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Abstract

Due to bond prices pull-to-par, zero coupon bonds historical returns are not stationary, as they tend to zero as time to maturity approaches. Given that the historical simulation method for computing Value at Risk (VaR) requires a stationary sequence of historical returns, zero coupon bonds historical returns can not be used to compute VaR by historical simulation. Their use would systematically overestimate VaR, resulting in invalid VaR sequences. In this paper we propose an adjustment of zero coupon bonds historical returns. We call the adjusted returns "pulled-to-par" returns. We prove that when the zero coupon bonds continuously compounded yields to maturity are stationary the adjusted pulled-to-par returns allow VaR computation by historical simulation. We first illustrate the VaR computation in a simulation scenario, then we apply it to real data on euro zone STRIPS.

1 Introduction

According to Basel II, although banks can develop their own internal models, Value at Risk (VaR) is still the minimum standard [Basel II(2006), 195]. This regulatory framework is expected to be in use until the end of 2019, time when the 2016 revised standard implementation [Basel II(2016), 4] is expected to replace it. Among the VaR models based on variance-covariance matrices, historical simulation, or Monte Carlo simulation, no particular type of model is prescribed. The survey [Mehta et al.(2012), 4] refers that 75 percent of banks use historical simulation. For a comprehensive literature review on VaR methodologies, strengths and limitations, we refer to [Abad, Benito and López(2014)].

Zero coupon bonds historical returns are not stationary, as they tend to zero as the time to maturity approaches – the so-called *pull-to-par effect*. Returns convergence to zero result from bond prices convergence to their par value at maturity[1]. Without this convergence bonds would mature at a price different from their payoff at maturity, leading to arbitrage opportunities[2], which is believed that does not happens in efficient markets.

The pull-to-par of bond prices is, thus, the key factor that distinguishes the dynamics of bond prices. Given the non-stationarity of zero coupon bonds historical returns, they cannot be used to compute VaR by historical simulation, because this method requires stationary historical returns.

In this paper we propose an *adjustment* of zero coupon bonds historical returns that allows computing VaR by historical simulation. The aim of our proposal is to compute VaR by historical simulation of portfolios with zero coupon bonds, keeping the same level of simplicity the historical simulation method allows for portfolios with stocks. The underlying ideas of the adjustment were first described in [Sousa et al.(2014)]. Intuitively, all historical returns are pulled to the dates relevant to the VaR computations. The goal of such adjustment is to correct the pull-to-par effect of bond prices while preserving historical market movements.

In our main theoretical results (Section 3), we prove that the proposed method applies whenever zero coupon bonds continuously compounded yields to maturity are stationary. We then illustrate the pull-to-par VaR computations and backtest them in a simulation scenario (Section 4). Finally, we apply the proposed method to euro bond STRIPS (Section 5).

2 Value at risk and historical simulation

Consider, the time instant t, an asset with value V_t, at time t, and an holding period Δ. The asset profit or loss $PL_{t+\Delta}$, at time $t+\Delta$, over the holding period Δ, is given by:

$$PL_{t+\Delta} = V_{t+\Delta} - V_t = V_t \left(\frac{V_{t+\Delta} - V_t}{V_t} \right)$$

where

$$\frac{V_{t+\Delta} - V_t}{V_t} = \frac{V_{t+\Delta}}{V_t} - 1 \tag{1}$$

is the asset return at time $t+\Delta$ over the period Δ.

[1]The bond par value is also known as the face value. For more details on the pull-to-par effect, see for instance [Fabozzi(2004), 50]

[2]For a formal definition of arbitrage see [Björk(2004), 92].

The VaR, at time t, with confidence level α and horizon Δ, is the loss L that is not exceeded with probability α, when holding the asset over the period Δ. Formally,

$$\mathbb{P}(PL_{t+\Delta} < L) = 1 - \alpha \tag{2}$$

where $\mathbb{P}$ denotes probability. The VaR value L is the $1 - \alpha$ quantile of the profits and losses distribution.

In the historical simulation method [Dowd(2007)] the VaR value L, in Equation (2), is computed over a distribution of simulated profits and losses. Each non-overlapping historical return is applied to the current asset value to build a simulated profit or loss, $\widetilde{PL}$. Denoting by s an historical time, $s \leq t - \Delta$

$$\widetilde{PL}_{t+\Delta} = V_t \left(\frac{V_{s+\Delta}}{V_s} - 1 \right) \tag{3}$$

The assumption is that the profits or losses process is stationary. Under this assumption the empirical distribution of simulated profits or losses converges to the real distribution of profits and losses [McNeil, Frey and Embrechts(2005), 50]. Therefore, the VaR value L, obtained from (2), is the $1 - \alpha$ quantile of the simulated profits and losses distribution.

In case of portfolios with several assets, the synchronized simulated profits or losses of all portfolio assets are added to obtain the simulated portfolio profits or losses distribution. The VaR value is obtained from the $1 - \alpha$ quantile of the portfolio simulated profits and losses distribution.

