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DELTA INVARIANTS OF SMOOTH CUBIC SURFACES

IVAN CHELTSOV AND KEWEI ZHANG

Abstract. We prove that δ-invariants of smooth cubic surfaces are at least $\frac{6}{5}$.

All varieties are assumed to be projective and defined over $\mathbb{C}$.

1. Introduction

The existence of Kähler-Einstein metrics on Fano manifolds is an important problem in complex geometry. By Yau–Tian–Donaldson conjecture (confirmed in [CDS15, T15]), we know that all K-stable Fano manifolds are Kähler-Einstein. Moreover, we also know explicit criteria that can be used to verify K-stability in many cases. One such criterion has been found by Tian in [T87] and later generalized by Fujita in [F16]. It is the following

Theorem 1.1 ([T87, F16])**.** Let X be a Fano manifold of dimension $n \geqslant 2$. If $\alpha(X) \geqslant \frac{n}{n+1}$, then X is K-stable.

Here, $\alpha(X)$ is the α-invariant defined in [T87]. By [CS08, Theorem A.3], one has

$$\alpha\big(X\big) = \sup\left\{\lambda \in \mathbb{Q}\ \middle|\ \begin{array}{l}\text{the log pair } \ (X, \lambda D) \ \text{ is log canonical}\\ \text{for every effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X\end{array}\right\}.$$

In [C08], the first author computed the α-invariants of two-dimensional Fano manifolds, known as del Pezzo surfaces. Namely, if S be a smooth del Pezzo surface, then

$$\alpha(S) = \begin{cases} \dfrac{1}{3} \text{ if } S \cong \mathbb{F}_1 \text{ or } K_S^2 \in \{7, 9\},\\ \dfrac{1}{2} \text{ if } S \cong \mathbb{P}^1 \times \mathbb{P}^1 \text{ or } K_S^2 \in \{5, 6\},\\ \dfrac{2}{3} \text{ if } K_S^2 = 4,\\ \dfrac{2}{3} \text{ if } S \text{ is a cubic surface in } \mathbb{P}^3 \text{ with an Eckardt point},\\ \dfrac{3}{4} \text{ if } S \text{ is a cubic surface in } \mathbb{P}^3 \text{ without Eckardt points},\\ \dfrac{3}{4} \text{ if } K_S^2 = 2 \text{ and } |-K_S| \text{ has a tacnodal curve},\\ \dfrac{5}{6} \text{ if } K_S^2 = 2 \text{ and } |-K_S| \text{ has no tacnodal curves},\\ \dfrac{5}{6} \text{ if } K_S^2 = 1 \text{ and } |-K_S| \text{ has a cuspidal curve},\\ 1 \text{ if } K_S^2 = 1 \text{ and } |-K_S| \text{ has no cuspidal curves}.\end{cases}$$

In particular, if $K_S^2 \leqslant 4$, then S is K-stable by Theorem 1.1, so that it is Kähler-Einstein. If $K_S^2 = 5$, then S is unique and $\mathrm{Aut}(S) \cong \mathfrak{S}_5$. In this case, we have $\alpha_{\mathfrak{S}_5}(S) = 2$ by [C08], where $\alpha_{\mathfrak{S}_5}(S)$ is a $\mathfrak{S}_5$-invariant α-invariant, which can be defined similarly to $\alpha(S)$. Now using an $\mathfrak{S}_5$-equivariant counterpart of Theorem 1.1 in [T87], we conclude that the surface S is also Kähler-Einstein. All remaining del Pezzo surfaces are toric, so that they are Kähler-Einstein if and only if their Futaki characters vanish [WZ04]. Together with Matsushima's obstruction, this give Tian's celebrated

Theorem 1.2 ([T90])**.** A smooth del Pezzo surface admits a Kähler-Einstein metric if and only if it is not a blow up of $\mathbb{P}^2$ at one or two points.

Note that smooth cubic surfaces form the hardest case in Tian's original proof of this result, which requires Cheeger–Gromov theory, Hörmander L^2 estimates, partial C^0 estimates and the lower semi-continuity of log canonical thresholds. In this paper, we will give another proof of Theorem 1.2 in this case using a new criterion for K-stability, which has been recently discovered by Fujita and Odaka in [FO18]. They stated it in terms of the so-called δ-invariant, which we describe now.

Fix a Fano manifold X. For a sufficiently large and sufficiently divisible integer k, consider a basis $s_1, \cdots, s_{d_k}$ of the vector space $H^0(\mathcal{O}_X(-kK_X))$, where $d_k = h^0(\mathcal{O}_X(-kK_X))$. For this basis, consider the $\mathbb{Q}$-divisor

$$\frac{1}{kd_k}\sum_{i=1}^{d_k}\big\{s_i = 0\big\} \sim_{\mathbb{Q}} -K_X.$$

Any $\mathbb{Q}$-divisor obtained in this way is called a k-basis type (anticanonical) divisor. Let

$$\delta_k\big(X\big) = \sup\left\{\lambda \in \mathbb{Q}\ \middle|\ \begin{array}{l}\text{the log pair } (X, \lambda D) \text{ is log canonical}\\ \text{for every } k\text{-basis type } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X\end{array}\right\}.$$

Then let

$$\delta(X) = \limsup_{k\in\mathbb{N}} \delta_k(X).$$

By [BJ17, Theorem A], one has

$$\frac{\dim(X)+1}{\dim(X)}\alpha\big(X\big) \leqslant \delta\big(X\big) \leqslant \big(\dim(X)+1\big)\alpha\big(X\big).$$

The number $\delta(X)$ is also referred to as the *stability threshold* (cf. [BJ17, BBJ18]), because of

Theorem 1.3 ([BJ17, Theorem B])**.** The following assertions hold:

(1) X is K-semistable if and only if $\delta(X) \geqslant 1$;
(2) X is uniformly K-stable if and only if $\delta(X) > 1$.

How to compute or at least estimate $\delta(X)$ effectively? In general this is not very easy. In [PW18], Park and Won estimated the δ-invariants of all smooth del Pezzo surfaces, which gave another proof of Tian's Theorem 1.2. But it seems unclear to us how to generalize their approach for higher-dimensional Fano manifolds. Motivated by this, in our recent joint work with Yanir Rubinstein [CRZ18], we developed new geometric tools to estimate δ-invariants of (log) del Pezzo surfaces, which enabled us to partially prove a conjecture proposed in [CR15]. In this paper, we will use the same methods to give a sharper estimate for the δ-invaraints of smooth cubic surfaces. To be precise, we prove

Theorem 1.4. Let S be a smooth cubic surface in $\mathbb{P}^3$. Then $\delta(S) \geqslant \frac{6}{5}$.

Corollary 1.5 ([T90, PW18])**.** All smooth cubic surfaces in $\mathbb{P}^3$ are uniformly K-stable, so that they are Kähler-Einstein.

For a smooth cubic surface S, it follows from [PW18, Theorem 4.9] that

$$\delta(S) \geqslant \frac{36}{31}.$$

Our bound $\delta(S) \geqslant \frac{6}{5}$ is slightly better ☺. Moreover, the proof of Theorem 1.4 is completely different from the proof of [PW18, Theorem 4.9]. The essential ingredient in our proof is a *vanishing order estimate* for basis type divisors (see Theorem 2.10). This estimate combined with the techniques from [C08] give us the desired lower bound for $\delta(S)$.

This paper is organized as follows. In Section 2, we present known results about divisors on smooth surfaces, and, as an illustration, we give a new proof of [PW18, Theorem 4.7]. In Section 3, we give various multiplicity estimates for basis type divisors on smooth cubic surfaces, which will be important to bound their δ-invariants in the proof of Theorem 1.4. These estimates also imply that δ-invariants of smooth cubic surfaces are at least $\frac{18}{17}$. In Section 4, we prove Theorem 1.4.

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2. Basic tools

In this section, we collect some basic notions and tools that will be used throughout this article. Let S be a smooth surface, and let P be a point in S. Let D be an effective divisor on S. Suppose that $f = 0$ is the local defining equation of D near the point P, then the multiplicity of D at P, is defined to be the vanishing order of f at P, which we denote by $\mathrm{mult}_P(D)$. Let $\pi\colon \widetilde{S} \to S$ be the blow up of the point P, and let E be the exceptional curve of π. Denote by $\widetilde{D}$ the proper transform of D via π. Then we have

$$\pi^*(D) = \widetilde{D} + \mathrm{mult}_P(D) \cdot E.$$

Definition 2.1. Let C_1 and C_2 be two irreducible curves on a surface S. Suppose that C_1 and C_2 intersect at P. Let $\mathcal{O}_P$ be the local ring of germs of holomorphic functions defined in some neighborhood of P. Then the local intersection number of C_1 and C_2 at the point P is defined by

$$\big(C_1 \cdot C_2\big)_P = \dim_{\mathbb{C}} \mathcal{O}_P/\langle f_1, f_2\rangle,$$

where $f_1 = 0$ and $f_2 = 0$ are local defining functions of C_1 and C_2 around the point P. The global intersection number $C_1 \cdot C_2$ is defined by

$$C_1 \cdot C_2 = \sum_{P \in C_1 \cap C_2} \big(C_1 \cdot C_2\big)_P.$$

This definition and the definition of $\mathrm{mult}_P(D)$ extends to $\mathbb{R}$-divisors by linearity. For instance, say we have a curve C and a $\mathbb{R}$-divisor $\Delta = \sum_i a_i Z_i$, where Z_i's are distinct prime divisors and $a_i \in \mathbb{R}$. Then

$$\big(C \cdot \Delta\big)_P = \sum_i a_i \big(C \cdot Z_i\big)_P,$$

where $(C.Z_i)_P = 0$ if Z_i does not pass through the point P.

In the following, let D be an effective $\mathbb{R}$-divisor on S. We will investigate how to express the singularity of the log pair (S, D) at the point P in terms of $\mathrm{mult}_P(\cdot)$ and $\big(\cdot\big)_P$.

Lemma 2.2 ([K97])**.** If (S, D) is not log canonical at P, then $\mathrm{mult}_P(D) > 1$.

Let C be an irreducible curve on S. Write

$$D = aC + \Delta,$$

where a is a non-negative real number that is also denoted as $\mathrm{ord}_C(D)$, and Δ is an effective $\mathbb{R}$-divisor on S whose support does not contain the curve C.

Lemma 2.3 ([CRZ18, Proposition 3.3])**.** Suppose that $a \leqslant 1$, the curve C is smooth at the point P, and $\mathrm{mult}_P(\Delta) \leqslant 1$. If (S, D) is not log canonical at P, then

$$\big(C \cdot \Delta\big)_P > 2 - a.$$

Corollary 2.4. If $a \leqslant 1$, the curve C is smooth at P, and the log pair (S, D) is not log canonical at P, then

$$\big(C \cdot \Delta\big)_P > 1.$$

Let $\pi\colon \widetilde{S} \to S$ be the blow up of the point P, and let E_1 be the exceptional curve of π. Denote by $\widetilde{D}$ the proper transform of D via π. Then

$$K_{\widetilde{S}} + \widetilde{D} + \big(\mathrm{mult}_P(D) - 1\big)E_1 \sim_{\mathbb{R}} \pi^*\big(K_S + D\big).$$

This implies

Corollary 2.5. The log pair (S, D) is log canonical at P if and only if the log pair

$$\Big(\widetilde{S}, \widetilde{D} + \big(\mathrm{mult}_P(D) - 1\big)E_1\Big)$$

is log canonical along the curve E_1.

