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The (logarithmic) least squares optimality of the arithmetic (geometric) mean of weight vectors calculated from all spanning trees for incomplete additive (multiplicative) pairwise comparison matrices

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Abstract

Complete and incomplete additive/multiplicative pairwise comparison matrices are applied in preference modelling, multi-attribute decision making and ranking. The equivalence of two well known methods is proved in this paper. The arithmetic (geometric) mean of weight vectors, calculated from all spanning trees, is proved to be optimal to the (logarithmic) least squares problem, not only for complete, as it was recently shown in Lundy, M., Siraj, S., Greco, S. (2017): The mathematical equivalence of the "spanning tree" and row geometric mean preference vectors and its implications for preference analysis, European Journal of Operational Research 257(1) 197–208, but for incomplete matrices as well. Unlike the complete case, where an explicit formula, namely the row arithmetic/geometric mean of matrix elements, exists for the (logarithmic) least squares problem, the incomplete case requires a completely different and new proof. Finally, Kirchhoff's laws for the calculation of potentials in electric circuits is connected to our results.

Keywords: decision analysis, multi-criteria decision making, incomplete pairwise comparison matrix, additive, multiplicative, least squares, logarithmic least squares, Laplacian matrix, spanning tree

1 Introduction

Preference modelling is a family of qualitative and quantitative approaches in order to support decisions, especially the choice of an alternative among a set of possible actions, or ranking them. Many real decision problems involve multiple and often competing criteria [30], therefore the weights of their importance are also taken into account. Pairwise comparisons are applicable in both single and multiple criteria decision making, as they divide complex problem into smaller tasks.

1.1 Incomplete multiplicative pairwise comparison matrices

Cardinal preferences of decision makers are often modelled and calculated by pairwise comparison matrices [45]. Questions 'How many times is a criterion more important than another one?' or 'How many times is a given alternative better than another one with respect to a fixed criterion?' are typical in multi-attribute decision problems. The numerical answers are collected into a

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multiplicative pairwise comparison matrix $\mathbf{A} = [a_{ij}]_{i,j=1\ldots n}$ fulfilling reciprocity, i.e., $a_{ij} = 1/a_{ji}$. A pairwise comparison matrix can be complete, as in the Analytic Hierarchy Process (AHP) [45], or incomplete [7, 13, 23, 31, 37, 40, 39, 42, 46, 47, 48, 51, 56]. A complete multiplicative pairwise comparison matrix $\mathbf{A} = [a_{ij}]$ is called consistent if cardinal transitivity, i.e., $a_{ij}a_{jk} = a_{ik}$ holds for all i, j, k. Otherwise, the matrix is inconsistent, and several inconsistency indices have been proposed, see [9, 11, 40, 45].

In this study incomplete means 'not necessarily complete', in other words, the number of missing elements is allowed to be zero.

Example 1.1. *Let* $\mathbf{A}$ *be a* 6×6 *incomplete multiplicative pairwise comparison matrix as follows:*

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix},$$

where $a_{ij} = 1/a_{ji}$ *for all the known elements.*

Incomplete pairwise comparison matrices can be applied not only in the same multiple criteria decision situations in which the complete matrices arise (hundreds of case studies are listed in, e.g., [33, 50, 57]), but also to larger decision and ranking problems. Bozóki, Csató and Temesi [6] proposed a ranking method for top tennis players based on their pairwise results, where incompleteness occurs in a natural way. Csató [19] constructed a 149×149 incomplete pairwise comparison matrix to rank the teams of the 39th Chess Olympiad 2010. Chao, Kou, Li and Peng [14] ranked 1544 Go players based on their matches played against each other, which naturally formed an incomplete pairwise comparison matrix. Duleba, Mishina and Shimazaki [21] applied small but incomplete matrices in developing a decision model for urban bus transportation supply. Benítez, Delgado-Galván, Izquierdo and Pérez-García [5] calculated the priorities from incomplete matrices in finding the best leakage control policy to minimize water loss. Krejči [36, Chapter 5] presents an incomplete pairwise comparison matrix based model for the evaluation of artistic performance.

1.2 The logarithmic least squares (LLS) problem for multiplicative matrices

The basic problem of finding the *best* weight vector usually includes an additional information on how *closeness* is defined or specified. The classical approaches apply metrics based on least squares [17], weighted least squares [17], logarithmic least squares [18, 29, 35, 44], just to name a few. Further weighting methods are discussed by Golany and Kress [28] and by Choo and Wedley [16]. Even the well-known eigenvector method [45] is proved to be a distance minimizing method [24, 25], although its metric seems to be rather artificial.

Definition 1.1. *The Logarithmic Least Squares (LLS) problem [37, 51] is defined as follows:*

$$\min \sum_{\substack{i,j: \\ a_{ij} \text{ is known}}} \left[\log a_{ij} - \log\left(\frac{w_i}{w_j}\right)\right]^2$$

$$\textit{subject to} \qquad w_i > 0, \qquad i = 1, 2, \ldots, n. \tag{1}$$

Originally, the LLS problem was defined for complete multiplicative pairwise comparison matrices, i.e., the sum in the objective function is taken for all i, j [18, 29, 35, 44]. In this special case, the LLS optimal solution is unique and it can be explicitly computed by taking the row-wise geometric mean [18, 35, 44]. Furthermore, in case of 3×3 complete pairwise comparison matrices, the eigenvector method and the LLS method are equivalent, they result in the same weight vector [18]. Several characterizations of the complete LLS weighting method (or equivalently, the row geometric mean) can be found in [3, 20, 24, 25].

The most common scalings are $\sum_{i=1}^{n} w_i = 1$ and $\prod_{i=1}^{n} w_i = 1$. Scaling $w_1 = 1$ (called *ideal-mode* in Lundy, Siraj and Greco [39]), can also be interpreted in the following way: the first object (criterion, alternative) is considered a reference point and all the others are expressed according to it, similar to SMART [22], if the first criterion is the least important one.

Given an (in)complete pairwise comparison matrix $\mathbf{A}$ of size $n \times n$, an undirected graph $G(V, E)$ is defined as follows: G has n nodes and the edge between nodes i and j is drawn if and only if the matrix element a_{ij} is known. The graph of the incomplete pairwise comparison matrix in Example 1.1 is given in Figure 1.

The graph-theoretic consideration makes it possible to represent the direct comparison a_{ij} between elements i and j, as well as the indirect ones, e.g., via paths of two (a_{ik}, a_{kj}), three $(a_{ik}, a_{k\ell}, a_{\ell j})$ or more edges [1, 3, 8, 27, 31, 32]. See also [23, Subsection 2.2] as well as all references on spanning trees in subsection 1.4 of this paper.