The problem with fixed income assets is that the stationarity assumption does not hold, by definition. In the following we focus on zero coupon bond prices, as they are key for valuation of any fixed income instrument.

2.1 Zero coupon bonds historical returns

Consider the market price $p(t,T)$, at time $t \leq T$, of a zero coupon bond paying 1 at maturity T. The pull-to-par convergence of the bond price to the par value ensures that at time T we have that $p(T,T) = 1$. Consider also holding the bond quantity q over the period Δ. Given that the VaR is computed at time t, the zero coupon bond historical return, at time $s + \Delta < t$, over the period Δ, is given by

$$\frac{V_{s+\Delta}}{V_s} - 1 = \frac{q\, p(s+\Delta,T)}{q\, p(s,T)} - 1 = \frac{p(s+\Delta,T)}{p(s,T)} - 1. \tag{4}$$

Zero coupon bonds historical returns do incorporate historical market movements that should be considered in VaR computation. However, due to bond prices pull-to-par they are not stationary as they tend to zero as time to maturity approaches. Therefore the returns at time $t > s + \Delta$ are systematically smaller than those at the historical times $s + \Delta$. Hence, the use of zero coupon bonds historical returns to compute VaR by historical simulation would systematically overestimate VaR resulting in invalid VaR sequences. This phenomenon will be clearly illustrated in the simulation study carried out in Section 4.

2.2 Zero coupon bonds "pulled -to-par" returns

In this paper we proposes[3] the following zero coupon bond historical return at time $s+\Delta$ for holding period Δ, adjusted (or pulled) to times to maturity[4] $T-t$ and $T-(t+\Delta)$, as follows

$$\frac{p(s+\Delta,T)^{1-\frac{t-s}{T-(s+\Delta)}}}{p(s,T)^{1-\frac{t-s}{T-s}}}-1. \tag{5}$$

We call the adjusted historical returns from (5) "pulled-to-par" returns.

3 On the "pulled-to-par" VaR method

Let $p(t,T)$ be the price process, defined on the probability space $(\Omega,\mathcal{A},\mathbb{P})$, of a zero coupon bond paying 1 at maturity T, from which a single realization is observed. Given $p(t,T)$ we can always define its continuously compounded yield-to-maturity as,

$$p(t,T)=e^{-y(t,T)(T-t)} \ . \tag{6}$$

This yield-to-maturity is the typical risk factor used for fixed income risk management purposes [McNeil, Frey and Embrechts(2005), 31]. It is, therefore, not surprising that the applicability of the pulled-to-par VaR method depends on (distributional) properties of this risk factor.

Our main result, in Theorem (3.1) below, shows that under strict sense stationarity of the continuously compounded yield-to-maturity, pulled-to-par returns do allow computing VaR by historical simulation for portfolios with zero coupon bonds.

Taking a back-testing point of view, i.e. we assuming that the zero coupon bond already matured and that the entire price sequence was observed. Then, any historical market price $p(s,T)$ implies a continuously compounded yield to maturity, $y(s,T)$, at time s, which satisfies

$$1=p(s,T)e^{y(s,T)(T-s)} \ , \tag{7}$$

and is, thus, defined as,

$$y(s,T)=-\frac{\log p(s,T)}{T-s} \ . \tag{8}$$

[3] Building on the discrete time intuition in [Sousa et al.(2014)], Theorem 3.2 shows that, for continuously compounded yields, (5) defines the zero coupon bond "pulled -to-par" returns.

[4] These are the relevant times to maturity at time VaR is computed.

Theorem 3.1. *Let $s \leq t + \Delta$. If the yield-to-maturity $y(s, T)$ is stationary, in the strict sense, the $1 - \alpha$ quantile of the pulled-to-par returns distribution equals the zero coupon bond value at risk at time t.*

Proof. Theorem (3.1) follows straight forward from writing the bond return of Equation (4) at time t, the time VaR is computed, and the proposed pulled-to-par return of Equation (5) as

$$\frac{p(t+\Delta,T)}{p(t,T)} - 1 = e^{-(T-t)(y(t+\Delta,T)-y(t,T))+\Delta y(t+\Delta,T)} - 1 \tag{9}$$

and

$$\frac{p(s+\Delta,T)^{1-\frac{t-s}{T-(s+\Delta)}}}{p(s,T)^{1-\frac{t-s}{T-s}}} - 1 = e^{-(T-t)(y(s+\Delta,T)-y(s,T))+\Delta y(s+\Delta,T)} - 1. \tag{10}$$