Thus, using Lemma 2.2 and Corollary 2.5, we obtain the following simple criterion.

Corollary 2.6. Suppose that

$$\mathrm{mult}_Q\big(\pi^*(D)\big) = \mathrm{mult}_P\big(D\big) + \mathrm{mult}_Q\big(\widetilde{D}\big) \leqslant 2$$

for every point $Q \in E_1$. Then (S, D) is log canonical at P.

If D is a Cartier divisor, then its volume is the number

$$\mathrm{vol}(D) = \limsup_{k\in\mathbb{N}} \frac{h^0(\mathcal{O}_S(kD)}{k^2/2!},$$

where the lim sup can be replaced by a limit (see [L04, Example 11.4.7]). Likewise, if D is a $\mathbb{Q}$-divisor, we can define its volume using the identity

$$\mathrm{vol}(D) = \frac{\mathrm{vol}\big(\lambda D\big)}{\lambda^2}$$

for an appropriate $\lambda \in \mathbb{Q}_{>0}$. Then the volume $\mathrm{vol}(D)$ only depends on the numerical equivalence class of the divisor D. Moreover, the volume function can be extended by continuity to $\mathbb{R}$-divisors. Furthermore, it is log-concave:

$$\sqrt{\mathrm{vol}(D_1+D_2)} \geqslant \sqrt{\mathrm{vol}(D_1)} + \sqrt{\mathrm{vol}(D_2)}. \tag{2.7}$$

for any pseudoeffective $\mathbb{R}$-divisors D_1 and D_2 on the surface S. For more details about volumes of $\mathbb{R}$-divisors, we refer the reader to [LM09, L04].

If D is not pseudoeffective, then $\mathrm{vol}(D) = 0$. If the divisor D is nef, then

$$\mathrm{vol}(D) = D^2.$$

This follows from the asymptotic Riemann–Roch theorem [L04]. If the divisor D is not nef, its volume can be computed using its Zariski decomposition [F79, P03]. Namely, if D is pseudoeffective, then there exists a nef $\mathbb{R}$-divisor N on the surface S such that

$$D \sim_{\mathbb{R}} N + \sum_{i=1}^{r} a_i C_i,$$

where each C_i is an irreducible curve on S with $N \cdot C_i = 0$, each a_i is a non-negative real number, and the intersection form of the curves $C_1, \ldots, C_r$ is negative definite. Such decomposition is unique, and it follows from [BKS04, Corollary 3.2] that

$$\mathrm{vol}\big(D\big) = \mathrm{vol}\big(N\big) = N^2.$$

This immediately gives

Corollary 2.8. Let $Z_1, \ldots, Z_s$ be irreducible curves on S such that $D \cdot Z_i \leqslant 0$ for every i, and the intersection form of the curves $Z_1, \ldots, Z_s$ is negative definite. Then

$$\mathrm{vol}(D) = \mathrm{vol}\Big(D - \sum_{i=1}^{s} b_i Z_i\Big),$$

where $b_1, \ldots, b_s$ are (uniquely defined) non-negative real numbers such that

$$\Big(D - \sum_{i=1}^{s} b_i Z_i\Big) \cdot Z_j = 0$$

for every j.

Corollary 2.9. Let Z be an irreducible curve on S such that $Z^2 < 0$ and $D \cdot Z \leqslant 0$. Then

$$\mathrm{vol}(D) = \mathrm{vol}\Big(D - \frac{D\cdot Z}{Z^2} Z\Big).$$

Let $\eta\colon \widehat{S}\to S$ be a birational morphism (possibly an identity) such that $\widehat{S}$ is smooth. Fix a (non necessarily η-exceptional) irreducible curve F in the surface $\widehat{S}$. Let

$$\tau(F)=\sup\Big\{x\in\mathbb{R}_{>0}\ \Big|\ \eta^*(D)-xF \text{ is numerically equivalent to an effective divisor}\Big\}.$$

This is called the *pseudo-effective threshold* of F.

Theorem 2.10. Suppose that S is smooth del Pezzo surface, and D is a k-basis type divisor with $k\gg 1$. Then

$$\mathrm{ord}_F\big(\eta^*(D)\big)\leqslant\frac{1}{(-K_S)^2}\int_0^{\tau(F)}\mathrm{vol}\big(\eta^*(-K_S)-xF\big)dx+\epsilon_k,$$

where ϵ_k is a small constant depending on k such that $\epsilon_k\to 0$ as $k\to\infty$.

Proof. This is a very special case of [FO18, Lemma 2.2]. □

In [BJ17, BBJ18], the quantity

$$S(F)=\frac{1}{(-K_S)^2}\int_0^{\tau(F)}\mathrm{vol}\big(\eta^*(-K_S)-xF\big)dx$$

is also called the *expected vanishing order* of anticanonical sections along the divisor F.

Theorem 2.10 plays a crucial role in the proof of Theorem 1.4. As a warm up, let us show how to use Theorem 2.10 to estimate δ-invariants of smooth del Pezzo surfaces of degree 1.

Theorem 2.11 ([PW18, Theorem 4.7])**.** Let S be a smooth del Pezzo surface of degree 1. Then $\delta(S)\geqslant\frac{3}{2}$.

Proof. Fix some rational number $\lambda<\frac{3}{2}$. Let D be a k-basis type divisor with $k\gg 1$, and let P be a point in S. We have to show that the log pair $(S,\lambda D)$ is log canonical at P. By Lemma 2.2, it is enough to prove that

$$\mathrm{mult}_P\big(D\big)\leqslant\frac{1}{\lambda}.$$

Applying Theorem 2.10 with $\widehat{S}=\widetilde{S}$, $\eta=\pi$ and $F=E_1$, we see that

$$\mathrm{mult}_P\big(D\big)\leqslant\int_0^{\tau(E_1)}\mathrm{vol}\big(\pi^*(-K_S)-xE_1\big)dx+\epsilon_k,$$

where ϵ_k is a constant depending on k such that $\epsilon_k\to 0$ as $k\to\infty$.

Let us compute $\tau(E_1)$. To do this, take a curve $C\in|-K_S|$ such that $P\in C$. Denote by $\widetilde{C}$ its proper transform on the surface $\widetilde{S}$. If C is smooth at P, then

$$\pi^*\big(-K_S\big)\sim_{\mathbb{Q}}\widetilde{C}+E_1 \text{ and } \widetilde{C}^2=C^2-1=0,$$

which implies that $\tau(E_1)=1$. In this case, we have

$$\mathrm{mult}_P\big(D\big)\leqslant\int_0^1\mathrm{vol}(\eta^*(-K_S)-xE_2)dx+\epsilon_k=$$
$$=\int_0^1\big((\pi^*(-K_S)-xE_1)\big)^2dx+\epsilon_k=\int_0^1\big(1-x^2\big)^2dx+\epsilon_k=\frac{2}{3}+\epsilon_k.$$

Therefore, if C is smooth at P, then the log pair $(S,\lambda D)$ is log canonical at P for $k\gg 1$.

To complete the proof, we may assume that C is singular at P. Then P is either nodal or cuspidial, so we have $\mathrm{mult}_P C=2$ and

$$\pi^*\big(-K_S\big)\sim\widetilde{C}+2E_1,$$

so that $\tau(E_1)=2$, since $\widetilde{C}^2=-3$. Using Corollary 2.9, we see that

$$\mathrm{vol}\big(\pi^*(-K_S)-xE_1\big)=\begin{cases}1-x^2, & 0\leqslant x\leqslant\frac{1}{2},\\ \frac{(x-2)^2}{3}, & \frac{1}{2}\leqslant x\leqslant 2.\end{cases}$$

so that $\mathrm{mult}_P(D)\leqslant\frac{5}{6}+\epsilon_k$. This gives $\delta(S)\geqslant\frac{6}{5}$. To get $\delta(S)\geqslant\frac{3}{2}$, we must work harder.

Fix a point $Q\in E_1$. By Corollary 2.6, to prove that $(S,\lambda D)$ is log canonical at P, it is enough to show that

$$\mathrm{mult}_Q\big(\pi^*(D)\big)=\mathrm{mult}_P\big(D\big)+\mathrm{mult}_Q\big(\widetilde{D}\big)\leqslant\frac{2}{\lambda}.$$

Let $\sigma\colon\widehat{S}\to\widetilde{S}$ be the blow up of the point Q. Denote by E_2 the exceptional curve of σ. Let $\eta=\pi\circ\sigma$. Applying Theorem 2.10 with $F=E_1$, we see that

$$\mathrm{mult}_Q\big(\pi^*(D)\big)\leqslant\int_0^{\tau(E_2)}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)dx+\varepsilon_k.$$

Here, as above, the term ε_k is a constant that depends on k such that $\varepsilon_k\to 0$ as $k\to\infty$.

Let $\widehat{C}$ and $\widehat{E}_1$ be the proper transforms on $\widehat{S}$ of the curves C and E_1, respectively. Then the intersection form of the curves $\widehat{C}$ and $\widehat{E}_1$ is negative definite. If $Q\in\widetilde{C}$, then

$$\eta^*(-K_S)\sim_{\mathbb{Q}}\widehat{C}+2\widehat{E}_1+3E_2,$$

so that $\tau(E_2)=3$. In this case, using Corollary 2.9, we see that

$$\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)=\mathrm{vol}\Big(\eta^*(-K_S)-xE_2-\frac{x}{2}\widehat{E}_1\Big)=\Big(\eta^*(-K_S)-xE_2-\frac{x}{2}\widehat{E}_1\Big)^2=1-\frac{x^2}{2}$$

provided that $0\leqslant x\leqslant\frac{2}{3}$. Likewise, if $\frac{2}{3}\leqslant x\leqslant 3$, then Corollary 2.8 gives

$$\begin{aligned}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)&=\mathrm{vol}\Big(\eta^*(-K_S)-xE_2-\frac{5x-1}{7}\widehat{E}_1-\frac{3x-2}{7}\widehat{C}\Big)=\\&=\Big(\eta^*(-K_S)-xE_2-\frac{5x-1}{7}\widehat{E}_1-\frac{3x-2}{7}\widehat{C}\Big)^2=\\\big(\eta^*(-K_S)-xE_2\big)&\Big(\eta^*(-K_S)-xE_2-\frac{5x-1}{7}\widehat{E}_1-\frac{3x-2}{7}\widehat{C}\Big)=\frac{(3-x)^2}{7}.\end{aligned}$$