The following theorem provides a method for solving the LLS problem (1).

Theorem 1.1. *(Bozóki, Fülöp, Rónyai [7, Section 4]) Let $\mathbf{A}$ be an incomplete or complete multiplicative pairwise comparison matrix such that its associated graph G is connected. Then the optimal solution $\mathbf{w} = \exp \mathbf{y}$ of the logarithmic least squares problem (1) is the unique solution of the following system of linear equations:*

$$(\mathbf{Ly})_i = \sum_{k:e(i,k)\in E(G)} \log a_{ik} \qquad \textit{for all } i = 1, 2, \ldots, n-1, n, \tag{2}$$

$$y_1 = 0. \tag{3}$$

where $\mathbf{L}$ denotes the Laplacian matrix of G (ℓ_{ii} is the degree of node i and $\ell_{ij} = -1$ if nodes i and j are adjacent).

$\mathbf{L}$ has rank $n-1$. Scaling (3), being equivalent to $w_1 = 1$, plays a technical role only. It can be replaced by, e.g., the commonly used $\prod_{i=1}^{n} w_i = 1$ ($\Leftrightarrow \sum_{i=1}^{n} y_i = 0$).

Example 1.2. *Let incomplete multiplicative pairwise comparison matrix $\mathbf{A}$ be the same as in Example 1.1. Equations (2) for $i = 1, 2, \ldots, 6$ form the following system of linear equations:*

$$\begin{pmatrix} 4 & -1 & 0 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ -1 & 0 & -1 & 3 & -1 & 0 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 (= 0) \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} = \begin{pmatrix} \log a_{12} + \log a_{14} + \log a_{15} + \log a_{16} \\ \log a_{21} + \log a_{23} \\ \log a_{32} + \log a_{34} \\ \log a_{41} + \log a_{43} + \log a_{45} \\ \log a_{51} + \log a_{54} \\ \log a_{61} \end{pmatrix},$$

where the matrix of coefficients above is the Laplacian matrix of the connected graph G in Figure 1, that corresponds to incomplete pairwise comparison matrix $\mathbf{A}$.

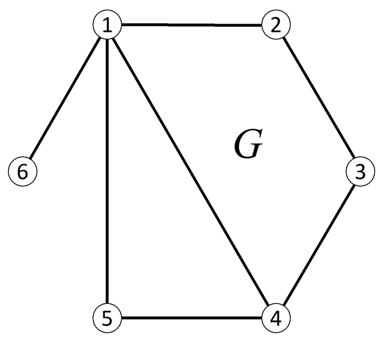


Figure 1. Graph G of Example 1.1

1.3 The least squares (LS) problem for additive matrices

Pairwise comparison matrices are relevant not only in multiplicative sense. An additive pairwise comparison matrix [1, 4] $\mathbf{B} = [b_{ij}]_{i,j=1\ldots n}$ fulfils skew-symmetry, i.e., $b_{ij} = -b_{ji}$. For every multiplicative pairwise comparison matrix $\mathbf{A}$, $\mathbf{B} = \log(\mathbf{A})$ (elementwise) is an additive pairwise comparison matrix and vice versa ($\mathbf{A} = \exp(\mathbf{B})$). The additive pairwise comparison matrix $\mathbf{B}$ is called consistent if $b_{ij} + b_{jk} = b_{ik}$ holds for all i, j, k. See [10, Subsection 4.1.1] for the applications of additive matrices in multi-criteria decision models like SMART [22] or REMBRANDT [38, Chapter 12],[43]. Additive pairwise comparison matrices can also be incomplete, similar to the multiplicative ones.

Example 1.3. *Recall Example 1.1. The incomplete additive pairwise comparison matrix* $\mathbf{B} = \log(\mathbf{A})$ *(elementwise, except for the missing ones) is as follows:*

$$\mathbf{B} = \begin{pmatrix} 0 & b_{12} & & b_{14} & b_{15} & b_{16} \\ -b_{12} & 0 & b_{23} & & & \\ & -b_{23} & 0 & b_{34} & & \\ -b_{14} & & -b_{34} & 0 & b_{45} & \\ -b_{15} & & & -b_{45} & 0 & \\ -b_{16} & & & & & 0 \end{pmatrix},$$

The least squares (LS) minimization problem defined for additive pairwise comparison matrices can be written as

$$\begin{aligned} \min \quad & \sum_{\substack{i,j: \\ b_{ij} \text{ is known}}} \left(b_{ij} - y_i + y_j\right)^2 \\ \text{subject to} \quad & y_1 = 0. \end{aligned} \tag{4}$$

The least squares minimization problem for additive matrices (4) is widely applied in multi-criteria decision making and preference modelling, see [1, 2, 4, 38].

The LS problem (4) can be traced back to Thurstone [52] and Horst [34].

LS is among the scoring models discussed by Chebotarev and Shamis [15, Section 8.1], or in the context of preference graphs by Čaklović and Kurdija [12, Section 2].

Note that Theorem 1.1 applies to the LS problem (4) too, with $\mathbf{A} = \exp(\mathbf{B})$.

Rewording the definition of consistency, $a_{ij}a_{jk} = a_{ik} \Leftrightarrow a_{ij}a_{jk}a_{ki} = 1$ (multiplicative) and $b_{ij} + b_{jk} = b_{ik} \Leftrightarrow b_{ij} + b_{jk} + b_{ki} = 0$ (additive), require that the product/sum of matrix elements in any 3-cycle must be 1/0. This leads to the more general definition of consistency that can be applied to both complete and incomplete pairwise comparison matrices.

Definition 1.2. *A multiplicative/additive (in)complete pairwise comparison matrix* $\mathbf{A}/\mathbf{B}$ *is called consistent, if* $a_{i_1 i_2} \cdot a_{i_2 i_3} \cdot \ldots \cdot a_{i_k i_1} = 1$ / $b_{i_1 i_2} + b_{i_2 i_3} + b_{i_k i_1} = 0$ *for any cycle* $i_1, i_2, \ldots i_k, i_1$ *in the graph of the matrix.*

Note that this definition is equivalent to that the incomplete matrix can be (fully) completed such that the complete matrix is consistent. Furthermore this completion is unique if and only if the graph is connected. It follows from the definition that an incomplete matrix with an acyclic graph (a tree or a disjoint union of trees) is consistent. Consistency is also equivalent to that the optimum value of the logarithmic least squares (1) / least squares (4) problem is 0. Again, the optimal solution is unique up to scaling if and only if the graph is connected.

