \mathcal{F}^* : r \leq r'\}$, is an idempotent closure operator.

Proof. (a) Let $r \leq r'$ for all $r' \in \mathcal{F}$. Then $r \leq \bigwedge\{r' \in \mathcal{F}^* : r \leq r'\} = c^{\mathcal{F}}_X(r)$.

(b) Let $r \leq s$ in sub$X$. Then $\{r' \in \mathcal{F}^* : s \leq r'\} \subseteq \{r' \in \mathcal{F}^* : r \leq r'\}$, hence $c^{\mathcal{F}}_X(r) = \bigwedge\{r' \in \mathcal{F}^* : r \leq r'\} \leq \bigwedge\{r' \in \mathcal{F}^* : s \leq r'\} = c^{\mathcal{F}}_X(s)$.

(c) Let $f : X \to Y \in \mathbb{C}$ and $n \in \mathsf{sub}Y$. Then

$$\begin{aligned} c^{\mathcal{F}}_X(f^*(n)) &= \bigwedge\{k \in \mathcal{F}^* : f^*(n) \leq k\} \\ &\leq \bigwedge\{f^*(n') \in \mathcal{F}^* : f^*(n) \leq f^*(n')\} \\ &\leq \bigwedge\{f^*(n') \in \mathcal{F}^* : n \leq n'\} \cong f^*(\bigwedge\{n' \in \mathcal{F}^* : n \leq n'\}) = f^*(c^{\mathcal{F}}_Y(n)). \end{aligned}$$

(d) Since $c^{\mathcal{F}}_X(r) \leq c^{\mathcal{F}}_X(r)$, one has $c^{\mathcal{F}}_X(c^{\mathcal{F}}_X(r)) = \bigwedge\{r' \in \mathcal{F}^* : c^{\mathcal{F}}_X(r) \leq r'\} \leq c^{\mathcal{F}}_X(r)$. Furthermore, $c^{\mathcal{F}}_X(r) \leq c^{\mathcal{F}}_X(c^{\mathcal{F}}_X(r))$ by (a). Consequently, $c^{\mathcal{F}}_X(c^{\mathcal{F}}_X(r)) \cong c^{\mathcal{F}}_X(r)$.

□

Note that $c^{\mathcal{F}}_X(r) \cong r$ for all $r \in \mathcal{F}$, that is each $r \in \mathcal{F}$ is $c^{\mathcal{F}}$-closed. Consequently, $c^{\mathcal{F}}$ is the largest closure operator satisfying this property.

Definition 2.3.16. [HŠ18] Let i be an interior operator. By considering the class of A^i-closed, B^i-closed, C^i-closed subobjects for the class $\mathcal{F}$, we define closure operators α^i, β^i and γ^i, respectively.

Consequently, one has the following.

Remark 2.3.17. [HŠ18]

(a) The maps $i \mapsto \beta^i$ and $i \mapsto \gamma^i$ are order reversing maps from $\mathsf{INT}(\mathbb{C}, \mathcal{M})$ to $\mathsf{CLOS}(\mathbb{C}, \mathcal{M})$.

(b) The map $i \mapsto \alpha^i$ does not respect order. Indeed, if we assume $\mathbb{C}$-morphisms reflect 0, then both the trivial t^{in} and discrete d^{in} interior operator induce the same closure operator given by $\alpha^{t^{\text{in}}}(r) \cong r \cong \alpha^{d^{\text{in}}}(r)$.

By assuming preimages commute with the joins in the category $\mathbb{C}$ one obtains an interior operator analogous to the construction of the above closure operator as follows (see also [HŠ18]).

Proposition 2.3.18. Let preimages commute with the joins in the category $\mathbb{C}$ and $\mathcal{F} \subseteq \mathcal{M}$. The family $(i_X^{\mathcal{F}})_{X \in \mathbb{C}}$, where $i_X^{\mathcal{F}}(r) := \bigvee\{r' \in \mathcal{F}^* : r' \leq r\}$, is an idempotent interior operator.

Observe that $i_X^{\mathcal{F}}(r) \cong r$ for all $r \in \mathcal{F}$, that is each $r \in \mathcal{F}$ is $i^{\mathcal{F}}$-open. Consequently, $i^{\mathcal{F}}$ is the smallest operator satisfying this property. By defining open subobjects with respect to a given closure operator one obtains three types of interior operators. To see this, we start by recalling the following definition from [HŠ18].

Definition 2.3.19. Given a closure operator c, an $\mathcal{M}$-subobject $r : R \to X$ is called:

1. $\mathcal{A}^c$-open if for all $s \in \text{sub}X$, $r \wedge c_X(s) \leq c_X(r \wedge s)$;
2. $\mathcal{B}^c$-open if for all $s \in \text{sub}X$ $r \wedge s \cong 0_X \Rightarrow r \wedge c_X(s) = 0_X$;
3. $\mathcal{C}^c$-open if r is pseudocomplemented and $r^c = c_X(r^c)$.

Definition 2.3.20. [HŠ18] Let c be a closure operator. By considering the class of $\mathcal{A}^c$-open, $\mathcal{B}^c$-open, $\mathcal{C}^c$-open subobjects for the class $\mathcal{F}$, we define interior operators $\mathfrak{a}^c$, $\mathfrak{b}^c$ and $\mathfrak{c}^c$, respectively.

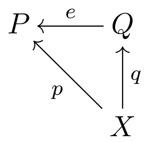
The maps $c \mapsto \mathfrak{b}^c$ and $c \mapsto \mathfrak{c}^c$ are order reversing while $c \mapsto \mathfrak{a}^c$ does not respect the order. Furthermore, the pairs $(\alpha, \mathfrak{a})$, $(\beta, \mathfrak{b})$ amd $(\gamma, \mathfrak{c})$ form Galois connections if the subobject lattices are Boolean algebras (see [HŠ18]).

2.4 Interior and dual closure operators

In this section we first discuss dual closure operators in an arbitrary category and we then compare them with interior operators. Dual closure operators are defined for an arbitrary category with a suitable quotient object structure. Throughout this section we consider a finitely cocomplete category $\mathbb{C}$ with $(\mathcal{E}, \mathcal{M})$-factorization systems for morphisms such that $\mathcal{E}$ is a fixed class of epimorphisms. We begin the section by defininig $\mathcal{E}$-quotient objects.The notion of quotient objects provides the categorical formulations for structures such as quotient sets in set theory, quotient groups in group theory, quotient rings in ring theory, quotient modules in module theory and quotient spaces in topology. In an arbitrary category we define dual closure operators on a suitable axiomatically defined class of quotient objects. Quotient objects are described by special morphisms in $\mathbb{C}$ which may be thought of as quotient maps. It is a dual concept to subobjects.

Definition 2.4.1. For a given $X \in \mathbb{C}$, quot$X := \{e \in \mathcal{E} |$ domain of e is $X\}$.

Remark 2.4.2. (a) quot$X$ is naturally ordered by $p \leq q \Leftrightarrow \exists e\ (p = e \circ q)$ as shown in the diagram below (see [AHS90]). The fact that q is epic ensures that e is unique. Also, since p is epic we have e is also epic. Geometrically this ordering makes sense as the codomain of p is "smaller" than the codomain of q.



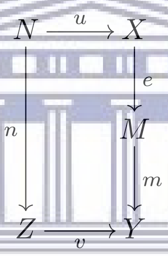
Indeed, the relation " $\leq$ ", which is defined above is reflexive and transitive. Hence quot$X$ is a preordered class.

(b) If $p \leq q$ and $q \leq p$ then codom$(p) \cong$ codom(q) and we write $p \cong q$. We do not distinguish between isomorphic quotient objects. We shall assume that quot$X$ is a set or a small class, i.e. $\mathbb{C}$ is $\mathcal{E}$-cowell-powered.

(c) Every object X has at least one $\mathcal{E}$-quotient object, namely 1_X. Since $p = p \circ 1_X \Leftrightarrow p \leq 1_X$ for all $p \in$ quot$X$, one has 1_X is the largest quotient object of X.

(d) In the categories of **Ab**, **Grp**, $\mathbf{Mod_R}$, and **Rng**, each quotient object $X \to P \cong X \to X/\text{ker}(X \to P)$. Indeed , this is due to the first isomorphism theorem in the respective categories.

Definition 2.4.3 (Left $\mathcal{E}$-factorization)**.** [DT95] Let $f : X \to Y$ be in $\mathbb{C}$ with $e : X \to M \in \mathcal{E}$ and $m : M \to Y$ in $\mathbb{C}$. Then any factorization $f = m \circ e$ such that for any commutative diagram



in $\mathbb{C}$ with $n \in \mathcal{E}$ there exists a unique $w : Z \to M$ with $v = m \circ w, e \circ u = w \circ n$ is called a left $\mathcal{E}$-factorization of f. The property of existence of w is called the diagonalization property of the factorization.

Remark 2.4.4. (a) A left $\mathcal{E}$-factorization of f in $\mathbb{C}$ is a right $\mathcal{E}$-factorization of f in $\mathbb{C}^{op}$. Indeed, we reverse the arrows and interchange the roles of e and m.

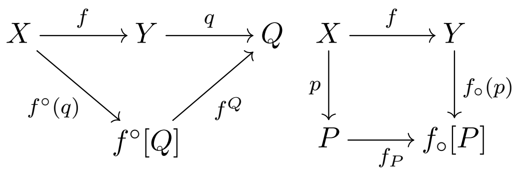
(b) Let $\mathcal{E}$ be closed under composition. Then $\mathbb{C}$ has $(\mathcal{E}, \mathcal{M})$ factorizations if and only if every morphism has a factorization which is simultaneously a left $\mathcal{E}$-factorization and a right $\mathcal{M}$-factorization.

(c) If $\mathbb{C}$ has $(\mathcal{E}, \mathcal{M})$-factorizations then $\mathbb{C}^{op}$ has $(\mathcal{M}, \mathcal{E})$-factorizations. Thus if a property holds for $\mathcal{M}$ then its dual is also true for $\mathcal{E}$ and vice versa.

The following is dual to the image-preimage definition given in Definition 1.3.3.

Definition 2.4.5 (co-image/co-preimage)**.** For $f : X \to Y$ in $\mathbb{C}$, $p \in$ quot$X$ and $q \in$ quot$Y$ we define the co-image $f^{\circ}(q) \in$ quot$X$ of q under f as the $\mathcal{E}$-component of the left $\mathcal{E}$ factorization of $q \circ f$, which is described by the left commutative diagram below, and the co-preimage $f_{\circ}(p) \in$ quot$Y$ of p under f

as the pushout of p along f, which is shown by the right commutative diagram below.



As a result of the co-image/co-preimage definition we get for every morphism $f : X \to Y$ there is co-image/co-preimage adjunction $f^\circ \dashv f_\circ$: quot$X \to$ quot$Y$. That is, for $q \in$ quot$Y$ and $p \in$ quot$X$ one has $f^\circ(q) \leq p$ if and only if $q \leq f_\circ(p)$. In the sequel we use $\mathcal{M}'$ to denote the class of morphisms in $\mathcal{M}$ that are stable under pushout along morphisms in $\mathcal{E}$ and $\mathcal{M}^*$ to denote the largest pushout stable class in $\mathcal{M}$. Consequenlty, one has the following dual results given in Remark 1.3.5.

Remark 2.4.6. Let $f : X \to Y$ be any morphism in $\mathbb{C}$ and $p, p_i \in$ quot$X$ and $q, q_i \in$ quot$Y$ for all $i \in I$ then since quot$X$ and quot$Y$ are preordered classes, one has the following properties:

(a) $q \leq f_\circ(f^\circ(q))$ and $f^\circ(f_\circ(p)) \leq p$;

(b) $f^\circ(\bigvee_{i\in I} q_i) = \bigvee_{i\in I} f^\circ(q_i)$ and $f_\circ(\bigwedge_{i\in I} p_i) = \bigwedge_{i\in I} f_\circ(p_i)$,

(c) $f \in \mathcal{M}$ if and only if $f^\circ(1_Y) \cong 1_X$;

(d) If $f \in \mathcal{E}$, or f is epic and $\mathcal{M}$ is stable under pushout along epimorphisms then $f_\circ(f^\circ(q)) \cong q$;

(d) If $f \in \mathcal{M}'$ then $f^\circ(f_\circ(p)) \cong p$;

(e) $f \in \mathcal{E}$ then $f^\circ(q) \cong q \circ f$. In particular, we have $p^\circ(1_P) \cong 1_P \circ p \cong p$.

Now we are ready to define dual closure operators. A dual closure operator d on $\mathbb{C}$ with respect to $(\mathcal{E}, \mathcal{M})$ factorization system is defined as a closure operator on $\mathbb{C}^{op}$ with respect to $(\mathcal{E}^{op}, \mathcal{M}^{op}) = (\mathcal{M}, \mathcal{E})$ factorization system. As a result we have the following definition.

Definition 2.4.7. [DT15] A dual closure operator d on $\mathbb{C}$ with respect to $\mathcal{E}$ is a family

$$(d_X : \text{quot}X \to \text{quot}X)_{X\in\mathbb{C}}$$

of functions which are

(D_1) extensive: $p \leq d_X(p)$,

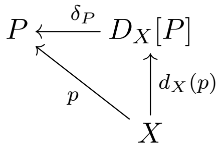
(D_2) monotone: if $p \leq p'$, then $d_X(p) \leq d_X(p')$,

(D_3) and which satisfy the continuity condition: $f^\circ(d_Y(q)) \leq d_X(f^\circ(q))$,

for all $f : X \to Y \in \mathbb{C}$ and $p, p' \in$ quot$X$ and $q \in$ quot$Y$.

The above definition of a dual closure operator is basically equivalent to the one which is given in [DT15] with the order relation reversed. Since the dual closure operator is acting on the quotient objects rather than subobjects the choice of our order makes sense geometrically, as mentioned at the beginning of this section. Let us also note that a very early attempt to introduce a notion of dual closure operators in the categories of groups was made by [Cas86] and it was termed coclosure operators.

The extension condition (D_1) implies that for every $\mathcal{E}$-quotient object $p : X \to P$ has a canonical factorization



That is: $p = p \circ 1_X = \delta_P \circ d_X(p)$.

Remark 2.4.8. Let $\text{sub}^{'}X$ be the subobjects of X in the opposite category. Then d is a family of functions $\{d_X : \text{sub}^{'}X \to \text{sub}^{'}X | X \in \mathbb{C}\}$ such that for all X in $\mathbb{C}$ we have:

(a) $(\forall p^{op} \in \text{sub}^{'}X)(p^{op} \leq d_X(p^{op}))$ if and only if $(\forall p \in \text{quot}X)(p \leq d_X(p))$;

(b) $(\forall p^{op}, t^{op} \in \text{sub}^{'}X)(p^{op} \leq t^{op} \Rightarrow d_X(p^{op}) \leq d_X(t^{op}))$ if and only if $(\forall p, t \in \text{quot}X)(p \leq t \Rightarrow d_X(p) \leq d_X(t))$ and

(c) $(\forall q^{op} \in \text{sub}^{'}Y$ and $\forall f^{op} : Y \to X \in \mathbb{C}^{op})(f^{op}(d_Y(q^{op})) \leq d_X(f(q^{op})))$ if and only if $(\forall q \in \text{quot}Y$ and $\forall f : X \to Y \in \mathbb{C})(f^{\circ}(d_Y(q)) \leq d_X(f^{\circ}(q)))$.

Lemma 2.4.9. The continuity condition of a dual closure operator d can be expressed as

$$f^{\circ}(d_Y(q)) \leq d_X(f^{\circ}(q)) \Leftrightarrow d_Y(f_{\circ}(p)) \leq f_{\circ}(d_X(p)) \text{ for all } q \in \text{quot}Y \text{ and } p \in \text{quot}X.$$

Proof. $(\Rightarrow)$ Let $q \in \text{quot}Y$ such that $f^{\circ}(d_Y(q)) \leq d_X(f^{\circ}(q))$. Then for $p \in \text{quot}X$ since $(f^{\circ}, f_{\circ})$ is a Galois connection we have $f_{\circ}(p) \in \text{quot}Y$ and hence we get $f^{\circ}(d_Y(f_{\circ}(p))) \leq d_X(f^{\circ}(f_{\circ}(p))) \leq d_X(p)$. Thus $d_Y(f_{\circ}(p)) \leq f_{\circ}(d_X(p))$.
$(\Leftarrow)$ Let $p \in \text{quot}X$ such that $d_Y(f_{\circ}(p)) \leq f_{\circ}(d_X(p))$. Then for $q \in \text{quot}Y$ since $(f^{\circ}, f_{\circ})$ is a Galois connection we have $f^{\circ}(q) \in \text{quot}X$ and hence we get $d_Y(q) \leq d_Y(f_{\circ}(f^{\circ}(q))) \leq f_{\circ}(d_X(f^{\circ}(q)))$. Thus $f^{\circ}(d_Y(q)) \leq d_X(f^{\circ}(q))$. □

Furthermore, one has the following:

Remark 2.4.10. The following are also equivalent descriptions of the continuity condition of a dual closure operator. Let $q \in \text{quot}Y$ and $p \in \text{quot}X$. Then $f^{\circ}(d_Y(q)) \leq d_X(f^{\circ}(q)) \Leftrightarrow d_Y(q) \leq f_{\circ}(d_X(f^{\circ}(q))) \Leftrightarrow f^{\circ}(d_Y(f_{\circ}(p)) \leq d_X(p)$.

Before we consider examples, let us recall the following terminologies from [DT95].

Definition 2.4.11. A preradical on **Ab** (**Grp**, $\mathbf{Mod_R}$, resp.) is a subfunctor of the identity functor of **Ab** (**Grp**, $\mathbf{Mod_R}$, resp.); that is, $\mathbf{r}X$ is a subgroup (subgroup, submodule, resp.) of X for every $X \in$ **Ab** (**Grp**, $\mathbf{Mod_R}$, resp.) and for every homomorphism $f : X \to Y$, $f(\mathbf{r}X) \leq \mathbf{r}Y$. $\mathbf{r}$ is called radical if $\mathbf{r}(X/\mathbf{r}X) = 0$, idempotent if $\mathbf{r}(\mathbf{r}X) = \mathbf{r}X$, hereditary if $\mathbf{r}M = M \cap \mathbf{r}X$, for every subgroup (subgroup, submodule, resp.) M of X and cohereditary if for every subgroup (subgroup, submodule) M of X, $\mathbf{r}(X/M) = (M + \mathbf{r}X)/M$.

Examples 2.4.12. [DT15]

(a) Consider the category **Ab** of abelian groups and group homomorphisms with (surjective homomorphisms, injective homorphisms)-factorization. We have the following prototypical dual closure operator d^t. Let $A \leq X \in$ **Ab** and $tA = \{a \in A : (\exists n \in Z^+)(na = 0)\}$ be the torsion subgroup of A. Define $d^t_X(X \to X/A) = X \to X/t\text{ker}(X \to X/A) = X \to X/tA$. Then

(i) Since $tA \leq A$ we have $X \to X/A \leq X \to X/tA = d^t_X(X \to X/A)$;

(ii) Suppose $X \to X/B \leq X \to X/A$ then $A \leq B$ and hence $tA \leq tB$. As a result, $d^t_X(X \to X/B) = X \to X/tB \leq X \to X/tA = d^t_X(X \to X/A)$;

(iii) Let $f : X \to Y$ be any homomorphism and $X \to X/A \in \text{quot}X$. Then since $f(tA) \leq tf(A)$ and the pushout of $X \to X/A$ is $Y \to Y/f(A)$, we have

$$\begin{aligned} d^t_Y(f_\circ(X \to X/A)) &= d^t_Y(Y \to Y/f(A)) = Y \to Y/tf(A) \\ &\leq Y \to Y/f(tA) = f_\circ(X \to X/tA) = f_\circ(d^t_X(X \to X/A)) \end{aligned}$$

Therefore by $(i), (ii), (iii)$ we have d^t is a dual closure operator on **Ab**.

(b) Consider the category $\mathbf{Mod_R}$ of $\mathbf{R}$ modules and $\mathbf{R}$-linear maps, for a commutative unital ring $\mathbf{R}$, with (surjective linear maps, injective linear maps)-factorization. One can have the following dual closure operators induced by preradicals. Recall that a preradical $\mathbf{r}$ in $\mathbf{Mod_R}$ is a subfunctor of the identity functor $1_{\mathbf{Mod_R}}$ of $\mathbf{Mod_R}$. That is, $\mathbf{r}$ is a functor on $\mathbf{Mod_R}$ such that $\mathbf{r}$ assigns to every $\mathbf{R}$-module $\mathbf{M}$ a submodule $\mathbf{r}M$ of M $(i.e.\ \mathbf{r}M \leq M)$ and $f(\mathbf{r}M) \leq \mathbf{r}N$ for every $\mathbf{R}$-linear map $f : M \to N$. Now, let $M \leq X \in \mathbf{Mod_R}$ and $\mathbf{r}$ be a preradical in $\mathbf{Mod_R}$. Define $(d_\mathbf{r})_X(X \to X/M) = X \to X/\mathbf{r}\text{ker}(X \to X/M) = X \to X/\mathbf{r}M$ and $d^\mathbf{r}_X(X \to X/M) = X \to X/\text{ker}(X \to X/M) \cap \mathbf{r}X = X \to X/M \cap \mathbf{r}X$. Then

(i) Since $\mathbf{r}M \leq M$ and $M \cap \mathbf{r}X \leq M$ one has
$X \to X/M \leq X \to X/\mathbf{r}M = (d_\mathbf{r})_X(X \to X/M)$ and
$X \to X/M \leq X \to X/M \cap \mathbf{r}X = d^\mathbf{r}_X(X \to X/M)$;

(ii) Suppose $X \to X/N \leq X \to X/M$. Then $M \leq N$ and hence for the injection $M \overset{i}{\hookrightarrow} N$ we get $\mathbf{r}M = i(\mathbf{r}M) \leq \mathbf{r}N$ and $M \cap \mathbf{r}X \leq N \cap \mathbf{r}X$. As a result,
$(d_\mathbf{r})_X(X \to X/N) = X \to X/\mathbf{r}N \leq X \to X/\mathbf{r}M = (d_\mathbf{r})_X(X \to X/M)$ and
$d^\mathbf{r}_X(X \to X/N) = X \to X/N \cap \mathbf{r}X \leq X \to X/M \cap \mathbf{r}X = d^\mathbf{r}_X(X \to X/M)$;

(iii) Let $f : X \to Y$ be any homomorphism and $X \to X/M \in \text{quot}X$. Then since
$f(\mathbf{r}M) \leq \mathbf{r}f(M)$ (because $M \overset{f}{\to} f(M)$ can be considered as linear map),
$f(M \cap \mathbf{r}X) \leq f(M) \cap f(\mathbf{r}X)) \leq f(M) \cap \mathbf{r}Y$ (because $M \cap \mathbf{r}X \leq M, \mathbf{r}X$ and $f(\mathbf{r}X) \leq \mathbf{r}Y$) and the pushout of $X \to X/M$ is $Y \to Y/f(M)$, hence

$$\begin{aligned} (d_\mathbf{r})_Y(f_\circ(X \to X/M)) &= (d_\mathbf{r})_Y(Y \to Y/f(M)) = Y \to Y/\mathbf{r}f(M) \\ &\leq Y \to Y/f(\mathbf{r}M) = f_\circ(X \to X/\mathbf{r}M) = f_\circ((d_\mathbf{r})_X(X \to X/M)) \text{ and} \end{aligned}$$

$$\begin{aligned} d^\mathbf{r}_Y(f_\circ(X \to X/M)) &= (d_\mathbf{r})_Y(Y \to Y/f(M)) = Y \to Y/f(M) \cap \mathbf{r}Y \\ &\leq Y \to Y/f(M \cap \mathbf{r}X) = f_\circ(X \to X/M \cap \mathbf{r}X) = f_\circ(d^\mathbf{r}_X(X \to X/M)). \end{aligned}$$

Therefore by $(i), (ii), (iii)$ we have $d_\mathbf{r}$ and $d^\mathbf{r}$ are a dual closure operators on $\mathbf{Mod_R}$.

(c) Consider the category **Grp** of groups and surjective group homomorphisms with (RegEpi, mono)-factorization. Let $N \trianglelefteq G \in$ **Grp**. Then

(i) $(d_{\mathbf{r}})_G(G \to G/N) = G \to G/\mathbf{r}\mathrm{ker}(G \to G/N) = G \to G/\mathbf{r}N$ and $d^{\mathbf{r}}_G(G \to G/N) = G \to G/\mathrm{ker}(G \to G/N) \cap \mathbf{r}G = G \to G/N \cap \mathbf{r}G$, where $\mathbf{r}$ is a preradical in **Grp**, are dual closure operators on **Grp**. Indeed, since $\mathbf{r}$ is a subfunctor of an identity functor $1_{\mathbf{Grp}}$ of **Grp** we have $\mathbf{r}G \leq G$ and for any automorphism of G, $f(\mathbf{r}G) \leq \mathbf{r}G$ and hence $\mathbf{r}G$ is a characterstic subgroup of G, denoted by $\mathbf{r}G$ char G, hence $\mathbf{r}G \trianglelefteq G$. Consequently, $\mathbf{r}N$ char $N \trianglelefteq G$ implies $\mathbf{r}N \trianglelefteq G$. Therefore, $(d_{\mathbf{r}})_G(G \to G/N) = G \to G/\mathbf{r}N$ and $d^{\mathbf{r}}_G(G \to G/N) = G \to G/N \cap \mathbf{r}G$ are well defined. Furthermore, for any surjective homomorphism $f : G \to H$, we have $N \trianglelefteq G \Rightarrow f(N) \trianglelefteq H$. We also have $f(\mathbf{r}G) \leq \mathbf{r}H$ which in turn implies $f(N \cap \mathbf{r}G) \leq f(N) \cap f(\mathbf{r}G) \leq f(N) \cap \mathbf{r}H$ and hence the continuity conditions of the two dual closure operators hold true. The extension and order preservation properties are trivially true for the two operators.

(ii) $(d_{\mathbf{c}})_G(G \to G/N) = G \to G/\mathbf{c}\mathrm{ker}(G \to G/N) = G \to G/\mathbf{c}N$ and $d^{\mathbf{c}}_G(G \to G/N) = G \to G/\mathrm{ker}(G \to G/N) \cap \mathbf{c}G = G \to G/N \cap \mathbf{c}G$, where $\mathbf{c}N$ is the commutator subgroup of N in **Grp**, are dual closure operators on **Grp**. In fact, $\mathbf{c}G \leq G$ and for any group homomorphism $f : G \to H$ we have $f(\mathbf{c}G) \leq \mathbf{c}H$. Hence, assigning to G its commutator subgroup $\mathbf{c}G$ defines a preradical of G. Therefore, by (a), $d_{\mathbf{c}}$ and $d^{\mathbf{c}}$ are dual closure operators.

(d) Consider the category **Rng** of unital rings and surjective ring homomorphisms with (RegEpi, mono)-factorization. Let $I \lhd R \in$ **Rng**. Then $d^n_R(R \to R/I) = R \to R/I^n$, where I^n is finite sums of n-fold products of elements in I, defines a dual closure operator. Here, $I = \mathrm{ker}(R \to R/I)$. Note that $I^n \lhd I$, $I_1 \subseteq I_2 \Rightarrow I_1^n \subseteq I_2^n$ and for any surjective ring homomorphism $f : R \to S$ we have $f(I) \lhd S$ and $f(I^n) \lhd (f(I))^n$.

(d) Consider the category **Top** with (quotient, injection)-factorization. Let $\mathbf{A}$ be the class of all non-empty connected subset of $X \in$ **Top** and $p : X \to P$ be a quotient map then $ew^{\mathbf{A}}(X \to P) = X \to X/\sim$, where $x \sim y \Leftrightarrow [p(x) = p(y)]\&[(\exists A \in \mathbf{A})(A \subseteq p^{-1}(\{p(x)\}))]$ with $x, y \in A$ defines a dual closure operator.The construction of this dual closure operator leads to ($\mathbf{A}$-monotone, $\mathbf{A}$-light)-factorization of morphisms whose codomain is $\mathbf{T_1}$.

We define closed and sparse quotient objects as in [DT15].

Definition 2.4.13. An $\mathcal{E}$-quotient object $p : X \to P$ of X is said to be

(a) d-closed if $p^{op} : P \to X$ is closed with respect to d as a closure operator in $\mathbb{C}^{op}$, that is: $d_X(p) \cong p \Leftrightarrow \delta_p : d_X[P] \to P$ is an isomorphism;

(b) d-sparse if p^{op} is dense with respect to d as a closure operator in $\mathbb{C}^{op}$, that is: $d_X(p) \cong 1_X \Leftrightarrow d_X(p) : X \to d_X[P]$ is an isomorphism.

Proposition 2.4.14. Let $f : X \to Y$ be a morphism in $\mathbb{C}$.

(a) If $p : X \to P \in \mathrm{quot}X$ is a d-closed then $f_\circ(p)$ is a d-closed quotient object of Y.

(b) If $q : Y \to Q \in \mathrm{quot}X$ is a d-sparse and $f \in \mathcal{M}$ then $f^\circ(q)$ is a d-sparse quotient object of X.

(c) If each $p_i : X \to P_i$ is a d-closed quotient object of X then $\bigwedge_{i \in I} p_i$ is a d-closed quotient object of X.

(d) If each $q_i : X \to Q_i$ is a d-sparse quotient object of Y then $\bigvee_{i \in I} q_i$ is a d-sparse quotient object of Y.

Proof. The dual proof can be found in [Cas03, Lemmas 4.11 and 4.13]. □

Remark 2.4.15. In the last section of the next chapter, we will see that a closed morphism in $\mathcal{E}$ is a closed quotient object.

Corollary 2.4.16. Let p be an epimorphism such that $q \circ p$ is a d-closed $\mathcal{E}$-quotient then q is d-closed.

Proof. Since p is an epimorphism the diagram

$$\begin{array}{ccc} \bullet & \xrightarrow{p} & \bullet \\ {\scriptstyle q\circ p}\downarrow & & \downarrow{\scriptstyle q} \\ \bullet & \xrightarrow[1]{} & \bullet \end{array}$$

is a pushout, hence $q \circ p$ is d-closed implies q is d-closed. □

Hereafter, we use C_d and S_d to denote the class of d-closed and the class of d-sparse morphisms in $\mathcal{E}$ respectively. The dual of the next definition is given in [DT95].

Definition 2.4.17. A morphism $f : X \to Y$ is d-sparse if $f^{\circ}(1_Y)$ is a d-sparse quotient object of X. That is, $d_X(f^{\circ}(1_Y)) \cong 1_X$.

Remark 2.4.18. (a) $f : X \to Y$ in $\mathbb{C}$ is d-sparse if $f^{op} : Y \to X$ in $\mathbb{C}^{op}$ is dense dense with respect to d as a closure operator in $\mathbb{C}^{op}$. That is, $f^{op}(1_Y)$ is dense in Y, which is equivalent to $d_X(f^{op}(1_Y)) \cong 1_X$.

(b) $\mathcal{M}$ is a subclass of S_d.

(c) S_d is closed under limits in $\mathbb{C}^2$. In particular, if for any morphisms p and q, $q \circ p \in S_d$, then $p \in S_d$.

Examples 2.4.19. [DT15]

(a) Consider the dual closure operator d^t defined by $d^t_X(X \to X/A) = X \to X/tA$, where $A \leq X \in$ **Ab**. Then

(i) If A is a torsion subgroup then $tA = A$ and hence $d^t_X(X \to X/A) = X \to X/tA = X \to X/A$. Therefore, $X \to X/A$ is d^t-closed quotient object of X.

(ii) If A is a torsion-free subgroup then $tA = \{0\}$ and hence $d^t_X(X \to X/A) = X \to X/tA = X \to X/\{0\} \cong X \to X \cong 1_X$. Consequently, $X \to X/A$ is d^t-sparse quotient object of X.

(b) Consider the dual closure operator $d_{\mathbf{c}}$ defined by $(d_{\mathbf{c}})_G(G \to G/N) = G \to G/\mathbf{c}N$, where $N \trianglelefteq G \in$ **Grp**. Then

(i) If N is a perfect subgroup then $\mathbf{c}N = N$. Consequently, $(d_{\mathbf{c}})_G(G \to G/N) = G \to G/\mathbf{c}N = G \to G/N$. Therefore, $G \to G/N$ is $d_{\mathbf{c}}$-closed quotient object of G.

(ii) If N is an abelian subgroup then $\mathbf{c}N = \{e_G\}$. Consequently, $(d_{\mathbf{c}})_G(G \to G/N) = G \to G/\mathbf{c}N = G \to G/\{e_G\} \cong G \to G \cong 1_G$. Therefore, $G \to G/N$ is $d_{\mathbf{c}}$-sparse quotient object of G.

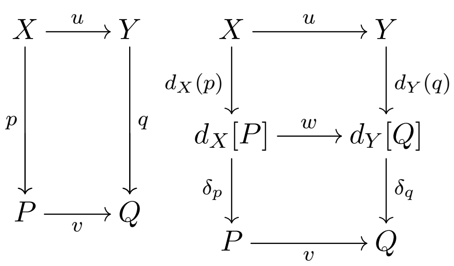
(c) Consider the dual closure operators$(d_{\mathbf{r}})_X(X \to X/M) = X \to X/\mathbf{r}M$ and $d^{\mathbf{r}}_X(X \to X/M) = X \to X/M \cap \mathbf{r}X$, where $M \leq X \in \mathbf{Mod_R}$. Then

(i) $X \to X/M$ is $d_{\mathbf{r}}$-closed ($d_{\mathbf{r}}$-sparse) quotient object of X if and only if $M \subseteq \mathbf{r}M$ ($\mathbf{r}M = \{0\}$), respectively;

(ii) $X \to X/M$ is $d^{\mathbf{r}}$-closed ($d^{\mathbf{r}}$-sparse) quotient object of X if and only if $M \subseteq \mathbf{r}X$ ($M \cap \mathbf{r}X = \{0\}$), respectively.

The following lemma is a dual of the Diagonalization Lemma given in [DT95].

Lemma 2.4.20 (Dual Diagonalization Lemma (DDL))**.** For any commutative left diagram below



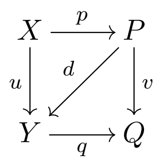
with $p, q \in \mathcal{E}$, there is a unique morphism $w : d_X[P] \to d_Y[Q]$ such that the above right diagram commutes.

Proof. By the diagonalization property of left $\mathcal{E}$-factorizations one has $u^{\circ}(q) \leq p$ in quot$X$. Hence by the order preservation and continuity condition of a dual closure operator we get $u^{\circ}(d_Y(q)) \leq d_X(p)$. Therefore, w is the composite $d_X[P] \to u^{\circ}[d_Y[Q]] \to d_Y[Q]$. □

Corollary 2.4.21. (a) If q in (DDL) is d-sparse then there exists a unique $t : d_X[p] \to Y$ such that $t \circ d_X(p) = u$ and $v \circ \delta_p = q \circ u$;

(b) If p in (DDL) is d-closed then there exists a unique $s : P \to d_Y[Q]$ such that $s \circ p = d_Y(q) \circ u$ and $v = \delta_q \circ s$;

(c) In (DDL) if q is d-sparse and p is d-closed then there exists a unique $d : P \to Y$ such that $d \circ p = u$ and $q \circ d = v$, that is: we have the following commutative diagram.



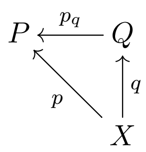
(d) $\mathbf{C}_d \cap \mathbf{S}_d$ is the class of isomorphisms in $\mathbb{C}$

Definition 2.4.22. [DT15] The dual closure operator d is

(a) idempotent if it is idempotent as a closure operator in $\mathbb{C}^{op}$, that is, $d_X(d_X(p)) \cong d_X(p)$ for all $p : X \to P \in \mathcal{E}$;

(b) weakly cohereditary (wch) if it is weakly hereditary as a closure operator in $\mathbb{C}^{op}$, that is, $\delta p : d_X[P] \to P$ is d-sparse, which is equivalent to $d_{d_X[P]}(\delta p) \cong 1_{d_X[P]}$ for all $p : X \to P \in \mathcal{E}$;

(c) coherediary (ch) if it is hereditary as a closure operator in $\mathbb{C}^{op}$, that is, for every pair of $\mathcal{E}$-quotient objects p, q of $X \in \mathbb{C}$ with $p \leq q$ one has $d_Q(p_q) \cong q_\circ(d_X(p))$, where p_q is the unique morphism such that $p = p_q \circ q$, shown in the commutative diagram below. We observe that $p_q \cong q_\circ(p)$.



(d) maximal if it is minimal as a closure operator in $\mathbb{C}^{op}$, that is, $d_X(q) \cong q \vee d_X(p)$ for all $p \leq q$ in quot$X$.

As a consequence of Definition 2.4.22 we have the following remarks, which are dual to the results given in [DT95].

Remark 2.4.23. (a) For an idempotent dual closure operator d, every morphism has a left C_d-factorization and S_d is closed under composition.

(b) For a weakly cohereditary dual closure operator d, every morphism has a right S_d-factorization and C_d is closed under composition.

Remark 2.4.24. The following assertions are equivalent for a dual closure operator d:

1. d is idempotent and wch;
2. d is idempotent and C_d is closed under composition;
3. d is wch and S_d is closed under composition;
4. $\mathbb{C}$ has $(\mathbf{C_d}, \mathbf{S_d})$-factorizations.

As a consequence one has:

Remark 2.4.25. Let d be a dual closure operator.

(a) d is ch if and only if d is wch and S_d is right cancellable with respect to $\mathcal{E}$ ($q \circ p \in S_d \Rightarrow q \in S_d$). Consequently, using Remark 2.4.24 we also have d is ch and idempotent if and only if d is wch, S_d is right cancellable with respect to $\mathcal{E}$ and S_d is closed under composition.

(b) d is maximal if and only if d is idempotent and C_d is left cancellable with respect to $\mathcal{E}$ ($q \circ p \in C_d \Rightarrow p \in C_d$). Consequently, using Remark 2.4.24 we also obtain d is maximal and wch if and only if d is idempotent, C_d is left cancellable with respect to $\mathcal{E}$ and C_d is closed under composition.

Examples 2.4.26. [DT15]

(a) Consider the dual closure operator d^t defined by $d^t_X(X \to X/A) = X \to X/tA$, where $A \leq X \in$ **Ab**. Then

(i) Since $t(tA) = tA$ we have $d^t(d^t_X(X \to X/A)) = d^t(X \to X/tA) = X \to X/t(tA) = X \to X/tA = d^t_X(X \to X/A)$ and hence d^t is idempotent.

(ii) Since ker$(X/tA \to X/A) = A/tA$ and $t(A/tA) = \{0_A\}$ we have $d_{X/tA}(X/tA \to X/A) \cong d_{X/tA}(X/tA \to (X/tA)\diagup(A/tA)) \cong X/tA \to (X/tA)\diagup t(A/tA) \cong X/tA \to (X/tA)\diagup\{0_A\} \cong X/tA \to X/tA \cong 1_{X/tA} \cong 1_{d_X[x \to X/A]}$. Thus d^t is a wch dual closure operator.

(iii) d^t is maximal by Remark 2.4.25(b). Indeed, by (ii), d^t is idempotent and for $A \leq B \leq X$ and $X \xrightarrow{p} X/A \xrightarrow{q} X/B$, if $q \circ p \in C_d$ then B is a torsion subgroup and hence A is a torsion subgroup and this in turn implies $p \in C_d$.

(iv) d^t is not cohereditary by Remark 2.4.25(a). Indeed, for $A \leq B \leq X$ and $X \xrightarrow{p} X/A \xrightarrow{q} X/B$, if $q \circ p \in S_d$ then B is torsion free and this may not imply ker$q = B/A$ is torsion free and hence $q \in S_d$.

(b) Consider the dual closure operators$(d_{\mathbf{r}})_X(X \to X/M) = X \to X/\mathbf{r}M$ and $d^{\mathbf{r}}_X(X \to X/M) = X \to X/M \cap \mathbf{r}X$, where $M \leq X \in \mathbf{Mod_R}$. Then

(i) $d^{\mathbf{r}}$ is idempotent and maximal;

(ii) $d_{\mathbf{r}}$ is idempotent if $\mathbf{r}$ is idempotent;

(c) $(d_{\mathbf{c}})_G(G \to G/N) = G \to G/\mathbf{c}N$, where $N \trianglelefteq G \in$ **Grp** is wch but not idempotent (hence not maximal).

(d) Let $H \leq G \in$ **Ab** and $m \in Z^+$ then $d^m_G(G \to G/H) = G \to G/mH$ is a dual closure operator, which is cohereditary (hence wch).

In what follows we compare dual closure operators with interior operators. In Definitions 2.1.1 and 2.4.7 we have seen that interior operators act on subobjects while dual closure operators act on the dual of subobjects, quotient objects rather than subobjects. We also know that the notion of dual closure operators is the categorical dual of the notion of closure operators and the notion of interior operator is an order dualization of the notion of closure operators. In fact, in any category where all the subobjects are normal (= their morphisms are equivalently described by their kernels), that is, when $\mathcal{E}$ is the class of regular epimorphisms of the category, for example in all abelian categories, one can redefine dual closure operators on subobjects rather than on quotient objects as given below.

Definition 2.4.27. A dual closure operator d on $\mathbb{C}$ with respect to $\mathcal{M}$ is a family

$$(d_X : \text{sub}X \to \text{sub}X)_{X \in \mathbb{C}}$$

of functions which are

(D_1) contractive: $d_X(r) \leq r$,

(D_2) monotone: if $r \leq s$, then $d_X(r) \leq d_X(s)$,

(D_3) and which satisfy the continuity condition: $d_X(f^\circ(n)) \leq f^\circ(d_Y(n))$,

for all $f : X \to Y \in \mathbb{C}$ and $r, s \in \text{sub}X$ and $n \in \text{sub}Y$.

Consequently, we can study the relationship between dual closure and interior operators. Indeed, they share two of the three characteristic properties (order preservation and contraction conditions), but the continuity condition of each is different, and this difference is significant: in all abelian categories we have abundant dual closure operators and contrary to this there is only one, discrete interior operator.

2.5 Some Examples of Interior operators

Most of the examples given in this section will be used throughout the thesis.

Examples 2.5.1. (a) [CR10, Cas11, CM13] Let $\mathbb{C}$ be the category **Top** of topological spaces and continuous maps with (Surjections, Embeddings)-factorization system and $R \subseteq X \in$ **Top**. We define the following interior operators on $\mathbb{C} =$**Top** with respect to $\mathcal{M} =$ Embeddings.

(i) The Kuratowski interior operator given by $k_X^{\mathsf{in}}(R) = \bigcup\{O$ open in $X : O \subseteq R\}$.

(ii) The inverse Kuratowski interior operator given by $k_X^{*\mathsf{in}}(R) = \bigcup\{C$ closed in $X : C \subseteq R\} = \{x \in R : k_X(\{x\}) \subseteq R\}$, where $k_X(\{x\})$ is the Kuratowski closure of $\{x\}$ in the topology of X.

(iii) The Θ^{in}-interior given by $\Theta_X^{\mathsf{in}}(R) = \{r \in R : \exists$ a open neighbourhood U_r of r in X such that $k_X(U_r) \subseteq R\}$, where $k_X(U_r)$ is the Kuratowski closure of U_r. With the trivial observations that $\Theta_X^{\mathsf{in}}(R) = \bigcup\{k_X^{\mathsf{in}}(C) : C \subseteq R, C$ closed in $X\}$.

(iv) The quasicomponent interior operator given by $q_X^{\mathsf{in}}(R) = \bigcup\{O$ clopen in $X : O \subseteq R\}$. Note that, by Proposition 2.3.11 and the Theorem in [DT95, p. 87], the quasicomponent interior operator q^{in} is the smallest proper interior operator of **Top** and satisfies $t^{\mathsf{in}} \leq s \leq q^{\mathsf{in}} \leq p^{\mathsf{in}} \leq d^{\mathsf{in}}$ for any proper interior operator p^{in} (see also Remark 2.1.6(b)).

(v) The b^{in}-interior (or front interior) defined by $b_X^{\mathsf{in}}(R) = \{r \in R : \exists$ a open neighbourhood U_r of r in X such that $k_X(\{r\}) \cap U_r \subseteq R\}$, where $k_X(\{r\})$ is the Kuratowski closure of $\{r\}$.

(vi) The l^{in}-interior defined by $l_X^{\mathsf{in}}(R) = \{r \in X : C_r \subseteq R\}$, where C_r denotes the connected component of $r \in X$.

(vii) The sequential interior operator given by $s_X(R) = \{r \in R :$ there does not exist a sequence (x_n) in $X \setminus R$ converging to $r\}$.