Thus, if $Q\in\widetilde{C}$, then

$$\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)=\begin{cases}1-\frac{x^2}{2}, & 0\leqslant x\leqslant\frac{2}{3},\\ \frac{(3-x)^2}{7}, & \frac{2}{3}\leqslant x\leqslant 3,\end{cases}$$

so that $\mathrm{mult}_Q(\pi^*(D))\leqslant\frac{2}{\lambda}$ for $k\gg 1$, because

$$\int_0^3\mathrm{vol}(\eta^*(-K_S)-xE_2)dx=\frac{11}{9}<\frac{2}{\lambda}.$$

Likewise, if $Q \notin \widetilde{C}$, then

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \widehat{C} + 2\widehat{E}_1 + 2E_2.$$

so that $\tau(E_2) = 2$. In this case, using Corollary 2.8, we deduce that

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 1 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 1, \\ \frac{(2-x)^2}{2}, & 1 \leqslant x \leqslant 2, \end{cases}$$

which implies that

$$\int_0^2 \mathrm{vol}\big(\eta^*(-K_S) - xE_2\big)dx = 1,$$

so that $\mathrm{mult}_Q(\pi^*(D)) \leqslant \frac{2}{\lambda}$ for $k \gg 1$. □

Remark 2.12. In the proof of Theorem 2.11, there is another way to treat the case when the curve C is singular at P, which relies on Lemma 2.3. Indeed, let S be a smooth del Pezzo surface of degree 1, let P be a point in S, and let C be a curve in $|-K_S|$ that passes trough P. Suppose that

$$\mathrm{mult}_P\big(C\big) = 2.$$

Let D be any k-basis type divisor such that $D \sim -K_S$ with $k \gg 1$, and let λ be a positive real number such that $\lambda < \frac{3}{2}$. Let us show that $(S, \lambda D)$ is log canonical at P. We argue by contradiction. Suppose that $(S, \lambda D)$ is not log canonical at P. Write

$$D = aC + \Delta,$$

where $a \geqslant 0$ and Δ is an effective $\mathbb{Q}$-divisor whose support does not contain C. Note that

$$a \leqslant \int_0^\infty \mathrm{vol}(-K_S - xC)dx + \epsilon_k = \frac{1}{3} + \epsilon_k,$$

where ϵ_k is a constant that depends on k such that $\varepsilon_k \to 0$ as $k \to \infty$. Let $m = \mathrm{mult}_P(\Delta)$. Then

$$1 = D \cdot C = (aC + \Delta) \cdot C \geqslant a + 2m,$$

so that $m \leqslant \frac{1-a}{2}$. Let $\pi\colon \widetilde{S} \to S$ be the blow up of the point P. Let E be the exceptional curve of π, and let $\widetilde{C}$ and $\widetilde{\Delta}$ be the proper transforms of C and Δ on $\widetilde{S}$, respectively. Then the log pair

$$\Big(\widetilde{S}, \lambda a\widetilde{C} + \lambda\widetilde{\Delta} + \big(\lambda(2a+m) - 1\big)E\Big)$$

is not log canonical at some point $Q \in E$. Note that $\lambda(2a+m) - 1 < 1$. But

$$E \cdot (\lambda\Delta) = \lambda m \leqslant \lambda\frac{1-a}{2} < \frac{3}{2} \cdot \frac{1}{2} < 1.$$

Thus, we have $Q \in E \cap \widetilde{C}$ by Corollary 2.4. On the other hand, for $k \gg 1$, we have

$$\mathrm{mult}_Q\big(\lambda\widetilde{\Delta} + (\lambda(2a+m) - 1)E\big) \leqslant 2\lambda(a+m) - 1 \leqslant \lambda \cdot (1 + \frac{1}{3} + \epsilon_k) - 1 \leqslant 1,$$

so that we can apply Lemma 2.3 to our pair at Q. This gives

$$\lambda C \cdot \Delta - 2m\lambda + 2\lambda(2a+m) - 2 = \widetilde{C} \cdot \big(\lambda\widetilde{\Delta} + (\lambda(2a+m) - 1)E\big) > 2 - \lambda a,$$

so that $\lambda(1+4a)>4$, and hence

$$\frac{3}{2}(1+4\cdot\frac{1}{3}+\epsilon_k)>4,$$

which is absurd for $\epsilon_k \ll 1$. This proves the desired log canonicity of our pair $(S,\lambda D)$.

The following (simple) result can be very handy.

Lemma 2.13. In the assumptions and notations of Theorem 2.10, one has

$$\int_{\mu}^{\tau(F)} \mathrm{vol}\big(\eta^*(-K_S)-xF\big)dx \leqslant \big(\tau(F)-\mu\big)\mathrm{vol}\big(\eta^*(-K_S)-\mu F\big)$$

for any $\mu\in[0,\tau(F)]$.

Proof. The assertion follows from the fact that $\mathrm{vol}(\eta^*(-K_S)-xF)$ is a non-increasing function on $x\in[0,\tau(F)]$. □

Using (2.7), this result can be improved as follows:

Lemma 2.14. In the assumptions and notations of Theorem 2.10, one has

$$\int_{\mu}^{\tau(F)} \mathrm{vol}\big(\eta^*(-K_S)-xF\big)dx \leqslant \frac{2}{3}\Big(\tau(F)-\mu\Big)\mathrm{vol}\big(\eta^*(-K_S)-\mu F\big)$$

for any $\mu\in[0,\tau(F)]$.

Proof. The required assertion follows from the proof of [F17, Proposition 2.1]. □

We will apply both Lemmas 2.13 and 2.14 to estimate the integral in Theorem 2.10 in the cases when it is not easy to compute.

3. Multiplicity estimates

Let S be a smooth cubic surface in $\mathbb{P}^3$, and let D be a k-basis type divisor with $k\gg 1$. The goal of this section is to bound multiplicities of the divisor D using Theorem 2.10. As in Theorem 2.10, we denote by ϵ_k a small number such that $\epsilon_k\to 0$ as $k\to\infty$.

Lemma 3.1. Let L be a line on S. Then

$$\mathrm{ord}_L(D)\leqslant\frac{5}{9}+\epsilon_k.$$

Proof. Let us use assumptions and notations of Theorem 2.10 with $\eta=\mathrm{Id}_S$ and $F=L$. Let H be a general hyperplane section of the surface S that contains L. Then $H=L+C$, where C is an irreducible conic. Since $C^2=0$, we have $\tau(F)=1$, so that

$$\mathrm{ord}_L(D)\leqslant\frac{1}{3}\int_0^1\mathrm{vol}(-K_S-xL)dx+\epsilon_k=\frac{1}{3}\int_0^1(-K_S-xL)^2dx+\epsilon_k=\frac{5}{9}+\epsilon_k$$

by Theorem 2.10. □

Fix a point $P \in S$. Let $\pi \colon \widetilde{S} \to S$ be the blowup of this point. Denote by E_1 the exceptional divisor of π. Fix a point $Q \in E_1$. Let $\sigma \colon \widehat{S} \to \widetilde{S}$ be the blowup of this point. Denote by E_2 the exceptional curve of σ. Let $\eta = \pi \circ \sigma$, then

$$\tau(E_2) = \sup\Big\{x \in \mathbb{R}_{>0} \ \Big| \ \eta^*(-K_S) - xE_2 \text{ is numerically equivalent to an effective divisor}\Big\}.$$

Applying Theorem 2.10, we get

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{1}{3}\int_0^{\tau(E_2)} \mathrm{vol}\big(\eta^*(-K_S) - xE_2\big)dx + \epsilon_k. \tag{3.2}$$

Let T_P be the unique hyperplane section of the surface S that is singular at the point P. Then we have the following four possibilities:

- $T_P = L_1 + L_2 + L_3$, where L_1, L_2 and L_3 are lines such that $P = L_1 \cap L_2 \cap L_3$;
- $T_P = L_1 + L_2 + L_3$, where L_1, L_2 and L_3 are lines such that $L_3 \not\ni P = L_1 \cap L_2$;
- $T_P = L + C$, where L is a line and C is a conic such that $P \in C \cap L$.
- T_P is an irreducible cubic curve.

We plan to bound the integral in (3.2) depending on the type of the curve T_P and on the position of the point $Q \in E_1$. First, we deal with the cases when Q is contained in the proper transform of the curve T_P. We start with

Lemma 3.3. Suppose that $T_P = L_1 + L_2 + L_3$, where L_1, L_2 and L_3 are lines passing through P. Let $\widetilde{L}_1$, $\widetilde{L}_2$ and $\widetilde{L}_3$ be the proper transforms on $\widetilde{S}$ of the lines L_1, L_2 and L_3, respectively. Suppose that $Q \in \widetilde{L}_1 \cap \widetilde{L}_2 \cap \widetilde{L}_3$. Then

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{17}{9} + \epsilon_k.$$

Proof. We may assume that $Q = \widetilde{L}_1 \cap E_1$. Denote by $\widehat{L}_1$, $\widehat{L}_2$, $\widehat{L}_3$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the curves $\widetilde{L}_1$, $\widetilde{L}_2$, $\widetilde{L}_3$ and E_1, respectively. Then the intersection form of the curves $\widehat{L}_1$, $\widehat{L}_2$, $\widehat{L}_3$ and $\widehat{E}_1$ is negative definite. Moreover, we have

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \widehat{L}_1 + \widehat{L}_2 + \widehat{L}_3 + 3\widehat{E}_1 + 4E_2.$$

Thus, we conclude that $\tau(E_2) = 4$. Now, using Corollary 2.8, we compute

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 1, \\ \frac{20-4x-x^2}{6}, & 1 \leqslant x \leqslant 2, \\ \frac{(4-x)^2}{3}, & 2 \leqslant x \leqslant 4. \end{cases}$$

Then the required result follows from (3.2). □

Lemma 3.4. Suppose that $T_P = L_1 + L_2 + L_3$, where L_1, L_2 and L_3 are lines such that $P = L_1 \cap L_2$ and $P \notin L_3$. Let $\widetilde{L}_1$ and $\widetilde{L}_2$ be the proper transforms on $\widetilde{S}$ of the lines L_1 and L_2, respectively. Suppose that $Q = \widetilde{L}_1 \cap E_1$ or $\widetilde{L}_2 \cap E_1$. Then

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{49}{27} + \epsilon_k.$$

Proof. Denote by $\widehat{L}_1$, $\widehat{L}_2$, $\widehat{L}_3$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the curves L_1, L_2, L_3 and E_1, respectively. Then

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \widehat{L}_1 + \widehat{L}_2 + \widehat{L}_3 + 2\widehat{E}_1 + 3E_2.$$

Since the intersection form of the curves $\widehat{L}_1$, $\widehat{L}_2$, $\widehat{L}_3$ and $\widehat{E}_1$ is semi-negative definite, we conclude that $\tau(E_2) = 3$. Then, using Corollary 2.8, we get

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 1, \\ \frac{20-4x-x^2}{6}, & 1 \leqslant x \leqslant 2, \\ \frac{12-4x}{3}, & 2 \leqslant x \leqslant 3. \end{cases}$$

Then the required result follows from (3.2). □

Lemma 3.5. Suppose that $T_P = L + C$, where L is a line, and C is an irreducible conic. Suppose that L and C meet transversally at P. Denote by $\widetilde{L}$ and $\widetilde{C}$ the proper transforms on $\widetilde{S}$ of the curves L and C, respectively. Suppose that $Q = \widetilde{L} \cap E_1$. Then