Equation (9) results from:

$$\begin{aligned} \frac{p(t+\Delta,T)}{p(t,T)} - 1 &= \frac{e^{-y(t+\Delta,T)(T-(t+\Delta))}}{e^{-y(t,T)(T-t)}} - 1 \qquad (11) \\ &= e^{-y(t+\Delta,T)(T-(t+\Delta))+y(t,T)(T-t)} - 1 \\ &= e^{-Ty(t+\Delta,T)+ty(t+\Delta,T)+\Delta y(t+\Delta,T)+Ty(t,T)-ty(t,T)} - 1 \\ &= e^{-T(y(t+\Delta,T)-y(t,T))+t(y(t+\Delta,T)-y(t,T))+\Delta y(t+\Delta,T)} - 1 \\ &= e^{-(T-t)(y(t+\Delta,T)-y(t,T))+\Delta y(t+\Delta,T)} - 1 \end{aligned}$$

while Equation (10) results from:

$$\begin{aligned} \frac{p(s+\Delta,T)^{1-\frac{t-s}{T-(s+\Delta)}}}{p(s,T)^{1-\frac{t-s}{T-s}}} - 1 &= \frac{\left(e^{-y(s+\Delta,T)(T-(s+\Delta))}\right)^{1-\frac{t-s}{T-(s+\Delta)}}}{\left(e^{-y(s,T)(T-s)}\right)^{1-\frac{t-s}{T-s}}} - 1 \\ &= \frac{e^{-y(s+\Delta,T)(T-(s+\Delta))+y(s+\Delta,T)(T-(s+\Delta))\frac{t-s}{T-(s+\Delta)}}}{e^{-y(s,T)(T-s)+y(s,T)(T-s)\frac{t-s}{T-s}}} - 1 \\ &= \frac{e^{-y(s+\Delta,T)(T-(s+\Delta))+y(s+\Delta,T)(t-s)}}{e^{-y(s,T)(T-s)+y(s,T)(t-s)}} - 1 \\ &= \underbrace{\frac{e^{-y(s+\Delta,T)(T-(s+\Delta))}}{e^{-y(s,T)(T-s)}}}_{\text{Equation (11) on time } s} \frac{e^{y(s+\Delta,T)(t-s)}}{e^{y(s,T)(t-s)}} - 1 \end{aligned}$$

$$
\begin{aligned}
&= e^{-(T-s)(y(s+\Delta,T)-y(s,T))+\Delta y(s+\Delta,T)} e^{y(s+\Delta,T)(t-s)-y(s,T)(t-s)} - 1 \\
&= e^{-(T-s)(y(s+\Delta,T)-y(s,T))+\Delta y(s+\Delta,T)} e^{(t-s)(y(s+\Delta,T)-y(s,T))} - 1 \\
&= e^{-(T-s)(y(s+\Delta,T)-y(s,T))+\Delta y(s+\Delta,T)+(t-s)(y(s+\Delta,T)-y(s,T))} - 1 \\
&= e^{(-(T-s)+(t-s))(y(s+\Delta,T)-y(s,T))+\Delta y(s+\Delta,T)} - 1 \\
&= e^{(-T+s+t-s)(y(s+\Delta,T)-y(s,T))+\Delta y(s+\Delta,T)} - 1 \\
&= e^{-(T-t)(y(s+\Delta,T)-y(s,T))+\Delta y(s+\Delta,T)} - 1.
\end{aligned}
$$

If the yield to maturity, $y(t,T)$, is stationary, in the strict sense, then [Papoulis(1984), 219-220], then denoting equality in distribution by $\overset{d}{=}$, we have

1. The probability distribution of $y(t,T)$ is independent of t. Let $y(T)$ be a random variable with the time independent yield to maturity distribution. We have that

$$
y(t+\Delta,T) \overset{d}{=} y(s+\Delta,T) \overset{d}{=} y(T); \tag{12}
$$

2. The probability distribution of $y(t-\Delta,T)-y(t,T)$ depends only on Δ. Let $y(\Delta,T)$ be a random variable with the Δ dependent yield to maturity distribution. We have that:

$$
y(t+\Delta,T) - y(t,T) \overset{d}{=} y(s+\Delta,T) - y(s,T) \overset{d}{=} y(\Delta,T). \tag{13}
$$

Given Equations (12) and (13) the returns in Equations (9) and (10) are given by:

$$
\begin{aligned}
\frac{p(t+\Delta,T)}{p(t,T)} - 1 &= e^{-(T-t)(y(t+\Delta,T)-y(t,T))+\Delta y(t+\Delta,T)} - 1 \\
&\overset{d}{=} e^{-(T-t)y(\Delta,T)+\Delta y(T)} - 1
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{p(s+\Delta,T)^{1-\frac{t-s}{T-(s+\Delta)}}}{p(s,T)^{1-\frac{t-s}{T-s}}} - 1 &= e^{-(T-t)(y(s+\Delta,T)-y(s,T))+\Delta y(s+\Delta,T)} - 1 \\
&\overset{d}{=} e^{-(T-t)y(\Delta,T)+\Delta y(T)} - 1.
\end{aligned}
$$

Both the bond return at time VaR is computed (Equation (9)) and the proposed pulled-to-par return of Expression (5) have the same probability distribution. Therefore, the $1-\alpha$ quantile of the proposed pulled-to-par return equals the VaR value L.