The close relation of Definition 1.2 to Kirchhoff's Voltage Law (the signed sum of the potential differences around any closed loop is zero) is recalled in Section 3.

1.4 Aggregations of weight vectors calculated from all spanning trees

The spanning tree approach by Tsyganok [53, 54] does not assume any distance function or measure of closeness. The basic idea is that the set of pairwise comparisons is considered as the union of minimal, connected subsets, or, in graph-theoretic terms, spanning trees. Let S denote the number of all spanning trees of graph G. Every spanning tree determines a unique weight vector fitting on the corresponding subset of matrix elements perfectly, as the incomplete pairwise comparison matrix associated to a spanning tree is consistent according to Definition 1.2. Given a spanning tree, the calculation of its associated weight vector requires $O(n)$ steps.

The number of spanning trees can be very large. In the special case of complete pairwise comparison matrices, the number of all spanning trees is $S = n^{n-2}$ by Cayley's theorem. Another extremal case is when the graph of the incomplete pairwise comparison matrix is itself a tree ($S = 1$).

The most natural candidates for the aggregation of weight vectors calculated from all spanning trees are the arithmetic [47, 48, 53, 54] and the geometric means [39, 55].

The following theorem connects two weighting methods.

Theorem 1.2. *(Lundy, Siraj and Greco [39]) The geometric mean of weight vectors calculated from all spanning trees is logarithmic least squares optimal in case of complete multiplicative pairwise comparison matrices.*

The rest of the paper is organized as follows. The proof of Theorem 1.2 is based on that an explicit formula (row geometric mean of matrix elements) exists for the complete LLS problem [18, 35]. As the incomplete LLS problem does not have such a closed form solution, only an implicit one according to equations (2), a new and essentially different approach is needed to extend the theorem to the case of missing elements. This theorem, the main result of the paper, stating that the geometric mean of weight vectors calculated from all spanning trees is logarithmic least squares optimal in both cases of incomplete and complete multiplicative pairwise comparison matrices, is given in Section 2. Equivalently, the arithmetic mean of weight vectors calculated from all spanning trees is least squares optimal for additive pairwise comparison matrices. Section 3 shows that spanning trees appear in a natural way in electric circuits, and the calculation of potentials with Kirchhoff's Rules is directly related to the least squares problem written for additive matrices. Section 4 concludes with computational complexity and open questions.

2 Main result: the arithmetic (geometric) mean of weight vectors calculated from all spanning trees is (logarithmic) least squares optimal

Theorem 2.1. ***(multiplicative)*** *Let* $\mathbf{A}$ *be an incomplete or complete multiplicative pairwise comparison matrix such that its associated graph is connected. Then the optimal solution of the logarithmic least squares problem (1) is equal, up to a scalar multiplier, to the geometric mean of weight vectors calculated from all spanning trees.*

Before proving, let us rephrase Theorem 2.1 with the elementwise logarithm of an incomplete or complete multiplicative pairwise comparison matrix, which is a(n incomplete) additive (skew symmetric) matrix, let us denote it by $\mathbf{B}$. An undirected graph G is associated to $\mathbf{B}$ as follows: it has n nodes and the edge between nodes i and j is drawn if and only if the matrix element b_{ij} is given. Let $T^1, T^2, \ldots, T^s, \ldots, T^S$ denote the spanning trees of G. Let $\mathbf{y}^s \in \mathbb{R}^n$, $s = 1, 2, \ldots, S$, be the weight vector calculated from spanning tree T^s and scaled by $y_1 = 0$.

Theorem 2.2. ***(additive)*** *Let* $\mathbf{B}$ *be an incomplete or complete additive (skew symmetric) matrix such that its associated graph is connected. Then the optimal solution of the least squares problem (4) is equal to the arithmetic mean of weight vectors calculated from all spanning trees, each one scaled by* $y_1^s = 0$.

Proof. Let G be the connected graph associated with the (in)complete multiplicative pairwise comparison matrix $\mathbf{A}$ and let $E(G)$ denote the set of edges. The edge between nodes i and j is denoted by $e(i,j)$. The Laplacian matrix of graph G is denoted by $\mathbf{L}$. Let $T^1, T^2, \ldots, T^s, \ldots, T^S$ denote the spanning trees of G, where S denotes the number of spanning trees. $E(T^s)$ denotes the set of edges in T^s. Hereafter, upper index s is also used for indexing a weight vector or a pairwise comparison matrix, associated to spanning tree T^s. Let $\mathbf{w}^s, s = 1, 2, \ldots, S$, denote the weight vector calculated from spanning tree T^s. Weight vector $\mathbf{w}^s$ is unique up to a scalar multiplier. For sake of simplicity we can assume that $w_1^s = 1$, but other ways of scaling, e.g., $\prod w_i = 1$ can also be chosen. Let $\mathbf{y}^s := \log \mathbf{w}^s$, $s = 1, 2, \ldots, S$, where the logarithm is taken element-wise. Let $\mathbf{w}^{LLS}$ denote the optimal solution to the LLS problem (scaled by $w_1^{LLS} = 1$) and $\mathbf{y}^{LS} := \log \mathbf{w}^{LLS}$. The formal statement of Theorem 2.1 is that

$$w_i^{LLS} = \sqrt[S]{\prod_{s=1}^{S} w_i^s}, \qquad i = 1, 2, \ldots, n,$$

that is, by taking the logarithm, equivalent to

$$\mathbf{y}^{LS} = \frac{1}{S} \sum_{s=1}^{S} \mathbf{y}^s,$$

(which is the statement of Theorem 2.2) that we shall prove. By Theorem 1.1,

$$\left(\mathbf{L}\mathbf{y}^{LS}\right)_i = \sum_{k: e(i,k) \in E(G)} b_{ik} \qquad \text{for all } i = 1, 2, \ldots, n,$$

where $b_{ik} = \log a_{ik}$ for all $(i,k) \in E(G)$. Since graph G is connected, vector $\mathbf{y}^{LS}$ is unique with the scaling $y_1^{LS} = 0$.