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{9}{5} + \epsilon_k.$$

Proof. Denote by $\widehat{L}$, $\widehat{C}$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the curves L, C and E_1, respectively. Then

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \widehat{L} + \widehat{C} + 2\widehat{E}_1 + 3E_2.$$

Since the intersection form of the curves $\widehat{L}$, $\widehat{C}$ and $\widehat{E}_1$ is negative definite, we conclude that $\tau(E_2) = 3$. Moreover, using Corollary 2.8, we get

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 1, \\ \frac{20-4x-x^2}{6}, & 1 \leqslant x \leqslant \frac{14}{5}, \\ 4(3-x)^2, & \frac{14}{5} \leqslant x \leqslant 3. \end{cases}$$

Now the required assertion follows from (3.2). □

Lemma 3.6. Suppose that $T_P = L + C$, where L is a line, and C is an irreducible conic. Suppose that L and C meet transversally at P. Denote by $\widetilde{L}$ and $\widetilde{C}$ the proper transforms on $\widetilde{S}$ of the curves L and C, respectively. Suppose that $Q = \widetilde{C} \cap E_1$. Then

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{5}{3} + \epsilon_k.$$

Proof. Denote by $\widehat{L}$, $\widehat{C}$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the curves L, C and E_1, respectively. Then

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \widehat{L} + \widehat{C} + 2\widehat{E}_1 + 3E_2.$$

Since the intersection form of the curves $\widehat{L}$, $\widehat{C}$ and $\widehat{E}_1$ is negative definite, we conclude that $\tau(E_2) = 3$. Moreover, using Corollary 2.8, we get

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 2, \\ (3-x)^2 & 2 \leqslant x \leqslant 3. \end{cases}$$

Now the required assertion follows from (3.2). □

Lemma 3.7. Suppose that $T_P = L + C$, where L is a line and C is an irreducible conic. Suppose that L and C meet tangentially at P. Denote by $\widetilde{L}$ and $\widetilde{C}$ the proper transforms on $\widetilde{S}$ of the curves L and C, respectively. Suppose that $Q = E_1 \cap \widetilde{L} \cap \widetilde{C}$. Then

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{17}{9} + \epsilon_k.$$

Proof. Denote by $\widehat{L}$, $\widehat{C}$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the curves $\widetilde{L}$, $\widetilde{L}$ and E_1, respectively. Then

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \widehat{L} + \widehat{C} + 2\widehat{E}_1 + 4E_2.$$

Since the intersection form of the curves $\widehat{L}$, $\widehat{C}$ and $\widehat{E}_1$ is negative definite, we conclude that $\tau(E_2) = 4$. Moreover, using Corollary 2.8, we get

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 1, \\ \frac{20-4x-x^2}{6}, & 1 \leqslant x \leqslant 2, \\ \frac{(4-x)^2}{3}, & 2 \leqslant x \leqslant 4. \end{cases}$$

Then the required result follows from (3.2). □

Lemma 3.8. Suppose that T_P is an irreducible cubic. Let $\widetilde{C}$ be the proper transform of the curve C on the surface $\widetilde{S}$. Suppose that $Q \in \widetilde{C}$. Then

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{5}{3} + \epsilon_k.$$

Proof. Denote by $\widehat{C}$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the curves $\widetilde{C}$ and E_1, respectively. Then

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \widehat{C} + 2\widehat{E}_1 + 3E_2.$$

This gives $\tau(E_2) = 3$, because the intersection form of the curves $\widehat{C}$ and $\widehat{E}_1$ is negative definite. Using Corollary 2.8, we get

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 2, \\ (3-x)^2, & 2 \leqslant x \leqslant 3. \end{cases}$$

Then the required result follows from (3.2). □

Now we consider the cases when Q is not contained in the proper transform of the singular curve T_P on the surface $\widetilde{S}$. We start with

Lemma 3.9. Suppose that $T_P = L_1 + L_2 + L_3$, where L_1, L_2 and L_3 are lines passing through P. Let $\widetilde{L}_1$, $\widetilde{L}_2$ and $\widetilde{L}_3$ be the proper transforms on $\widetilde{S}$ of the lines L_1, L_2 and L_3, respectively. Suppose that $Q \notin \widetilde{L}_1 \cup \widetilde{L}_2 \cup \widetilde{L}_3$. Then

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{5}{3} + \epsilon_k.$$

Proof. Denote by $\widehat{L}_1$, $\widehat{L}_2$, $\widehat{L}_3$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the curves $\widetilde{L}_1$, $\widetilde{L}_2$, $\widetilde{L}_3$ and E_1, respectively. Then

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \widehat{L}_1 + \widehat{L}_2 + \widehat{L}_3 + 3\widehat{E}_1 + 3E_2.$$

This gives $\tau(E_2)=3$, because the intersection form of the curves $\widehat{L}_1$, $\widehat{L}_2$, $\widehat{L}_3$ and $\widehat{E}_1$ is negative definite. Using Corollary 2.8, we get

$$\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)=\begin{cases}3-\frac{x^2}{2}, & 0\leqslant x\leqslant 2,\\ (3-x)^2, & 2\leqslant x\leqslant 3.\end{cases}$$

Then the required result follows from (3.2). $\square$

In the remaining cases, the pseudoeffective threshold $\tau(E_2)$ is not (always) easy to compute. There is a (birational) reason for this. To explain it, recall from [D12] that the linear system $|-K_{\widetilde{S}}|$ is free from base points and gives a morphism $\phi\colon\widetilde{S}\to\mathbb{P}^2$. Taking its Stein factorization, we obtain a commutative diagram

$$\begin{tikzcd}\widetilde{S} \arrow[r, "\alpha"] \arrow[d, "\pi"'] \arrow[dr, "\phi"] & \overline{S} \arrow[d, "\beta"] \\ S \arrow[r, dashed, "\rho"'] & \mathbb{P}^2\end{tikzcd}$$

where α is a birational morphism, β is a double cover branched over a (possibly singular) quartic curve, and ρ is a linear projection from the point P. Here, the surface $\overline{S}$ is a (possibly singular) del Pezzo surface of degree 2. Note that the morphism α is biregular if and only if the curve T_P is irreducible. Moreover, if T_P is reducible, then α-exceptional curves are proper transforms of the lines on S that pass through P.

Let ι be the Galois involution of the double cover β. Then its action lifts to $\widetilde{S}$. On the other hand, this action does not always descent to a (biregular) action of the surface S. Nevertheless, we can always consider ι as a birational involution of the surface S. This involution is known as Geiser involution (see [D12]). It is biregular if and only if P is an Eckardt point of the surface. In this case, the curve E_1 is ι-invariant. However, if P is not an Eckardt point, then $\iota(E_1)$ is the proper transform of the (unique) irreducible component of the curve T_P that is not a line passing through P. In both cases, there exists a commutative diagram

$$\begin{tikzcd} & \widetilde{S} \arrow[dl, "\pi"'] \arrow[dr, "\nu"] & \\ S \arrow[rr, dashed, "\psi"'] & & S'\end{tikzcd}$$

where S' is a smooth cubic surface in $\mathbb{P}^3$, which is isomorphic to the surface S via the involution τ, the morphism ν is the contraction of the curve $\iota(E_1)$, and ψ is a birational map given by the linear subsystem in $|-2K_S|$ consisting of all curves having multiplicity at least 3 at the point P.

Let $Q'=\nu(Q)$ and $P'=\nu(\iota(E_1))$. Denote by T'_Q the unique hyperplane section of the cubic surface S' that is singular at Q'. If P is not an Eckardt point and Q is not contained in the proper transform of the curve T_P, then $Q'\neq P'$. In this case, the number $\tau(E_2)$ can be computed using T'_Q. This explains why the remaining cases are (slightly) more complicated.

Lemma 3.10. Suppose that $T_P = L_1 + L_2 + L_3$, where L_1, L_2 and L_3 are lines such that $P = L_1 \cap L_2$ and $P \notin L_3$. Let $\widetilde{L}_1$, $\widetilde{L}_2$ and $\widetilde{L}_3$ be the proper transforms on $\widetilde{S}$ of the lines L_1, L_2 and L_3, respectively. Suppose that $Q \notin \widetilde{L}_1 \cup \widetilde{L}_2$. Then

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{5}{3} + \epsilon_k.$$

Proof. Denote by $\widehat{L}_1$, $\widehat{L}_2$, $\widehat{L}_3$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the curves L_1, L_2, L_3 and E_1, respectively. Then

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \widehat{L}_1 + \widehat{L}_2 + \widehat{L}_3 + 2\widehat{E}_1 + 2E_2,$$

which implies that $\tau(E_2) \leqslant 2$. Using Corollary 2.9, we see that

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = 3 - \frac{x^2}{2}$$

provided that $0 \leqslant x \leqslant 2$. However, we have $\tau(E_2) > 2$, because the intersection form of the curves $\widehat{L}_1$, $\widehat{L}_2$, $\widehat{L}_3$ and $\widehat{E}_1$ is not semi-negative definite. This also follows from the fact that $\mathrm{vol}(\eta^*(-K_S) - 2E_2) > 0$.

Recall that $\nu\colon \widetilde{S} \to S'$ is the contraction of the curve $\widetilde{L}_3$. We let $L_1' = \nu(\widetilde{L}_1)$, $L_2' = \nu(\widetilde{L}_2)$ and $E_1' = \nu(E_1)$. Then L_1', L_2' and E_1' are coplanar lines on S'.

Since $Q' \in E_1'$, the line E_1' is an irreducible component of the curve T_Q'. Thus, either T_Q' consists of three lines, or T_Q' is a union of the line E_1' and an irreducible conic.

Suppose that $T_Q' = E_1' + Z'$, where Z' is an irreducible conic on S'. Then $Q' \in E_1' \cap Z'$ and $Z' \sim L_1' + L_2'$, which implies that the conic Z' does not meet the lines L_1' and L_2'. Denote by $\widehat{Z}$ the proper transform of the conic Z' on the surface $\widehat{S}$. We have

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \frac{1}{2}\Big(\widehat{Z} + \widehat{L}_1 + \widehat{L}_2\Big) + 2\widehat{E}_1 + \frac{5}{2}E_2.$$

This implies that $\tau(E_2) = \frac{5}{2}$, because the intersection form of the curves $\widehat{Z}$, $\widehat{L}_1$, $\widehat{L}_2$ and $\widehat{E}_1$ is semi-negative definite. Using this $\mathbb{Q}$-rational equivalence and Corollary 2.8, we compute

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 2, \\ 5 - 2x, & 2 \leqslant x \leqslant \frac{5}{2}. \end{cases}$$

Thus, a direct computation and (3.2) give

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{59}{36} + \epsilon_k < \frac{5}{3} + \epsilon_k,$$

which gives the required assertion.

To complete the proof, we may assume that $T_Q' = E_1' + M' + N'$, where M' and N' are two lines on S' such that $Q' = E_1' \cap M'$. Then $M' + N' \sim L_1' + L_2'$, which implies that the lines M' and N' do not meet the lines L_1' and L_2'. Denote by $\widehat{M}$ and $\widehat{N}$ the proper transforms on the surface $\widehat{S}$ of the lines M' and N', respectively.