$(viii)$ The zero-interior operator z^{in} defined by $z_X^{\mathsf{in}}(R) = \bigcup\{Q \subseteq X : Q \subseteq R$ and $X \setminus Q$ is a zero set$\}$. Note that a zero set S is the preimage of zero under a continuous function from X to $\Re$, that is: $S = f^{-1}(0) = \{x \in X : f(x) = 0\}$, with $f : X \to \Re$ continuous.

(b) Let $\mathbb{C}$ be the category **SGph** of directed spatial graphs and graph homomorphisms with the (Surjective homorphisms, Embeddings)-factorization system, where a directed spatial graph (G, R) consists of a set G of vertices and a reflexive relation $R \subseteq G \times G$ of edges of the graph (there is a loop at each vertex/point of the directed graph). We use $g \to g^{'} \Leftrightarrow (g, g^{'}) \in R$ to describe there is an edge from g to g'. A graph homomorphism $f : (G, R) \to (G^{'}, R^{'})$ is an edge preserving map, that is, $f : G \to G^{'}$ such that $g \to g^{'}$ implies $f(g) \to f(g^{'})$ for all $g, g^{'} \in G$. For $H \subseteq G$, we have $(H, R \cap (H \times H))$ is a subgraph (an embedding) of a directed spatial graph (G, R) and for the class $\mathcal{M}$ of all embeddings, **SGph** is $\mathcal{M}$-complete and hence finitely $\mathcal{M}$-complete; see [DT95]. For each directed spatial graph (G, R) and a subset H of G, consider the assignments $\uparrow^{\mathsf{in}}$ and $\downarrow^{\mathsf{in}}$ given as follows:

(i) the up-interior $\uparrow_G^{\mathsf{in}}(H) = \{h \in H : \nexists g \in G \setminus H$ such that $g \to h\} = \{h \in H :$ $(\forall g \in G \setminus H)$ there is no edge $g \to h\}$;

(ii) the down-interior $\downarrow_G^{\mathsf{in}}(H) = \{h \in H : \nexists g \in G \setminus H$ such that $h \to g\} = \{h \in H :$ $(\forall g \in G \setminus H)$ there is no edge $h \to g\}$;

Both $\uparrow^{\text{in}}$ and $\downarrow^{\text{in}}$ are standard and non-idempotent interior operators of **SGph**.

(c) Let $\mathbb{C}$ be the category **Unif** of uniform spaces and uniformly continuous maps with (Surjections, Uniform embeddings)-factorization system, where a uniform space $(X,\mathcal{U})$ comprises a set X and a filter $\mathcal{U}$ of reflexive relations on X such that for every $U \in \mathcal{U}$ there exists $V, W \in \mathcal{U}$ with $V^{-1} = \{(y,x) : (x,y) \in V\} \subseteq U$, $W \circ W = \{(x,w) : (\exists y \in X)(x,y),(y,w) \in W\} \subseteq U$. A uniformly continuous map $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is a map $f : X \to Y$ such that for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $f : (X,U) \to (Y,V)$ is a spatial graph homomorphism, that is: $(f \times f)(U) \subseteq V$ and for the class $\mathcal{M}$ of all uniform embeddings, **Unif** is $\mathcal{M}$-complete; see [DT95]. For each uniform space $(X,\mathcal{U})$ and subset R of X define $q^{\text{uin}}_X(R) = \bigcup\{U \subseteq X : U \subseteq R, X \setminus U \text{ uniformly clopen}\}$. Then $q^{\text{uin}} = (q^{\text{uin}}_X)_{X \in \textbf{Unif}}$ is a standard, idempotent and additive interior operator. Note that a subset R of a uniform space X is called uniformly clopen if the characterstic function $\chi_R : X \to D = \{0,1\}$ with D discrete uniform space is uniformly continuous.

(d) Let $\mathbb{C}$ be the category **QUnif** of quasi-uniform spaces and uniformly continuous maps with (Surjections, Quasi-uniform embeddings)-factorization system, where a uniform space $(X,\mathcal{U})$ comprises of a set X and a filter $\mathcal{U}$ of reflexive relations on X such that for every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ with $V \circ V \subseteq U$. A uniformly continuous map $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is a map $f : X \to Y$ such that for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $f : (X,U) \to (Y,V)$ is a spatial graph homomorphism and for the class $\mathcal{M}$ of all quasi-uniform embeddings, **QUnif** is $\mathcal{M}$-complete; see [FL82, DT95, DK00]. Let $(X,\mathcal{U})$ be a uniform space and $R \subseteq X$. Then $\alpha^{\text{in}}_X(R) = \bigcup\{X \setminus U^{-1}[X \setminus R] : U \in \mathcal{U}\}$, $\beta^{\text{in}}_X(R) = \bigcup\{X \setminus (U \cap U^{-1})[X \setminus R] : U \in \mathcal{U}\}$ and $\gamma^{\text{in}}_X(R) = \bigcup\{X \setminus U[X \setminus R] : U \in \mathcal{U}\}$, where $U[R]$ denotes $\{y \in X : (x,y) \in U \text{ for some } x \in R\}$, are standard, idempotent and additive interior operators on **QUnif** with respect to $\mathcal{M}$. Moreover, for a subcategory $\mathbb{A}$ of **QUnif**, $\text{regint}^{\mathbb{A}}_X(R) = \bigcup\{\text{Sep}(f,g) \subseteq R : f,g : X \to A, A \in \mathbb{A}\}$, where $\text{Sep}(f,g) = \{x \in X : f(x) \neq g(x)\}$ is an idempotent interior operator. In fact, there are abundant interior operators of **QUnif**.

The above examples of interior operators all are obtained from their corresponding well known closure operators by applying Proposition 2.3.11. Indeed, by Proposition 2.3.13 every closure operator gives rise to an interior operator on a category $\mathbb{C}$ having a categorical transformation operator.

(e) [Cas11] Let $\mathbb{C}$ be the category **Grp** of groups and group homomorphisms with the (Surjective homomorphisms, Injective homomorphisms)-factorization system. Let H be a subgroup of $G \in$**Grp**. Define $n_G(H) = \bigvee\{N \trianglelefteq G : N \leq H\}$ with $N \leq H$ standing for "N subgroup of H" and $N \trianglelefteq G$ for "N normal subgroup of H", which is the subgroup generated by all the normal subgroups in G contained in H. Then for a group homomorphism $f : G_1 \to G_2$ and K subgroup of G_2 we obtain $n_{G_2}(K) \trianglelefteq G_2$ as the subgroup generated by the family of normal subgroups is normal. Also, since the inverse image of a normal subgroup is normal we get $f^{-1}(n_{G_2}(K))$ is a normal subgroup contained in $f^{-1}(K)$. Hence $f^{-1}(n_{G_2}(K)) \leq n_{G_1}(f^{-1}(K))$. We can easily verify the other two conditions of interior operators. Therefore, $n_G(H)$ is an interior operator in **Grp** and n is called the normal interior operator. Note that if G is a Dedekind group (a group in which every subgroup is normal), then the normal and discrete interior operators coincide.

(f) [Cas16] Consider the category **Rng** of rings and ring homomorphisms with the (Surjective homomorphisms, Injective homomorphisms)-factorization system. The ideal operator J, defined for $S \leq R \in$ **Rng** by $j_R(S) = \bigvee\{I \text{ an ideal of } R : I \leq S\}$ is an interior operator on **Rng** with respect to injective homorphisms since ideals are preserved under suprema and preimages. Note that even if R has a unity, its subrings need not contain this. Let us also note that if R is a cyclic ring (a ring in which its additive group is cyclic, hence each of its subrings is an ideal), then the ideal and discrete interior operators coincide.

(g) [DT15] In the category R-**Mod** of modules over a commutative ring R and module homomorphisms with the (Surjective homomorphisms, Injective homomorphisms)-factorization system, one only has the discrete interior operator. Indeed, for $N \leq M \in R$-**Mod** let $i_M : \text{sub}M \to \text{sub}M$ be an interior operator. Then for any module homomorphism $f : M \to M'$ and $N' \leq M'$, by the continuity condition (I_3), $f^{-1}(i_{M'}(N')) \leq i_M(f^{-1}(N'))$. In particular, for the quotient module homomorphism $f : M \to M/N$, one has $N =$ kernel of $f = f^{-1}(0_{M/N}) = f^{-1}(i_{M/N}(0_{M/N})) \leq i_M(f^{-1}(\{0_{M/N}\})) = i_M(N)$. Therefore, $i_M(N) = N$ for all $N \leq M$, that is, the interior operator i is discrete.

Remark 2.5.2. (a) [DT15] As a generalization of the Example (g), in any category where all the subobjects are normal (= their morphisms are equivalently described by their kernels), for example in all abelian categories including the category **Ab** of abelian groups and group homorphism, the category F-**Vect** of vector spaces over a field F and linear transformations, R-**Mod** and so on, one only has the discrete interior operator.

(b) The category **Set** with $\mathcal{M}$ the class of injective maps has no proper interior operators. Indeed, this follows readily from Proposition 2.3.11 and the Lemma in [DT95, p. 87].

By considering a pseudofunctor $\mathcal{S} : \mathbb{C} \to$ **Pos**, where **Pos** is the category of partially ordered sets and monotone maps, such that for any $f : X \to Y$ in $\mathbb{C}$ one can assign an adjoint pair

$$\mathcal{S}Y \underset{f_*(-)}{\overset{f^*(-)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{S}X \,.$$

one can have further examples. As pointed out in [RH17], considering a category $\mathbb{C}$ with the pseudofunctor $\mathcal{S}$ is opening up a wider range of examples of interior operators operating on $\mathcal{S}X$ instead of sub$X$ for each $X \in \mathbb{C}$ without requiring the structure of a factorization system.

Examples 2.5.3. [RH17] Let $\mathbb{C} =$**Loc** be the category of locales and locale maps. Then its dual **Frm** is the category of frames and frame homomorphisms. Indeed, for each locale map $f : X \to Y$ one has an adjunction

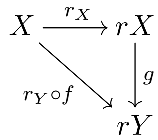
$$\mathcal{O}Y \underset{f_*(-)}{\overset{f^*(-)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{O}X \,.$$

where $\mathcal{O}X$ is the lattice of "open sets" of a locale X and f^* is the frame homomorphism. Since f^* preserves joins it has right adjoint f_* given by $f_*(x) = \bigvee\{y : f^*(y) \leq x\}$. Therefore $\mathcal{S} = \mathcal{O}$ is the required pseudofunctor. Define $i_X(m) = \bigvee\{x \in \mathcal{O}X : x \prec m\}$, where $\prec$ is the rather below or completely below or proximity relation on frames. Then as a result of the properties of $\prec$ we have $i = \{i_X : X \in$ **Loc**$\}$ is an interior operator on $\mathbb{C}$ with respect to $\mathcal{O}$.

Next we study interior operators induced by reflections.

Proposition 2.5.4. Let $\mathbb{S}$ be a full reflective subcategory of $\mathbb{C}$, $X \in \mathbb{C}$ and $X \overset{r_X}{\to} rX$ be the reflection morphism. Then $j^{\mathbb{S}}_X(m) = r^*_X((r_X)_*(m)))$ defines an idempotent and standard interior operator $j^{\mathbb{S}}$ on $\mathbb{C}$ with respect to $\mathcal{M}$.

Proof. It is obvious that $j^{\mathbb{S}}$ is contractive and monotone and $j^{\mathbb{S}}_X(1_X) = 1_X$. To show the remaining property, let $f : X \to Y$ be a morphism in $\mathbb{C}$ and $n \in$ sub$Y$. Then there exist reflection morphisms $r_X : X \to rX$ and $r_Y : Y \to rY$ with $rX, rY \in \mathbb{S}$ such that for morphism $X \overset{r_Y \circ f}{\to} r_Y, \exists! g : rX \to rY$ in $\mathbb{S}$ such that the diagram



commutes. That is, $g \circ r_X = r_Y \circ f$. Consequently,
$f^*(j^{\mathbb{S}}_Y(n)) = f^*(r^*_Y((r_Y)_*(n))) \leq f^*(r^*_Y((r_Y)_*(f_*(f^*(n))))) = (r_Y \circ f)^*((r_Y \circ f)_*(f^*(n))) =$
$(g \circ r_X)^*((g \circ r_X)_*(f^*(n))) = r^*_X(g^*(g_*((r_X)_*(f^*(n))))) \leq r^*_X((r_X)_*(f^*(n)))) = j^{\mathbb{S}}_X(f^*(n))$.
Thus $j^{\mathbb{S}}$ is an interior operator. Moreover, since $r^*_X \dashv (r_X)_*$ one obtains
$j^{\mathbb{S}}_X(m) = r^*_X((r_X)_*(m))) \leq r^*_X(r_X)_*(r^*_X((r_X)_*(m))))) = r^*_X(r_X)_*(j^{\mathbb{S}}_X(m)) = j^{\mathbb{S}}_X(j^{\mathbb{S}}_X(m))$. Therefore, $j^{\mathbb{S}}$ is idempotent. □

In fact, if $\mathbb{S}$ is $\mathcal{M}$-reflective subcategory then $j^{\mathbb{S}}$ coincides with a discrete interior operator.

Remark 2.5.5. For a full reflective subcategory $\mathbb{S}$ of $\mathbb{C}$ with $X \in \mathbb{S}$ and $m \in$sub$X$ one has 1_X as reflection morphism. Consequently, $j^{\mathbb{S}}_X(m) \cong m \cong d(\mathbb{S})_X(m)$.

As a generalization of Proposition 2.5.4 one has:

Proposition 2.5.6 (Lifting of an interior operator)**.** Let $\mathbb{S}$ be a reflective subcategory of $\mathbb{C}$ and i be an interior operator on $\mathbb{S}$ with respect to $\mathcal{M}$. If preimages commute with arbitrary joins in the category $\mathbb{C}$ then the family $(i(\mathbb{S})_X :$ sub$X \to$ sub$X)_{X \in \mathbb{C}}$ such that for all $X \in \mathbb{C}$, $i(\mathbb{S})_X(m) = r^*_X(i_{rX}((r_X)_*(m)))$, where $X \overset{r_X}{\to} rX$ is the $\mathbb{S}$-reflection morphism, is an interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$.

Proof. (a) Let $m \in$ sub$X$. Then $r^*_X(i_{rX}((r_X)_*(m))) \leq r^*_X((r_X)_*(m))) \leq m$;

(b) Let $m, n \in$ sub$X$ such that $m \leq n$ then $(r_X)_*(m) \leq (r_X)_*(n)$. Hence $i_{rX}((r_X)_*(m)) \leq i_{rX}((r_X)_*(n))$. Therefore $i(\mathbb{S})_X(m) = r^*_X(i_{rX}((r_X)_*(m))) \leq r^*_X(i_{rX}((r_X)_*(n))) = i(\mathbb{S})_X(n)$.

(c) Let $f : X \to Y$ be a morphism in $\mathbb{C}$ and $n \in$ sub$Y$. Then since $r_X : X \to rX$ and $r_Y : Y \to rY$ are reflection morphisms $\exists! g : rX \to rY$ in $\mathbb{S}$ such that $g \circ r_X = r_Y \circ f$. Consequently, $f^*(i(\mathbb{S})_Y(n)) = f^*(r^*_Y(i_{rY}((r_Y)_*(n)))) = r^*_X(g^*(i_{rY}((r_Y)_*(n)))) \leq r^*_X(i_{rX}(g^*((r_Y)_*(n)))) \leq r^*_X(i_{rX}(g^*((r_Y)_*(f_*(f^*(n)))))) = r^*_X(i_{rX}(g^*(g_*((r_X)_*(f^*(n)))))) \leq r^*_X(i_{rX}((r_X)_*(f^*(n)))) = i(\mathbb{S})_X(f^*(n))$.

□

We call the interior operator $i(\mathbb{S})$ a lifted interior operator on $\mathbb{C}$ from $\mathbb{S}$. The lifted interior operator is a counterpart of the lifted closure operator, found in [Cas03].

Corollary 2.5.7. Let i be an interior operator on a reflective subcategory $\mathbb{S}$ of the category $\mathbb{C}$ such that preimages commute with arbitrary joins in $\mathbb{C}$ and $i(\mathbb{S})$ be a lifted interior operator on $\mathbb{C}$ from $\mathbb{S}$. Then the following assertions hold.

(a) An $\mathcal{M}$-subobject m is $i(\mathbb{S})$-open in X implies $(r_X)_*(m)$ is i-open $\mathcal{M}$-subobject in rX, provided that $\mathbb{S}$ is $\mathcal{E}'$-reflective.

(b) An $\mathcal{M}$-subobject n is i-open in rX implies $r_X^*(n)$ is $i(\mathbb{S})$-open in X.

(c) 1_{rX} is i-open in rX implies 1_X is $i(\mathbb{S})$-open in X. Furthermore, if $\mathbb{S}$ is $\mathcal{E}'$-reflective then the converse is true.

(d) If i is idempotent and $\mathbb{S}$ is $\mathcal{E}'$-reflective then $i(\mathbb{S})$ is idempotent.

(e) If i is additive then so is $i(\mathbb{S})$. Moreover, if i is standard then so is $i(\mathbb{S})$.

Proof. (a) Suppose m is $i(\mathbb{S})$-open in X and $r_X \in \mathcal{E}'$ then
$i_{rX}((r_X)_*(m)) \cong (r_X)_*(r_X^*(i_{rX}((r_X)_*(m)))) \cong (r_X)_*(i(\mathbb{S})_X(m)) \cong (r_X)_*(m)$.

(b) Assume n is i-open in rX then $r_X^*(n) \cong r_X^*(i_{rX}(n)) \leq r_X^*(i_{rX}((r_X)_*(r_X^*(n)))) \cong i(\mathbb{S})_X(r_X^*(n))$.

(c) Suppose 1_{rX} is i-open in rX then
$i(\mathbb{S})_X(1_X) \cong r_X^*(i_{rX}((r_X)_*(1_X))) \cong r_X^*(i_{rX}(1_{rX})) \cong r_X^*(1_{rX}) \cong 1_X$. On the other hand assume 1_X is $i(\mathbb{S})$-open in X and $r_X \in \mathcal{E}'$ then

$$\begin{aligned} i_{rX}(1_{rX}) &\cong (r_X)_*(r_X^*(i_{rX}((r_X)_*(1_{rX})))) \\ &\cong (r_X)_*(r_X^*(i_{rX}((r_X)_*((r_X)_*(1_X))))) \cong (r_X)_*(i(\mathbb{S})_X(1_X)) \cong (r_X)_*(1_X) \cong 1_{rX}. \end{aligned}$$

(d) Suppose i is idempotent and $r_X \in \mathcal{E}'$ then for all $m \in \text{sub}X$,

$$\begin{aligned} i(\mathbb{S})_X(i(\mathbb{S})_X(m)) &\cong r_X^*(i_{rX}((r_X)_*(i(\mathbb{S})_X(m)))) \\ &\cong r_X^*(i_{rX}((r_X)_*(r_X^*(i_{rX}((r_X)_*(m)))))) \\ &\cong r_X^*(i_{rX}(i_{rX}((r_X)_*(m)))) \cong r_X^*(i_{rX}((r_X)_*(m))) \cong i(\mathbb{S})_X(m). \end{aligned}$$

(e) This follows from the fact that both r_X^* and $(r_X)_*$ commute with arbitray meets (hence with binary meets) and additivity of i. In fact, the standardness of $i(\mathbb{S})$ follows from Remarks 1.3.5(f) and 1.4.3(c).

□

3. Special Morphisms with Respect to an Interior Operator

In the first three sections of this chapter, as in the previous chapters we work with an $\mathcal{M}$-complete category $\mathbb{C}$ equipped with $(\mathcal{E}, \mathcal{M})$-factorization structure for morphisms such that $\mathcal{M}$ is a fixed class of monomorphisms (see Remark 1.2.8) and we consider an interior operator i on $\mathbb{C}$ with respect to $\mathcal{M}$.

3.1 Classes of morphisms with respect to an interior operator

In this section we investigate the four classes of morphisms (initial, closed, open and final morphisms) with respect to a given interior operator. We study cancellation and stability (under composition and pullback) properties of each of the four classes of morphisms. We also discuss the interrelationships between these morphisms. These classes of morphisms are essential tools for understanding topological constructions. Indeed, in the coming chapters we use these morphisms to investigate categorical notions of connectedness and compactness. To this end, as in the case of Section 1.4, we further assume that the preimage $f^*(-)$ preserves arbitrary joins for every morphism f in the category $\mathbb{C}$. Consequently, as already mentioned in Section 1.4, the preimage $f^*(-)$ has a right adjoint $f_*(-) : \text{sub}X \to \text{sub}X$, as the right adjoint of a frame homomorphism. Then, by looking at the equivalent descriptions of the i-continuity condition given in Proposition 2.1.7(a), (b), (c) and (d) one would ask when do we have "$\cong$" instead of "$\leq$". In such cases we obtain the notions of open, closed, initial and final morphisms with respect to i, respectively. These notions are studied for closure and neighbourhood operators; see, for example, [DT95, GT00, CGT01, Raz13, RH17]. In [Cas15], the notion of open morphism with respect to an interior operator is introduced.

Definition 3.1.1. A $\mathbb{C}$-morphism $f : X \to Y$ is called

(a) [Cas15] i-open if $i_X(f^*(n)) \cong f^*(i_Y(n))$ for all $n \in \text{sub}Y$, that is: the preimage $f^*(-)$ commutes with the interior operator i;

(b) i-closed if $f_*(i_X(m)) \cong i_Y(f_*(m))$ for all $m \in \text{sub}X$, that is: the dual image $f_*(-)$ commutes with the interior operator i;

(c) [LTOC11] i-initial if $i_X(m) \cong f^*(i_Y(f_*(m)))$ for all $m \in \text{sub}X$;

(d) i-final if $f_*(i_X(f^*(n))) \cong i_Y(n)$ for all $n \in \text{sub}Y$.

Of course, as is studied in [Cas15], the notion of i-open morphism does not need the assumption that each preimage preserves arbitrary joins. In [HIR16], it is shown that i-open morphisms are precisely $\sqsubseteq^i$-strict morphisms. From now on we use $\mathcal{O}(i), \mathcal{K}(i), \mathcal{I}(i)$, and $\mathcal{F}(i)$ to denote the class of all i-open, i-closed, i-initial, and i-final morphisms, respectively.

Remark 3.1.2. Let $f : X \to Y$ be a morphism in $\mathbb{C}$, $m \in \text{sub}X$ and $n \in \text{sub}Y$. Then the equivalent formulations of the continuity condition of i yield:

(a) $f \in \mathcal{O}(i) \Leftrightarrow i_X(f^*(n)) \leq f^*(i_Y(n))$;

(b) $f \in \mathcal{K}(i) \Leftrightarrow f_*(i_X(m)) \leq i_Y(f_*(m))$;

(c) $f \in \mathcal{I}(i) \Leftrightarrow i_X(m) \leq f^*(i_Y(f_*(m)))$;

(d) $f \in \mathcal{F}(i) \Leftrightarrow f_*(i_X(f^*(n))) \leq i_Y(n)$.

Using the relationship between interior operator i and left neighborhoud operator ν given in Proposition 2.2.10, one has i-initial (i-open, i-closed, i-final, resp.) morphisms are precisely ν^i-initial (ν^i-open, ν^i-closed, ν^i-final, resp.) morphisms (see [Raz13, RH17]). Moreover, since here the preimage functor for any given morphism is assumed to preserve arbitrary joins (hence binary joins), Lemma 1.4.7(b) and Remark 2.3.12 yield the following:

Proposition 3.1.3. Let sub$X$ be a Boolean algebra for every $\mathbb{C}$-object X, c be a closure operator and i^c be the induced interior operator from c given by $i^c_X(m) = \overline{c_X(\overline{m})}$, where $\overline{m}$ denotes the complement of m. Then a morphism f is i^c-initial (i^c-open, i^c-closed, i^c-final, resp.) if and only if f is c-initial (c-open, c-closed, c-final, resp.); see for example [GT00, CGT01].

Next we discuss some basic properties of i-closed morphisms and their "duals", i-open morphisms. The class $\mathcal{K}(i)$ of i-closed morphisms has the following fundamental properties which are called basic stability properties.

Proposition 3.1.4. The class $\mathcal{K}(i)$

(a) is stable under composition,

(b) is left-cancellable with respect to $\mathcal{M}$, that is: if $g \circ f \in \mathcal{K}(i)$ and $g \in \mathcal{M}$ then $f \in \mathcal{K}(i)$,

(c) contains all the isomorphisms and

(d) is right-cancellable with respect to $\mathcal{E}'$, that is: if $g \circ f \in \mathcal{K}(i)$ and $f \in \mathcal{E}'$ then $g \in \mathcal{K}(i)$.

Proof. Consider morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathbb{C}$.

(a) Suppose $f, g \in \mathcal{K}(i)$. Then for any $m \in$ sub$X$ we have

$$\begin{aligned}
(g \circ f)_*(i_X(m)) &\cong (g_* \circ f_*)(i_X(m)) \\
&\cong g_*(f_*(i_X(m))) \\
&\cong g_*(i_Y(f_*(m))) && (f\ i\text{-closed}) \\
&\cong i_Z(g_*(f_*(m))) && (g\ i\text{-closed}) \\
&\cong i_Z((g \circ f)_*(m)).
\end{aligned}$$

(b) Suppose $g \circ f \in \mathcal{K}(i)$ and $g \in \mathcal{M}$. Then for any $m \in$ sub$X$ we have

$$\begin{aligned}
f_*(i_X(m)) &\cong g^*(g_*(f_*(i_X(m)))) && (g \in \mathcal{M}) \\
&\cong g^*((g \circ f)_*(i_X(m))) \\
&\cong g^*(i_Z((g \circ f)_*(m))) && (g \circ f \in \mathcal{K}(i)) \\
&\leq i_Y(g^*(g_*(f_*(m)))) && (g \text{ is } i\text{-continuous }) \\
&\cong i_Y(f_*(m)) && (g \in \mathcal{M}).
\end{aligned}$$

(c) If $f : X \to Y$ is an isomorphism with inverse $f^{-1} : Y \to X$ then $f^{-1} \circ f = 1_X$ is obviously i-closed and $f^{-1} \in \mathcal{M}$. Consequently, (b) implies $f \in \mathcal{K}(i)$.

(d) Suppose $g \circ f \in \mathcal{K}(i)$ and $f \in \mathcal{E}'$. Then for any $n \in \text{sub}Y$ we have

$$\begin{aligned} g_*(i_Y(n)) &\cong g_*(f_*(f^*(i_Y(n)))) \\ &\cong (g \circ f)_*(f^*(i_Y(n))) \\ &\leq (g \circ f)_*(i_X(f^*(n))) && (f \text{ is } i\text{-continuous }) \\ &\cong i_Z((g \circ f)_*(f^*(n)) && (g \circ f \ i\text{-closed}) \\ &\cong i_Z(g_*(n)). \end{aligned}$$

□

A straight application of Proposition 3.1.4 gives the following corollary:

Corollary 3.1.5. Let $f = m \circ e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}'$. $f \in \mathcal{K}(i)$ if and only if $m, e \in \mathcal{K}(i)$.

Remark 3.1.6. In the category of **Top**, closed morphisms with respect to the Kuratowski closure operator k are exactly the closed maps (see [GT00]). Consequently, by Proposition 3.1.3 the k^{in}-closed morphisms are precisely the closed maps.

There is the following interaction of an i-closed morphism with i-open subobjects.

Proposition 3.1.7. Let $f : X \to Y$ be an i-closed morphism. Then the right adjoint f_* of the preimage f^* maps i-open $\mathcal{M}$-subobjects into i-open $\mathcal{M}$-subobjects. Moreover, if i is idempotent then the converse is true.

Proof. Let $f : X \to Y$ be an i-closed morphism such that m is an i-open subobject of X. Then

$$\begin{aligned} f_*(m) &\cong f_*(i_X(m)) && (m \ i\text{-open in } X) \\ &\cong i_Y(f_*(m)) && (f \in \mathcal{K}(i)). \end{aligned}$$

Therefore, $f_*(m)$ is an i-open subobject of Y. Conversely, if i is an idempotent and f_* maps i-open subobjects into i-open subobjects then $i_X(m)$ is an i-open subobject of X and hence $f_*(i_X(m))$ is an i-open subobject of Y. Consequently, $f_*(i_X(m)) \cong i_Y(f_*(i_X(m))) \leq i_Y(f_*(m))$. Hence f is i-closed. □

Corollary 3.1.8. Let $f : X \to Y$ be an i-closed morphism in $\mathcal{M}$. Then every i-open subobject m of X is of the form $f^*(n)$ for some i-open subobject n of Y.

Proof. Let m be an i-open subobject of X. Then by the above proposition, $f_*(m)$ is an i-open subobject of Y. Consequently, with Remark 2.1.2(b), $m \cong f^*(f_*(m))$ is an i-open subobject n of Y since $f \in \mathcal{M}$. Therefore, $m \cong f^*(n)$ with $n = f_*(m)$ i-open subobject of Y. □

Corollary 3.1.9. Let $f : X \to Y$ be an i-closed morphism in $\mathcal{E}'$. Then n is an i-open subobject of Y if and only if $f^*(n)$ is an i-open subobject of X.

Proof. The necessary part is true by the Remark 2.1.4 (b). To show the sufficient condition let $f^*(n)$ is be an i-open subobject of X. Then since $f \in \mathcal{E}'$, by Proposition 3.1.7, one has $n \cong f_*(f^*(n))$ is an i-open subobject of Y. □

The following is a characterization of closed morphisms with respect to an interior operator.

Proposition 3.1.10. Let i be idempotent. A morphism $f : X \to Y \in \mathcal{K}(i)$ if and only if for every i-open subobject m of X and for every subobject n of Y such that $f^*(n) \leq m$, there exists an i-open subobject k of Y such that $n \leq k$ and $f^*(k) \leq m$.

Proof. $(\Rightarrow)$ Suppose $f : X \to Y \in \mathcal{K}(i)$, $n \in \text{sub}Y$ and m is an i-open subobject of X such that $f^*(n) \leq m$. Then, there exists $k = f_*(m)$ such that $n \leq f_*(m) = k$ and $k = f_*(m)$ is an i-open subobject of Y by Proposition 3.1.7. Moreover, $f^*(k) = f^*(f_*(m)) \leq m$.

$(\Leftarrow)$ Suppose $f : X \to Y$ satisfies the condition in the proposition and let m be an i-open subobject of X. Then, for $n = f_*(m)$, one has $f^*(n) = f^*(f_*(m)) \leq m$. Consequently, there exists an i-open subobject k of Y such that $[n = f_*(m) \leq k$ and $f^*(k) \leq m] \Leftrightarrow [n = f_*(m) \leq k$ and $k \leq f_*(m)]$. As a result, $f_*(m) \cong k$ and hence $f_*(m)$ is an i-open subobject of Y. Therefore, by Proposition 3.1.7, f is i-closed.

□

Similar to the class of i-closed morphisms we have the following properties of the class of i-open morphisms.

Proposition 3.1.11. [Cas15] The class $\mathcal{O}(i)$

(a) is stable under composition,

(b) is left-cancellable with respect to $\mathcal{M}$, that is: if $g \circ f \in \mathcal{O}(i)$ and $g \in \mathcal{M}$ then $f \in \mathcal{O}(i)$,

(c) contains all the isomorphisms and

(d) is right-cancellable with respect to $\mathcal{E}'$, that is: if $g \circ f \in \mathcal{O}(i)$ and $f \in \mathcal{E}'$ then $g \in \mathcal{O}(i)$.

Proof. Given the role of adjunctions, the proof is computationally analogous to that of Proposition 3.1.4. □

From Proposition 3.1.11 one deduces the following corollary:

Corollary 3.1.12. Let $f = m \circ e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}'$. $f \in \mathcal{O}(i)$ if and only if $m, e \in \mathcal{O}(i)$.

In the following proposition we discuss interaction of an i-open morphism with i-open subobjects. Indeed, for an idempotent interior operator i, i-open morphisms are characterized by preservation of i-open subobjects.

Proposition 3.1.13. Recently in [Cas15] it was proved that:

(a) $f \in \mathcal{O}(i)$ if and only if $f(i_X(m)) \leq i_Y(f(m))$ for all $m \in \text{sub}X$.

(b) If $f : X \to Y$ is i-open morphism then f maps i-open $\mathcal{M}$-subobjects into i-open $\mathcal{M}$-subobjects. Moreover, if i is idempotent then the converse is true.

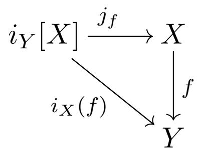
Consequently, with Lemma 2.1.13, one has:

Proposition 3.1.14. Let i be any interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$. Then the following statements hold for an $\mathcal{M}$-morphism $f : X \to Y$:

(a) $f \in \mathcal{O}(i) \Leftrightarrow f(i_X(m)) \cong i_Y(f(m))$ for all $m \in \text{sub}X$. That is: the image $f(-)$ commutes with the interior i for any i-open $\mathcal{M}$-morphism f;

(b) If $f \in \mathcal{O}(i)$ then every i-open subobject m of X is of the form $f^*(n)$ for some i-open subobject n of Y;

(c) If $f \in \mathcal{O}(i)$ and i is idempotent then $j_f : i_Y[X] \to X$ is an i-open subobject (see [Cas15]).



Proof. (a) Let $m \in \text{sub}X$. Since $f \in \mathcal{O}(i)$, from the above proposition one obtains $f(i_X(m)) \leq i_Y(f(m))$, and since $f \in \mathcal{M}$, one has $i_Y(f(m)) \leq f(i_X(m))$ by Lemma 2.1.13. Consequently, $f(i_X(m)) \cong i_Y(f(m))$. In fact, the converse is true by the above proposition.

(b) Let m be an i-open subobject of X. Then by the above proposition, $f(m)$ is i-open subobject of Y. Consequently, with Remarks 1.3.10(a) and 2.1.4(b), $m \cong f^*(f(m))$ is i-open subobject n of Y since $f \in \mathcal{M}$. Therefore, $m \cong f^*(n)$ with $n = f(m)$ is an i-open subobject of Y.

(c) Let f be an i-open morphism in $\mathcal{M}$. Then (a) and idempotency of i yield $f \circ i_X(j_f) \cong f(i_X(j_f)) \cong i_Y(f(j_f)) \cong i_Y(f \circ j_f) \cong i_Y(i_Y(f)) \cong i_Y(f) \cong f \circ j_f$. Consequently, $i_X(j_f) \cong j_f$ since f is monic.

□

Proposition 3.1.14(a) states that i-open $\mathcal{M}$-morphisms are the morphisms whose image commutes with the interior i.

Corollary 3.1.15. Let $r_s : R \to S$ be an i-open $\mathcal{M}$-subobject and $s : S \to X$ be an i-open morphism in $\mathcal{M}$. Then $r = s \circ r_s$ is an i-open $\mathcal{M}$-subobject of X.

Proof. Since s is an i-open morphism and r_s is an i-open subobject, Proposition 3.1.13 (b) yields $s(r_s)$ is an i-open subobject. Consequently, with Remark 1.3.6 one derives $r = s \circ r_s \cong s(r_s) \cong i_X(s(r_s)) \cong i_X(s \circ r_s) \cong i_X(r)$. □

Similar to Corollary 3.1.9 we also have the following.

Corollary 3.1.16. Let $f : X \to Y$ be an i-open morphism in $\mathcal{E}'$. Then n is an i-open subobject of Y if and only if $f^*(n)$ is an i-open subobject of X.

Proof. The necessary part is true by the Remark 2.1.4 (b). To show the sufficient condition let $f^*(n)$ be an i-open subobject of X. Since $f \in \mathcal{E}'$, by Proposition 3.1.13 (b), one has $n \cong f(f^*(n))$ is an i-open subobject of Y. □

The following is a characterization of open morphisms with respect to an interior operator.

Proposition 3.1.17. Let i be idempotent. A morphism $f : X \to Y \in \mathcal{O}(i)$ if and only if for every i-open subobject m of X and for every subobject n of Y such that $m \leq f^*(n)$, there exists an i-open subobject k of Y such that $k \leq n$ and $m \leq f^*(k)$.

Proof. ($\Rightarrow$) Suppose $f : X \to Y \in \mathcal{O}(i)$, $n \in \text{sub}Y$ and m is an i-open subobject of X such that $m \leq f^*(n)$. Then, there exists $k = f(m)$ such that $k = f(m) \leq n$ and $k = f(m)$ is i-open subobject of Y by Proposition 3.1.13(b). Moreover, $m \leq f^*(f(m)) = f^*(k)$.

($\Leftarrow$) Suppose $f : X \to Y$ satisfies the condition in the proposition and let m be an i-open subobject of X. Then, for $n = f(m)$, one has $m \leq f^*(f(m)) = f^*(n)$. Consequently, there exists an i-open subobject k of Y such that $k \leq f(m) = n$ and $m \leq f^*(k)] \Leftrightarrow [k \leq f(m)$ and $n = f(m) \leq k]$. As a result, $f(m) \cong k$ and hence $f(m)$ is an i-open subobject of Y. Therefore, by Proposition 3.1.13(b), f is i-open.

□

We note that for the normal interior operator n, which is idempotent on the category **Grp** of groups and group homomorphisms with the (surjective homomorphism, injective homorphisms)-factorization system, the n-open morphisms are precisely group homomorphisms which preserve normal subobjects. In fact surjective group homomorphisms preserve normal subobjects.

Remark 3.1.18. Let i be a standard interior operator and $f : X \to Y \in \mathcal{O}(i)$. Then $f(1_X)$ is an i-open subobject of Y. Indeed, by Proposition 3.1.13(a), $f(1_X) \cong f(i_X(1_X)) \leq i_Y(f(1_X))$. This is of course an immediate consequence of Proposition 3.1.13(b). Note that for a standard interior operator i, 1_X is an i-open subobject of X.

Remark 3.1.19. Let i be an interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$ and

$$\begin{array}{ccc} M & \xrightarrow{u} & N \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ X & \xrightarrow[f]{} & Y \end{array}$$

with $m, n \in \mathcal{M}$ be a commutative diagram. Then:

(a) $f \in \mathcal{O}(i)$ if and only if there is a uniquely determined morphism $w : i_X[M] \to i_X[N]$ making the diagram

$$\begin{array}{ccc} i_X[M] & \xrightarrow{w} & i_X[N] \\ {\scriptstyle j_m}\downarrow & & \downarrow{\scriptstyle j_n} \\ M & \xrightarrow{u} & N \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ X & \xrightarrow[f]{} & Y \end{array}$$

commutative: that is: the preservation property of i holds if and only if $f \in \mathcal{O}(i)$ (see [Cas15, Cas16]).

(b) If $f \in \mathcal{O}(i)$ and n is i-open subobject, then there is a uniquely determined morphism $s : i_X[M] \to N$ with $s = 1_N \circ s = u \circ j_m$ and $n \circ s = f \circ i_X(m)$.

(c) If $f \in \mathcal{O}(i)$ and m is i-codense subobject, then there is a uniquely determined morphism $t : O_M \to i_X[N]$ with $u \circ 0_M = j_m \circ t$ and $f \circ 0_X = i_X(m)$.

(d) If $f \in \mathcal{O}(i)$, m is i-codense subobject and n is i-open subobject, then there is a uniquely determined morphism $d : O_M \to N$ with $u \circ 0_M = d$ and $n \circ d = f \circ 0_X$.

Indeed, $(b), (c)$ and (d) are direct consequences of (a).

We now turn to some properties of i-initial morphisms and their "duals", i-final morphisms. The class $\mathcal{I}(i)$ of i-initial morphisms behaves as follows.

Proposition 3.1.20. The class $\mathcal{I}(i)$

(a) is stable under composition,

(b) is left-cancellable, that is: if $g \circ f \in \mathcal{I}(i)$ then $f \in \mathcal{I}(i)$,

(c) contains all the isomorphisms and

(d) is right-cancellable with respect to $\mathcal{E}'$, that is: if $g \circ f \in \mathcal{I}(i)$ and $f \in \mathcal{E}'$ then $g \in \mathcal{I}(i)$.

Proof. Consider morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathbb{C}$.

(a) Suppose $f, g \in \mathcal{I}(i)$. Then for any $m \in \text{sub}X$ we have

$$\begin{aligned} i_X(m) &\cong f^*(i_Y(f_*(m))) && (f \text{ is } i\text{-initial}) \\ &\cong f^*(g^*(i_Z(g_*(f_*(m))))) && (g \text{ is } i\text{-initial}) \\ &\cong (g \circ f)^*(i_Z((g \circ f)_*(m))). \end{aligned}$$

(b) Suppose $g \circ f \in \mathcal{I}(i)$. Then for any $m \in \text{sub}X$ we have

$$\begin{aligned} i_X(m) &\cong (g \circ f)^*(i_Z((g \circ f)_*(m))) && (g \circ f \text{ is } i\text{-initial}) \\ &\cong f^*(g^*(i_Z(g_*(f_*(m))))) \\ &\leq f^*(i_Y(g^*(g_*(f_*(m))))) && (g \text{ is } i\text{-continuous}) \\ &\leq f^*(i_Y(f_*(m))) && (g^* \dashv g_*). \end{aligned}$$

(c) If $f : X \to Y$ is an isomorphism with inverse $f^{-1} : Y \to X$ then $f^{-1} \circ f = 1_X$ is obviously i-initial. Consequently, (b) implies $f \in \mathcal{I}(i)$.

(d) Suppose $g \circ f \in \mathcal{I}(i)$ and $f \in \mathcal{E}'$. Then for any $n \in \text{sub}Y$ we have

$$\begin{aligned} i_Y(n) &\leq f_*(f^*(i_Y(n))) && (f^* \dashv f_*) \\ &\leq f_*(i_X(f^*(n))) && (f \text{ is } i\text{-continuous}) \\ &\cong f_*((g \circ f)^*(i_Z((g \circ f)_*(f^*(n))))) && (g \circ f \text{ is } i\text{-initial}) \\ &\cong f_*(f^*(g^*(i_Z(g_*(f_*(f^*(n))))))) \\ &\cong g^*(i_Z(g_*(n))) && (f \in \mathcal{E}'). \end{aligned}$$

□

As an immediate consequence of Proposition 3.1.20 we obtain:

Corollary 3.1.21. Let i be an interior operator.

(a) Let $f = m \circ e$ with $e \in \mathcal{E}'$. $f \in \mathcal{I}(i)$ if and only if $m, e \in \mathcal{I}(i)$.

(b) Every section or split monomorphism is i-initial.

Proof. (a) It follows immediately from Proposition 3.1.20.

(b) Let $s : X \to Y$ be a section. Then $\exists r : Y \to X$ such that $r \circ s = 1_X$. But since $1_X \in \mathcal{I}(i)$ then by Proposition 3.1.20 (b) we have that $s \in \mathcal{I}(i)$.

□

Remark 3.1.22. (a) Let $\Gamma_f = \langle 1_X, f\rangle : X \to X \times Y$ be the graph of $f : X \to Y$ in $\mathbb{C}$. Then $\pi_X \circ \Gamma_f = 1_X$, hence Γ_f is a section. Consequently, by Corollary 3.1.21(b), Γ_f is i-initial. That is: $i_X(m) \cong \Gamma_f^*(i_{X\times Y}((\Gamma_f)_*(m)))$ for all $m \in$ sub$X$.

(b) Let $\delta_X = \langle 1_X, 1_X\rangle : X \to X \times X$ be the diagonal of $X \in \mathbb{C}$. Then $\pi_X \circ \delta_X = 1_X$, hence δ_X is a section. Consequently, by Corollary 3.1.21(b), δ_X is i-initial. That is: $i_X(m) \cong \delta_X^*(i_{X\times X}((\delta_X)_*(m)))$ for all $m \in$ sub$X$.

Note that from the adjointness property, one has $f : X \to Y \in \mathcal{I}(i) \Leftrightarrow f(i_X(m)) \leq i_Y(f_*(m)$. Let i be an idempotent interior operator. Then we have the following characterization of i-initial morphisms in an arbitrary category.

Proposition 3.1.23. Let i be idempotent. A morphism $f : X \to Y$ in $\mathbb{C}$ is i-initial if and only if for every i-open subobject m of X, there exists an i-open subobject n of Y such that $m \cong f^*(n)$.

Proof. $(\Rightarrow)$ Suppose f is an i-initial morphism and m is an i-open subobject of X. Then $m \cong i_X(m) \cong f^*(i_Y(f_*(m))) \cong f^*(n)$, where n is $i_Y(f_*(m))$ such that $i_Y(n) = i_Y(i_Y((f_*(m)))) \cong i_Y(f_*(m)) = n$. Thus $m \cong f^*(n)$ with n as i-open subobject of Y.