It is therefore sufficient to show that

$$\left(\mathbf{L}\frac{1}{S}\sum_{s=1}^{S}\mathbf{y}^s\right)_i = \sum_{k:e(i,k)\in E(G)} b_{ik} \qquad \text{for all } i = 1,2,\ldots,n. \tag{5}$$

Observe that the Laplacian matrices of any two spanning trees are different, therefore 'intermediate' incomplete multiplicative pairwise comparison matrices are needed. Consider an arbitrary spanning tree T^s. Then $\frac{w_i^s}{w_j^s} = a_{ij}$ for all $e(i,j) \in E(T^s)$. Introduce the incomplete multiplicative pairwise comparison matrix $\mathbf{A}^s$ by $a_{ij}^s := a_{ij}$ for all $e(i,j) \in E(T^s)$ and $a_{ij}^s := \frac{w_i^s}{w_j^s}$ for all $e(i,j) \in E(G)\backslash E(T^s)$. The incomplete multiplicative pairwise comparison matrix $\mathbf{A}^s$ is consistent according to Definition 1.2 for all $s = 1,2,\ldots,S$. Again, $b_{ij}^s := \log a_{ij}^s (= y_i^s - y_j^s)$. Now the Laplacian matrices of $\mathbf{A}$ and $\mathbf{A}^s$ are the same ($\mathbf{L}$). Since the weight vector $\mathbf{w}^s$ is generated by the matrix elements belonging to spanning tree T^s, it is also the optimal solution of the LLS problem regarding $\mathbf{A}^s$ (furthermore, the optimum value is zero, because $a_{ij}^s = \frac{w_i^s}{w_j^s}$ for all $e(i,j) \in E(G)$). Equivalently, the following system of linear equations holds.

$$(\mathbf{L}\mathbf{y}^s)_i = \sum_{k:e(i,k)\in E(T^s)} b_{ik} + \sum_{k:e(i,k)\in E(G)\backslash E(T^s)} b_{ik}^s \qquad \text{for all } i = 1,2,\ldots,n. \tag{6}$$

Lemma 2.1.

$$\sum_{s=1}^{S}\left(\sum_{k:e(i,k)\in E(T^s)} b_{ik} + \sum_{k:e(i,k)\in E(G)\backslash E(T^s)} b_{ik}^s\right) = S\sum_{k:e(i,k)\in E(G)} b_{ik}. \tag{7}$$

Proof. Let i be fixed arbitrarily and consider node i in all spanning trees. There is nothing to do with edges $e(i,k) \in E(T^s)$. Since T^s is a spanning tree, for every edge $e(i,k) \in E(G)\backslash E(T^s)$ there exists a unique path
$P = \{e(i,k_1), e(k_1,k_2), \ldots, e(k_\ell,k)\} \subseteq E(T^s)$. $P \cup e(i,k)$ is a cycle and

$$b_{ik}^s = b_{ik_1} + b_{k_1k_2} + \ldots + b_{k_\ell k}. \tag{8}$$

Consider the following spanning tree: $T^{s'_{i,k,k_1}} := (T^s\backslash e(i,k_1)) \cup e(i,k)$ as in Figure 2.

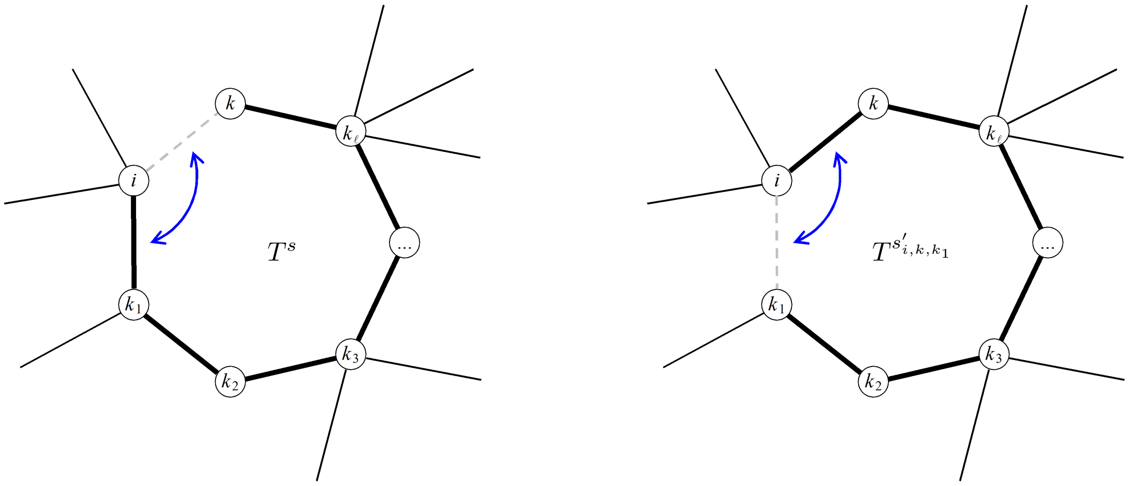


Figure 2. The replacement of edge $e(i, k_1)$ in spanning tree T^s by edge $e(i, k)$ results in spanning tree $T^{s'_{i,k,k_1}}$.

Spanning trees T^s and $T^{s'_{i,k,k_1}}$ differ in one edge only and

$$b_{ik_1}^{s'_{i,k,k_1}} = b_{ik} + b_{kk_\ell} + \ldots + b_{k_2k_1}. \tag{9}$$

Adding up equations (8) and (9) results in

$$b_{ik}^s + b_{ik_1}^{s'_{i,k,k_1}} = b_{ik} + b_{ik_1}, \tag{10}$$

all intermediate terms vanish due to the reciprocal property of pairwise comparison matrices. Now let us continue this process and go through all edges $e(i, k) \in E(G)\backslash E(T^s)$ for all k and s. The remarkable symmetry of the set of all spanning trees implies that every edge occurs in exactly one pair. Summing all these equations like (10), the statement of Lemma 2.1 follows. $\square$

We can now complete the proof of Theorem 2.1: add up equations in Eq. (6) for all $s = 1, 2, \ldots, S$, then divide by S, then the left hand side becomes the left hand side of Eq. (5). The identity of the right hand sides follows from Lemma 2.1, therefore Eq. (5) is proved. It implies $\mathbf{y}^{LS} = \frac{1}{S}\sum_{s=1}^{S}\mathbf{y}^s$, and, equivalently, $w_i^{LLS} = \sqrt[S]{\prod_{s=1}^{S} w_i^s}$, $i = 1, 2, \ldots, n$, which is the statement of Theorem 2.1. $\square$

Remark. Complete pairwise comparison matrices ($S = n^{n-2}$) are included in Theorems 2.1 and as a special case. The proof of Theorem 2.1 can also be considered as a second and shorter proof of Theorem 1.2.