Suppose that Q' is also contained in the line N'. This simply means that Q' is an Eckardt point of the surface S'. Then

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \frac{1}{2}\Big(\widehat{M} + \widehat{N} + \widehat{L}_1 + \widehat{L}_2\Big) + 2\widehat{E}_1 + 3E_2.$$

This gives $\tau(E_2)\geqslant 3$. In fact, we have $\tau(E_2)=3$ here, because the intersection form of the curves $\widehat{M}$, $\widehat{N}$, $\widehat{L}_1$, $\widehat{L}_2$, $\widehat{E}_1$ is negative definite. Using Corollary 2.8, we get

$$\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)=\begin{cases}3-\frac{x^2}{2}, & 0\leqslant x\leqslant 2,\\ (3-x)^2 & 2\leqslant x\leqslant 3.\end{cases}$$

Now, direct computations and (3.2) give the required inequality.

To complete the proof the lemma, we have to consider the case $Q'\notin N'$. Then

$$\eta^*(-K_S)\sim_{\mathbb{Q}}\frac{1}{2}\Big(\widehat{M}+\widehat{N}+\widehat{L}_1+\widehat{L}_2\Big)+2\widehat{E}_1+\frac{5}{2}E_2.$$

In particular, we see that $\tau(E_2)\geqslant\frac{5}{2}$. Using this $\mathbb{Q}$-rational equivalence and Corollary 2.8, we compute

$$\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)=\begin{cases}3-\frac{x^2}{2}, & 0\leqslant x\leqslant 2,\\ 7-4x+\frac{x^2}{2}, & 2\leqslant x\leqslant\frac{5}{2}.\end{cases}$$

Thus, in particular, we have $\tau(E_2)>\frac{5}{2}$, since

$$\mathrm{vol}\Big(\eta^*(-K_S)-\frac{5}{2}E_2\Big)=\frac{1}{8}.$$

As in the previous cases, we can find $\tau(E_2)$ and compute $\mathrm{vol}(\eta^*(-K_S)-xE_2)$ for $x>\frac{5}{2}$. However, we can avoid doing this. Namely, note that the divisor $\widehat{E}_1+2\widehat{N}+\widehat{M}$ is nef and

$$\Big(\widehat{E}_1+2\widehat{N}+\widehat{M}\Big)\cdot\Big(\eta^*(-K_S)-xE_2\Big)=6-2x,$$

so that $\tau(E_2)\leqslant 3$. Therefore, using (3.2) and Lemma 2.13, we see that

$$\begin{aligned}\mathrm{mult}_Q\big(\pi^*(D)\big)&\leqslant\frac{1}{3}\int_0^{\tau(E_2)}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\epsilon_k=\\ &=\frac{1}{3}\int_0^{\frac{5}{2}}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\frac{1}{3}\int_{\frac{5}{2}}^{\tau(E_2)}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\epsilon_k=\\ &=\frac{79}{48}+\frac{1}{3}\int_{\frac{5}{2}}^{\tau(E_2)}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\epsilon_k\leqslant\frac{79}{48}+\frac{\tau(E_2)-\frac{5}{2}}{3}\mathrm{vol}\Big(\eta^*(-K_S)-\frac{5}{2}E_2\Big)+\epsilon_k=\\ &=\frac{79}{48}+\frac{\tau(E_2)-\frac{5}{2}}{24}+\epsilon_k\leqslant\frac{79}{48}+\frac{1}{48}+\epsilon_k=\frac{5}{3}+\epsilon_k.\end{aligned}$$

This finish the proof of the lemma. $\square$

Lemma 3.11. Suppose that $T_P=L+C$, where L is a line and C is an irreducible conic. Denote by $\widetilde{L}$ and $\widetilde{C}$ the proper transforms on $\widetilde{S}$ of the curves L and C, respectively. Suppose that $Q\notin\widetilde{L}\cup\widetilde{C}$. Then

$$\mathrm{mult}_Q(\pi^*(D))\leqslant\frac{5}{3}+\epsilon_k.$$

Proof. Denote by $\widehat{L}$, $\widehat{C}$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the curves L, $\widetilde{C}$ and E_1, respectively. Then

$$\eta^*(-K_S)\sim_{\mathbb{Q}}\widehat{L}+\widehat{C}+2\widehat{E}_1+2E_2,$$

so that $\tau(E_2)\geqslant 2$. Using Corollary 2.9, we see that

$$\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)=3-\frac{x^2}{2}$$

provided that $0\leqslant x\leqslant 2$. Since $\mathrm{vol}(\eta^*(-K_S)-2E_2)>0$, we see that $\tau(E_2)>2$.

Recall that $\nu\colon\widetilde{S}\to S'$ is the contraction of the curve $\widetilde{C}$. Let $L'=\nu(\widetilde{L})$ and $E_1'=\nu(E_1)$. Then L' is a line and E_1' is a conic on S' such that $P'\in L'\cap E_1'$.

First, we suppose that T_Q' is irreducible. Denote by $\widehat{T}_Q$ the proper transform of the cubic T_Q' on the surface $\widehat{S}$. Then $\widehat{T}_Q\cdot\widehat{E}_1=0$ and

$$\widehat{T}_Q\cdot\widehat{L}=\widehat{E}_1\cdot\widehat{L}=1.$$

Since $\widehat{L}^2=\widehat{E}_1^2=-2$ and $\widehat{T}_Q^2=-1$, we see that the intersection form of the curves $\widehat{L}$, $\widehat{T}_Q$ and $\widehat{E}_1$ is negative definite. On the other hand, we have

$$\eta^*(-K_S)\sim_{\mathbb{Q}}\frac{1}{2}\Big(\widehat{T}_Q+\widehat{L}\Big)+\frac{3}{2}\widehat{E}_1+\frac{5}{2}E_2.$$

This shows that $\tau(E_2)=\frac{5}{2}$. Hence, using Corollary 2.8, we get

$$\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)=\begin{cases}3-\frac{x^2}{2}, & 0\leqslant x\leqslant 2,\\ \frac{44-8x-4x^2}{12}, & 2\leqslant x\leqslant\frac{17}{7},\\ 4(5-2x)^2, & \frac{17}{7}\leqslant x\leqslant\frac{5}{2}.\end{cases}$$

Then a direct calculation and (3.2) give

$$\mathrm{mult}_Q\big(\pi^*(D)\big)\leqslant\frac{103}{63}+\epsilon_k<\frac{5}{3}+\epsilon_k.$$

Now we suppose that $T_Q'=\ell'+Z'$, where ℓ' is a line, and Z' is an irreducible conic. Denote by $\widehat{\ell}$ and $\widehat{Z}$ the proper transforms on $\widehat{S}$ of the curves ℓ' and Z', respectively. We get

$$\eta^*(-K_S)\sim_{\mathbb{Q}}\frac{1}{2}\Big(\widehat{\ell}+\widehat{Z}+\widehat{L}\Big)+\frac{3}{2}\widehat{E}_1+\frac{5}{2}E_2.$$

which implies that $\tau(E_2)\geqslant\frac{5}{2}$. Using Corollary 2.8, we get

$$\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)=\begin{cases}3-\frac{x^2}{2}, & 0\leqslant x\leqslant 2,\\ \frac{34-16x+x^2}{6}, & 2\leqslant x\leqslant\frac{5}{2}.\end{cases}$$

In particular, we have

$$\mathrm{vol}\Big(\eta^*(-K_S)-\frac{5}{2}E_2\Big)=\frac{1}{24},$$

which implies that $\tau(E_2)>\frac{5}{2}$. Observe that the divisor $\widehat{\ell}+2\widehat{Z}+\widehat{L}$ is nef and

$$\big(\widehat{\ell}+2\widehat{Z}+\widehat{L}\big)\cdot\big(\eta^*(-K_S)-xE_2\big)=9-3x,$$

which implies that $\tau(E_2)\leqslant 3$. Thus, using (3.2) and Lemma 2.13, we get

$$\mathrm{mult}_Q\big(\pi^*(D)\big)\leqslant\frac{1}{3}\int_0^{\tau(E_2)}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\epsilon_k=$$
$$=\frac{1}{3}\int_0^{\frac{5}{2}}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\frac{1}{3}\int_{\frac{5}{2}}^{\tau(E_2)}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\epsilon_k=$$
$$=\frac{709}{432}+\frac{1}{3}\int_{\frac{5}{2}}^{\tau(E_2)}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\epsilon_k\leqslant\frac{709}{432}+\frac{\tau(E_2)-\frac{5}{2}}{3}\mathrm{vol}\Big(\eta^*(-K_S)-\frac{5}{2}E_2\Big)+\epsilon_k=$$
$$=\frac{709}{432}+\frac{\tau(E_2)-\frac{5}{2}}{48}+\epsilon_k\leqslant\frac{709}{432}+\frac{1}{96}+\epsilon_k=\frac{89}{54}+\epsilon_k<\frac{5}{3}+\epsilon_k.$$

To complete the proof of the lemma, we may assume that $T'_Q=\ell'+M'+N'$, where ℓ', M' and N' are lines such that $Q'\in M'\cap N'$. Since E_1' is a conic passing through Q', we conclude that Q' is not contained in the line ℓ'. Note that $\ell'\neq L'$, and the lines ℓ', M' and N' do not pass through P'.

Denote by $\widehat{\ell}$, $\widehat{M}$ and $\widehat{N}$ the proper transforms on $\widehat{S}$ of the lines ℓ', M' and N', respectively. We get

$$\eta^*(-K_S)\sim_{\mathbb{Q}}\frac{1}{2}\Big(\widehat{\ell}+\widehat{M}+\widehat{N}+\widehat{L}\Big)+\frac{3}{2}\widehat{E}_1+\frac{5}{2}E_2,$$

which implies that $\tau(E_2)\geqslant\frac{5}{2}$. In fact, we have $\tau(E_2)>\frac{5}{2}$, because the intersection form of the curves $\widehat{\ell}$, $\widehat{M}$, $\widehat{N}$, $\widehat{L}$ and $\widehat{E}_1$ is not semi-negative definite. Nevertheless, we can use Corollary 2.8 to compute

$$\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)=\begin{cases}3-\frac{x^2}{2}, & 0\leqslant x\leqslant 2,\\ \frac{92-56x+8x^2}{12}, & 2\leqslant x\leqslant\frac{5}{2},\end{cases}$$

so that, in particular, we have

$$\mathrm{vol}\Big(\eta^*(-K_S)-\frac{5}{2}E_2\Big)=\frac{1}{6}.$$

Observe that the divisor $2\widehat{\ell}+\widehat{M}+\widehat{N}$ is nef and

$$\big(2\widehat{\ell}+\widehat{M}+\widehat{N}\big)\cdot\big(\eta^*(-K_S)-xE_2\big)=6-2x,$$

which implies that $\tau(E_2)\leqslant 3$. Thus, using (3.2) and Lemma 2.14, we get

$$\mathrm{mult}_Q\big(\pi^*(D)\big)\leqslant\frac{1}{3}\int_0^{\tau(E_2)}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\epsilon_k=$$
$$=\frac{1}{3}\int_0^{\frac{5}{2}}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\frac{1}{3}\int_{\frac{5}{2}}^{\tau(E_2)}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\epsilon_k=$$
$$=\frac{89}{54}+\frac{1}{3}\int_{\frac{5}{2}}^{\tau(E_2)}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\epsilon_k\leqslant\frac{89}{54}+\frac{2}{9}\Big(\tau(E_2)-\frac{5}{2}\Big)\mathrm{vol}\Big(\eta^*(-K_S)-\frac{5}{2}E_2\Big)+\epsilon_k=$$
$$=\frac{89}{54}+\frac{2}{54}\Big(\tau(E_2)-\frac{5}{2}\Big)+\epsilon_k\leqslant\frac{89}{54}+\frac{1}{54}+\epsilon_k=\frac{5}{3}+\epsilon_k.$$

The proof is complete. □

Lemma 3.12. Suppose that T_P is an irreducible cubic curve. Let $\widetilde{C}$ be its proper transform on the surface $\widetilde{S}$. Suppose that $Q \notin \widetilde{C}$. Then

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{5}{3} + \epsilon_k.$$

Proof. Denote by $\widehat{C}$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the curves $\widetilde{C}$ and E_1, respectively. Then

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \widehat{C} + 2\widehat{E}_1 + 2E_2.$$

Thus, using Corollary 2.9, we get $\mathrm{vol}(\eta^*(-K_S) - xE_2) = 3 - \frac{x^2}{2}$ provided that $0 \leqslant x \leqslant 2$.