$(\Leftarrow)$ From idempotency of i, for all $m \in$ sub$X$, $i_X(m)$ is an i-open subobject of X and as a result there exists an i-open subobject n of Y such that $i_X(m) \cong f^*(n)$. Hence,

$$\begin{aligned}
i_X(m) &\cong f^*(n) && \text{(Assumption and } i\text{-idempotent)}\\
&\cong f^*(i_Y(n)) && (n \quad i\text{-open })\\
&\leq f^*(i_Y(f_*(f^*(n)))) && (f^* \dashv f_*)\\
&\cong f^*(i_Y(f_*(i_X(m)))) && (i_X(m) \cong f^*(n))\\
&\leq f^*(i_Y(f_*(m))) && (i\text{-contractive}).
\end{aligned}$$

Therefore, $f \in \mathcal{I}(i)$. □

Definition 3.1.24. An interior operator i is said to be initial with respect to a reflective subcategory $\mathbb{S}$ of $\mathbb{C}$ if for every $X \in \mathbb{C}$ the reflection morphism $X \overset{r_X}{\to} rX$ is i-initial, that is, $i_X(m) \cong r_X^*(i_{rX}((r_X)_*(m)))$ for all $m \in$sub$X$.

Consequently, one has the next remark.

Remark 3.1.25. Let $\mathbb{S}$ and $\mathbb{S}'$ be reflective subcategories of $\mathbb{C}$ and $i, i' \in INT(\mathbb{C}, \mathcal{M})$.

(a) If i and i' are initial with respect to $\mathbb{S}$ and $i \leq i'$ on $\mathbb{S}$, then $i \leq i'$.

(b) If $\mathbb{S} \subseteq \mathbb{S}'$ and i is $\mathbb{S}$-initial then i is also $\mathbb{S}'$-initial. Indeed, for any $X \in \mathbb{C}$ there exist $\mathbb{S}$-reflection morphism $X \overset{r_X}{\to} rX$ and $\mathbb{S}'$-reflection morphism $X \overset{r'_X}{\to} r'X$. Besides, the fact that $\mathbb{S} \subseteq \mathbb{S}'$ implies

$\exists g : r'X \to rX$ such that $r_X = g \circ r'_X$. Consequently,

$$\begin{aligned} i_X(m) &\cong r_X^*(i_{rX}((r_X)_*(m))) && (i\ \mathbb{S}\text{-initial}) \\ &\cong r_X'^*(g^*(i_{rX}(g_*((r'_X)_*(m))))) \\ &\leq r_X'^*(i_{r'X}(g^*(g_*((r'_X)_*(m))))) && (g\ i\text{-continuous}) \\ &\leq r_X'^*(i_{r'X}((r'_X)_*(m))) && (g^* \dashv g_*). \end{aligned}$$

Analogous to the class of i-initial morphisms we have the following properties of the class of i-final morphisms.

Proposition 3.1.26. The class $\mathcal{F}(i)$

(a) is stable under composition,

(b) is right-cancellable, that is: if $g \circ f \in \mathcal{F}(i)$ then $g \in \mathcal{F}(i)$,

(c) contains all the isomorphisms and

(d) is left-cancellable with respect to $\mathcal{M}$, that is: if $g \circ f \in \mathcal{F}(i)$ and $g \in \mathcal{M}$ then $f \in \mathcal{F}(i)$.

Proof. Consider morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathbb{C}$.

(a) Suppose $f, g \in \mathcal{F}(i)$. Then for any $p \in$ sub$Z$ we have

$$\begin{aligned} (g \circ f)_*(i_X((g \circ f)^*(p))) &\cong g_*(f_*(i_X(f^*(g^*(p))))) \\ &\cong g_*(i_Y(g^*(p))) && (f\ i\text{-final}\ ) \\ &\cong i_Z(p) && (g\ i\text{-final}\ ). \end{aligned}$$

Therefore, $g \circ f \in \mathcal{F}(i)$.

(b) Suppose $g \circ f \in \mathcal{F}(i)$. Then for any $p \in$ sub$Z$ we have

$$\begin{aligned} g_*(i_Y(g^*(p))) &\leq g_*(f_*(f^*(i_Y(g^*(p))))) && (f^* \dashv f_*) \\ &\leq g_*(f_*(i_X(f^*(g^*(p))))) && (f\ i\text{-continuous}) \\ &\cong (g \circ f)_*(i_X((g \circ f)^*(p))) \\ &\cong i_Z(p) && (g \circ f\ i\text{-final}). \end{aligned}$$

(c) If $f : X \to Y$ is an isomorphism with inverse $f^{-1} : Y \to X$ then $f \circ f^{-1} = 1_X$ is obviously i-final. Consequently, (b) implies $f \in \mathcal{F}(i)$.

(d) Suppose $g \circ f \in \mathcal{F}(i)$ and $g \in \mathcal{M}$. Then for any $n \in$ sub$Y$ we have

$$\begin{aligned} f_*(i_X(f^*(n))) &\cong g^*(g_*(f_*(i_X(f^*(g^*(g_*(n))))))) && (g \in \mathcal{M}) \\ &\cong g^*(g \circ f)_*(i_X((g \circ f)^*(g_*(n)))) \\ &\cong g^*(i_Z(g_*(n))) && (g \circ f\ i\text{-final}) \\ &\leq i_Y(g^*(g_*(n))) && (g\ i\text{-continuous}) \\ &\cong i_Y(n) && (g \in \mathcal{M}). \end{aligned}$$

□

Of course, given the role of adjunctions, the proof of the above proposition is computationally the same as that of Proposition 3.1.20.

Corollary 3.1.27. (a) Let $f = m \circ e$ with $m \in \mathcal{M}$. $f \in \mathcal{F}(i)$ if and only if $m, e \in \mathcal{F}(i)$.

(b) Every retraction or split epimorphism is i-final.

Proof. (a) It is a consequence of Proposition 3.1.26.

(b) Let $r : Y \to X$ be a section. Then $\exists s : X \to Y$ such that $r \circ s = 1_X$. But since $1_X \in \mathcal{F}(i)$ then by Proposition 3.1.26 (b) we have that $r \in \mathcal{F}(i)$.

□

Proposition 3.1.28. Let i be an interior operator.

(a) $f : X \to Y \in \mathcal{F}(i) \Leftrightarrow f_*(i_X(m)) \leq i_Y(f(m))$.

(b) If i is a standard interior operator then $\mathcal{F}(i) \subseteq \mathcal{E}$.

(c) If i is a standard interior operator then $\mathcal{M} \cap \mathcal{F}(i)$ is a class of isomorphisms.

Proof. (a) If $f : X \to Y \in \mathcal{F}(i)$ then for $n = f(m)$, where $n \in$ sub$Y$ and $m \in$ sub$X$ the inequality $f_*(i_X(f^*(n))) \leq i_Y(n)$ implies $f_*(i_X(f^*(f(m)))) \leq i_Y(f(m))$. This turns out to be $f_*(i_X(m)) \leq f_*(i_X(f^*(f(m)))) \leq i_Y(f(m))$, since $f \dashv f^*$. Hence, $f_*(i_X(m)) \leq i_Y(f(m))$. Conversely, for $m = f^*(n)$ the inequality $f_*(i_X(m)) \leq i_Y(f(m))$ gives that $f_*(i_X(f^*(n))) \leq i_Y(f(f^*(n))) \leq i_Y(n)$ and hence $f_*(i_X(f^*(n))) \leq i_Y(n)$. Therefore, $f_*(i_X(f^*(n))) \leq i_Y(n)$ if and only if $f_*(i_X(m)) \leq i_Y(f(m))$.

(b) Let $f \in \mathcal{F}(i)$. Then by (a) we get $f_*(i_X(m)) \leq i_Y(f(m))$. In particular, for $m = 1_X$ one has $f_*(i_X(1_X)) \leq i_Y(f(1_X))$. This turns out to be $1_Y = f_*(1_X) = f_*(i_X(1_X)) \leq i_Y(f(1_X)) \leq f(1_X) \leq 1_Y$ and hence $1_Y \cong f(1_X)$. Thus $f \in \mathcal{E}$.

(c) This is an immediate consequence of (b).

□

In the following proposition we give a partial characterization of final morphisms.

Proposition 3.1.29. Let $f : X \to Y$ be an i-final morphism. Then a subobject n of Y is i-open if and only if $f^*(n)$ is i-open in X.

Proof. Since the necessary part is well known we focus on the sufficient condition. Suppose $f^*(n)$ is i-open. Then since f is i-final and $f^* \dashv f_*$ we obtain $n \leq f_*(f^*(n)) \leq f_*(i_X(f^*(n))) \leq i_Y(n)$, as desired. □

Definition 3.1.30. A $\mathbb{C}$-morphism $f : X \to Y$ is called weakly i-final if $n \wedge f_*(i_X(f^*(n))) \cong i_Y(n)$ for all $n \in$ sub$Y$.

We use $\mathcal{WF}(i)$ to denote the class of all weakly i-final morphisms. Clearly, finality implies weak finality since if $n \in$ sub$Y$ then $i_Y(n) \leq n$, hence $n \wedge i_Y(n) \cong i_Y(n)$ and consequently, for $f : X \to Y \in \mathcal{F}(i)$, one has $f_*(i_X(f^*(n))) \cong i_Y(n)$. Therefore, $n \wedge f_*(i_X(f^*(n))) \cong n \wedge i_Y(n) \cong i_Y(n)$.Thus $f \in \mathcal{WF}(i)$.

Remark 3.1.31. Let $f : X \to Y$ in $\mathbb{C}$ and $n \in$ sub$Y$.

(a) $f \in \mathcal{WF}(i) \Leftrightarrow n \wedge f_*(i_X(f^*(n))) \leq i_Y(n)$. Indeed, since $i_Y(n) \leq n$ and $i_Y(n) \leq i_Y(f_*(f^*(n))) \leq f_*(i_X(f^*(n)))$, one always has $i_Y(n) \leq n \wedge f_*(i_X(f^*(n)))$.

(b) $\mathcal{WF}(i) \cap \mathcal{E}' \subseteq \mathcal{F}(i)$. Indeed, if $f \in \mathcal{E}'$ then $f_*(i_X(f^*(n))) \leq f_*(f^*(n)) \cong n$. Consequently, with $f \in \mathcal{WF}(i)$, one has $f_*(i_X(f^*(n))) \cong n \wedge f_*(i_X(f^*(n))) \leq i_Y(n)$.

It is also routine to verify the following.

Proposition 3.1.32. The class $\mathcal{WF}(i)$

(a) is stable under composition,

(b) is right-cancellable, that is: if $g \circ f \in \mathcal{WF}(i)$ then $g \in \mathcal{F}(i)$,

(c) contains all the isomorphisms and

(d) is left-cancellable with respect to $\mathcal{M}$, that is: if $g \circ f \in \mathcal{WF}(i)$ and $g \in \mathcal{M}$ then $f \in \mathcal{WF}(i)$.

We obtain the following corollary from Proposition 3.1.32.

Corollary 3.1.33. Let $f = m \circ e$ with $m \in \mathcal{M}$. $f \in \mathcal{WF}(i)$ if and only if $m, e \in \mathcal{WF}(i)$.

Definition 3.1.34. Let i be an interior operator.

(a) A cone $\mathbb{S} := (f_i : X \to X_i)_{i \in I}$ is called i-initial (or the f_i's are jointly i-initial) if $i_X(m) \cong \bigvee_{i \in I} f_i^*(i_{X_i}((f_i)_*(m)))$ for all $m \in$sub$X$.

(b) A cocone $(f_i : X_i \to X)_{i \in I}$ is called weakly i-final (or the f_i's are jointly weakly i-final) if $i_X(m) \cong m \wedge \bigwedge_{i \in I} (f_i)_*(i_{X_i}(f_i^*(n)))$ for all $m \in$sub$X$.

As a generalization of Propositions 3.1.20 and 3.1.32 we have the following.

Proposition 3.1.35. (a) Let $\mathbb{S} := (f_i : X \to X_i)_{i \in I}$ be a cone and $u : Z \to X$ be a morphism such that $(f_i \circ u : Z \to X_i)_{i \in I}$ is i-initial. Then

(i) u is i-initial;

(ii) $(f_i : X \to X_i)_{i \in I}$ is i-initial provided that $u \in \mathcal{E}'$.

(b) Let $(f_i : X_i \to X)_{i \in I}$ be a cocone and $u : X \to Z$ be a morphism such that $(u \circ f_i : X_i \to Z)_{i \in I}$ is weakly i-final. Then

(i) u is weakly i-final;

(ii) $(f_i : X_i \to X)_{i \in I}$ is weakly i-final provided that $u \in \mathcal{M}$.

The following proposition shows connections of i-initial morphisms with the other three morphism classes.

Proposition 3.1.36. Let i be an interior operator.

(a) $\mathcal{K}(i) \cap \mathcal{M} \subseteq \mathcal{I}(i)$.

(b) $\mathcal{O}(i) \cap \mathcal{M} \subseteq \mathcal{I}(i)$.

(c) $\mathcal{I}(i) \cap \mathcal{E}' \subseteq \mathcal{K}(i) \cap \mathcal{O}(i) \cap \mathcal{F}(i)$.

Proof. (a) Suppose $f : X \to Y \in \mathcal{K}(i) \cap \mathcal{M}$. Then for any $m \in$ sub$X$ one has

$$\begin{aligned} i_X(m) &\cong f^*(f_*(i_X(m))) && (f \in \mathcal{M}) \\ &\cong f^*(i_Y(f_*(m))) && (f \ i\text{-closed}) \\ &\Rightarrow f \in \mathcal{I}(i). \end{aligned}$$

(b) Suppose $f : X \to Y \in \mathcal{O}(i) \cap \mathcal{M}$. Then for any $m \in$ sub$X$ one has

$$\begin{aligned} i_X(m) &\cong i_X(f^*(f_*(m))) && (f \in \mathcal{M}) \\ &\cong f^*(i_Y(f_*(m))) && (f \ i\text{-open}) \\ &\Rightarrow f \in \mathcal{I}(i). \end{aligned}$$

(c) Suppose $f : X \to Y \in \mathcal{I}(i) \cap \ \mathcal{E}'$. Then for any $m \in$ sub$X$ and $n \in$ sub$Y$ one has

$$\begin{aligned} f_*(i_X(m)) &\cong f_*(f^*(i_Y(f_*(m)))) && (f \in \mathcal{I}(i)) \\ &\cong i_Y(f_*(m)) && (f \in \mathcal{E}') \\ &\Rightarrow f \in \mathcal{K}(i), \end{aligned}$$

$$\begin{aligned} i_X(f^*(n)) &\cong f^*(i_Y(f_*(f^*(n)))) && (f \in \mathcal{I}(i)) \\ &\cong f^*(i_Y(n)) && (f \in \mathcal{E}') \\ &\Rightarrow f \in \mathcal{O}(i) \text{ and} \end{aligned}$$

$$\begin{aligned} f_*(i_X(f^*(n))) &\cong f_*(f^*(i_Y(f_*(f^*(n))))) && (f \in \mathcal{I}(i)) \\ &\cong i_Y(n) && (f \in \mathcal{E}') \\ &\Rightarrow f \in \mathcal{F}(i). \end{aligned}$$

□

Similar to the proposition above there are the following immediate connections between i-final and the other three morphism classes.

Proposition 3.1.37. Let i be an interior operator.

(a) $\mathcal{K}(i) \cap \mathcal{E}' \subseteq \mathcal{F}(i)$.

(b) $\mathcal{O}(i) \cap \mathcal{E}' \subseteq \mathcal{F}(i)$.

(c) $\mathcal{F}(i) \cap \mathcal{M} \subseteq \mathcal{K}(i) \cap \mathcal{O}(i) \cap \mathcal{I}(i)$.

Proof. (a) Let $f : X \to Y \in \mathcal{K}(i) \cap \ \mathcal{E}'$. Then $f_*(i_X(f^*(n))) \cong i_Y(f_*(f^*(n))) \cong i_Y(n)$ for all $n \in$ sub$Y$. Therefore, $f \in \mathcal{F}(i)$.

(b) Let $f : X \to Y \in \mathcal{O}(i) \cap \ \mathcal{E}'$. Then $f_*(i_X(f^*(n))) \cong f_*(f^*(i_Y(n))) \cong i_Y(n)$. Consequently, $f \in \mathcal{F}(i)$.

(c) Suppose $f : X \to Y \in \mathcal{F}(i) \cap \mathcal{M}$. Then for any $m \in \mathsf{sub}X$ and $n \in \mathsf{sub}Y$ one has

$$\begin{aligned} f_*(i_X(m)) &\cong f_*(i_X(f^*(f_*(m)))) && (f \in \mathcal{M}) \\ &\cong i_Y(f_*(m)) && (f \in \mathcal{F}(i)) \\ &\Rightarrow f \in \mathcal{K}(i)\ , \end{aligned}$$

$$\begin{aligned} i_X(f^*(n)) &\cong f^*(f_*(i_X(f^*(n)))) && (f \in \mathcal{M}) \\ &\cong f^*(i_Y(n)) && (f \in \mathcal{F}(i)) \\ &\Rightarrow f \in \mathcal{O}(i) \text{ and} \end{aligned}$$

$$\begin{aligned} i_X(m) &\cong f^*(f_*(i_X(f^*(f_*(m))))) && (f \in \mathcal{M}) \\ &\cong f^*(i_Y(f_*(m))) && (f \in \mathcal{F}(i)) \\ &\Rightarrow f \in \mathcal{I}(i). \end{aligned}$$

□

In the remainder of this section we focus on the pullback behaviour of i-closed, i-open, i-initial and i-final morphisms. We begin with the following equivalent descriptions of the Beck-Chevalley Property (BCP) using the right adjoints of preimages.

Lemma 3.1.38. For a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y \end{array}$$

one always has:

(a) For all t in $\mathsf{sub}B$, $f^*(b_*(t)) \leq a_*(g^*(t))$

(b) For all m in $\mathsf{sub}X$, $b^*(f_*(m)) \leq g_*(a^*(m))$

Proof. (a) Let t in $\mathsf{sub}B$. Then

$$\begin{aligned} f^*(b_*(t)) &\leq a_*(a^*(f^*(b_*(t)))) && (a^* \dashv a_*) \\ &\cong a_*((f \circ a)^*(b_*(t))) \\ &\cong a_*((b \circ g)^*(b_*(t))) && (f \circ a = b \circ g) \\ &\cong a_*(g^*(b^*(b_*(t)))) \\ &\leq a_*(g^*(t)) && (b^* \dashv b_*). \end{aligned}$$

(b) follows from (a).

□

Remark 3.1.39. [DT95]

(a) We say that the commutative diagram in Lemma 3.1.38 satisfies the Beck-Chevalley's Property (BCP) if for every $m \in \text{sub}X$, $g(a^*(m)) \cong b^*(f(m))$. In fact, we also have for every $n \in \text{sub}B$, $a(g^*(n)) \cong f^*(b(n))$.

(b) Every pullback diagram in $\mathbb{C}$ satisfies the Beck-Chevalley Property (BCP) if and only if $\mathcal{E}$ is stable under pullback. Indeed, consider the pullback diagram in Lemma 3.1.38 with $f \in \mathcal{E}$. Then by Remark 1.3.6, $1_B \cong b^*(1_Y) \cong b^*(f(1_X)) \cong g(a^*(1_X)) \cong g(1_A)$. Consequently, $g \in \mathcal{E}$. The converse follows from the pullback property. Further, let the diagram above be a pullback with $b \in \mathcal{M}$ and $\mathcal{E}$ be stable under pullback along $\mathcal{M}$-morphisms then the diagram satisfies (BCP).

Lemma 3.1.40. If the commutative diagram in Lemma 3.1.38 is a pullback and satisfies the Beck-Chevalley's Property (BCP) then

(a) $a_*(g^*(t)) \cong f^*(b_*(t))$ for all t in $\in \text{sub}B$ and

(b) $g_*(a^*(m)) \cong b^*(f_*(m))$ for all m in sub$X$.

Proof. (a) Let $t \in \text{sub}B$. Then

$$\begin{aligned}
a_*(g^*(t)) &\leq f^*(f(a_*(g^*(t)))) && (f \dashv f^*)\\
&\leq f^*(b_*(b^*(f(a_*(g^*(t)))))) && (b^* \dashv b_*)\\
&\cong f^*(b_*(g(a^*(a_*(g^*(t)))))) && (BCP)\\
&\leq f^*(b_*(g(g^*(t)))) && (a^* \dashv a_*)\\
&\leq f^*(b_*(t)) && (g \dashv g^*)\\
&\leq a_*(g^*(t)) && (\text{Lemma 3.1.38}).
\end{aligned}$$

Therefore, $a_*(g^*(t)) \cong f^*(b_*(t))$.

(b) This follows from (a).

□

Remark 3.1.41. The converse of Lemma 3.1.40 holds, that is: if (a) or (b) is true then the pullback diagram of Lemma 3.1.40 satisfies the Beck-Chevalley's Property (BCP). Indeed, suppose $a_*(g^*(t)) \cong f^*(b_*(t))$ for all $t \in \text{sub}B$. Then for any $m \in \text{sub}X$ one has:

$$\begin{aligned}
b^*(f(m)) &\leq b^*(f(a_*(a^*(m)))) && (a^* \dashv a_*)\\
&\leq b^*(f(a_*(g^*(g(a^*(m)))))) && (g \dashv g^*)\\
&\cong b^*(f(f^*(b_*(g(a^*(m)))))) && (a_*(g^*(t)) \cong f^*(b_*(t)))\\
&\leq b^*(b_*(g(a^*(m)))) && (f \dashv f^*)\\
&\leq g(a^*(m)) && (b^* \dashv b_*).
\end{aligned}$$

Therefore, $g(a^*(m)) \cong b^*(f(m))$ for all $m \in \text{sub}X$ since one always has $g(a^*(m)) \leq b^*(f(m))$. Furthermore, a similar argument yields that $a(g^*(t)) \cong f^*(b(t)) \Rightarrow b^*(f_*(m)) \cong g_*(a^*(m))$ for all $t \in \text{sub}B$ and $m \in \text{sub}X$.

The following theorem shows i-initial, i-open, i-closed and i-final morphisms ascend along i-initial morphisms and they descend along i-final morphisms.

Theorem 3.1.42 (Pullback ascent and descent)**.** *Assume that $\mathcal{E}$ is stable under pullback and consider the pullback diagram*

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y \end{array}.$$

Then:

(a) $[a \in \mathcal{I}(i)$ *and* $f \in \mathcal{I}(i)(\mathcal{O}(i), \mathcal{K}(i), \mathcal{F}(i),$ *resp.*$)] \Rightarrow g \in \mathcal{I}(i)(\mathcal{O}(i), \mathcal{K}(i), \mathcal{F}(i),$ *resp.*$)$. *Moreover,* $b \in \mathcal{I}(i)$.

(b) $[b \in \mathcal{F}(i)$ *and* $g \in \mathcal{F}(i)(\mathcal{O}(i), \mathcal{K}(i), \mathcal{I}(i),$ *resp.*$)] \Rightarrow f \in \mathcal{F}(i)(\mathcal{O}(i), \mathcal{K}(i), \mathcal{I}(i),$ *resp.*$)$. *Moreover,* $a \in \mathcal{F}(i)$.

(c) $[a \in \mathcal{I}(i), b \in \mathcal{M}$ *and* $f \in \mathcal{WF}(i)] \Rightarrow g \in \mathcal{WF}(i)$.

(d) $[b \in \mathcal{WF}(i), f \in \mathcal{M}$ *and* $g \in \mathcal{I}(i)] \Rightarrow f \in \mathcal{I}(i)$.

Proof. Note that the assumption $\mathcal{E}$ is stable under pullback implies the above pullback diagram satisfies the Beck-Chevalley Property (BCP) by Remark 3.1.39(b).

(a) Suppose a is i-initial.

(i-initial): If f is i-initial then

$$\begin{aligned} a, f \in \mathcal{I}(i) &\Rightarrow f \circ a \in \mathcal{I}(i) \\ &\Rightarrow b \circ g \in \mathcal{I}(i) && (f \circ a = b \circ g). \end{aligned}$$

Therefore, by Proposition 3.1.20(b), $g \in \mathcal{I}(i)$.

(i-open): If f is i-open and $t \in \text{sub}A$ then

$$\begin{aligned} g(i_A(t)) &\cong g(a^*(i_X(a_*(t)))) && (a \in \mathcal{I}(i)) \\ &\cong b^*(f(i_X(a_*(t) && (\text{ BCP }) \\ &\cong b^*(i_Y(f(a_*(t)))) && (f \in \mathcal{O}(i)) \\ &\leq i_B(b^*(f(a_*(t)))) && (b\ i\text{-continuous}) \\ &\cong i_B(g(a^*(a_*(t)))) && (\text{ BCP}) \\ &\leq i_B(g(t)) && (a^* \dashv a_*). \end{aligned}$$

Hence, $g \in \mathcal{O}(i)$.

(i-closed): If f is i-closed and $t \in \text{sub}A$ then

$$\begin{aligned} g_*(i_A(t)) &\leq g_*(a^*(i_X(a_*(t)))) && (a \in \mathcal{I}(i)) \\ &\cong b^*(f_*(i_X(a_*(t) && (\text{ Lemma 3.1.40 }) \\ &\cong b^*(i_Y(f_*(a_*(t)))) && (f \in \mathcal{K}(i)) \\ &\leq i_B(b^*(f_*(a_*(t)))) && (b\ i\text{-continuous}) \\ &\cong i_B(g_*(a^*(a_*(t)))) && (\text{ Lemma 3.1.40 }) \\ &\leq i_B(g_*(t)) && (a^* \dashv a_*). \end{aligned}$$

Hence, $g \in \mathcal{K}(i)$.

(i-final): If f is i-final and $p \in$ sub$B$ then

$$\begin{aligned} g_*(i_A(g^*(p))) &\cong g_*(a^*(i_X(a_*(g^*(p))))) && (a \in \mathcal{I}(i)) \\ &\cong b^*(f_*(i_X(a_*(g^*(p))))) && (\text{ Lemma } 3.1.40 \text{ }) \\ &\cong b^*(f_*(i_X(f^*(b_*(p))))) && (\text{ Lemma } 3.1.40 \text{ }) \\ &\cong b^*(i_Y(b_*(p))) && (f \in \mathcal{F}(i)) \\ &\leq i_B(b^*(b_*(p))) && (b \text{ } i\text{-continuous}) \\ &\leq i_B(p) && (b^* \dashv b_*). \end{aligned}$$

Hence, $g \in \mathcal{F}(i)$.

Furthermore, b is i-initial. Indeed, since $g^* \dashv g_*$ and g is i-continuous we get

$$\begin{aligned} i_B(p) \leq i_B(g_*(g^*(p))) \leq g_*(i_A(g^*(p))) &\\ \cong g_*(a^*(i_X(a_*(g^*(p))))) &\\ \cong b^*(f_*(i_X(a_*(g^*(p))))) &\\ \cong b^*(f_*(i_X(f^*(b_*(p))))) &\\ \cong b^*(i_Y(b_*(p))). & \end{aligned}$$

(b) Suppose b is i-final.

(i-final): If g is i-final then

$$\begin{aligned} b, g \in \mathcal{F}(i) &\Rightarrow b \circ g \in \mathcal{F}(i) \\ &\Rightarrow f \circ a \in \mathcal{F}(i) && (f \circ a = b \circ g). \end{aligned}$$

Therefore, by Proposition 3.1.26 (b), $f \in \mathcal{F}(i)$.

(i-open): If g is i-open and $m \in$ sub$X$ then

$$\begin{aligned} f(i_X(m)) &\leq b_*(b^*(f(i_X(m)))) && (b^* \dashv b_*) \\ &\cong b_*(g(a^*(i_X(m)))) && (\text{ BCP }) \\ &\leq b_*(g(i_A(a^*(m)))) && (a \text{ } i\text{-continuous}) \\ &\cong b_*(i_B(g(a^*(m)))) && (g \in \mathcal{O}(i)) \\ &\cong b_*(i_B(b^*(f(m)))) && (\text{ BCP}) \\ &\cong i_Y(f(m)) && (b \in \mathcal{F}(i)). \end{aligned}$$

Hence, $f \in \mathcal{O}(i)$.

(i-closed): If g is i-closed and $m \in$ sub$X$ then

$$\begin{aligned} f_*(i_X(m)) &\leq b_*(b^*(f_*(i_X(m)))) && (b^* \dashv b_*) \\ &\cong b_*(g_*(a^*(i_X(m)))) && (\text{ Lemma } 3.1.40 \text{ }) \\ &\leq b_*(g_*(i_A(a^*(m)))) && (a \text{ } i\text{-continuous}) \\ &\cong b_*(i_B(g_*(a^*(m)))) && (g \in \mathcal{K}(i)) \\ &\cong b_*(i_B(b^*(f_*(m)))) && (\text{ Lemma } 3.1.40 \text{ }) \\ &\cong i_Y(f_*(m)) && (b \in \mathcal{F}(i)). \end{aligned}$$

Hence, $f \in \mathcal{K}(i)$.

(i-initial): If g is i-initial and $m \in$ sub$X$ then

$$\begin{aligned}
i_X(m) &\leq i_X(a_*(a^*(m))) && (a^* \dashv a_*)\\
&\leq a_*(i_A(a^*(m))) && (a\ i\text{-continuous})\\
&\cong a_*(g^*(i_B(g_*(a^*(m))))) && (g \in \mathcal{I}(i))\\
&\cong f^*(b_*(i_B(b^*(f_*(m))))) && (\text{ Lemma } 3.1.40\ )\\
&\cong f^*(i_Y(f_*(m))) && (b \in \mathcal{F}(i)).
\end{aligned}$$

Hence, $f \in \mathcal{I}(i)$.

Furthermore, a is i-final. Indeed, since $f^* \dashv f_*$ and f is i-continuous we get

$$\begin{aligned}
a_*(i_A(a^*(m))) &\cong a_*(g^*(i_B(g_*(a^*(m)))))\\
&\cong f^*(b_*(i_B(b^*(f_*(m)))))\\
&\cong f^*(i_Y(f_*(m)))\\
&\leq i_X(f^*(f_*(m))) \leq i_X(m).
\end{aligned}$$

(c) Let $p \in$ sub$B$.

$$\begin{aligned}
p \wedge g_*(i_A(g^*(p))) &\cong p \wedge g_*(a^*(i_X(a_*(g^*(p))))) && (a \in \mathcal{I}(i))\\
&\cong b^*(b_*(p)) \wedge b^*(f_*(i_X(a_*(g^*(p))))) && (b \in \mathcal{M} \text{ and Lemma } 3.1.40\ )\\
&\cong b^*(b_*(p) \wedge f_*(i_X(f^*(b_*(p))))) && (b^* \dashv b_* \text{ and Lemma } 3.1.40\ )\\
&\cong b^*(i_Y(b_*(p))) && (f \in \mathcal{WF}(i))\\
&\leq i_B(b^*(b_*(p))) && (b\ i\text{-continuous})\\
&\cong i_B(p) && (b \in \mathcal{M}).
\end{aligned}$$

Hence, $g \in \mathcal{WF}(i)$.

(d) Let $m \in$ sub$X$.

$$\begin{aligned}
i_X(m) &\leq i_X(a_*(a^*(m))) && (a^* \dashv a_*)\\
&\leq a_*(i_A(a^*(m))) && (a\ i\text{-continuous})\\
&\cong a_*(g^*(i_B(g_*(a^*(m))))) && (g \in \mathcal{I}(i))\\
&\cong f^*(b_*(i_B(b^*(f_*(m))))) && (\text{ Lemma } 3.1.40\ ).
\end{aligned}$$

Consequently,

$$\begin{aligned}
i_X(m) &\leq m \wedge f^*(b_*(i_B(b^*(f_*(m)))))\\
&\cong f^*(f_*(m)) \wedge f^*(b_*(i_B(b^*(f_*(m)))))\\
&\cong f^*(f_*(m) \wedge b_*(i_B(b^*(f_*(m)))))\\
&\cong f^*(i_Y(f_*(m))) && (b \in \mathcal{WF}(i)).
\end{aligned}$$

Hence, $f \in \mathcal{I}(i)$.

□

In **Top**, k^{in}-open morphisms (or the usual open maps) are stable under pullback, hence the assumption that $a \in \mathcal{I}(k^{\text{in}})$ is not needed in Theorem 3.1.42 (see [DT95]).

Examples 3.1.43. (a) Consider the Kuratowski interior operator k^{in} on the category **Top** of topological spaces with (surjective, embedding)-factorization structure and a continuous function $f : X \to Y$.

(i) f is k^{in}-initial if and only if $M \subseteq X$ is open if and only if $M = f^{-1}(N)$ for some $N \subseteq Y$ is open. Since k^{in} is idempotent and by Proposition 3.1.23, f is k^{in}-initial if and only if X carries the initial topology with respect to f.

(ii) f is k^{in}-final if and only if f is surjective and for any $N \subseteq Y$ one has $M \subseteq N$ is open if and only if $f^{-1}(M) \subseteq f^{-1}(N)$ is open (every subspace $N \subseteq Y$ carries the final topology with respect to the restriction $f^{-1}(N) \to N$ of f). That is, the k^{in}-final maps are precisely the hereditary quotient maps. Note that hereditary quotient maps are surjective maps $f : X \to Y$ for which every restriction $f^{-1}(N) \to N$ of f with $N \subseteq Y$ is a quotient map.

(iii) $f \in \mathcal{O}(k^{\text{in}})$ if and only if f is an open map, that is: $O \subseteq X$ is open $\Rightarrow f(O) \subseteq Y$ is open.

(b) Consider the up-interior $\uparrow^{\text{in}}_{G}(H) = \{h \in H : (\forall g \in G \setminus H)$ there is no edge $g \to h\}$ on the category **SGph** of spatial graphs with (surjective, embedding)-factorization structure.
The $\uparrow^{\text{in}}$-initial morphisms $f : G \to G^{'}$ are characterized by the condition $g \to g^{'} \Leftrightarrow f(g) \to f(g^{'})$ for all $g, g^{'} \in G$ and the $\uparrow^{\text{in}}$-final morphisms are precisely the surjections $f : G \to G^{'}$ such that $k \to k^{'} \Leftrightarrow \exists g, g^{'} \in G$ with $g \to g^{'}, f(g) = k$ and $f(g^{'}) = k^{'}$.

(c) Let i be an interior operator on a reflective subcategory $\mathbb{S}$ of the category $\mathbb{C}$ such that preimages commute with arbitrary joins in $\mathbb{C}$ and let $i(\mathbb{S})$ be a lifted interior operator on $\mathbb{C}$ from $\mathbb{S}$. Then

(i) Each $\mathbb{S}$-reflection morphism is $i(\mathbb{S})$-initial,

(ii) Each $\mathbb{S}$-reflection morphism is $i(\mathbb{S})$-open, $i(\mathbb{S})$-closed and $i(\mathbb{S})$-final provided that $\mathbb{S}$ is $\mathcal{E}'$-reflective.

(d) Let t^{in} be the trivial interior operator on $\mathbb{C}$ such that $\mathbb{C}$-morphisms reflect 0. Then $f^{*}(0_Y) \cong 0_X$, hence for $f \in \mathcal{E}^{'}$, one has $f_{*}(0_X) \cong f_{*}(f^{*}(0_Y)) \cong 0_Y$. Consequently, every morphism is both t^{in}-initial and t^{in}-open and every morphism in $\mathcal{E}^{'}$ is both t^{in}-closed and t^{in}-final. Let d^{in} be the discrete interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$. Then every morphism is both d^{in}-open and d^{in}-closed, every morphism in $\mathcal{M}$ is d^{in}-initial and every morphism in $\mathcal{E}^{'}$ is d^{in}-final.

In this section, due to the role of the adjunctions, we have seen that a lot of results (and proofs) are mirror to each other. Moreover, since we assume that the preimage $f^{*}(-)$ preserves arbitrary joins for every morphism f in the category $\mathbb{C}$, this allows $f^{*}(-)$ to have both left $f(-)$ and right $f_{*}(-)$ adjoints. Consequently, some of the results (and proofs) with respect to interior operators are similar to that of closure operators. Indeed, this is should not come as a surprise since by Theorem 2.3.8 we know that there is a natural way of moving from closure to interior operators and vice versa. Our results provide interior-theoretic descriptions of the notions. Furthermore, there are new insights and importantly some things that can only be done with interior operators. Note that the assumption that arbitrary joins are preserved by each preimage is essential and enables us to explicitly define the notions of closed, initial and final morphisms in terms of dual images. In fact, one can not deal with these notions without having this assumption.

3.2 Quasi open, codense morphisms with respect to an interior operator

In this section we introduce a notion of quasi open morphisms with respect to an interior operator i on an arbitrary category $\mathbb{C}$ and discuss some of their properties. In particular, it is shown that the quasi i-open morphisms of $\mathbb{C}$ are characterized as the morphisms which reflect i-codensity. We also introduce a general notion of i-codense subobjects. We begin with the following definition and proposition which motivate the notion of quasi i-open morphisms.

Definition 3.2.1. A morphism $f : X \to Y$ is said to reflect i-codensity if $f^*(-)$ maps i-codense subobjects of Y to i-codense subobjects of X.

Consequently, every i-open morphism reflects i-codensity:

Proposition 3.2.2. Suppose $f : X \to Y \in \mathcal{O}(i)$ reflects the least subobject. Then f reflects i-codensity.

Proof. Let n be an i-codense subobject of Y. Then, one has

$$\begin{aligned} i_X(f^*(n)) &\cong f^*(i_Y(n)) && (f \in \mathcal{O}(i)) \\ &\cong f^*(0_Y) && (n \ i\text{-codense}) \\ &\cong 0_X && (f \text{ reflects } 0_Y). \end{aligned}$$

□

Note that in any category in which the preimage functor for any given morphism preserves arbitrary joins (in particular, in topological categories $\mathbb{C}$ over **Set**), each morphism reflects the least subobject (see Remark 1.4.3(c)).

Remark 3.2.3. (a) Let sub$X$ be a Boolean algebra for every $\mathbb{C}$-object X and suppose complements are preserved by preimages. Let c be a closure operator and i^c be the induced interior operator from c given by $i^c_X(m) = \overline{c_X(\overline{m})}$ for all $m \in$ sub$X$, where $\overline{m}$ denotes the complement of m. Then a $\mathbb{C}$-morphism f reflects i^c-codensity if and only if it reflects c-density.

(b) A morphism which reflects i-codensity need not be i-open. Indeed, in **Top** the embedding r of $[0, 1]$ into $\Re$ reflects codensity with respect to the Kuratowski interior operator k^{in} induced by the Euclidean topology but r is not k^{in}-open map (see [CGT04]).

In the above proposition we showed that every i-open morphism which reflects the least subobject reflects i-codensity. However, a morphism which reflects i-codensity may not be i-open by Remark 3.2.3(b). These observations motivate the following notion:

Definition 3.2.4. A morphism $f : X \to Y$ is said to be quasi i-open if the interior of each subobject of X is the least subobject of X whenever the interior of its image under f is the least subobject of Y, that is: $(\forall m \in \text{sub}X)\,(i_Y(f(m)) \cong 0_Y \Rightarrow i_X(m) \cong 0_X)$.

In **Top**, the quasi open morphisms with respect to the Kuratowski interior operator k^{in} are precisely the quasi open maps studied in [MP62, Kao83, Kim98]. Such maps are also called semi-open in [HS68].

Consequently, the above definition provides a generalization to an arbitrary category $\mathbb{C}$ of the notion of quasi open maps in topology in terms of interior operators.

The following is a handy characterization of quasi i-open morhisms in terms of i-codensity.

Proposition 3.2.5. For a morphism $f : X \to Y$ in $\mathbb{C}$, the following are equivalent:

(a) f is quasi i-open;

(b) each subobject of X is i-codense in X whenever its image under f is i-codense in Y, that is: $(\forall m \in \mathrm{sub}X)\,(f(m)$ is i-codense in $Y \Rightarrow m$ is i-codense in $X)$;

(c) f reflects i-codensity, that is: if n is i-codense in Y then $f^*(n)$ is i-codense in X.

Proof. $(a) \Rightarrow (b)$ follows immediately from the definitions.

$(b) \Rightarrow (c)$ Let n be an i-codense in Y. Since $f(f^*(n)) \leq n$, one has $f(f^*(n))$ is i-codense in Y by Remark 2.1.15(c). Consequently, $f^*(n)$ is i-codense in X.

$(c) \Rightarrow (a)$ Let $m \in \mathrm{sub}X$ such that $i_Y(f(m)) \cong 0_Y$. Then $f(m)$ is i-codense in Y, hence $f^*(f(m))$ is i-codense in X by hypothesis. Consequently, by Remark 2.1.15(c), m is i-codense in X since $m \leq f^*(f(m))$. Therefore, $i_X(m) \cong 0_X$.

□

Proposition 3.2.5 states that the quasi i-open morphisms of $\mathbb{C}$ are precisely the morphisms which reflect i-codensity. Next we show that quasi i-open morphisms are a generalization of i-open morphisms.

Proposition 3.2.6. If f is an i-open morphism and reflects the least subobject, then f is a quasi i-open.

Proof. Let $m \in \mathrm{sub}X$ such that $i_Y(f(m)) \cong 0_Y$. Since f is an i-open morphism, one has $f(i_X(m)) \leq i_Y(f(m)) \cong 0_Y$. Consequently, $i_X(m) \leq f^*(0_Y) \cong 0_X$ since f reflects 0_Y. Therefore, f is a quasi i-open morphism. □

Of course, the above proposition is a direct consequence of Propositions 3.2.2 and 3.2.5(c).

Corollary 3.2.7. Every i-open morphism in the class $\mathcal{M}$ is quasi i-open.

Proof. follows from the above proposition since each subobject morphism reflects the least subobject. □

Remark 3.2.8. (a) The class C^i of i-codense $\mathcal{M}$-subobjects is stable under pullback along $\mathcal{QO}(i)$-morphisms since, by Proposition 3.2.5, quasi i-open morphisms reflect i-codensity.

(b) The class C^i of i-codense $\mathcal{M}$-subobjects is left-cancellable with respect to the class of i-open morphisms in $\mathcal{M}$. Indeed, for $s, t \in \mathcal{M}$ such that $s \circ t \in C^i$ and $s \in \mathcal{O}(i)$, one has the pullback diagram

$$\begin{array}{ccc} \cdot & \xrightarrow{1} & \cdot \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s\circ t} \\ \cdot & \xrightarrow[s]{} & \cdot \end{array}$$

with $s \in \mathcal{QO}(i)$ by Corollary 3.2.7. Consequently, by (a), $t \in C^i$ since t is a pullback of $s \circ t \in C^i$ along $s \in \mathcal{QO}(i)$.

As a consequence of Propositions 2.1.14 and 3.2.5 the following corollaries are now evident.

Corollary 3.2.9. The following statements are equivalent for an $\mathcal{M}$-morphism $f : X \to Y$:

(a) f is quasi i-open;

(b) $(\forall m \in \text{sub}X)\,(f(m)$ is i-codense in $Y \Leftrightarrow m$ is i-codense in $X)$

Corollary 3.2.10. The following statements are equivalent for an $\mathcal{E}'$-morphism $f : X \to Y$:

(a) f is quasi i-open;

(b) $(\forall n \in \text{sub}Y)\,(f^*(n)$ is i-codense in $X \Leftrightarrow n$ is i-codense in $Y)$

In what follows we use $\mathcal{QO}(i)$ to denote the class of quasi i-open morphisms. The class $\mathcal{QO}(i)$ has the following stability properties:

Proposition 3.2.11. The class $\mathcal{QO}(i)$

(a) is closed under composition,

(b) is left-cancellable with respect to $\mathcal{M}$, that is: if $g \circ f \in \mathcal{QO}(i)$ and $g \in \mathcal{M}$ then $f \in \mathcal{QO}(i)$,

(c) contains all the isomorphisms and

(d) is right-cancellable with respect to $\mathcal{E}'$, that is: if $g \circ f \in \mathcal{QO}(i)$ and $f \in \mathcal{E}^*$ then $g \in \mathcal{QO}(i)$.

Proof. Consider the morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathbb{C}$.

(a) Suppose $f, g \in \mathcal{QO}(i)$. Let $m \in \text{sub}X$ such that $(g \circ f)(m)$ is i-codense in Z. Then:

$g(f(m))$ is i-codense in Z	$((g \circ f)(m) \cong g(f(m)))$
$\Rightarrow f(m)$ is i-codense in Y	$(g \in \mathcal{QO}(i))$
$\Rightarrow m$ is i-codense in X	$(f \in \mathcal{QO}(i))$.

Therefore, $g \circ f \in \mathcal{QO}(i)$.

(b) Suppose $g \circ f \in \mathcal{QO}(i)$ and $g \in \mathcal{M}$. Let $m \in \text{sub}X$ such that $f(m)$ is i-codense in Y. Then:

$g(f(m))$ is i-codense in Z	(Proposition 2.1.14(a))
$\Rightarrow (g \circ f)(m)$ is i-codense in Z	$((g \circ f)(m) \cong g(f(m)))$
$\Rightarrow m$ is i-codense in X	$(g \circ f \in \mathcal{QO}(i))$.

Therefore, $f \in \mathcal{QO}(i)$.

