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On Array Noncomputable Degrees, Maximal Pairs and Simplicity Properties

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Meiner Mutter, in Liebe und Dankbarkeit

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Abstract

In this thesis, we give contributions to topics which are related to array noncomputable (a.n.c.) Turing degrees, maximal pairs and to simplicity properties. The outline is as follows. In Chapter 2, we introduce a subclass of the a.n.c. Turing degrees, the so called *completely array noncomputable* (c.a.n.c. for short) Turing degrees. Here, a computably enumerable (c.e.) Turing degree $\mathbf{a}$ is c.a.n.c. if any c.e. set $A \in \mathbf{a}$ is weak truth-table (wtt) equivalent to an a.n.c. set. We show in Section 2.3 that these degrees exist (indeed, there exist infinitely many low c.a.n.c. degrees) and that they cannot be high. Moreover, we apply some of the ideas used to show the existence of c.a.n.c. Turing degrees to show the stronger result that there exists a c.e. Turing degree whose c.e. members are halves of maximal pairs in the c.e. computably Lipschitz (cl) degrees, thereby solving the first part of the first open problem given in the paper by Ambos-Spies, Ding, Fan and Merkle [ASDFM13].

In Chapter 3, we present an approach to extending the notion of array noncomputability to the setting of almost-c.e. sets (these are the sets which correspond to binary representations of left-c.e. reals). This approach is initiated by the Heidelberg Logic Group and it is worked out in detail in an upcoming paper by Ambos-Spies, Losert and Monath [ASLM18], in the thesis of Losert [Los18] and in [ASFL$^+$]. In [ASLM18], the authors introduce the class of sets with the *universal similarity property* (u.s.p. for short; throughout this thesis, sets with the u.s.p. will shortly be called u.s.p. sets) which is a strong form of array noncomputability in the setting of almost-c.e. sets and they show that sets with this property exist precisely in the c.e. not totally ω-c.e. degrees. Then it is shown that, using u.s.p. sets, one obtains a simplified method for showing the existence of almost-c.e. sets with a property $\mathcal{P}$ (for certain properties $\mathcal{P}$) that are contained in c.e. not totally ω-c.e. degrees, namely by showing that u.s.p. sets have property $\mathcal{P}$. This is demonstrated by showing that u.s.p. sets are computably bounded random (CB-random), thereby extending a result from Brodhead, Downey and Ng [BDN12]. Moreover, it is shown that the c.e. not totally ω-c.e. degrees can be characterized as those c.e. degrees which contain an almost-c.e. set which is not cl-reducible to any complex almost-c.e. set. This

affirmatively answers a conjecture by Greenberg.

For the if-direction of the latter result, we prove a new result on maximal pairs in the almost-c.e. sets by showing the existence of locally almost-c.e. sets which are halves of maximal pairs in the almost-c.e. sets such that the second half can be chosen to be c.e. and arbitrarily sparse. This extends Yun Fan's result on maximal pairs [Fan09]. By our result, we also get a new proof of one of the main results in Barmpalias, Downey and Greenberg [BDG10], namely that in any c.e. a.n.c. degree there is a left-c.e. real which is not cl-reducible to any ML-random left-c.e. real.

In this thesis, we give an overview of some of the results from [ASLM18] and sketch some of the proofs to illustrate this new methodology and, subsequently, we give a detailed proof of the above maximal pair result.

In Chapter 4, we look at the interaction between a.n.c. wtt-degrees and the most commonly known simplicity properties by showing that there exists an a.n.c. wtt-degree which contains an r-maximal set. By this result together with the result by Ambos-Spies [AS18] that no a.n.c. wtt-degree contains a dense simple set, we obtain a complete characterization which of the classical simplicity properties may hold for a.n.c. wtt-degrees.

The guiding theme for Chapter 5 is a theorem by Barmpalias, Downey and Greenberg [BDG10] in which they characterize the c.e. not totally ω-c.e. degrees as the c.e. degrees which contain a c.e. set which is not wtt-reducible to any hypersimple set. So Ambos-Spies asked what the above characterization would look like if we replaced hypersimple sets by maximal sets in the above theorem. In other words, what are the c.e. Turing degrees that contain c.e. sets which are not wtt-reducible to any maximal set. We completely solve this question on the set level by introducing the new class of *eventually uniformly wtt-array computable* (e.u.wtt-a.c.) sets and by showing that the c.e. sets with this property are precisely those c.e. sets which are wtt-reducible to maximal sets. Indeed, this characterization can be extended in that we can replace wtt-reducible by ibT-reducible and maximal sets by dense simple sets. By showing that the c.e. e.u.wtt-a.c. sets are closed downwards under wtt-reductions and under the join operation, it follows that the c.e. wtt-degrees containing e.u.wtt-a.c. sets form an ideal in the upper semilattice of the c.e. wtt-degrees and, further, we obtain a characterization of the c.e. wtt-degrees which contain c.e. sets that are not wtt-reducible to any maximal set. Moreover, we give upper and lower bounds (with respect to $\subseteq$) for the class of the c.e. e.u.wtt-a.c. sets. For the upper bound, we show that any c.e. e.u.wtt-a.c. set has array computable wtt-degree. For the lower bound, we introduce the notion of a *wtt-superlow* set and show that any wtt-superlow c.e. set is e.u.wtt-a.c. Besides, we show that the wtt-superlow c.e. sets can be characterized as the c.e. sets whose bounded jump is ω-computably approximable (ω-c.a. for short); hence, they are precisely the bounded low sets as introduced in the paper by Anderson, Csima and Lange [ACL17]. Furthermore, we prove a hierarchy theorem for the wtt-superlow c.e. sets and we show that there exists a Turing complete set which lies in the intersection of that hierarchy. Finally, it is shown that the above bounds are strict, i.e., there exist c.e. e.u.wtt-a.c. sets which are not wtt-superlow and that there exist c.e. sets whose wtt-degree is array computable and which are not e.u.wtt-a.c. (where here, we obtain the separation even on the level of Turing degrees). The results from Chapter 5 will be included in a paper which is in preparation by Ambos-Spies, Downey and Monath [ASDM19].

Zusammenfassung

In dieser Arbeit leisten wir Beiträge die im Zusammenhang mit array noncomputable (a.n.c.) Turinggraden, maximalen Paaren und Simplizitätseigenschaften stehen. Die Gliederung ist wie folgt. In Kapitel 2 führen wir eine Teilklasse der a.n.c. Turinggrade ein, die sogenannten *completely array noncomputable* (kurz c.a.n.c.) Turinggrade, ein. Hierbei ist ein rekursiv aufzählbarer (r.a.) Turinggrad $\mathbf{a}$ c.a.n.c. falls jede r.a. Menge $A \in \mathbf{a}$ weak truth-table (wtt)-äquivalent zu einer a.n.c. Menge ist. Wir zeigen in Abschnitt 2.3, dass solche Grade existieren (in der Tat gibt es unendlich viele niedrige c.a.n.c. Grade) and dass sie nicht hoch sein können. Außerdem wenden wir einige der Ideen, die wir nutzen, um die Existenz von c.a.n.c. Turinggraden zu zeigen, an, um das stärkere Ergebnis zu zeigen, dass es einen r.a. Turinggrad gibt, dessen r.a. Mengen Hälften von maximalen Paaren in den r.a. cl-Graden sind, womit wir zugleich den ersten Teil des ersten offenen Problems im Paper von Ambos-Spies, Ding, Fan und Merkle [ASDFM13] lösen.

In Kapitel 3 präsentieren wir einen Zugang um den Begriff der array noncomputability auf den Kontext der almost-c.e. Mengen (diese Mengen entsprechen gerade den Binärdarstellungen linksberechenbarer reeller Zahlen) zu erweitern. Dieser Zugang wurde initiiert von der Heidelberger Logik-Gruppe und ist im Detail in den Arbeiten von Ambos-Spies, Losert und Monath [ASLM18], in der Doktorarbeit von Losert [Los18] und in [ASFL$^+$] ausgearbeitet. In [ASLM18] führen die Autoren die Klasse der Mengen mit der *universal similarity property* (kurz u.s.p.; in der gesamten Arbeit werden Mengen mit der u.s.p. einfach als u.s.p. Mengen bezeichnet) ein welche eine starke Form der array noncomputability im Kontext der almost-c.e. Mengen darstellt und sie zeigen, dass die Mengen mit dieser Eigenschaft gerade in den r.a. not totally ω-c.e. Turinggraden existieren. Anschließend wird gezeigt, dass man mit Hilfe von u.s.p. Mengen eine vereinfachte Methode erhält, um die Existenz von almost-c.e. Mengen mit einer Eigenschaft $\mathcal{P}$ (für gewisse Eigenschaften $\mathcal{P}$) nachzuweisen, die in r.a. not totally ω-c.e. Graden enthalten sind, nämlich, indem man zeigt, dass u.s.p. Mengen die Eigenschaft $\mathcal{P}$ haben. Dies wird demonstriert, indem gezeigt wird, dass u.s.p. Mengen computably bounded zufällig (CB-zufällig) sind, womit zugleich ein Ergebnis von Brodhead, Downey and Ng [BDN12], verschärft wird. Weiterhin

wird gezeigt, dass man die r.a. not totally ω-c.e. Grade als diejenigen r.a. Grade charakterisieren kann, welche eine almost-c.e. Menge enthalten, die nicht cl-reduzierbar auf eine komplexe almost-c.e. Menge ist. Dies bejaht eine Vermutung von Greenberg.

Für die Wenn-dann-Richtung des letztgenannten Resultats beweisen wir ein neues Ergebnis über maximale Paare in den almost-c.e. Mengen, indem wir zeigen, dass es lokal almost-c.e. Mengen gibt, die Hälfte eines maximalen Paares in den almost-c.e. Mengen sind, sodass die zweite Hälfte als r.a. und beliebig dünn gewählt werden kann. Dies erweitert Yun Fans Ergebnis über maximale Paare [Fan09]. Mit unserem Ergebnis bekommen wir zudem einen neuen Beweis eines der Hauptresultate in Barmpalias, Downey und Greenberg [BDG10], nämlich dass es in jedem r.a. a.n.c. Grad eine linksberechenbare reelle Zahl gibt, die nicht cl-reduzierbar auf eine ML-zufällige linksberechenbare reelle Zahl ist.

In dieser Arbeit geben wir einen Überblick zu einigen der Resultate von [ASLM18] und skizzieren einige der Beweise, um diese neue Methodik zu illustrieren und geben anschließend einen detaillierten Beweis des obigen Ergebnisses über maximale Paare an.

In Kapitel 4 betrachten wir die Interaktion zwischen a.n.c. wtt-Graden und den allgemein bekannten Simplizitätseigenschaften, indem wir zeigen, dass es a.n.c. wtt-Grade gibt, welche r-maximale Mengen enthalten. Zusammen mit dem Ergebnis von Ambos-Spies [AS18], dass kein a.n.c. wtt-Grad Mengen enthalten kann, die dense simple sind, erhalten wir eine vollständige Charakterisierung welche der klassischen Simplizitätseigenschaften für a.n.c. wtt-Grade gelten können.

In Kapitel 5 ist das Leitthema ein Theorem von Barmpalias, Downey und Greenberg [BDG10], in dem sie die r.a. nicht total ω-c.e. Grade als diejenigen r.a. Grade charakterisieren, welche eine r.a. Menge enthalten die nicht wtt-reduzierbar auf eine hypersimple Menge ist. So stellte Ambos-Spies die Frage wie obige Charakterisierung aussähe wenn man im obigen Theorem hypersimple durch maximal ersetzt. Mit anderen Worten, welches sind die r.a. Turinggrade die eine r.a. Menge enthalten, die nicht wtt-reduzierbar auf eine maximale Menge sind. Wir beantworten diese Frage vollständig auf der Ebene der Mengen, indem wir die neue Klasse der *eventually uniformly wtt-array computable* (e.u.wtt-a.c.)

Mengen einführen und indem wir zeigen, dass Mengen mit dieser Eigenschaft gerade diejenigen Mengen sind, welche auf maximale Mengen wtt-reduzierbar sind. In der Tat lässt sich diese Charakterisierung dahingehend erweitern, dass wir wtt-reduzierbar durch ibT-reduzierbar and maximale Mengen durch dense simple Mengen ersetzen können. Indem wir zeigen, dass die r.a. e.u.wtt-a.c. Mengen nach unten gegen wtt-Reduktionen und gegen die Join-Operation abgeschlossen sind, folgt, dass die r.a. wtt-Grade, die eine e.u.wtt-a.c. Menge enthalten, ein Ideal im oberen Halbverband der r.a. wtt-Grade bilden und weiterhin erhalten wir eine Charakterisierung der r.a. wtt-Grade, die r.a. Mengen enthalten, die nicht wtt-reduzierbar auf maximale Mengen sind.

Außerdem geben wir obere und untere Schranken (bzgl. $\subseteq$) für die Klasse der r.a. e.u.wtt-a.c. Mengen an. Für die obere Schranke zeigen wir, dass jede r.a. e.u.wtt-a.c. Menge array computable wtt-Grad hat. Für die untere Schranke führen wir den Begriff der *wtt-superlow* Mengen ein und zeigen, dass jede wtt-superlow r.a. Menge e.u.wtt-a.c. ist. Außerdem zeigen wir, dass die wtt-superlow r.a. Mengen als diejenigen r.a. Mengen charaktersiert werden können, deren bounded jump ω-computably approximable (kurz ω-c.a.) ist. Somit sind sie gerade die bounded low Mengen, so wie sie in der Arbeit von Anderson, Csima und Lange [ACL17] eingeführt werden. Weiterhin beweisen wir einen Hierarchiesatz für die wtt-superlow r.a. Mengen und wir zeigen, dass es eine Turing-vollständige Menge gibt, die im Durchschnitt dieser Hierarchie liegt. Schließlich wird noch gezeigt, dass die obigen Schranken strikt sind, d.h., dass es r.a. e.u.wtt-a.c. Mengen gibt, die nicht wtt-superlow sind und dass es r.a. Mengen gibt, deren wtt-Grad array computable ist, aber nicht e.u.wtt-a.c. sind (wobei wir hier die Trennung sogar auf der Ebene der Turing Grade erhalten). Die Ergebnisse aus Kapitel 5 werden in einem Paper, das in zusammen von Ambos-Spies, Downey und Monath [ASDM19] vorbereitet wird, enthalten sein.

Contents

Chapter 1

Preliminaries

In this chapter, we provide the notation and the definitions that we use throughout this thesis. More specific notation and definitions are introduced in the respective chapters. We assume the reader to be familiar with the basic concepts of computability theory as it is given, e.g., in the book of Soare [Soa87] or in the book of Downey and Hirschfeldt [DH10]. Most of our notation is adapted from these two sources. In particular, we assume familiarity with the concept of (oracle) Turing machines and that of partial computable functions.

The set of all natural numbers is denoted by ω and natural numbers are denoted by lower case letters such as e, i, j or k, l, m, n, or x, y, z. Throughout this thesis, we are mainly dealing with natural numbers, so unless otherwise stated, when we refer to numbers we always mean natural numbers. In constructions or approximations, we use lower case letters like s, t to refer to the stage number or to the number of steps of approximation, respectively. Subsets of ω are denoted by upper case letters like A, B, C, or V, W, X, Y, Z. A set $A \subseteq \omega$ is identified with its characteristic function, so we have $A = A(0)A(1)\dots$, where we write $A(x) = 1$ for $x \in A$ and $A(x) = 0$, otherwise. In particular, we identify the power set of ω with 2^ω, the set of all infinite binary sequences. The set of all finite binary sequences is denoted by $2^{<\omega}$. Elements of $2^{<\omega}$ are called *(binary) strings* or sometimes *nodes*, where the latter term is mainly used when we deal with tree constructions. Strings are denoted by lower case Greek letters such as $\alpha, \beta, \gamma, \delta$, or σ, τ. The empty string is denoted by λ. For any string σ, we write

$|\sigma|$ for the length of σ and $\sigma(i)$, where $i < |\sigma|$ for the $(i+1)$st bit of σ. For any two strings σ, τ, we write $\sigma \sqsubseteq \tau$ if σ is an initial segment of τ and we write $\sigma \sqsubset \tau$ if $\sigma \sqsubseteq \tau$ and $\sigma \neq \tau$ hold. The concatenation of two strings σ, τ is denoted by $\sigma\tau$. The lexicographical ordering on strings is denoted by $\leq_{lex}$ and $\sigma \leq_{lex} \tau$ holds for two strings σ, τ if and only if either $\sigma \sqsubseteq \tau$ holds or if $\sigma(i) < \tau(i)$ holds for the least $i < \min\{|\sigma|, |\tau|\}$ such that $\sigma(i) \neq \tau(i)$. For any set A and any number $n \in \omega$, we write $A \upharpoonright n$ for the unique string σ of length n such that $\sigma(i) = A(i)$ holds for all $i < n$. We often identify strings with numbers, where a string α may be identified with the $n \in \omega$ if and only if 1α equals the binary representation of $n + 1$. The set of all finite sequences of natural numbers is denoted by $\omega^{<\omega}$. We identify $\omega^{<\omega}$ with the disjoint union $\bigcup_{k\in\omega} \omega^k$, where $\omega^0 = \{\lambda\}$ (that is, the empty sequence of $\omega^{<\omega}$ is also denoted by λ), where we identify ω^1 with ω and where ω^k denotes the set of all tuples of numbers of length k for $k \geq 2$.

We denote the standard pairing function by $\langle \cdot, \cdot \rangle : \omega^2 \to \omega$, i.e., it holds that $\langle x, y \rangle = \frac{(x+y)(x+y+1)}{2} + x$. Note that $\langle \cdot, \cdot \rangle$ is a computable bijection which is strictly increasing in both arguments. Moreover, $\langle \cdot, \cdot \rangle$ induces for any $k \geq 1$ computable bijections $\langle \cdot, \ldots, \cdot, \rangle_k : \omega^k \to \omega$ and a computable bijection $\langle \ldots \rangle_* : \omega^{<\omega} \to \omega$ by letting $\langle \cdot \rangle_1$ be equal to the identity function, by letting $\langle x_0, \ldots, x_k \rangle_{k+1} = \langle \langle x_0, \ldots, x_{k-1} \rangle_k, x_k \rangle$ for all $k \in \omega$ and all $x_0, \ldots, x_k$ and by letting $\langle \lambda \rangle_* = 0$ and by letting $\langle x_0, \ldots, x_{n-1} \rangle_* = \langle n-1, \langle x_0, \ldots, x_{n-1} \rangle_n \rangle + 1$ for all $(x_0, \ldots, x_{n-1}) \in \omega^{<\omega}$ with $n > 0$. By abuse of notation, we write $\langle x_0, \ldots, x_{n-1} \rangle$ instead of $\langle x_0, \ldots, x_{n-1} \rangle_n$ if n is clear from the context and, in the same way, we write $\langle x_0, \ldots, x_{n-1} \rangle$ instead of $\langle x_0, \ldots, x_{n-1} \rangle_*$ if it is clear from the context that the length of $(x_0, \ldots, x_{n-1})$ varies. For any $e \in \omega$, we let $\omega^{[e]} = \{\langle e, x \rangle : x \in \omega\}$.

Partial functions are denoted by lower case Greek letters like φ, ψ and total functions are denoted by lower case letters such as f, g, h. For a partial function φ, we let $\mathrm{dom}(\varphi)$ be the domain of φ and we write $\varphi(x) \downarrow$ if $x \in \mathrm{dom}(\varphi)$ and $\varphi(x) \uparrow$, otherwise. Turing functionals are denoted by upper case Greek letters like Φ, Ψ. As usual, for a Turing functional Φ, a set A and a number x, we write $\Phi^A(x)$ instead of $\Phi(A, x)$, where A is called the *oracle* in this context. The use functional of a Turing functional is denoted by its corresponding lower case Greek letter. So given a Turing functional Φ and an oracle A, the use function of Φ^A is denoted by φ^A, where we let $\varphi^A(x)$ be the greatest oracle query in the computation of $\Phi^A(x)$ on input x. Given a Turing functional Φ or a partial computable function φ we let, for any oracle A and any $x, s \in \omega$, $\Phi^A_s(x)$, $\varphi^A_s(x)$ and $\varphi_s(x)$ denote the approximation of $\Phi^A(x)$, $\varphi^A(x)$ and $\varphi(x)$ within s steps of computation, respectively.