Recall that $\nu\colon \widetilde{S} \to S'$ is the contraction of the curve $\widetilde{C}$. Let $E' = \nu(E_1)$. Then E_1' is an irreducible cubic curve that is singular at P'. Thus, the curve E_1' is smooth at the point Q', so that $T_Q' \neq E_1'$. One can easily check that T_Q' does not contain P'.

Suppose that T_Q' is an irreducible cubic. Denote by $\widehat{T}_Q$ the proper transform of the curve T_Q' on the surface $\widehat{S}$. We get $\widehat{E}_1^2 = -2$, $\widehat{T}_Q^2 = -1$, $\widehat{E}_1 \cdot \widehat{T}_Q = 1$ and

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \frac{1}{2}\widehat{T}_Q + \frac{3}{2}\widehat{E}_1 + \frac{5}{2}E_2,$$

which implies that $\tau(E_2) = \frac{5}{2}$. Using Corollary 2.8, we get

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant \frac{12}{5}, \\ 3(5-2x)^2, & \frac{12}{5} \leqslant x \leqslant \frac{5}{2}. \end{cases}$$

Then (3.2) and direct calculations give

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{49}{30} + \epsilon_k < \frac{5}{3} + \epsilon_k.$$

Now we suppose that $T_Q' = \ell' + Z'$, where ℓ' is a line and Z' is an irreducible conic. Denote by $\widehat{\ell}$ and $\widehat{Z}$ the proper transforms on $\widehat{S}$ of the curves ℓ_Q' and Z', respectively. We get

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \frac{1}{2}\Big(\widehat{\ell} + \widehat{Z}\Big) + \frac{3}{2}\widehat{E}_1 + \frac{5}{2}E_2.$$

Since the intersection form of the curves $\widehat{\ell}$, $\widehat{Z}$ and $\widehat{E}_1$ is semi-negative definite, we conclude that $\tau(E_2) = \frac{5}{2}$. Using Corollary 2.8, we get

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 2, \\ 5 - 2x, & 2 \leqslant x \leqslant \frac{5}{2}. \end{cases}$$

Hence, using (3.2), we see that

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{59}{36} + \epsilon_k < \frac{5}{3} + \epsilon_k.$$

To complete the proof, we may assume that $T_Q' = \ell' + M' + N'$, where ℓ', M' and N' are lines such that $Q' \in M' \cap N'$. Denote by $\widehat{\ell}$, $\widehat{M}$ and $\widehat{N}$ the proper transforms on $\widehat{S}$ of the lines ℓ', M' and N', respectively. If Q' is contained in the line ℓ', then

$$\eta^*(-K_S)\sim_{\mathbb{Q}}\frac{1}{2}\Big(\widehat{\ell}+\widehat{M}+\widehat{N}\Big)+\frac{3}{2}\widehat{E}_1+3E_2,$$

and the intersection form of the curves $\widehat{\ell}$, $\widehat{M}$, $\widehat{N}$ and $\widehat{E}_1$ is negative definite, which implies that $\tau(E_2)=3$. In this case, Corollary 2.8 gives

$$\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)=\begin{cases}3-\frac{x^2}{2}, & 0\leqslant x\leqslant 2,\\ (3-x)^2, & 2\leqslant x\leqslant 3,\end{cases}$$

which implies the required inequality by (3.2).

To complete the proof, we may assume that Q' is not contained in ℓ'. Then the intersection form of the curves $\widehat{\ell}$, $\widehat{M}$, $\widehat{N}$ and $\widehat{E}_1$ is not semi-negative definite. Since

$$\eta^*(-K_S)\sim_{\mathbb{Q}}\frac{1}{2}\Big(\widehat{\ell}+\widehat{M}+\widehat{N}\Big)+\frac{3}{2}\widehat{E}_1+\frac{5}{2}E_2,$$

we conclude that $\tau(E_2)>\frac{5}{2}$. Moreover, using Corollary 2.8, we get

$$\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)=\begin{cases}3-\frac{x^2}{2}, & 0\leqslant x\leqslant 2,\\ \frac{x^2-8x+14}{2}, & 2\leqslant x\leqslant \frac{5}{2}.\end{cases}$$

In particular, we have

$$\mathrm{vol}\Big(\eta^*(-K_S)-\frac{5}{2}E_2\Big)=\frac{1}{8}.$$

Observe that the divisor $2\widehat{\ell}+\widehat{M}+\widehat{N}$ is nef and

$$\big(2\widehat{\ell}+\widehat{M}+\widehat{N}\big)\cdot\big(\eta^*(-K_S)-xE_2\big)=6-2x,$$

which implies that $\tau(E_2)\leqslant 3$. Thus, using (3.2) and Lemma 2.13, we get

$$\begin{aligned}\mathrm{mult}_Q\big(\pi^*(D)\big)&\leqslant\frac{1}{3}\int_0^{\tau(E_2)}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\epsilon_k=\\ &=\frac{1}{3}\int_0^{\frac{5}{2}}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\frac{1}{3}\int_{\frac{5}{2}}^{\tau(E_2)}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\epsilon_k=\\ &=\frac{79}{48}+\frac{1}{3}\int_{\frac{5}{2}}^{\tau(E_2)}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\epsilon_k\leqslant\frac{79}{48}+\frac{\tau(E_2)-\frac{5}{2}}{3}\mathrm{vol}\Big(\eta^*(-K_S)-\frac{5}{2}E_2\Big)+\epsilon_k=\\ &=\frac{79}{48}+\frac{\tau(E_2)-\frac{5}{2}}{24}+\epsilon_k\leqslant\frac{79}{48}+\frac{1}{48}+\epsilon_k=\frac{5}{3}+\epsilon_k.\end{aligned}$$

This completes the proof of the lemma. □

Using Corollary 2.6 and Lemmas 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9, 3.10, 3.11, 3.12, we immediately get

Corollary 3.13. We have $\delta(S)\geqslant\frac{18}{17}$.

4. Proof of the main result

In this section, we prove Theorem 1.4. Let S be a smooth cubic surface. We have to prove that $\delta(S) \geqslant \frac{6}{5}$. Fix a positive rational number $\lambda < \frac{6}{5}$. Let D be a k-basis type divisor. To prove Theorem 1.4, it is enough to show that, the log pair $(S, \lambda D)$ is log canonical for $k \gg 1$. Suppose that this is not the case. Then there exists a point $P \in S$ such that $(S, \lambda D)$ is not log canonical at P for $k \gg 1$. Let us seek for a contradiction using results obtained in Section 3.

Let $\pi\colon \widetilde{S} \to S$ be the blowup of the point P, and let E_1 be the exceptional divisor of the blow up π. Denote by $\widetilde{D}$ the proper transform of D via π. Then

$$K_{\widetilde{S}} + \lambda\widetilde{D} + \big(\lambda\mathrm{mult}_P(D) - 1\big)E_1 \sim_{\mathbb{Q}} \pi^*\big(K_S + \lambda D\big).$$

By Corollary 2.5, the log pair $(\widetilde{S}, \lambda\widetilde{D} + (\lambda\mathrm{mult}_P(D) - 1)E_1)$ is not log canonical at some point $Q \in E_1$. Thus, using Lemma 2.2, we see that

$$\mathrm{mult}_Q\big(\pi^*\big(D\big)\big) = \mathrm{mult}_P\big(D\big) + \mathrm{mult}_Q\big(\widetilde{D}\big) > \frac{2}{\lambda} > \frac{5}{3}. \tag{4.1}$$

Let $\sigma\colon \widehat{S} \to \widetilde{S}$ be the blowup of the point Q, and let E_2 be the exceptional curve of σ. Denote by $\widehat{D}$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the divisors $\widetilde{D}$ and E_1, respectively. By Corollary 2.5, the log pair

$$\Big(\widehat{S}, \lambda\widehat{D} + \big(\lambda\mathrm{mult}_P(D) - 1\big)\widehat{E}_1 + \big(\lambda\mathrm{mult}_P(D) + \lambda\mathrm{mult}_Q(\widetilde{D}) - 2\big)E_2\Big)$$

is not log canonical at some point $O \in E_2$.

Let T_P be the hyperplane section of the surface S that is singular at P. Then T_P must be reducible. This follows from (4.1) and Lemmas 3.8 and 3.12.

Denote by $\widetilde{T}_P$ the proper transform of the curve T_P on the surface $\widetilde{S}$. Then $Q \in \widetilde{T}_P$. This follows from (4.1) and Lemmas 3.10 and 3.11.

In the remaining part of this section, we will deal with the following four cases:

(1) T_P is a union of three lines passing through P;
(2) T_P is a union of three lines and only two of them pass through P;
(3) T_P is a union of line and a conic that intersect transversally at P;
(4) T_P is a union of line and a conic that intersect tangentially at P.

We will treat each of them in a separate subsection. We start with

4.1. Case 1.

We have $T_P = L_1 + L_2 + L_3$, where L_1, L_2 and L_3 are lines passing through the point P. We write

$$\lambda D = a_1L_1 + a_2L_2 + a_3L_3 + \Omega,$$

where a_1, a_2 and a_3 are nonnegative rational numbers, and Ω is an effective $\mathbb{Q}$-divisor whose support does not contain L_1, L_2 or L_3. Then

$$L_1 \cdot \Omega = \lambda + a_1 - a_2 - a_3. \tag{4.2}$$

Denote by $\widetilde{L}_1$, $\widetilde{L}_2$ and $\widetilde{L}_3$ the proper transforms on $\widetilde{S}$ of the lines L_1, L_2 and L_3, respectively. We know that $Q \in \widetilde{L}_1 \cup \widetilde{L}_2 \cup \widetilde{L}_3$, so that we may assume that $Q = \widetilde{L}_1 \cap E_1$.