(c) Let $f : X \to Y$ be an isomorphism. Then f has an inverse $f^{-1} : Y \to X$ such that $f^{-1} \circ f = 1_X$. Consequently, (b) implies $f \in \mathcal{QO}(i)$ since 1_X is obviously quasi i-open and $f^{-1} \in \text{Iso}(\mathbb{C}) \subseteq \mathcal{M}$.

(d) Suppose $g \circ f \in \mathcal{QO}(i)$ and $f \in \mathcal{E}'$. Let $n \in \text{sub}Y$ such that $g(n)$ is i-codense in z. Since $f \in \mathcal{E}'$, one has $g(f(f^*(n))) \cong g(n)$. Hence $g(f(f^*(n))) \cong (g \circ f)(f^*(n))$ is i-codense in Z. Consequently, $f^*(n)$ is i-codense in X since $g \circ f \in \mathcal{QO}(i)$. This in turn implies n is i-codense in Y since $f \in \mathcal{E}'$. Therefore, $g \in \mathcal{QO}(i)$. $\square$

As an immediate consequence of the above proposition, one has:

Corollary 3.2.12. Let $f = m \circ e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}'$. $f \in \mathcal{QO}(i)$ if and only if $m, e \in \mathcal{QO}(i)$.

Corollary 3.2.13. Let i be any interior operator and $f : X \to Y \in \mathcal{QO}(i)$. If $m : M \to X \in \mathcal{M}$ is an i-open morphism, then $f \circ m \in \mathcal{QO}(i)$.

Proof. This is an immediate consequence of Corollary 3.2.7 and Proposition 3.2.11(a). □

Let us denote by $\mathcal{QO}(i)^*$ the class $\{f \in \mathbb{C} :$ every pullback of f reflects i-codensity$\}$. Then, by Proposition 3.2.11 and properties of pullbacks, $\mathcal{QO}(i)^*$ satisfies the following fundamental stability properties.

Proposition 3.2.14. The class $\mathcal{QO}(i)^*$

(a) contains all the isomorphisms, is closed under composition and stable under pullback,

(b) is left-cancellable with respect to $\mathcal{M}$, that is: if $g \circ f \in \mathcal{QO}(i)^*$ and $g \in \mathcal{M}$ then $f \in \mathcal{QO}(i)^*$ and

(c) is right-cancellable with respect to $\mathcal{E}^*$, that is: if $g \circ f \in \mathcal{QO}(i)^*$ and $f \in \mathcal{E}^*$ then $g \in \mathcal{QO}(i)^*$.

In the remainder of this section we introduce a notion of codense morphisms with respect to an interior operator i, which are generalizations of i-codense subobjects.

Definition 3.2.15. A morphism $f : X \to Y$ is an i-codense if the $\mathcal{M}$-part $f(1_X)$ of the $(\mathcal{E}, \mathcal{M})$-factorization of f is an i-codense subobject of Y, that is: $i_Y(f(1_X)) \cong 0_Y$.

We use $\mathcal{CD}(i)$ to denote the class of i-codense morphisms.

Remark 3.2.16. (a) i-codense $\mathcal{M}$-morphisms are precisely i-codense $\mathcal{M}$-subobjects.

(b) For a standard interior operator i, $\mathcal{E}$-morphisms with non-trivial codomain can not be i-codense. Note that a trivial object Y is an object in $\mathbb{C}$ with $0_Y \cong 1_Y$, that is: 0_Y is an isomorphism, hence sub$Y$ having exactly one member, up to isomorphism.

(c) A morphism $f : X \to Y$ is i-codense if and only if $f(m)$ is an i-codense subobject of Y for all $m \in$ sub$X$. Indeed, this follows from $f(m) \leq f(1_X)$.

(d) Let preimages commute with arbitrary joins in the category $\mathbb{C}$. If $f : X \to Y \in \mathcal{CD}(i)$ and $g : Y \to Z \in \mathcal{E}'$ then $g_*(f(1_X))$ is an i-codense subobject of Z. Indeed, by the above definition $f(1_X)$ is i-codense in Y. Consequently, by Remark 2.1.15(d), $g_*(f(1_X))$ is i-codense in Z.

Examples 3.2.17. (a) In the category **Top** each non-surjective continuous function with indiscrete topological space codomain is a codense morphism with respect to k^{in}.

(b) In the category **Grp**, the i-codense morphisms are exactly the non-surjective group homomorphisms $f : G \to H$ for which their image $f(G)$ do not contain proper normal subgroups of H. In particular,

(i) the trivial group homorphism $f : G \to H$ given by $f(g) = e_H$ for all $g \in G$ is a codense morphism with respect to any interior operator i on **Grp**.

(ii) any non-surjective group homomorphism $f : G \to S$, where S is a simple group is codense with respect to the normal interior operator on **Grp**.

(c) Any non-surjective ring homomorphism $f : R \to S$, where S is a cyclic ring is codense with respect to the ideal interior operator on **Rng**.

(d) In the category of R-**Mod**, the zero maps are d^{in}-codense morphisms.

Consequently, one has the following stability properties of $\mathcal{CD}(i)$.

Proposition 3.2.18. The class $\mathcal{CD}(i)$

(a) is stable under composition with $\mathbb{C}$-morphisms from the right, that is: if $g \in \mathcal{CD}(i)$ and f in $\mathbb{C}$ then $g \circ f \in \mathcal{CD}(i)$,

(b) is right-cancellable with respect to $\mathcal{E}$, that is: if $g \circ f \in \mathcal{CD}(i)$ and $f \in \mathcal{E}$ then $g \in \mathcal{CD}(i)$,

(c) is left-cancellable with respect to $\mathcal{QO}(i)$, that is: if $g \circ f \in \mathcal{CD}(i)$ and $g \in \mathcal{QO}(i)$ then $f \in \mathcal{CD}(i)$,

(d) is stable under composition with $\mathcal{M}$-morphisms from the left, that is: if $g \in \mathcal{M}$ and $f \in \mathcal{CD}(i)$ then $g \circ f \in \mathcal{CD}(i)$.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathbb{C}$ such that:

(a) $g \in \mathcal{CD}(i)$. Then $g(1_Y)$ is an i-codense subobject of Z. Consequently, since $(g \circ f)(1_X) \cong g(f(1_X)) \leq g(1_Y))$, Remark 2.1.15$(c)$ yields $(g \circ f)(1_X)$ is an i-codense subobject of Z. Therefore, $g \circ f \in \mathcal{CD}(i)$.

(b) $g \circ f \in \mathcal{CD}(i)$ and $f \in \mathcal{E}$. Then Remarks 1.3.6(b) and 1.3.9 and Definition 3.2.15 yield $i_Z(g(1_Y)) \cong i_Z(g(f(1_X))) \cong i_Z((g \circ f)(1_X)) \cong 0_Z$.

(c) $g \circ f \in \mathcal{CD}(i)$ and $g \in \mathcal{QO}(i)$. Then $g(f(1_X)) \cong (g \circ f)(1_X)$ is an i-codense subobject of Z. Hence, by Proposition 3.2.5, $g^*(g(f(1_X)))$ is an i-codense subobject of Y. Consequently, by Remark 2.1.15(c), $f(1_X)$ is an i-codense subobject of Y since $f(1_X) \leq g^*(g(f(1_X)))$. Therefore, $f \in \mathcal{CD}(i)$.

(d) This follows from Proposition 2.1.14(a).

□

3.3 Quotient maps with respect to an interior operator

In the category **Top** of topological spaces and continuous maps, a quotient map is just an epimorphism $f : X \to Y$ for which $B \subseteq Y$ is open whenever $f^*(B)$ is open. In this section we make use of this concept and the idea of the paper [CGT01] to introduce and study the notion of quotient maps with respect to an interior operator in the category $\mathbb{C}$. We use these maps to investigate a notion of connectedness with respect to a given interior operator in Section 5.2. We start with the next definition.

Definition 3.3.1. A morphism $f : X \to Y$ in $\mathbb{C}$ is said to reflect i-open subobjects if each subobject n of Y is i-open provided that $f^*(n)$ is i-open in X.

Definition 3.3.2. A morphism $f : X \to Y$ in $\mathbb{C}$ is said to be an i-quotient if it lies in $\mathcal{E}$ and reflects i-open subobjects.

Remark 3.3.3. Let $\mathcal{Q}(i)$ be the class of all i-quotient morphisms. Then

(a) In Definition 3.3.2 we can replace the statement $f^*(n)$ is i-open in X implies n is i-open in Y by $f^*(n)$ is i-open in X if and only if n is i-open in Y. Indeed, if n is i-open in Y then one always has $f^*(n)$ is i-open in X.

(c) Let i be a standard interior operator. Then as a consequence of Proposition 3.1.28(b) and Proposition 3.1.29 one always has that an i-final morphism is an i-quotient, with the converse statement failing already for $\mathbb{C} =$ **Top**, $i = k^{\text{in}}$ (see [CGT01]).

We discuss some important properties of i-quotient maps in the following proposition.

Proposition 3.3.4. The class $\mathcal{Q}(i)$

(a) is stable under composition,

(b) is right-cancellable, that is: if $g \circ f \in \mathcal{Q}(i)$ then $g \in \mathcal{Q}(i)$,

(c) contains all the isomorphisms and

(d) is left-cancellable with respect to $\mathcal{M}$ provided that $\mathcal{E} \subseteq \mathcal{E}'$, that is: if $g \circ f \in \mathcal{Q}(i)$ and $g \in \mathcal{M}$ and $\mathcal{E} \subseteq \mathcal{E}'$ then $f \in \mathcal{Q}(i)$.

Proof. (a) Suppose $f : X \to Y, g : Y \to Z \in \mathcal{Q}(i)$. Then $f, g \in \mathcal{E}$ and both f and g reflect i-open subobjects. Hence, $g \circ f \in \mathcal{E}$ and for $u \in \text{sub}Z$ since f is an i-quotient. Consequently, $(g \circ f)^*(u) = f^*(g^*(u))$ is i-open in X implies $g^*(u)$ is i-open in Y. This in turn implies u is i-open in Z as g is also an i-quotient map. Therefore, $g \circ f \in \mathcal{E}$ and $(g \circ f)^*(u)$ is i-open in X implies u is i-open in Z, as desired.

(b) Suppose for $f : X \to Y, g : Y \to Z, g \circ f \in \mathcal{Q}(i)$. Then $g \circ f \in \mathcal{E}$ and as result $g \in \mathcal{E}$. And also $g \circ f$ reflects i-open subobjects. Now, consider an i open subobject $g^*(n)$ of Y. Then

$f^*(g^*(n))$ is i-open in X	(i-open subobjects are stable under pullback)
$\Rightarrow (g \circ f)^*(n)$ is i-open in X	$(g \circ f)^*(n) = f^*(g^*(n))$
$\Rightarrow n$ i-open in Z	$(g \circ f \in \mathcal{Q}(i))$.

Therefore, $g \circ f \in \mathcal{E}$ and $g^*(n)$ is i-open in Y implies n is i-open in Z, as desired.

(c) If $f : X \to Y$ is an isomorphism with inverse $f^{-1} : Y \to X$ then $f \circ f^{-1} = 1_X$ is obviously i-quotient. Consequently, (b) implies $f \in \mathcal{Q}(i)$.

(d) Suppose for $f : X \to Y, g : Y \to Z, g \circ f \in \mathcal{Q}(i)$. Then $g \circ f \in \mathcal{E}$ and $g \circ f$ reflects i-open subobjects. Since $g \in \mathcal{M}$ and $\mathcal{E}$ is stable under pullback along $\mathcal{M}$-morphisms , $g \circ f \in \mathcal{E} \Rightarrow f \in \mathcal{E}$. Also consider an i open subobject $f^*(n)$ of X. Then

$f^*(n)$ is i-open in X
$\Rightarrow (g \circ f)^*(g(n)) \cong f^*(g^*(g(n))) \cong f^*(n)$ is i-open in X, since $g \in \mathcal{M}$
$\Rightarrow g(n)$ is i-open in Z, since $g \circ f \in \mathcal{Q}(i)$
$\Rightarrow n \cong g^*(g(n))$ is i-open in Y, since i-open subobjects are stable under pullback and $g \in \mathcal{M}$.

Thus, $f \in \mathcal{E}$ and $f^*(n)$ is i-open in X implies n is i-open in $Y,$ as desired.

□

The following two propositions show the connection between i-quotient maps with i-open (or i-closed) morphisms.

Proposition 3.3.5. (a) $\mathcal{O}(i) \cap \mathcal{E}^{'} \subseteq \mathcal{Q}(i)$.

(b) Let preimages commute with joins in the category $\mathbb{C}$. Then $\mathcal{K}(i) \cap \mathcal{E}^{'} \subseteq \mathcal{Q}(i)$.

That is, every i-open (or i-closed) morphism in $\mathcal{E}^{'}$ is an i-quotient.

Proof. (a) Suppose $f : X \to Y$ and $f \in \mathcal{O}(i) \cap\ \mathcal{E}^{'}$ such that $f^*(n)$ is i-open in X. Then

$$\begin{aligned} n &\cong f(f^*(n)) && (f \in \mathcal{E}^{'}) \\ &\cong f(i_X(f^*(n))) && (f^*(n)\ \ i\text{-open }) \\ &\cong f(f^*(i_Y(n))) && (f \in \mathcal{O}(i)) \\ &\cong i_Y(n) && (f \in \mathcal{E}^{'}) \\ &\Rightarrow n\ \ i\text{-open in } Y. \end{aligned}$$

Therefore, $f^*(n)$ is i-open in X implies n is i-open in Y and clearly $\mathcal{E}^{'} \subseteq \mathcal{E}$. Hence, $f \in \mathcal{E}$ and f reflects i-open subobjects. Consequently, $f \in \mathcal{Q}(i)$.

(b) Suppose $f : X \to Y$ and $f \in \mathcal{K}(i) \cap \mathcal{E}^{'}$ such that $f^*(n)$ is i-open in X. Then

$$\begin{aligned} n &\cong f_*(f^*(n)) && (f \in \mathcal{E}^{'}) \\ &\cong f_*(i_X(f^*(n))) && (f^*(n)\ \ i\text{-open }) \\ &\cong i_Y(f_*(f^*(n))) && (f \in \mathcal{K}(i)) \\ &\cong i_Y(n) && (f \in \mathcal{E}^{'}) \\ &\Rightarrow n \text{ is } i\text{-open in } Y. \end{aligned}$$

Therefore, $f \in \mathcal{E}$ and $f^*(n)$ is i-open in X implies n is i-open in Y and clearly $\mathcal{E}^{'} \subseteq \mathcal{E}$. Hence, $f \in \mathcal{E}$ and f reflects i-open subobjects. Consequently, $f \in \mathcal{Q}(i)$.

□

Proposition 3.3.6. Let $f : X \to Y$ be an i-quotient morphism.

(a) If f is an i-open morphism then for each i-open subobject m of X, the subobject $f^*(f(m))$ is i-open in X. Moreover, if i is idempotent the converse is true.

(b) Let preimages commute with joins in the category $\mathbb{C}$. If f is an i-closed morphism then for each i-open subobject m of X, the subobject $f^*(f_*(m))$ is i-open in X. Moreover, if i is idempotent the converse is true.

Proof. (a) Let $f : X \to Y$ be an i-open morphism such that m is i-open subobject of X. Then by Remark 3.1.13(b), $f(m)$ is an i-open subobject of Y. Hence, $f(m) \cong i_Y(f(m))$. Consequently, since $f \in \mathcal{O}(i)$), one obtains $f^*(f(m)) \cong f^*(i_Y(f(m))) \cong i_X(f^*(f(m)))$. Therefore, $f^*(f(m))$ is i-open in X. On the other hand assume that i is idempotent and that for each i-open subobject m of X, the subobject $f^*(f(m))$ is i-open in X. Then, since i is idempotent, we have that

$i_X(m)$ is i-open in X and hence by assumption, $f^*(f(i_X(m)))$ is i-open in X. Consequently, $f(i_X(m))$ is i-open in Y, since $f \in \mathcal{Q}(i)$. That is, $i_Y(f(i_X(m))) \cong f(i_X(m))$. Therefore, $f(i_X(m)) \cong i_Y(f(i_X(m))) \leq i_Y(f(m))$, since $i_X(m) \leq m$ and hence $f \in \mathcal{O}(i)$. In fact we can apply the converse part of Remark 3.1.13(b).

(b) Let $f : X \to Y$ be an i-closed morphism such that m is an i-open subobject of X.

$$\begin{aligned} f^*(f_*(m)) &\cong f^*(f_*(i_X(m))) && (m \ \ i\text{-open }) \\ &\cong f^*(i_Y(f_*(m))) && (f \in \mathcal{K}(i)) \\ &\leq i_X(f^*(f_*(m))) && (f \ i\text{-continuous }) \\ &\Rightarrow i_X(f^*(f_*(m))) \cong f^*(f_*(m)) && (i\text{-contractive }). \end{aligned}$$

Thus, $f^*(f_*(m))$ is i-open in X. On the other hand assume that i is idempotent and that for each i-open subobject m of X, the subobject $f^*(f_*(m))$ is i-open in X. Then, since i is idempotent, we have that $i_X(m)$ is i-open in X and hence by assumption, $f^*(f_*(i_X(m)))$ is i-open in X. Consequently, $f_*(i_X(m))$ is i-open in Y, since $f \in \mathcal{Q}(i)$. That is, $i_Y(f_*(i_X(m))) \cong f_*(i_X(m))$. Therefore, $f_*(i_X(m)) \cong i_Y(f_*(i_X(m))) \leq i_Y(f_*(m))$, since $i_X(m) \leq m$ and hence $f \in \mathcal{K}(i)$. In fact we can apply Proposition 3.1.7.

□

We now turn to pullbacks of i-quotients and show that the class of i-quotient maps ascends along both i-open and i-closed monomorphism.

Theorem 3.3.7. *Given a pullback diagram*

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y \end{array}$$

such that a and b are monomorphisms and $\mathcal{E}$ is stable under pullback along monomorphisms, one has:

(a) *If f is an i-quotient morphism and a is an i-open morphism, then g is an i-quotient morphism. Furthermore, b is an i-open morphism provided that i is idempotent.*

(b) *Let preimages commute with joins in the category $\mathbb{C}$. If f is an i quotient morphism and a is an i-closed morphism, then g is an i-quotient morphism. Furthermore, b is an i-closed morphism provided that i is idempotent.*

Proof. The fact that $f \in \mathcal{E}$ and $\mathcal{E}$ is stable under pullback along monomorphisms implies $g \in \mathcal{E}$. Moreover,

(a) Suppose for $t \in \text{sub}B$ we have that $g^*(t)$ is i-open subobject of A. Then

$$\begin{aligned} a(g^*(t)) &\cong a(i_A(g^*(t))) && (g^*(t) \text{ is } i\text{-open in } A) \\ &\cong i_X(a(g^*(t))) && (a \in \mathcal{O}(i)) \\ &\Rightarrow i_X(a(g^*(t))) \cong a(g^*(t)) && \\ &\Rightarrow f^*(b(t)) \cong a(g^*(t)) \text{ is } i\text{-open in } X && (\text{BCP}) \\ &\Rightarrow b(t) \text{ is } i\text{-open in } Y && (f \in \mathcal{Q}(i)) \\ &\Rightarrow i_Y(b(t)) \cong b(t). && \end{aligned}$$

Consequently, $t \leq b^*(b(t)) \cong b^*(i_Y(b(t))) \leq i_X(b^*(b(t))) \cong i_X(t)$, since $b \dashv b^*, b$ is i-continuous, b is monic and $\mathcal{E}$ is stable under pullback along monomorphisms. Therefore, $i_X(t) \cong t$ and hence t is i-open in B. As a result $g \in \mathcal{Q}(i)$. On the other hand suppose t is i-open in B. Then $g^*(t)$ is i-open in A. Consequently, the fact that $a \in \mathcal{O}(i)$ together with Remark 3.1.13 implies $a(g^*(t))$ is i-open in X. As a result of BCP, we have that $f^*(b(t)) \cong a(g^*(t))$ is i-open in X. This implies $b(t)$ is i-open in X, since $f \in \mathcal{Q}(i)$. Consequently by Remark 3.1.13(b) we have that b is i-open morphism.

(b) Suppose for $t \in \text{sub}B$ we have that $g^*(t)$ is an i-open subobject of A. Then

$$\begin{aligned} a_*(g^*(t)) &\cong a_*(i_A(g^*(t))) && (g^*(t) \text{ is } i\text{-open in } A) \\ &\cong i_X(a_*(g^*(t))) && (a \in \mathcal{K}(i)) \\ &\Rightarrow i_X(a_*(g^*(t))) \cong a_*(g^*(t)) \\ &\Rightarrow f^*(b_*(t)) \cong a_*(g^*(t)) \text{ is } i\text{-open in } X && (\text{Lemma 3.1.40}) \\ &\Rightarrow b_*(t) \text{ is } i\text{-open in } Y && (f \in \mathcal{Q}(i)) \\ &\Rightarrow i_Y(b_*(t)) \cong b(t). \end{aligned}$$

Therefore, $t \cong b^*(b_*(t)) \cong b^*(i_Y(b_*(t))) \leq i_X(b^*(b_*(t))) \leq i_X(t)$, since $b^* \dashv b_*, b$ is i-continuous, b is monic and $\mathcal{E}$ is stable under pullback along monomorphisms. Hence $i_X(t) \cong t$, that is, t is i-open in B. Thus $g \in \mathcal{Q}(i)$. On the other hand suppose t is i-open in B. Then $g^*(t)$ is i-open in A and hence $a_*(g^*(t))$ is i-open in X, since $a \in \mathcal{K}(i)$ and Proposition 3.1.7. As a result of Lemma 3.1.40, we have that $f^*(b_*(t)) \cong a_*(g^*(t))$ is i-open in X. This implies $b_*(t)$ is i-open in X, since $f \in \mathcal{Q}(i)$. Consequently by Proposition 3.1.7 we have that b is an i-closed morphism.

□

Examples 3.3.8. (a) In the category Top of topological spaces with the usual (surjective, embedding) factorization structure, surjective k^{in}-closed (or k^{in}-open) morphisms are i-quotient morphisms.

(b) Let n be the normal interior operator on the category **Grp** of groups and group homomorphisms with the (surjective homomorphism, injective homorphisms)-factorization system then the n-quotient morphisms are precisely surjective group homomorphisms. Note that surjective group homomorphisms preserve normal subobjects.

3.4 Some remarks on the classes of a dual closure operator

In this section similar to what we have done in Section 3.1 we study four classes of morphisms with respect to a dual closure operator. We discuss their behaviour under composition, cancellation and pushout. In order to do this as in the Section 2.4 we consider a finitely cocomplete category $\mathbb{C}$ with $(\mathcal{E}, \mathcal{M})$-factorization systems for morphisms such that $\mathcal{E}$ is a fixed class of epimorphisms. Analogous to the class of morphisms with respect to an interior operator we define d-closed, d-open, d-initial and d-final morphisms by replacing $\leq$ by $\cong$ in the equivalent descriptions of the continuity condition with respect to a dual closure operator given in Lemma 2.4.9 and Remark 2.4.10. Of course this notion is a dual notion to classes of morphisms with respect to closure operators (see [GT00, CGT01]).

Definition 3.4.1. A morphism $f : X \to Y \in \mathbb{C}$ is said to be

(a) d-closed if for all $q \in$ quot$Y$ one has $d_X(f^\circ(q)) \cong f^\circ(d_Y(q))$, that is: f is d-closed if the left adjoint commutes with the dual closure operator;

(b) d-open if for all $p \in$ quot$X$ one has $f_\circ(d_X(p)) \cong d_Y(f_\circ(p))$, that is: f is d-open if the right adjoint commutes with the dual closure operator;

(c) d-final if for all $p \in$ quot$X$ one has $d_X(p) \cong f^\circ(d_Y(f_\circ(p)))$;

(d) d-initial if for all $q \in$ quot$Y$ one has $f_\circ(d_X(f^\circ(q))) \cong d_Y(q)$.

For the remainder of this section, we use $\mathcal{O}(d), \mathcal{K}(d), \mathcal{I}(d)$ and $\mathcal{F}(d)$ to denote the class of all d-open, d-closed, d-initial and d-final respectively.

Remark 3.4.2. The formulas for d-closed, d-open, d-final and d-initial morphisms are the same as the formulas for i-open, i-closed, i-initial and i-final morphisms, respectively, except for the fact that the former ones act on quotientobjects while the latter ones act on subobjects.

Consequently, Propositions 3.1.11, 3.1.13 (b), 3.1.4, 3.1.20, and 3.1.26 yield the following propositions, respectively.

Proposition 3.4.3. (a) $\mathcal{K}(d)$ is stable under composition and contains all the isomorphisms.

(b) $g \circ f \in \mathcal{K}(d)$ and $g \in \mathcal{M}' \Rightarrow f \in \mathcal{K}(d)$.

(c) $g \circ f \in \mathcal{K}(d)$ and $f \in \mathcal{E} \Rightarrow g \in \mathcal{K}(d)$.

Proposition 3.4.4. Let $f : X \to Y$ be a d-closed morphism. Then f° maps d-closed $\mathcal{E}$-quotient objects into d-closed $\mathcal{E}$-quotient objects. Moreover, if d is idempotent then the converse is true.

Corollary 3.4.5. A d-closed morphism $p : X \to P$ in $\mathcal{E}$ gives a d-closed quotient object. The converse is true if d is weakly cohereditary.

Proof. If p in $\mathcal{E}$ is d-closed morphism then $d_X(p^\circ(q)) \cong p^\circ(d_P(q))$ for all $q : P \to Q$. In particular for $q = 1_P$ we obtain $d_X(p) \cong d_X(1_P \circ p) \cong d_X(p^\circ(1_P)) \cong p^\circ(d_P(1_P)) \cong p^\circ(1_P) \cong 1_P \circ p \cong p$. By the Dual Diagonalization Lemma the converse is also true. □

Proposition 3.4.6. (a) $\mathcal{O}(d)$ is stable under composition and contains all the isomorphisms.

(b) $g \circ f \in \mathcal{O}(d)$ and $g \in \mathcal{M}' \Rightarrow f \in \mathcal{O}(d)$.

(c) $g \circ f \in \mathcal{O}(d)$ and $f \in \mathcal{E} \Rightarrow g \in \mathcal{O}(d)$.

Proposition 3.4.7. Let d be a dual closure operator.

(a) $\mathcal{F}(d)$ is stable under composition and contains all the isomorphisms.

(b) $g \circ f \in \mathcal{F}(d) \Rightarrow f \in \mathcal{F}(d)$.

(c) $g \circ f \in \mathcal{F}(d)$ and $f \in \mathcal{E} \Rightarrow g \in \mathcal{F}(d)$.

Remark 3.4.8. (a) Every section is d-final. Indeed, this follows from Proposition 3.4.7(b).

(b) Every d-final morphism belongs to $\mathcal{M}$. Indeed, $1_X \cong d_X(1_X) \cong f^\circ(d_Y(f_\circ(1_X))) \cong f^\circ(d_Y(1_Y)) \cong f^\circ(1_Y)$.

Proposition 3.4.9. Let d be a dual closure operator.

(a) $\mathcal{I}(d)$ is stable under composition and contains all the isomorphisms.

(b) $g \circ f \in \mathcal{I}(d) \Rightarrow g \in \mathcal{I}(d)$.

(c) $g \circ f \in \mathcal{I}(d)$ and $g \in \mathcal{M}' \Rightarrow f \in \mathcal{I}(d)$.

Remark 3.4.10. (a) Every retraction is d-initial. Indeed, this is an immediate consequence of Proposition 3.4.9(b).

(b) The dual closure operator is cohereditary if and only if every morphism in $\mathcal{E}$ is d-initial. Indeed, for $e : X \to P \in$ quot$X$ and $p : E \to P \in$ quot$E$, one has $d_E(p) \cong e_\circ(d_X(p \circ e)) \cong e_\circ(d_X(e^\circ(p)))$.

Analogous to Proposition 3.1.29 we have the following partial characterization of d-final morphisms.

Proposition 3.4.11. Let $f : X \to Y$ be a d-final morphism. Then a quotient object p of X is d-closed if and only if $f_\circ(p)$ is d-closed in Y.

Proof. ($\Rightarrow$) Suppose p is d-closed in X. Then the continuity condition of dual closure operator implies $d_Y(f_\circ(p)) \leq f_\circ(d_X(p)) \cong f_\circ(p)$.

($\Leftarrow$) Suppose $f_\circ(p)$ is d-closed in Y. Then $d_Y(f_\circ(p)) \cong f_\circ(p)$. Hence, d-finality of f and $f^\circ \dashv f_\circ$ implies $d_X(p) \leq f^\circ(d_Y(f_\circ(p))) \cong f^\circ(f_\circ(p)) \leq p$. Therefore, the right adjoint $f_\circ$ preserves d-closed quotient objects.

□

Let d be an idempotent dual closure operator. Then similar to Proposition 3.1.23 we have the following characterization of d-initial morphisms in an arbitrary category.

Proposition 3.4.12. Let d be idempotent. Then $f : X \to Y$ is d-initial if and only if for every d-closed quotient object q of Y there exists a d-closed quotient object p of X such that $q \cong f_\circ(p)$.

Proof. ($\Leftarrow$) Suppose f is a d-initial morphism and q is a d-closed quotient object of Y. Then $q \cong d_Y(q) \cong f_\circ(d_X(f^\circ(q))) \cong f_\circ(p)$, where $p \cong d_X(f^\circ(q))$ such that $d_X(p) \cong d_X(d_X(f^\circ(q))) \cong d_X(f^\circ(q)) \cong p$. Thus $q \cong f_\circ(p)$ with p as a d-closed quotient object of X.

($\Leftarrow$) From idempotency of d, for all $q \in$ quot$Y$, $d_Y(q)$ is a d-closed quotient object of Y and as a result there exists a d-closed quotient object p of X such that $d_Y(q) \cong f_\circ(p)$. Hence,

$$\begin{aligned} f_\circ(d_X(f^\circ(q))) &\leq f_\circ(d_X(f^\circ(d_Y(q)))) && (d\text{-expansive}) \\ &\leq f_\circ(d_X(f^\circ(f_\circ(p)))) && (d_Y(q) \cong f_\circ(p) \text{ and } d\text{-idempotent}) \\ &\leq f_\circ(d_X(p)) && (f^\circ \dashv f_\circ) \\ &\cong f_\circ(p) && (p \;\; d\text{-closed}). \end{aligned}$$

Therefore, $f \in \mathcal{I}(d)$.

□

The following proposition shows connections of d-final morphisms with the other three morphism classes and immediately follows from Proposition 3.4.7 and Remark 3.4.2.

Proposition 3.4.13. (a) $\mathcal{O}(d) \cap \mathcal{M}' \subseteq \mathcal{F}(d)$.

(b) $\mathcal{K}(d) \cap \mathcal{M}' \subseteq \mathcal{F}(d)$.

(c) $\mathcal{I}(d) \cap \mathcal{M}' \subseteq \mathcal{F}(d)$.

(d) $\mathcal{F}(d) \cap \mathcal{E} \subseteq \mathcal{K}(d) \cap \mathcal{O}(d)$.

Corollary 3.4.14. Let $f : X \to Y$ be a d-closed (or open) morphism in $\mathcal{M}'$. Then $p \in \text{quot}X$ is d-closed if and only if $f_\circ(p) \in \text{quot}Y$ is d-closed.

The following proposition shows us some additional properties of d-initial morphisms and follows from Proposition 3.4.9 and Remark 3.4.2.

Proposition 3.4.15. (a) $\mathcal{O}(d) \cap \mathcal{E} \subseteq \mathcal{I}(d)$.

(b) $\mathcal{K}(d) \cap \mathcal{E} \subseteq \mathcal{I}(d)$.

(c) $\mathcal{F}(d) \cap \mathcal{E} \subseteq \mathcal{I}(d)$.

(d) $\mathcal{I}(d) \cap \mathcal{M}' \subseteq \mathcal{K}(d) \cap \mathcal{O}(d)$.

We now deal with the pushout behaviour of d-closed, d-open, d-initial and d-final morphisms.

Lemma 3.4.16. For a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow[g]{} & B \end{array}$$

one always has:

(a) For all r in quot$A$, $b^\circ(g_\circ(r)) \leq f_\circ(a^\circ(r))$.

(b) For all s in quot$B$, $g^\circ(b_\circ(s)) \leq a_\circ(f^\circ(s))$.

Proof. (a)

$$\begin{aligned} b^\circ(g_\circ(r)) &\leq f_\circ(f^\circ(b^\circ(g_\circ(r)))) && (f^\circ \dashv f_\circ) \\ &\cong f_\circ((b \circ f)^\circ(g_\circ(r))) \\ &\cong f_\circ((g \circ a)^\circ(g_\circ(r))) && (b \circ f = g \circ a) \\ &\cong f_\circ(a^\circ(g^\circ(g_\circ(r)))) \\ &\leq f_\circ(a^\circ(r)) && (g^\circ \dashv g_\circ). \end{aligned}$$

(b) follows from (a).

□

We say that the commutative diagram in Lemma 3.4.16 satisfies the dual Beck-Chevalley's Property (BCP) if for every $r \in \text{quot}A$, $b^{\circ}(g_{\circ}(r)) \cong f_{\circ}(a^{\circ}(r))$. In fact, we also have for every $s \in \text{quot}B$, $g^{\circ}(b_{\circ}(s)) \cong a_{\circ}(f^{\circ}(s))$. The following theorem shows that d-open, d-closed and d-initial morphims ascends along d-final morphisms and d-open, d-closed and d-final morphisms descend along d-initial morphisms. This is again an immediate consequence of Theorem 3.1.42 and Remark 3.4.2.

Theorem 3.4.17 (Pushout ascent and descent). *Let d be a dual closure operator on $\mathbb{C}$ with respect to $\mathcal{E}$ and consider the above pushout diagram satisfying the dual Beck-Chevalley's Property (BCP). That is, if $\mathcal{M} \subseteq \mathcal{M}'$ then*

(a) $[a \in \mathcal{F}(d)$ *and* $g \in \mathcal{F}(d)(\mathcal{O}(d), \mathcal{K}(d), \mathcal{I}(d)$, *resp.*$)] \Rightarrow f \in \mathcal{F}(d)(\mathcal{O}(d), \mathcal{K}(d), \mathcal{I}(d)$, *resp.*$)$

(b) $[b \in \mathcal{I}(d)$ *and* $f \in \mathcal{I}(d)(\mathcal{O}(d), \mathcal{K}(d), \mathcal{F}(d)$, *resp.*$)] \Rightarrow g \in \mathcal{I}(d)(\mathcal{O}(d), \mathcal{K}(d), \mathcal{F}(d)$, *resp.*$)$

Corollary 3.4.18. Let d be a cohereditary dual closure operator (or $f_{\circ}(p) : Y \to f_{\circ}[P]$ is a retraction). Then the coresriction $P \to f_{\circ}[P]$ of d-initial (d-open, d-closed, d-final, resp.) along $p : X \to P$ in $\mathcal{E}$ is d-initial (d-open, d-closed, d-final, resp.).

Examples 3.4.19. (a) Consider the dual closure operator d^t defined by $d^t_X(X \to X/A) = X \to X/tA$, where $A \leq X \in$ **Ab**. Then a homomorphism $f : X \to Y$ is

(i) d^t-closed if $f^{-1}(tB) = tf^{-1}(B)$ for all $B \leq Y$;

(ii) d^t-open if $f(tA) = tf(A)$ for all $A \leq X$;

(iii) d^t-final if $tA = f^{-1}(tf(A))$ for all $A \leq X$;

(iv) d^t-initial if $f(tf^{-1}(B)) = tB$ for all $B \leq Y$.

(b) Consider the dual closure operator $(d_{\mathbf{r}})_X(X \to X/M) = X \to X/\mathbf{r}M$, where $M \leq X \in \mathbf{Mod_R}$. Then an R-linear map $f : X \to Y$ is

(i) $d_{\mathbf{r}}$-closed if $f^{-1}(\mathbf{r}N) = \mathbf{r}f^{-1}(N)$ for all $N \leq Y$;

(ii) $d_{\mathbf{r}}$-open if $f(\mathbf{r}M) = \mathbf{r}f(M)$ for all $M \leq X$;

(iii) $d_{\mathbf{r}}$-final if $\mathbf{r}M = f^{-1}(\mathbf{r}f(M))$ for all $M \leq X$;

(iv) $d_{\mathbf{r}}$-initial if $f(\mathbf{r}f^{-1}(N)) = \mathbf{r}N$ for all $N \leq Y$.

(c) Consider the dual closure operator $(d^{\mathbf{r}})_X(X \to X/M) = X \to X/(M \cap \mathbf{r}X)$ where $M \leq X \in \mathbf{Mod_R}$. Then an R-linear map $f : X \to Y$ is

(i) $d^{\mathbf{r}}$-closed if f is $\mathbf{r}$-reflecting, that is: $f^{-1}(\mathbf{r}X) = \mathbf{r}Y$;

(ii) $d^{\mathbf{r}}$-open if and only if $d^{\mathbf{r}}$-closed;

(iii) $d^{\mathbf{r}}$-final if f is injective and $\mathbf{r}$-reflecting;

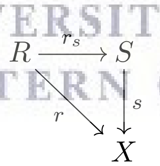
(iv) $d^{\mathbf{r}}$-initial if f is $\mathbf{r}$-preserving, that is: $f(\mathbf{r}X) = \mathbf{r}Y$.

4. Hereditary Interior Operators

Hereditary closure operators were introduced by Dikranjan and Giuli in [DG87] on an arbitrary category and have been investigated and used by several authors; see [DT95, Cas03]. More recently, hereditary interior operators have been introduced by Castellini in [Cas11]. His notion of hereditary interiors is a direct translation of the hereditary behaviour of the usual interior operator induced by the topology, but does not lend itself to a natural and general notion in an arbitrary category. In this chapter, we begin by introducing a notion of hereditary interior operators using the right adjoint of the preimage of a given morphism by assuming that each pullback commutes with the join of subobjects, as in [LTOC11]. In particular, we show that these operators behave as well as hereditary closure operators, discuss some of their basic properties and present some examples. Moreover, we prove that specific interior operators of these kind are Castellini's hereditary interior operators. We then introduce a concept of dense morphisms with respect to an interior operator and study their basic properties. In particular, we show that the class of dense morphisms with respect to a hereditary interior operator is left cancellable with respect to the class $\mathcal{M}$ and for an idempotent interior operator i, the class $\mathcal{E}^i$ of i-dense morphisms is closed under composition. We conclude the chapter by providing a few remarks on maximal interior operators. In order to be able to develop the theory of hereditary interior operators, as in the previous chapter, we consider an $\mathcal{M}$-complete category $\mathbb{C}$ with $(\mathcal{E},\mathcal{M})$-factorization systems for morphisms such that $\mathcal{M}$ is a fixed class of monomorphisms and further assume (with the exception of Section 4.3) that the preimage $f^*(-)$ preserves arbitrary joins for every morphism f in the category $\mathbb{C}$.

4.1 Heredity

Recall that a categorical closure operator c in the sense of [DG87] is hereditary if $s \wedge c_X(r) \cong s \circ c_S(r_s) \Leftrightarrow s^*(c_X(r)) \cong s^*(c_X(s(r_s))) \cong c_S(r_s) \Leftrightarrow s^*(c_X(s(r_s))) \leq c_S(r_s)$ for any two subobjects $r : R \to X$ and $s : S \to X$ of X such that $r \leq s \Leftrightarrow \ r = s \circ r_s$.



On the other hand, hereditary interior operators have been introduced more recently in [Cas11]. The author calls an interior operator i hereditary if $s \circ i_S(r_s) \wedge i_X(s) \leq i_X(r) \Leftrightarrow s \circ i_S(r_s) \wedge i_X(s) \cong i_X(r)$ for any two subobjects $r : R \to X$ and $s : S \to X$ of X such that $r \leq s$. This property, which we term C-hereditary, is a direct translation of the hereditary behaviour of the usual interior operator on a topological space. In fact, in the category **Top** of topological spaces and continuous maps, i is C-hereditary if $i_S(R) \cap i_X(S) = i_X(R)$ for any $X \in$ **Top** and $R \subseteq S \subseteq X$, in particular, the Kuratowski interior operator k^{in} enjoys this property. Consequently, these operators do not lend themselves to a natural and general notion in an arbitrary category and do not behave as well as hereditary closure operators (see in particular, [Cas11, Examples 3.8.(b) and (c)], [Cas15, Corollary 2] and they can not be characterized as in Proposition 4.1.16).

Moreover, one might be tempted to define a hereditary interior operator i by replacing c by i and reversing the order (replacing " $\leq$ " by " $\geq$ ") in the definition of hereditary closure operator c, that is: i is "hereditary" if it satisfies the property $i_S(r_s) \leq s^*(i_X(s(r_s))) \cong s^*(i_X(r)) \Leftrightarrow s \circ i_S(r_s) \leq s \circ s^*(i_X(r)) \cong s \wedge i_X(r) \cong i_X(r) \Leftrightarrow s \circ i_S(r_s) \cong i_X(r)$. However, this definition does not give the right notion for heredity. The Kuratowski interior operator k^{in}, which is obtained by set complementation from the Kuratowski closure operator k (which is hereditary) on the category **Top**, does not satisfy the property. In fact, for $R = \{2\} \subseteq \{2,3\} = S \subseteq (X = \{1,2,3\}, \tau_X = \{\emptyset, \{1\}, \{1,2\}, X\}) \in$ **Top** we have $k^{\text{in}}_S(R) = R \not\subseteq \emptyset = k^{\text{in}}_X(R)$. In both categories **Top** and **Grp**, only the discrete interior operator enjoys this property. These interior operators are called strongly hereditary and were studied in [Cas16]. Note that C-hereditary interior operators are obtained by modifying the above property.

In this section, we introduce and study a general notion of hereditary interior operators using the right adjoint of the preimage of a given morphism in an arbitray category $\mathbb{C}$. In particular, we prove that hereditary interior operators behave as well as hereditary closure operators. The notions of initiality, finality, openness and closedness with respect to a hereditary interior operator behave in a similar fashion to the respective notions with resect to a hereditary closure operator in [GT00]. Indeed, we obtain a characterization of heredity of a given interior operator i in terms of "initial embeddings" with respect to i. Moreover, we study the relationship between our hereditary and Castellini's (strongly) hereditary interior operators. To this purpose we start with the following observation:

Remark 4.1.1. Let $r : R \to X$ and $s : S \to X$ be subobjects of $X \in \mathbb{C}$ such that $r \leq s$. Then the continuity of s with respect to an interior operator i in terms of the dual image s_* implies that $s^*(i_X(s_*(r_s))) \leq i_S(s^*(s_*(r_s))) \cong i_S(r_s)$.

Remark 4.1.1 together with the above definition of hereditary closure operators motivates the following definition:

Definition 4.1.2. An interior operator i on $\mathbb{C}$ with respect to $\mathcal{M}$ is said to be hereditary if for all $r \leq s$ in sub$X$ and $X \in \mathbb{C}$, one has

$$i_S(r_s) \cong s^*(i_X(s_*(r_s))).$$

Since from the above observation one always has $s^*(i_X(s_*(r_s))) \leq i_S(r_s)$, to prove that i is hereditary, it is sufficient to show that $i_S(r_s) \leq s^*(i_X(s_*(r_s)))$. But with the adjointness property this is equivalent to $s \circ i_S(r_s) \leq i_X(s_*(r_s))$. In fact, in a topological category $\mathbb{C}$ over **Set**, i is hereditary if $i_S(R) \subseteq i_X(R \cup (X \setminus S))$ for all $R \subseteq S \subseteq X \in \mathbb{C}$.

The following remark will be useful in deriving some of the results that we are going to present.

Remark 4.1.3. Let i be an interior operator on $\mathbb{C}$, $r \leq s$ in sub$X$ and $X \in \mathbb{C}$. Then one has the following properties of i:

(a) By Proposition 1.4.4(a) one has $r = s \circ r_s \cong s(r_s) \leq s_*(r_s)$. Consequently, the monotonicity property of i yields $i_X(r) \leq i_X(s_*(r_s))$.

(b) By Proposition 1.4.4(a), one has $s^*(s_*(r_s)) \cong r_s$, hence $s \wedge s_*(r_s) \cong s \circ s^*(s_*(r_s)) \cong s \circ r_s = r$. Therefore, $i_X(s \wedge s_*(r_s)) \cong i_X(r)$.

(c) The contraction and monotonicity property of i imply $i_X(r) \leq i_X(s) \leq s$. Thus, with the continuity condition of i, (a) and (b), one has $i_X(r) \cong s \wedge i_X(r) \cong s \circ s^*(i_X(r)) \leq s \circ s^*(i_X(s_*(r_s))) \leq s \circ i_S(s^*(s_*(r_s))) \cong s \circ i_S(r_s)$. Note that $s \circ i_S(r_s) \leq s$ trivially.

(d) From (a) and (c) one obtains $i_X(r) \leq i_X(s) \wedge i_X(s_*(r_s))$.