□

Theorem (3.2) shows that the pulled-to.par returns of Expression (5) are computed from historical prices at times s and $s+\Delta$, pulled to times t and $t+\Delta$, using the continuously compounded implied historical yields to maturity, at times s and $s+\Delta$. The pull-to-par effect is thus corrected, while preserving historical market movements.

Theorem 3.2. *The zero coupon bonds pulled-to-par returns of Expression* (5) *computed from historical prices $p(s,T)$ and $p(s+\Delta,T)$ are given by*

$$\frac{p_{s+\Delta}(t+\Delta,T)}{p_s(t,T)} - 1, \tag{14}$$

where:

- *$p_s(t,T)$ is the bond valuation at time t given the historical market price $p(s,T)$ and the corresponding continuously compounded implied yield to maturity $y(s,T)$;*
- *and $p_{s+\Delta}(t+\Delta,T)$ is the bond valuation at time $t+\Delta$ given the historical market price $p(s+\Delta,T)$ and the corresponding continuously compounded implied yield to maturity $y(s+\Delta,T)$.*

Proof. Given Equation (7), $p_s(t,T)$ is given by

$$p_s(t,T) = p(s,T)e^{y(s,T)(t-s)}. \tag{15}$$

Substituting $y(s,T)$, given by Equation (8)

$$\begin{aligned}
p_s(t,T) &= p(s,T)e^{-\frac{\log p(s,T)}{T-s}(t-s)} \\
&= p(s,T)e^{-\log p(s,T)\frac{t-s}{T-s}} \\
&= p(s,T)e^{\log \frac{1}{p(s,T)}\frac{t-s}{T-s}} \\
&= p(s,T)\left(e^{\log \frac{1}{p(s,T)}}\right)^{\frac{t-s}{T-s}} \\
&= p(s,T)\left(\frac{1}{p(s,T)}\right)^{\frac{t-s}{T-s}} \\
&= p(s,T)p(s,T)^{-\frac{t-s}{T-s}} \\
&= p(s,T)^{1-\frac{t-s}{T-s}}.
\end{aligned}$$

Repeating a similar manipulation for $p_{s+\Delta}(t+\Delta,T)$

$$
\begin{aligned}
p_{s+\Delta}(t+\Delta,T) &= p(s+\Delta,T)e^{y(s+\Delta,T)((t+\Delta)-(s+\Delta))} \\
&= p(s+\Delta,T)e^{-\frac{\log p(s+\Delta,T)}{T-(s+\Delta)}(t-s)} \\
&= p(s+\Delta,T)p(s+\Delta,T)^{-\frac{t-s}{T-(s+\Delta)}} \\
&= p(s+\Delta,T)^{1-\frac{t-s}{T-(s+\Delta)}}.
\end{aligned}
$$

Substituting $p_s(t,T)$ and $p_{s+\Delta}(t+\Delta,T)$ in Equation (14) gives

$$
\frac{p_{s+\Delta}(t+\Delta,T)}{p_s(t,T)} - 1 = \frac{p(s+\Delta,T)^{1-\frac{t-s}{T-(s+\Delta)}}}{p(s,T)^{1-\frac{t-s}{T-s}}} - 1
$$

which equals Expression (5).

□

4 Simulation

The main purpose of this section is to provide a controlled environment where the assumption of Theorem (3.1) holds, namely, where the yields to maturity of zero coupon bonds are stationary.

We use this controlled environment to illustrate Zero Coupon Bonds VaR computation with pulled-to-par returns and compare it with the VaR computation using historical returns.

Given that the historical simulation performance depends on the size of the available historical sequence [McNeil, Frey and Embrechts(2005), 51] we also evaluate the ratio of valid VaR sequences obtained for the selected simulation scenario.

4.1 Simulation scenario

Our guideline for simulating prices of zero coupon bonds was to replicate the conditions of our real data database.

Regarding the availability of prices, our real data database, has prices from 2006-01-03 to 2018-06-01, spanning approximately twelve and a half years. In order to get a similar time spanning we simulated sequences of 4533 daily prices. We also simulated weekends, because the non-availability of prices at weekends decreases the number of returns available [5] and the historical simulation method depends on the number of returns available. We simulated weekends by removing the last two prices of each non-overlapping 7 prices sequence.

[5] As an example, in a sequence of 7 days, starting at a Monday, there are only 4 one day returns available, instead of the potentially 6 if there were prices at weekends.

As for the prices simulation, Theorem (3.1) requires stationary yields to maturity. Looking at the prices in our database negative yields to maturity occurrences can be observed. To accommodate both scenarios we simulate the yields to maturity with a constant mean uniformly distributed between -1.0% and 1%. In each time instant, this constant mean is added with an independent uniform distribution between 0.0% and 0.1%. The resulting sequence is smoothed by a five instants moving average to add some time structure.