Let $\widetilde{\Omega}$ be the proper transform of the divisor Ω on the surface $\widetilde{S}$, and let $m = \mathrm{mult}_P(\Omega)$. Then the log pair

$$\Big(\widetilde{S}, a_1\widetilde{L}_1 + \widetilde{\Omega} + \big(a_1 + a_2 + a_3 + m - 1\big)E_1\Big)$$

is not log canonical at the point Q.

By Lemma 3.1, we have

$$a_1 \leqslant \Big(\frac{5}{9} + \varepsilon_k\Big)\lambda < 1, \tag{4.3}$$

where ε_k is a small constant depending on k such that $\varepsilon_k \to 0$ as $k \to \infty$. Thus, applying Corollary 2.4, we see that

$$L_1 \cdot \Omega + a_1 + a_2 + a_3 - 1 = \widetilde{L}_1 \cdot \Big(\widetilde{\Omega} + \big(a_1 + a_2 + a_3 + m - 1\big)E_1\Big) > 1,$$

which gives $L_1 \cdot \Omega > 2 - a_1 - a_2 - a_3$. Combining this with (4.2), we get

$$a_1 > \frac{2 - \lambda}{2}. \tag{4.4}$$

Let $\widetilde{m} = \mathrm{mult}_Q(\widetilde{\Omega})$. Then by Lemma 3.3, we have

$$2a_1 + a_2 + a_3 + m + \widetilde{m} \leqslant \Big(\frac{17}{9} + \epsilon_k\Big)\lambda, \tag{4.5}$$

where ϵ_k is a small constant depending on k such that $\epsilon_k \to 0$ as $k \to \infty$. Then using (4.4) and $m \geqslant \widetilde{m}$, we deduce that

$$\widetilde{m} < \Big(\frac{13}{9} + \frac{\epsilon_k}{2}\Big)\lambda - 1 < 1. \tag{4.6}$$

Denote by $\widehat{L}_1$ and $\widehat{\Omega}$ the proper transforms on $\widehat{S}$ of the divisors $\widetilde{L}_1$ and $\widetilde{\Omega}$, respectively. Then the log pair

$$\Big(\widehat{S}, a_1\widehat{L}_1 + \widehat{\Omega} + \big(a_1 + a_2 + a_3 + m - 1\big)\widehat{E_1} + \big(2a_1 + a_2 + a_3 + m + \widetilde{m} - 2\big)E_2\Big)$$

is not log canonical at the point O.

We claim that $O \in \widehat{L}_1 \cup \widehat{E}_1$. Indeed, we have $(2a_1 + a_2 + a_3 + m + \widetilde{m} - 2) < 1$ by (4.5). Thus, if $O \notin \widehat{L}_1 \cup \widehat{E}_1$, then Corollary 2.4 gives

$$\widetilde{m} = \widehat{\Omega} \cdot E_2 \geqslant \big(\widehat{\Omega} \cdot E_2\big)_O > 1,$$

which is impossible by (4.6). Thus, we have $O \in \widehat{L}_1 \cup \widehat{E}_1$.

If $O \in \widehat{E}_1$, then the log pair

$$\Big(\widehat{S}, \widehat{\Omega} + \big(a_1 + a_2 + a_3 + m - 1\big)\widehat{E_1} + \big(2a_1 + a_2 + a_3 + m + \widetilde{m} - 2\big)E_2\Big)$$

is not log canonical at the point O. Then Corollary 2.4 gives $a_1 + a_2 + a_3 + m + \widetilde{m} > 2$, so that (4.4) and (4.5) gives

$$\Big(\frac{17}{9} + \epsilon_k\Big)\lambda \geqslant 2a_1 + a_2 + a_3 + m + \widetilde{m} > 2 + a_1 > 3 - \frac{\lambda}{2},$$

which is impossible, since $\lambda < \frac{6}{5}$ and $\epsilon_k \to 0$ as $k \to \infty$.

Thus, we see that $O \in \widehat{L}_1$. Then the log pair

$$\Big(\widehat{S}, a_1\widehat{L}_1 + \widehat{\Omega} + \big(2a_1 + a_2 + a_3 + m + \widetilde{m} - 2\big)E_2\Big)$$

is not log canonical at the point O. Now, using (4.5) and (4.6), we have

$$\mathrm{mult}_O\Big(\widehat{\Omega}+\big(2a_1+a_2+a_3+m+\widetilde{m}-2\big)E_2\Big) = 2a_1+a_2+a_3+m+2\widetilde{m}-2 < \Big(\frac{10}{3}+\frac{3\epsilon_k}{2}\Big)\lambda-3 < 1,$$

since $\lambda < \frac{6}{5}$ and $k \gg 1$. Thus, Lemma 2.3 gives

$$L_1 \cdot \Omega + 2a_1 + a_2 + a_3 - 2 = \widehat{L}_1 \cdot \Big(\widehat{\Omega} + \big(2a_1 + a_2 + a_3 + m + \widetilde{m} - 2\big)E_2\Big) > 2 - a_1,$$

so that $L_1 \cdot \Omega + 3a_1 + a_2 + a_3 > 4$. Using (4.2) we get $\lambda + 4a_1 > 4$. Using (4.3), we get

$$\Big(\frac{29}{9} - \varepsilon_k\Big)\lambda > 4,$$

which is impossible, since $\lambda < \frac{6}{5}$ and $\varepsilon_k \to 0$ as $k \to \infty$.

4.2. **Case 2.** We have $T_P = L_1 + L_2 + L_3$, where L_1, L_2 and L_3 are coplanar lines such that $P = L_1 \cap L_2$ and $P \notin L_3$. As in the previous case, we write

$$\lambda D = a_1 L_1 + a_2 L_2 + \Omega,$$

where a_1 and a_2 are nonnegative rational numbers, and Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the lines L_1 and L_2. Then

$$L_1 \cdot \Omega = \lambda + a_1 - a_2. \tag{4.7}$$

Denote by $\widetilde{L}_1$ and $\widetilde{L}_2$ the proper transforms on $\widetilde{S}$ of the lines L_1 and L_2, respectively. We know that $Q \in \widetilde{L}_1 \cup \widetilde{L}_2$, so that we may assume that $Q = \widetilde{L}_1 \cap E_1$. Let $\widetilde{\Omega}$ be the proper transform of the divisor Ω on the surface $\widetilde{S}$, and let $m = \mathrm{mult}_P(\Omega)$. Then the log pair

$$\Big(\widetilde{S}, a_1\widetilde{L}_1 + \widetilde{\Omega} + \big(a_1 + a_2 + a_3 + m - 1\big)E_1\Big)$$

is not log canonical at the point Q.

By Lemma 3.1, we have

$$a_1 \leqslant \Big(\frac{5}{9} + \varepsilon_k\Big)\lambda < 1, \tag{4.8}$$

where ε_k is a small constant depending on k such that $\varepsilon_k \to 0$ as $k \to \infty$. Thus, using Corollary 2.4, we obtain $L_1 \cdot \Omega > 2 - a_1 - a_2$. Then, using (4.7), we deduce

$$a_1 > \frac{2-\lambda}{2}. \tag{4.9}$$

Let $\widetilde{m} = \mathrm{mult}_Q(\widetilde{\Omega})$. By Lemma 3.4, we have

$$2a_1 + a_2 + m + \widetilde{m} \leqslant \Big(\frac{49}{27} + \epsilon_k\Big)\lambda. \tag{4.10}$$

where ϵ_k is a small constant depending on k such that $\epsilon_k \to 0$ as $k \to \infty$. Thus, using (4.9) and $\widetilde{m} \leqslant m$, we deduce

$$\widetilde{m} < \Big(\frac{38}{27} + \frac{\epsilon_k}{2}\Big)\lambda - 1 < 1. \tag{4.11}$$

Denote by $\widehat{L}_1$ and $\widehat{\Omega}$ the proper transforms on $\widehat{S}$ of the divisors $\widetilde{L}_1$ and $\widetilde{\Omega}$, respectively. Then the log pair

$$\Big(\widehat{S}, a_1\widehat{L}_1 + \widehat{\Omega} + \big(a_1 + a_2 + m - 1\big)\widehat{E}_1 + \big(2a_1 + a_2 + m + \widetilde{m} - 2\big)E_2\Big)$$

is not log canonical at the point O. Then $2a_1 + a_2 + m + \widetilde{m} - 2 < 1$ by (4.10). Thus, using (4.11) and arguing as in Subsection 4.1, we see that $O \in \widehat{L}_1 \cup \widehat{E}_1$.

If $O \in \widehat{E}_1$, then the log pair

$$\Big(\widehat{S}, \widehat{\Omega} + \big(a_1 + a_2 + m - 1\big)\widehat{E}_1 + \big(2a_1 + a_2 + m + \widetilde{m} - 2\big)E_2\Big)$$

is not log canonical at the point O, so that $a_1 + a_2 + m + \widetilde{m} > 2$ by Corollary 2.4. Hence, using (4.9) and (4.10), we get

$$\Big(\frac{49}{27} + \epsilon_k\Big)\lambda \geqslant 2a_1 + a_2 + m + \widetilde{m} > 2 + a_1 > 3 - \frac{\lambda}{2},$$

which is impossible, since $\lambda < \frac{6}{5}$ and $\epsilon_k \to 0$ as $k \to \infty$.

We see that $O \in \widehat{L}_1$. Then the log pair

$$\Big(\widehat{S}, a_1\widehat{L}_1 + \widehat{\Omega} + \big(2a_1 + a_2 + m + \widetilde{m} - 2\big)E_2\Big)$$

is not log canonical at the point O. Now, using (4.10) and (4.11), we deduce

$$\mathrm{mult}_O\Big(\widehat{\Omega} + \big(2a_1 + a_2 + m + \widetilde{m} - 2\big)E_2\Big) = 2a_1 + a_2 + m + 2\widetilde{m} - 2 < \Big(\frac{29}{9} + \frac{3\epsilon_k}{2}\Big)\lambda - 3 < 1,$$

because $\lambda < \frac{6}{5}$ and $k \gg 1$. Then we may apply Lemma 2.3 to get

$$L_1 \cdot \Omega + 2a_1 + a_2 - 2 = \widehat{L}_1 \cdot \Big(\widehat{\Omega} + \big(2a_1 + a_2 + m + \widetilde{m} - 2\big)E_2\Big) > 2 - a_1,$$

so that $L_1 \cdot \Omega + 3a_1 + a_2 > 4$. Using (4.7) we get $\lambda + 4a_1 > 4$. Then, by (4.8), we have

$$\Big(\frac{29}{9} - \varepsilon_k\Big)\lambda > 4,$$

which is impossible, since $\lambda < \frac{6}{5}$ and $\varepsilon_k \to 0$ as $k \to \infty$.