(e) $s \wedge i_X(s_*(r_s)) \cong s \circ s^*(i_X(s_*(r_s))) \leq s \circ i_S(s^*(s_*(r_s))) \cong s \circ i_S(r_s)$. Indeed, this follows from the continuity condition of i and (b).

(f) It follows from (e) that $i_X(s) \wedge i_X(s_*(r_s)) \cong i_X(s) \wedge s \wedge i_X(s_*(r_s)) \leq i_X(s) \wedge s \circ i_S(r_s)$.

(g) Since $s \dashv s^* \dashv s_*$ one has $s \circ i_S(r_s) \leq i_X(s_*(r_s)) \Leftrightarrow i_S(r_s) \leq s^*(i_X(s_*(r_s))) \Leftrightarrow s \circ i_S(r_s) \leq s \wedge i_X(s_*(r_s))$.

As a consequence of Definition 4.1.2 and the previous remark one directly obtains the following handy characterizations of heredity.

Proposition 4.1.4. Let i be an interior operator then the following are equivalent:

(a) i is hereditary;

(b) $s \circ i_S(r_s) \leq i_X(s_*(r_s))$ for all $r \leq s$ in sub$X$ and $X \in \mathbb{C}$;

(c) $s \circ i_S(r_s) \leq s \wedge i_X(s_*(r_s))$ for all $r \leq s$ in sub$X$ and $X \in \mathbb{C}$;

(d) $s \circ i_S(r_s) \cong s \wedge i_X(s_*(r_s))$ for any pair of $\mathcal{M}$-subobjects r, s of X in $\mathbb{C}$ such that $r \leq s$;

(e) s^* preserves the interior of $s_*(r_s)$, that is: $i_S(s^*(s_*(r_s))) \cong s^*(i_X(s_*(r_s)))$ for all $r \leq s$ in sub$X$ and $X \in \mathbb{C}$.

As mentioned before, a strongly hereditary (C-hereditary, resp.) interior operator i is just an interior operator such that $s \circ i_S(r_s) \cong i_X(r)$ $(i_X(s) \wedge s \circ i_S(r_s),$ resp.) for all $r \leq s$ in sub$X$ and $X \in \mathbb{C}$. With these definitions in mind, we investigate the relationship between hereditary, C-hereditary and strongly hereditary interior operators as follows.

Proposition 4.1.5. We have the following relations:

(a) Every strongly hereditary interior operator is in fact hereditary.

(b) Every additive and hereditary interior operator is C-hereditary.

Proof. Let $r, s \in$ sub$X$ such that $r \leq s$ and i be an interior operator on $\mathbb{C}$.

(a) Suppose i is strongly hereditary. Then $s \circ i_S(r_s) \cong i_X(r)$. Consequently, with Remark 4.1.3(a) one obtains $s \circ i_S(r_s) \cong i_X(r) \leq i_X(s_*(r_s))$. Therefore, i is hereditary.

(b) Assume i is an additive and hereditary interior operator. Then

$$\begin{aligned}
i_X(s) \wedge s \circ i_S(r_s) &\cong i_X(s) \wedge s \wedge i_X(s_*(r_s)) && (i \text{ hereditary})\\
&\cong i_X(s) \wedge i_X(s_*(r_s)) && (i \text{ contractive})\\
&\cong i_X(s \wedge s_*(r_s)) && (i \text{ additive})\\
&\cong i_X(r) && (\text{Remark 4.1.3}(b)).
\end{aligned}$$

□

In the following result, we give a condition which ensures that a C-hereditary interior operator is (strongly) hereditary.

Proposition 4.1.6. Let $r \leq s$ in sub$X$ and $X \in \mathbb{C}$. If i is a C-hereditary interior operator on $\mathbb{C}$ satisfying the property $s \circ i_S(r_s) \leq i_X(s)$ then i is (strongly) hereditary.

Proof. Let $r \le s$ in sub$X$ and $X \in \mathbb{C}$. Then

$$\begin{aligned} s \circ i_S(r_s) &\cong s \circ i_S(r_s) \wedge i_X(s) && \text{(hypothesis on } i) \\ &\cong i_X(r) && (i \text{ C-hereditary }). \end{aligned}$$

Consequently, i is strongly hereditary and it turns out that by Proposition 4.1.5(a) i is hereditary. □

The following result provides a partial characterization of hereditary and C-hereditary interior operators.

Proposition 4.1.7. Consider the following properties of an interior operator i on $\mathbb{C}$:

(a) i is hereditary.

(b) i is C-hereditary.

(c) $i_X(s) \wedge i_X(s_*(r_s)) \cong i_X(s) \wedge (s \circ i_S(r_s))$ for all $r \le s$ in sub$X$ and $X \in \mathbb{C}$.

Then $(a) \Rightarrow (c)$ and $(b) \Rightarrow (c)$. Furthermore, if i is additive then one also has $(c) \Rightarrow (b)$.

Proof. $(a) \Rightarrow (c)$: Let i be an hereditary interior operator. Then $s \circ i_S(r_s) \cong s \wedge i_X(s_*(r_s))$. Consequently, $i_X(s) \wedge s \circ i_S(r_s) \cong i_X(s) \wedge s \wedge i_X(s_*(r_s)) \cong i_X(s) \wedge i_X(s_*(r_s))$ by the contraction property of i.

$(b) \Rightarrow (c)$: Let i be a C-hereditary interior operator. Then $i_X(s) \wedge s \circ i_S(r_s) \cong i_X(r)$. Hence $i_X(s) \wedge s \circ i_S(r_s) \cong i_X(r) \le i_X(s) \wedge i_X(s_*(r_s))$ follows with Remark 4.1.3(d). Consequently, with Remark 4.1.3(f) the property in (c) holds.

$(c) \Rightarrow (b)$: Let i be an additive interior operator satisfying the property in (c). Then for all $r \le s$ in sub$X$ and $X \in \mathbb{C}$ one obtains:

$$\begin{aligned} i_X(s) \wedge s \circ i_S(r_s) &\cong i_X(s) \wedge i_X(s_*(r_s)) && ((c)) \\ &\cong i_X(s \wedge s_*(r_s)) && (i \text{ additive}) \\ &\cong i_X(r) && (\text{Remark 4.1.3}(b)). \end{aligned}$$

Therefore, i is C-hereditary.

□

In fact, strongly hereditary interior operators satisfy the above property (c) since they are hereditary by Proposition 4.1.5(a).

As a consequence of Lemma 1.4.7(b) and Definition 4.1.2, one has:

Remark 4.1.8. Let i be an interior operator on a topological category $\mathbb{C}$ over **Set** and $r: R \to X, s: S \to X$ be embeddings in $\mathbb{C}$. Then we may assume $R \subseteq S \subseteq X \in \mathbb{C}$. Furthermore,

(a) Since $s_*(r_s) = \overline{s(\overline{r_s})}$, where $r = s \circ r_s$ and $\overline{r_s}$ denotes the complement of r_s, the domain of $s_*(r_s)$ is $X \setminus s(S \setminus R) = R \cup (X \setminus S)$.

(b) i is hereditary if $i_S(R) \subseteq i_X(R \cup (X \setminus S))$.

(c) If $i_S(R) \subseteq i_X(R)$ then i is strongly hereditary and hence hereditary.

We now present some of the motivating examples of hereditary interior operators. Further examples will appear later.

Examples 4.1.9. (a) The prototypical example of a hereditary interior operator is the Kuratowski interior operator k^{in} in the category **Top**, which assigns the usual topological interior R° to each subspace R of a topological space X, that is: $k^{\mathsf{in}}_X(R) = \bigcup\{O \text{ open in } X : O \subseteq R\}$. Indeed, let $r \in k^{\mathsf{in}}_S(R)$. Then there exists $U = S \cap O \in \tau_S$, where $O \in \tau_X$, such that $r \in U = S \cap O \subseteq R$. But $S \cap O \subseteq R$ implies that $O \cup (X \setminus S) = X \cap (O \cup (X \setminus S)) = (S \cup (X \setminus S)) \cap (O \cup (X \setminus S)) = (S \cap O) \cup (X \setminus S) \subseteq R \cup (X \setminus S)$. Consequently, $r \in O \subseteq O \cup (X \setminus S) \subseteq R \cup (X \setminus S)$. This yields $r \in k^{\mathsf{in}}_X(R \cup (X \setminus S))$, hence k^{in} is hereditary. Furthermore, since k^{in} is additive, it follows from Proposition 4.1.5(b) that it is C-hereditary.

(b) The inverse Kuratowski interior operator in the category **Top** which is given by $k^{*\mathsf{in}}_X(R) = \bigcup\{C \text{ closed in } X : C \subseteq R\} = \{x \in R : k_X(\{x\}) \subseteq R\}$, where $k_X(\{x\})$ is the Kuratowski closure of $\{x\}$ in the topology of X, is hereditary. Indeed, let $x \in k^{*\mathsf{in}}_S(R) = \{r \in R : k_S(\{x\}) \subseteq R\}$, where $k_S(\{x\})$ is the Kuratowski closure of $\{x\}$ in the topology of S. Then $x \in R$ and $k_S(\{x\}) \subseteq R$. We claim that $k_X(\{x\}) \subseteq k_S(\{x\})$. In fact, let $y \in k_X(\{x\})$. Then every τ_X-open set O_y containing y has a nonempty intersection with $\{x\}$. That is, any open set in X containing y contains x. Now let U_y be any τ_S-open set containing y. Then $y \in U_y = S \cap O$ for some $O \in \tau_X$. This in turn implies $y \in S$ and $y \in O \in \tau_X$. Consequently, $x \in O$. Hence $x \in R \subseteq S$ and $x \in O$. This turns out that $x \in S \cap O = U_y$ and hence $U_y \cap \{x\} \neq \emptyset$. Therefore, $y \in k_S(\{x\})$. Thus, $k_X(\{x\}) \subseteq k_S(\{x\})$. As a result $x \in R$ and $k_S(\{x\}) \subseteq R$ implies $x \in R$ and $k_X(\{x\}) \subseteq k_S(\{x\}) \subseteq R \subseteq R \cup (X \setminus S)$. Hence $x \in k^{*\mathsf{in}}_X(R)$. Therefore, $k^{*\mathsf{in}}_S(R) \subseteq k^{*\mathsf{in}}_X(R \cup (X \setminus S))$ and hence $k^{*\mathsf{in}}$ is hereditary. Furthermore, since $k^{*\mathsf{in}}$ is additive, it follows from Proposition 4.1.5(b) that it is C-hereditary.

Remark 4.1.10. Recall from [Cas11] that the composition of interior operators is an interior operator. However, heredity is not stable under composition of interior operators, that is: the composite of two hereditary interior operators need not be hereditary. Indeed, in the category of **Top** with (Surjections, Embeddings)-factorization system we have seen that the Kuratowski interior operator k^{in} and the inverse Kuratowski interior operator $k^{*\mathsf{in}}$ are hereditary but the composition $k^{*\mathsf{in}} \circ k^{\mathsf{in}}$ fails to be hereditary. To see this, let $R = \{2\} \subseteq S = \{1,2\} \subseteq X = \{1,2,3\}$ with $\tau_X = \{\emptyset, \{1\}, \{2\}, \{1,2\}, X\}$ then $k^{\mathsf{in}}_S(R) = R = k^{*\mathsf{in}}_S(R)$. Consequently, $k^{*\mathsf{in}}_S(k^{\mathsf{in}}_S(R)) = k^{*\mathsf{in}}_S(R) = R$. But $k^{*\mathsf{in}}_X(k^{\mathsf{in}}_X(R \cup (X \setminus S))) = k^{*\mathsf{in}}_X(k^{\mathsf{in}}_X(\{2,3\})) = k^{*\mathsf{in}}_X(\{2\}) = \emptyset$. Therefore, $k^{*\mathsf{in}}_S(k^{\mathsf{in}}_S(R)) \nsubseteq k^{*\mathsf{in}}_X(k^{\mathsf{in}}_X(R \cup (X \setminus S)))$. Hence $k^{*\mathsf{in}} \circ k^{\mathsf{in}}$ is not hereditary.

In what follows we establish a natural relationship between hereditary interior and hereditary closure operators.

Proposition 4.1.11. Suppose sub$X$ is a Boolean algebra for every $\mathbb{C}$-object X. If c is a hereditary closure operator then the induced interior operator i^c given by $i^c_X(m) = \overline{c_X(\overline{m})}$ for all $m \in$ sub$X$, where $\overline{m}$ denotes the complement of m, is hereditary. Similarly, if i is a hereditary interior operator then the closure operator c^i given in the previous proposition is hereditary.

Proof. Let $r : R \to X$ and $s : S \to X$ be subobjects of X such that $r \leq s$. Assume c is a hereditary closure operator. Then since here the preimage functor for any given morphism is assumed to preserve arbitrary joins (hence binary joins), Lemma 1.4.7 and Remark 2.3.12 yield

$$\begin{aligned}
& s^*(c_X(s(\overline{r_s}))) \leq c_S(\overline{r_s}) \\
& \Rightarrow \overline{c_S(\overline{r_s})} \leq \overline{s^*(c_X(s(\overline{r_s})))} \cong s^*(\overline{c_X(s(\overline{r_s}))}) \\
& \Rightarrow i^c_S(r_s) \leq s^*(i_X(\overline{s(\overline{r_s})})) \cong s^*(i^c_X(s_*(r_s))) \\
& \Rightarrow i^c \text{ is hereditary.}
\end{aligned}$$

Analogously if i is a hereditary interior operator then c^i is hereditary. □

The above proposition enables us to establish a bijective correspondence between hereditary closure and hereditary interior operators. Consequently, with the examples of hereditary closure operators which are found in [DT95] and [Cas03] we obtain the following additional examples.

Examples 4.1.12. (a) Let $\mathbb{C}$ be the category **Top** with (Surjections, Embeddings)-factorization system and $R \subseteq S \subseteq X \in$ **Top**.

- (i) The Θ^{in}-interior operator given by $\Theta^{\text{in}}_X(R) = \{x \in R : \exists$ an open neighbourhood U_x of x in X such that $k_X(U_x) \subseteq R\}$, where $k_X(U_x)$ is the Kuratowski closure of U_x, is not hereditary. This interior operator can be obtained from Θ-closure operator, which is not hereditary, via set-theoretic complementation. Hence by Proposition 4.1.11 Θ^{in}-interior is not hereditary. On the other hand this interior operator is shown to be C-hereditary (see [Cas11]).
- (ii) The quasicomponent interior operator $q^{\text{in}}_X(R) = \bigcup\{O$ clopen in $X : O \subseteq R\}$ is not hereditary. This interior operator can be obtained from quasicomponent closure operator, which is not hereditary, via set-theoretic complementation. Hence by Proposition 4.1.11 the quasicomponent interior is not hereditary. On the other hand this interior operator is not known whether C-hereditary or not, that is, this problem is unsettled in [Cas11].

(b) Let $\mathbb{C}$ be the category **PreTop** of pretopological spaces and continuous functions with the (Surjections, Embeddings)-factorization system. $\check{c}^{\text{in}}_X(R) = \bigcup\{O$ open in $X : O \subseteq R\}$ is hereditary. This operator is called the Čech interior operator.

(c) Let $\mathbb{C}$ be the category **SGph** of directed spatial graphs and graph homomorphisms with the *(Surjective homorphisms, Embeddings)*-factorization system and let (G, R) be a directed spatial graph and $H \subseteq G$. Then both the up-interior given by $\uparrow^{\text{in}}_G (H) = \{h \in H :\ (\forall g \in G \setminus H)$ there is no edge $g \to h\}$ and the down-interior given by $\downarrow^{\text{in}}_G (H) = \{h \in H :\ (\forall g \in G \setminus H)$ there is no edge $h \to g\}$ are hereditary interior operators of **SGph**.

In the following lemma we prove that heredity is stable under arbitrary meet and join.

Lemma 4.1.13. Let $(i_k)_{k\in K} \subseteq INT(\mathbb{C}, \mathcal{M})$ be a nonempty family such that each i_k is hereditary. Then so are $\bigwedge\limits_{k\in K} i_k$ and $\bigvee\limits_{k\in K} i_k$.

Proof. Let $r : R \to X$ and $s : S \to X$ be subobjects of X such that $r \leq s$. Suppose each interior operator $i_k, k \in K$ is hereditary. Then $(i_k)_S(r_s) \cong s^*((i_k)_X(s_*(r_s)))$. Now if we set $i^\star = \bigwedge\limits_{k\in K} i_k$ and $i^\diamond = \bigvee\limits_{k\in K} i_k$ then

(a) $i^{\star}_S(r_s) = (\bigwedge_{k\in K} i_k)_S(r_s) \cong \bigwedge_{k\in K}(i_k)_S(r_s) \cong \bigwedge_{k\in K} s^*((i_k)_X(s_*(r_s))) \cong s^*(\bigwedge_{k\in K}(i_k)_X(s_*(r_s))) \cong s^*((\bigwedge_{k\in K} i_k)_X(s_*(r_s))) \cong s^*(i^{\star}_X(s_*(r_s)))$. Indeed, this follows from the fact that " limits commute with limits", hence meets commute with preimages.

(b) Since we assumed that each preimage preserves arbitrary joins we have that $i^{\diamond}_S(r_s) = (\bigvee_{k\in K} i_k)_S(r_s) \cong \bigvee_{k\in K}(i_k)_S(r_s) \cong \bigvee_{k\in K} s^*((i_k)_X(s_*(r_s))) \cong s^*(\bigvee_{k\in K}(i_k)_X(s_*(r_s))) \cong s^*((\bigvee_{k\in K} i_k)_X(s_*(r_s))) \cong s^*(i^{\diamond}_X(s_*(r_s)))$.

□

Consequently, each interior operator has both a hereditary core and hull as shown below. To this end, let HEINT$(\mathbb{C},\mathcal{M})$ denote the conglomerate of hereditary interior operators on $\mathbb{C}$ with respect to $\mathcal{M}$. The previous lemma motivates the following definition:

Definition 4.1.14. The hereditary hull $\hat{i}$ and hereditary core $\overset{\vee}{i}$ of an interior operator i is defined by $\hat{i} = \bigwedge\{j \in \mathsf{HEINT}(\mathbb{C},\mathcal{M}) : i \sqsubseteq j\}$ and $\overset{\vee}{i} = \bigvee\{j \in \mathsf{HEINT}(\mathbb{C},\mathcal{M}) : j \sqsubseteq i\}$, respectively.

As a consequence, the following result is obtained.

Theorem 4.1.15. *The conglomerate HEINT$(\mathbb{C},\mathcal{M})$ is both reflective and coreflective in $INT(\mathbb{C},\mathcal{M})$. The reflection and coreflection of $i \in INT(\mathbb{C},\mathcal{M})$ are its hereditary hull $\hat{i}$ and its hereditary core $\overset{\vee}{i}$, respectively.*

Proof. Let $i \in \mathsf{INT}(\mathbb{C},\mathcal{M})$. Since $\hat{i}$ and $\overset{\vee}{i}$ are hereditary hull and hereditary core one has that $i \sqsubseteq \hat{i}$ and $\overset{\vee}{i} \sqsubseteq i$, respectively. Consequently, we do have the following Galois connections.

$$\mathsf{HEINT}(\mathbb{C},\mathcal{M}) \underset{\longrightarrow}{\overset{\longleftarrow}{\bot}} \mathsf{INT}(\mathbb{C},\mathcal{M}) \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathsf{HEINT}(\mathbb{C},\mathcal{M})$$

□

In what follows we want to investigate the notions of initial, final, open and closed morphisms with respect to a hereditary interior operator. We begin with the following characterization of heredity in terms of the notion of initiality.

Proposition 4.1.16. An interior operator i is hereditary if and only if every morphism in $\mathcal{M}$ is i-initial.

Proof. Suppose i is hereditary. Let $s : S \to X$ be a morphism in $\mathcal{M}$ and $t : T \to S$ be a subobject of S. Then $i_S(t) \cong s^*(i_X(s_*(t)))$ since $s \circ t \leq s$. Therefore, s is i-initial.

On the other hand, assume that every morphism in $\mathcal{M}$ is i-initial. Let $r \leq s \in \mathsf{sub}X \subseteq \mathcal{M}$ and $X \in \mathbb{C}$. Then s is i-initial, hence $i_S(r_s) \cong s^*(i_X(s_*(r_s)))$. □

The following lemma provides conditions of heredity for free.

Lemma 4.1.17. Let i be any interior operator. Then $i_S(r_s) \cong s^*(i_X(s_*(r_s)))$ for all $r \leq s$ in sub$X$ and $X \in \mathbb{C}$ with s

(a) an i-open morphism or;

(b) an i-closed morphism or;

(c) an i-final morphism or;

(d) a section.

Proof. It follows from the fact that any section is i-initial and each i-open(or i-closed or i-final) morphism in $\mathcal{M}$ is i-initial. □

We recall the following definition from [Cas16].

Definition 4.1.18. An interior operator i on $\mathbb{C}$ is said to be modal if and only if every morphism in $\mathbb{C}$ is i-open.

Consequently, with Lemma 4.1.17(a) we obtain the following result:

Proposition 4.1.19. Every modal interior operator is hereditary.

Proof. Let i be a modal interior operator, $r : R \to X$ and $s : S \to X$ be subobjects of $X \in \mathbb{C}$ such that $r \leq s$. Since every morphism is i-open, s is an open morphism. Hence, by Lemma 4.1.17(a), one has $i_S(r_s) \cong s^*(i_X(s_*(r_s)))$. Consequently, i is hereditary. □

The previous proposition can also be obtained as a consequence of [Cas16, Proposition $3.15.(a)$] and Proposition 4.1.5(a).

Remark 4.1.20. For a modal interior operator i, the class of i-codense subobjects is stable under pullback. Indeed, let $X \overset{f}{\to} Y$ be a morphism in $\mathbb{C}$ and n be an i-codense subobject of Y. Since i is modal, f is i-open. Consequently, $i_X(f^*(n)) \cong f^*(i_Y(n)) \cong f^*(0_Y) \cong 0_X$ since each morphism in $\mathbb{C}$ reflects the least subobject. Therefore, $f^*(n)$ is i-codense in X.

The following result shows that for additive hereditary interior operator i, the class of i-open subobjects is closed under composition.

Proposition 4.1.21. For any additive and hereditary interior operator i, composites of i-open subobjects are i-open.

Proof. Let i be an additive hereditary interior operator, $t : T \to S$ and $s : S \to X$ be i-open subobjects of S and X, respectively. Since $s \circ t \leq s$, by Remark 4.1.3(b), one has $s \circ t \cong s \circ i_S(t) \cong s \wedge i_X(s_*(t)) \cong i_X(s) \wedge i_X(s_*(t)) \cong i_X(s \wedge s_*(t)) \cong i_X(s \circ t)$. Therefore, the composite $s \circ t$ is i-open. □

We now show:

Lemma 4.1.22. Let i be an interior operator and $s : S \to X$ be a subobject of $X \in \mathbb{C}$. Then the following statements hold.

(a) Suppose that i is additive. If s is an i-open subobject and i-initial then it is an i-open morphism.

(b) [Cas15] Suppose that i is standard. If s is an i-open morphism then it is an i-open subobject.

(c) If s is an i-open morphism then it is i-initial.

(d) Suppose that i is both standard and additive. s is an i-open subobject and i-initial if and only if it is an i-open morphism.

Proof. (a) Let $t \in \text{sub}S$. Then

$$\begin{aligned} s(i_S(t)) &\cong s \circ i_S(t) && \text{(Remark 1.3.6}(a)) \\ &\cong s \circ s^*(i_X(s_*(t))) && (s\text{-initial}) \\ &\cong s \wedge i_X(s_*(t)) && \\ &\cong i_X(s) \wedge i_X(s_*(t)) && (s\ i\text{-open subobject}) \\ &\cong i_X(s \wedge s_*(t)) && (i \text{ additive}) \\ &\cong i_X(s \circ t) && \text{(Remark 4.1.3}(b)) \\ &\leq i_X(s(t)) && (s \circ t = s(t) \circ e, e \in \mathcal{E}). \end{aligned}$$

(b) $s = s \circ 1_S \cong s \circ i_S(1_S) \cong s(i_S(1_S)) \leq i_X(s(1_S)) \cong i_X(s \circ 1_S) \cong i_X(s)$.

(c) $i_S(t) \cong i_S(s^*(s_*(t))) \cong s^*(i_X(s_*(t)))$ for all $t \in \text{sub}S$.

(d) This is just (a), (b) and (c) together.

□

The following are properties of open subobjects with respect to a hereditary interior operator.

Proposition 4.1.23. Let i be an additive interior operator and $s : S \to X$ be a subobject of $X \in \mathbb{C}$. Then the following statements hold.

(a) Suppose that i is hereditary. If s is an i-open subobject then it is an i-open morphism.

(b) Suppose that i is standard and hereditary. s is an i-open morphism if and only if it is an i-open subobject.

Proof. This follows from Proposition 4.1.16 and Lemma 4.1.22. □

Corollary 4.1.24. Let i be an additive and hereditary interior operator. Then:

(a) If m is an i-open subobject of X and $f : X \to Y \in \mathcal{QO}(i)$, then $f \circ m \in \mathcal{QO}(i)$.

(b) The class of i-codense $\mathcal{M}$-subobjects is left-cancellable with respect to the class of i-open subobjects.

Proof. (a) is an immediate consequence of Corollary 3.2.13 and Proposition 4.1.23(a).

(b) follows from Remark 3.2.8(b) and Proposition 4.1.23(a).

□

In what follows we deal with the pullback behaviour of open, closed, initial and final morphisms with respect to a hereditary interior operator. Recall that each of the notions of initial, open, closed and final morphism with respect to an interior operator i ascends along i-initial morphisms and descends along i-final morphisms. In particular, each of the class of initial, final, open, closed morphisms with respect to a hereditary interior operator is stable under pullback along $\mathcal{M}$-morphisms. More precisely:

Proposition 4.1.25. Let i be a hereditary interior operator and $\mathcal{E}$ be stable under pullback along $\mathcal{M}$-morphisms. Then for every $n : N \to Y$ in $\mathcal{M}$ the restriction $f^*(N) \to N$ (which is understood to be the pullback of f along n, i.e., the $\mathbb{C}$-morphism $\hat{f}$ in the pullback diagram below) of the i-initial (i-final, i-open, i-closed, resp.) morphism $f : X \to Y$ is i-initial (i-final, i-open, i-closed, resp.). In fact, heredity is not needed if the pullback of the given M-morphism is a section.

$$\begin{array}{ccc} f^*[N] & \xrightarrow{\hat{f}} & N \\ {\scriptstyle f^*(n)}\downarrow & & \downarrow{\scriptstyle n} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Proof. This is a consequence of Proposition 4.1.16, Remark 3.1.39(b) and Theorem 3.1.42. □

As an immediate consequence of Propositions 3.2.6 and 4.1.25, one obtains:

Corollary 4.1.26. Let i be a hereditary interior operator and $\mathcal{E}$ be stable under pullback along $\mathcal{M}$-morphisms. If f is an i-open morphism then any of its pullback along $\mathcal{M}$-morphisms reflects i-codensity.

Note that since we assume each preimage commutes with joins in the category $\mathbb{C}$, each morphism reflects the least subobject.

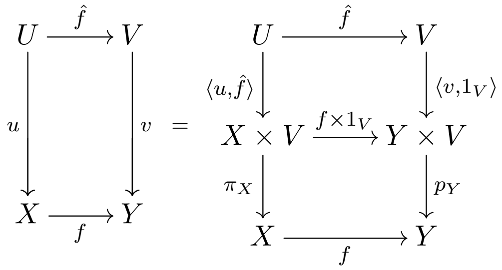
Definition 4.1.27. Let i be an interior operator. A morphism f is stably i-closed (i-open, i-initial, i-final, resp.) if every pullback of f is i-closed (i-open, i-initial, i-final, resp.).

Remark 4.1.28. Let i be an additive and hereditary interior operator and $s : S \to X$ be a subobject of $X \in \mathbb{C}$. If s is an i-open subobject then s is stably i-open morphism. Indeed, this is a consequence of the fact the class of all i-open $\mathcal{M}$-subobject is stable under pullback and Proposition 4.1.23.

In the following proposition, for a given hereditary interior operator i we provide sufficient conditions on the objects involved for each of the classes of i-morphisms to be stable under pullback.

Proposition 4.1.29. Let i be a hereditary interior operator and let $\mathcal{E}$ be stable under pullback along $\mathcal{M}$-morphisms. A morphism $f : X \to Y$ in $\mathbb{C}$ is stably i-closed (i-open, i-initial, i-final, resp.) if and only if $f \times 1_V$ is i-closed (i-open, i-initial, i-final, resp.) for every object $V \in \mathbb{C}$.

Proof. Since the verifications of the statements for i-initiality, i-finality and i-openness are very similar to the proof of the assertion for i-closedness, the following proves the claim. Suppose $f \times 1_V : X \times V \to Y \times V$ is i-closed for all $V \in \mathbb{C}$. Let $\hat{f} : U \to V$ be a pullback of f along $v : V \to Y$, as in the left diagram below. Then one can factorize this pullback diagram, as in the right diagram below with both the outer rectangle and the lower square pullbacks.



Consequently, the upper square of the right above diagram is a pullback, hence $\hat{f}$ is a pullback of $f \times 1_V$. The fact that $\mathcal{M}$ is stable under pullback and $\langle v, 1_V\rangle$ is a section (hence a regular monomorphism) implies $\langle v, 1_V\rangle \in \mathcal{M}$. Consequently, by Proposition 4.1.25, $\hat{f} \in \mathcal{K}(i)$. The other part follows from straightforward verification. □

Proposition 4.1.16 leads us to the following definition:

Definition 4.1.30. Given an interior operator i, we call i cohereditary if and only if every morphism in $\mathcal{E}$ is i-final.

Consequently, with Proposition 3.1.42 one deduces the following:

Proposition 4.1.31. Let i be a cohereditary interior operator. If the pullback $g : A \to B$ of $f : X \to Y$ along $b : B \to Y \in \mathcal{E}$ is an i-initial (i-final, i-open, i-closed, resp.) morphism then f itself is i-initial (i-final, i-open, i-closed, resp.). In fact, coheredity is not needed if b is a retraction.

In the remainder of this section we shall deal with weakly hereditary interior operators. Recall from [DG87] that a closure operator c is weakly hereditary if $c_{c_X[M]}(j_m) \cong 1_{c_X[M]}$ for all $m : M \to X \in \mathcal{M}$ with $m = c_X(m) \circ j_m$. This is equivalent to $c_{c_X[M]}(j_m) \cong (c_X(m))^*(c_X(m)) = (c_X(m))^*(c_X(c_X(m) \circ j_m))$ for all $m : M \to X \in \mathcal{M}$ with $m = c_X(m) \circ j_m$. In fact, this is also equivalent to the property $c_S(r_s) \cong s^*(c_X(s(r_s)))$ holds for $r = m \leq s = c_X(m)$ in sub$X$ and $X \in \mathbb{C}$, that is: c satisfies the heredity condition for $r = m \in$ sub$X$ and $s = c_X(m)$ with $m \in$ sub$X$ and $X \in \mathbb{C}$. On the other hand, for a hereditary interior operator i, one has $i_M(j_m) \cong m^*(i_X(m_*(j_m)))$ for all $m \in$ sub$X$ and $X \in \mathbb{C}$ since the subobjects r and s in Definition 4.1.2 are arbitary which satisfy $r \leq s$ one may take $r = i_X(m)$ and $s = m \in$ sub$X$ with $m \in$ sub$X$ and $X \in \mathbb{C}$. This motivates the following definition:

Definition 4.1.32. An interior operator i is weakly hereditary if $i_R(j_r) \cong r^*(i_X(r_*(j_r)))$ for all $r \in$ sub$X$ and $X \in \mathbb{C}$.

Clearly, heredity implies weak heredity and hence all examples of hereditary interior operators are weakly hereditary. The adjunction $r^* \dashv r_*$ and continuity condition of i give the following:

Remark 4.1.33. (*a*) An interior operator i is weakly hereditary if $r \circ i_R(j_r) \leq i_X(r_*(j_r)) \Leftrightarrow r \circ i_R(j_r) \leq$ (or $\cong$) $r \wedge i_X(r_*(j_r))$ for all $r \in$ sub$X$.

(*b*) Let i be an interior operator on topological category $\mathbb{C}$ over **Set** then i is weakly hereditary if $i_R(i_X(R)) \subseteq i_X(i_X(R) \cup (X \setminus R))$ for all $R \subseteq X \in \mathbb{C}$.

Weak heredity is stable under both arbitrary meet and join analogously to heredity. More precisely:

Lemma 4.1.34. Let $(i_k)_{k\in K} \subseteq INT(\mathbb{C},\mathcal{M})$ be a nonempty family such that each i_k is weakly hereditary. Then so are $\bigwedge\limits_{k\in K} i_k$ and $\bigvee\limits_{k\in K} i_k$.

Consequently, each interior operator has both a weakly hereditary core and hull as shown below. To this end, let $\mathsf{WHEINT}(\mathbb{C},\mathcal{M})$ denote the conglomerate of weakly hereditary interior operators on $\mathbb{C}$ with respect to $\mathcal{M}$. The previous lemma motivates the following definition:

Definition 4.1.35. A weakly hereditary hull $\overset{\diamond}{i}$ and weakly hereditary core $\overset{\oplus}{i}$ of an interior operator i is defined by $\overset{\diamond}{i} = \bigwedge\{j \in \mathsf{WHEINT}(\mathbb{C},\mathcal{M}) : i \sqsubseteq j\}$ and $\overset{\oplus}{i} = \bigvee\{j \in \mathsf{WHEINT}(\mathbb{C},\mathcal{M}) : j \sqsubseteq i\}$, respectively.

As a consequence, the following result is obtained.

Theorem 4.1.36. *The conglomerate $WHEINT(\mathbb{C},\mathcal{M})$ of weakly hereditary interior operators on $\mathbb{C}$ with respect to $\mathcal{M}$ is both reflective and coreflective in $INT(\mathbb{C},\mathcal{M})$. The reflection and coreflection of $i \in INT(\mathbb{C},\mathcal{M})$ are its hereditary hull $\overset{\diamond}{i}$ and its hereditary core $\overset{\oplus}{i}$, respectively.*

4.2 Dense morphisms with respect to an interior operator

In this section we consider an interior operator i on $\mathbb{C}$ with respect to $\mathcal{M}$ and introduce a notion of dense morphisms with respect to i. We use these morphisms in the second section of the next chapter to investigate a notion of connectedness with respect to i. In [DT95], a subobject $m : M \to X$ is dense with respect to a closure operator c if $c_X(m \circ 1_M) \cong c_X(m) \cong 1_X$. Now, if we assume sub$X$ is a Boolean algebra for every $\mathbb{C}$-object X then $m : M \to X$ is c-dense if $i_X(m_*(0_M)) \cong 0_X$. This observation yields the following definition.

Definition 4.2.1. An $\mathcal{M}$-subobject $m : M \to X$ is called i-dense in X if $i_X(m_*(0_M)) \cong 0_X$.

Remark 4.2.2. In the category of **Top** if $m : M \to X$ is a dense subobject with respect to the Kuratowski closure operator k then m is a dense subobject with respect to the Kuratowski interior operator k^*.

As a generalization of the above definition a morphism $f : X \to Y$ in an arbitrary category is dense with respect to a closure operator c if $c_Y(f(1_X)) \cong 1_Y$ (see [DT95]). Now, if we assume sub$X$ is a Boolean algebra for every $\mathbb{C}$-object X then by Remark 2.3.12 we have that $(\forall m \in \text{sub}X)(i_X(m) = \overline{c_X(\overline{m})})$, where $\overline{m}$ denotes the complement of m. Consequently, f is c-dense if $i_Y(f_*(0_X)) \cong 0_Y$. Indeed, this motivates us to have the following definition.

Definition 4.2.3. A morphism $f : X \to Y$ is said to be i-dense if $f_*(0_X)$ is i-dense in Y. That is, $i_Y(f_*(0_X)) \cong 0_Y$.

Remark 4.2.4. (a) If $m : M \to X$ in $\mathcal{M}$ is an i-dense morphism then m is i-dense as a subobject of X.

(b) Let $f : X \to Y$ be a k-dense morphism in the category of **Top**, where k is the Kuratowski closure operator. Then $k_Y(f(X)) = Y \Leftrightarrow k^*_Y(f_*(\emptyset)) = \emptyset$. Hence, f is k^*-dense.

(c) Let $f : X \to Y$ be a dense morphism in the category of **Loc** and $(i_X : \mathcal{O}X \to \mathcal{O}X)_{X\in\mathbf{Loc}}$ be an interior operator. Then since the right adjoint of a dense frame homorphisms maps the bottom element to the bottom we have that f is i-dense.

Remark 4.2.5. Let $\mathcal{E}^i$ be the class of i-dense morphisms in $\mathbb{C}$. Then $\mathcal{E}'$ is a subclass of $\mathcal{E}^i$. Indeed, let $f \in \mathcal{E}'$. Then by Proposition 1.4.4(c) one has that $i_Y(f_*(0_X)) \cong i_Y(0_Y) \cong 0_Y$.

This leads us to the following observation.

Proposition 4.2.6. Let $f : X \to Y$ be a morphism in $\mathcal{E}'$ and $m : M \to X$ be an i-dense subobject of X. Then $f \circ m$ is an i-dense morphism.

Proof. $i_Y((f \circ m)_*(0_M)) \cong i_Y(f_*(m_*(0_M))) \leq f_*(i_X(m_*(0_M))) \cong f_*(0_X) \cong 0_Y$, by Proposition 1.4.4$(c)$. □

In the next result we discuss stability and cancellation properties of the class $\mathcal{E}^i$.

Proposition 4.2.7. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathbb{C}$. Then

(a) $\mathcal{E}^i$ is right cancellable, that is: $g \circ f \in \mathcal{E}^i \Rightarrow g \in \mathcal{E}^i$.

(b) $g \in \mathcal{E}', f \in \mathcal{E}^i \Rightarrow g \circ f \in \mathcal{E}^i$, that is: $\mathcal{E}^i$ is stable under composition with the class $\mathcal{E}'$ from the left.

(c) $f \in \mathcal{E}', g \in \mathcal{E}^i \Rightarrow g \circ f \in \mathcal{E}^i$, that is: $\mathcal{E}^i$ is stable under composition with the class $\mathcal{E}'$ from the right.

(d) If $g \circ f \in \mathcal{E}^i, g \in \mathcal{M}$ and i is hereditary then $f \in \mathcal{E}^i$, that is: for hereditary interior operators the class $\mathcal{E}^i$ is left cancellable with respect to $\mathcal{M}$.

Proof. (a) $i_Z(g_*(0_Y)) \leq i_Z(g_*(f_*(f^*(0_Y)))) \cong i_Z(g \circ f)_*(0_X) \cong 0_Z$.

(b) $i_Z((g \circ f)_*(0_X)) \cong i_Z(g_*(f_*(0_X))) \leq g_*(i_Y(f_*(0_X))) \cong g_*(0_Y) \cong 0_Z$.

(c) $i_Z((g \circ f)_*(0_X)) \cong i_Z(g_*(f_*(0_X))) \cong i_Z(g_*(0_Y)) \cong 0_Z$.

(d) Since i is hereditary then by the Proposition 4.1.16 we have that every morphism in $\mathcal{M}$ is i-initial. In particular, g is i-initial here. Consequently, since $f_*(0_X) \in \mathrm{sub}Y$ one has that $i_Y(f_*(0_X)) \cong g^*(i_Z(g_*(f_*(0_X)))) \cong g^*(i_Z((g \circ f)_*(0_X))) \cong g^*(0_Z) \cong 0_Y$.

□

The following proposition shows that for a hereditary interior operator i the class of i-dense subobjects is left cancellable with respect to $\mathcal{M}$.

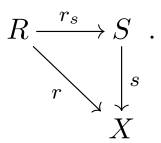
Proposition 4.2.8. Let i be a hereditary interior operator and $r \leq s$ in sub$X$. If r is an i-dense subobject of X, then r_s is an i-dense subobject of S.

Proof. The fact that i is hereditary implies every morphism in $\mathcal{M}$, in particular s, is i-initial. Moreover, subobjects (and hence s) reflect least subobjects. Consequently, $i_S((r_s)_*(0_R)) \cong s^*(i_X(s_*((r_s)_*(0_R)))) \cong s^*(i_X((s \circ r_s)_*(0_R))) \cong s^*(i_X(r_*(0_R))) \cong s^*(0_X) \cong 0_S$. □

Of course, the above Proposition can be considered as a corollary to Proposition 4.2.7(d).

Proposition 4.2.9. Let i be an idempotent interior operator. Then composites of i-dense subobjects are i-dense.

Proof. Suppose $r, s \in$sub$X$ such that the diagram



commutes, r_s is an i-dense subobject of S and s is an i-dense subobject of X. Then the continuity condition of i yields $i_X(r_*(0_R)) \cong i_X((s \circ r_s)_*(0_R)) \cong i_X(s_*((r_s)_*(0_R))) \leq s_*(i_S((r_s)_*(0_R))) \cong s_*(0_S)$. This combined with the idempotency property of i produce $i_X(r_*(0_R)) \cong i_X(i_X(r_*(0_R))) \leq i_X(s_*(0_S)) \cong 0_X$. Thus, $i_X(r_*(0_R)) \cong 0_X$. Therefore, r is an i-dense subobject of X. □

The above Proposition can be generalized as follows.

Proposition 4.2.10. For any idempotent interior operator i, the class $\mathcal{E}^i$ of i-dense morphisms in $\mathbb{C}$ is stable under composition.

Proof. Let $X \overset{f}{\to} Y \overset{g}{\to} Z$ such that both $f, g \in \mathcal{E}^i$. Then since $f \in \mathcal{E}^i$ we have that $i_Z((g \circ f)_*(0_X)) \cong i_Z(g_*(f_*(0_X))) \leq g_*(i_Y(f_*(0_X))) \cong g_*(0_Y)$. This together with idempotency of i, monotoncity of i and $g \in \mathcal{E}^i$ implies $i_Z((g \circ f)_*(0_X)) \cong i_Z(i_Z((g \circ f)_*(0_X))) \leq i_Z(g_*(0_Y)) \cong 0_Z$. Therefore, $i_Z((g \circ f)_*(0_X)) \cong 0_Z$ and hence $g \circ f \in \mathcal{E}^i$. □

4.3 Maximal interior operators

In this section we study maximal interior operators. To do this, let i be an interior operator throughout this section and consider the following stability property of O^i under composition with $\mathcal{M}$ from the right:

(RCO) For all $r_s : R \to S$ and $s : S \to X$ in $\mathcal{M}$, if s is i-open then $r = s \circ r_s$ is i-open, that is: $O^i \circ \mathcal{M} \subseteq O^i$. Consequently, we have the following lemma:

Lemma 4.3.1. For any idempotent interior operator i, (RCO) yields that $r \leq i_X(1_X)$ if and only if r is i-open.

Proof. $(\Rightarrow)$: Suppose $r \leq i_X(1_X)$. Then since i is idempotent, one has $i_X(1_X)$ is i-open. Hence (RCO) gives $r = i_X(1_X) \circ r_{i_X(1_X)}$ is i-open.

$(\Leftarrow)$: Suppose r is i-open. Then the fact that $r \leq 1_X$ implies $r \cong i_X(r) \leq i_X(1_X)$.

□

Lemma 4.3.2. If i is an idempotent interior operator and (RCO) holds for $X \in \mathbb{C}$ then
$i_X(r) \cong r \wedge i_X(1_X)$.

Proof. Since $r \wedge i_X(1_X) \leq i_X(1_X)$ then idempotency of i and (RCO) imply $r \wedge i_X(1_X)$ is i-open. Consequently, one has $r \wedge i_X(1_X) \cong i_X(r \wedge i_X(1_X)) \leq i_X(r) \wedge i_X(i_X(1_X)) \cong i_X(r) \wedge i_X(1_X) \cong i_X(r)$. Moreover, one always has $i_X(r) \leq r$ and $r \leq 1_X$. As a result, $i_X(r) \leq r \wedge i_X(1_X)$. Therefore, $i_X(r) \cong r \wedge i_X(1_X)$. □

Consequently, one has the following definition.

Definition 4.3.3. An interior operator i is called maximal if $i_X(r) \cong r \wedge i_X(1_X)$ for all $r \in \mathrm{sub}X$ and $X \in \mathbb{C}$.

Proposition 4.3.4. Let $i \in \mathrm{INT}(\mathbb{C}, \mathcal{M})$. i is maximal if and only if $i_X(r) \cong r \wedge i_X(s)$ for all $r \leq s$ in $\mathrm{sub}X$ and $X \in \mathbb{C}$.

Proof. ($\Rightarrow$) Suppose i is maximal and $r \leq s$ in $\mathrm{sub}X$. Then $i_X(r) \cong r \wedge i_X(1_X)$ and $i_X(s) \cong s \wedge i_X(1_X)$. Consequently, $r \wedge i_X(s) \cong r \wedge s \wedge i_X(1_X) \cong r \wedge i_X(1_X) \cong i_X(r)$.