Finally we selected maturities distributed in the year ahead of our period of analysis because the maturities of our real data database follow this same criterion. In sequences lengths terms the maturities were simulated to be uniformly distributed between 4533 plus 1 and 4533 plus 365.

Appendix A illustrates the negative yield to maturity simulated sequence along with the corresponding zero coupon bond prices.

4.2 Simulated zero coupon bond VaR

In this section we illustrate the VaR computation of a simulated zero coupon bond using pulled-to-par returns.

Following the Basel recommendations [Basel II(2006)] we start to compute VaR one year after the beginning of our period of analysis so that the VaR is computed with a minimum of one year of historical data. Then, we compute VaR at each instant using all the available historical data until that instant.

Figure 1 illustrates the historical returns and the VaR value obtained using the historical simulation method over the proposed pull-to-par returns, for the 12.8 maturity zero coupon bond whose prices are illustrated in Appendix A. The horizon and confidence level is $\Delta = 1$ day and $\alpha = 0.99$, respectively. The returns that violate the VaR value are also showed.

It is clear from Figure 1 that the pulled-to-par VaR value follows the historical returns pattern, diminishing towards zero as time to maturity approaches.

Just for illustration purposes Figure 1 also includes the $1 - 0.99$ quantile of the historical returns. It is clear from Figure 1 that this quantile can not be used to compute VaR. It does not follow the returns diminishing pattern. As a consequence there are almost no violations of this quantile which would result in an invalid VaR sequence.

In order to validate the obtained VaR sequences we applied both the standard Bernoulli coverage test, for the level of VaR violations, and the text of [Christoffersen(1998)] for violations of independence. Implementation of both tests in several programming languages is provided in [Danelsson(2015)].

Table 1 shows both statistical tests p-values for the VaR sequence of Figure 1. Using the common p-value threshold of 0.05, it confirms that the Var sequence of Figure 1 is valid. The returns VaR violations in Figure 1 (orange excedences) of the pulled-to-par VaR (green line), occur as required at with 1% frequency and are independent.

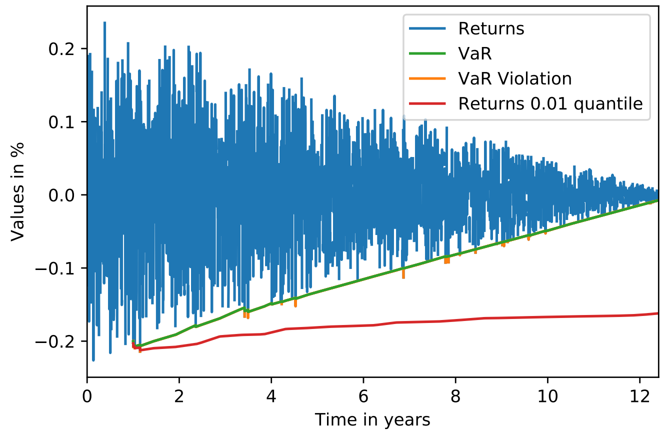

Figure 1: One day horizon, 0.99 confidence level, 12.8 maturity zero coupon bond pulled-to-par VaR example.

	p-value
Level test	0.402
Independence test	0.414

Table 1: Figure 1 VaR violations statistical tests p-values.

4.3 VaR computation performance

The historical simulation method is based on the convergence of the empirical returns distribution to the real distribution. This convergence takes place as the number of returns tends to infinity. In practice the number of returns is limited. In this section we evaluate the performance of the pulled-to-par VaR computation, with pulled in the simulation scenario previously described.

Ideally we would consider the confidence levels from Basel recommendations, namely, 97.5% and 99% [Basel II(2006), Basel II(2016)], as well as the recommended 10 day time horizon. However, in terms of the time horizon, and since it also allowed shorter horizons to be scaled to the 10 days period, here e evaluate 1 day periods. This maximizes the number of returns available and allow for a better performance evaluation.

We repeat the VaR computation 1000 times for each confidence level. These number of repetitions was empirically determined by observing that the performance ratios remain almost unchanged around 500 repetitions. We just doubled this number and observed that the ratios did not change.

Table 2 illustrates the performance ratios obtained. Under the described simulation scenario the performance of the one day VaR computation for a period slightly exceeding 10 years is always above 90%.

	Violations level p-value > 0.05	Volialtions independence p-value > 0.05	Valid VaR sequences
$\alpha = 0.975$ confidence level	949	955	909
$\alpha = 0.99$ confidence level	918	981	900

Table 2: One day VaR computation performance over 1000 repetitions.