Example 2.1. *(An illustration of the proof of Theorem 2.1)*

Let incomplete multiplicative pairwise comparison matrix $\mathbf{A}$ *be the same as in Example 1.1. The associated graph* G *and its* ($S = 11$) *spanning trees* $T^1, T^2, \dots, T^{11}$ *are shown in Figure 3. Consider spanning tree* T^1 *having edges* $e(1,5), e(1,6), e(2,3), e(3,4), e(4,5), e(5,6)$. *Simple calculation results in its weight vector*

$$\mathbf{w}^1 = \begin{pmatrix} 1 \\ a_{23}a_{34}a_{45}/a_{15} \\ a_{34}a_{45}/a_{15} \\ a_{45}/a_{15} \\ 1/a_{15} \\ 1/a_{16} \end{pmatrix}.$$

Ratios $\frac{w_i^1}{w_j^1} = a_{ij}$ *for all* i, j *such that* $e(i,j) \in E(T^1)$. *In order to write the incomplete multiplicative pairwise comparison matrix* $\mathbf{A}^1$, *we need edges* $e(1,2), e(1,4) \in E(G) \backslash E(T^1)$ *and the corresponding equations* $a_{12}^1 := \frac{w_1^1}{w_2^1}$ *and* $a_{14}^1 := \frac{w_1^1}{w_4^1}$. *Then*

$$\mathbf{A}^1 = \begin{pmatrix} 1 & a_{15}/(a_{23}a_{34}a_{45}) & & a_{15}/a_{45} & a_{15} & a_{16} \\ a_{23}a_{34}a_{45}/a_{15} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{45}/a_{15} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}.$$

Then equations (6) for $s = 1$ *are as follows:*

$$\begin{pmatrix} 4 & -1 & 0 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ -1 & 0 & -1 & 3 & -1 & 0 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ b_{23}+b_{34}+b_{45}-b_{15} \\ b_{34}+b_{45}-b_{15} \\ b_{45}-b_{15} \\ -b_{15} \\ -b_{16} \end{pmatrix} = \begin{pmatrix} b_{15}+b_{16} \\ b_{23} \\ -b_{23}+b_{34} \\ -b_{34}+b_{45} \\ -b_{15}+b_{45} \\ -b_{16} \end{pmatrix} + \begin{pmatrix} b_{12}^1+b_{14}^1 \\ b_{21}^1 \\ 0 \\ b_{41}^1 \\ 0 \\ 0 \end{pmatrix},$$

where $b_{12}^1 = b_{15} - b_{23} - b_{34} - b_{45}$, $b_{21}^1 = -b_{12}^1 = -b_{15} + b_{23} + b_{34} + b_{45}$ *and* $b_{41}^1 = b_{45} - b_{15}$.

We have that weight vector $\mathbf{w}^1$ *is the unique solution to both of the LLS problems*

$$\min \sum_{\substack{i,j: \\ e(i,j) \in E(T^1)}} \left[\log a_{ij} - \log\left(\frac{w_i}{w_j}\right)\right]^2$$

$$\text{subject to} \quad w_i > 0, \quad i = 1,2,\dots,6,$$
$$w_1 = 1,$$

and

$$\begin{aligned} \min & \sum_{\substack{i,j:\\ e(i,j)\in E(G)}} \left[\log a^1_{ij} - \log\left(\frac{w_i}{w_j}\right)\right]^2 \\ \textit{subject to} \quad & w_i > 0, \qquad i = 1,2,\ldots,6, \\ & w_1 = 1, \end{aligned}$$

and the optimum values are zeros in both cases.

Now let us focus on Lemma 2.1 with node $i = 1$. *Edges adjacent to node 1 are missing 12 times (and they are not missing 32 times) in the whole set of spanning trees, hence we can identify 6 pairs. They induce 6 pairs of equations, that are labelled in Figure 3. In tree* T^1,

$$b^1_{12} = b_{15} + b_{54} + b_{43} + b_{32}. \tag{11}$$

Note that equation (11), as well as the forthcoming ones, is labelled on the corresponding edges in Figure 3. Now $s = 1, k = 2, k_1 = 5$ *and* $s'_{1,2,5} = 4$, *because the replacement of edge* $e(1,5)$ *in tree* T^1 *by edge* $e(1,2)$ *results in tree* T^4. *Here*

$$b^4_{15} = b_{12} + b_{23} + b_{34} + b_{45}. \tag{12}$$

The sum of equations (11) and (12) confirms (10).

Let us continue by edge $e(1,4)$ *in tree* T^1.

$$b^1_{14} = b_{15} + b_{54}, \tag{13}$$
$$b^2_{15} = b_{14} + b_{45}. \tag{14}$$

The remaining four pairs of edges and their equations are listed below.

$$b^2_{12} = b_{14} + b_{43} + b_{32}, \tag{15}$$
$$b^4_{14} = b_{12} + b_{23} + b_{34}, \tag{16}$$

$$b^3_{12} = b_{14} + b_{43} + b_{32}, \tag{17}$$
$$b^7_{14} = b_{12} + b_{23} + b_{34}, \tag{18}$$

$$b^5_{14} = b_{15} + b_{54}, \tag{19}$$
$$b^8_{15} = b_{14} + b_{45}, \tag{20}$$

$$b^6_{14} = b_{15} + b_{54}, \tag{21}$$
$$b^9_{15} = b_{14} + b_{45}. \tag{22}$$

Lemma 2.1 is now confirmed for $i = 1$:

$$\sum_{s=1}^{11}\left(\sum_{k:e(1,k)\in E(T^s)} b_{1k} + \sum_{k:e(1,k)\in E(G)\setminus E(T^s)} b^s_{1k}\right) = 11 \sum_{k:e(1,k)\in E(G)} b_{1k} = 11(b_{12} + b_{14} + b_{15} + b_{16}).$$

Let us move to node 2. *Three pairs of equations can be obtained:*

$$b^1_{21} = b_{23} + b_{34} + b_{45} + b_{51}, \tag{23}$$
$$b^5_{23} = b_{21} + b_{15} + b_{54} + b_{43}, \tag{24}$$

$$b^2_{21} = b_{23} + b_{34} + b_{41}, \tag{25}$$
$$b^8_{23} = b_{21} + b_{14} + b_{43}, \tag{26}$$

$$b^3_{21} = b_{23} + b_{34} + b_{41}, \tag{27}$$
$$b^{10}_{23} = b_{21} + b_{14} + b_{43}. \tag{28}$$

Lemma 2.1 is now confirmed for $i = 2$:

$$\sum_{s=1}^{11} \left(\sum_{k:e(2,k)\in E(T^s)} b_{2k} + \sum_{k:e(2,k)\in E(G)\setminus E(T^s)} b^s_{2k} \right) = 11 \sum_{k:e(2,k)\in E(G)} b_{2k} = 11(b_{21} + b_{23}).$$

Cases related to the remaining nodes can be treated likewise.