4.3. **Case 3.** We have $T_P = L + C$, where L is a line and C is an irreducible conic such that they intersect transversally at P. As in the previous cases, we write

$$\lambda D = aL + bC + \Omega,$$

where a and b are nonnegative rational numbers, and Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the curves L and C. Then Lemma 3.1 gives us

$$a \leqslant \Big(\frac{5}{9} + \varepsilon_k\Big)\lambda < 1, \tag{4.12}$$

where ε_k is a small constant depending on k such that $\varepsilon_k \to 0$ as $k \to \infty$. And also, we have

$$L \cdot \Omega = \lambda + a - 2b. \tag{4.13}$$

Denote by $\widetilde{L}$ and $\widetilde{C}$ the proper transforms on $\widetilde{S}$ of the curves L and C, respectively. We know that $Q \in \widetilde{L} \cup \widetilde{C}$. Moreover, using (4.1) and Lemma 3.6, we see that $Q = \widetilde{L} \cap E_1$.

Denote by $\widetilde{\Omega}$ the proper transforms on $\widetilde{S}$ of the divisor Ω. Let $m = \mathrm{mult}_P(\Omega)$. Then the log pair

$$\Big(\widetilde{S}, a\widetilde{L} + \widetilde{\Omega} + \big(a+b+m-1\big)E_1\Big)$$

is not log canonical at Q. Since $a<1$, we can apply Corollary 2.4 to this log pair and the curve $\widetilde{L}$. This gives $L\cdot\Omega > 2-a-b$. Combining this with (4.13), we have $\lambda+2a-b>2$, so that

$$a > \frac{2+b-\lambda}{2} \geqslant \frac{2-\lambda}{2}. \tag{4.14}$$

Let $\widetilde{m} = \mathrm{mult}_Q(\widetilde{\Omega})$. Then Lemma 3.5 gives

$$2a+b+m+\widetilde{m} \leqslant \Big(\frac{9}{5}+\epsilon_k\Big)\lambda, \tag{4.15}$$

where ϵ_k is a small constant depending on k such that $\epsilon_k \to 0$ as $k\to\infty$. Thus, using (4.14) and $\widetilde{m}\leqslant m$, we deduce that

$$\widetilde{m} < \Big(\frac{7}{5}+\frac{\epsilon_k}{2}\Big)\lambda - 1 < 1. \tag{4.16}$$

Denote by $\widehat{L}$ and $\widehat{\Omega}$ the proper transforms on $\widehat{S}$ of the divisors $\widetilde{L}$ and $\widetilde{\Omega}$, respectively. Then the log pair

$$\Big(\widehat{S}, a\widehat{L} + \widehat{\Omega} + \big(a+b+m-1\big)\widehat{E}_1 + \big(2a+b+m+\widetilde{m}-2\big)E_2\Big)$$

is not log canonical at the point O. Note that $2a+b+m+\widetilde{m}-2<1$ by (4.15). Thus, using (4.16) and arguing as in Subsection 4.1, we see that $O\in\widehat{L}\cup\widehat{E_1}$.

If $O\in\widehat{E_1}$, then the log pair

$$\Big(\widehat{S}, \widehat{\Omega} + \big(a+b+m-1\big)\widehat{E}_1 + \big(2a+b+m+\widetilde{m}-2\big)E_2\Big)$$

is not log canonical at O. Applying Corollary 2.4 again, we obtain $a+b+m+\widetilde{m}>2$, so that (4.14) and (4.15) give

$$\Big(\frac{9}{5}+\epsilon_k\Big)\lambda \geqslant 2a+b+m+\widetilde{m} > 2+a > 3-\frac{\lambda}{2},$$

which is impossible, since $\lambda<\frac{6}{5}$ and $\epsilon_k\to 0$ as $k\to\infty$.

We see that $O\in\widehat{L}$. Then the log pair

$$\Big(\widehat{S}, a\widehat{L} + \widehat{\Omega} + \big(2a+b+m+\widetilde{m}-2\big)E_2\Big)$$

is not log canonical at the point O. Now using (4.15) and (4.16), we obtain

$$\mathrm{mult}_O\Big(\widehat{\Omega} + \big(2a+b+m+\widetilde{m}-2\big)E_2\Big) = 2a+b+m+2\widetilde{m}-2 < \Big(\frac{12}{5}+\frac{3\epsilon_k}{2}\Big)\lambda - 3 < 1,$$

because $\lambda<\frac{6}{5}$ and $\epsilon_k\to 0$ as $k\to\infty$. Thus, applying Lemma 2.3, we get

$$L\cdot\Omega + 2a+b-1 = \widehat{L}\cdot\Big(\widehat{\Omega} + \big(2a+b+m+\widetilde{m}-2\big)E_2\Big) > 2-a,$$

which gives $L\cdot\Omega+3a+b>4$. Using (4.13), we get $\lambda+4a>4+b\geqslant 4$, so that (4.12) implies that

$$\Big(\frac{29}{9}-\varepsilon_k\Big)\lambda>4,$$

which is impossible, since $\lambda<\frac{6}{5}$ and $\varepsilon_k\to 0$ as $k\to\infty$.

4.4. **Case 4.** We have $T_P=L+C$, where L is a line, and C is an irreducible conic that tangents L at the point P. We write

$$\lambda D=aL+bC+\Omega,$$

where a and b are nonnegative rational numbers, and Ω is an effective $\mathbb{Q}$-divisor whose support does not contain L and C. Let $m=\mathrm{mult}_P(\Omega)$. Then

$$a+b+m>1 \tag{4.17}$$

by Lemma 2.2. Meanwhile, it follows from Lemma 3.1 that

$$a\leqslant\Big(\frac{5}{9}+\varepsilon_k\Big)\lambda<1, \tag{4.18}$$

where ε_k is a small constant depending on k such that $\varepsilon_k\to 0$ as $k\to\infty$. And also, we have

$$L\cdot\Omega=\lambda+a-2b. \tag{4.19}$$

Denote by $\widetilde{L}$ and $\widetilde{C}$ the proper transforms on $\widetilde{S}$ of the curves L and C, respectively. We know that $Q=\widetilde{L}\cap\widetilde{C}$. Denote by $\widetilde{\Omega}$ the proper transforms on $\widetilde{S}$ of the divisor Ω. Then the log pair

$$\Big(\widetilde{S},a\widetilde{L}+b\widetilde{C}+\widetilde{\Omega}+\big(a+b+m-1\big)E_1\Big)$$

is not log canonical at the point Q. Since $a<1$ by (4.18), we may apply Corollary 2.4 to this log pair at Q with respect to the curve $\widetilde{L}$. This gives

$$L\cdot\Omega>2-a-2b.$$

Combining this with (4.19), we get $\lambda+2a>2$, so that

$$a>\frac{2-\lambda}{2}. \tag{4.20}$$

Let $\widetilde{m}=\mathrm{mult}_Q(\widetilde{\Omega})$. Then Lemma 3.7 gives

$$2a+2b+m+\widetilde{m}=\lambda\cdot\mathrm{mult}_Q(\pi^*(D))\leqslant\Big(\frac{17}{9}+\epsilon_k\Big)\lambda. \tag{4.21}$$

where ϵ_k is a small constant depending on k such that $\epsilon_k\to 0$ as $k\to\infty$. Thus, using (4.20) and $\widetilde{m}\leqslant m$, we deduce that

$$\widetilde{m}<\Big(\frac{13}{9}+\frac{\epsilon_k}{2}\Big)\lambda-1<1. \tag{4.22}$$

Denote by $\widehat{L}$, $\widehat{C}$ and $\widehat{\Omega}$ the proper transforms on $\widehat{S}$ of the divisors $\widetilde{L}$, $\widetilde{C}$ and $\widetilde{\Omega}$, respectively. Then the log pair

$$\Big(\widehat{S},a\widehat{L}+b\widehat{C}+\widehat{\Omega}+\big(a+b+m-1\big)\widehat{E}_1+\big(2a+2b+m+\widetilde{m}-2\big)E_2\Big)$$

is not log canonical at O. Moreover, it follows from (4.21) that $2a+2b+m+\widetilde{m}-2<1$. Thus, using (4.22) and arguing as in Subsection 4.1, we see that $O\in\widehat{L}\cup\widehat{C}\cup\widehat{E_1}$.

If $O\in\widehat{E_1}$, then the log pair

$$\Big(\widehat{S},\widehat{\Omega}+\big(a+b+m-1\big)\widehat{E}_1+\big(2a+2b+m+\widetilde{m}-2\big)E_2\Big)$$

is not log canonical at O. In this case, Corollary 2.4 applied to this log pair (and the curve E_2) gives $a+b+m+\widetilde{m}>2$, so that (4.20) and (4.15) give

$$\Big(\frac{17}{9}+\epsilon_k\Big)\lambda\geqslant 2a+2b+m+\widetilde{m}>2+a+b>3-\frac{\lambda}{2},$$

which is impossible, since $\lambda<\frac{6}{5}$ and $\epsilon_k\to 0$ as $k\to\infty$.

If $O\in\widehat{C}$, then the log pair

$$\Big(\widehat{S},b\widehat{C}+\widehat{\Omega}+\big(2a+2b+m+\widetilde{m}-2\big)E_2\Big)$$

is not log canonical at O. In this case, if we apply Corollary 2.4 to this log pair with respect to E_2, we get $b+\widetilde{m}>1$, so that (4.21) gives

$$2a+b+m+1<\Big(\frac{17}{9}+\epsilon_k\Big)\lambda-1.$$

Combining this with (4.17)), we see that $a<(\frac{17}{9}+\epsilon_k)\lambda-2$, so that (4.20) gives

$$\Big(\frac{43}{18}+\epsilon_k\Big)\lambda>3,$$

which is impossible, since $\lambda<\frac{6}{5}$ and $\epsilon_k\to 0$ as $k\to\infty$.

We see that $O\in\widehat{L}$. Then the log pair

$$\Big(\widehat{S},a\widehat{L}+\widehat{\Omega}+\big(2a+2b+m+\widetilde{m}-2\big)E_2\Big)$$

is not log canonical at the point O. Now using (4.21), (4.22) and $\lambda<\frac{6}{5}$, we deduce that

$$\mathrm{mult}_O\Big(\widehat{\Omega}+\big(2a+2b+m+\widetilde{m}-2\big)E_2\Big)=2a+2b+m+2\widetilde{m}-2<\Big(\frac{10}{3}+\frac{3\epsilon_k}{2}\Big)\lambda-3<1.$$

since $\lambda<\frac{6}{5}$ and $k\to\infty$. Then we may apply Lemma 2.3 to get

$$L\cdot\Omega+2a+2b-2=\widehat{L}\cdot\Big(\widehat{\Omega}+\big(2a+2b+m+\widetilde{m}-2\big)E_2\Big)>2-a,$$

which gives $L\cdot\Omega+3a+2b>4$. Using (4.19), we see that $\lambda+4a>4$, so that (4.18) gives

$$\Big(\frac{29}{9}-\varepsilon_k\Big)\lambda>4,$$

which is impossible, since $\lambda<\frac{6}{5}$ and $\varepsilon_k\to 0$ as $k\to\infty$.

The proof of Theorem 1.4 is complete.

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Ivan Cheltsov
University of Edinburgh, United Kingdom
HSE University, Russian Federation
`I.Cheltsov@ed.ac.uk`

Kewei Zhang
Peking University, Beijing, China
`kwzhang@pku.edu.cn`