Figure 1 together with Table 2 summarize the overall idea of this paper. It is clear from Figure 1 that bond historical returns tend to zero as maturity approaches (blue line). This is the direct consequence of the pull-to-par convergence of bond prices to the par value at maturity. It is also clear that the quantile of the historical returns (red line) systematically overestimates VaR as there are almost no historical returns violations of this value. On the other hand, the pulled-to-par VaR value proposed (green line) is indeed adjusted for the pull-to-par pattern as it clearly follow the diminishing pattern of the historical returns. Table 2 shows that the historical returns violations of this value do confirm that it is a valid VaR sequence for over more than 90% of the simulation repetitions.

4.4 Computational overhead

Expression (5) implies recalculation of pulled-to-par returns of each time t VaR is computed. This results in a computational overhead compared to the historical simulation of portfolios of stocks, where no (pull-to-par) adjustment is needed. However the pulled to par returns for each time t can be computed independently of each other. This means that they can be parallelized. Given the current trend of multicore GPU's[6] we anticipate that this overhead will easily overcome.

5 Real data

In this section we apply the pulled-to-par VaR method to real zero coupon bonds traded in the market.

The long term zero coupon bond market is almost nonexistent compared to the huge market of long term coupon bonds. However, investment houses detach coupons and principal from coupon bonds and trade STRIPS[7] independently of the original bonds. As these STRIPS have only one cash-flow, for all practical purposes, they are nothing but zero coupon bonds.

Our database contains sovereign eurozone STRIPS daily prices from 2006-01-03 to 2018-06-01. The eurozone is a natural region of choice has it allows

[6]As an example the NVIDEA GEFORCE GTX 1080 Ti GPU has 3584 computing cores.
[7]Separate Trading of Registered Interest and Principal of Securities

trading of STRIPS of different countries, with very different risk profiles, under the same currency. All the STRIPS were issued before 2006-01-03 and all were alive at 2018-06-01. The maturities range from 2018-06-01 to 2019-06-01. One year ahead of the last prices date. The choice of this near to maturity scenario was intentional. It provides the presence the vanishing returns near maturity. There are 19 STRIPS. Two from Germany, 3 from France, 7 from Italy, 1 from Belgium, 1 from Netherlands, 3 from Austria and 2 from Spain. A preliminary analysis allowed noticing that the STRIPS from the same country exhibited highly correlated price sequences. Almost indistinguishable. Therefore we kept only one STRIPS from each country. We choose the one with the shorter maturity. The list of the 7 STRIPS used is in Appendix B.

We use STRIPS both individually and in portfolios. The results are for 0.99 confidence level and 1 day horizon.

5.1 Individual STRIPS

Table 3 shows level and independence VaR violations tests p-values for the individual STRIPS. It can be observed a huge difference regarding our simulation scenario, where the stationarity assumption was guaranteed. Given that the method proposed in this paper assumes that yields are stationary we attribute the generally poor performance of the method on real data to nonstationary yields [Afonso et al.(2015)]. Nevertheless it should be highlighted that the analysis period includes a severe European sovereign debt crisis. Under this extremely adverse conditions there is one VaR valid sequence. Figure 2 shows the prices of this individual STRIPS and Figure 3 shows the corresponding returns, VaR sequence and VaR violations. For comparison purposes Figures 4 and 5 show the same sequences for and invalid VaR STRIPS.

	Level	Independence	VaR sequence
GG7292384	0.041	0.480	INVALID
GG7088238	0.113	0.065	VALID
EC5586903	0.536	0.001	INVALID
CP5051463	0.012	0.001	INVALID
GG7150772	0.000	0.868	INVALID
GG7292699	0.290	0.000	INVALID
EC4900568	0.000	0.004	INVALID

Table 3: Level and independence VaR violations tests p-values of individual STRIPS.

Comparing Figures 2 and 4, which illustrate a real bond price sequences, with Figure 7, which illustrates a simulated price sequence, a lack regularity in the evolution of the real bonds prices can be observed. This lack of regularity is characterized by high volatility periods mixed with small volatility periods as well as huge price drops (remarkably clear in Figure 4). This kind of evolution

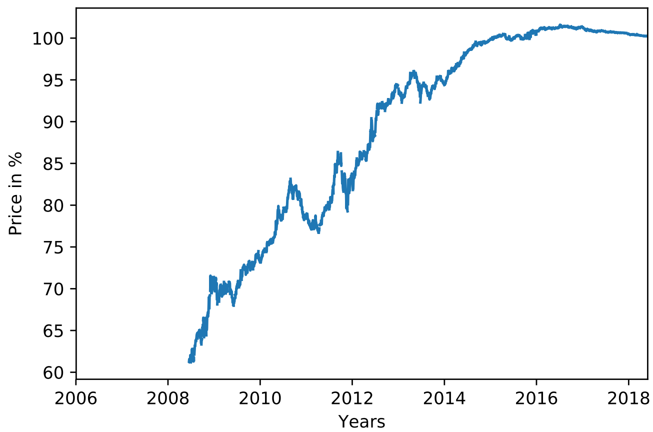

Figure 2: French GG7088238 STRIPS, 25-10-2018 maturity, historical prices.