We fix a Goedel numbering of all Turing functionals $\{\Phi_e\}_{e\in\omega}$ and a Goedel numbering of all partial computable functions $\{\varphi_e\}_{e\in\omega}$ and let $\{W_e\}_{e\in\omega}$, where $W_e = \mathrm{dom}(\varphi_e)$, denote the induced standard enumeration of all computably enumerable (c.e.) sets. For given $e, s \in \omega$, we let $W_{e,s} = \mathrm{dom}(\varphi_{e,s})$. We adapt the now commonly used Lachlan notation for approximations of computations, i.e., given a Turing functional Φ and a set A which is approximated by a sequence of sets $\{A_s\}_{s\in\omega}$ in the limit we let $\Phi^A(x)[s] = \Phi^{A_s}_s(x)$. By approximations in the limit, we refer to the discrete topology of the natural numbers, i.e., a sequence of sets $\{A_s\}_{s\in\omega}$ (a sequence of functions $\{f_s\}_{s\in\omega}$) approximates a set A (a function f) in the limit if, for every x, $A_s(x) = A(x)$ ($f_s(x) = f(x)$) holds for almost all stages, i.e., for all but finitely many stages. We follow the usual convention on converging computations, i.e., for any oracle A and any numbers $e, x, y, s \in \omega$ if $\Phi^A_{e,s}(x) \downarrow= y$ then $\max\{e, x, y, \varphi^A_e(x)\} < s$ and, similarly, if $\varphi_{e,s}(x) \downarrow= y$ then $\max\{e, x, y\} < s$; in particular, it holds that $W_{e,s} \subseteq \omega \restriction s$.

Let us recall the use principle for Turing functionals.

Theorem 1.0.1 (Use Principle). *Let A and B be two given sets, let Φ be a Turing functional and let $x, y \in \omega$. Then $\Phi^A(x) \downarrow= y$ and $B \restriction \varphi^A(x)+1 = A \restriction \varphi^A(x)+1$ imply that $\Phi^B(x) \downarrow= y$ and $\varphi^B(x) = \varphi^A(x)$.*

When we attempt to define a Turing functional Ψ in stages s along with a computable enumeration $\{A_s\}_{s\in\omega}$ of a c.e. set A, we follow the standard procedure as it is described, e.g., in [DH10, p.29 f.]. In the following, we shortly recall how this is done. Let $x \in \omega$ be given. Initially, we set $\Psi^A(x)[0] \uparrow$. At some stage $s+1$ of the construction, we may declare $\Psi^A(x)[s+1] \downarrow= y$ and declare its use to be $\psi^A(x)[s+1] = z$ for some values $y, z \in \omega$ (in some cases, we have $z = s$). Then for any stage $t > s$, we keep $\Psi^A(x)[t+1] = \Psi^A(x)[t]$ and $\psi^A(x)[t+1] = \psi^A(x)[t]$ unless we enumerate a number $n \leq z$ into A at stage $t+1$. In the latter case, we may let $\Psi^A(x)[t+1] \uparrow$ and henceforth repeat the procedure after stage $t+1$. In order to make sure that eventually $\Psi^A(x) \downarrow$ holds, we have to make sure that

$\psi^A(x)$ stabilizes, i.e., that there exists a stage s such that $\psi^A(x)[t] = \psi^A(x)[s]$ holds for all $t > s$. If A and the computable enumeration $\{A_s\}_{s\in\omega}$ are given then we may also follow this procedure with the restriction that, for any stage s such that $\Psi^A(x)[s]\downarrow$ and $\psi^A(x)[s]\downarrow$ holds, we may declare $\Psi^A(x)[s+1]\uparrow$ only if $A_{s+1} \restriction \psi^A(x)[s]+1 \neq A_s \restriction \psi^A(x)[s]+1$ holds.

Next, we summarize the reducibilities which are considered in this thesis.

Definition 1.0.2. *Let A and B be any two sets. Then we say that*

(i) A is Turing reducible *to B, denoted by $A \leq_T B$, if there exists $e \in \omega$ such that $A = \Phi_e^B$.*

(ii) A is weak truth-table reducible (wtt) *to B, denoted by $A \leq_{wtt} B$, if there exists $e \in \omega$ and a computable function $f : \omega \to \omega$ such that $A = \Phi_e^B$ and such that $\varphi_e^B(x) \leq f(x)$ holds for all x. More precisely, given a computable function f, we write $A \leq_{f-T} B$ and say that A* is f-bounded Turing reducible to B *if $A \leq_{wtt} B$ holds and f witnesses this fact.*

(iii) A is truth-table reducible (tt) *to B, denoted by $A \leq_{tt} B$, if there exist computable functions $g : \omega \to \omega^{<\omega}$ and $h : \omega \times 2^{<\omega} \to \{0,1\}$ such that, for any x, it holds that $A(x) = h(x, B(y_0)\dots B(y_{n-1}))$, where $g(x) = (y_0, \dots, y_{n-1})$.*

(iv) A is computable Lipschitz (cl) reducible *to B, denoted by $A \leq_{cl} B$, if there exists $e \in \omega$ such that $A = \Phi_e^B$ and if there exists a constant $c \in \omega$ such that $\varphi_e^B(x) \leq x + c$ holds for all x.*

(v) A is identity bounded Turing (ibT) reducible *to B, denoted by $A \leq_{ibT} B$, if $A \leq_{\mathrm{id}-T} B$, where* id *is the identity function.*

(vi) A is many-one reducible *to B, denoted by $A \leq_m B$, if there exists a computable function $f : \omega \to \omega$ such that $A(x) = B(f(x))$ holds for any x.*

(vii) A is one-one reducible *to B, denoted by $A \leq_1 B$, if there exists a computable one-one function $f : \omega \to \omega$ such that $A(x) = B(f(x))$ holds for any x.*

For any of the above reducibilities r, we say that A and B are r-equivalent *and denote it by $A \equiv_r B$ if $A \leq_r B$ and $B \leq_r A$ hold. We write $A <_r B$ if $A \leq_r B$ and $B \not\leq_r A$ hold, and we let* $\deg_r(A) = \{B \subseteq \omega : A \equiv_r B\}$ *be the* r-degree *of A.*

For any of the above reducibilities r, we denote the r-degrees by lower case bold face letters such as $\mathbf{a}$, $\mathbf{b}$, etc. (where we tacitly assume that the reducibility is clear from the context). An r-degree $\mathbf{a}$ is c.e. if it contains a c.e. set. For any two r-degrees $\mathbf{a}$ and $\mathbf{b}$, we write $\mathbf{a} \leq \mathbf{b}$ if there exist sets $A \in \mathbf{a}$ and $B \in \mathbf{b}$ such that $A \leq_r B$ holds and we write $\mathbf{a} < \mathbf{b}$ if $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$ holds. Since all of the above reducibilities are reflexive and transitive, it follows that the induced relation $\equiv_r$ is an equivalence relation, the r-degrees are the equivalence classes of $\equiv_r$ and the induced $\leq$-relation is a partial ordering on the set of all r-degrees. Moreover, we write $\mathbf{a} \vee \mathbf{b}$ ($\mathbf{a} \wedge \mathbf{b}$) for the least upper bound (greatest lower bound) of $\mathbf{a}$ and $\mathbf{b}$ for two r-degrees $\mathbf{a}$ and $\mathbf{b}$, if it exists. Note that the least upper bound always exists for two r-degrees $\mathbf{a}$ and $\mathbf{b}$ for all reducibilities $r \in \{m, tt, wtt, T\}$ and it is given by $\deg_r(A \oplus B)$ for any $A \in \mathbf{a}$ and $B \in \mathbf{b}$, where $A \oplus B$ is the join of the sets A and B which is given by $A \oplus B = \{2x : x \in A\} \cup \{2x+1 : x \in B\}$; and $\mathbf{a} \vee \mathbf{b}$ exists for $r \in \{ibT, cl\}$ if there exist disjoint c.e. sets $A \in \mathbf{a}$ and $B \in \mathbf{b}$ in which case it is given by $\deg_r(A \cup B)$.

By definition, r-degrees only contain sets. However, we may extend the definition of Turing reducibility to functions and sets by saying that a function f is *Turing reducible* to a function g (f is *Turing reducible* to a set A) and denote it by $f \leq_T g$ ($f \leq_T A$), too, if there exists $e \in \omega$ such that $f = \Phi_e^{\mathrm{graph}(g)}$ ($f = \Phi_e^A$) holds, where $\mathrm{graph}(g) = \{\langle x, g(x) \rangle : x \in \omega\}$. Then, for any function f and any two sets A, B, it holds that $f \leq_T A$ and $A \leq_T B$ imply that $f \leq_T B$. Thus, we may write $f \leq_T \mathbf{a}$ for a function f and a Turing degree $\mathbf{a}$ meaning that $f \leq_T A$ holds for some set $A \in \mathbf{a}$.

Chapter 2

Completely Array Noncomputable Degrees

2.1 Introduction

Array noncomputable sets are introduced by Downey, Jockusch and Stob in 1990 [DJS90] as the class of computably enumerable (c.e.) sets that comprehend a certain type of permitting, called *multiple permitting*. *Permitting* is a technique in computability theory that is used to construct a c.e. set A which is Turing below a c.e. set B, where B may be given or may be under construction as well. In a simple permitting argument, for instance, we may put a number x into A only if a number $y \leq x$ enters B at the same stage or later. In this case we say that x is *permitted by* B. Note that simple permitting ensures that $A \leq_{ibT} B$.

So in any construction where the set A has to meet an infinite sequence of requirements $\{\mathcal{R}_e\}_{e\in\omega}$ such that each $\mathcal{R}_e$ may be eventually satisfied by enumerating one number into A (for example, consider the requirements for making A simple), we can argue that simple permitting can be guaranteed by any noncomputable c.e. set B in order to meet all requirements. To wit, suppose that we have appointed numbers $x_0 < x_1 < \ldots$, so called *followers*, at stages $s_0 < s_1 < \ldots$, respectively for the sake of a fixed requirement $\mathcal{R}_e$, each of them awaiting to be permitted by B. Then, by effectivity of the construction, the strictly increasing sequences $\{x_i\}_{i\in\omega}$ and $\{s_i\}_{i\in\omega}$ are computable. So, by noncomputability of the set B, eventually one of them is permitted; hence, while waiting for a stage s such that one of the $x_0 < \cdots < x_n$ is permitted by B, it suffices to appoint a new follower x_{n+1} at a stage $s_{n+1} > s_n$ to finally meet $\mathcal{R}_e$.

In a *multiple permitting* argument, the requirements become more intricate in such a way that a follower x needs to be permitted more than once (for example, x may be associated with enumerating many numbers $x_1, \ldots, x_n$ into A for the sake of one requirement). Then B being merely noncomputable may not be sufficient any more to permit all of $x_1, \ldots, x_n$ to enter A. Downey, Jockusch and Stob define the notion of array noncomputable (a.n.c.) sets as a property of c.e. sets such that the sets with this property capture such multiple permitting arguments, where the number of permissions needed for x is bounded by $f(x)$, where f is a fixed computable function which does not depend on the requirement. In [DJS90], the authors show that any non-low$_2$ degree is a.n.c., i.e., contains an a.n.c. set, that the a.n.c. degrees are closed upwards and that there is a low a.n.c. set. Moreover, Downey, Jockusch and Stob characterize the a.n.c. degrees as those degrees $\mathbf{a}$ which, given a computable function f, can compute a function $g \leq_T \mathbf{a}$ which is not f-c.e., i.e., does not have a computable approximation $\{g_s\}_{s\in\omega}$ such that the number of stages s such that $g_{s+1}(x) \neq g_s(x)$ is bounded by $f(x)$ (Note that the term f-c.e. is nowadays replaced by f-computably approximable (f-c.a.). Since we refer here to results in the literature where mainly the former term is used, we stick to the former term as well in this and the following chapter.). In a follow up of [DJS90], Downey, Jockusch and Stob [DJS96] extend the definition of a.n.c. degrees to the class of all degrees via a domination property (see Theorem 2.2.4 below; here and in the following, we refer to c.e. a.n.c. degrees as a.n.c. degrees) and show that this definition coincides with the definition of a.n.c. degrees in the class of c.e. degrees.

In the literature, there are many examples that demonstrate that the multiple permitting technique as provided by the a.n.c. degrees comprehends the combinatorics of a wide class of constructions in computability theory. For instance, a.n.c. degrees may be characterized as those degrees which contain maximal pairs both in the c.e. and the left-c.e. computable Lipschitz (cl) degrees ([ASDFM13] and [BDG10], respectively), they can be characterized as the c.e. sets which have infinitely often the highest Kolmogorov complexity possible among all c.e. sets

([Kum96]) and a.n.c. degrees are precisely those c.e. degrees which contain left-c.e. reals that are not cl-computable by any 1-random left-c.e. real ([BDG10]), to mention but a few results in this respect. For more on cl-maximal pairs in the c.e. sets, we refer the reader to Section 2.5 and, for cl-maximal pairs in the left-c.e. reals, we refer the reader to Chapter 3.

In this chapter, we contribute further results to the class of a.n.c. degrees by introducing a subclass of the a.n.c. degrees, called the *completely array noncomputable degrees*. The outline of this chapter is as follows. In Section 2.2, we recall the definitions of a.n.c. sets and degrees and recall basic properties about them. Then in Section 2.3, we give the definition of completely array noncomputable degrees and we show that such degrees exist (Theorem 2.3.3). In Section 2.4, we prove that they form a proper subclass of the array noncomputable degrees by showing that any high c.e. degree does not have this property (Theorem 2.4.2). Finally, in Section 2.5 we review the relation between a.n.c. degrees and maximal pairs and strengthen Theorem 2.3.3 by showing that there exists a Turing degree whose c.e. members are all halves of maximal pairs (Theorem 2.4.2). The latter result gives an affirmative answer to the first part of the first open problem in [ASDFM13].

2.2 Preliminaries

We begin this section with the basic definitions of a.n.c. sets and degrees and state some of the basic results about a.n.c. degrees. We start with the notion of a very strong array.

Definition 2.2.1 ([DJS90])**.** *A* very strong array *(*v.s.a. *for short) is a sequence of finite sets $\mathcal{F} = \{F_n\}_{n\in\omega}$ such that there exists a computable function $f : \omega \to \omega$ such that for all n, it holds that $F_n = D_{f(n)}$ (i.e., $f(n)$ is the canonical index of F_n), $0 < |F_n| < |F_{n+1}|$ and $F_m \cap F_n = \emptyset$ holds for all $m \neq n$. A v.s.a. $\mathcal{F} = \{F_n\}_{n\in\omega}$ is called* complete *if $\bigcup_{n\in\omega} F_n = \omega$ holds.*

Based on very strong arrays, array noncomputability is defined for c.e. sets as follows.

Definition 2.2.2. *Given a v.s.a.* $\mathcal{F} = \{F_n\}_{n\in\omega}$*, a c.e. set* A *is* $\mathcal{F}$-array noncomputable *(*$\mathcal{F}$-a.n.c.*) if, for every c.e. set* B*,*

$$\exists n \; (A \cap F_n = B \cap F_n) \tag{2.1}$$

holds. A *is called* array noncomputable *(*a.n.c.*) if it is* $\mathcal{F}$*-a.n.c. for some v.s.a.* $\mathcal{F}$*; and a c.e. (wtt-) degree* **a** *is* array noncomputable *if* **a** *contains an array noncomputable set. Finally, a c.e. set* A *and a c.e. (wtt-) degree* **a** *are called* array computable *(*a.c.*) if they are not array noncomputable.*

Note that in the original definition of very strong arrays given in [DJS90], it is required that every v.s.a. is complete. Definition 2.2.1 follows the one given in [DH10]. However, this does not affect the notion of array noncomputability for wtt-degrees. Namely, as shown in [ASFL$^+$], every a.n.c. set A in the sense of Definition 2.2.2 is $\mathcal{F}$-a.n.c. for a complete v.s.a. $\mathcal{F} = \{F_n\}_{n\in\omega}$. Moreover, since c.e. sets are closed under finite variation, (2.1) is equivalent to

$$\exists^\infty n \; (A \cap F_n = B \cap F_n). \tag{2.2}$$

Downey, Jockusch and Stob show in [DJS90, Theorem 2.5] that a.n.c. Turing degrees are closed upwards and that array noncomputability for wtt-degrees does not depend on the choice of the very strong array. We summarize this in the following theorem.

Theorem 2.2.3 ([DJS90])**.** *Let* $r \in \{wtt, T\}$*, let* $\mathcal{F}$ *be a v.s.a. and let* A, B *be c.e. sets such that* A *is a.n.c. and such that* $A \leq_r B$ *holds. Then there exists an* $\mathcal{F}$*-a.n.c. set* C *such that* $C \equiv_r B$ *holds.*

In their later sequel paper [DJS96], Downey, Jockusch and Stob define a.n.c. degrees in terms of a domination property which extends the definition of a.n.c. degrees to the class of all degrees and they show that this definition is equivalent on the level of c.e. degrees. In the following theorem, for future reference, we subsumize this and the following characterization which classifies the a.n.c. degrees in terms of the Ershov hierarchy and which becomes an important connection when we look at stronger forms of multiple permitting in Chapter 3.

Theorem 2.2.4 (Theorem 1.6 of [DJS90] and Proposition 1.4 of [DJS96])**.** *For a c.e. degree degree* $\mathbf{a}$*, the following are equivalent.*

1. $\mathbf{a}$ *is a.n.c.*
2. *For every computable function* f *there exists a function* $g \leq_T \mathbf{a}$ *such that* g *is not* f*-c.e.*
3. *For every* $f \leq_{wtt} \mathbf{0}'$ *there exists a function* $g \leq_T \mathbf{a}$ *such that* $g(n) \geq f(n)$ *for infinitely many* $n \in \omega$*.*

2.3 Completely Array Noncomputable Degrees Exist

Next, let us focus on the c.e. wtt-degrees inside a given a.n.c. degree. For that, recall the following result from Downey, Jockusch and Stob ([DJS90, Corollary 3.14]).

Theorem 2.3.1. *The c.e. array computable wtt-degrees form an ideal in the class of the c.e. wtt-degrees.*

By Downey and Stob [DS93, Theorem 9.5 (1)], a.n.c. Turing degrees cannot be contiguous, i.e., they must contain at least two c.e. wtt-degrees. Hence, we may conclude by Theorem 2.2.3 and by the density of the c.e. wtt-degrees ([LS75]) that any a.n.c. Turing degree contains infinitely many a.n.c. wtt-degrees. However, this leaves the question open whether all the c.e. wtt-degrees inside a given a.n.c. Turing degree may be a.n.c., too. This inspires the following definition.

Definition 2.3.2. *A c.e. Turing degree* $\mathbf{a}$ *is called* completely array noncomputable *(*completely a.n.c. *or* c.a.n.c. *for short) if, for any c.e. set* $W \in \mathbf{a}$ *there is an array noncomputable c.e. set* $B \equiv_{wtt} W$*.*

In the following, we prove that such degrees indeed exist.

Theorem 2.3.3. *There exists a completely array noncomputable degree* $\mathbf{a}$.

Proof. In the following, fix a complete v.s.a. $\mathcal{F} = \{F_n\}_{n\in\omega}$. For the proof of Theorem 2.3.3 we construct a c.e. set A and auxiliary c.e. sets B_e ($e \in \omega$) in stages s, where A_s and $B_{e,s}$ denote the finite set of numbers that are enumerated into A and B_e by stage s, respectively such that A and the sets B_e meet for all $m \in \omega$ the requirements

$$\mathcal{R}_m : (A = \Phi_{e_1}^{W_{e_0}} \;\&\; W_{e_0} = \Phi_{e_2}^{A}) \Rightarrow (B_e \leq_{wtt} W_{e_0} \;\&\; \exists n\ (B_e \cap F_n = W_d \cap F_n)),$$

where $m = \langle e, d\rangle$ and $e = \langle e_0, e_1, e_2\rangle$.