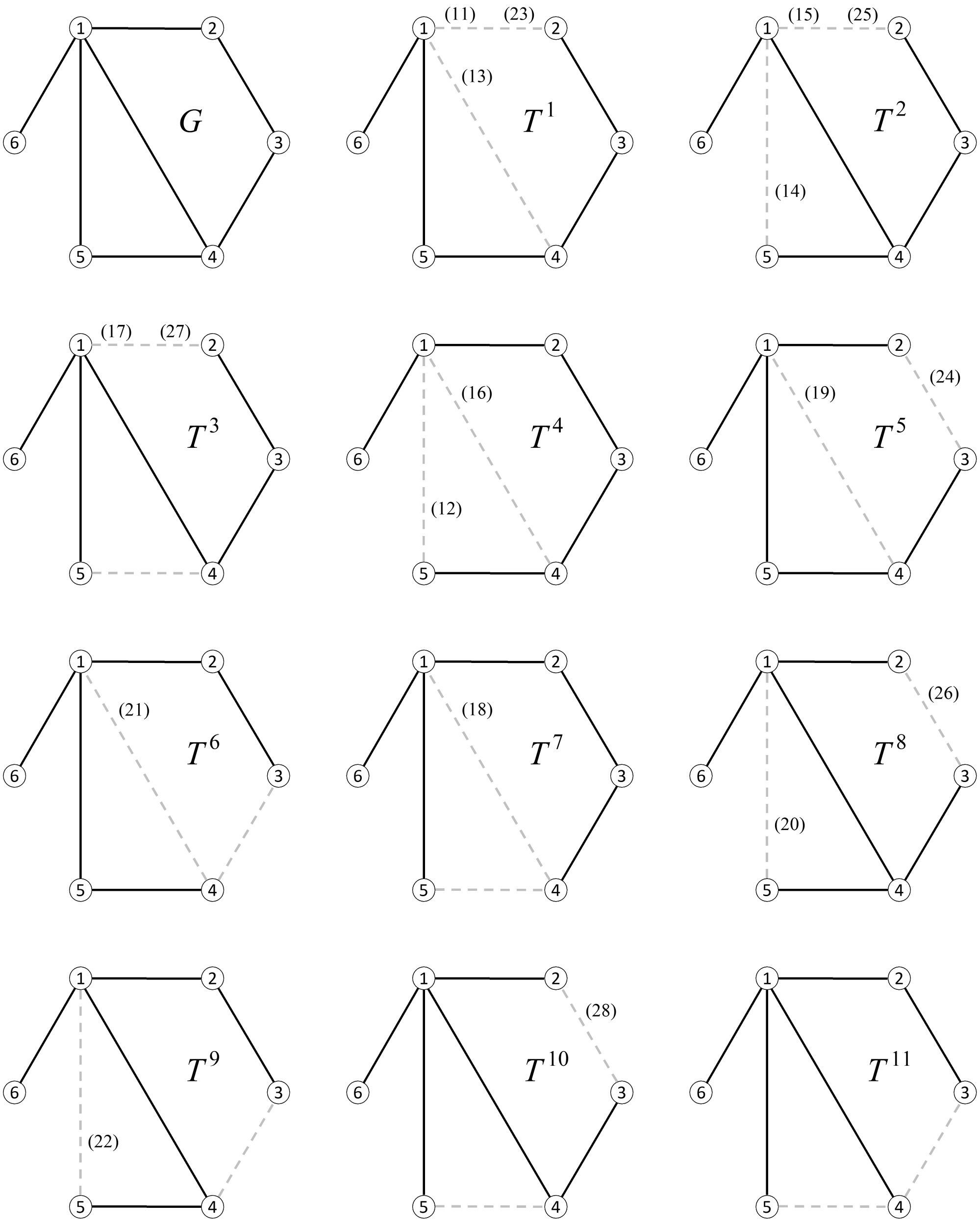


Figure 3. Graph G of Example 2.1 and its spanning trees $T^1, T^2, \ldots, T^{11}$

3 Electric circuits and potentials

The least squares problem for additive matrices (4) occurs in a natural way not only in decision theory, but in physics as well. Energy minimization and potentials in electric circuits are discussed in this section, namely, the least squares problem (4) and Theorem 2.2 are illustrated by an example.

Example 3.1. *Consider the following electric circuit on four nodes.*

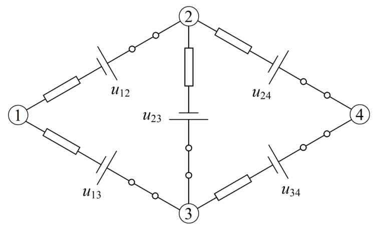


Figure 4. The electric circuit on four nodes in Example 3.1

Every resistor has the same resistance R. The values of $u_{12}, u_{13}, u_{23}, u_{24}, u_{34}$ are arbitrary real numbers. The aim is to calculate the potentials U_1, U_2, U_3, U_4 of nodes 1,2,3,4 such that the total energy (power) of the system is minimal. The objective function follows from a physical law by nature. The total energy is the sum of electrical powers ($V \cdot I = \frac{V^2}{R}$) of the resistors, where V denotes the potential difference (voltage drop) across the given resistor and I denotes the current through it. For a resistor between nodes i and j, $V = u_{ij} - U_i + U_j$. Since resistance R is assumed to be constant, the objective function to be minimized is the sum (for all edges (i,j) in the graph) of terms $(u_{ij} - U_i + U_j)^2$. We have the optimization problem (4) with the incomplete additive (skew symmetric) matrix

$$\mathbf{B} = \begin{pmatrix} 0 & u_{12} & u_{13} & \\ -u_{12} & 0 & u_{23} & u_{24} \\ -u_{13} & -u_{23} & 0 & u_{34} \\ & -u_{24} & -u_{34} & 0 \end{pmatrix}$$

and variables $\mathbf{y} = (U_1 = 0, U_2, U_3, U_4)^\top$. It is worth noting that if (and only if) matrix $\mathbf{B}$ is consistent according to Definition 1.2, then currents are zeros and $U_i^* - U_j^* = u_{ij}$ for all edges (i,j), the total power of the circuit is zero.