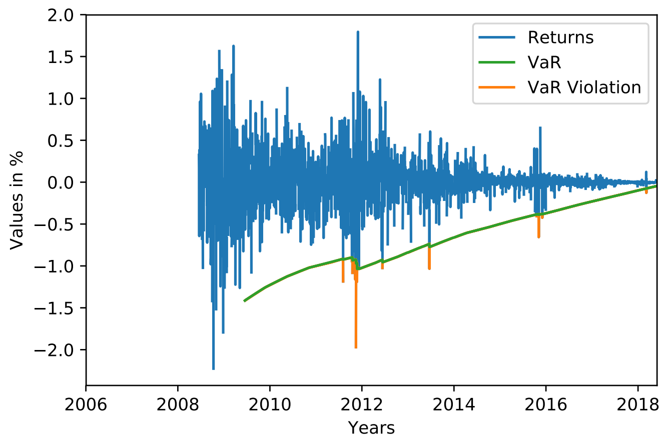

Figure 3: French GG7088238 STRIPS, 25-10-2018 maturity, one day horizon, 0.99 confidence level, returns, VaR and Var violations.

is a clear sign of nonstationarity of yields and compromises the results of the method application in a real data scenario.

This volatility pattern, can be observed again in Figures 3 and 5 reflected in the evolution of the historical returns. Instead of the smooth pattern of diminishing historical returns as maturity approaches, observed in Figure 1, the real data sequence exhibits periods of diminishing returns followed by periods of large returns.

Despite all this adverse observations it can also be observed in Figures 3 and

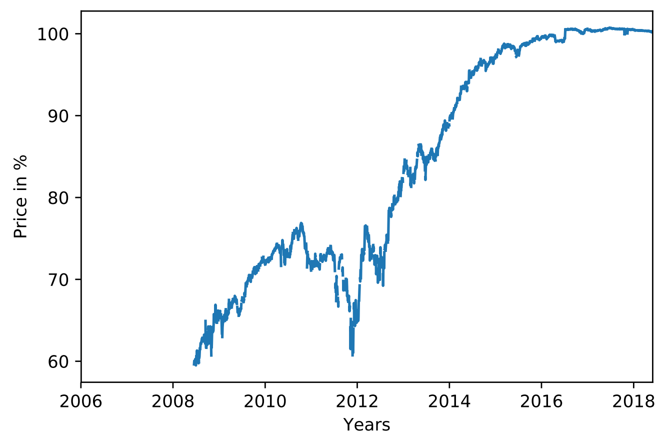

Figure 4: Italy EC5586903 STRIPS, 01-08-2018 maturity, historical prices.

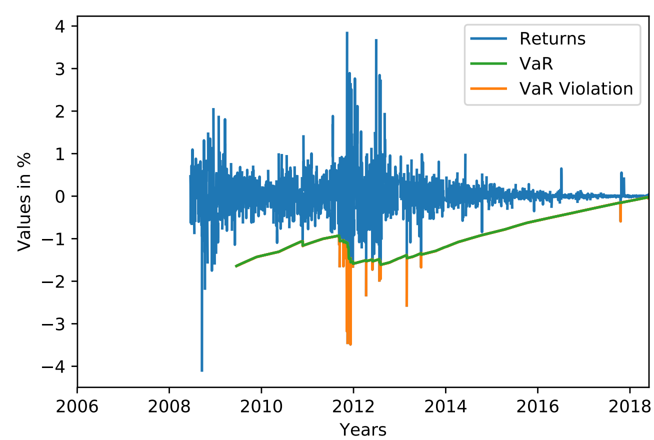

Figure 5: Italy EC5586903 STRIPS, 01-08-2018 maturity, one day horizon, 0.99 confidence level, returns, VaR and Var violations.

5, that the VaR value, computed we the proposed pull-to-par returns, is in fact adjusted to the pull-to-par diminishing pattern of the historical returns, leading to one valid VaR sequence. That of the French STRIPS GG7088238.

The overall conclusion of this discussion is that the stationarity assumption seems to be too strong for the real data scenario leading to a general poor performance in the case of individual STRIPS. Just one in the seven STRIPS tested resulted in a valid VaR sequence.

5.2 Portfolios

With the 7 STRIPS available we constructed all the possible portfolios with the number of STRIPS varying from 1 to 7 (the portfolios with 1 STRIPS equals the individual case described in the preceding section but are repeated here for means of comparison). Table 4 shows the number of valid portfolios VaR. Despite the poor general performance it can be observed that the diversification reached with portfolios with such a small number of portfolios allows the number of valid VaR sequences to increase.

Number of Portfolios	Number of STRIPS per Portfolio	Number of Valid VaR	Valid VaR Percentage
7	1	1	14%
21	2	5	24%
35	3	7	20%
35	4	8	23%
21	5	5	24%
7	6	2	29%
1	7	0	0%

Table 4: Number of valid Portfolios VaR sequences.