By Theorem 2.2.3, meeting all requirements ensures that $\mathbf{a} = deg_T(A)$ has the desired properties. Before we give the formal construction, let us first give the strategy of a single requirement. It is then easy to see that all requirements can be met by a straightforward finite injury argument, where the priority ordering is as usual, i.e., $\mathcal{R}_m$ has *higher priority than* $\mathcal{R}_{m'}$, denoted by $\mathcal{R}_m < \mathcal{R}_{m'}$ if and only if $m < m'$. In the following, fix m and let $e, d \in \omega$ be such that $m = \langle e, d\rangle$ and e_i ($i \leq 2$) be such that $e = \langle e_0, e_1, e_2\rangle$.

At a stage $s + 1$ when the strategy for $\mathcal{R}_m$ starts we appoint a set F_n with $B_{e,s} \cap F_n = \emptyset$, assign it to $\mathcal{R}_m$ and initialize all requirements of lower priority, i.e., if $\mathcal{R}_{m'}$ is assigned $F_{n'}$ at stage s then this assignment is cancelled and $\mathcal{R}_{m'}$ has to choose a set $F_{n''}$ with $n'' > n'$ later. More specifically, we assign F_s to $\mathcal{R}_m$ at the stage $s + 1$ when the strategy for $\mathcal{R}_m$ starts. By making sure that only one requirement may be assigned a set F_n at each stage, this ensures that $B_{e,s} \cap F_n = \emptyset$ holds and, further, that the assignment of sets to requirements is strictly increasing with respect to the priority ordering, that it is is nondecreasing with respect to the stage number and that the question whether a set F_n ever becomes assigned to some requirement $\mathcal{R}_m$ in the course of the construction is decidable. Namely, if F_n is not assigned to any requirement by stage $n + 1$ then it is never assigned to any requirement (in particular, $F_n \cap B_e = \emptyset$ holds in the latter case).

Then at stages $t > s$ we attempt to make $B_e \cap F_n = W_d \cap F_n$ by letting B_e copy W_d, i.e., if $B_{e,t} \cap F_n \neq W_{d,t} \cap F_n$ holds then we let $B_{e,t+1} = B_{e,t} \cup (W_{d,t+1} \cap F_n)$. Obviously, this strategy ensures that $B_{e,t} \cap F_n \subseteq W_{d,t} \cap F_n$ holds for all $t > s$ and $B_e \cap F_n = W_d \cap F_n$ holds in the limit.

In order to guarantee that at the same time $B_e \leq_{wtt} W_{e_0}$ holds, we have to define a computable function f_e such that, for given n and stage t, if B_e changes on F_n at stage $t+1$ then W_{e_0} changes below $f_e(n)$ at a stage $t'+1 \geq t+1$. Then we can argue that $B_e(x)$ can be computed from $W_{e_0} \upharpoonright f_e(n)$ for all $x \in F_n$ uniformly in n; hence, $B_e \leq_{g_e-T} W_{e_0}$ holds where $g_e(x) = f_e(h(x))$ and $h(x)$ is the unique index such that $x \in F_{h(x)}$ holds (note that $h(x)$ is computable since $\mathcal{F}$ is complete).

The idea behind the value $f_e(n)$ for given n is based on the following three observations. First, we do not need any permission from W_{e_0} if F_n is never assigned to any requirement of the form $\mathcal{R}_{\langle e,d'\rangle}$ for some $d' \in \omega$. So it suffices to let $f_e(n) = 0$ in the latter case. For the remainder of the argument, w.l.o.g. we may assume that F_n is assigned to $\mathcal{R}_m$ in the course of the construction, say at stage $s+1$ as above.

Then second, we need at most $|F_n|$ permissions from W_{e_0} for the sake of meeting $\mathcal{R}_m$ since there are at most $|F_n|$ many numbers that may be enumerated into $W_d \cap F_n$.

Third, we only have to make sure that $B_e \leq_T W_{e_0}$ holds if the hypothesis of $\mathcal{R}_m$ (hence, for all requirements $\mathcal{R}_{\langle e,d'\rangle}$ with $d' \in \omega$) is true. So in the following, w.l.o.g., we may assume that the hypothesis of all requirements $\mathcal{R}_{\langle e,d'\rangle}$ ($d' \in \omega$) holds. However, since the question whether the hypothesis of $\mathcal{R}_m$ holds is not decidable, we approximate this question in the course of the construction by the following *length of agreement function*,

$$l(e,s) = \mu x(A_s(x) \neq \Phi_{e_1}^{W_{e_0}}(x)[s] \text{ or } W_{e_0,s}(x) \neq \Phi_{e_2}^{A}(x)[s]). \tag{2.3}$$

Then in order to get the desired permissions from W_{e_0} for numbers $x \in B_e \cap F_n$ when needed, we wait until a stage $s' > s$ appears such that there exists a triple of sequences of numbers $(\{x_i\}_{i<|F_n|}, \{y_i\}_{i<|F_n|}, \{z_i\}_{i<|F_n|})$ such that, for all $i < |F_n|$,

(i) $x_i \in \omega^{[m]}$,

(ii) $z_{|F_n|-1} < l(e,s')$,

(iii) $x_i < y_i < z_i$ and $z_i < x_{i+1}$ if $i < |F_n| - 1$,

(iv) $y_i > \varphi_{e_1}^{W_{e_0}}(x_i)[s']$,

(v) $z_i > \max\{\varphi_{e_2}^{A}(u)[s'] :\ u \leq y_i\}$, and

(vi) $\{x_0, \ldots, x_{|F_n|-1}\} \cap A_{s'} = \emptyset$

hold. Indeed, such a stage $s' > s$ exists if the hypothesis of $\mathcal{R}_m$ is true. Then at the least such stage $s' > s$, we assign such a triple $(\{x_i\}_{i<|F_n|}, \{y_i\}_{i<|F_n|}, \{z_i\}_{i<|F_n|})$ to $\mathcal{R}_m$ at stage $s'+1$ where, in the following, the numbers of the form x_i in $(\{x_i\}_{i<|F_n|}, \{y_i\}_{i<|F_n|}, \{z_i\}_{i<|F_n|})$ of sequences are called *followers of* $\mathcal{R}_m$. Then we delay the above copying strategy for B_e on F_n as follows.

Call a stage $t+1$ *critical (w.r.t.* $\mathcal{R}_m$*)* if $t > s'$ and it holds that $B_{e,t} \cap F_n \neq W_{d,t} \cap F_n$ and $l(e,t) > z_{|F_n|-1}$. Then at any such stage $t+1$, we choose the largest $x_i \notin A_t$, enumerate it into A and correct B_e on F_n by copying $W_d \cap F_n$. Then, by the use principle, W_{e_0} has to enumerate a number $\leq \varphi_{e_1}^{W_{e_0}}(x_i)[t]$ after stage t in case that the hypothesis of $\mathcal{R}_m$ holds. However, by (v), W_{e_0} cannot change below y_i between s' and t since we put followers in reversed order into A. Hence, by (iv), it follows that $\varphi_{e_1}^{W_{e_0}}(x_i)[t] = \varphi_{e_1}^{W_{e_0}}(x_i)[s']$. So, by (ii) and by convention on converging computations, it suffices to let $f_e(n) = s'$.

Note that the above argument only applies if we assume that no other requirement enumerates a number below z_i by stage t into A. So, in order to make the strategies for different requirement compatible, we define a restraint function $r : \omega^2 \to \omega$, where $r(m,s)$ denotes the restraint which is imposed on $\mathcal{R}_m$ by higher priority requirements. It is initially set to zero and it is increased at any stage such that $\mathcal{R}_m$ is initialized (by convention on converging computations, it suffices to let $r(m,s+1) = s$ in this case). In particular, every time $\mathcal{R}_m$ is initialized it has to wait to be assigned a new triple $(\{x_i\}_{i<|F_n|}, \{y_i\}_{i<|F_n|}, \{z_i\}_{i<|F_n|})$ along with a new set F_n.

For any stage s, we say that a triple $(\{x_i\}_{i<|F_n|}, \{y_i\}_{i<|F_n|}, \{z_i\}_{i<|F_n|})$ is *suitable for* $\mathcal{R}_m$ *and* F_n *at stage* $s+1$ (or *suitable* for short) if (i)–(vi) hold for $(\{x_i\}_{i<|F_n|}, \{y_i\}_{i<|F_n|}, \{z_i\}_{i<|F_n|})$ and such that $r(m,s) \leq x_0$ holds.

This explains the basic concept of how to define sets A and B_e that meet all requirements. We now turn to the formal construction.

Construction.

Stage 0. Let $A_0 = B_{e,0} = \emptyset$ and $r(m,s) = 0$ for all $e, m \in \omega$.

Stage $s+1$. Let A_s and $B_{e,s}$ be given for all $e \in \omega$. We say that $\mathcal{R}_m$ *requires attention at stage* $s+1$ if $m \leq s$ and either

(I) no set is assigned to $\mathcal{R}_m$, or

(II) F_n is assigned to $\mathcal{R}_m$, no follower is assigned to $\mathcal{R}_m$ and there exists a suitable triple $(\{x_i\}_{i<|F_n|}, \{y_i\}_{i<|F_n|}, \{z_i\}_{i<|F_n|})$ for $\mathcal{R}_m$ and F_n at stage $s+1$, or

(III) F_n is assigned to $\mathcal{R}_m$, $(\{x_i\}_{i<|F_n|}, \{y_i\}_{i<|F_n|}, \{z_i\}_{i<|F_n|})$ is assigned to $\mathcal{R}_m$ and $s+1$ is critical.

Let m be minimal such that $\mathcal{R}_m$ requires attention at stage $s+1$ and let $e, d \in \omega$ such that $m = \langle e, d\rangle$. Say that $\mathcal{R}_m$ *receives attention* and act according to the clause via which $\mathcal{R}_m$ requires attention.

If (I) holds, assign F_s to $\mathcal{R}_m$.

If (II) holds, let $(\{x_i\}_{i<|F_n|}, \{y_i\}_{i<|F_n|}, \{z_i\}_{i<|F_n|})$ be the least suitable triple for $\mathcal{R}_m$ and F_n (w.r.t. to the lexicographical ordering of triples of numbers, where we assume that sequences of numbers are also ordered lexicographically). Assign $(\{x_i\}_{i<|F_n|}, \{y_i\}_{i<|F_n|}, \{z_i\}_{i<|F_n|})$ to $\mathcal{R}_m$.

If (III) holds, let $i < |F_n|$ be largest such that $x_i \notin A_s$. Let $A_{s+1} = A_s \cup \{x_i\}$, $B_{e,s+1} = B_{e,s} \cup (W_{d,s+1} \cap F_n)$ and, for all $e' \neq e$, let $B_{e',s+1} = B_{e',s}$.

In any case, *initialize* all requirements $\mathcal{R}_{m'}$ with $m' > m$, i.e., cancel their assigned set and their assigned sequence (if any). Moreover, for all $m' > m$, let $r(m', s+1) = s$ and, for all $m' < m$, let $r(m', s+1) = r(m', s)$.

This ends the formal construction.

Verification

We prove in a series of claims that A and the sets B_e meet the requirements. Before, let us give some general remarks on the construction which will be tacitly used in the proofs of the claims below. Unless otherwise stated, they can be proven by a straightforward induction on the stage number s.

First of all, the construction is effective and $\{A_s\}_{s\in\omega}$ and $\{B_{e,s}\}_{s\in\omega}$ are computable enumerations of A and B_e for all $e \in \omega$, respectively. Hence, A and all sets B_e are c.e. sets. For any stage s there exists a unique $m \leq s$ such that $\mathcal{R}_m$ receives attention at stage $s+1$. So for any s, there is at most one number that is enumerated into A at stage $s+1$ and there is at most one e such that $B_{e,s+1} \neq B_{e,s}$ holds.

More precisely, a number x may be enumerated into A or a set B_e at a stage $s+1$ only if there exists a requirement $\mathcal{R}_{\langle e',d\rangle}$ which receives attention via (III) at stage $s+1$ (where $e' = e$ if x enters B_e). In any case, $s+1$ is critical w.r.t. $\mathcal{R}_{\langle e',d\rangle}$, $\mathcal{R}_{\langle e',d\rangle}$ is assigned a set F_n and a sequence $\{x_i\}_{i<|F_n|}$ and, if x enters A then there exists $i < |F_n|$ such that $x = x_i$ and it holds that $x < l(e', s)$; and if x enters B_e then $x \in F_n \cap W_{d,s+1} \setminus W_{d,s}$. In particular, $A_s \cup B_{e,s} \subseteq \omega \restriction s$ for all $e \in \omega$.

Furthermore, any requirement $\mathcal{R}_m$ is assigned at most one set of $\mathcal{F}$ and at most one sequence at any stage, if F_n is the assigned set to $\mathcal{R}_m$ at a stage s then $m \leq n < s$, if $\mathcal{R}_m$ gets F_n assigned via (I) at stage $s+1$ then $n = s$; and if $\mathcal{R}_m$ is assigned a sequence then it is also assigned a set F_n and the sequence assigned to $\mathcal{R}_m$ has length $|F_n|$. In particular, if F_n is assigned to $\mathcal{R}_{\langle e,d\rangle}$ then $B_{e,n} \cap F_n = \emptyset$. The assignment of sets to requirements is strictly increasing in the index of the requirement and nondecreasing in the stage number i.e., if $\mathcal{R}_m$ is assigned F_n and $\mathcal{R}_{m'}$ is assigned $F_{n'}$ at stage s then $m < m'$ implies $n < n'$ and if $\mathcal{R}_m$ is assigned F_n at stage s and $F_{n'}$ at stage s' then $s \leq s'$ implies $n \leq n'$. Moreover, if $n < n'$ holds in the latter case then $\mathcal{R}_m$ is initialized at a stage $t \in (s, s')$.

Thus, from the way sets F_n are assigned to requirements and at which stages numbers may enter A or a set B_e, we may deduce that for any numbers $e, d, n \in \omega$ and any stage s such that $\mathcal{R}_{\langle e,d\rangle}$ is assigned F_n at stage s, it holds that

$$B_{e,s} \cap F_n = W_{d,\tilde{s}} \cap F_n, \tag{2.4}$$

where $\tilde{s}$ is the largest stage less than or equal to s such that $\mathcal{R}_{\langle e,d\rangle}$ receives attention via (III) at stage $\tilde{s}$.

Besides, if a requirement $\mathcal{R}_m$ gets a sequence $\{x_i\}_{i<|F_n|}$ assigned at a stage $s+1$, it gets $\{x_i\}_{i<|F_n|}$ assigned via (II). So $\{x_i\}_{i<|F_n|}$ is suitable for $\mathcal{R}_m$ and F_n at stage $s+1$. So by (i) in the definition of a suitable sequence, only $\mathcal{R}_m$ may enumerate one of the followers x_i into A.

Finally, the function $r(m, s)$ is computable and nondecreasing in s, $r(m, s) \leq s$ holds for all m and s and $r(m, s+1) \neq r(m, s)$ holds only if $\mathcal{R}_m$ is initialized by a higher priority requirement at stage $s+1$.

Now the first claim states that the action of each requirement is finitary and that the restraint functions reach a finite limit.

Claim 1. Every requirement $\mathcal{R}_m$ requires attention only finitely often, it eventually gets a permanent set assigned and $r^*(m) = \lim_{s\to\infty} r(m, s)$ exists.

Proof. The proof is by induction on m. Fix $m \in \omega$ and suppose the claim to be true for all $m' < m$. Then by inductive hypothesis, there is a stage s_0 such that no requirement $\mathcal{R}_{m'}$ with $m' < m$ requires attention after stage s_0. So $r(m, s) = r(m, s_0)$ for all $s \geq s_0$ by construction; hence, $r^*(m)$ exists. Moreover, after stage s_0, $\mathcal{R}_m$ is not initialized so if it requires attention at a stage $s \geq s_0$, it receives attention and acts. Thus, any set or sequence that is assigned to $\mathcal{R}_m$ after stage s_0 is permanently assigned to $\mathcal{R}_m$. If $\mathcal{R}_m$ is not assigned a set at stage s_0 already, $\mathcal{R}_m$ requires attention at stage s_0+1 via (I) so $\mathcal{R}_m$ gets a permanent set assigned. Then after stage s_0+1, $\mathcal{R}_m$ may require attention at most once via (II) and at most $|F_n|$ many times via (III) after stage s_0. Hence, the claim holds for $\mathcal{R}_m$. $\square$

By construction, the set $D \subseteq \omega$ which consists of all numbers n such that F_n gets assigned to a requirement in the course of the construction is computable and infinite. So we may fix a computable enumeration $\{n_k\}_{k\in\omega}$ of the elements of D in order of magnitude and, for $k \in \omega$, let e_k and d_k be such that F_{n_k} is assigned to $\mathcal{R}_{\langle e_k,d_k\rangle}$ at stage n_k+1. By computability of D and by effectivity of the construction, the sequences $\{e_k\}_{k\in\omega}$ and $\{d_k\}_{k\in\omega}$ are computable, too.

For the next two claims, fix e such that the hypothesis of $\mathcal{R}_{\langle e,d\rangle}$ holds for all d. We show that there exists a computable function f_e such that $B_e(x)$ can be computed from $W_{e_0} \upharpoonright f_e(n)$ for all $x \in F_n$ uniformly in n. To that end, let P be the binary predicate such that, for any $m, s \in \omega$, $P(m,s)$ holds if and only if either $\mathcal{R}_m$ is initialized or $\mathcal{R}_m$ receives attention via (II) at stage $s+1$. Then for any $n \in \omega$, let

$$f_e(n) = \begin{cases} 0 & \text{if } \forall k \leq n \ (n_k = n \ \Rightarrow \ e_k \neq e), \\ \mu s > n(P(\langle e, d_k\rangle, s)) & \text{if } \exists k \leq n \ (n_k = n \ \& \ e_k = e) \end{cases} \tag{2.5}$$

Then we first show that f_e is indeed total and computable.

Claim 2. f_e is total and computable.

Proof. First note that $f_e(n)$ is well defined since for any $n \in D$ there is a unique k such that $n = n_k$. So by computability of D and computability of the sequences $\{e_k\}_{k\in\omega}$ and $\{d_k\}_{k\in\omega}$, it suffices to show that given n such that F_n is assigned to a requirement of the form $\mathcal{R}_{\langle e,d\rangle}$ for some $d \in \omega$, either $\mathcal{R}_{\langle e,d\rangle}$ is initialized or it requires attention via (II) after stage n.

For a proof by contradiction, suppose that F_n is permanently assigned to $\mathcal{R}_{\langle e,d\rangle}$ and $\mathcal{R}_{\langle e,d\rangle}$ does not require attention via (II) after stage n. So $r^*(\langle e,d\rangle) = r(\langle e,d\rangle, n) \leq n$ since $\mathcal{R}_{\langle e,d\rangle}$ is not initialized after stage n. Then since the hypothesis of $\mathcal{R}_{\langle e,d\rangle}$ holds, let $\{x_i\}_{i<|F_n|}$, $\{y_i\}_{i<|F_n|}$ and $\{z_i\}_{i<|F_n|}$ be defined by induction on $i < |F_n|$ such that

$$\begin{aligned} x_0 &= \langle \langle e,d\rangle, n+1\rangle, \\ y_i &= \max\{\varphi_{e_1}^{W_{e_0}}(x_i), x_i+1\}, \\ z_i &= \max(\{\varphi_{e_2}^{A}(u) : u \leq y_i\} \cup \{y_i+1\}), \text{ and} \\ x_{i+1} &= \langle \langle e,d\rangle, z_i+1\rangle \ \ (i < |F_n|-1) \end{aligned}$$

holds and let $s > n$ be the least stage such that $\varphi_{e_1}^{W_{e_0}}(x_i)[s] = \varphi_{e_1}^{W_{e_0}}(x_i)$ and $\varphi_{e_2}^{A}(u)[s] = \varphi_{e_2}^{A}(u)$ holds for all $u \leq y_i$ and all $i < |F_n|$ and such that $z_{|F_n|-1} < l(e,s)$. By definition, $r^*(m) < x_0$ and $x_i \notin A_s$ for all $i < |F_n|$ since by $x_i \in \omega^{[\langle e,d\rangle]}$, none of the x_i is ever enumerated into A after stage n as they may be enumerated into A only if $\mathcal{R}_{\langle e,d\rangle}$ receives attention via (III) after stage n. But then $\{x_i\}_{i<|F_n|}$, $\{y_i\}_{i<|F_n|}$ and $\{z_i\}_{i<|F_n|}$ satisfy (i)–(vi) at stage $s+1$; hence, $\{x_i\}_{i<|F_n|}$ is suitable for $\mathcal{R}_{\langle e,d\rangle}$ and F_n at stage $s+1$ so $\mathcal{R}_{\langle e,d\rangle}$ does require attention via (II) at stage $s+1$, contrary to assumption. □

Next, we show that based on f_e, it holds that $B_e \leq_{g_e-T} W_{e_0}$ with $g_e(x) = f_e(h(x))$, where recall that $h(x) \in \omega$ is such that $x \in F_{h(x)}$ holds for all $x \in \omega$.