($\Leftarrow$) Let $X \in \mathbb{C}$ and $r \in \mathrm{sub}X$. Then since $r \leq 1_X$ we obtain $i_X(r) \cong r \wedge i_X(1_X)$.

□

An interior operator i is fully additive if $i_X(\bigwedge_{i\in I} m_i) \cong \bigwedge_{i\in I} i_X(m_i)$ for all $m_i \in \mathrm{sub}X$, $X \in \mathbb{C}$ and $i \in I \neq \emptyset$. Consequently, with Proposition 4.3.4 one has:

Corollary 4.3.5. Every maximal interior operator is fully additive.

Proof. Suppose i is a maximal interior operator. Then since for all $i \in I$, $\bigwedge_{i\in I} m_i \leq m_i$, one has
$\bigwedge_{i\in I} i_X(m_i) \cong \bigwedge_{i\in I} i_X(m_i) \wedge \bigwedge_{i\in I} m_i \leq i_X(m_i) \wedge \bigwedge_{i\in I} m_i \cong i_X(\bigwedge_{i\in I} m_i)$. □

Proposition 4.3.6. Every maximal interior operator is idempotent.

Proof. Let i be a maximal interior operator and $m \in \mathrm{sub}X$. Since $i_X(m) \leq m$ then setting $r = i_X(m)$ and $s = m$ Proposition 4.3.4 yields $i_X(i_X(m)) \cong i_X(m) \wedge i_X(m)$. □

Proposition 4.3.7. If i is maximal then O^i satisfies (RCO).

Proof. Let $s : S \to X \in O^i$ and $r_s : R \to S \in \mathcal{M}$. Since $r = s \circ r_s \leq s$, maximality of i gives $i_X(r) \cong r \wedge i_X(s) \cong r \wedge s \cong r$. Consequently, $r = s \circ r_s \in O^i$. Therefore, $O^i \circ \mathcal{M} \subseteq O^i$, hence O^i satisfies (RCO). □

Theorem 4.3.8. *An interior operator i is maximal if and only if i is idempotent and (RCO) holds for all $X \in \mathbb{C}$.*

Proof. The necessary conditions hold by Propositions 4.3.6 and 4.3.7. Conversely, assume that i is idempotent and (RCO) holds for all $X \in \mathbb{C}$. Then for $r \leq s$ in $\mathrm{sub}X$ and $X \in \mathbb{C}$, one has $i_X(s)$ is i-open. Consequently, by (RCO), $r \wedge i_X(s)$ is i-open since $r \wedge i_X(s) \leq i_X(s)$. Hence, $r \wedge i_X(s) \cong i_X(r \wedge i_X(s)) \leq i_X(r) \wedge i_X(i_X(s)) \cong i_X(r) \wedge i_X(s) \cong i_X(r)$. Moreover, one always has $i_X(r) \leq r$ and $i_X(r) \leq i_X(s)$ and hence $i_X(r) \leq r \wedge i_X(s)$. Thus $i_X(r) \cong r \wedge i_X(s)$. □

Proposition 4.3.9. Let $i, j \in \mathrm{INT}(\mathbb{C}, \mathcal{M})$ such that j is maximal. Then $j \circ i \cong j \wedge i$.

Proof. Let $r \in \mathsf{sub}X$. Since $i_X(r) \le r$ maximality of j gives $(j \circ i)_X(r) \cong j_X(i_X(r)) \cong i_X(r) \wedge j_X(r) \cong (j \wedge i)_X(r)$. Therefore, $j \circ i \cong j \wedge i$. □

Proposition 4.3.10. Maximality is stable under arbitrary meet, that is: for a non-empty family $(i_k)_{k \in K}$ with each i_k maximal, one has $\bigwedge\limits_{k \in K} i_k$ is maximal.

Proof. Let $r \le s$ in $\mathsf{sub}X$. Since "meets commute with meets" one has:
$(\bigwedge\limits_{k \in K} i_k)_X(r) \cong \bigwedge\limits_{k \in K} (i_k)_X(r) \cong \bigwedge\limits_{k \in K} r \wedge (i_k)_X(s) \cong r \wedge \bigwedge\limits_{k \in K} (i_k)_X(s) \cong r \wedge (\bigwedge\limits_{k \in K} i_k)_X(s)$. Hence $\bigwedge\limits_{k \in K} i_k$ is maximal. □

Let $\mathsf{MAXINT}(\mathbb{C}, \mathcal{M})$ denote the conglomerate of maximal interior operators on $\mathbb{C}$ with respect to $\mathcal{M}$. Since the arbitrary meet of maximal interior operators is maximal one obtains the following.

Definition 4.3.11. Given an interior operator i, its maximal hull i^{ma} is defined by

$$i^{\mathsf{ma}} := \bigwedge \{ j \in \mathsf{MAXINT}(\mathbb{C}, \mathcal{M}) : i \sqsubseteq j \}.$$

As a result,

Theorem 4.3.12. *The conglomerate MAXINT$(\mathbb{C}, \mathcal{M})$ is reflective in INT$(\mathbb{C}, \mathcal{M})$ and the reflection of $i \in$ INT$(\mathbb{C}, \mathcal{M})$ is its maximal hull.*

Proof. $\mathsf{INT}(\mathbb{C}, \mathcal{M}) \underset{j \leftarrow j}{\overset{i \mapsto i^{\mathsf{ma}}}{\rightleftarrows}} \mathsf{MAXINT}(\mathbb{C}, \mathcal{M})$. □

Proposition 4.3.13. Assume in the category $\mathbb{C}$ that preimages commute with arbitrary joins. If both i and j are maximal interior operators then $i \vee j$ is maximal.

Proof. Let $r \le s$ in $\mathsf{sub}X$. Since by Remark 1.4.8 each $\mathsf{sub}X$, where $X \in \mathbb{C}$, is a frame (hence a distributive lattice), one has $(i \vee j)_X(r) \cong i_X(r) \vee j_X(r) \cong (r \wedge i_X(s)) \vee (r \wedge j_X(s)) \cong r \wedge (i_X(s) \vee j_X(s)) \cong r \wedge (i \vee j)_X(s)$. □

Proposition 4.3.14. In any category, the only maximal standard interior operator is the discrete one.

Proof. Let i be a maximal and standard interior operator. Then for any $r \in \mathsf{sub}X$, one has $i_X(r) \cong r \wedge i_X(1_X) \cong r \wedge 1_X \cong r$. Hence i is discrete. □

Recall that in [Cas16], it was shown that the non-trivial examples of interior operators in the categories **Top** and **Grp** are standard. Therefore, by Proposition 4.3.14, in **Top** only the trivial and discrete interior operators are maximal while the discrete one is the only maximal interior operator in **Grp**. Even though, we are not sure whether non-trivial examples of maximal interior operators exist in other categories, from a theoretical point of view maximal interior operators may look interesting.

5. Connectedness via an Interior Operator

The foundation of the general theory of topological connectedness was begun with Preuß in [Pre71] and Herrlich in [Her68]. Thereafter the categorical notion of connectedness on an arbitrary category has been studied by using closure and neighbourhood operators; see [CH94, CT97, Cas01, Cle01, CH03b, Šla09, Raz12]. In most of these papers, the property that every morphism $X \to D$ with D discrete object relative to a given closure c has to be constant is taken as a definition for the object X to be c-connected. More recently, in [CR10], Castellini and Ramos studied the notion of connectedness in the category of topological spaces and continuous maps by using interior operators. In this chapter, we use the concept of categorical interior operators to study two possible general notions of "connectedness" in an arbitrary category. To this end, as in the case of the previous two chapters, we work in an $\mathcal{M}$-complete category $\mathbb{C}$ supplied with an $(\mathcal{E}, \mathcal{M})$-factorization structure for morphisms such that $\mathcal{M}$ is a fixed class of monomorphisms and assume that the preimage $f^*(-)$ preserves arbitrary joins for every morphism f in the category $\mathbb{C}$. We also consider an interior operator i on $\mathbb{C}$ with respect to $\mathcal{M}$.

5.1 Connectedness via constant morphisms

In this section, following the ideas of [Cle01] we first introduce the concept of coarse and fine objects with respect to a given interior operator and a relative notion of constant morphisms. We then use these notions to investigate the notions of connectedness and disconnectedness with respect to interior operators on $\mathbb{C}$ in a more general categorical setting. Our notion generalizes the work given in [CR10], extending the concept to a suitable arbitrary category. Furthermore, under mild conditions on $\mathbb{C}$, we construct a commutative diagram of Galois connections between the conglomerate of all interior operators on $\mathbb{C}$ with the reverse order, the conglomerate of all full subcategories of $\mathbb{C}$ and the dual of the conglomerate of all full subcategories of $\mathbb{C}$ to relate our notions to the Herrlich-Preuß-Arhangel'skii-Wiegandt (HPAW) connectedness-disconnectedness Galois connection. In the sequel we denote the terminal object by 1 and the unique terminal morphism $X \to 1$, where $X \in \mathbb{C}$, by $!_X$. Let us recall the following definition from [CT97].

Definition 5.1.1. An object $X \in \mathbb{C}$ is preterminal if $!_X : X \to 1$ is monic.

1 is a preterminal object, since $!_1 : 1 \to 1$ is an isomorphism we have that $!_1 : 1 \to 1$ is monic. In the category of **Sets**, **Top**, **Pos**, objects with the empty underlying set and the one element underlying set are preterminal and in a poset considered as a category every object is preterminal. We use $\mathcal{P}$ to denote the full subcategory of preterminal objects of $\mathbb{C}$.

Remark 5.1.2. Let $\mathcal{E}$ be stable under pullback along monomorphisms. Then

$$\mathcal{P} \subseteq \{X \in \mathbb{C} : !_X^*((!_X)_*(m)) \cong m \text{ for all } m \in \text{sub}X\}.$$

The following result describes the construction of an interior operator of interest associated with $\mathcal{P}$.

Proposition 5.1.3. The operator $j = (j_X : \text{ sub}X \to \text{sub}X)_{X \in \mathbb{C}}$ defined by

$$j_X(r) = \bigvee\{g^*(g_*(r)) : X \xrightarrow{g} P, P \in \mathcal{P}\}$$

for all $r \in \text{sub}X$ is a standard and idempotent interior operator on $\mathbb{C}$.

Proof. One can easily verify that j satisfies the contractiveness and monotonocity property. To prove the continuity condition, let $f : X \to Y$ be a morphism in $\mathbb{C}$. Then for all $n \in$ sub$Y$ one obtains

$$\begin{aligned}
f^*(j_Y(n)) &\cong f^*(\bigvee\{h^*(h_*(n)) : Y \overset{h}{\to} P, P \in \mathcal{P}\}) \\
&\cong \bigvee\{f^*(h^*(h_*(n))) : Y \overset{h}{\to} P, P \in \mathcal{P}\} \\
&\cong \bigvee\{(h \circ f)^*(h_*(n)) : Y \overset{h}{\to} P, P \in \mathcal{P}\} \\
&\leq \bigvee\{(h \circ f)^*(h_*(f_*(f^*(n)))) : Y \overset{h}{\to} P, P \in \mathcal{P}\} \\
&\cong \bigvee\{(h \circ f)^*((h \circ f)_*(f^*(n))) : Y \overset{h}{\to} P, P \in \mathcal{P}\} \\
&\cong \bigvee\{(h \circ f)^*((h \circ f)_*(f^*(n))) : X \overset{h \circ f}{\to} P, P \in \mathcal{P}\} \\
&\leq \bigvee\{g^*(g_*(f^*(n))) : X \overset{g}{\to} P, P \in \mathcal{P}\} \cong j_X(f^*(n)).
\end{aligned}$$

To show idempotency, let $g : X \to Q, Q \in \mathcal{P}$. Then for all $r \in$ sub$X$ we have that

$$\begin{aligned}
&g_*(r) \cong 1_Q^*((1_Q)_*(g_*(r))) \leq \bigvee\{f^*(f_*(g_*(r))) : Q \overset{f}{\to} P, P \in \mathcal{P}\} = j_Q(g_*(r)) \\
&\Rightarrow g_*(r) \cong j_Q(g_*(r)) \\
&\Rightarrow g^*(g_*(r)) \cong g^*(j_Q(g_*(r))) \leq g^*(g_*(j_X(r))) \\
&\Rightarrow j_X(r) = \bigvee\{g^*(g_*(r)) : X \overset{g}{\to} Q, Q \in \mathcal{P}\} \leq \bigvee\{g^*(g_*(j_X(r))) : X \overset{g}{\to} Q, Q \in \mathcal{P}\} = j_X(j_X(r)) \\
&\Rightarrow j_X(r) \cong j_X(j_X(r)).
\end{aligned}$$

We now show that j is standard. Let $g : X \to P, P \in \mathcal{P}$. Then $g^*(g_*(1_X)) \cong g^*(1_P) \cong 1_X$. Consequently, $j_X(1_X) \cong \bigvee\{g^*(g_*(1_X)) : X \overset{g}{\to} P, P \in \mathcal{P}\} \cong \bigvee\{1_X\} \cong 1_X$. Therefore, $j_X(1_X) \cong 1_X$. □

Note that the fact that j is standard implies j is different from the trivial interior operator $t_X(r) \cong 0_X$ for all $r \in$ sub$X$.

Definition 5.1.4. The operator j in Proposition 5.1.3 is called the indiscrete (coarse) interior operator.

Remark 5.1.5. (a) Let $P \in \mathcal{P}$. Then $j_P(r) \cong r$ for all $r \in$ sub$X$. Indeed, since $P \in \mathcal{P}$ we have that $P \overset{1_P}{\to} P$ is one of the g's in the class $\{P \overset{g}{\to} Q, P \in \mathcal{P}\}$. So, $r = 1_P^*((1_P)_*(r)) \leq \bigvee\{g^*(g_*(r)) : P \overset{g}{\to} Q, Q \in \mathcal{P}\} = j_P(r)$. Hence, $r \cong j_P(r)$;

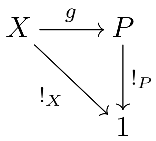
(b) In a category $\mathbb{C}$ in which the preimage functor for any given morphism preserves arbitrary joins, the indiscrete interior operator j always exists (see Proposition 5.1.3);

(c) The discrete interior (or fine) operator d can be described by $d_X(r) \cong r \wedge \bigwedge\{f_*(j_P(f^*(r))) : P \overset{f}{\to} X, P \in \mathcal{P}\} \cong r \wedge \bigwedge\{f_*(f^*(r)) : P \overset{f}{\to} X, P \in \mathcal{P}\}$ for all $X \in \mathbb{C}$ and $r \in$ sub$X$.

The following result depicts under what condition the indiscrete interior operator is induced by the terminal object 1.

Proposition 5.1.6. Let $\mathcal{E}$ be stable under pullback along monomorphisms. Then $j_X(r) \cong !_X^*((!_X)_*(r))$ for all $X \in \mathbb{C}$ and $r \in$ sub$X$.

Proof. Let $r \in$ sub$X$ and $g : X \to P$ with $P \in \mathcal{P}$ be any morphism. Then the diagram

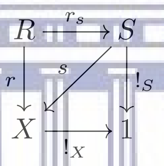


commutes and $!_P$ is monic and since $\mathcal{E}$ is stable under pullback along monomorphisms we have that $!_P^*((!_P)_*(g_*(r))) \cong g_*(r)$. Consequently, $!_X^*((!_X)_*(r)) \cong (!_P \circ g)^*((!_P \circ g)_*(r)) \cong g^*(!_P^*((!_P)_*(g_*(r)))) \cong g^*(g_*(r))$. This in turn implies $j_X(r) = \bigvee\{g^*(g_*(r)) : X \overset{g}{\to} P, P \in \mathcal{P}\} \cong \bigvee\{!_X^*((!_X)_*(r))\} \cong !_X^*((!_X)_*(r))$. □

Throughout the remainder of this chapter unless otherwise specified, we assume that $\mathcal{E}$ is stable under pullback along monomorphisms and j denotes the indiscrete interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$.

Corollary 5.1.7. The indiscrete interior operator j is hereditary.

Proof. Let $r \leq s$ in sub$X$. Then the diagram



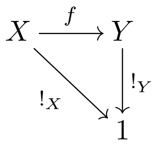
commutes. Consequently, by Proposition 5.1.6, $j_S(r_s) \cong !_S^*((!_S)_*(r_s)) \cong (!_X \circ s)^*((!_X \circ s)_*(r_s)) \cong s^*(!_X^*((!_X)_*(s_*(r_s)))) \cong s^*(j_X(s_*(r_s)))$. □

Proposition 5.1.8. Let $f : X \to Y$ be any morphism in $\mathbb{C}$. Then

(a) $f \in \mathcal{I}(j)$;

(b) $f \in \mathcal{E} \Rightarrow f \in \mathcal{F}(j) \cap \mathcal{K}(j) \cap \mathcal{O}(j)$.

Proof. (a) Let $f : X \to Y$ be any morphism in $\mathbb{C}$ and $r \in$ sub$X$. Then the diagram



commutes and since $\mathcal{E}$ is stable under pullback along monomorphisms we have by Proposition 5.1.6, $j_X(r) \cong !_X^*((!_X)_*(r)) \cong (!_Y \circ f)^*((!_Y \circ f)_*(r)) \cong f^*(!_Y^*((!_Y)_*(f_*(r)))) \cong f^*(j_Y(f_*(r)))$.

(b) Let $f : X \to Y$ be a morphism in $\mathcal{E}$, $r \in$ sub$X$ and $k \in$ sub$Y$. Then, since $\mathcal{M}$ is a subset of the class of monomorphisms, we have that stability of $\mathcal{E}$ under pullback along monomorphisms implies that $\mathcal{E}$ is stable under pullback along $\mathcal{M}$ (and hence along subobjects). Hence, $f_*(f^*(k)) \cong k$ for all $k \in$ sub$Y$. Consequently, we have by Proposition 5.1.6, $f_*(j_X(f^*(k))) \cong f_*(!_X^*((!_X)_*(f^*(k)))) \cong f_*((!_Y \circ f)^*((!_Y \circ f)_*(f^*(k)))) \cong f_*(f^*(!_Y^*((!_Y)_*(f_*(f^*(k)))))) \cong !_Y^*((!_Y)_*(k)) \cong$

$j_Y(k)$. Therefore, $f \in \mathcal{F}(j)$ and one also has $f_*(j_X(r)) \cong f_*(!_X^*((!_X)_*(r))) \cong f_*((!_Y \circ f)^*((!_Y \circ f)_*(r))) \cong f_*(f^*(!_Y^*((!_Y)_*(f_*(r))))) \cong !_Y^*((!_Y)_*(f_*(r))) \cong j_Y(f_*(r))$, hence $f \in \mathcal{K}(j)$. Moreover, $f \in \mathcal{O}(j)$, since $j_X(f^*(k)) \cong !_X^*((!_X)_*(f^*(k))) \cong f^*(!_Y^*((!_Y)_*(f_*(f^*(k))))) \cong f^*(!_Y^*((!_Y)_*(k))) \cong f^*(j_Y(k))$. Of course, this is true by (a). Indeed, f is j-initial and hence being in $\mathcal{E}$ implies f is j-final, closed and open by Proposition 3.1.36(c).

□

One can easily observe that any morphism in $\mathbb{C}$ is open and closed with respect to a discrete interior operator d, and morphisms in $\mathcal{M}$ are d-initial while morphisms in $\mathcal{E}$ are d-final. Next, we use the discrete interior operator and a newly defined indiscrete interior operator to introduce our notion of i-fine and i-coarse objects with respect to a given interior operator i.

Definition 5.1.9. Let $i \in INT(\mathbb{C}, \mathcal{M})$. Then an object $X \in \mathbb{C}$ is

(a) i-coarse (or i-indiscrete) object if $i_X \leq j_X$;

(b) i-fine (or i-discrete) object if $d_X \leq i_X$.

Let $i \in INT(\mathbb{C}, \mathcal{M})$. Then in the sequel we use $J(i)$ and $D(i)$ to denote the class of all i-coarse and i-fine objects, respectively, that is: $J(i) := \{X \in \mathbb{C} : i_X \leq j_X\}$ and $D(i) := \{X \in \mathbb{C} : d_X \leq i_X\} = \{X \in \mathbb{C} : i_X(m) \cong m$ for all $m \in$ sub$X\}$.

Proposition 5.1.10. For any interior operator i and any morphism $f : X \to Y \in \mathcal{F}(j)$ with $X \in J(i)$ we have that $Y \in J(i)$, that is: the subcategory $J(i)$ is closed under j-final morphisms.

Proof. Let $X \in J(i)$ and $n \in$ sub$Y$. Then $i_Y(n) \leq j_Y(n)$. Thus, the continuity condition of i and the j-finality of f yield $i_Y(n) \leq i_Y(f_*(f^*(n))) \leq f_*(i_X(f^*(n))) \leq f_*(j_X(f^*(n))) \cong j_Y(n)$. □

Corollary 5.1.11. Let i be any interior operator and $f : X \to Y$ be any morphism in $\mathcal{E}$. Then $X \in J(i) \Rightarrow Y \in J(i)$, that is: $J(i)$ is closed under $\mathcal{E}$-images.

Proof. Let $f : X \to Y \in \mathcal{E}$. Then the stability of $\mathcal{E}$ under pullback along monomorphisms and Proposition 5.1.8 implies $f \in \mathcal{F}(j)$. Furthermore, Proposition 5.1.10 provides $X \in J(i) \Rightarrow Y \in J(i)$. □

Proposition 5.1.12. Let i be any interior operator and $f : X \to Y$ be any morphism in $\mathcal{I}(i)$. Then $Y \in J(i) \Rightarrow X \in J(i)$.

Proof. Let $m \in$ sub$X$. Then since $f \in \mathcal{I}(i)$ and $Y \in J(i)$ we obtain $i_X(m) \cong f^*(i_Y(f_*(m))) \leq f^*(j_Y(f_*(m))) \cong f^*(!_Y^*((!_Y)_*(f_*(m)))) \cong !_X^*((!_X)_*(m)) \cong j_X(m)$. Hence $i_X(m) \leq j_X(m)$ for all $m \in$ sub$X$. □

Proposition 5.1.13. Let i be any interior operator and $f : X \to Y$ be any morphism in $\mathcal{I}(d^{\text{in}})$. Then $Y \in D(i) \Rightarrow X \in D(i)$.

Proof. Let $Y \in D(i)$ and $r \in$ sub$X$. Then d^{in}-initiality of f and the continuity condition of i implies $d_X^{\text{in}}(r) \cong f^*(d_Y^{\text{in}}(f_*(r))) \leq f^*(i_Y(f_*(r))) \leq i_X(f^*(f_*(r))) \leq i_X(r)$. □

Corollary 5.1.14. For any interior operator i, $D(i)$ is closed under $\mathcal{M}$-subobjects.

Proof. Let $M \overset{m}{\to} X$ be in sub$X$ such that $X \in D(i)$. Then since the discrete interior operator d^{in} is hereditary we have that every morphism in $\mathcal{M}$, in particular, m is d^{in}-initial. One obtains with Proposition 5.1.13, $M \in D(i)$. □

Remark 5.1.15. Let i be an interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$ and $f : X \to Y$ be a morphism in $\mathbb{C}$. If $Y \in D(i)$ then $f^*(n)$ is i-open in X for all $n \in$ sub$Y$. Indeed, let $Y \in D(i)$. Then each subobject n of Y is i-open in Y. Consequently, by Remark 2.1.4(b), $f^*(n)$ is i-open in X. This in turn of course implies the above corollary. In fact, for $m : M \to X \in$ sub$X$ with $X \in D(i)$, one has $t \cong m^*(m(t))$ is i-open in M for all $t \in$ sub$M$. Therefore, $M \in D(i)$.

Proposition 5.1.16. Let i be an interior operator and a morphism $f : X \to Y \in \mathcal{F}(i)$. Then $X \in D(i) \Rightarrow Y \in D(i)$.

Proof. Let $n \in$ sub$Y$. Then the i-initiality of f and $X \in D(i)$ implies $d_Y(n) \leq d_Y(f_*(f^*(n))) \leq f_*(d_X(f^*(n))) \leq f_*(i_X(f^*(n))) \cong i_Y(n)$. □

Remark 5.1.17. Let i be an interior operator. Then $J(i) \cap D(i) \subseteq \{X \in \mathbb{C} : \ j_X = d_X\}$. Indeed, let $X \in J(i) \cap D(i)$. Then $d_X \leq i_X \leq j_X$. So, $d_X = j_X$, since one always has $j \leq d$.

The following result shows that one can construct two interior operators $j(\mathbb{A})$ and $d(\mathbb{B})$ of interest which are induced by subcategories $\mathbb{A}$ and $\mathbb{B}$, respectively. A particular case of the construction of $j(\mathbb{A})$ appears in [CR10].

Proposition 5.1.18. Let $\mathbb{A}$ and $\mathbb{B}$ be any two subcategories of $\mathbb{C}$ then

a. $j(\mathbb{A})_X(r) := \bigvee\{g^*(g_*(r)) : X \overset{g}{\to} A, A \in \mathbb{A}\}$ for all $X \in \mathbb{C}$ and $r \in$ sub$X$ defines an interior operator on $\mathbb{C}$. Moreover, $j(\mathbb{A})$ is always an idempotent and standard interior operator;

b. $d(\mathbb{B})_X(r) := r \wedge \bigwedge\{f_*(f^*(r)) : B \overset{f}{\to} X, B \in \mathbb{B}\}$ for all $X \in \mathbb{C}$ and $r \in$ sub$X$ is defines an interior operator on $\mathbb{C}$.

Proof. Similar to the proof of Proposition 5.1.3. □

We call $j(\mathbb{A})$ and $d(\mathbb{B})$ discrete and indiscrete interior operators with respect to subcategories $\mathbb{A}$ and $\mathbb{B}$, respectively and these notations will be used in the rest of this section.

Recall that constant morphisms have played a significant role in the study of the notions of connectedness and disconnectedness in an arbitrary category (see [Her68, Pre71, AW75]). Next we present a relative notion of constant morphisms in order to investigate the general concepts of connectedness and disconnectedness in the category $\mathbb{C}$. To this purpose, we consider a class

$$\mathbb{S}_o = \{X \in \mathbb{C} : !^*_X((!_X)_*(m)) \cong m \text{ for all } m \in \text{sub}X\}$$

(see [Cle95, Raz12]). The following lemma describes properties of the class $\mathbb{S}_o$.

Lemma 5.1.19. From the definition of $\mathbb{S}_o$ one has the following.

(a) $\mathbb{S}_o$ is closed under $\mathcal{M}$-subobjects, that is, if $R \overset{r}{\to} X \in \mathcal{M}$ and $X \in \mathbb{S}_o$, then $R \in \mathbb{S}_o$;

(b) Let $\mathcal{E}$ be stable under pullback along $\mathcal{M}$-morphisms. Then $\mathbb{S}_o$ is closed under $\mathcal{E}$-images, that is, if $X \overset{q}{\to} Q \in \mathcal{E}$ and $X \in \mathbb{S}_o$ then $Q \in \mathbb{S}_o$.

Proof. (a) Let $t : T \to R$ be a subobject of R such that $r \in \mathcal{M}$ and $X \in \mathbb{S}_o$. Then we have a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{r} & X \\ & {\scriptstyle !_R}\searrow & \downarrow{\scriptstyle !_X} \\ & & 1 \end{array}$$

and hence $!_R \cong !_X \circ r$ and $r^*(r_*(t)) \cong t$. As a consequence,
$!_R((!_R)_*(t)) \cong (!_X \circ r)^*((!_X \circ r)_*(t)) \cong r^*(!_X^*((!_X)_*(r_*(t)))) \cong r^*(r_*(t)) \cong t$. Therefore, $R \in \mathbb{S}_o$.

(b) Let $n : N \to Q$ be in sub$Q$ such that $q \in \mathcal{E}$ and $X \in \mathbb{S}_o$. Then the diagram

$$\begin{array}{ccc} X & \xrightarrow{q} & Q \\ & {\scriptstyle !_X}\searrow & \downarrow{\scriptstyle !_Q} \\ & & 1 \end{array}$$

commutes and hence $!_X = !_Q \circ q$. Consequently, $n \cong q_*(q^*(n)) \cong q_*(!_X^*((!_X)_*(q^*(n)))) \cong q_*((!_Q \circ q)^*((!_Q \circ q)_*(q^*(n)))) \cong q_*(q^*(!_Q^*((!_Q)_*(q_*(q^*(n)))))) \cong !_Q^*((!_Q)_*(n))$. Thus, $Q \in \mathbb{S}_o$.

□

Remark 5.1.20. We also note that the followings.

a. $\mathbb{S}_o$ is a non-empty subcategory of $\mathbb{C}$. Indeed, $!_1 : 1 \to 1 \in \mathcal{M}$, as $\mathcal{M}$ contains all isomorphisms, and hence $!_1^*((!_1)_*(m)) \cong m$ for all $m \in$ sub1. Therefore, $1 \in \mathbb{S}_o$.

b. We call any object X in $\mathbb{S}_o$ a constant object.

This remark enables us to define constant morphisms with respect to $\mathbb{S}_o$ and $(\mathcal{E}, \mathcal{M})$-factorization system as follows.

Definition 5.1.21. A morphism $f : X \to Y$ is constant if the domain $f[X]$ of the $\mathcal{M}$-part of the $(\mathcal{E}, \mathcal{M})$-factorization of f is a constant object, that is, $f[X] \in \mathbb{S}_o$.

Note that $f[X]$ is the domain of $f(1_X)$, the image of f. One can also easily observe that:
If $X \xrightarrow{e} f[X] \xrightarrow{m} Y$ is the $(\mathcal{E}, \mathcal{M})$-factorization of $f \cong f \circ 1_X$, shown in the diagram below, then f is constant if and only if m is constant if and only if e is constant.

$$\begin{array}{ccccc} X & \xrightarrow{1_X} & X & \xrightarrow{f} & Y \\ & {\scriptstyle e}\searrow & & \nearrow{\scriptstyle m \cong f(1_X)} & \\ & & f[X] & & \end{array}$$

Remark 5.1.22. Let $X \in \mathbb{C}$. Then $X \xrightarrow{!_X} 1$ is constant. Indeed, let (e, m) be the $(\mathcal{E}, \mathcal{M})$-factorization of $!_X : X \to !_X[X] \to 1$. Then the fact that $1 \in \mathbb{S}_o$ (see Remark 5.1.20 (a)) and $\mathbb{S}_o$ is closed under $\mathcal{M}$-subobjects (see Lemma 5.1.19(a)), one has $!_X[X] \in \mathbb{S}_o$.

Examples 5.1.23. ([Cle95])

(a) Let $\mathbb{C}$ be the category **Sets** with (Surjections, Injections)-factorization system and $f : X \to Y$ be any function. Then $f[X] \in \mathbb{S}_o \Leftrightarrow !^*_{f[X]}((!_{f[X]})_*(N)) = N \ \forall N \subseteq f[X] \Leftrightarrow f[X]$ is a singleton set. Hence, the constant morphisms are the constant maps. In fact $\mathbb{S}_o$ contains only the empty set and the singletons.

(b) Let $\mathbb{C}$ be the category **Top** with (Surjections, Embeddings)-factorization system. Then $\mathbb{S}_o$ contains only the empty space and the singleton spaces. Hence, the constant morphisms are precisely the constant maps.

(c) Let $\mathbb{C}$ be the category **SGph** of directed spatial graphs and graph homomorphisms with the (Surjective homorphisms, Embeddings)-factorization system. Then $\mathbb{S}_o$ contains only the null graph (a graph with no vertices nor edges) and the graphs with a single vertex and one loop. Hence, the constant morphisms are the constant maps.

Lemma 5.1.24. Let $\mathcal{E}$ be stable under pullback along $\mathcal{M}$-morphisms. Then any morphism that can be factored through a constant object is constant.

Proof. Let $f : X \to Y$ be a morphism that is factored through $S \in \mathbb{S}_o$. Then $\exists\ X \xrightarrow{r} S \xrightarrow{s} Y$ such that $f = s \circ r$. Now, let $S \xrightarrow{e'} s[S] \xrightarrow{m'} Y$ and $X \xrightarrow{e} f[X] \xrightarrow{m} Y$ be the $(\mathcal{E}, \mathcal{M})$-factorization of s and f respectively. Then we obtain the commutative diagram below.

$$\begin{array}{ccc} X & \xrightarrow{e} & f[X] \\ {\scriptstyle e' \circ r}\downarrow & & \downarrow{\scriptstyle m} \\ s[S] & \xrightarrow[m']{} & Y \end{array}$$

Hence, by the diagonalization property $\exists! d : f[X] \to s[S]$ such that $d \circ e = e' \circ r$ and $m = m' \circ d$. But since $\mathcal{M}$ is left cancellable with respect to itself ($g \circ f \in \mathcal{M}$ and $g \in \mathcal{M}$ implies $f \in \mathcal{M}$) we have that $d \in \mathcal{M}$. Furthermore, $S \in \mathbb{S}_o$ and $e' : S \to s[S] \in \mathcal{E}$ and $\mathcal{E}$ is stable under pullback along $\mathcal{M}$-morphisms. Then Lemma 5.1.19(b) implies $s[S] \in \mathbb{S}_o$. Consequently, Lemma 5.1.19(a) implies $f[X] \in \mathbb{S}_o$. □

The following result summarizes some properties of constant morphisms.

Proposition 5.1.25. Let $\mathbb{S}_m$ be the class of constant morphisms.

(a) Let $\mathcal{E}$ be stable under pullback along $\mathcal{M}$-morphisms. Then $f \in \mathbb{S}_m \Rightarrow g \circ f \circ h \in \mathbb{S}_m$;

(b) $\mathbb{S}_m$ is left cancellable with respect to $\mathcal{M}$, that is, $g \circ f \in \mathbb{S}_m$ and $g \in \mathcal{M} \Rightarrow f \in \mathbb{S}_m$;

(c) $\mathbb{S}_m$ is right cancellable with respect to $\mathcal{E}$, that is, $g \circ f \in \mathbb{S}_m$ and $f \in \mathcal{E} \Rightarrow g \in \mathbb{S}_m$.

Proof. (a) Let $W \xrightarrow{h} X \xrightarrow{f} Y \xrightarrow{g} Z$ be a morphism such that f is constant. Then $f[X] \in \mathbb{S}_o$. Thus, $g \circ f \circ h = \ W \xrightarrow{h} X \xrightarrow{e} f[X] \xrightarrow{m} Y \xrightarrow{g} Z \ = \ W \xrightarrow{eoh} f[X] \xrightarrow{gom} Z$ is factored through a constant object $f[X]$. Therefore, Lemma 5.1.24 implies $g \circ f \circ h$ is constant.

(b) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a morphism such that $g \circ f$ is constant and $g \in \mathcal{M}$. Then $(g \circ f)[X] \in \mathbb{S}_o$. Let (e, m) and (e', m') be the $(\mathcal{E}, \mathcal{M})$-factorization of $g \circ f$ and f respectively. Then (e, m)

and $(e^{'}, g \circ m^{'})$ are the $(\mathcal{E}, \mathcal{M})$-factorization of $g \circ f$. Then there exists a unique isomorphism $d : (g \circ f)[X] \to f[X]$, such that the diagram.

$$\begin{array}{ccc} X & \xrightarrow{e} & (g \circ f)[X] \\ {\scriptstyle e^{'}}\downarrow & \overset{d}{\swarrow} & \downarrow{\scriptstyle m} \\ f[X] & \xrightarrow[g \circ m^{'}]{} & Z \end{array}$$

commutes. Consequently, $f[X] \cong (g \circ f)[X] \in \mathbb{S}_o$. Therefore, f is constant.

(c) Let $X \overset{f}{\to} Y \overset{g}{\to} Z$ be a morphism such that $g \circ f$ is constant and $f \in \mathcal{E}$. Then $(g \circ f)[X] \in \mathbb{S}_o$. Let (e, m) and $(e^{'}, m^{'})$ be the $(\mathcal{E}, \mathcal{M})$-factorization of $g \circ f$ and g respectively. Then (e, m) and $(e^{'} \circ f, m^{'})$ are the $(\mathcal{E}, \mathcal{M})$-factorization of $g \circ f$. Then there exists a unique isomorphism $d : (g \circ f)[X] \to f[X]$, such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{e} & (g \circ f)[X] \\ {\scriptstyle e^{'} \circ f}\downarrow & \overset{d}{\swarrow} & \downarrow{\scriptstyle m} \\ g[X] & \xrightarrow[m^{'}]{} & Z \end{array}$$

commutes. As a result, $g[X] \cong (g \circ f)[X] \in \mathbb{S}_o$. Therefore, g is constant.

□

Let X and Y be any two objects of $\mathbb{C}$. Then we use $X \parallel Y$ to denote every morphism $X \to Y$ is constant.

Definition 5.1.26. Let $\mathbb{A}$ and $\mathbb{B}$ be any two full subcategories of $\mathbb{C}$. Then we define

(a) the left-constant subcategory of $\mathbb{A}$ by $l(\mathbb{A}) = \{X \in \mathbb{C} : (\forall Y \in \mathbb{A})(X \parallel Y)\}$ and

(b) the right-constant subcategory of $\mathbb{B}$ by $r(\mathbb{B}) = \{Y \in \mathbb{C} : (\forall X \in \mathbb{B})(X \parallel Y)\}$.

Throughout the remainder of this section, we use $S(\mathbb{C})$ to denote the conglomerate of all full subcategories of $\mathbb{C}$ that are ordered by inclusion $\subseteq$ and $S(\mathbb{C})^{op}$ to denote its dual with the reverse order $\supseteq$. We also use $\preceq$ to denote the order of the dual $INT(\mathbb{C}, \mathcal{M})^{op}$ of the conglomerate of all interior operators of $\mathbb{C}$ with respect to $\mathcal{M}$ ordered by $\leq$. With these notations, the Galois connection introduced in [Her68, Pre78] can be described in the following proposition:

Proposition 5.1.27. Let $\mathbb{A}$ and $\mathbb{B}$ be any two subcategories of $\mathbb{C}$. Then

$$S(\mathbb{C}) \underset{l}{\overset{r}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} S(\mathbb{C})^{op} \text{ is a Galois connection.}$$

Proof. Let $\mathbb{A}$ and $\mathbb{B}$ be in $S(\mathbb{C})$. Then we need to show that $r(\mathbb{B}) \supseteq \mathbb{A} \Leftrightarrow \mathbb{B} \subseteq l(\mathbb{A})$.

(a) Suppose $r(\mathbb{B}) \supseteq \mathbb{A}$. Then $\mathbb{A} \subseteq r(\mathbb{B})$. Now, let $X \in \mathbb{B}$. Then $X \parallel Y$ for all $Y \in \mathbb{A}$. So, $X \in l(\mathbb{A})$ and hence $\mathbb{B} \subseteq l(\mathbb{A})$.

(b) Suppose $\mathbb{B} \subseteq l(\mathbb{A})$. Then for any $Y \in \mathbb{A}$ we have $X \parallel Y$ for all $X \in \mathbb{B}$. Therefore, $Y \in l(\mathbb{B})$ and hence $r(\mathbb{B}) \supseteq \mathbb{A}$.

□

The above Galois connectection is known as the Herrlich-Preuß-Arhangel'skiĭ-Wiegandt (left-constant, right-constant) correspondence. As a consequence of Proposition 5.1.25 one obtains the following result.

Proposition 5.1.28. Let $\mathbb{A}$ and $\mathbb{B}$ be any two full subcategories of $\mathbb{C}$.

(a) $l(\mathbb{A})$ is closed under $\mathcal{E}$-images;

(b) $r(\mathbb{B})$ is closed under $\mathbb{M}$-subobjects.

Definition 5.1.29. Let i be a given interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$. We say that $X \in \mathbb{C}$ is

(a) i-connected if $X \in l(D(i))$, that is, for every i-discrete object Y, any $\mathbb{C}$-morphism $X \to Y$ is constant ;

(b) i-disconnected if $X \in r(J(i))$, that is, for every i-indiscrete object Y, any $\mathbb{C}$-morphism $Y \to X$ is constant.

Proposition 5.1.30. Let i be a given interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$.

(a) Let $R \overset{r}{\to} X$ be in $\mathcal{M}$ and X is i-disconnected. Then R is i-disconnected;

(b) Let $X \overset{f}{\to} Y$ be in $\mathcal{E}$ and X is i-connected. Then Y is i-connected.

Proof. Apply Proposition 5.1.28. □

Corollary 5.1.31. Let i be a given interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$ and suppose $\mathbb{C}$ admits arbitrary products. Let $X = \prod_{i\in I} X_i$ be an i-connected product in $\mathbb{C}$ such that each projections $p_i : X \to X_i \in \mathcal{E}'$. Then each X_i is i-connected.

Proposition 5.1.32. Let i be a given interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$. Then a right constant subcategory is closed under monosources.

Proof. Let $\mathbb{A}$ be a full subcategory of $\mathbb{C}$, $r(\mathbb{A})$ be the right-constant subcategory of $\mathbb{A}$ and $(\pi_i : Y \to Y_i)_{i\in I}$ be a monosource with $Y_i \in r(\mathbb{A})$. Let $X \in \mathbb{A}$ and (e, m) and (e_f, m_f) be the $(\mathcal{E}, \mathcal{M})$-factorization of f and $\pi_i \circ f$, respectively. Then by the unique diagonalization property there exists a unique $d : f[X] \to (\pi_i \circ f)[X]$ such that $\pi_i \circ m_f = m \circ d$ and $e = d \circ e_f$. Suppose $!_{f[X]} \circ u = !_{f[X]} \circ v$ for any two pair morphisms u and v with codomain $f[X]$. Consequently, $!_{(\pi_i \circ f)[X]} \circ d \circ u = !_{(\pi_i \circ f)[X]} \circ d \circ v$. But since $\pi_i \circ f$ is constant, one has $d \circ u = d \circ v$. This in turn implies $m \circ d \circ u = m \circ d \circ v$. Hence,

$$\begin{aligned}
&\pi_i \circ m_f \circ u = \pi_i \circ m_f \circ v \text{ for all } i && (m \circ d = \pi_i \circ m_f)\\
&\Rightarrow m_f \circ u = m_f \circ v && (\pi_i' \text{s jointly monic})\\
&\Rightarrow u = v && (m_f \text{ monic})\\
&\Rightarrow !_{f[X]} \text{is monic.}
\end{aligned}$$

Thus, by Proposition 1.4.4(a), $!^*_{f[X]}((!_{f[X]})_*(m)) \cong m$ for all $m \in \text{sub}f[X]$, that is: $f[X] \in \mathbb{S}_o$. Therefore, f is constant and hence $X \parallel Y$ for all $X \in \mathbb{A}$. As a consequence, $Y \in r(\mathbb{A})$. □

The right constant subcategory contains the preterminal objects. Indeed, if $X \in \mathcal{P}$ then $!_X : X \to 1$ is monic. But since $!_X : X \to 1$ is constant (see Remark 5.1.22), one has $1 \in r(\mathbb{B})$. Consequently, $X \in r(\mathbb{B})$ since the right constant subcategory is closed under monosources and hence under mono.

Corollary 5.1.33. Let i be a given interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$ and assume that $\mathbb{C}$ admits arbitrary products. Then the product of a family of i-disconnected objects is i-disconnected.

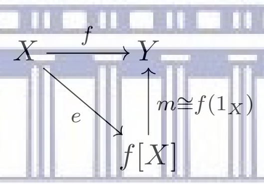
Proof. Let $(p_i : X = \prod_{i\in I} X_i \to X_i)_{i\in I}$ be a product with each X_i being i-disconnected. Then each X_i belongs to $r(J(i))$. Consequently, by Proposition 5.1.32, one has $X = \prod_{i\in I} X_i \in r(J(i))$ since products are monosources. Therefore, $X = \prod_{i\in I} X_i$ is disconnected. □

Lemma 5.1.34. Let i be a given interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$. Then $J(i) \cap D(i) \subseteq \mathbb{S}_o$.

Proof. Let $X \in J(i) \cap D(i)$. Then $d_X \le i_X \le j_X$ and hence $d_X \cong j_X$. Consequently, the stability of $\mathcal{E}$ under pullback along monomorphisms implies $r \cong d_X(r) \cong j_X(r) \cong !^*_X((!_X)_*(r))$ for all $r \in$sub$X$. Therefore, $X \in \mathbb{S}_o$. □

Proposition 5.1.35. Let i be a given interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$ and $f : X \to Y$ be any morphism in $\mathbb{C}$ with $X \in J(i)$ and $Y \in D(i)$. Then f is constant.