Assume two loop currents I_a and I_b around loops 1231 and 2432 and write Kirchhoff's Voltage Law (the directed sum of the potential differences around any closed loop is zero, (compare to Definition 1.2)):

$$RI_a + u_{12} + R(I_a - I_b) + u_{23} - u_{13} + RI_a = 0$$
$$RI_b + u_{24} - u_{34} + RI_b - u_{23} + R(I_b - I_a) = 0$$

that results in

$$I_a = \frac{-3u_{12} + 3u_{13} - 2u_{23} - u_{24} + u_{34}}{8R}$$
$$I_b = \frac{-u_{12} + u_{13} + 2u_{23} - 3u_{24} + 3u_{34}}{8R}.$$

Assume without loss of generality that $U_1 = 0$. Then

$$
\begin{aligned}
U_2 &= U_1 + RI_a + u_{12} = \frac{5}{8}u_{12} + \frac{3}{8}u_{13} - \frac{1}{4}u_{23} - \frac{1}{8}u_{24} + \frac{1}{8}u_{34}\\
U_3 &= U_1 - RI_a + u_{13} = \frac{3}{8}u_{12} + \frac{5}{8}u_{13} + \frac{1}{4}u_{23} + \frac{1}{8}u_{24} - \frac{1}{8}u_{34}\\
U_4 &= U_2 + RI_b + u_{24} = \frac{1}{2}u_{12} + \frac{1}{2}u_{13} + \frac{1}{2}u_{24} + \frac{1}{2}u_{34}
\end{aligned}
\tag{29}
$$

Kirchhoff's Current Law (the signed sum of currents is zero for every node) can be also verified.

Now let us consider the spanning tree approach. Graph G has 8 spanning trees shown in Figure 5, the corresponding circuits are given in Figure 6.

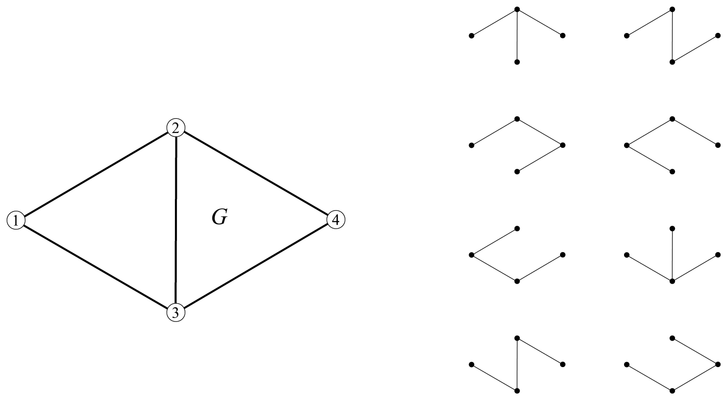


Figure 5. Graph G of Example 3.1 and its 8 spanning trees

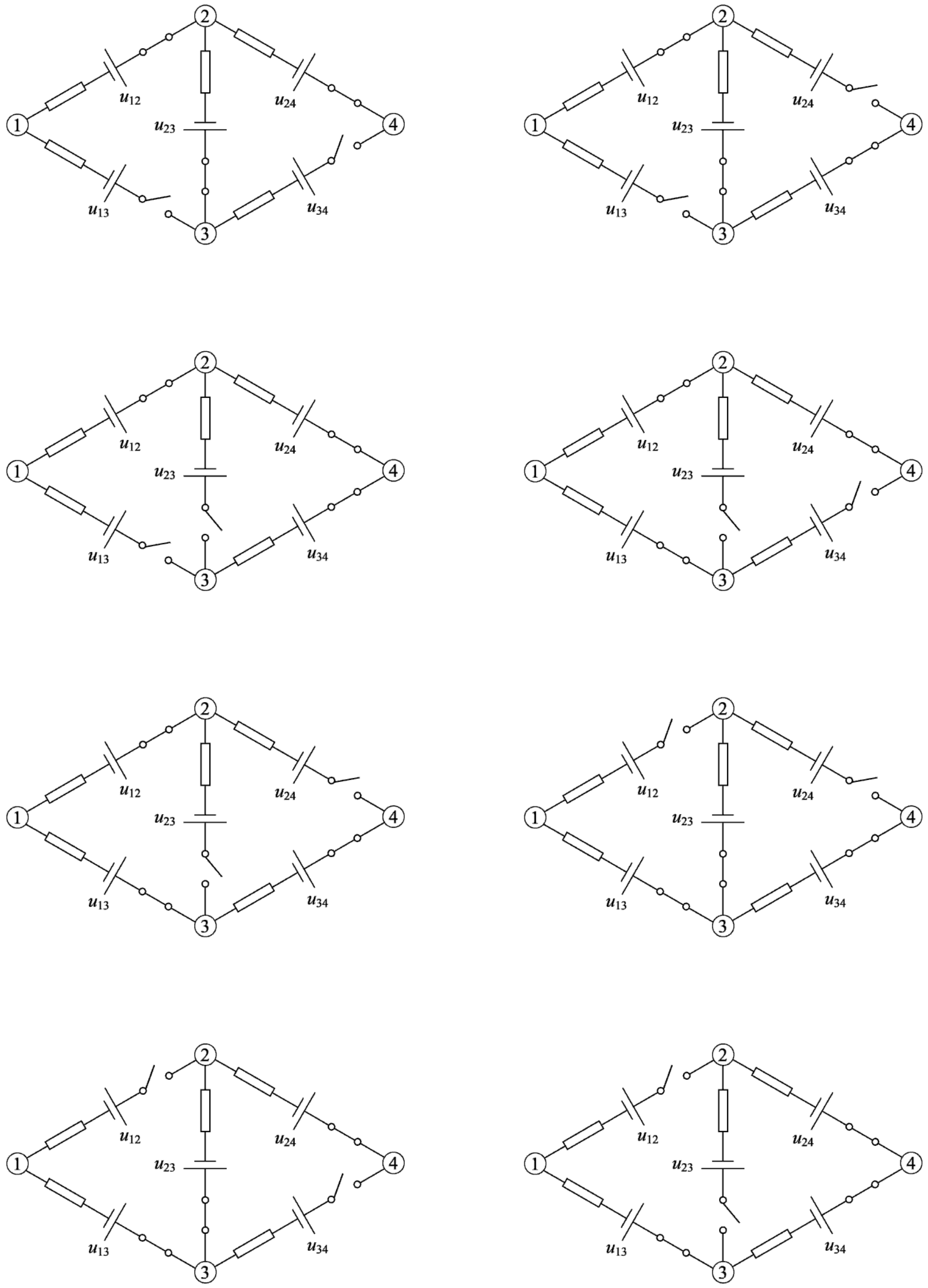


Figure 6. Circuits corresponding to the 8 spanning trees of Example 3.1

We shall apply Theorem 2.2, without loss of generality we assume again that $U_1 = 0$. *The* *calculation of the potentials is elementary for every spanning tree, because the (signed) voltages along the unique path from node 1 to another node are summed:*