Claim 3. $B_e \leq_{g_e-T} W_{e_0}$.

Proof. Let $x \in \omega$ be arbitrary and let $n = h(x)$. It suffices to show that if x is enumerated into B_e at a stage $t+1$ then there exists a stage $t' \geq t$ and a number $y < f_e(n)$ such that y enters W_{e_0} at stage $t'+1$. So suppose that x enters B_e at stage $t+1$. Then as $B_{e,t+1} \cap F_n \neq B_{e,t} \cap F_n$, F_n is assigned to a requirement $\mathcal{R}_{\langle e,d\rangle}$, $\mathcal{R}_{\langle e,d\rangle}$ is assigned a sequence $\{x_i\}_{i<|F_n|}$ and $\mathcal{R}_{\langle e,d\rangle}$ receives attention via (III) at stage $t+1$. So $t+1$ is critical (w.r.t. to $\mathcal{R}_{\langle e,d\rangle}$) and we may fix the $i < |F_n|$ such that x_i is enumerated into A at stage $t+1$ and, furthermore, we may fix the stage $s+1 < t+1$ such that $\mathcal{R}_{\langle e,d\rangle}$ gets $\{x_i\}_{i<|F_n|}$ assigned. In particular, $s > n$ and $\mathcal{R}_{\langle e,d\rangle}$ receives attention via (II) at stage $s+1$. So $\{x_i\}_{i<|F_n|}$ is suitable for $\mathcal{R}_{\langle e,d\rangle}$ and F_n at stage $s+1$.

By assumption, $\mathcal{R}_{\langle e,d\rangle}$ is assigned F_n at any stage $s' \in (n, t+1]$. So $\mathcal{R}_{\langle e,d\rangle}$ is not initialized at any such stage. Thus, s is the least stage greater than n such that $\mathcal{R}_{\langle e,d\rangle}$ receives attention via (II) at stage $s+1$; hence, $f_e(n) = s$ by (2.5). Now by construction, we can argue that there exists a sequence of critical stages $s < t_{|F_n|-1} < t_{|F_n|-2} < \cdots < t_i = t$ such that, for any j with $i \leq j < |F_n|$, $\mathcal{R}_{\langle e,d\rangle}$ receives attention via (III) and x_j is enumerated into A at stage t_j+1.

Since $t+1$ is critical, it holds that $x_i < l(e,t)$. So by the fact that the hypothesis of $\mathcal{R}_{\langle e,d\rangle}$ holds, there exists a number $y < \varphi_{e_1}^{W_{e_0}}(x_i)[t]$ and a stage $t' \geq t$ such that y enters W_{e_0} at stage $t'+1$. Thus, the claim follows if we show that $\varphi_{e_1}^{W_{e_0}}(x_i)[t] = \varphi_{e_1}^{W_{e_0}}(x_i)[s]$ holds since $\varphi_{e_1}^{W_{e_0}}(x_i)[s] < f_e(n)$.

To that end, observe that for any $m > \langle e,d\rangle$, $r(m, s+1) = s$ since $\mathcal{R}_m$ is initialized by $\mathcal{R}_{\langle e,d\rangle}$ at stage $s+1$. So after stage s, no number below s is enumerated into A by any requirement of lower priority than $\mathcal{R}_{\langle e,d\rangle}$. Moreover, since $\mathcal{R}_{\langle e,d\rangle}$ is not initialized at any stage $s' \in (n, t+1]$, no requirement of higher priority enumerates a number into A at any such stage. Hence, by (iii) and since followers are enumerated in reversed order, it follows that

$$A_t \upharpoonright z_i = A_s \upharpoonright z_i$$

which, by (v), implies that

$$W_{e_0,t} \upharpoonright y_i = W_{e_0,s} \upharpoonright y_i,$$

which, by (iv) implies $\varphi_{e_1}^{W_{e_0}}(x_i)[t] = \varphi_{e_1}^{W_{e_0}}(x_i)[s]$. □

Finally, we show that all requirements are met. Fix $m \in \omega$ in the following.

Claim 4. $\mathcal{R}_m$ is met.

Proof. For a proof by contradiction, suppose that $\mathcal{R}_m$ is not met. Let $e, d \in \omega$ be such that $m = \langle e, d\rangle$. Then the hypothesis of $\mathcal{R}_{\langle e,d'\rangle}$ holds for all $d' \in \omega$ but either $B_e \leq_{wtt} W_{e_0}$ or $\exists n\ (B_e \cap F_n = W_d \cap F_n)$ fail. By Claim 3, $B_e \leq_{wtt} W_{e_0}$ holds so $B_e \cap F_n \neq W_d \cap F_n$ holds for all n. By Claim 1, fix the set F_n which is permanently assigned to $\mathcal{R}_m$ (so $\mathcal{R}_m$ is not initialized after stage n). By (2.4), $B_{e,t} \cap F_n = W_{d,\tilde{t}} \cap F_n$ for any $t > n$ where $\tilde{t}$ is the largest stage less than or equal to t such that $\mathcal{R}_m$ receives attention at stage $\tilde{t}+1$. By Claim 3, f_e is total and computable. So since $\mathcal{R}_m$ is not initialized after stage n, by (2.5), we may fix the least stage $s > n$ such that $\mathcal{R}_m$ receives attention via (II) at stage $s+1$. So $\mathcal{R}_m$ is permanently assigned a sequence $\{x_i\}_{i<|F_n|}$ after stage s since a sequence assigned to a requirement is cancelled only if the corresponding set which is assigned to the requirement is cancelled as well.

Now since the hypothesis of $\mathcal{R}_m$ holds, $x_{|F_n|-1} < l(e,t)$ and $B_{e,t} \cap F_n \neq W_{d,t} \cap F_n$ hold for sufficiently large t. So since $\{x_i : i < |F_n|\} \cap A_s = \emptyset$ by (vi), we can argue that there exists a sequence of stages $t_0 < \cdots < t_{|F_n|-1}$ such that $s < t_0$ and such that $\mathcal{R}_m$ receives attention via (III) at stage $t_i + 1$ for each $i < |F_n|$. Hence, by definition of (III) and by (2.4), it follows that $W_{d,t_0} \cap F_n \neq \emptyset$ and $W_{d,t_{i+1}} \cap F_n \neq W_{d,t_i} \cap F_n$ for all $i < |F_n| - 1$. In particular, $W_{t_{|F_n|-1}} \cap F_n = F_n$ as W_d is a c.e. set. However, for the least stage $t > t_{|F_n|-1}$

such that $B_{e,t} \cap F_n \neq W_{d,t} \cap F_n$ (which exists by assumption), we infer that $|W_d \cap F_n| > |F_n|$ which is impossible. □

Since every requirement is met by Claim 4 and since meeting the requirements ensures that A is completely array noncomputable, this completes the proof of Theorem 2.3.3. □

By Theorem 2.3.1, it is not hard to show that if a c.a.n.c. degree is the join of two c.e. degrees then one of them must be c.a.n.c., too.

Lemma 2.3.4. *Let* $\mathbf{a_0}$ *and* $\mathbf{a_1}$ *be c.e. degrees such that* $\mathbf{a_0} \vee \mathbf{a_1}$ *is c.a.n.c. Then either* $\mathbf{a_0}$ *or* $\mathbf{a_1}$ *is c.a.n.c.*

Proof. Suppose not. Then we may choose c.e. sets $A_i \in \mathbf{a_i}$ for each $i \leq 1$ such that the wtt-degree of A_i is array computable. But then $\deg_{wtt}(A_0 \oplus A_1) \subseteq \mathbf{a_0} \vee \mathbf{a_1}$ is c.e. and array computable by Theorem 2.3.1, contrary to assumption. □

Now we get the following two results as immediate corollaries using Sacks' splitting theorem [Sac63].

Corollary 2.3.5. *For every c.a.n.c. degree* $\mathbf{a}$ *there exists a low c.a.n.c. degree* $\mathbf{b}$ *such that* $\mathbf{b} < \mathbf{a}$.

Corollary 2.3.6. *There exists infinitely many low c.a.n.c. degrees.*

2.4 High Degrees Are Not Completely Array Noncomputable

We now turn to the question which c.e. degrees may be completely a.n.c. and which may not. As a first observation which follows from [DJS90], we may derive that the complete Turing degree is not completely array noncomputable.

Theorem 2.4.1. $\mathbf{0}'$ *is not completely array noncomputable.*

Proof. By [DJS90, Theorem 3.15], $\mathbf{0}'$ is the join of two c.e. array computable degrees $\mathbf{a}_1$ and $\mathbf{a}_2$. So for any two c.e. sets $A_1 \in \mathbf{a}_1$ and $A_2 \in \mathbf{a}_2$, $A_1 \oplus A_2$ is Turing complete and has array computable wtt-degree by Theorem 2.3.1. □

As our next result we show that no high c.e. degree is be completely a.n.c. Before we give the proof we would like to notice that this result is shown independently by other people. For instance, Downey and Ng [DN18] show that any high c.e. degree is the join of two array computable c.e. degrees, thereby strengthening the above result from [DJS90]; hence, by Theorem 2.3.1, any high c.e. degree contains an array computable wtt-degree. Moreover, Ambos-Spies shows in [AS18] that no a.n.c. wtt-degree contains a dense simple set. Hence, since a c.e. degree is high iff it contains a dense simple set by Martin [Mar66], we get the same result again.

Here, we give a direct proof of this statement.

Theorem 2.4.2. *Let* $\mathbf{a}$ *be a high c.e. Turing degree. Then there exists a c.e. set* $B \in \mathbf{a}$ *such that* $deg_{wtt}(B)$ *is array computable.*

Proof. For the proof of Theorem 2.4.2, fix a high c.e. set $A \in \mathbf{a}$ and a very strong array $\mathcal{F} = \{F_n\}_{n\in\omega}$. We construct B and auxiliary c.e. sets $\{V_e\}_{e\in\omega}$ in stages s, where B_s and $V_{e,s}$ denote the finite set of numbers which are enumerated into B and V_e by stage s, respectively, such that B and $\{V_e\}_{e\in\omega}$ meet the requirements for all $e, n \in \omega$,

$$\mathcal{R}_{\langle e,n\rangle}: \; W_{e_0} = \Phi_{e_1}^{B,\varphi_{e_2}} \;\Rightarrow\; W_{e_0} \cap F_n \neq V_e \cap F_n,$$

where $e = \langle e_0, e_1, e_2\rangle$ and $e < n$ and such that B meets the global requirement $B \equiv_T A$. Here, we let, for any oracle X and any $e_0, e_1, y \in \omega$,

$$\Phi_{e_0}^{X,\varphi_{e_1}}(y) = \begin{cases} \Phi_{e_0}^X(y) & \text{if } \Phi_{e_0}^X(y)\downarrow,\ \varphi_{e_1}(y)\downarrow \text{ and } \varphi_{e_0}^X(y) \leq \varphi_{e_1}(y), \\ \uparrow & \text{otherwise.} \end{cases}$$

Since a.n.c. wtt-degrees are closed upwards and since, by Theorem 2.2.3, array noncomputability for wtt-degrees is independent of the choice of very strong array, making sure that for all e, almost all requirements $\mathcal{R}_{\langle e,n\rangle}$ are met and that $A \equiv_T B$ holds implies that B has the desired properties. Before giving the formal construction, let us first present the idea behind it and introduce some of the concepts to be used in the construction.

We begin with the strategy for making $B \equiv_T A$. For this purpose, we have to ensure both $A \leq_T B$ and $B \leq_T A$. For the former, we define a total computable function $\gamma : \omega^2 \to \omega$ (called a *marker function*) such that, for any $x, s \in \omega$,

$$\gamma(x,s) < \gamma(x+1,s), \tag{2.6}$$

$$\gamma(x,s) \leq \gamma(x,s+1), \tag{2.7}$$

$$\gamma^*(x) = \lim_{s\to\infty} \gamma(x,s) \text{ exists}, \tag{2.8}$$

$$\gamma(x,s) \neq \gamma(x,s+1) \;\Rightarrow\; B_{s+1} \upharpoonright \gamma(x,s)+1 \neq B_s \upharpoonright \gamma(x,s)+1, \tag{2.9}$$

$$x \in A_{s+1} \setminus A_s \;\Rightarrow\; \gamma(x,s) \neq \gamma(x,s+1). \tag{2.10}$$

In the following, numbers of the form $\gamma(x,s)$ are called *markers*. From a marker function γ as above, we can compute A using B as an oracle as follows. Given x, compute with oracle B the least stage s such that $B \upharpoonright \gamma(x,s)+1 = B_s \upharpoonright \gamma(x,s)+1$ holds. Such a stage exists by (2.8). Then for any stage $t > s$, $\gamma(x,t) = \gamma(x,s)$ holds by (2.9); hence, $A(x) = A_s(x)$ holds by (2.10). Note that we did not use (2.6) and (2.7) for the definition of a Turing reduction from A to B; however, in the construction we make sure that only markers enter B (see the definition of B_s below). So by (2.6) and (2.7), we can argue that $\gamma(x,s) \notin B_s$ holds for all x, s.

For the definition of γ, we fix a computable enumeration $\{A_s\}_{s\in\omega}$ of A such that $|A_{s+1} \setminus A_s| = 1$ and let a_s be the unique element that enters A at stage $s+1$ (note that such that a computable enumeration of A exists since high sets are noncomputable hence infinite). Then the idea is to define a computable sequence of numbers $\{x_s\}_{s\in\omega}$ such that $x_s \leq a_s$ holds for all s, to let, for all $x, s \in \omega$,

$$\begin{aligned} \gamma(x,0) &= \langle x, 0\rangle \\ \gamma(x,s+1) &= \begin{cases} \gamma(x,s) & \text{if } x < x_s, \\ \langle x, s+1\rangle & \text{otherwise.} \end{cases} \end{aligned} \tag{2.11}$$

and to let $B_s = \{\gamma(x_t, t) : t < s\}$. Then $\{B_s\}_{s\in\omega}$ is a computable enumeration of B, (2.10) follows by the fact that $x_s \leq a_s$ holds and (2.6), (2.7) and (2.9) follow directly from the definition of γ and $\{B_s\}_{s\in\omega}$. So it remains to define $\{x_s\}_{s\in\omega}$

such that (2.8) holds and, for the sake of $B \leq_T A$, to make sure that A can compute, given x, the least stage s such that $x_t > x$ holds for all $t \geq s$. Namely, by (2.6) and by definition of B this implies that $B(x) = B_s(x)$ holds; hence, $B \leq_T A$. Since this will immediately follow from the strategy of how to meet the requirements $\mathcal{R}_{\langle e,n\rangle}$, we describe next how the latter are met.

In the following, fix e, n with $e < n$. Then the strategy for $\mathcal{R}_{\langle e,n\rangle}$ starts at a stage $s+1$ such that $W_{e_0,s}(z) = \Phi_{e_1}^{B,\varphi_{e_2}}(z)[s]$ holds for all $z \in F_n$ and such that $F_n \not\subseteq V_{e,s}$ and $W_{e_0,s} \cap F_n = V_{e,s} \cap F_n$ hold. At stage $s+1$ we put the least number $x \in F_n \setminus V_{e,s}$ into V_e. Note that if $s+1$ fails to exist then, since we ensure that $V_{e,0} = \emptyset$ holds, either the hypothesis of $\mathcal{R}_{\langle e,n\rangle}$ does not hold or $W_{e_0} \cap F_n \neq V_e \cap F_n$ holds. In both cases, $\mathcal{R}_{\langle e,n\rangle}$ is met trivially. Hence, in the following, we may assume that $s+1$ exists. Now obviously, the above strategy does not prevent W_{e_0} from copying V_e on F_n. The important observation is that if $s' > s_0 + 1$ is the least stage such that $W_{e_0,s'}(z) = \Phi_{e_1}^{B,\varphi_{e_2}}(z)[s']$ holds for all $z \in F_n$ and such that $F_n \not\subseteq V_{e,s'}$ and $W_{e_0,s'} \cap F_n = V_{e,s'} \cap F_n$ hold then there must be a number $y \leq \max\{\varphi_{e_2}(z) : z \in F_n\}$ and a stage $t \in [s, s')$ such that y enters B at stage $t+1$. So the above strategy for $\mathcal{R}_{\langle e,n\rangle}$ fails only if there exist more than $|F_n|$ many numbers that enter B below $\max\{\varphi_{e_2}(z) : z \in F_n\}$ after stage $s+1$.

Thus, as $|F_n| \geq n+1$ holds and since only markers may enter B, the idea is to bound the number of markers below $\max\{\varphi_{e_2}(z) : z \in F_n\}$ by n before we start the strategy for $\mathcal{R}_{\langle e,n\rangle}$ by ensuring that $x_{s_0} \leq \min\{n, a_{s_0}\}$ holds at the first stage s_0+1 such that $W_{e_0,s_0}(z) = \Phi_{e_1}^{B,\varphi_{e_2}}(z)[s_0]$ holds for all $z \in F_n$ (in particular, we make sure that $s_0 < s$ holds). In this way, by (2.11) and by convention on converging computations, all markers $\gamma(x, s_0)$ with $x \geq n$ are moved to a greater value than $\max\{\varphi_{e_2}(z) : z \in F_n\}$. Thus, by performing the above strategy at stages $t \geq s$ such that $W_{e_0,t}(z) = \Phi_{e_1}^{B,\varphi_{e_2}}(z)[t]$ holds for all $z \in F_n$ and such that $F_n \not\subseteq V_{e,t}$ and $W_{e_0,t} \cap F_n = V_{e,t} \cap F_n$ hold, we make sure that $\mathcal{R}_{\langle e,n\rangle}$ is met.

Now in order to combine this strategy with the requirement to make $B \leq_T A$ we make sure that, given x, A can compute a stage s such that $x_t > x$ for all $t > s$. For that, we make use of the fact that A is high. By Martin's characterization of high c.e. degrees [Mar66], we may fix a dominating function $f^* \leq_T A$, i.e., for every total, computable $h : \omega \to \omega$, $h(n) < f^*(n)$ holds for almost every n. So fix a Turing functional Φ such that $f^* = \Phi^A$ holds. Then, for any $x, s \in \omega$, we define

$$f(x,s) = \begin{cases} \Phi^A(x)[s] & \text{if } \Phi^A(x)[s]\downarrow, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $f(x,s)$ approximates f^* in the limit and that the least modulus of convergence of f,

$$m^*(x) = \mu s(\forall t \geq s\ (f(x,t) = f(x,s)))$$

is computable by A since A is c.e. and it is also a dominating function since, for any computable function h such that $m^*(n) < h(n)$ holds for infinitely many n, it follows that the function $n \mapsto f(n, h(n))$ is a total computable function which is not dominated by f^*. In the construction, we make sure that the markers $\gamma(x,s)$ with $x \geq n$ are moved above $\max\{\varphi_{e_2}(z) : z \in F_n\}$ at a stage $s+1$ only if there exists $n' \leq n$ such that $f(n', s+1) \neq f(n', s)$. So if the hypothesis of $\mathcal{R}_{\langle e,n\rangle}$ holds, it follows that φ_{e_2} is total; hence, the function h_e which maps k to the least stage s such that $F_k \subset \mathrm{dom}(\varphi_{e_2,s})$ holds is total and computable.