Proof. Let (e, m) be the $(\mathcal{E}, \mathcal{M})$-factorization of f, shown in the diagram



Then the fact that $X \in J(i)$ and $X \xrightarrow{e} f[X] \in \mathcal{E}$ we have by Corollary 5.1.11 $f[X] \in J(i)$. Besides, since $f[X] \xrightarrow{m} Y \in \mathcal{M}$ and $Y \in D(i)$ then Corollary 5.1.14 obtains $f[X] \in D(i)$. Thus, $f[X] \in J(i) \cap D(i)$. This in turn implies $f[X] \in \mathbb{S}_o$, by Lemma 5.1.34. Therefore, f is constant. □

Corollary 5.1.36. Let i be any given interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$. Then the i-indiscrete objects are i-connected objects.

Proof. Let $X \in J(i)$. Then by Proposition 5.1.35, any morphism $f : X \to Y$ with $Y \in D(i)$ must be constant. Hence, $J(i) \subseteq l(D(i))$. □

Following the definitions of J and D one has:

Remark 5.1.37. For interior operators i and i' on $\mathbb{C}$ such that $i \preceq i'$ one has $J(i) \subseteq J(i')$ and $D(i) \supseteq D(i')$. That is, J and D can be seen as functors from $INT(\mathbb{C}, \mathcal{M})^{op}$ to that of $S(\mathbb{C})$ and $S(\mathbb{C})^{op}$, respectively.

From the definitions of $j(\mathbb{A})$ and $d(\mathbb{B})$, where $\mathbb{A}$, $\mathbb{B} \in S(\mathbb{C})$ one also obtains the following:

Remark 5.1.38. Let $\mathbb{A}$ and $\mathbb{B}$ be any two full subcategories of $\mathbb{C}$. Then

(a) $\mathbb{A} \subseteq \mathbb{B} \Rightarrow j(\mathbb{A}) \le j(\mathbb{B})$;

(b) $\mathbb{A} \subseteq \mathbb{B} \Rightarrow d(\mathbb{A}) \preceq d(\mathbb{B})$;

(c) $j(\mathbb{A})_A = d_A$ for all $A \in \mathbb{A}$, that is, $j(\mathbb{A})$ is discrete in $\mathbb{A}$;

(d) $d(\mathbb{B})_B = j_B$ for all $B \in \mathbb{B}$, that is, $d(\mathbb{B})$ is indiscrete in $\mathbb{B}$.

As a consequence of the above remark d and j may be interpreted as functors from $S(\mathbb{C})$ and $S(\mathbb{C})^{op}$ to $INT(\mathbb{C}, \mathcal{M})^{op}$, respectively.

Lemma 5.1.39. Let i be any interior operator. Then $d(J(i)) \preceq i$.

Proof. Let $r \in \text{sub}X$. Then $r \leq g_*(g^*(r))$ for all $g : A \to X$ with $A \in J(i)$. Hence one has $r \leq \bigwedge\{g_*(g^*(r)) : \ g : A \to X \text{ with } A \in J(i)\}$. Consequently,

$$\begin{aligned} i_X(r) &\leq i_X(\bigwedge\{g_*(g^*(r)) : A \xrightarrow{g} X, A \in J(i)\}) \\ &\leq \bigwedge\{i_X(g_*(g^*(r))) : A \xrightarrow{g} X, A \in J(i)\} \\ &\leq \bigwedge\{g_*(i_A(g^*(r))) : A \xrightarrow{g} X, A \in J(i)\} \cong d(J(i))_X(r). \end{aligned}$$

□

Proposition 5.1.40. $S(\mathbb{C}) \underset{J}{\overset{d}{\rightleftarrows}} INT(\mathbb{C}, \mathcal{M})^{op}$ ($d \dashv J$) is a Galois connection.

Proof. Let $\mathbb{B} \in S(\mathbb{C})$ and $i \in INT(\mathbb{C}, \mathcal{M})$. We need to show that $d(\mathbb{B}) \preceq i \Leftrightarrow \mathbb{B} \subseteq J(i)$.

($\Rightarrow$) Suppose $d(\mathbb{B}) \preceq i$. Then $i \leq d(\mathbb{B})$ and hence Remark 5.1.38 provides $i_B \leq d(\mathbb{B})_B = j_B$ for all $B \in \mathbb{B}$. Consequently, $B \in J(i)$. Therefore, $\mathbb{B} \subseteq J(i)$.

($\Leftarrow$) Assume that $\mathbb{B} \subseteq J(i)$. Then Remark 5.1.38 implies $d(J(i)) \leq d(\mathbb{B})$. Consequently, Lemma 5.1.39 yields $i \leq d(J(i)) \leq d(\mathbb{B})$. Therefore, $d(\mathbb{B}) \preceq i$.

□

Lemma 5.1.41. Let i be any interior operator. Then $i \preceq j(D(i))$.

Proof. Let $r \in \text{sub}X$ and $X \xrightarrow{f} P, P \in D(i)$. Then the continuity condition of i provides

$$\begin{aligned} &f^*(f_*(r)) \cong f^*(d_P(f_*(r))) \leq f^*(i_P(f_*(r))) \leq i_X(f^*(f_*(r))) \leq i_X(r) \\ &\Rightarrow j(D(i))_X(r) = \bigvee\{f^*(f_*(r)) : X \xrightarrow{f} P, P \in D(i)\} \leq i_X(r) \\ &\Rightarrow j(D(i)) \leq i. \end{aligned}$$

□

Proposition 5.1.42. $INT(\mathbb{C}, \mathcal{M})^{op} \underset{j}{\overset{D}{\rightleftarrows}} S(\mathbb{C})^{op}$ ($D \dashv j$) is a Galois connection.

Proof. Let $\mathbb{A} \in S(\mathbb{C})$ and $i \in INT(\mathbb{C}, \mathcal{M})$. We need to show that $D(i) \supseteq \mathbb{A} \Leftrightarrow i \preceq j(\mathbb{A})$.

($\Rightarrow$) Suppose $D(i) \supseteq \mathbb{A}$. Then $\mathbb{A} \subseteq D(i)$ and hence Remark 5.1.38 provides $j(\mathbb{A}) \leq j(D(i))$. Consequently, Lemma 5.1.41 yields $j(\mathbb{A}) \leq j(D(i)) \leq i$. Therefore, $j(\mathbb{A}) \leq i$ and hence $i \preceq j(\mathbb{A})$.

($\Leftarrow$) Assume that $i \preceq j(\mathbb{A})$. Then $j(\mathbb{A}) \leq i$. Using Remark 5.1.38 we have that $d_A = j(\mathbb{A})_A \leq i_A$ for all $A \in \mathbb{A}$. Therefore, $\mathbb{A} \subseteq D(i) \Leftrightarrow D(i) \supseteq \mathbb{A}$.

□

Propositions 5.1.35, 5.1.40 and 5.1.42 give:

Proposition 5.1.43. Let $\mathbb{A} \in S(\mathbb{C})$. Then

(a) $(J \circ j)(\mathbb{A}) \subseteq l(\mathbb{A})$;

(b) $(D \circ d)(\mathbb{A}) \subseteq r(\mathbb{A})$.

Proof. (a) Let $X \in J(j(\mathbb{A}))$ and $f : X \to Y$ with $Y \in \mathbb{A}$ be any morphism. Since by Proposition 5.1.42 $D \dashv j : S(\mathbb{C})^{op} \to INT(\mathbb{C}, \mathcal{M})^{op}$ we have $D(j(\mathbb{A})) \supseteq \mathbb{A}$, that is: $\mathbb{A} \subseteq D(j(\mathbb{A}))$. Hence, every object of $\mathbb{A}$ belongs to $D(j(\mathbb{A}))$, in particular $Y \in D(j(\mathbb{A}))$. As a result $f : X \to Y$ is a morphism with $X \in J(j(\mathbb{A}))$ and $Y \in D(j(\mathbb{A}))$. Consequently, by Proposition 5.1.35 f must be constant and hence $X \in l(\mathbb{A})$.

(b) Let $Y \in D(d(\mathbb{A}))$ and $f : X \to Y$ with $X \in \mathbb{A}$ be any morphism. Since by Proposition 5.1.40 $d \dashv J : INT(\mathbb{C}, \mathcal{M})^{op} \to S(\mathbb{C})$ we have $\mathbb{A} \subseteq J(d(\mathbb{A}))$. Hence every object of $\mathbb{A}$ belongs to $J(d(\mathbb{A}))$, in particular $X \in J(d(\mathbb{A}))$. Thus $f : X \to Y$ is a morphism with $X \in J(d(\mathbb{A}))$ and $Y \in D(d(\mathbb{A}))$. Consequently, by Proposition 5.1.35 f must be constant and hence $Y \in r(\mathbb{A})$.

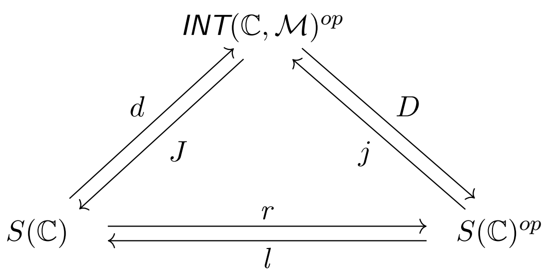
□

Now we are ready to show that the Herrlich-Preuß-Arhangel'skii-Wiegandt (HPAW) (left-constant, right-constant) correspondence is the composition of the adjunctions defined in Propositions 5.1.40 and 5.1.42. As a consequence of Propositions 5.1.40 and 5.1.42 one has the Galois correspondences

$$S(\mathbb{C}) \underset{d}{\overset{J}{\leftrightarrows}} INT(\mathbb{C}, \mathcal{M})^{op} \underset{j}{\overset{D}{\rightleftarrows}} S(\mathbb{C})^{op}$$

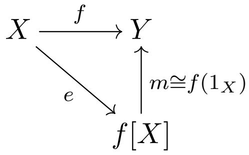
Furthermore, one has:

Theorem 5.1.44. *Let $\mathcal{P} = \mathbb{S}_o$. Then the Galois connection $S(\mathbb{C}) \underset{l}{\overset{r}{\rightleftarrows}} S(\mathbb{C})^{op}$ factors through $INT(\mathbb{C}, \mathcal{M})^{op}$ via the Galois connections $S(\mathbb{C}) \underset{J}{\overset{d}{\rightleftarrows}} INT(\mathbb{C}, \mathcal{M})^{op}$ and $INT(\mathbb{C}, \mathcal{M})^{op} \underset{j}{\overset{D}{\rightleftarrows}} S(\mathbb{C})^{op}$, that is: there is a commutative triangle*



of adjunctions, factoring r through D and l through J, that is, $r = D \circ d$ and $l = J \circ j$.

Proof. Note that one can easily observe that $D \circ d$ and $J \circ j$ give rise to a Galois connection between $S(\mathbb{C})$ and $S(\mathbb{C})^{op}$. Due to the uniqueness of the adjoints in a Galois connection if we show one equality then the other equality follows. Hence it is sufficient to show that $D(d(\mathbb{B})) = r(B)$. To this end, let $Y \in r(\mathbb{B})$ and $n \in$sub$Y$. Then $X \parallel Y$ for all $X \in \mathbb{B}$, that is $X \stackrel{f}{\to} Y$ with $X \in \mathbb{B}$ is constant and hence its image $f[X] \in \mathbb{S}_o = \mathcal{P}$.



Consequently, the stability of $\mathcal{E}$ under pullback along monomorphisms (and hence $\mathcal{M}$-morphisms) implies

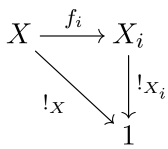
$$j_X(f^*(n)) = \bigvee\{g^*(g_*(f^*(n))) : X \stackrel{g}{\to} P, P \in \mathcal{P}\} \geq e^*(e(f^*(n))) \cong e^*(e(m^*(n))) \cong e^*(m^*(n)) \cong f^*(n).$$

This in turn implies $n \leq f_*(f^*(n)) \leq f_*(j_X(f^*(n)))$, hence $n \leq \bigwedge\{f_*(j_X(f^*(n))) : X \stackrel{f}{\to} Y, X \in \mathbb{B}\}$. As a consequence, $n \cong n \wedge \bigwedge\{f_*(j_X(f^*(n))) : X \stackrel{f}{\to} Y, X \in \mathbb{B}\} \cong d(\mathbb{B})_Y(n)$, that is: $Y \in D(d(\mathbb{B}))$. On the other hand, let $Y \in D(d(\mathbb{B}))$, $X \stackrel{f}{\to} Y$ with $X \in \mathbb{B}$ and $T \stackrel{t}{\to} f[X]$ be in sub$f[X]$.Then the stability of $\mathcal{E}$ under pullback along monomorphisms (and hence $\mathcal{M}$-morphisms) and Proposition 5.1.8 imply $j_{f[X]}(t) \cong e_*(e^*(j_{f[X]}(t))) \cong e_*(j_X(e^*(t))) \cong e_*(d(\mathbb{B})_X(e^*(n))) \geq d(\mathbb{B})_{f[X]}(e_*(e^*(n))) \cong d(\mathbb{B})_{f[X]}(n) \cong n$. Therefore, $f[X] \in \mathbb{S}_o$ and hence $X \parallel Y$ for all $X \in \mathbb{B}$. So, $Y \in r(B)$. Therefore, the Herrlich-Preuß-Arhangel'skii-Wiegandt (left-constant, right-constant) correspondence is the composition of the adjunctions (D, j) and (d, J). We note that $S(\mathbb{C}) \underset{J \circ j}{\overset{D \circ d}{\rightleftarrows}} S(\mathbb{C})^{op}$ is called the connectedness-disconnectedness Galois connection. □

In the following results of this section by assuming the category $\mathbb{C}$ admits products we would like to explore the behaviour of i_X on products. Consequently, with Definition 3.1.34 and Proposition 5.1.6, one has the following result.

Proposition 5.1.45. Let i be a given interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$ and $\mathbb{S} := (f_i : X \to X_i)_{i \in I}$ be an i-initial cone. If $X_i \in J(i)$ for all $i \in I$ then $X \in J(i)$.

Proof. Suppose $\mathbb{S}$ is i-initial and $X_i \in J(i)$. Let $m : M \to X \in$ sub$X$. Then the diagram



commutes. Hence, Definition 3.1.34 and Proposition 5.1.6 imply

$$\begin{aligned} i_X(m) &\leq \bigvee_{i \in I} f_i^*(i_{X_i}((f_i)_*(m))) \\ &\leq \bigvee_{i \in I} f_i^*(j_{X_i}((f_i)_*(m))) \cong \bigvee_{i \in I} f_i^*(!_{X_i}^*((!_{X_i})_*((f_i)_*(m))))) \cong \bigvee_{i \in I} !_X^*((!_X)_*(m)) \cong !_X^*((!_X)_*(m)) \cong j_X(m). \end{aligned}$$

□

Consequently, one obtains the following corollary.

Corollary 5.1.46. Let i be a given interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$. The product of i-indiscrete objects is i-indiscrete provided that the projections are jointly i-initial.

Proof. Put $X = \prod_{i\in I} X_i$ and then apply Proposition 5.1.45. □

Corollary 5.1.47. Let $\mathbb{A} \in S(\mathbb{C}), \mathcal{P} = \mathbb{S}_o$ and $(p_i : X = \prod_{i\in I} X_i \to X_i)_{i\in I}$ be $j(\mathbb{A})$-initial. If $X_i \in l(\mathbb{A})$ for all $i \in I$ then $X = \prod_{i\in I} X_i \in l(\mathbb{A})$.

Proof. Let $X_i \in l(\mathbb{A})$. Then by Proposition 5.1.44 we have that $l(\mathbb{A}) = J(j(\mathbb{A}))$ and hence $X_i \in J(j(\mathbb{A}))$ for all $i \in I$. But since each projection is $j(\mathbb{A})$-initial then by Corollary 5.1.46 one obtains that $X = \prod_{i\in I} X_i \in J(j(\mathbb{A})) = l(\mathbb{A})$. □

In the above corollary if $\mathbb{A} = D(i)$ then we have the product of i-connected objects is i-connected.

In [AW75], a topological space X is T_0 if and only if $X \in r(\mathbb{B}_0)$ (disconnected subclasses), where $\mathbb{B}_0$ is the class of indiscrete topological spaces. This motivates the following definition.

Definition 5.1.48. Given an interior operator i, we say that an object $X \in \mathbb{C}$ is T_0 with respect to i if and only if $X \in r(J(i))$, that is: X is i-disconnected.

T_0 may be interpreted as a functor $T_0 = r \circ J : INT(\mathbb{C}, \mathcal{M})^{op} \to S(\mathbb{C})^{op}$.

Proposition 5.1.49. Let i be any given interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$. Then $D(i) \subseteq T_0(i)$.

Proof. Let $Y \in D(i)$. Then by Proposition 5.1.35, any morphism $f : X \to Y$ with $X \in J(i)$ must be constant. Therefore, $Y \in r(J(i)) = T_0(i)$. Consequently, the i-discrete objects are T_0 objects with respect to i. □

Examples 5.1.50. (a) Let $\mathbb{C}$ be the category **Top** with (Surjections, Embeddings)-factorization system and $R \subseteq X \in$ **Top**. Since $\mathcal{E}$ = the class of surjections is stable under pullback along monomorphisms and $\mathcal{P} = \mathbb{S}_o$, one has $j_X(X) = X$ and $j_X(R) = \emptyset$ for every proper subset R of X. Thus X is i-indiscrete if and only if $i_X \leq J_X$ if and only if $i_X(R) = \emptyset$ for every proper subset R of X if and only if every proper subset R of X is i-codense. Hence, in the category **Top**, our notions of connectedness and disconnectedness with respect to i coincides with the notions which are presented in [CR10]. Therefore, our approach generalizes the work of [CR10].

(b) Let $\mathbb{C}$ be the category **SGph** of directed spatial graphs and graph homomorphisms with the *(Surjective homorphisms, Embeddings)*-factorization system. For each directed spatial graph (G, R) and a subset $H \subseteq G$, consider the up-interior operator $\uparrow^\circ_G (H) = \{h \in H : (\forall g \in G \setminus H)$ there is no edge $h \to g\}$; then surjective homorphisms are stable under pullback along monomorphisms and $\mathcal{P} = \mathbb{S}_o :=$ class of empty graphs with empty edge and one point graphs with a loop. The discrete objects are graphs whose edges are only loops, which are dicrete graphs while a graph G is indiscrete if and only if for any $g, h \in G$ one has $g \to h$. Consequently, a graph G is $\uparrow^\circ$-connected if and only if for all $h, g \in G$ either $h \to g$ or $g \to h$.

In this section the assumption that arbitrary joins are preserved by each preimage is crucial in the development of the theory of the notions of connectedness and disconnectedness with respect to a given interior operator. It enables us to introduce the concept of coarse and fine objects with respect to a given interior operator. Consequently, the notions of connectedness and disconnectedness with respect to interior operators on $\mathbb{C}$ in a more general categorical setting are introduced in such a way that the notions generalize the work of [CR10], extending the concept to a suitable arbitrary category. On the other hand, since each preimage has both left and right adjoints, some of the results (and proofs) with respect to interior operators are analogous to that of closure operators. Indeed, this should not come as a surprise since by Theorem 2.3.8 we know that there is a natural way of moving from closure to interior operators and vice versa. Our results provide an interior-theoretic descriptions of the notions. Moreover, there are new insights and importantly some things that can only be done with interior operators.

5.2 Connectedness via partitions

In this section, by considering a lattice structure of subobjects with pseudocomplements we investigate the notion of connectedness with respect to an interior operator similar to what has been done for closure operators in [Šla09]. We will see that this notion is a direct translation from classical topology and show that it is a generalization of connectedness of topological spaces. We start the section with the following remark.

Remark 5.2.1. Recall that

(a) A subobject $m^c \in$ sub$X$ is the pseudocomplement of $m \in$ sub$X$ if it holds that $r \leq m^c \Leftrightarrow r \wedge m \cong 0_X$ for all $r \in$ sub$X$. If m^c exists, m is said to be pseudocomplemented. If sub$X$ is a Boolean algebra then the pseudocomplements are precisely the complements and

(b) for a morphism $f : X \to Y$ in $\mathbb{C}$, $m \in$ sub$X$ and $n \in$ sub$Y$ then the equation $f(m \wedge f^*(n)) \cong f(m) \wedge n$ is known as the Frobenius Reciprocity Law (FRL). One can observe that $\mathcal{E} \subseteq \mathcal{E}' \Leftrightarrow$ FRL holds for all morphisms in $\mathbb{C}$. That is, $\mathcal{E}$ is stable under pullback along $\mathcal{M}$-morphisms is equivalent to FRL holds for all morphisms in $\mathbb{C}$ (see [CGT96]).

Following [Šla09], let us now define a partition of an object $X \in \mathbb{C}$ as follows.

Definition 5.2.2. Let m be a subobject of X with pseudocomplement m^c. We call the pair (m, m^c) a partition of X. If both m and its pseudocomplement m^c are i-open then the pair (m, m^c) is called an i-open partition of X.

Consequently, we have the following two lemmas.

Lemma 5.2.3. Subobjects satisfy the FRL given in Remark 5.2.1(b).

Proof. Let $m : M \to X$ be in $\mathcal{M}$ such that $t \in$ sub$M$ and $r \in$ sub$X$ then since composition is associative in any category one has $m(t \wedge m^*(r)) \cong m \circ (t \wedge m^*(r)) \cong m \circ (t \circ t^*(m^*(r))) \cong (m \circ t) \circ t^*(m^*(r)) \cong (m \circ t) \circ (m \circ t)^*(r) \cong (m \circ t) \wedge r \cong m(t) \wedge r$. $\square$

Lemma 5.2.4. Let $m : M \to X$ be an $\mathcal{M}$ suboject. Then $m^*(-) :$ sub$X \to$ sub$M$ preserves both pseudocomplements and i-open partitions.

Proof. Let (r, r^c) be a partition of X. Then by the previous lemma FRL holds for m and by Remark 5.2.1 every morphism (in particular, each subobject) reflects the least subobject. As a consequence, $t \leq m^*(r^c) \Leftrightarrow m(t) \leq r^c \Leftrightarrow m(t) \wedge r \cong 0_X \Leftrightarrow m(t \wedge m^*(r)) \cong 0_X \Leftrightarrow t \wedge m^*(r) \cong 0_M \Leftrightarrow t \leq (m^*(r))^c$ for all $t \in \text{sub}M$. Therefore, $(m^*(r))^c \cong m^*(r^c)$. Hence, $(m^*(r), m^*(r^c))$ is a partition of M. Moreover, if (r, r^c) is an i-open partition of X then since the pullback of an i-open subobject is i-open we have $(m^*(r), m^*(r^c))$ is an i-open partition of M. □

Recall from [Šla09] a partition (m, m^c) of X is said to be trivial if $m \cong 0_X$ or $m^c \cong 0_X$.

Definition 5.2.5. A $\mathbb{C}$-object X is i-connected if X has no non-trivial i-open partition, that is, if every i-open partition of X is trivial.

Remark 5.2.6. Let $\mathbb{C}$ be the category **Top** supplied with (continuous surjections, embeddings)-factorization structures for morphisms and k^{in} be the Kuratowski interior operator. Then one can observe that k^i-connectedness is precisely the well known topological connectedness. Hence, the above definition is a natural way of extending the notion of connectedness in general topology.

The following is a generalized result on connectedness using the notion of i-dense subobject.

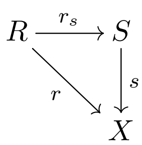
Theorem 5.2.7. *Let $m : M \to X$ be an i-dense subobject of X such that its domain M is i-connected. Then X is i-connected. That is, any object having an i-dense subobject with i-connected domain is i-connected.*

Proof. Let (r, r^c) be an i-open partition of X. Then by Lemma 5.2.4 $(m^*(r), m^*(r^c))$ is an i-open partition of M. Since M is i-connected we have that either $m^*(r) \cong 0_M$ or $m^*(r^c) \cong 0_M$. If $m^*(r) \cong 0_M$ then $m_*(m^*(r)) \cong m_*(0_M)$. Since m is i-dense subobject then $r \cong i_X(r) \leq i_X(m_*(m^*(r))) \cong i_X(m_*(0_M)) \cong 0_X$. Thus $r \cong 0_X$. Similarly, if $m^*(r^c) \cong 0_M$ then $m_*(m^*(r^c)) \cong m_*(0_M)$. Consequently, $r^c \cong i_X(r) \leq i_X(m_*(m^*(r^c))) \cong i_X(m_*(0_M)) \cong 0_X$ since m is i-dense subobject. Hence $r^c \cong 0_X$. So, either $r \cong 0_X$ or $r^c \cong 0_X$. Therefore, X has no non-trivial i-open partition, that is: X is i-connected. □

The above theorem yields a classical result on connectedness in topological spaces as a special case.

Corollary 5.2.8. Let i be a hereditary interior operator, $r : R \to X$ and $s : S \to X$ be subobjects of X such that $r \leq s$. If r is i-dense with i-connected domain R then S is i-connected.

Proof. Let $r \leq s$. Then the diagram below commutes.



Since i is hereditary and $r \cong s \circ r_s$ is i-dense with $s \in \mathcal{M}$ then by Proposition 4.2.7 we have that $r_s : R \to S$ is an i-dense subobject of S having i-connected domain R. Consequently, by Theorem 5.2.7, S is i-connected. □

The following lemma is a generalization of the Lemma 5.2.4.

Lemma 5.2.9. [Šla09] Let $\mathcal{E} \subseteq \mathcal{E}'$. Then any morphism in $\mathbb{C}$ reflects both pseudocomplements and i-open partitions.

Proof. Let $f : X \to Y$ be a morphism in $\mathbb{C}$. Then as is mentioned in the Remark 5.2.1, it reflects the least subobject. On the other hand, since $\mathcal{E}$ is stable under pullback along $\mathcal{M}$-morphisms then by Remark 5.2.1, FRL holds. Consequently, for any $m \in \text{sub}X$ one has $m \leq f^*(n^c) \Leftrightarrow f(m) \leq n^c \Leftrightarrow f(m) \wedge n \cong 0_Y \Leftrightarrow f(m \wedge f^*(n)) \cong 0_Y \Leftrightarrow m \wedge f^*(n) \cong 0_X \Leftrightarrow m \leq (f^*(n))^c$. Therefore, $(f^*(n))^c \cong f^*(n^c)$. Moreover, if (n, n^c) is an i-open partition of Y then since the pullback of an i-open subobject is i-open we have $(f^*(n), f^*(n^c))$ is an i-open partition of X. □

The following result describes preservation of connectedness under a natural condition.

Proposition 5.2.10. Let $f : X \to Y$ be a morphism in $\mathcal{E}'$ such that X is i-connected. Then Y is i-connected.

Proof. Let (n, n^c) be an i-open partition of Y. Then by Lemma 5.2.9 we obtain that $(f^*(n), f^*(n^c))$ is an i-open partition of X. Since X is i-connected we have that $f^*(n) \cong 0_X$ or $f^*(n^c) \cong 0_X$. Consequently, by Proposition 1.4.4 (c) one obtains $n \cong f_*(f^*(n)) \cong f_*(0_X) \cong 0_Y$ or $n^c \cong f_*(f^*(n^c)) \cong f_*(0_X) \cong 0_Y$. Therefore, (n, n^c) is a trivial i-open partition of Y and hence Y is i-connected. □

Corollary 5.2.11. Let $X = \prod_{i \in I} X_i$ be an i-connected product in $\mathbb{C}$ such that each projection $p_i : X \to X_i \in \mathcal{E}'$. Then each X_i is i-connected.

Hereafter, we use sub^+X to denote the class of all non trivial subobjects of X, that is: $\text{sub}^+X := \{m \in \text{sub}X : 0_X < m\}$. Following [Šla09] we have the notion of i-monotone which is described below.

Definition 5.2.12. A morphism $X \xrightarrow{f} Y$ is i-monotone if for all n in sub^+Y there exists $Q \xrightarrow{q} Y$ in sub^+Y such that $0_Y < q \leq n$ and $f^*[Q]$ is i-connected. That is, $X \xrightarrow{f} Y$ is i-monotone if for every non-zero subobject n of Y there exists a non-zero subobject q of Y which is smaller or equal to n such that the domain of the pullback of q along f is i-connected.

Lemma 5.2.13. Let $X \xrightarrow{f} Y$ be an i-monotone morphism in $\mathcal{E}'$. Then f takes an i-open partition of X to a partition of Y. Moreover, if (m, m^c) is an i-open partition of X then $m^c \cong f^*(f(m^c))$. Besides, if for each $\mathbb{C}$-object X, $\text{sub}X$ is a Boolean algebra then $m \cong f^*(f(m))$.

Proof. Let (m, m^c) be an i-open partition of X. To prove that $f(m^c) \cong f(m)^c$ we need to show that $n \wedge f(m) \cong 0_X \Leftrightarrow n \leq f(m^c)$.

($\Rightarrow$) Let $n \in \text{sub}Y$ such that $n \wedge f(m) \cong 0_Y$. Then $f^*(n) \wedge m \leq f^*(n) \wedge f^*(f(m)) \cong f^*(n \wedge f(m)) \cong f^*(0_Y) \cong 0_X$. Hence $f^*(n) \wedge m \cong 0_X$. Thus $f^*(n) \leq m^c$. As a consequence, $n \cong f(f^*(n)) \leq f(m^c)$ since $f \in \mathcal{E}'$ and $f(-)$ is order preserving.

($\Leftarrow$) Let $n \in \text{sub}Y$ such that $n \leq f(m^c)$. In order to show $n \wedge f(m) \cong 0_Y$, we use proof by contradiction. To this end, suppose $0_Y < n \wedge f(m)$. Since f is i-monotone, $\exists\, Q \xrightarrow{q} Y$ in sub^+Y such that $0_Y < q \leq n \wedge f(m)$ and $f^*[Q]$ is i-connected. Consequently, $0_Y < q \leq n \leq f(m^c)$ and $0_Y < q \leq f(m)$. Moreover, since $f \in \mathcal{E}'$, f satisfies FRL given in Remark 5.2.1(b). Hence, $0_Y < q \cong f(m^c) \wedge q \cong f(m^c \wedge f^*(q))$ and $0_Y < q \cong f(m) \wedge q \cong f(m \wedge f^*(q))$. Thus, $0_X < m^c \wedge f^*(q)$ and $0_X < m \wedge f^*(q)$. But since by Lemma 5.2.4 $((f^*(q))^*(m), (f^*(q))^*(m^c))$ is an i-open partition of i-connected $f^*[Q]$, one has $(f^*(q))^*(m) \cong 0_{f^*[Q]}$ or $(f^*(q))^*(m^c) \cong 0_{f^*[Q]}$ since $f^*[Q]$ is i-connected. Hence, $m \wedge f^*(q) \cong f^*(q) \circ (f^*(q))^*(m) \cong f^*(q) \circ 0_{f^*[Q]} \cong 0_X$ or $m^c \wedge f^*(q) \cong f^*(q) \circ (f^*(q))^*(m^c) \cong f^*(q) \circ 0_{f^*[Q]} \cong 0_X$, which is contradiction. This leads us to conclude that $n \wedge f(m) \cong 0_Y$.

Moreover, if (m, m^c) is an i-open partition of X then $f(m^c) \cong f(m)^c$, as shown above. Hence $(f(m), f(m^c))$ is a partion of Y. Consequently, $0_Y \cong f(m) \wedge f(m^c) \cong f(m \wedge f^*(f(m^c))) \Leftrightarrow m \wedge f^*(f(m^c)) \cong 0_X \Leftrightarrow f^*(f(m^c)) \leq m^c$ since f satisfies FRL given in Remark 5.2.1(b). Therefore, $m^c \cong f^*(f(m^c))$. On the other hand, if sub$X$ is a Boolean algebra then from FRL one obtains that $0_Y \cong f(m^c) \wedge f(m) \cong f(m^c \wedge f^*(f(m))) \Leftrightarrow m^c \wedge f^*(f(m)) \cong 0_X \Leftrightarrow f^*(f(m)) \leq m^{cc} \cong m$. Therefore, $m \cong f^*(f(m))$. □

Proposition 5.2.14. Let $f : X \to Y$ be an i-open and monotone morphism in $\mathcal{E}'$. Then X is i-connected if and only if Y is i-connected.

Proof. By Proposition 5.2.10 the necessary part holds. To show that the sufficiency part let (m, m^c) be an i-open partition of X. Since f is i-open then both $f(m)$ and $f(m^c)$ are i-open subobjects of Y. On the other hand, f is an i-monotone morphism in $\mathcal{E}'$ and hence by Lemma 5.2.13, $(f(m), f(m^c))$ is an i-open partion of Y. Thus, $f(m) \cong 0_Y$ or $f(m^c) \cong 0_Y$, since Y is i-connected. Consequently, we obtain $m \leq f^*(f(m)) \cong f^*(0_Y) \cong 0_X$ or $m^c \leq f^*(f(m^c)) \cong f^*(0_Y) \cong 0_X$. Hence, $m \cong 0_X$ or $m^c \cong 0_X$. Therefore, X is i-connected. □

Proposition 5.2.15. Let $f : X \to Y$ be an i-quotient and monotone morphism in $\mathcal{E}'$ and suppose for each $\mathbb{C}$-object X, sub$X$ is a Boolean algebra. Then X is i-connected if and only if Y is i-connected.

Proof. The necessary part is clear by Proposition 5.2.10. It remains to show the sufficiency part. To this end, let (m, m^c) be an i-open partition of X. Since f is an i-monotone morphism in $\mathcal{E}'$ and hence by Lemma 5.2.13, $(f(m), f(m^c))$ is a partion of Y. On the other hand, since f is an i-quotient and both $m \cong f^*(f(m))$ and $m^c \cong f^*(f(m^c))$ are i-open subobject of X we have that both $f(m)$ and $f(m^c)$ are i-open subobject of Y. Note that i-quotient morphisms reflect i-open subobjects. Thus, $(f(m), f(m^c))$ is an i-open partion of Y. Hence, $f(m) \cong 0_Y$ or $f(m^c) \cong 0_Y$, since Y is i-connected. Consequently, by Lemma 5.2.13, we obtain $m \cong f^*(f(m)) \cong f^*(0_Y) \cong 0_X$ or $m^c \cong f^*(f(m^c)) \cong f^*(0_Y) \cong 0_X$. Therefore, X is i-connected. □

Proposition 5.2.16. Suppose $f : X \to Y$ is an i-final and monotone morphism in $\mathcal{E}'$ and suppose for each $\mathbb{C}$-object X, sub$X$ is a Boolean algebra. Then X is i-connected if and only if Y is i-connected.

Proof. Similar to the proof of Proposition 5.2.15. □

Proposition 5.2.17. Let i be a hereditary interior operator, $f : X \to Y$ be an i-monotone morphism in $\mathcal{E}'$ and $\hat{f} : f^*(N) \to N$ be the restriction of f along the subobject $n : N \to Y$. Then $f^*(N)$ is i-connected if and only if N is i-connected provided that

(a) $f \in \mathcal{O}(i)$ or

(b) $f \in \mathcal{F}(i)$ and for each $\mathbb{C}$-object X, sub$X$ is a Boolean algebra.

Proof. ($\Rightarrow$) Suppose $f^*(N)$ is i-connected. Since $f \in \mathcal{E}'$ we have that $g : f^*(N) \to N$ is in $\mathcal{E}'$. Therefore, by Proposition 5.2.10, N is i-connected.

($\Rightarrow$) Suppose N is i-connected. Let $f \in \mathcal{F}(i) \cup \mathcal{O}(i)$. Since $n \in \mathcal{M}$, one has $\hat{f} \in \mathcal{F}(i) \cup \mathcal{O}(i)$ by Proposition 4.1.25. On the other hand, since f is i-monotone the pullback propery yields $\hat{f}$ is i-monotone. And also $f \in \mathcal{E}^{'}$ implies $\hat{f} \in \mathcal{E}^{'}$. Thus, $\hat{f} : f^{*}(N) \to N$ is an i-monotone morphism in $\mathcal{E}^{'} \cap (\mathcal{F}(i) \cup \mathcal{O}(i))$. Therefore, by Propositions 5.2.14 and 5.2.16, we obtain that $f^{*}(N)$ is i-connected.

$\square$

6. Compactness with Respect to an Interior operator

Inspired by the Kuratowski-Mròwka theorem (see Theorem 6.1.9) the categorical theory of compactness with respect to a closure operator was started by Manes in [Man74]. In [HSS87], Herrlich, Salicrup and Strecker studied a notion of compactness with respect to $\mathcal{M}$ in a concrete category over **Set** equipped with $(\mathcal{E}, \mathcal{M})$-factorization structure for morphisms, which is a generalization of the classical compactness for topological spaces. Afterwards, the notion of compactness of objects of an arbitrary category by using closure operators was investigated by a number of authors; see, for example, [Cas90, Cle96, CGT96, Tho99, CG05, GŠ09, Hol09], turning the Kuratowski-Mròwka theorem into a definition. In this chapter, by using the notion of interior operator we present two interior-theoretic approaches to compactness in an arbitrary category. We show that a special case of each theory produces classical results of compactness in general topology. To this end, as in the case of the previous three chapters, we work in an $\mathcal{M}$-complete category $\mathbb{C}$ equipped with $(\mathcal{E}, \mathcal{M})$-factorization structure for morphisms such that $\mathcal{M}$ is a fixed class of monomorphisms and assume that the preimage $f^*(-)$ preserves arbitrary joins for every morphism f in $\mathbb{C}$. We also assume that $\mathcal{M}$ contains all regular monomorphisms whenever it is needed and consider an interior operator i on $\mathbb{C}$ with respect to $\mathcal{M}$.

6.1 Compactness via closed morphisms

Tholen in [Tho99] then together with Clementino and Giuli in [CGT04] presented a categorical approach to topological properties such as compactness and Hausdorff separation by providing a category $\mathbb{C}$ with an additional structure given by a distinguished class of morphisms that behave almost like the class $\mathcal{K}(i)$ which is generated in Section 3.1. This approach generalizes the work of [CGT96]. In this section, by considering the suitable class $\mathcal{K}(i)$ of i-closed morphisms we first study stably i-closed morphisms. We then investigate a notion of compactness with respect to i and show that this notion behaves almost analogously to compactness via closure operators presented in [CGT96]. We conclude this section by presenting a notion of Hausdorff separation relative to i and mention a property which connects i-compact and i-Hausdorff objects. In order to do this we begin with the following fact:

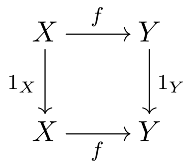
Remark 6.1.1. The pullback of an i-closed morphism need not be an i-closed morphism. Indeed, the map $\Re \to 1$, where 1 is a one-point space, is $(k^{\text{in}}-)$ closed but the pullback $\Re \times \Re \to \Re$ of this map along itself, as in

$$\begin{array}{ccc} \Re \times \Re & \longrightarrow & \Re \\ \downarrow & & \downarrow \\ \Re & \longrightarrow & 1 \end{array}$$

is not $(k^{\text{in}}-)$ closed since both projections of $\Re \times \Re$ map the closed set $S = \left\{(a, \frac{1}{a}) : a \neq 0\right\}$ to $\Re \backslash \{0\}$ which is not a closed subset of $\Re$.

Recall from Definition 4.1.27 that a morphism $f : X \to Y$ in $\mathbb{C}$ is stably closed with respect to an interior operator i (stably i-closed) if any pullback of f is i-closed. We use $\mathcal{K}(i)^*$ to denote the class of stably i-closed morphisms.

Remark 6.1.2. $\mathcal{K}(i)^*$ is a subclass of $\mathcal{K}(i)$. Indeed, the diagram below is a pullback, hence if f is stably i-closed then f is i-closed. Therefore, $\mathcal{K}(i)^* \subseteq \mathcal{K}(i)$.

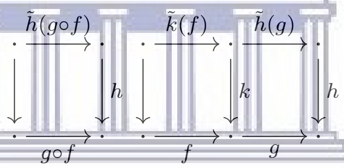


As a consequence the class $\mathcal{K}(i)^*$ satisfies the following fundamental stability properties.

Proposition 6.1.3. The class $\mathcal{K}(i)^*$

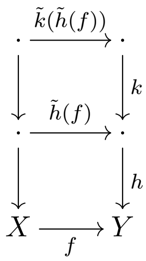
- (a) is stable under composition,
- (b) is stable under pullback,
- (c) is left-cancellable with respect to $\mathcal{M}$, that is: if $g \circ f \in \mathcal{K}(i)^*$ and $g \in \mathcal{M}$ then $f \in \mathcal{K}(i)^*$,
- (d) contains all the isomorphisms and
- (e) is right-cancellable with respect to $\mathcal{E}^*$, that is: if $g \circ f \in \mathcal{K}(i)^*$ and $f \in \mathcal{E}^*$ then $g \in \mathcal{K}(i)^*$.

Proof. (a) Let $f, g \in \mathcal{K}(i)^*$ and $\tilde{h}(g \circ f)$ be the pullback of $g \circ f$ along any h, as in the left diagram below. Then factorize this pullback diagram, as in the right diagram below



with $\tilde{h}(g)$ and $\tilde{k}(f)$ as pullbacks of g and f, respectively. Then both $\tilde{h}(g)$ and $\tilde{k}(f)$ belong to $\mathcal{K}(i)$, hence the pullback $\tilde{h}(g \circ f)$ of $g \circ f$ along h which is given by $\tilde{h}(g \circ f) \cong \tilde{h}(g) \circ \tilde{k}(f) \in \mathcal{K}(i)$ by Proposition 3.1.4(a).

(b) Let $f : X \to Y \in \mathcal{K}(i)^*$. Then the pullback of f along any morphism is an i-closed morphism. In particular, any pullback $\tilde{h}(f)$ of f along h, shown in the diagram below, is an i-closed morphism. Moreover, any pullback $\tilde{k}(\tilde{h}(f))$ of $\tilde{h}(f)$ along k, as in



is also an i-closed morphism since $\tilde{k}(\tilde{h}(f)) \cong \widetilde{h \circ k}(f)$, where $\widetilde{h \circ k}(f)$ is the pullback of f along $h \circ k$, is a pullback of f. Therefore, by Definition 4.1.27 $\tilde{h}(f) \in \mathcal{K}(i)^*$. As is explained above, one also has $\mathcal{K}(i)^* \subseteq \mathcal{K}(i)$.

(c) Let $g \circ f \in \mathcal{K}(i)^*$ and $g \in \mathcal{M}$. Then the diagram

$$\begin{array}{ccc} \cdot & \xrightarrow{1} & \cdot \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g\circ f} \\ \cdot & \xrightarrow[g]{} & \cdot \end{array}$$

is a pullback. Consequently, with (b), $f \in \mathcal{K}(i)^*$.

(d) It follows from the fact that the class Iso($\mathbb{C}$) of isomorphisms is stable under pullback and Proposition 3.1.4(c).

(e) Let $g \circ f \in \mathcal{K}(i)^*$. Let $\tilde{h}(g)$ be the pullback of g along any morphism h, as in the left diagram below. Then the right diagram below

$$\begin{array}{ccc} \cdot & \xrightarrow{\tilde{h}(g)} & \cdot \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle h} \\ \cdot & \xrightarrow[g]{} & \cdot \end{array} \qquad \begin{array}{ccccc} \cdot & \xrightarrow{\tilde{k}(f)} & \cdot & \xrightarrow{\tilde{h}(g)} & \cdot \\ \downarrow & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle h} \\ \cdot & \xrightarrow[f]{} & \cdot & \xrightarrow[g]{} & \cdot \end{array}$$

is a pullback, where $\tilde{k}(f)$ is the pullback of f along k. Then $\tilde{k}(f) \in \mathcal{E}^* \subseteq \mathcal{E}'$ and $\tilde{h}(g) \circ \tilde{k}(f) \in \mathcal{K}(i)$. Consequently, by Proposition 3.1.4(d), $\tilde{h}(g) \in \mathcal{K}(i)$.

□

We obtain the following corollary from Proposition 6.1.3.

Corollary 6.1.4. (a) Let $f = m \circ e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}^*$. $f \in \mathcal{K}(i)^*$ if and only if $m, e \in \mathcal{K}(i)^*$.

(b) If $f : X \to Y \in \mathcal{K}(i)^*$, so is the restriction $\hat{f} : f^*(N) \to N$ for all $n : N \to Y$ in sub$Y$.

Lemma 6.1.5. $f : X \to Y \in \mathcal{K}(i)^* \Leftrightarrow f \times 1_V : X \times V \to Y \times V \in \mathcal{K}(i)^*$ for all $V \in \mathbb{C}$.

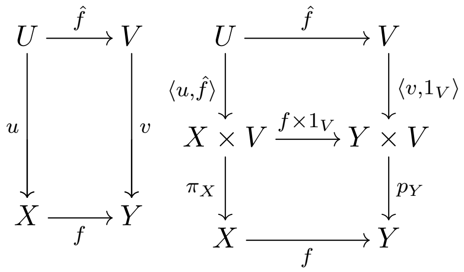
Proof. ($\Rightarrow$) Suppose $f \in \mathcal{K}(i)^*$. Using the fact that $f \times 1_V$ is a pullback of f, that is: the diagram

$$\begin{array}{ccc} X \times V & \xrightarrow{f\times 1_V} & Y \times V \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a pullback and $\mathcal{K}(i)^*$ is stable under pullback (see Proposition 6.1.3(b)), one has $f \times 1_V \in \mathcal{K}(i)^*$.