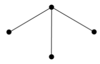
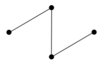
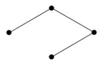
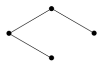
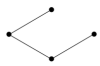
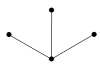
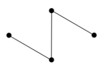
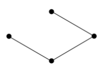
spanning tree	U_1	U_2	U_3	U_4
	*0*	u_{12}	$u_{12}+u_{23}$	$u_{12}+u_{24}$
	*0*	u_{12}	$u_{12}+u_{23}$	$u_{12}+u_{23}+u_{34}$
	*0*	u_{12}	$u_{12}+u_{24}-u_{34}$	$u_{12}+u_{24}$
	*0*	u_{12}	u_{13}	$u_{12}+u_{24}$
	*0*	u_{12}	u_{13}	$u_{13}+u_{34}$
	*0*	$u_{13}-u_{23}$	u_{13}	$u_{13}+u_{34}$
	*0*	$u_{13}-u_{23}$	u_{13}	$u_{13}-u_{23}+u_{24}$
	*0*	$u_{13}+u_{34}-u_{24}$	u_{13}	$u_{13}+u_{34}$
arithmetic mean	*0*	$\frac{5}{8}u_{12}+\frac{3}{8}u_{13}-\frac{1}{4}u_{23}$ $-\frac{1}{8}u_{24}+\frac{1}{8}u_{34}$	$\frac{3}{8}u_{12}+\frac{5}{8}u_{13}+\frac{1}{4}u_{23}$ $+\frac{1}{8}u_{24}-\frac{1}{8}u_{34}$	$\frac{1}{2}u_{12}+\frac{1}{2}u_{13}$ $+\frac{1}{2}u_{24}+\frac{1}{2}u_{34}$

Table 1. Potentials calculated from the 8 spanning trees of Example 3.1

The arithmetic means in Table 1 are the same as the ones derived from Kirchhoff's laws given in (29).

According to Theorem 2.2 the arithmetic means in Table 1 satisfy the following system (in an analogous way to (2)-(3)):

$$\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{5}{8}u_{12}+\frac{3}{8}u_{13}-\frac{1}{4}u_{23}-\frac{1}{8}u_{24}+\frac{1}{8}u_{34} \\ \frac{3}{8}u_{12}+\frac{5}{8}u_{13}+\frac{1}{4}u_{23}+\frac{1}{8}u_{24}-\frac{1}{8}u_{34} \\ \frac{1}{2}u_{12}+\frac{1}{2}u_{13}+\frac{1}{2}u_{24}+\frac{1}{2}u_{34} \end{pmatrix} = \begin{pmatrix} u_{12}+u_{13} \\ -u_{12}+u_{23}+u_{24} \\ -u_{13}-u_{23}+u_{34} \\ -u_{24}-u_{34} \end{pmatrix},$$

where the matrix above is the Laplacian of G, and the right hand side is the vector of row elements' sum in $\mathbf{B}$.

4 Conclusions

It was shown in this paper that two weighting methods, based on rather different principles and approaches, are equivalent not only for complete pairwise comparison matrices, as it was recently proved by Lundy, Siraj and Greco [39], but also for incomplete ones. The arithmetic (geometric) mean of weight vectors calculated from all spanning trees was proved to be (logarithmic) least squares optimal. The proof of the complete case [39] cannot be extended to the incomplete case, due to that the incomplete (L)LS optimal solution does not have an explicit formula. However, the implicit formula (2) was still applicable to operations with spanning trees.

The advantages rooted in the definition of the two methods, namely the clear interpretation of taking all spanning trees into account and the optimality by a widely analyzed objective functions (LLS, LS), are now united. Spanning trees not only unfold the graph of comparisons, but their corresponding weight vectors also provide an expressive decomposition of the (logarithmic) least squares optimal weight vector. An important consequence of the paper is that future analyses of weighting methods should not distinguish between the incomplete LLS/LS and the geometric/arithmetic mean of weight vectors from all spanning trees.

There is a significant difference in computational complexity. The (logarithmic) least squares problem can be solved from a single system of linear equations (the coefficient matrix is the Laplacian), requiring at most $O(n^{2.376})$ steps in theory [49]. However, recent approximate and iterative algorithms optimized for large and sufficiently sparse matrices run in nearly linear time [49, 58]. The enumeration of all spanning trees with the algorithm of Gabow and Myers [26], requires $O(n+m+nS)$ steps, where m denotes the number of edges in G. The computational complexity of calculating all weight vectors, associated to the spanning trees, is $\max\{O(nS), O(n+m+nS)\}$ steps, where S, the number of spanning trees, is between 1 and n^{n-2}. We can conclude that, except for special matrices whose associated graph has a small number of spanning trees, the (logarithmic) least squares problem is faster to solve.

Certain applications apply the spanning trees enumeration, but not necessarily together with the aggregation by the geometric mean. The approach of spanning trees enumeration is used in determining the consistency to build the distribution of expert estimates based on the matrix [41]. Such problems offer further research possibilities.

The possible equivalence of some mean of weight vectors, calculated from all spanning trees and other weighting methods, is still an open problem.

Taking weights into consideration in (logarithmic) least squares problem (see, e.g., [1] and [38, Chapter 6]) is a possible extension. In group decision making, weights represent the voting powers of the individual decision makers. Multiple comparisons for the same pairs, or considering information quality and source credibility also lead to weighted models with objective functions $\sum v_{ij} \left[\log a_{ij} - \log\left(\frac{w_i}{w_j}\right)\right]^2$ or $\sum v_{ij} \left(b_{ij} - y_i + y_j\right)^2$. An extension of Theorems 2.1 and 2.2 to the weighted case is more than inspiring. Note that the weighted variant of the corresponding representation with electric circuits and potentials in Section 3 leads to non-identical resistances.

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