Thus, given e such that the hypothesis of some (hence any) requirement $\mathcal{R}_{\langle e,n\rangle}$ holds we can argue that, for almost all n, it holds that $h_e(n) < m(n)$; whence, for almost all n, there exists a stage such s that all markers $\gamma(x,s)$ with $x \geq n$ are moved to a greater value than $\max\{\varphi_{e_2}(z) : z \in F_n\}$ at stage $s+1$. Hence, by the above strategy, $\mathcal{R}_{\langle e,n\rangle}$ is met for almost all n for any such number e.

This explains the basic strategy of how to define sets B and V_e with the desired properties. We now proceed to the formal construction (recall that B_s is determined by γ and in turn that γ is determined by $\{x_s\}_{s\in\omega}$ by (2.11)).

Construction.

Stage 0. Let $V_{e,0} = \emptyset$ for all $e \in \omega$.

Stage $s+1$. Let $x_{s'}$ for all $s' < s$ (hence B_s) and $V_{e,s}$ be given for all $e \in \omega$. We say that $\mathcal{R}_m$ *requires attention at stage* $s+1$ if $m \leq s$

and for the unique e, n such that $e < n$ and $m = \langle e, n \rangle$, it holds that $\forall z \in F_n\ (\varphi_{e_2}(z)[s]\downarrow)$ and either

(I) $\gamma(n, s) \leq \max\{\varphi_{e_2}(z) : z \in F_n\}$ and $f(n, s+1) \neq f(n, s)$, or

(II) $\max\{\varphi_{e_2}(z) : z \in F_n\} < \gamma(n, s)$, $\forall z \in F_n\ (W_{e_0,s}(z) = \Phi_{e_1}^{B,\varphi_{e_2}}(z)[s])$, $W_{e_0,s} \cap F_n = V_{e,s} \cap F_n$ and $F_n \not\subseteq V_{e,s}$.

Let $M_{(\mathrm{I})}$ ($M_{(\mathrm{II})}$) be the set of all numbers m such that $\mathcal{R}_m$ requires attention via (I) ((II)). Then we let $x_s = \min(\{a_s\} \cup \{n : \exists e < n\ (\langle e, n \rangle \in M_{(\mathrm{I})})\})$ and for all $e < n$ such that $\langle e, n \rangle \in M_{(\mathrm{II})}$ holds, we let $V_{e,s+1} = V_{e,s} \cup \{\min(F_n \setminus V_{e,s})\}$, and we let $V_{e,s+1} = V_{e,s}$, otherwise.

This ends the formal construction.

Verification

We prove in a series of claims that the so constructed sets B and V_e ($e \in \omega$) have the desired properties. Before that, let us give some general remarks about the construction that will be tacitly assumed in the proofs below. Unless otherwise stated, they can be shown by induction on the stage number.

First of all, the construction is effective so $\{x_s\}_{s\in\omega}$ is a computable sequence of numbers and $\{V_{e,s}\}_{s\in\omega}$ is a computable enumeration of V_e. So, by (2.11), γ is a total computable function, V_e is a c.e. set for all e and $\{B_s\}_{s\in\omega}$ is a computable enumeration of B; hence, B is a c.e. set as well.

Furthermore, we note that at any stage $s+1$ such that a requirement $\mathcal{R}_{\langle e,n\rangle}$ requires attention, it follows that $\varphi_{e_2,s}(z)\downarrow$ holds for all $z \in F_n$. Then in the first claim, we show that any requirement requires attention at most finitely often

Claim 1. Every requirement requires attention only finitely often.

Proof. Fix numbers e, n with $e < n$. Then on the one hand, for any stage s such that $\mathcal{R}_{\langle e,n\rangle}$ requires attention via (I) at stage $s+1$, it holds that $x_s \leq n$ by construction. So, by (2.11) and by convention on converging computations, it follows that $\max\{\varphi_{e_2}(z) : z \in F_n\} < \gamma(n, s)$; hence, $\mathcal{R}_{\langle e,n\rangle}$ may require attention via (I) at most once. On the other hand, by construction, $\mathcal{R}_{\langle e,n\rangle}$ may require attention via (II) at most $|F_n|$ many times. $\square$

Then based on Claim 1, we can easily show that γ satisfies (2.8).

Claim 2. For all x there exist at most finitely many stages s such that $x_s \leq x$. In fact, given x, A can compute a stage s such that $x_t > x$ holds for all $t \geq s$. In particular, (2.8) holds.

Proof. Let $x \in \omega$ be given. By Claim 1, fix the least stage s such that no requirement $\mathcal{R}_{\langle e,y\rangle}$ with $e < y \leq x$ requires attention via (I) at any stage $t \geq s$ and such that $A \upharpoonright x+1 = A_s \upharpoonright x+1$ holds. We claim that $x_t > x$ holds for all $t \geq s$ and that A can compute s. For the former, suppose for a contradiction that $t \geq s$ is such that $x_t \leq x$. Then we distinguish between the following cases. If $x_t = a_t$ holds then $A_{t+1} \upharpoonright x+1 \neq A_s \upharpoonright x+1$, contrary to choice of s. Otherwise, by construction, it follows that there exists $e < x_t$ such that $\mathcal{R}_{\langle e,x_t\rangle}$ requires attention via (I) at stage $t+1$, again contrary to choice of s.

So in order to complete the proof, we have to show that A can compute s given x. So let x be given. Now by definition of s, it suffices to argue that A can compute the least stage s_0 such that no requirement $\mathcal{R}_{\langle e,y\rangle}$ with $e < y \leq x$ requires attention via (I) at any stage $t \geq s_0$ since then it follows that $s = \max\{s_0, s_1\}$ holds, where s_1 is the least stage such that $A \upharpoonright x+1 = A_{s_1} \upharpoonright x+1$ holds. However, it holds that $s_0 \leq \max\{m^*(y) : y \leq x\}$ since any requirement $\mathcal{R}_{\langle e,y\rangle}$ may only require attention via (I) at a stage $t+1$ if $f(y,t+1) \neq f(y,t)$ holds. So since m^* is computable from A, it follows that s_0 is computable from A, too, by effectivity of the construction. $\square$

So since $x_s \leq a_s$ holds for all stages s by definition of $\{x_s\}_{s\in\omega}$, it follows by Claim 2 that (2.8) holds; hence $A \leq_T B$ holds since (2.6), (2.7), (2.9) and (2.10) hold by definition of γ and by definition of B. Next, we show that $B \leq_T A$ holds, too.

Claim 3. $B \leq_T A$.

Proof. Let $x \in \omega$ be given. Then by Claim 2, compute with oracle A the least stage s such that $x_t > x$ holds for all $t \geq s$. But then it follows from (2.6) and by definition of B that $B(x) = B_s(x)$ holds. $\square$

Finally, we can show that for any e, almost all requirements $\mathcal{R}_{\langle e,n\rangle}$ $(e < n)$ are met. In the following, fix e.

Claim 4. For almost all n, $\mathcal{R}_{\langle e,n\rangle}$ is met.

Proof. If the hypothesis of $\mathcal{R}_{\langle e,n\rangle}$ does not hold for some $n > e$, it does not hold for any $n > e$. Thus, $\mathcal{R}_{\langle e,n\rangle}$ is met for all $n > e$. So w.l.o.g, we may assume that the hypothesis of $\mathcal{R}_{\langle e,n\rangle}$ holds for all $n > e$. For a proof by contradiction, suppose that $\mathcal{R}_{\langle e,n\rangle}$ is not met for infinitely many $n > e$. We claim that there is a computable function which is not dominated by f^*. For given n, let $h_e(n)$ be the least stage s such that $\varphi_{e_2,s}(z)\downarrow$ for all $z \in F_n$ (note that h_e is total and computable).

In the following, fix $n > e$ such that $\mathcal{R}_{\langle e,n\rangle}$ is not met. We claim that $\mathcal{R}_{\langle e,n\rangle}$ never requires attention via (II). Otherwise, let s_0 be the least and, by Claim 1, let s_1 be the last stage s such that $\mathcal{R}_{\langle e,n\rangle}$ requires attention via (II) at stage $s+1$. Then $s_0 \geq h_e(n)$ since requiring attention requires that $\varphi_{e_2,s}(z)\downarrow$ holds for all $z \in F_n$. Moreover, by construction, it holds that $V_{e_0,s_0} \cap F_n = \emptyset$, $\gamma(n, s_0) > \max\{\varphi_{e_2}(z) : z \in F_n\}$ and $V_e \cap F_n = V_{e_1,s_1} \cap F_n$ since $\{F_n\}_{n\in\omega}$ is a very strong array and no other requirement enumerates numbers into $V_e \cap F_n$.

Now if $F_n \not\subseteq V_e$, we claim that $\mathcal{R}_{\langle e,n\rangle}$ is met. Namely, by definition of (II), the enumeration of $\min(F_n \setminus V_{e,s_1})$ into V_e at stage $s_1 + 1$ yields $V_{e,s_1+1} \cap F_n \neq W_{e_0,s_1} \cap F_n$. However, as $W_{e_0} \cap F_n = V_e \cap F_n$ and $W_{e_0,s_1} \cap F_n = V_{e,s_1} \cap F_n$ hold, it follows by the assumption that $F_n \not\subseteq V_e$ that $\mathcal{R}_{\langle e,n\rangle}$ requires attention after stage s_1 contrary to choice of s_1.

Hence, in order to show that $\mathcal{R}_{\langle e,n\rangle}$ never requires attention via (II), it suffices to show that $F_n \not\subseteq V_e$. For that, we argue as follows. For any stage s such that $\mathcal{R}_{\langle e,n\rangle}$ requires attention via (II) at stage $s+1$, there exists a stage $t \in (s, s_1]$ and a number $\gamma(x,t) \leq \max\{\varphi_{e_2}(z) : z \in F_n\}$ that enters B at stage $t+1$, because

W_{e_0} enumerates $y = \min(F_n \setminus V_{e,s})$ at a stage $t' \in (s,t]$ and because it holds that

$$\Phi^{B_s}_{e_1,s}(y) = W_{e_0,s}(y) = 0 \neq W_{e_0,s_1}(y) = \Phi^{B_{s_1}}_{e_1,s_1}(y).$$

However, since only markers enter B, by (2.6) and (2.7) and by choice of s_0, it holds that $|\{x : \gamma(x,s_0) \leq \max\{\varphi_{e_2}(z) : z \in F_n\}\}| \leq n$ since requiring attention via (II) entails that $\max\{\varphi_{e_2}(z) : z \in F_n\} < \gamma(n,s_0)$. So in particular, at most n numbers below $\max\{\varphi_{e_2}(z) : z \in F_n\}$ may enter B after stage s_0. Hence, W_{e_0} changes at most n times on F_n since $W_{e_0,s_0} \cap F_n = V_{e,s_0} \cap F_n = \emptyset$ and, for each change of W_{e_0} on F_n, it enumerates at most one number into $W_{e_0} \cap F_n$ since we do so by construction. By the fact that $\mathcal{R}_{\langle e,n\rangle}$ is not met, this implies

$$|V_e \cap F_n| = |W_{e_0} \cap F_n| \leq n < |F_n|$$

since $\{F_n\}_{n\in\omega}$ is a very strong array.

So $\mathcal{R}_{\langle e,n\rangle}$ never requires attention via (II). In particular, $W_{e_0} \cap F_n = V_e \cap F_n = \emptyset$ holds. But then we conclude that $\gamma(n,s) \leq \max\{\varphi_{e_2}(z) : z \in F_n\}$ holds for all stages s and that $\mathcal{R}_{\langle e,n\rangle}$ never requires attention via (II). Otherwise, since the hypothesis of $\mathcal{R}_{\langle e,n\rangle}$ holds, it follows by construction that there exists a stage $s' > s$ such that $\mathcal{R}_{\langle e,n\rangle}$ requires attention via (II), contrary to what we have just proven.

Summarizing, if $\mathcal{R}_{\langle e,n\rangle}$ is not met then for all stages $s \geq h(n)$, it holds that $\gamma(n,s) \leq \max\{\varphi_{e_2}(z) : z \in F_n\}$ and $W_{e_0} \cap F_n = V_e \cap F_n = \emptyset$ and $\mathcal{R}_{\langle e,n\rangle}$ never requires attention via any clause. However, this implies that $f(n,s) = f(n,h_e(n))$ holds for all stages $s \geq h_e(n)$; hence, $f^*(n) = f(n,h_e(n))$ holds for any n such that $\mathcal{R}_{\langle e,n\rangle}$ is not met. However, as there are infinitely many such n by assumption, f^* does not dominate the computable function mapping n to $f(n,h_e(n))$, contrary to choice of f^*. □

Since $A \equiv_T B$ holds by Claims 2 and 3 and by construction and since for all e, almost all requirements $\mathcal{R}_{\langle e,n\rangle}$ are met by Claim 4, it follows that B has the desired properties. This completes the proof of Theorem 2.4.2. □

We complete this section by giving a yet different proof of Theorem 2.4.2 which is due to an idea of Merkle and which uses the notion of anti-complex sets which is due to Franklin, Greenberg and Stephan [FGSW13]. A set A is called *anti-complex* if, for every computable order h and almost all numbers n it holds that $C(A \upharpoonright h(n)) \leq n$, where C denotes the plain Kolmogorov complexity. In [FGSW13], Franklin et al. show that anti-complex sets can be characterized as those sets which have c.e. traceable wtt-degree (Theorem 1.3 in [FGSW13]) and that, for c.e. wtt-degrees, c.e. traceability coincides with array computability (Theorem 1.5 in [FGSW13]). So it suffices to prove the following.

Theorem 2.4.3 (Merkle: private communication)**.** *Every high c.e. degree* **a** *contains a c.e. anti-complex set A.*

Proof. Fix a high c.e. set $B \in \mathbf{a}$. We first claim that there exists a dominating function f^* which has a computable approximation $f : \omega^2 \to \omega$ such that f^* and f have the following properties:

$$f(x,s) \leq f(x,s+1), \tag{2.12}$$

$$m^*_f(x) \leq f^*(x), \text{ and} \tag{2.13}$$

$$f^* \equiv_T B, \tag{2.14}$$

where $m^*_f(x) = \mu s(\forall t \geq s\ (f(x,t) = f(x,s))$. Such functions f^* and f can be obtained as follows. By [Mar66], let $g^* \leq_T B$ be a dominating function. Fix a Turing functional Φ such that $g^* = \Phi^B$ and a computable enumeration $\{B_s\}_{s\in\omega}$ of B, where w.l.o.g., we may assume that $\varphi^B(x) \geq x$ holds for all $x \in \omega$. Let

$$m^*(x) = \mu s(\Phi^B(x)[s]\downarrow\ \&\ B \upharpoonright \varphi^B(x)[s] + 1 = B_s \upharpoonright \varphi^B(x)[s] + 1) \tag{2.15}$$

and

$$g(x,s) = \begin{cases} \Phi^B(x)[s] & \text{if } \Phi^B(x)[s]\downarrow, \\ 0 & \text{otherwise}, \end{cases},$$

$$m(x,s) = \begin{cases} \mu t < s(P(x,t,s)) & \text{if } \exists t < s\ (P(x,t,s)), \\ s & \text{otherwise}, \end{cases}$$

where $P \subseteq \omega^3$ is the computable ternary predicate such that, for all $x, t, s \in \omega$,

$P(x,t,s)$ holds iff $\Phi^B(x)[t]\downarrow$ and $B_s \upharpoonright \varphi^B(x)[t]+1 = B_t \upharpoonright \varphi^B(x)[t]+1$. Note that g, m are computable, m^* is total by totality of g^*, $m(x,s)$ is nondecreasing in s by construction and m and g approximate m^* and g^* in the limit, respectively. More precisely, $g(x,s) = g^*(x)$, $m(x,s) = m^*(x)$ and, moreover, $\varphi^B(x)[s] = \varphi^B(x)$ hold for all $s \geq m^*(x)$ by (2.15); hence, $B_s(x) = B(x)$ holds for all $s \geq m^*(x)$ as $\varphi^B(x) \geq x$. It follows that $B \leq_T m^*$ holds since $B(x) = B_{m^*(x)}(x)$ and $m^* \leq_T B$ holds since, given x, B can compute a stage s such that the inner clause of (2.15) which defines $m^*(x)$ holds. We then let

$$\begin{aligned} f(x,s) &= \max(\{g(x,t) : t \leq s\} \cup \{m(x,s)\}), \\ f^*(x) &= \lim_{s\to\infty} f(x,s) \end{aligned}$$

and claim that f and f^* have the required properties. First, $f(x,s)$ is computable given, it is nondecreasing in s (so (2.12) holds) and $f^*(x)$ exists since $f(x,s) = f(x, m^*(x))$ holds for all $s \geq m^*(x)$. This implies that $m^*_f(x) \leq m^*(x)$ by definition of m^*_f (so (2.13) holds) which in turn implies that $f^* \leq_T m^*$. In addition $m^* \leq_T f^*$ holds as f^* majorizes m^*; hence, (2.14) holds.

Given f and f^* as above, we let $A = \{\langle x,y\rangle : y \leq f^*(x)\}$. Then A is c.e. by (2.12) and $A \equiv_T B$ holds by (2.14). So it remains to show that A is anti-complex. For that, it suffices to show that, for all computable orders h and almost all numbers n, it holds that $C(A \upharpoonright h(n)) \leq n + O(1)$. Namely, suppose the latter and fix constants c_i $(i \leq 1)$ such that $C(A \upharpoonright h(2n+i)) \leq n + c_i$ holds for almost all n and all $i \leq 1$. Then for $c = \max\{c_0, c_1\}$, it follows that $C(A \upharpoonright h(n)) \leq \frac{n}{2} + c$; hence, $C(A \upharpoonright h(n)) \leq n$ holds for almost all n.

Now fix a computable order h. Then since f^* is dominating, let n_0 be such that $f(n) \geq h(n)$ holds for all $n \geq n_0$. Now fix any number $n \geq n_0$. Then the fact that n bits of information are enough to describe $A \upharpoonright h(n)$ follows from the following observation. Let $k < h(n)$ be given and let x, y be such that $k = \langle x,y\rangle$. Then we distinguish between the following cases:

(1) $x \geq n$. This implies $y \leq k < h(n) \leq h(x) \leq f^*(x)$ since $x \geq n_0$; hence $k \in A$.

(2) $x < n$. Then we distinguish between the following subcases.

(a) $h(n) \leq f^*(x)$. Then $k \in A$ follows analogously as in(1).

(b) $f^*(x) < h(n)$. Then by (2.13), it follows that $k \in A$ iff $y \leq f(x, h(n))$ holds.

Using this case distinction, the idea is to let M be a Turing machine which is defined as follows. On input $\sigma \in \{0,1\}^{<\omega}$, the ouput $M(\sigma)$ is a binary string of length $h(|\sigma|)$ such that, for all x, y with $\langle x, y\rangle < h(|\sigma|)$,

$$M(\sigma)(\langle x, y\rangle) = \begin{cases} 1 & \text{if either } x \geq |\sigma|, \text{ or} \\ & x < |\sigma| \text{ and } \sigma(x) = 1, \text{ or} \\ & x < |\sigma|,\ \sigma(x) = 0 \text{ and } y \leq f(x, h(|\sigma|)), \\ 0 & \text{otherwise.} \end{cases}$$

By the above case distinction, $M(\sigma_n) = A \upharpoonright h(n)$ holds for all $n \geq n_0$, where, for given n, σ_n is the binary string of length n such that, for all $x < n$, $\sigma_n(x) = 1$ iff $h(n) \leq f^*(x)$ holds. It follows that $C(A \upharpoonright h(n)) \leq n + O(1)$ for all $n \geq n_0$ by definition of C. □

2.5 Completely Array Noncomputable Degrees and Maximal Pairs

In this section, we strengthen Theorem 2.3.3 by showing that there exists a c.e. degree such that all c.e. sets in that degree are halves of (cl-)maximal pairs. Recall that a pair of c.e. sets (A, B) is a *cl-maximal pair in the c.e. sets* if there is no c.e. set C such that $A \leq_{cl} C$ and $B \leq_{cl} C$ holds and that a c.e. set A is *half of a cl-maximal pair* if there exists a c.e. set B such that (A, B) is a cl-maximal pair.