($\Leftarrow$) Suppose $f \times 1_V : X \times V \to Y \times V \in \mathcal{K}(i)^*$ for all $V \in \mathbb{C}$. Let $\hat{f} : U \to V$ be a pullback of f along $v : V \to Y$, as in the left diagram below. Then one factorizes this pullback diagram, as in the right diagram below with both the outer rectangle and the lower square pullbacks. Consequently, the upper square of the right diagram below is a pullback, hence $\hat{f}$ is a pullback of

$f \times 1_V$. Therefore, $\hat{f} \in \mathcal{K}(i)$ since $f \times 1_V$ is a stably i-closed morphism.



□

As an immediate consequence of the above lemma one has:

Remark 6.1.6. (a) $f : X \to Y \in \mathcal{K}(i)^* \Leftrightarrow 1_V \times f : V \times X \to V \times Y \in \mathcal{K}(i)^*$ for all $V \in \mathbb{C}$.

(b) $!_X : X \to 1 \in \mathcal{K}(i)^* \Leftrightarrow \pi_Y : X \times Y \to Y \in \mathcal{K}(i)^*$ for any object $Y \in \mathbb{C}$ since $1 \times Y \cong Y$ for the terminal object 1.

Proposition 6.1.7. $\mathcal{K}(i)^*$ is closed under finite products.

Proof. Let $f : X \to Y$ and $g : Z \to W$ be morphisms in $\mathcal{K}(i)^*$. Then by Lemma 6.1.5, both $1_Y \times g$ and $f \times 1_Z$ belong to $\mathcal{K}(i)^*$. Consequently, by Proposition 6.1.3(a), $f \times g = (1_Y \times g) \circ (f \times 1_Z) \in \mathcal{K}(i)^*$. Hence, $\mathcal{K}(i)^*$ is closed under binary products, that is: the product of two stably i-closed morphisms is stably i-closed morphism. Inductively, one shows the closure of $\mathcal{K}(i)^*$ under finite products. □

Recall from Corollary 4.1.25 that if $\mathcal{E}$ is stable under pullback along $\mathcal{M}$-morphisms and i is a hereditary interior operator, then any pullback of an i-closed morphism along $\mathcal{M}$-morphisms is i-closed. As a consequence, with Lemma 6.1.5, one has $f : X \to Y \in \mathcal{K}(i)^*$ if and only if $f \times 1_V : X \times V \to Y \times V \in \mathcal{K}(i)$ for all $V \in \mathbb{C}$ for a hereditary interior operator i with the class $\mathcal{E}$ which is stable under pullback along $\mathcal{M}$-morphisms; see Proposition 4.1.29. This in turn implies:

Remark 6.1.8. In **Top** with (Surjections, Embeddings)-factorization system, the stably k^{in}-closed morphisms are proper maps in the sense of [Bou66]. Indeed, k^{in} is a hereditary interior operator by Example 4.1.12(a), $\mathcal{E}$ = Surjections are stable under pullback.

Theorem 6.1.9 (Kuratowski and Mròwka). *A topological space X is compact iff the projection $p_2 : X \times Y \to Y$ is closed for all $Y \in$* **Top** *if and only if the unique map $!_X : X \to 1$, where 1 is a one-point space (the terminal object in* **Top***), is stably closed.*

This motivates the following definition:

Definition 6.1.10. An object $X \in \mathbb{C}$ is i-compact if the unique morphism $!_X : X \to 1$ of X into the terminal object 1 is a stably i-closed morphism, that is: if $!_X \in \mathcal{K}(i)^*$.

We use Comp(i) to denote the full subcategory of i-compact objects in $\mathbb{C}$.

Remark 6.1.11. $1 \in$ Comp(i). In fact, $\mathcal{K}(i)^*$ contains all isomorphisms, in particular $1 \to 1 \in \mathcal{K}(i)^*$.

As a consequence of Propositions 3.1.4 and 6.1.3 one can obtain the following characterization of i-compact objects.

Proposition 6.1.12. For any object X in $\mathbb{C}$, the following assertions are equivalent:

- (a) for any object $Y \in \mathbb{C}$, the projection $\pi_Y : X \times Y \to Y \in \mathcal{K}(i)$;
- (b) $X \in \mathsf{Comp}(i)$;
- (c) for any object $Y \in \mathbb{C}$, the projection $\pi_Y : X \times Y \to Y \in \mathcal{K}(i)^*$.

Proof. $(a) \Rightarrow (b)$: Suppose $\pi_Y : X \times Y \to Y \in \mathcal{K}(i)$ for all $Y \in \mathbb{C}$. Let $\hat{!_X} : Z \to Y$ be a pullback of $!_X$ along $!_Y$, as in the left diagram below. Then factorizing this pullback diagram, as in the right diagram below

$$\begin{array}{ccc} Z & \xrightarrow{\hat{!_X}} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle !_Y} \\ X & \xrightarrow[!_X]{} & 1 \end{array} \qquad \begin{array}{ccc} Z & \xrightarrow{\hat{!_X}} & Y \\ {\scriptstyle \langle u, \hat{!_X} \rangle}\downarrow & & \downarrow{\scriptstyle 1_Y} \\ X \times Y & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle !_Y} \\ X & \xrightarrow[!_X]{} & 1 \end{array}$$

with both the outer rectangle and the lower square pullbacks. Then the upper square of the right diagram above is a pullback, hence $\hat{!_X}$ is a pullback of π_Y. The fact that $1_Y \in \mathsf{Iso}(\mathbb{C})$ and $\mathsf{Iso}(\mathbb{C})$ is stable under pullback imply the pullback $\langle u, \hat{!_X} \rangle$ of 1_Y belongs to $\mathsf{Iso}(\mathbb{C})$, that is: $\langle u, \hat{!_X} \rangle \in \mathsf{Iso}(\mathbb{C})$. Consequently, by Proposition 3.1.4(c), $\langle u, \hat{!_X} \rangle$ is i-closed. Therefore, $\hat{!_X} = \pi_Y \circ \langle u, \hat{!_X} \rangle \in \mathcal{K}(i)$, by Proposition 3.1.4$(a)$.

$(b) \Rightarrow (c)$: Let X and $Y \in \mathbb{C}$. Since the projection $\pi_Y : X \times Y \to Y$ is the pullback of $!_X$ along $!_Y$, that is: the diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle !_Y} \\ X & \xrightarrow[!_X]{} & 1 \end{array}$$

is a pullback. Consequently, if $X \in \mathsf{Comp}(i)$ then $!_X \in \mathcal{K}(i)^*$, hence $\pi_Y : X \times Y \to Y \in \mathcal{K}(i)^*$ since $\mathcal{K}(i)^*$ is stable under pullback.

$(c) \Rightarrow (a)$: follows from Proposition 6.1.3(b). Of course, this is equivalent to (b) by Remark 6.1.6(b).

□

Remark 6.1.13. Let $\mathbb{A}$ and $\mathbb{B}$ be any two subcategories of $\mathbb{C}$. Then, the class $\mathcal{K}(i)$ induces a Galois connection $S(\mathbb{C}) \underset{p}{\overset{c}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} S(\mathbb{C})^{op}$, where $c(\mathbb{A}) = \{X \in \mathbb{C} : \ (\forall Y \in \mathbb{A})(\pi_Y : X \times Y \to Y \in \mathcal{K}(i))\}$ and $p(\mathbb{B}) = \{Y \in \mathbb{C} : \ (\forall X \in \mathbb{B})(\pi_Y : X \times Y \to Y \in \mathcal{K}(i))\}$.

As a consequence of Propositions 6.1.3 and 6.1.12, we provide a series of properties of compact objects with respect to an interior operator i. These properties generalize properties of the classical topological compactness.

Proposition 6.1.14. Let $f : X \to Y \in \mathcal{E}^*$ with $X \in \text{Comp}(i)$. Then $Y \in \text{Comp}(i)$.

Proof. Let $X \in \text{Comp}(i)$ and $f \in \mathcal{E}^*$ then $!_Y \circ f = !_X \in \mathcal{K}(i)^*$, hence as a consequence of Proposition 6.1.3(e), $!_Y \in \mathcal{K}(i)^*$. □

A direct application of the above proposition yields the following:

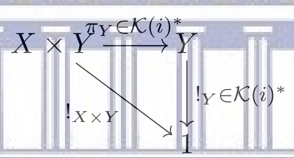
Corollary 6.1.15. Let $(X_i)_{i \in I}$ be a family of $\mathbb{C}$-objects such that each projection $p_i : \prod\limits_{j \in J} X_j \to X_i$ belongs to the class $\mathcal{E}$, which is stable under pullback and $X = \prod_{j \in J} X_j \in \text{Comp}(i)$. Then each factor X_j is compact relative to i.

Proposition 6.1.16. Comp(i) is closed under stably i-closed embeddings.

Proof. Let $m : M \to X \in \mathcal{M} \cap \mathcal{K}(i)^*$ such that $X \in \text{Comp}(i)$ then $!_X \in \mathcal{K}(i)^*$ and $!_M = !_X \circ m$ Consequently, Proposition 6.1.3(a) gives $!_M = !_X \circ m \in \mathcal{K}(i)^*$. Therefore, $M \in \text{Comp}(i)$. □

Proposition 6.1.17. Comp(i) is closed under finite products in $\mathbb{C}$.

Proof. Let $X, Y \in \text{Comp}(i)$. The fact that X is compact relative to i implies $\pi_Y : X \times Y \to Y \in \mathcal{K}(i)^*$ by Proposition 6.1.12. But since $Y \in \text{Comp}(i)$, one also has $!_Y \in \mathcal{K}(i)^*$, as in the commutative diagram

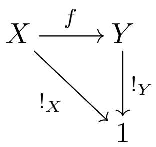


Consequently, by Proposition 6.1.3(a), $!_{X \times Y} = !_Y \circ \pi_Y \in \mathcal{K}(i)^*$. Hence, $X \times Y \in \text{Comp}(i)$, that is: Comp$(i)$ is closed under binary products. Indeed, the principle of mathematical induction produces the required result. □

The following is a characterization of compact objects with respect to an interior operator i.

Proposition 6.1.18. For $X, Y \in \mathbb{C}$, $X \in \text{Comp}(i) \Leftrightarrow \exists f : X \to Y \in \mathcal{K}(i)^*$ with $Y \in \text{Comp}(i)$.

Proof. Suppose $X \in \text{Comp}(i)$. Then $!_X : X \to 1 \in \mathcal{K}(i)^*$ with $1 \in \text{Comp}(i)$. Therefore, the required f is $!_X$ with $Y = 1 \in \text{Comp}(i)$. Conversely, if there exists $f : X \to Y \in \mathcal{K}(i)^*$ with $Y \in \text{Comp}(i)$ then the diagram below commutes with both f and $!_Y$ belong to $\mathcal{K}(i)^*$. Consequently, by Proposition 6.1.3, $!_X = !_Y \circ f \in \mathcal{K}(i)^*$. Therefore, $X \in \text{Comp}(i)$.



□

Corollary 6.1.19. (a) Let

$$\begin{array}{ccc} D & \xrightarrow{\hat{f}} & Z \\ {\scriptstyle \hat{g}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

be a pullback diagram with $f \in \mathcal{K}(i)^*$ and $Z \in \mathsf{Comp}(i)$ then $D \in \mathsf{Comp}(i)$. In particular, any fibre of $f \in \mathcal{K}(i)^*$ is i-compact, where a fibre D of f occurs in the above pullback diagram with $Z = 1$, the terminal object of $\mathbb{C}$.

(b) The preimage of any subobject n of Y with i-compact domain N under stably i-closed morphism $f : X \to Y$ is i-compact.

Proof. (a) Suppose $f \in \mathcal{K}(i)^*$. Then Proposition 6.1.3(b) yields $\hat{f} \in \mathcal{K}(i)^*$. Consequently, with the hypothesis $Z \in \mathsf{Comp}(i)$ and Proposition 6.1.18 one concludes $D \in \mathsf{Comp}(i)$. The particular case follows from the fact that $1 \in \mathsf{Comp}(i)$.

(b) The diagram below is a pullback with $f \in \mathcal{K}(i)^*$ and $N \in \mathsf{Comp}(i)$ then (a) yields $f^*[N] \in \mathsf{Comp}(i)$.

$$\begin{array}{ccc} f^*[N] & \longrightarrow & N \\ {\scriptstyle f^*(n)}\downarrow & & \downarrow{\scriptstyle n} \\ X & \xrightarrow[f]{} & Y \end{array}$$

$\square$

In what follows we study Hausdorffness (separatedness) with respect to an interior operator i on $\mathbb{C}$ with respect to $\mathcal{M}$.

Remark 6.1.20. A topological space Y is a Hausdorff (or a separated) space if the diagonal map $\delta_Y = \langle 1_Y, 1_Y \rangle : Y \to Y \times Y$ is closed if and only if δ_Y is stably k^{in}-closed.

This motivates the following:

Definition 6.1.21. An object Y of $\mathbb{C}$ is called i-Hausdorff (or i-separated) if the diagonal map $\delta_Y : Y \to Y \times Y$ is stably i-closed, that is: $\delta_Y \in \mathcal{K}(i)^*$.

In **Top**, the k^{in}-Hausdorffness yields the usual notion of Hausdorff separation. We use $\mathsf{Haus}(i)$ to denote the full subcategory of i-Hausdorff objects of $\mathbb{C}$. Trivially $1 \in \mathsf{Haus}(i)$.

Remark 6.1.22. (a) Let sub$X$ be a Boolean algebra for every $\mathbb{C}$-object X, i be a hereditary interior operator and c^i be the induced closure operator given by $c^i_X(m) = \overline{i_X(\overline{m})}$, where $\overline{m}$ denotes the complement of m. Then by Proposition 3.1.3 and Theorem 4.1.11, X is i-Hausdorff if and only if X is c^i-Hausdorff (see [CGT96]). Note that for a given closure operator c, a c-closed morphism in $\mathcal{M}$ is a c-closed subobject. Moreover, if c is hereditary (hence weakly hereditary) then a c-closed subobject is a c-closed morphism (see [CGT96]).

(b) Let $f : X \to Y$ be any morphism with $Y \in \mathsf{Haus}(i)$. Then the graph of f, $\Gamma_f = \langle 1_X, f \rangle = X \to X \times Y \in \mathcal{K}(i)^*$. Indeed, since Γ_f is the pullback $(f \times 1_Y)^*(\delta_Y)$ of the diagonal morphism δ_Y

along the morphism $f \times 1_Y$, as in

$$\begin{array}{ccc} X & \xrightarrow{\Gamma_f} & X \times Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times 1_Y} \\ Y & \xrightarrow[\delta_Y]{} & Y \times Y \end{array}$$

and $\mathcal{K}(i)^*$ is stable under pullback, one has $\Gamma_f = \langle 1_X, f\rangle \in \mathcal{K}(i)^*$. Moreover, any pullback of Γ_f belongs to $\mathcal{K}(i)^*$.

Definition 6.1.21 and Remark 6.1.22 yield:

Proposition 6.1.23. Let i be a hereditary interior operator on **Top**. Then our notion of Hausdorff separation with respect to i coincides with the notion of separation with respect to i which is studied in [CM13].

As a consequence of Definition 6.1.21 and Proposition 6.1.3 one has:

Proposition 6.1.24. (a) Let $Y \in \mathsf{Haus}(i)$. Then, $f : X \to Y \in \mathcal{K}(i)^* \Leftrightarrow f \times 1_V : X \times V \to Y \times V \in \mathcal{K}(i)$ for all $V \in \mathbb{C}$.

(b) Let $f : X \to Y \in \mathcal{K}(i)^*$ and $g : X \to Z$ be a morphism with $Z \in \mathsf{Haus}(i)$. Then $\langle f, g\rangle : X \to Y \times Z \in \mathcal{K}(i)^*$.

(c) Let $f : X \to Y$ and $h : Y \to Z$ be morphisms in $\mathbb{C}$ such that $h \circ f \in \mathcal{K}(i)^*$ and $Y \in \mathsf{Haus}(i)$. Then $f \in \mathcal{K}(i)^*$.

(d) Let $f : X \to Y \in \mathcal{K}(i)^* \cap \mathcal{E}^*$ with $X \in \mathsf{Haus}(i)$. Then $Y \in \mathsf{Haus}(i)$.

(e) If $X, Y \in \mathsf{Haus}(i)$ then $X \times Y \in \mathsf{Haus}(i)$.

(f) $\mathsf{Haus}(i)$ is closed under $\mathcal{M}$-morphisms, that is: if $m : M \to X \in \mathcal{M}$ with $X \in \mathsf{Haus}(i)$ then $M \in \mathsf{Haus}(i)$.

Proof. (a) follows from the above definition, Lemma 6.1.5, Remark 6.1.22(b) and the fact that $\mathcal{K}(i)^*$ is left cancellable with respect to $\mathcal{M}$.

(b) since $\langle f, g\rangle$ factors as $X \xrightarrow{\langle 1_X, g\rangle} X \times Z \xrightarrow{f \times 1_Z} Y \times Z$ with $\langle 1_X, g\rangle, f \times 1_Z \in \mathcal{K}(i)^*$ and $\mathcal{K}(i)^*$ is stable under composition, one has $\langle f, g\rangle : X \to Y \times Z \in \mathcal{K}(i)^*$.

(c) Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Gamma_f} & X \times Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h \circ f \times 1_Y} \\ Y & \xrightarrow[\langle h, 1_Y\rangle]{} & Z \times Y \end{array}$$

Then, since $h \circ f \in \mathcal{K}(i)^*$, one has $h \circ f \times 1_Y \in \mathcal{K}(i)^*$ by Lemma 6.1.5, and since $Y \in \mathsf{Haus}(i)$, one has $\Gamma_f \in \mathcal{K}(i)^*$ by Remark 6.1.22(b). Consequently, with Proposition 6.1.3(a) one obtains $\langle h, 1_Y\rangle \circ f = h \circ f \times 1_Y \circ \Gamma_f \in \mathcal{K}(i)^*$. Therefore, by Proposition 6.1.3(c), $f \in \mathcal{K}(i)^*$ since $\langle h, 1_Y\rangle \in \mathcal{M}$, as it is a section (hence a regular monomorphism).

(*d*) Let $f : X \to Y \in \mathcal{K}(i)^*$ and $X \in \mathsf{Haus}(i)$. One has a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \delta_X}\downarrow & & \downarrow{\scriptstyle \delta_Y} \\ X \times X & \xrightarrow[f\times f]{} & Y \times Y \end{array}$$

with $\delta_X \in \mathcal{K}(i)^*$ and $f \times f \in \mathcal{K}(i)^*$ (by Proposition 6.1.7). Consequently, Proposition 6.1.3(a) yields $\delta_Y \circ f = f \times f \circ \delta_X \in \mathcal{K}(i)^*$. Hence, by Proposition 6.1.3(e), $\delta_Y \in \mathcal{K}(i)^*$ since $f \in \mathcal{E}^*$.

(*e*) is a consequence of $\delta_{X\times Y} \cong \delta_X \times \delta_Y$ and Proposition 6.1.7.

(*f*) is a consequence of the stability of $\mathcal{K}(i)^*$ under composition and $\delta_M : M \to M \times M$ is a pullback of $\Gamma_m : M \to M \times X$ and hence a pullback of $\delta_X : X \to X \times X$.

□

The following is a property which relates i-compact objects to i-Hausdorff objects.

Proposition 6.1.25. For X i-compact and Y i-Hausdorff, every morphism $f : X \to Y \in \mathcal{K}(i)^*$.

Proof. Since f factors as $X \xrightarrow{\langle 1_X, f\rangle} X \times Y \xrightarrow{\pi_Y} Y$ with $\langle 1_X, f\rangle, \pi_Y \in \mathcal{K}(i)^*$. Consequently, Proposition 6.1.3(a) gives $f \in \mathcal{K}(i)^*$. □

As a consequence of the above Proposition and Proposition 6.1.3(a) one obtains:

Corollary 6.1.26. Let $Y \in \mathsf{Comp}(i) \cap \mathsf{Haus}(i)$. Then $f : X \to Y \in \mathcal{K}(i)^* \Leftrightarrow X \in \mathsf{Comp}(i)$.

Remark 6.1.27. (*a*) Due to the Kuratowski and Mròwka theorem, for $\mathbb{C} =$ **Top**, the k^{in}-compact objects are the usual compact spaces.

(*b*) Let sub$X$ be a Boolean algebra for every $\mathbb{C}$-object X, i be an interior operator and c^i be the induced closure operator given by $c^i_X(m) = \overline{i_X(\overline{m})}$, where $\overline{m}$ denotes the complement of m. Then by Proposition 3.1.3, X is compact relative to i if and only if X is compact with respect to c^i (see [CGT96]).

(*c*) We may deduce that i-compactness is well defined since its basic properties are shown to be similar to the classical compactness of topological spaces.

In this section the assumption that arbitrary joins are preserved by each preimage is essential in the development of the theory of the notions of compactness and separatedness with respect to a given interior operator. It allows us to introduce stably closed morphisms with respect to a given interior operator. Consequently, the notions of compactness and separatedness with respect to a given interior operator on $\mathbb{C}$ are introduced. Our notion of separatedness coincides with the one given in [CM13] for a given hereditary interior operator. On the other hand, since each preimage has both left and right adjoints, some of the results (and proofs) with respect to interior operators are analogous to that of closure operators. Indeed, this should not come as a surprise, as observed before. In fact, some of our results are specific cases of [CGT04]. Our results provide an interior-theoretic description of the notions. Moreover, there are new insights and importantly some things that can only be done with interior operators.

6.2 Compactness via covers

In this section, following the Borel-Lebesgue definition of compact spaces we introduce the notion of compactness with respect to an interior operator, similarly to what has been done for closure operators in [Cle96].

Definition 6.2.1. An object X of a category $\mathbb{C}$ is said to be Borel-Lebesgue compact with respect to an interior operator i if for every family $(r_k)_{k\in K}$ of subobjects of X such that $\bigvee_{k\in K} i_X(r_k) \cong i_X(1_X)$, there exists a finite subset J of K such that $i_X(1_X) \leq \bigvee_{j\in J} r_j$.

Since $r_k \leq 1_X$ for all $k \in K$, one has $i_X(r_k) \leq i_X(1_X)$ for all $k \in K$, hence $\bigvee_{k\in K} i_X(r_k) \leq i_X(1_X)$. Consequently, we can replace $\bigvee_{k\in K} i_X(r_k) \cong i_X(1_X)$ by $i_X(1_X) \leq \bigvee_{k\in K} i_X(r_k)$ in the above definition. We also note that, a concept of Borel-Lebesgue compact relative to an interior operator generalizes the classical notion of compactness for topological spaces, since if $\mathbb{C} =$ **Top** and i is the interior operator induced by the topology then the Borel-Lebesgue compact objects are the compact topological spaces.

Remark 6.2.2. An object $X \in \mathbb{C}$ is Borel-Lebesgue compact with respect to a standard interior operator i if for every family $(r_k)_{k\in K}$ of subobjects of X such that $1_X \leq \bigvee_{k\in K} i_X(r_k)$ (or $1_X \cong \bigvee_{k\in K} i_X(r_k)$), there exists a finite subset J of K such that $1_X \leq \bigvee_{j\in J} r_j$ (or $1_X \cong \bigvee_{j\in J} r_j$).

We use BLComp(i) to denote the subcategory of Borel-Lebesgue compact objects with respect to an interior operator i of the category $\mathbb{C}$. In **Top**, this notion of compactness coincides with the notion of compactness via closed morphisms.

Proposition 6.2.3. (Image of compact space is compact). Let i be a standard interior operator and $f : X \to Y \in \mathcal{E}'$. If $X \in$ BLComp(i) then so is Y.

Proof. Suppose $X \in$ BLComp(i). Let $(n_k)_{k\in K}$ be a family of subobjects of Y such that $\bigvee_{k\in K} i_Y(n_k) \cong i_Y(1_Y)$. Then the fact that i is standard and f^* commutes with the join of subobjects implies $i_X(1_X) \cong 1_X \cong f^*(1_Y) \cong f^*(i_Y(1_Y)) \cong f^*(\bigvee_{k\in K} i_Y(n_k)) \cong \bigvee_{k\in K} f^*(i_Y(n_k)) \leq \bigvee_{k\in K} i_X(f^*(n_k))$. Hence, $(f^*(n_k))_{k\in K}$ is a family of subobjects of X such that $\bigvee_{k\in K} i_X(f^*(n_k)) \cong i_X(1_X)$. But since $X \in$ BLComp(i) we have that there exists a finite subset J of K such that $i_X(1_X) \leq \bigvee_{j\in J} f^*(n_j)$. Consequently, the fact that $f \in \mathcal{E}'$ implies $i_Y(1_Y) \cong 1_Y \cong f_*(1_X) \cong f_*(i_X(1_X)) \leq f_*(\bigvee_{j\in J} f^*(n_j)) \cong f_*(f^*(\bigvee_{j\in J} n_j))) \cong \bigvee_{j\in J} n_j$. □

As an immediate consequence of Proposition 6.2.3, we obtain the following fact:

Corollary 6.2.4. Let i be a standard interior operator and $X = \prod_{i\in I} X_i$ be a product in $\mathbb{C}$ such that each projection $p_i : X \to X_i \in \mathcal{E}'$. If $X \in$ BLComp(i) then each $X_i \in$ BLComp(i).

As a generalization of Proposition 6.2.3, one has the following.

Proposition 6.2.5. Let i be a standard interior operator and $f : X \to Y$ be any morphism in $\mathbb{C}$ such that $f^*(n) \cong 1_X \Leftrightarrow n \cong 1_Y$. If $X \in \mathsf{BLComp}(i)$ then so is Y.

Proof. Let $(n_k)_{k\in K}$ be a family of subobjects of Y such that $\bigvee_{k\in K} i_Y(n_k) \cong i_Y(1_Y)$. Since i is standard and f^* commutes with the join of subobjects, one has $i_X(1_X) \cong 1_X \cong f^*(1_Y) \cong f^*(i_Y(1_Y)) \cong f^*(\bigvee_{k\in K} i_Y(n_k)) \cong \bigvee_{k\in K} f^*(i_Y(n_k)) \leq \bigvee_{k\in K} i_X(f^*(n_k))$. Hence, $(f^*(n_k))_{k\in K}$ is a family of subobjects of X such that $\bigvee_{k\in K} i_X(f^*(n_k)) \cong i_X(1_X)$. But since X belongs to $\mathsf{BLComp}(i)$ we have that there exists a finite subset J of K such that $1_X \cong i_X(1_X) \leq \bigvee_{j\in J} f^*(n_j)$. Consequently, $1_X \cong \bigvee_{j\in J} f^*(n_j)) \cong f^*(\bigvee_{j\in J} n_j)$. This together with the hypothesis $f^*(n) \cong 1_X \Leftrightarrow n \cong 1_Y$ for all $n \in \mathsf{sub}Y$ yields $\bigvee_{j\in J} n_j \cong 1_Y \cong i_Y(1_Y)$. □

The following is a converse of Proposition 6.2.3.

Proposition 6.2.6. Let i be a standard interior operator and $f : X \to Y \in \mathcal{I}(i) \cap \mathcal{E}'$. If $Y \in \mathsf{BLComp}(i)$ then so is X.

Proof. Suppose $Y \in \mathsf{BLComp}(i)$. Let $(r_k)_{k\in K}$ be a family of subobjects of X such that $i_X(1_X) \cong \bigvee_{k\in K} i_X(r_k)$. Then

$$
\begin{aligned}
i_Y(1_Y) \cong 1_Y \cong f_*(1_X) \cong f_*(i_X(1_X)) &\cong f_*(\bigvee_{k\in K} i_X(r_k)) && (i \text{ standard})\\
&\cong f_*(\bigvee_{k\in K} f^*(i_Y(f_*(r_k)))) && (f \in \mathcal{I}(i))\\
&\cong f_*(f^*(\bigvee_{k\in K} i_Y(f_*(r_k)))) && (f^* \text{commutes with the joins})\\
&\cong \bigvee_{k\in K} i_Y(f_*(r_k)) && (f \in \mathcal{E}').
\end{aligned}
$$

Thus $(f_*(r_k))_{k\in K}$ is a family of subobjects of Y such that $i_Y(1_Y) \cong \bigvee_{k\in K} i_Y(f_*(r_k))$. Consequently, compactness of Y implies there exists a finite subset J of K such that $i_Y(1_Y) \leq \bigvee_{j\in J} f_*(r_j)$. Hence, $i_X(1_X) \cong 1_X \cong f^*(1_Y) \cong f^*(i_Y(1_Y)) \leq f^*(\bigvee_{j\in J} f_*(r_j)) \cong \bigvee_{j\in J} f^*(f_*(r_j)) \leq \bigvee_{j\in J} r_j$. □

Next we show that the domain of a closed embedding of a compact object is compact. For that we first prove the following lemma.

Lemma 6.2.7. Let $\mathsf{sub}X$ be a Boolean algebra for each object $X \in \mathbb{C}$ and $f \in \mathcal{M}$ then the right adjoint f_* of f^* commutes with the join of subobjects.

Proof. Let $(r_k)_{k\in K}$ be a family of subobjects of X. Then by the fact that $f \in \mathcal{M}$ and Lemma 1.4.7 (b) one obtains: $f_*\left(\bigvee_{k\in K} r_k\right) \cong \overline{f\left(\overline{\bigvee_{k\in K} r_k}\right)} \cong \overline{f\left(\bigwedge_{k\in K} \overline{r_k}\right)} \cong \overline{\bigwedge_{k\in K} f(\overline{r_k})} \cong \bigvee_{k\in K} \overline{f(\overline{r_k})} \cong \bigvee_{k\in K} f_*(r_k)$. □

Proposition 6.2.8. (Closed subspace of a compact space is compact). Let sub$X$ be a Boolean algebra for each object $X \in \mathbb{C}$ and $r : R \to X$ be a closed morphism in $\mathcal{M}$ and i be an interior operator. If $X \in \mathsf{BLComp}(i)$ then so is R.

Proof. Let $(r_k)_{k\in K}$ be a family of subobjects of R such that $i_R(1_R) \cong \bigvee_{k\in K} i_R(r_k)$. Then

$$\begin{aligned} i_X(1_X) \cong i_X(r_*(1_R)) &\leq r_*(i_R(1_R)) && (r\ i\text{-continuous}) \\ &\cong r_*\left(\bigvee_{k\in K} i_R(r_k)\right) \\ &\cong \bigvee_{k\in K} r_*(i_R(r_k)) && (\text{Lemma } 6.2.7) \\ &\cong \bigvee_{k\in K} i_X(r_*(r_k)) && (r \in \mathcal{K}(i)). \end{aligned}$$

But since $X \in \mathsf{BLComp}(i)$, there exists $J \subseteq K$ finite such that $i_X(1_X) \leq \bigvee_{j\in J} r_*(r_j)$. Consequently, with Proposition 3.1.36 one has $i_R(1_R) \cong r^*(i_X(r_*(1_R))) \cong r^*(i_X(1_X)) \leq r^*(\bigvee_{j\in J} r_*(r_j)) \cong \bigvee_{j\in J} r^*(r_*(r_j)) \leq \bigvee_{j\in J} r_j$. Therefore, $R \in \mathsf{BLComp}(i)$. □

Examples 6.2.9. (a) Let $\mathbb{C}$ be the category **Top** of topological spaces and continuous functions with (Surjections, Embeddings)-factorization system.

(i) The Borel-Lebesgue compact objects with respect to the Kuratowski interior operator k^{in} are the compact spaces. Indeed, for every family $(R_i)_{i\in I}$ of subspaces of a topological space X such that $\bigcup_{i\in I} k^{\mathsf{in}}_X(R_i) = X$ we have a family $(k^{\mathsf{in}}_X(R_i))_{i\in I}$ of open subspaces which covers X. Consequently, the fact that X is compact implies there is a finite subcover of X. That is, there is a finite subset J of I such that $X = \bigcup_{j\in J} k^{\mathsf{in}}_X(R_j) \subseteq \bigcup_{j\in J} R_j$.

(ii) The Borel-Lebesgue compact objects with respect to the discrete interior operator d^{in} are the finite topological spaces.

(iii) Any topological space X is a Borel-Lebesgue compact object with respect to the trivial interior operator t^{in}.

(iv) The Borel-Lebesgue compact objects with respect to b^{in}-interior are topological spaces in which the topology induced by the b^{in} is compact.

(b) Let $\mathbb{C}$ be the category **PreTop** of pretopological spaces and continuous functions with (Surjections, Embeddings)-factorization system. Then the Borel-Lebesgue compact objects with respect to the Čech interior operator $\check{\mathsf{c}}^{\mathsf{in}}$ are precisely the compact pretopological spaces.

(c) Let $\mathbb{C}$ be the category **Loc** of locales with (onto localic maps, one to one localic maps)-factorization system. Then the Borel-Lebesgue compact objects with respect to the interior operator which assigns to each sublocale the largest open sublocale contained in it are the compact locales.

The notion of compactness via covers is trivial in all abelian categories (such as the category of modules over a commutative ring, the category of all abelian groups, the category of vector spaces over a field). Indeed, such categories only have the discrete interior operator (see Chapeter 2) and the Borel-Lebesgue compact objects with respect to this operator are the finite ones.

References

[AH19a] F. Assfaw and D. Holgate. Hereditary interior operators. *Topology Appl. (under review)*, 2019.

[AH19b] F. Assfaw and D. Holgate. Openness and codenseness with respect to an interior operator. *In preparation*, 2019.

[AHS90] J. Adámek, H. Herrlich, and G. Strecker. *Abstract and concrete categories*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1990.

[AW75] A. V. Arkhangel'skiĭ and R. Wiegandt. Connectednesses and disconnectednesses in topology. *General Topology and Appl.*, 5:9–33, 1975.

[Bir37] G. Birkhoff. The meaning of completeness. *Ann. of Math. (2)*, 38(1):57–60, 1937.

[Bir40] G. Birkhoff. *Lattice Theory*. American Mathematical Society, New York, 1940.

[Bou66] N. Bourbaki. *Elements of mathematics. General topology. Part 1.* Hermann, Paris; Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1966.

[Cas86] G. Castellini. Closure operators, monomorphisms and epimorphisms in categories of groups. *Cahiers Topologie Géom. Différentielle Catég.*, 27(2):151–167, 1986.

[Cas90] G. Castellini. Compact objects, surjectivity of epimorphisms and compactifications. *Cahiers Topologie Géom. Différentielle Catég.*, 31(1):53–65, 1990.

[Cas01] G. Castellini. Connectedness with respect to a closure operator. *Appl. Categ. Structures*, 9(3):285–302, 2001.

[Cas03] G. Castellini. *Categorical closure operators*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 2003.

[Cas11] G. Castellini. Interior operators in a category: idempotency and heredity. *Topology Appl.*, 158(17):2332–2339, 2011.

[Cas15] G. Castellini. Interior operators, open morphisms and the preservation property. *Appl. Categ. Structures*, 23(3):311–322, 2015.

[Cas16] G. Castellini. Some remarks on interior operators and the functional property. *Quaest. Math.*, 39(2):275–287, 2016.

[Čec37] E. Čech. On bicompact spaces. *Ann. of Math. (2)*, 38(4):823–844, 1937.

[CG05] G. Castellini and E. Giuli. U-closure operators and compactness. *Appl. Categ. Structures*, 13(5-6):453–467, 2005.

[CGT96] M.M. Clementino, E. Giuli, and W. Tholen. Topology in a category: compactness. *Portugal. Math.*, 53(4):397–433, 1996.

[CGT01] M.M. Clementino, E. Giuli, and W. Tholen. What is a quotient map with respect to a closure operator? *Appl. Categ. Structures*, 9(2):139–151, 2001.

[CGT04] M.M. Clementino, E. Giuli, and W. Tholen. A functional approach to general topology. In *Categorical foundations*, volume 97 of *Encyclopedia Math. Appl.*, pages 103–163. Cambridge Univ. Press, Cambridge, 2004.

[CH94] G. Castellini and D. Hajek. Closure operators and connectedness. *Topology Appl.*, 55(1):29–45, 1994.

[CH03a] G. Castellini and D. Holgate. Closure operator constructions depending on one parameter. *Quaest. Math.*, 26(3):289–305, 2003.

[CH03b] G. Castellini and D. Holgate. A link between two connectedness notions. *Appl. Categ. Structures*, 11(5):473–486, 2003.

[Cle95] M.M. Clementino. Constant morphisms and constant subcategories. *Appl. Categ. Structures*, 3(2):119–137, 1995.

[Cle96] M.M. Clementino. On categorical notions of compact objects. *Appl. Categ. Structures*, 4(1):15–29, 1996.

[Cle01] M.M. Clementino. On connectedness via closure operators. *Appl. Categ. Structures*, 9(6):539–556, 2001.

[CM13] G. Castellini and E. Murcia. Interior operators and topological separation. *Topology Appl.*, 160(12):1476–1485, 2013.

[CR10] G. Castellini and J. Ramos. Interior operators and topological connectedness. *Quaest. Math.*, 33(3):290–304, 2010.

[Csá63] Á. Császár. *Foundations of general topology*. A Pergamon Press Book. The Macmillan Co., New York, 1963.

[CT97] M.M. Clementino and W. Tholen. Separation versus connectedness. *Topology Appl.*, 75(2):143–181, 1997.

[DF04] D.S. Dummit and R.M. Foote. *Abstract algebra*. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2004.

[DG84] D. Dikranjan and E. Giuli. Epimorphisms and cowellpoweredness of epireflective subcategories of Top. In *Proceedings of the 12th winter school on abstract analysis (Srní, 1984)*, number Suppl. 6, pages 121–136, 1984.

[DG85] D. Dikranjan and E. Giuli. Closure operators induced by topological epireflections. In *Topology, theory and applications (Eger, 1983)*, volume 41 of *Colloq. Math. Soc. János Bolyai*, pages 233–246. North-Holland, Amsterdam, 1985.

[DG87] D. Dikranjan and E. Giuli. Closure operators. I. In *Proceedings of the 8th international conference on categorical topology (L'Aquila, 1986)*, volume 27, pages 129–143, 1987.

[DGT89] D. Dikranjan, E. Giuli, and W. Tholen. Closure operators. II. In *Categorical topology and its relation to analysis, algebra and combinatorics (Prague, 1988)*, pages 297–335. World Sci. Publ., Teaneck, NJ, 1989.

[Dik92] D. Dikranjan. Semiregular closure operators and epimorphisms in topological categories. *Rend. Circ. Mat. Palermo (2) Suppl.*, (29):105–160, 1992.

[DK00] D. Dikranjan and H.-P. Künzi. Separation and epimorphisms in quasi-uniform spaces. *Appl. Categ. Structures*, 8(1-2):175–207, 2000.

[DP02] B.A. Davey and H.A. Priestley. *Introduction to lattices and order*. Cambridge University Press, New York, second edition, 2002.

[DT95] D. Dikranjan and W. Tholen. *Categorical structure of closure operators*, volume 346 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1995.

[DT15] D. Dikranjan and W. Tholen. Dual closure operators and their applications. *J. Algebra*, 439:373–416, 2015.

[Eng89] R. Engelking. *General topology*, volume 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Berlin, second edition, 1989.

[FL82] P. Fletcher and W. F. Lindgren. *Quasi-uniform spaces*, volume 77 of *Lecture Notes in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1982.

[GŠ05] E. Giuli and J. Šlapal. Raster convergence with respect to a closure operator. *Cah. Topol. Géom. Différ. Catég.*, 46(4):275–300, 2005.

[GŠ09] E. Giuli and J. Šlapal. Neighborhoods with respect to a categorical closure operator. *Acta Math. Hungar.*, 124(1-2):1–14, 2009.

[GT00] E. Giuli and W. Tholen. Openness with respect to a closure operator. *Appl. Categ. Structures*, 8(3):487–502, 2000.

[Her22] P. Hertz. Über axiomensysteme für beliebige satzsysteme. *Mathematische Annalen*, 87(3-4):246–269, 1922.

[Her68] H. Herrlich. *Topologische Reflexionen und Coreflexionen*. Lecture Notes in Mathematics, No. 78. Springer-Verlag, Berlin-New York, 1968.

[HIR16] D. Holgate, M. Iragi, and A. Razafindrakoto. Topogenous and nearness structures on categories. *Appl. Categ. Structures*, 24(5):447–455, 2016.

[Hol09] D. Holgate. A generalisation of the functional approach to compactness. *Topology Appl.*, 156(12):2101–2108, 2009.

[HS68] H. Herrlich and G. E. Strecker. H-closed spaces and reflective subcategories. *Math. Ann.*, 177:302–309, 1968.

[HŠ11] D. Holgate and J. Šlapal. Categorical neighborhood operators. *Topology Appl.*, 158(17):2356–2365, 2011.

[HŠ18] D. Holgate and J. Šlapal. Closure, interior and neighbourhood in a category. *Hacettepe Journal of Mathematics and Statistics*, 47(6):1512–1520, 2018.

[HSS87] H. Herrlich, G. Salicrup, and G.E. Strecker. Factorizations, denseness, separation, and relatively compact objects. In *Proceedings of the 8th international conference on categorical topology (L'Aquila, 1986)*, volume 27, pages 157–169, 1987.

[Ira16] M. Iragi. Topogenous structures on categories. *Master thesis, University of the Western Cape*, 2016.

[Kao83] K. S. Kao. A note on M_1-spaces. *Pacific J. Math.*, 108(1):121–128, 1983.

[Kim98] J. W. Kim. A note on quasi-open maps. *J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math.*, 5(1):1–3, 1998.

[Kur22] C. Kuratowski. Sur l'opération a de l'analysis situs. *Fundamenta Mathematicae*, 3(1):182–199, 1922.

[LTOC11] J. Luna-Torres and C. O. Ochoa C. Interior operators and topological categories. *Adv. Appl. Math. Sci.*, 10(2):189–206, 2011.

[Man74] E.G. Manes. Compact Hausdorff objects. *General Topology and Appl.*, 4:341–360, 1974.

[Moo09] E. H. Moore. *On a form of general analysis with applications to linear differential and integral equations*. Tipografia della R. Accademia dei Lincei, proprietà del cav. V. Salviucci, 1909.

[MP62] S. Mardešić and P. Papić. Continuous images of ordered compacta, the Suslin property and diadic compacta. *Glasnik Mat.-Fiz. Astronom. Drukštvo Mat. Fiz. Hrvatske SER. II*, 17:3–25 (1963), 1962.

[Pie72] R.S. Pierce. Closure spaces with applications to ring theory. pages 565–616. Lecture Notes in Math., Vol. 246, 1972.

[Pre71] G. Preuß. Eine Galois-Korrespondenz in der Topologie. *Monatsh. Math.*, 75:447–452, 1971.

[Pre78] G. Preuß. Relative connectednesses and disconnectednesses in topological categories. *Quaestiones Math.*, 2(1-3):297–306, 1977/78.

[Raz12] A. Razafindrakoto. On coarse and fine neighbourhood operators. *Topology Appl.*, 159(13):3067–3079, 2012.

[Raz13] A. Razafindrakoto. *Neighbourhood operators on categories*. PhD thesis, Stellenbosch: Stellenbosch University, 2013.

[RH14] A. Razafindrakoto and D. Holgate. Interior and neighbourhood. *Topology Appl.*, 168:144–152, 2014.

[RH17] A. Razafindrakoto and D. Holgate. A lax approach to neighbourhood operators. *Appl. Categ. Structures*, 25(3):431–445, 2017.

[Rie09] F. Riesz. *Stetigkeitsbegriff und abstrakte Mengenlehre*. Tip. della R. Accademia dei Lincei, 1909.

[Sal76] S. Salbany. Reflective subcategories and closure operators. pages 548–565. Lecture Notes in Math., Vol. 540, 1976.

[Šla08] J. Šlapal. Convergence on categories. *Appl. Categ. Structures*, 16(4):503–519, 2008.

[Šla09] J. Šlapal. Another approach to connectedness with respect to a closure operator. *Appl. Categ. Structures*, 17(6):603–612, 2009.

[Šla11] J. Šlapal. Neighborhoods and convergence with respect to a closure operator. *Math. Slovaca*, 61(5):717–732, 2011.

[Tar29] A. Tarski. Les fondements de la géométrie des corps. *Ksiega Pamiatkowa Pierwszego Polskiego Zjazdu Matematycznego*, pages 29–33, 1929.

[Tho99] W. Tholen. A categorical guide to separation, compactness and perfectness. *Homology Homotopy Appl.*, 1:147–161, 1999.

[Vor00] S.J.R. Vorster. Interior operators in general categories. *Quaest. Math.*, 23(4):405–416, 2000.