This can be observed right on the first two lines of Table 4, where the number of STRIPS in the portfolios was increased from 1 to 2. The percentage of valid VaR sequences in portfolios with just one STRIPS is 14%. But the percentage of valid VaR sequences in portfolios with two STRIPS increases to 24%. Given that the pulled-to-par VaR method requires stationary yields this increase can be explained by the contributions of the diversification to the portfolio stationarity yield.

6 Conclusions

In this paper we propose adjusting zero coupon bonds historical returns in such a way that allows VaR computation for portfolios, using the historical simulation method. The *pulled-to-par VaR method*.

We prove that the proposed VaR method leads to accurate VaR computations whenever zero coupon bonds yields to maturity are stationary.

Simulation results show that one day VaR computation performance under Basel recommendations for a 10 years period is above 90%.

Real data performance on individual STRIPS is poor due the nonstationarity of bond rates in the period under analysis, [Afonso et al.(2015)]. Our sample includes the European sovereign debt crisis of 2010-2014. Despite this, the diversification effect in portfolios with a very small number of STRIPS, such as 3 and 4, do allow for increasing performance.

We identify the following strengths of the proposed method. The portfolio specific VaR is computed while using the market as the only source of information. The only information need is market prices. Neither subjective risk factors mapping [Alexander(2009)], risk factors correlations, standard maturities interpolation, interest rate and credit risk separation, nor ratings, are needed.

Regarding weaknesses, the proposed method inherits all the known weaknesses of the historical simulation method, namely, the need for stationary of the historical returns used (or adjusted historical returns in our case), and the need of large amounts of synchronized historical data for all securities in the portfolio, [McNeil, Frey and Embrechts(2005)].

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A Simulation Example

In this section we illustrate in Figure 6 a negative yield to maturity simulated sequence along with the corresponding zero coupon bond prices in Figure 7.

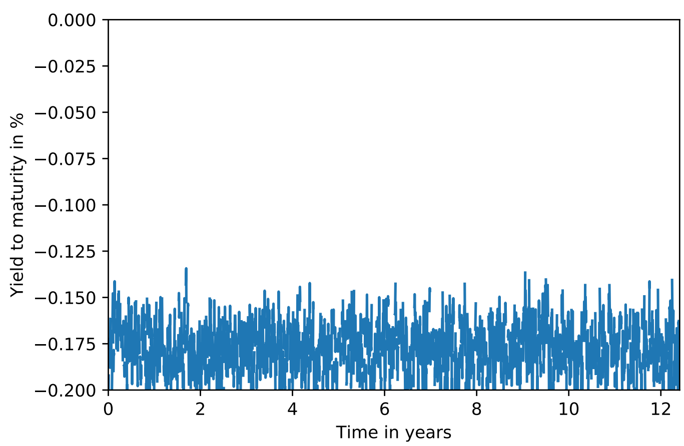

Figure 6: Negative yield to maturity simulated sequence.

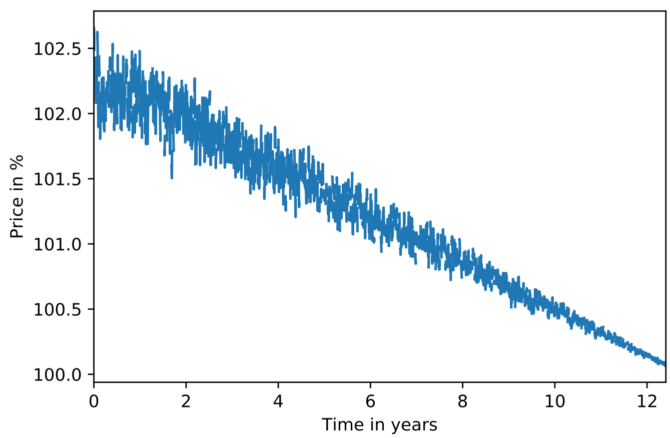

Figure 7: 12.8 maturity zero coupon bond price sequence. The corresponding yields to maturity sequence is the one in Figure 6.

B STRIPS list

Table 5: STRIPS Database.

Bloomberg ID	Maturity	Issuer Name
GG7292384	04-07-2018	Deutsche Bundesrepublik Coupon STRIPS
GG7088238	25-10-2018	French Republic Government Bond OAT Coupon STRIPS
EC5586903	01-08-2018	Italy Buoni Poliennali del Tesoro Coupon STRIPS
CP5051463	28-03-2019	Kingdom of Belgium Government Bond Coupon STRIPS
GG7150772	15-01-2019	Netherlands Government Bond Coupon STRIPS
GG7292699	15-07-2018	Republic of Austria Government Bond Coupon STRIPS
EC4900568	30-07-2018	Spain Government Bond Coupon STRIPS

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