The existence of cl-maximal pairs in the c.e. sets is shown independently by Barmpalias [Bar05] and Fan and Lu [FL05] and, as mentioned before, Ambos-Spies, Ding, Fan and Merkle [ASDFM13] show that halves of maximal pairs can be found in exactly the a.n.c. degrees. Moreover, they show that c.e. cl-maximal pairs coincide with the corresponding maximal pairs in the c.e. identity bounded Turing (ibT-) degrees using bounded shifts, where ibT-reducibility is introduced by Soare [Soa04]. So here and in the following chapters, we may refer to cl- or ibT-maximal pairs simply as maximal pairs.

Ambos-Spies ([AS16]) improves the above result from [ASDFM13] by showing that halves of maximal pairs of c.e. sets occur exactly in the a.n.c. wtt-degrees. In fact, for one direction, he shows the following.

Theorem 2.5.1 (Theorem 2.1 in [AS16])**.** *If A is half of a maximal pair then there exists an a.n.c. set $C \equiv_{ibT} A$.*

However, the converse is not true.

Theorem 2.5.2 (Theorem 3.1 in [AS16])**.** *There exists an a.n.c. set A which is not half of a maximal pair.*

In view of Theorem 2.3.3, this raises the question whether there exists a Turing degree whose c.e. members are all halves of maximal pairs. We can give an affirmative answer, thereby solving the first part of the first open problem in [ASDFM13].

Theorem 2.5.3. *There exists a c.e. Turing degree* $\mathbf{a}$ *such that every c.e. set* $A \in \mathbf{a}$ *is half of a maximal pair.*

Hence, by Theorem 2.5.1, we immediately get the existence of completely a.n.c. degrees as corollary.

Corollary 2.5.4. *There exists a completely a.n.c. degree* $\mathbf{a}$.

Remark 2.5.5. *The idea of the proof of Theorem 2.5.3 bears some analogy to the one that is used in Theorem 2.3.3 and combines it with the strategy for constructing a maximal pair as demonstrated in [ASDFM13]. So let us first recall the strategy for constructing a maximal pair (A, B) as presented there. The requirements are of the form*

$$\mathcal{R}: \; A \neq \hat{\Phi}^W, \text{ or } B \neq \hat{\Psi}^W,$$

where A and B are the c.e. sets under construction, $\hat{\Phi}, \hat{\Psi}$ are ibT-functionals and W is a given c.e. set. Then the strategy for meeting $\mathcal{R}$ is as follows. We *assign an interval* $[a, b]$ *to* $\mathcal{R}$ *such that* A *and* B *are empty on* $[a, b]$ *when the attack on* $\mathcal{R}$ *starts and such that* $b \geq 2a$ *holds. Then at stages* $s + 1$ *such that*

$$\forall x \in [a, b]\ (A_s(x) = \hat{\Phi}^W(x)[s]\ \&\ B_s(x) = \hat{\Psi}^W(x)[s]), \tag{2.16}$$

and $[a, b] \not\subseteq A_s \cup B_s$*, we enumerate a number* $x \in [a, b] \setminus A_s$ *into* A *at stage* $s + 1$ *in case* $[a, b] \not\subseteq A_s$ *holds; otherwise, we choose a number* $x \in [a, b] \setminus B_s$ *and enumerate it into* B *at stage* $s + 1$*. The claim is that this strategy meets* $\mathcal{R}$*. For a proof by contradiction, suppose that* $\mathcal{R}$ *is not met. Then for every* x *that is enumerated into one of the sets* A *or* B*,* W *enumerates a number* $y \leq x$ *at a later stage since* $\hat{\Phi}$ *and* $\hat{\Psi}$ *are ibT-functionals. However, by choice of* b *there are* $2|[a, b]| = 2(b - a + 1) > b + 1$ *many numbers in* $[a, b]$ *that may be enumerated into* A *or* B *while* W *may choose at most* $b + 1$ *numbers to respond to each attack.*

Now the proof of Theorem 2.5.3 is as follows.

Proof of Theorem 2.5.3. For the proof of Theorem 2.5.3, fix a computable numbering $\{\hat{\Phi}_e\}_{e\in\omega}$ of the ibT-functionals. Then we construct a c.e. set A and auxiliary c.e. sets V_e ($e \in \omega$) in stages s, where A_s and $V_{e,s}$ denote the finite sets of numbers that are enumerated into A and V_e by stage s, respectively, such that A and the sets V_e meet the requirements

$$\mathcal{R}_m :\ (A = \Phi_{e_1}^{W_{e_0}}\ \&\ W_{e_0} = \Phi_{e_2}^{A})\ \Rightarrow\ W_{e_0} \neq \hat{\Phi}_{d_1}^{W_{d_0}} \text{ or } V_e \neq \hat{\Phi}_{d_2}^{W_{d_0}}, \tag{2.17}$$

for all $m \in \omega$, where $m = \langle e, d\rangle$, $e = \langle e_0, e_1, e_2\rangle$ and $d = \langle d_0, d_1, d_2\rangle$. Clearly, meeting all the requirements ensures that $\mathbf{a} = \deg_T(A)$ has the desired properties. Before we give the formal construction, let us first introduce the idea how to meet a single requirement. Based on this, it is easy to see that all requirements can be met by a finite injury argument, where a requirement $\mathcal{R}_m$ has *higher priority than* $\mathcal{R}_{m'}$ if and only if $m < m'$. So in the following, fix m and let e, d be such that $m = \langle e, d\rangle$ and let e_i and d_i ($i \leq 2$) be such that $e = \langle e_0, e_1, e_2\rangle$ and $d = \langle d_0, d_1, d_2\rangle$.

Now in order to meet a requirement $\mathcal{R}_m$, the idea is to apply the above maximal pair strategy to V_e and W_{e_0}. However, this is not possible a priori since we do not have any control over the numbers that are put into W_{e_0}. For this purpose, we take some of the ideas used in the proof of Theorem 2.3.3 and make them compatible with the above maximal pair strategy.

More precisely, we say that, for a stage s, an interval $[a, b]$ and a triple of sequences $(\{x_i\}_{i\leq a}, \{y_i\}_{i\leq a}, \{z_i\}_{i\leq a})$ are *suitable for* $\mathcal{R}_m$ *at stage* $s+1$ (or *suitable* for short) if, for all $i \leq a$, it holds that

(i) $x_i \in \omega^{[e]}$,

(ii) $r(m, s) \leq a \leq x_i < y_i < z_i$ and $z_i < x_{i+1}$ if $i < a$,

(iii) $z_a \leq b < l_0(e, s)$,

(iv) $y_i > \varphi_{e_1}^{W_{e_0}}(x_i)[s]$,

(v) $z_i > \max\{\varphi_{e_2}^{A}(u)[s] : u \leq y_i\}$,

(vi) $\{x_0, \dots, x_a\} \cap A_s = \emptyset$, and

(vii) $[a, b] \cap V_{e,s} = \emptyset$,

where $l_0 : \omega^2 \to \omega$ is defined as

$$l_0(e, s) = \mu x(A_s(x) \neq \Phi_{e_1}^{W_{e_0}}(x)[s] \text{ or } W_{e_0,s}(x) \neq \Phi_{e_2}^{A}(x)[s]) \tag{2.18}$$

and where $r(m, s)$ denotes the restraint which is imposed on $\mathcal{R}_m$ by higher priority requirements. As in Theorem 2.3.3, $r(m, s)$ is increased at any stage $s+1$ such that there is a requirement of higher priority than $\mathcal{R}_m$ that acts at stage $s+1$ (where it is sufficient to let $r(m, s+1) = s$ in this case).

Call a stage $t+1$ *critical (w.r.t.* $\mathcal{R}_m$*)* if $\mathcal{R}_m$ is assigned an interval $[a, b]$ and a triple of sequences $(\{x_i\}_{i\leq a}, \{y_i\}_{i\leq a}, \{z_i\}_{i\leq a})$ and it holds that $\{x_i : i \leq a\} \not\subseteq A_t$ or $[a, b] \not\subseteq V_{e,t}$ and it holds that $b < \min\{l_0(e, s), l_1(m, s)\}$, where

$$l_1(m, s) = \mu x(W_{e_0,s}(x) \neq \hat{\Phi}_{d_1}^{W_{d_0}}(x)[s], \text{ or } V_{e,s}(x) \neq \hat{\Phi}_{d_2}^{W_{d_0}}(x)[s]). \tag{2.19}$$

Then the strategy is as follows. At the least stage $s+1$ such that there exists an interval $[a, b]$ and a triple of sequences $(\{x_i\}_{i\leq a}, \{y_i\}_{i\leq a}, \{z_i\}_{i\leq a})$ which are suitable for $\mathcal{R}_m$ at stage $s+1$, we assign $[a, b]$ and $(\{x_i\}_{i\leq a}, \{y_i\}_{i\leq a}, \{z_i\}_{i\leq a})$

to $\mathcal{R}_m$ at stage $s+1$ and initialize all lower priority requirements, i.e., for any $m' > m$ the triple of sequences and the interval which are assigned to $\mathcal{R}_{m'}$ (if any) are cancelled and $\mathcal{R}_{m'}$ has to choose a new suitable interval and triple of sequences after stage $s+1$. Then at critical stages $t+1 > s+1$, we put the largest $x_i \notin A_t$ into A if $\{x_i : i \leq a\} \not\subseteq A_t$ holds and otherwise we put the least $x \in [a,b] \setminus V_{e,t}$ into V_e and again initialize all requirements of lower priority at stage $t+1$.

Note that the notion of suitability for an interval $[a,b]$ and a triple of sequences $(\{x_i\}_{i \leq a}, \{y_i\}_{i \leq a}, \{z_i\}_{i \leq a})$ for a requirement $\mathcal{R}_{\langle e,d \rangle}$ only depends on e and that the restraint function $r(m,s)$ is nondecreasing in m, i.e., $r(m,s) \leq r(m',s)$ holds for all $m \leq m'$. Thus, any interval and triple of sequences which is suitable for a requirement $\mathcal{R}_{\langle e,d \rangle}$ is also suitable for any requirement $\mathcal{R}_{\langle e,d' \rangle}$ with $d' < d$; hence, we can argue that $\mathcal{R}_{\langle e,d \rangle}$ gets a suitable interval and triple of sequences assigned only if all $\mathcal{R}_{\langle e,d' \rangle}$ with $d' < d$ are assigned suitable intervals and triples of sequences. This becomes important when we have to argue that any requirement $\mathcal{R}_{\langle e,d \rangle}$ whose hypothesis is true eventually gets a permanent sequence assigned.

Hence, if the hypothesis of $\mathcal{R}_{\langle e,d \rangle}$ holds then, by definition of critical stages, it is not hard to show that the interval $[a,b]$ and the triple of sequences $(\{x_i\}_{i \leq a}, \{y_i\}_{i \leq a}, \{z_i\}_{i \leq a})$ meet $\mathcal{R}_{\langle e,d \rangle}$ since we have $b-a+1+a+1 > b+1$ many numbers to put into V_e and A, respectively, while W_d may respond at most $b+1$ many times.

This describes the basic idea for constructing c.e. sets A and V_e which meet all requirements $\mathcal{R}_m$. We now turn to the formal construction.

Construction.

Stage 0. Let $A_0 = V_{e,0} = \emptyset$ and $r(m,s) = 0$ for all $e, m \in \omega$.

Stage $s+1$. Let A_s and $V_{e,s}$ be given for all $e \in \omega$. We say that $\mathcal{R}_m$ *requires attention at stage* $s+1$ if $m \leq s$ and for the unique e, d such that $m = \langle e,d \rangle$, either

(I) no triple of sequences and no interval are assigned to $\mathcal{R}_m$ and there exists an interval $[a,b]$ and a triple of sequences $(\{x_i\}_{i \leq a}, \{y_i\}_{i \leq a}, \{z_i\}_{i \leq a})$

which are suitable for $\mathcal{R}_m$, or

(II) $[a, b]$ and $(\{x_i\}_{i \leq a}, \{y_i\}_{i \leq a}, \{z_i\}_{i \leq a})$ are assigned to $\mathcal{R}_m$ and $s + 1$ is critical.

If there is no requirement that requires attention at stage $s + 1$, let $A_{s+1} = A_s$, $V_{e,s+1} = V_{e,s}$ for all $e \in \omega$ and $r(m, s + 1) = r(m, s)$ for all $m \in \omega$. Otherwise, let m be minimal such that $\mathcal{R}_m$ requires attention at stage $s + 1$. Say that $\mathcal{R}_m$ *receives attention* and act according to the clause via which $\mathcal{R}_m$ requires attention.

If (I) holds, let a be the least number such that there exists $b \in \omega$ and a triple of sequences $(\{x_i\}_{i \leq a}, \{y_i\}_{i \leq a}, \{z_i\}_{i \leq a})$ such that $[a, b]$ and $(\{x_i\}_{i \leq a}, \{y_i\}_{i \leq a}, \{z_i\}_{i \leq a})$ are suitable for $\mathcal{R}_m$ at stage $s+1$. Then, for this a, let $(\{x_i\}_{i \leq a}, \{y_i\}_{i \leq a}, \{z_i\}_{i \leq a})$ be the least triple of sequences (w.r.t. to the lexicographical ordering of triples of numbers, where we assume that sequences of numbers are also ordered lexicographically) such that there exists b such that $[a, b]$ and $(\{x_i\}_{i \leq a}, \{y_i\}_{i \leq a}, \{z_i\}_{i \leq a})$ are suitable for $\mathcal{R}_m$ at stage $s + 1$ and, for this a and this triple of sequences $(\{x_i\}_{i \leq a}, \{y_i\}_{i \leq a}, \{z_i\}_{i \leq a})$, choose the least b such that $[a, b]$ and $(\{x_i\}_{i \leq a}, \{y_i\}_{i \leq a}, \{z_i\}_{i \leq a})$ are suitable for $\mathcal{R}_m$ at stage $s+1$. Assign $[a, b]$ and $(\{x_i\}_{i \leq a}, \{y_i\}_{i \leq a}, \{z_i\}_{i \leq a})$ to $\mathcal{R}_m$.

If (II) holds, distinguish between the following two cases. If $\{x_i : i \leq a\} \nsubseteq A_s$ holds, let $i \leq a$ be largest such that $x_i \notin A_s$ and let $A_{s+1} = A_s \cup \{x_i\}$ and $V_{e,s+1} = V_{e,s}$ for all $e \in \omega$. Otherwise, let $V_{e,s+1} = V_{e,s} \cup \{x\}$ for the least $x \in [a, b] \setminus V_{e,s}$ and let $A_{s+1} = A_s$. In either case, let $V_{e',s+1} = V_{e',s}$ for all $e' \neq e$.

In any case, *initialize* all requirements $\mathcal{R}_{m'}$ with $m' > m$, i.e., cancel their assigned interval and their assigned triple of sequences (if any). Moreover, for all $m' > m$, let $r(m', s + 1) = s$ and for all $m' \leq m$, let $r(m', s + 1) = r(m', s)$.

This ends the formal construction.

Verification

We prove in a series of claims that the so construced sets A and V_e meet the requirements. Before that, let us give some general remarks about the construction which will be tacitly used in the proofs of the claims below. Unless otherwise stated, they can be shown by an easy induction on the stage number.

First of all, the construction is effective and $\{A_s\}_{s\in\omega}$ and $\{V_{e,s}\}_{s\in\omega}$ are computable enumerations of A and V_e, respectively, for all $e \in \omega$. Hence, A and all sets V_e are c.e. sets. The restraint function r is computable, $r(m, s)$ is nondecreasing in m and in s, $r(m, s) \leq s$ holds for all m, s and $r(m, s + 1) \neq r(m, s)$ holds only if a requirement $\mathcal{R}_{m'}$ with $m' < m$ requires attention at stage $s + 1$.

At any stage $s+1$ at most one requirement receives attention and numbers are enumerated into A or V_e only if a requirement requires attention via (II) (where a requirement $\mathcal{R}_m$ may enumerate numbers into V_e only if there exists $d \in \omega$ such that $m = \langle e, d\rangle$) and, in any case, at any stage, a number is enumerated into at most one of the sets A or V_e for at most one $e \in \omega$.

More precisely, if x is enumerated into A or V_e at stage $t + 1$ then for the unique $m \in \omega$ such that $\mathcal{R}_m$ receives attention via (II) at stage $t + 1$, $\mathcal{R}_m$ is assigned an interval $[a, b]$ and a triple of sequences $(\{x_i\}_{i\leq a}, \{y_i\}_{i\leq a}, \{z_i\}_{i\leq a})$, where, at the stage $s+1 < t+1$ when $\mathcal{R}_m$ gets $(\{x_i\}_{i\leq a}, \{y_i\}_{i\leq a}, \{z_i\}_{i\leq a})$ and $[a, b]$ assigned, $[a, b]$ and $(\{x_i\}_{i\leq a}, \{y_i\}_{i\leq a}, \{z_i\}_{i\leq a})$ are suitable for $\mathcal{R}_m$. So $r(m, s) \leq x < \min\{l_0(e, t), l_1(\langle e, d\rangle, t)\}$ and $x \in [a, b]$ hold. So, since $\mathcal{R}_m$ initializes all requirements of lower priority at stage $s + 1$, only $\mathcal{R}_m$ may enumerate x into A at stage $t + 1$ and, after stage $t + 1$, any requirement $\mathcal{R}_{m'}$ with $m < m'$ may enumerate only numbers into A or V_e which are greater than b. Hence, by convention on converging computations, it holds that $A_s \cup V_{e,s} \subseteq \omega \upharpoonright s$ for all $e \in \omega$.

Finally, by the weak monotonicity of the restraint $r(m, s)$ in m, it follows that if a requirement $\mathcal{R}_{\langle e,d\rangle}$ requires attention via (I) at stage $s + 1$ then all requirements $\mathcal{R}_{\langle e,d'\rangle}$ with $d' < d$ which are not assigned an interval and a triple of sequences at stage s also require attention via (I). So if $\mathcal{R}_{\langle e,d\rangle}$ is assigned an interval and a triple of sequences at a stage s then all requirements $\mathcal{R}_{\langle e,d'\rangle}$ with $d' < d$ are assigned an interval and a triple of sequences, too.

Now the first claim states that the action of each requirement is finitary and that the restraint reaches a finite limit.

Claim 1. Every requirement $\mathcal{R}_m$ requires attention only finite often and $r^*(m) = \lim_{s\to\infty} r(m,s)$ exists.

Proof. For a proof by induction on m, suppose the claim to be true for all $m' < m$. By inductive hypothesis, let s_0 be a stage such that $\mathcal{R}_m$ is not initialized after stage s_0. So $r^*(m) = r(m, s_0)$ exists and it holds that whenever $\mathcal{R}_m$ requires attention after stage s_0, it receives attention and acts. Then after stage s_0, $\mathcal{R}_m$ may require attention at most once via (I). If it does not require attention via (I) it neither does require attention via (II) so the claim holds in this case. Otherwise, if $\mathcal{R}_m$ does require attention via (I) at a stage $s_1 > s_0$ then it gets an interval $[a, b]$ and a triple of sequences $(\{x_i\}_{i\leq a}, \{y_i\}_{i\leq a}, \{z_i\}_{i\leq a})$ assigned. Thus, by definition of (II), $\mathcal{R}_m$ may require attention at most $b + 2$ many times after stage s_1. So the claim holds in this case, too. □

Based on Claim 1, we can show that all requirements are met.

Claim 2. $\mathcal{R}_m$ is met.

Proof. For a proof by contradiction, suppose that $\mathcal{R}_m$ is not met and let e, d be such that $m = \langle e, d\rangle$. Then the hypothesis holds for $\mathcal{R}_m$ but the conclusion of $\mathcal{R}_m$ fails. Then let us first show that $\mathcal{R}_m$ is eventually permanently assigned an interval and a triple of sequences. For that, by Claim 1, suppose s_0 is a stage such that $\mathcal{R}_m$ is not initialized at any stage $s \geq s_0$. So $\lim_{s\to\infty} r(m,s) = r(m, s_0)$ and whenever $\mathcal{R}_m$ requires attention after stage s_0 it receives attention and acts. If $\mathcal{R}_m$ is assigned an interval and a triple of sequences at stage s_0 then this interval and triple of sequences are permanently assigned to $\mathcal{R}_m$ by choice of s_0. So w.l.o.g., we may assume that $\mathcal{R}_m$ is not assigned an interval and a triple of sequences at stage s_0. Hence, it suffices to show that there is a stage $s \geq s_0$ such that $\mathcal{R}_m$ requires attention via (I) at stage $s + 1$.

Suppose that this is not the case. Let $a = r(m, s_0)$ and let $\{x_i\}_{i\leq a}$, $\{y_i\}_{i\leq a}$

and $\{z_i\}_{i\leq a}$ be defined by induction on $i \leq a$ such that

$$\begin{aligned} x_0 &= \langle e, a\rangle, \\ y_i &= \max\{\varphi_{e_1}^{W_{e_0}}(x_i), x_i+1\}, \\ z_i &= \max(\{\varphi_{e_2}^{A}(u) : u \leq y_i\} \cup \{y_i+1\}), \\ x_{i+1} &= \langle e, z_i+1\rangle \ \ (i < a), \end{aligned}$$

let $b = z_a$ and let $s \geq s_0$ be the least stage such that both $\varphi_{e_1}^{W_{e_0}}(x_i)[s] = \varphi_{e_1}^{W_{e_0}}(x_i)$ and $\varphi_{e_2}^{A}(u)[s] = \varphi_{e_2}^{A}(u)$ hold for all $u \leq y_i$ and all $i \leq a$ and such that $z_a < l_0(e, s)$ (note that such an interval $[a, b]$ and such sequences $\{x_i\}_{i\leq a}$, $\{y_i\}_{i\leq a}$ and $\{z_i\}_{i\leq a}$ exist since, by assumption, the hypothesis of $\mathcal{R}_m$ holds). Then, by definition, we may argue that $(\{x_i\}_{i\leq a}, \{y_i\}_{i\leq a}, \{z_i\}_{i\leq a})$ satisfies (i)–(v). Moreover, by construction, all numbers that are enumerated into A or V_e by requirements of higher priority than $\mathcal{R}_m$ must be less than $r(m, s_0)$ and, for all requirements $\mathcal{R}_{\langle e',d'\rangle}$ of lower priority than $\mathcal{R}_m$, we can distinguish between the following two cases. If $e' \neq e$ then neither any of the x_i's may be enumerated into A by the action of $\mathcal{R}_{\langle e',d'\rangle}$ by (i) in the definition of suitability nor any number may be enumerated into V_e by the action of $\mathcal{R}_{\langle e',d'\rangle}$. Otherwise, $d < d'$ holds since $\mathcal{R}_{\langle e',d'\rangle}$ has lower priority than $\mathcal{R}_m$. However, in the latter case, $\mathcal{R}_{\langle e',d'\rangle}$ may be assigned a suitable interval and triple of sequences only after $\mathcal{R}_m$ is already assigned such objects. So it follows that (vi) and (vii) hold, too. Hence, $[a, b]$ and $(\{x_i\}_{i\leq a}, \{y_i\}_{i\leq a}, \{z_i\}_{i\leq a})$ are suitable for $\mathcal{R}_m$ at stage $s+1$; hence, $\mathcal{R}_m$ requires attention via (I) after stage s_0, contrary to assumption.

Thus, we may fix the least stage $s \geq s_0$ such that $\mathcal{R}_m$ is permanently assigned an interval $[a, b]$ and a triple of sequences $(\{x_i\}_{i\leq a}, \{y_i\}_{i\leq a}, \{z_i\}_{i\leq a})$ at stage $s+1$. Since $V_{e,s} \cap [a, b] = \{x_i : i \leq a\} \cap A_s = \emptyset$ and $b < \min\{l_0(e, t), l_1(m, t)\}$ holds for sufficiently large stages t, we can argue that there exists a sequence of stages $s < t_0 < t_1 < \cdots < t_{b+1}$ such that $\mathcal{R}_m$ receives attention via (II) at stage $t_i + 1$ (hence each $t_i + 1$ is a critical stage). So, by construction, for any $i \leq a$, x_i is enumerated into A at stage $t_{a-i} + 1$ and, for any $i \in [a, b]$, i is enumerated into V_e at stage $t_i + 1$.

As in the proof of Claim 3 of Theorem 2.3.3, we can argue that if x_i is enumerated into A at stage $t_{a-i} + 1$ ($i \leq a$) then there exists a $y_i' < y_i$ and a stage $t' > t_i$ such that y_i' enters W_{e_0} at stage $t' + 1$. Since there may be at most $b + 2$ many critical stages after stage s by construction, it follows that t_{i+1} is the least critical stage $> t_i$ for any $i \leq b$. Thus, it holds that $t' + 1 \leq t_{i+1}$. We can argue by the use principle (using the fact that $\hat{\Phi}_{d_1}$ is an ibT-functional) that $W_{d_0,t_i} \upharpoonright y_i' + 1 = W_{d_0,t_{i+1}} \upharpoonright y_i' + 1$ cannot hold; hence, there exists $y_i'' \leq y_i'$ and $t'' < t_{i+1}$ such that y_i'' enters W_{d_0} at stage $t'' + 1$. Likewise, since $\hat{\Phi}_{d_2}$ is an ibT-functional, for any $i \in [a, b]$, we can argue that the enumeration of i into V_e at stage $t_i + 1$ yields a change of W_{d_0} below i after stage t_i. Hence, since W_{d_0} is a c.e. set, for the least stage $t > t_{b+1}$ such that $b < \min\{l_0(e, t), l_1(m, t)\}$, it follows that $|W_{d_0,t} \upharpoonright b + 1| > b + 1$ which is impossible. □

Since all requirements are met by Claim 2 and since meeting all requirements ensures that $\deg_T(A)$ is as desired this completes the proof. □

Chapter 3

Totally ω-C.E. Degrees and Maximal Pairs

3.1 Introduction

Totally ω-computably enumerable (totally ω-c.e.) degrees are introduced by Downey, Greenberg and Weber in [DGW07] (Nowadays, it seems that the term totally ω-computably approximable (ω-c.a.) is preferred for these sets and degrees, as e.g., suggested by [DG19]. However, since the former term is the one which is widely used in the literature, we stick to the originally introduced notion in this chapter.). A degree $\mathbf{a}$ is totally ω-c.e. if every function $g \leq_T \mathbf{a}$ has a computable approximation $\{g_s\}_{s\in\omega}$ such that the number of mind changes of $g_s(x)$ is bounded by a computable order h, where recall that a function $h : \omega \to \omega$ is an order if it is nondecreasing and has unbounded range. Since by 2. of Theorem 2.2.4, a c.e. degree $\mathbf{a}$ is array computable iff there exists a computable order h such that any $g \leq_T \mathbf{a}$ is h-c.e., it can be easily seen that the class of c.e. not totally ω-c.e. degrees is a subclass of the a.n.c. degrees. Indeed, the motivation for introducing not totally ω-c.e. degrees is that these degrees provide a stronger form of multiple permitting, called *not totally ω-c.e. permitting* which is also introduced in [DGW07].

The authors demonstrate the not totally ω-c.e. permitting technique by showing that the c.e. not totally ω-c.e. degrees can be characterized as those c.e.

degrees which bound a critical triple, where recall that three incomparable c.e. degrees $\mathbf{a_0}, \mathbf{a_1}$ and $\mathbf{b}$ form a *critical triple* if $\mathbf{a_0} \vee \mathbf{b} = \mathbf{a_1} \vee \mathbf{b}$ and every c.e. degree $\mathbf{c}$ which is below $\mathbf{a_0}$ and $\mathbf{a_1}$ is below $\mathbf{b}$, thereby showing that the totally ω-c.e. degrees are definable in the Turing degrees. Later on, more instances of the fact that not totally ω-c.e. permitting captures the combinatorics of a wide class of constructions were given. For example, Barmpalias, Downey and Greenberg [BDG10] prove that any not totally ω-c.e. degree contains a set which is not wtt-reducible to any hypersimple set; Brodhead, Downey and Ng [BDN12] show that any not totally ω-c.e. contains a computably bounded random set. Furthermore, Downey and Greenberg [DG19] transfer the idea behind totally ω-c.e. degrees to certain ordinals higher than ω and show that a c.e. degree bounds an embedding of the nondistributive lattice M_3 iff it is not totally $< \omega^\omega$-c.e.

In this chapter, we extend the notion of a.n.c. sets to the setting of almost-c.e. sets which is in the style of Definition 2.2.2, where recall that a set A is almost-c.e. if it has a computable approximation $\{A_s\}_{s\in\omega}$ such that, for every number x and every stage s, x may either be enumerated into A at stage $s+1$ or it may be removed from A, but in the latter case some number $y < x$ must be put into A at the same stage (so, when identifying sets with real numbers from the unit interval $[0, 1]$, almost-c.e. sets just correspond to the left-c.e. reals, see Proposition 3.2.2 below).

The problem is that one cannot simply define array noncomputability for almost-c.e. sets in the straightforward way. Namely, as shown in [ASFL$^+$], given a very strong array $\mathcal{F} = \{F_n\}$ and an almost-c.e. set A there exists an almost-c.e. set B such that A and B are not $\mathcal{F}$-similar, where we say for two sets A and B and a v.s.a. $\mathcal{F}$ that A and B are $\mathcal{F}$-similar if (2.2) holds.

The solution is to require for the almost-c.e. set A to be $\mathcal{F}$-similar only to those almost-c.e. sets B for which there exists a computable almost enumeration which is compatible with $\mathcal{F}$ (such sets are called $\mathcal{F}$-a.c.e.; see Definition 3.3.1). Then it is not hard to show that almost-c.e. sets which are array noncomputable in this sense (such sets are called $\mathcal{F}$-a.c.e-a.n.c.; see Definition 3.3.2) exist precisely in the a.n.c. wtt-degrees (see Theorem 3.3.4 and Corollary 3.3.5).

It is a well known fact that no c.e. set is $\mathcal{F}$-a.n.c. with respect to all v.s.a. $\mathcal{F}$ (see Theorem 2.2 in [DJS90]). In contrast to this, Losert shows in [Los18] that there are almost-c.e. sets which are $\mathcal{F}$-similar to all $\mathcal{F}$-a.c.e. sets *for all* very strong arrays $\mathcal{F}$. If an almost-c.e. set A has this property then we say that A has the *universal similarity property* (u.s.p. for short). Now crucially, it is shown in [Los18] that u.s.p. sets exist precisely in the not totally ω-c.e. degrees. So u.s.p. sets are the generic sets for the not totally ω-c.e. degrees as a.n.c. sets are generic for the a.n.c. (wtt-) degrees.

The decisive advantage of introducing the notions of array noncomputability for almost-c.e. and that of u.s.p. sets is that they provide a much more modularized approach for investigating properties of almost-c.e. sets contained in a.n.c. (or not totally ω-c.e.) degrees. Namely, in all of the above mentioned results and the ones from Chapter 2 that show that any a.n.c. degree (c.e. not totally ω-c.e. degree) contains a c.e. or an almost-c.e. set with a certain property $\mathcal{P}$, the constructions are arranged in more or less the same way; a construction, possibly taken from elsewhere in computability theory showing that an almost-c.e. set A with property $\mathcal{P}$ exists is combined either with multiple or not totally ω-c.e. permitting (which ensures that $A \leq_T \mathbf{a}$ holds) and with coding (so $\mathbf{a} \leq_T A$ holds). So usually one has to take into account that both the permitting and the coding are made compatible in order to make the construction work.

Now using the above definition of array noncomputability (and that of the universal similarity property) for almost-c.e. sets enables us to avoid the permitting part in the following sense. If we attempt to show that any a.n.c. degree contains an almost-c.e. set A with a certain property $\mathcal{P}$, where $\mathcal{P}$ can be ensured by meeting a certain (not necessarily effective) list of requirements $\mathcal{R}_e$ ($e \in \omega$) then by the above result it suffices to prove that there exists a v.s.a. $\mathcal{F}$ such that, for every e, there exists an $\mathcal{F}$-a.c.e. set A_e such that every set which is $\mathcal{F}$-similar to A_e meets $\mathcal{R}_e$. So $\mathcal{P}$ holds for any $\mathcal{F}$-a.c.e.-a.n.c. set. Similarly, if we can show that, for every e, there exists a v.s.a. $\mathcal{F}_e$ and an $\mathcal{F}_e$-a.c.e. set A_e such that every set which is $\mathcal{F}_e$-similar to A_e meets $\mathcal{R}_e$ then every u.s.p. set has property $\mathcal{P}$; hence, any c.e. not totally ω-c.e. degree has an almost-c.e. set with property $\mathcal{P}$. An in-depth elaboration of the properties that are in this sense forced by u.s.p. sets is given in [ASFL$^+$].

Thus, by introducing the concept of array noncomputability for almost-c.e. sets, we benefit in three aspects. First of all, we obtain a new characterization of the c.e. not totally ω-c.e. degrees in terms of a uniform form of multiple permitting. Second, we get simpler proofs of some of the above mentioned results. For instance, we can strengthen the result by Brodhead, Downey and Ng [BDN12] by showing that any u.s.p. set is computably bounded random (CB-random). So any c.e. not totally ω-c.e. degree contains a left-c.e. real which is CB-random (Theorem 3.4.4 and Corollary 3.4.5).

Moreover, we also get new results by showing that a c.e. degree is not totally ω-c.e. if and only if it contains an almost-c.e. set which is not cl-reducible to any complex almost-c.e. set, thereby affirmatively answering a conjecture by Greenberg (Theorem 3.4.19). For the if direction, we prove a new and quite involved result on maximal pairs in the almost-c.e. sets which extends a result of Yun Fan [Fan09] (see Lemma 3.4.7). With the latter result we are also able to give a different proof of one the main results by Barmpalias, Downey and Greenberg [BDG10], namely that any c.e. a.n.c. degree contains a left-c.e. real which is not cl-reducible to any Martin-Löf random left-c.e. real (see Theorem 3.4.10).

All the results presented below will be published in the upcoming paper by Ambos-Spies, Losert and Monath [ASLM18]. For more on notions related to the universal similarity property and its relations to a.n.c. and not totally ω-c.e. degrees, we refer the reader to [Los18] and [ASL].

The outline of this chapter is as follows. In Section 3.2, we give the basic definitions needed for this chapter. Then in Section 3.3, we give the definitions of array noncomputability in the context of almost-c.e. sets and that of the universal similarity property and state (without proof) the main result in this respect that the Turing degrees of u.s.p. sets capture the class of the c.e. not totally ω-c.e. degrees.

In Section 3.4, we demonstrate the above described modularized approach on how to find sets in not totally ω-c.e. degrees with certain properties using u.s.p. sets with two examples. First, in Subsection 3.4.1, we roughly sketch the idea of how to show that u.s.p. sets are computably bounded (CB-) random. Second, in Subsection 3.4.2, we state and prove in detail that u.s.p. sets are halves of maximal pairs in the almost-c.e. sets, where the second halves may be chosen to be c.e. and a subset of a given infinite, computable set D.

In Subsection 3.4.3, we show how this result on maximal pairs forms the basis to prove that u.s.p. sets are not cl-reducible to any complex almost-c.e. set; hence, any c.e. not totally ω-c.e. degree contains a set which is not cl-reducible to any complex almost-c.e. set. Moreover, we also cite the result (without proof) that the c.e. not totally ω-c.e. degrees are in fact characterized by the property to contain an almost-c.e. set which is not cl-reducible to any complex almost-c.e. set. Finally, in Section 3.5, we give a short summary on the results of this chapter.

3.2 Preliminaries

We start with the definition of almost-c.e. sets and degrees.

Definition 3.2.1. *A* computable almost-enumeration *of a set A is a strong array $\{A_s\}_{s\in\omega}$ such that $\lim_{s\to\infty} A_s(x) = A(x)$ $(x \in \omega)$ and such that $A_0 = \emptyset$ and*

$$\forall\ s\ \forall\ x\ (x \in A_s \setminus A_{s+1}\ \Rightarrow\ \exists\ y < x\ (y \in A_{s+1} \setminus A_s)), \tag{3.1}$$

or, equivalently,

$$\forall\ s \forall\ x\ (A_s \restriction x \leq_{lex} A_{s+1} \restriction x) \tag{3.2}$$

holds. A set A is almost computably enumerable *(*almost-c.e. *or* a.c.e. *for short) if there is a computable almost-enumeration of A.*

Then it is not hard to show that almost-c.e. sets are just the analogue of left-c.e. reals on the set level. The proof of the following result may be found in [DH10, Theorem 5.1.7].

Proposition 3.2.2 ([CHKW98])**.** *A real $\alpha \in [0,1]$ is left-c.e. iff $\alpha = 0.A$ for an almost-c.e. set A.*

In order to define array noncomputability for almost-c.e. sets, we need a few auxiliary notions. Recall from Definition 2.2.1 that an infinite sequence of finite sets $\mathcal{F} = \{F_n\}$ is a very strong array (v.s.a.) if the sets F_n are uniformly given by their canonical index such that they are mutually disjoint, nonempty and growing in size. For our purposes, it is sometimes convenient to restrict ourselves to very strong arrays where the components of the array are intervals which are nicely ordered and which cover all of ω.

Definition 3.2.3. *An infinite sequence of finite sets $\mathcal{F} = \{F_n\}_{n\in\omega}$ is called a* very strong array of intervals *(*v.s.a.i. *for short) if $\mathcal{F}$ is a very strong array such that all sets F_n are intervals and such that $\max(F_n) < \min(F_{n+1})$ holds for all $n \in \omega$. $\mathcal{F}$ is called a* complete very strong array of intervals *(*c.v.s.a.i.*) if $\mathcal{F}$ is a very strong array of intervals such that $\mathcal{F}$ is complete, i.e., $\bigcup_{n\in\omega} F_n = \omega$ holds.*

Moreover, when defining array noncomputability for almost-c.e. sets, the following notion will be useful.

Definition 3.2.4. *Let $\mathcal{F} = \{F_n\}_{n\in\omega}$ be a v.s.a. and let A and B be any sets. A and B are $\mathcal{F}$-*similar *($A \sim_{\mathcal{F}} B$ for short) if* (2.2) *holds for A and B, i.e.,*

$$\exists^\infty n \ (A \cap F_n = B \cap F_n).$$

So in particular, a c.e. set A is a.n.c. iff there exists a v.s.a. $\mathcal{F}$ such that A is $\mathcal{F}$-similar to all c.e. sets.

3.3 Array Noncomputability and Almost-C.E. Sets

By a result of Ambos-Spies et al. [ASFL$^+$], no almost-c.e. set is $\mathcal{F}$-similar to all almost-c.e. sets in the straightforward sense. So the idea is to require that the computable almost-enumeration of a set is compatible with $\mathcal{F}$ in the following sense.

Definition 3.3.1 ([ASLM18])**.** *Let $\mathcal{F} = \{F_n\}_{n\in\omega}$ be a very strong array. A computable almost-enumeration $\{A_s\}_{s\in\omega}$ is $\mathcal{F}$-*compatible *if, for any $n, s \in \omega$,*

$$A_s \cap F_n \leq_{lex} A_{s+1} \cap F_n$$

(i.e., to be more precise, $A_s(x_0)\dots A_s(x_m) \leq_{lex} A_{s+1}(x_0)\dots A_{s+1}(x_m)$ where $x_0, \dots, x_m$ are the elements of F_n in increasing order) and, for any $x \notin \bigcup_{n\in\omega} F_n$

and any $s \in \omega$, $A_s(x) \leq A_{s+1}(x)$. A set A is $\mathcal{F}$-compatibly almost-c.e. *($\mathcal{F}$-*almost-c.e. *or* $\mathcal{F}$-a.c.e. *for short) if there is an $\mathcal{F}$-compatible computable almost-enumeration $\{A_s\}_{s\in\omega}$ of A. And A is* purely $\mathcal{F}$-almost-c.e. *if A is $\mathcal{F}$-almost-c.e. and $A \subseteq \bigcup_{n\in\omega} F_n$.*

Then we are ready to define what it means for an almost-c.e. set (degree) to be array noncomputable.

Definition 3.3.2 ([ASLM18]). *(a) Let $\mathcal{F} = \{F_n\}_{n\in\omega}$ be a very strong array. A set A is* $\mathcal{F}$-array noncomputable for the $\mathcal{F}$-almost-c.e. sets *(*$\mathcal{F}$-a.c.e.-a.n.c. *for short) if A is almost-c.e. and, for all $\mathcal{F}$-almost-c.e. sets B, $A \sim_{\mathcal{F}} B$, i.e.,* (2.2) *holds.*

(b) A set A is array noncomputable for the almost-c.e. sets *(*a.c.e.-a.n.c. *for short) if A is $\mathcal{F}$-a.c.e.-a.n.c. for some v.s.a. $\mathcal{F}$.*

(c) A degree **a** *is* array noncomputable for the almost-c.e. sets *(*a.c.e.-a.n.c. *for short) if there is an a.c.e.-a.n.c. set A in* **a**.

Note that, just as in the case of c.e. sets, it does not matter for Definition 3.3.2 whether we require that (2.1) or (2.2) holds.

Proposition 3.3.3 ([ASLM18]). *Let $\mathcal{F} = \{F_n\}_{n\in\omega}$ be a very strong array. An almost-c.e. set A is $\mathcal{F}$-a.c.e.-a.n.c. if and only if, for all $\mathcal{F}$-almost-c.e. sets B,* (2.1) *holds.*

Now, by the following theorem, we may deduce that a.c.e.-a.n.c. degrees and a.n.c. degrees coincide, where note that this also holds for wtt-degrees. Hence, the class of a.c.e.-a.n.c. (wtt-) degrees is closed upwards.

Theorem 3.3.4 ([ASLM18]). *Let $\mathcal{F} = \{F_n\}_{n\in\omega}$ be a v.s.a. Then the following hold.*

1. *For any a.c.e.-a.n.c. set A there is an $\mathcal{F}$-a.n.c. set B such that $A \equiv_{wtt} B$. In fact, for any a.c.e.-a.n.c. set A and any a.c.e. set $\hat{B}$ such that $A \leq_{\mathrm{wtt}} \hat{B}$, there is an $\mathcal{F}$-a.n.c. set B such that $\hat{B} \equiv_{wtt} B$.*

2. *For any a.n.c. set A there is an $\mathcal{F}$-a.c.e.-a.n.c. set B such that $A \equiv_{wtt} B$. Moreover, B can be chosen to be a purely $\mathcal{F}$-a.c.e. set. In fact, for any a.n.c. set A and any a.c.e. set $\hat{B}$ such that $A \leq_{\mathrm{wtt}} \hat{B}$, there is a purely $\mathcal{F}$-a.c.e. set B such that B is $\mathcal{F}$-a.c.e.-a.n.c. and $\hat{B} \equiv_{wtt} B$.*

The following corollaries are immediate.

Corollary 3.3.5 ([ASLM18]). *Let* $\mathbf{a}$ *be a wtt-degree (Turing degree). Then* $\mathbf{a}$ *is a.n.c. iff* $\mathbf{a}$ *is a.c.e.-a.n.c.*

Corollary 3.3.6 ([ASLM18]). *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be c.e. wtt-degrees (Turing degrees) such that* $\mathbf{a}$ *is a.c.e.-a.n.c. and* $\mathbf{a} \leq \mathbf{b}$. *Then* $\mathbf{b}$ *is a.c.e.-a.n.c.*

Corollary 3.3.7 ([ASLM18]). *Let $\mathcal{F}$ be a v.s.a. and let A be a.c.e.-a.n.c. There is a purely $\mathcal{F}$-a.c.e. set B such that B is $\mathcal{F}$-a.c.e.-a.n.c. and wtt-equivalent to A.*

3.3.1 The Universal Similarity Property

We proceed to give the definition of the universal similarity property for an almost-c.e. set A.

Definition 3.3.8 ([ASLM18]). *An almost-c.e. set A has the* universal similarity property *if A is $\mathcal{F}$-a.c.e.-a.n.c. for all very strong arrays $\mathcal{F}$, i.e., if, for any v.s.a. $\mathcal{F}$ and any $\mathcal{F}$-a.c.e. set B, A is $\mathcal{F}$-similar to B.*

The justification of why it often suffices to consider (complete) very strong arrays of intervals instead of all very strong arrays is given by the following proposition for which we say that a v.s.a. $\hat{\mathcal{F}} = \{\hat{F}_n\}_{n\in\omega}$ *dominates* a v.s.a. $\mathcal{F} = \{F_n\}_{n\in\omega}$ if, for any number n, there is a number m such that $F_m \subseteq \hat{F}_n$.

Proposition 3.3.9 ([ASLM18]). *(a) For any v.s.a. $\mathcal{F} = \{F_n\}_{n\in\omega}$ there is a complete very strong array of intervals $\hat{\mathcal{F}} = \{\hat{F}_n\}_{n\in\omega}$ which dominates $\mathcal{F}$.*

(b) Let $\hat{\mathcal{F}} = \{\hat{F}_n\}_{n\in\omega}$ and $\mathcal{F} = \{F_n\}_{n\in\omega}$ be very strong arrays such that $\hat{\mathcal{F}}$ dominates $\mathcal{F}$. Then any $\hat{\mathcal{F}}$-a.c.e.-a.n.c. set ($\hat{\mathcal{F}}$-a.n.c. set) is $\mathcal{F}$-a.c.e.-a.n.c. ($\mathcal{F}$-a.n.c.).

Note that, as shown in Theorem 2.2 of [DJS90], the analogue of Definition 3.3.8 in the setting of c.e. sets does not hold. To wit, w.l.o.g., we may assume that A is noncomputable (hence infinite) since computable sets are never $\mathcal{F}$-similar to all c.e. sets for any given v.s.a. $\mathcal{F}$. So we may choose an infinite computable subset B of A. Then, based on B, we may define a v.s.a.i. $\mathcal{F} = \{F_n\}$ such that $B \cap F_n \neq \emptyset$ holds for all n; hence, A is not $\mathcal{F}$-similar to the empty set. Now for the construction of sets with the universal similarity property, it is useful to note that, by Proposition 3.3.9, it suffices to consider complete very strong arrays of intervals.

Proposition 3.3.10 ([ASLM18])**.** *Let A be an almost-c.e. set such that, for any complete v.s.a.i. $\mathcal{F}$, A is $\mathcal{F}$-a.c.e.-a.n.c. Then A has the universal similarity property.*

Then in order to formulate the main existence result for u.s.p. sets, recall the definition of totally ω-c.e. degrees.

Definition 3.3.11. *A function g is* ω-c.e. *if there is a computable approximation $\{g_s\}_{s\in\omega}$ of g and a computable function h such that, for any $x \in \omega$,*

$$|\{s : g_{s+1}(x) \neq g_s(x)\}| < h(x).$$

A set A is totally ω-c.e. *if any function $g \leq_{\mathrm{T}} A$ is ω-c.e. A Turing degree is* totally ω-c.e. *if it contains a totally ω-c.e. set.*

Then the main existence result is as follows.

Theorem 3.3.12 ([ASLM18])**.** *For a c.e. Turing degree* **a** *the following are equivalent.*

(i) There is an almost-c.e. set A with the universal similarity property in **a**.

(ii) **a** *is not totally ω-c.e.*

3.4 Properties of Sets with the Universal Similarity Property

We continue to present two applications of Theorem 3.3.12 which show that c.e. not totally ω-c.e. degrees contain almost-c.e. sets with certain properties.

3.4.1 U.S.P. Sets and CB-Randomness

For the first application for which we state the results without proof, we need to recall the definitions of Martin-Löf randomness and that of computably bounded randomness.

Definition 3.4.1 ([ML66]). *1. A* Martin-Löf test *(or* ML-test *for short) is a uniform c.e. sequence* $\{U_n\}_{n\in\omega}$ *of c.e. sets* $U_n \subseteq \{0,1\}^*$ *such that, for* $n \in \omega$, $\mu([U_n]) < 2^{-n}$.

2. *A real* $\alpha \in \{0,1\}^\omega$ passes *an ML-test* $\{U_n\}_{n\in\omega}$ *if* $\alpha \notin \bigcap_{n\in\omega}[U_n]$*; and* $\{U_n\}_{n\in\omega}$ covers α *otherwise.*

3. *A set A* passes *the ML-test* $\{U_n\}_{n\in\omega}$ *(is* covered *by* $\{U_n\}_{n\in\omega}$*) if the characteristic sequence* α *of A passes the ML-test* $\{U_n\}_{n\in\omega}$ *(is covered by* $\{U_n\}_{n\in\omega}$*).*

4. *A real* α *(set A) is* Martin-Löf random *(or* ML-random *for short) if* α *(A) passes all ML-tests.*

Definition 3.4.2 ([BDN12]). *1. For any function* $f : \omega \to \omega$*, an ML-test* $\{U_n\}_{n\in\omega}$ *is* f-bounded *(or an* f-test *for short) if, for* $n \in \omega$, $|U_n| \leq f(n)$.

2. *A Martin-Löf test* $\{U_n\}_{n\in\omega}$ *is* computably-bounded *(or a* CB-test *for short) if* $\{U_n\}_{n\in\omega}$ *is f-bounded for some computable function f.*

3. *A real* α *(set A) is* f-Martin-Löf random *(or* f-ML-random *for short) if* α *(A) passes all f-tests.*

4. *A real* α *(set A) is* computably-bounded random *(or* CB-random *for short) if* α *(A) passes all CB-tests (i.e., if* α *(A) is f-ML-random for all computable functions f).*

Then in order to show that any u.s.p. set A is CB-random, we need the following lemma.

Lemma 3.4.3 ([ASLM18]). *Let f be a computable function. There is a c.v.s.a.i. $\mathcal{F} = \{F_n\}_{n\in\omega}$ such that any $\mathcal{F}$-a.c.e.-a.n.c. set A is f-ML-random.*

Then it is easy to prove the following theorem.

Theorem 3.4.4 ([ASLM18]). *Any almost-c.e. set with the universal similarity property is CB-random.*

Proof. Fix an almost-c.e. set A with the universal similarity property. Then, for any $e \in \omega$, consider the requirement

$$\mathcal{R}_e : A \text{ is } f_e\text{-ML-random}$$

where f_e denotes the $(e+1)$st total computable function in a list of all total computable functions $\{f_e\}_{e\in\omega}$. Clearly, by Definition 3.4.2, meeting all requirements $\mathcal{R}_e$ ensures that A is CB-random. Then, given e, we may fix a c.v.s.a.i. $\mathcal{F}_e$ as given by Lemma 3.4.3. Now since A is $\mathcal{F}_e$-a.c.e.-a.n.c., the claim immediately follows. □

Then from Theorems 3.3.12 and 3.4.4, we may deduce the following corollary.

Corollary 3.4.5 ([ASLM18]). *Let $\mathbf{a}$ be a c.e. Turing degree which is not totally ω-c.e. There is an almost-c.e. set $A \in \mathbf{a}$ such that A is CB-random.*

3.4.2 U.S.P. Sets and Maximal Pairs

Having introduced maximal pairs for the class of the c.e. sets in Section 2.5, we continue to consider maximal pairs in the almost-c.e. sets. In contrast to halves of maximal pairs in the c.e. sets which, by [ASDFM13], exist precisely in the a.n.c. (wtt-) degrees, Fan and Yu [FY11] show that any noncomputable almost-c.e. set is half of an ibT-maximal pair in the almost-c.e. sets. The Fan-Yu result implies that there is an ibT-maximal pair in the almost-c.e. sets where one of the halves is c.e. This fact is previously shown by Fan [Fan09] already, by using a more direct argument. By the following observation of Downey and Hirschfeldt, however, no pair of c.e. sets is ibT-maximal in the almost-c.e. sets.

Lemma 3.4.6 (Downey and Hirschfeldt [DH10], Theorem 9.14.6). *If A is c.e. and B is a Martin-Löf random almost-c.e. set then $A \leq_{\mathrm{ibT}} B$.*

Here we strengthen Fan's theorem that there is a maximal pair (A, B) in the almost-c.e. sets where B is c.e. in two directions. We show that the c.e. set B can be chosen to be arbitrarily sparse, i.e., to be a subset of any given infinite computable set D and that it suffices to let A be any $\mathcal{F}$-a.c.e.-a.n.c. set (where the choice of $\mathcal{F}$ depends on D). As we point out this gives an alternative proof of Barmpalias, Downey and Greenberg's result that in any a.n.c. degree there is an almost-c.e. set which is not cl-reducible to any random almost-c.e. set. Moreover, in the next section we use our maximal pair result to characterize the Turing degrees which contain almost-c.e. sets which cannot be reduced to any complex almost-c.e. sets.

Lemma 3.4.7 (First Maximal Pair Lemma). *Let D be an infinite computable set. There are a c.v.s.a.i. $\mathcal{F} = \{F_n\}_{n\in\omega}$ and a c.e. set $B \subseteq D$ such that, for any $\mathcal{F}$-a.c.e.-a.n.c. set A, (A, B) is an* ibT*-maximal pair in the almost-c.e. sets.*

This lemma immediately implies

Lemma 3.4.8 (Second Maximal Pair Lemma). *Let A be an almost-c.e. set with the universal similarity property and let D be any infinite computable set. There is a c.e. set $B \subseteq D$ such that (A, B) is an* ibT*-maximal pair in the almost-c.e. sets.*

The proof of the First Maximal Pair Lemma is based on the following technical lemma.

Lemma 3.4.9. *Let D be an infinite computable set. There is a computable function l such that the following hold. For any ibT-functionals $\hat{\Phi}$ and $\hat{\Psi}$, any almost-c.e. set V and any number $a \in \omega$, there are uniformly (in $\hat{\Phi}$, $\hat{\Psi}$, V and a) almost-c.e. sets*

$$A_a^{\hat{\Phi},\hat{\Psi},V} \subseteq [a, a + l(a)]$$

and uniformly (in $\hat{\Phi}$, $\hat{\Psi}$, V and a) c.e. sets

$$B_a^{\hat{\Phi},\hat{\Psi},V} \subseteq [a, a + l(a)] \cap D$$

such that

$$\exists\, x \in [a, a + l(a)]\ (A_a^{\hat{\Phi},\hat{\Psi},V}(x) \neq \hat{\Phi}^V(x) \text{ or } B_a^{\hat{\Phi},\hat{\Psi},V}(x) \neq \hat{\Psi}^V(x)) \tag{3.3}$$

holds.

We next show how the First Maximal Pair Lemma follows from this lemma and defer the quite involved proof of Lemma 3.4.9 to the next subsection.

Proof of Lemma 3.4.7 assuming Lemma 3.4.9. In the following, fix computable numberings $\{\hat{\Phi}_e\}_{e\in\omega}$ and $\{V_e\}_{e\in\omega}$ of the ibT-functionals and the almost-c.e. sets, respectively.

The required c.v.s.a.i. $\mathcal{F} = \{F_n\}_{n\in\omega}$ and c.e. set B are defined as follows. Fix l as in Lemma 3.4.9 – where w.l.o.g. we may assume that l is strictly increasing – and let $\mathcal{I} = \{I_n\}_{n\in\omega}$ be the unique c.v.s.a.i. such that $I_n = [\min I_n, \min I_n + l(\min I_n)]$. Let F_n^k $(k \leq n)$ be defined by

$$F_0^0 = I_0;\ F_1^0 = I_1, F_1^1 = I_2;\ F_2^0 = I_3, F_2^1 = I_4, F_2^2 = I_5;\ etc.,$$

and, for $k \leq n$, let $x_n^k = \min F_n^k$ (hence $F_n^k = [x_n^k, x_n^k + l(x_n^k)]$). Then $\mathcal{F}$ is defined by letting $F_n = F_n^0 \cup \cdots \cup F_n^n$ $(n \in \omega)$, and the desired set B is obtained by letting

$$B \cap F_n^k = B_{x_n^k}^{\hat{\Phi}_{k_0},\hat{\Phi}_{k_1},V_{k_2}}$$

for $k = \langle k_0, k_1, k_2\rangle \leq n$. Since the sets $B_a^{\hat{\Phi},\hat{\Psi},V}$ are uniformly c.e. and contained in D, it follows that B is c.e. and $B \subseteq D$. So, in order to complete the proof, given any $\mathcal{F}$-a.c.e.-a.n.c. set A, it suffices to show that the pair (A, B) is ibT-maximal in the a.c.e. sets.

For a contradiction assume not. Then there is an almost-c.e. set C such that $A \leq_{\mathrm{ibT}} C$ and $B \leq_{\mathrm{ibT}} C$. Fix $k = \langle k_0, k_1, k_2\rangle$ such that $C = V_{k_2}$, $A = \hat{\Phi}_{k_0}^C$ and $B = \hat{\Phi}_{k_1}^C$. Define the set $\hat{A} \subseteq \bigcup_{n\geq k} F_n^k$ by letting

$$\hat{A} \cap F_n^k = A_{x_n^k}^{\hat{\Phi}_{k_0},\hat{\Phi}_{k_1},V_{k_2}}.$$

Then $\hat{A}$ is $\mathcal{I}$-a.c.e., hence $\mathcal{F}$-a.c.e. Moreover, by (3.3) and by definition of $\hat{A}$ and

B, for any $n \geq k$,

$$\exists\, x \in F_n^k \subseteq F_n\ (\hat{A}(x) \neq \hat{\Phi}_{k_0}^{V_{k_2}}(x) \text{ or } B(x) \neq \hat{\Phi}_{k_1}^{V_{k_2}}(x))$$

holds. Since A is $\mathcal{F}$-a.c.e.-a.n.c., it follows that there is a number x such that $A(x) \neq \hat{\Phi}_{k_0}^{V_{k_2}}(x)$ or $B(x) \neq \hat{\Phi}_{k_1}^{V_{k_2}}(x)$. But, by $C = V_{k_2}$, this implies that $A \neq \hat{\Phi}_{k_0}^{C}$ or $B \neq \hat{\Phi}_{k_1}^{C}$ contrary to choice of k_0 and k_1. □

We close this subsection by pointing out that the First Maximal Pair Lemma provides an alternative proof of Barmpalias, Downey and Greenberg's result in [BDG10] that in any a.n.c. degree there is an almost-c.e. set which is not cl-reducible to any random almost-c.e. set. By Corollaries 3.3.5 and 3.3.7, it suffices to show the following.

Theorem 3.4.10. *There is a v.s.a. $\mathcal{F}$ such that no $\mathcal{F}$-a.c.e.-a.n.c. set is cl-reducible to any ML-random almost-c.e. set.*

Proof. Let $D = \omega$. Fix a v.s.a. $\mathcal{F}$ and a c.e. set B as in Lemma 3.4.7. We claim that $\mathcal{F}$ has the required properties. For a contradiction assume that A is $\mathcal{F}$-a.c.e-a.n.c., C is ML-random and almost-c.e., and $A \leq_{\mathrm{cl}} C$. Then, for any number m, the m-bounded left-shift $C_{-m} = \{x : x + m \in C\}$ of C is ML-random and almost-c.e., too, and there exists m_0 such that $A \leq_{\mathrm{ibT}} C_{-m_0}$ holds. By the former and by Lemma 3.4.6, $B \leq_{\mathrm{ibT}} C_{-m_0}$. So the pair (A, B) is not ibT-maximal in the almost-c.e. sets. But this contradicts the choice of $\mathcal{F}$ and B. □

Proof of Lemma 3.4.9

Before we turn to the proof of Lemma 3.4.9 we start with some general remarks on the strategies for constructing maximal pairs (A, B). All of these strategies are based on the same idea but their complexity greatly differs. In any case we have to meet the requirements

$$R:\ A \neq \hat{\Phi}^V \text{ or } B \neq \hat{\Psi}^V$$

where $\hat{\Phi}$ and $\hat{\Psi}$ are ibT-functionals and V is either a c.e. set or an almost-c.e. set (depending on whether we consider maximal pairs in the c.e. sets or the almost-c.e. sets). Then, in order to meet these requirements, an interval $[a, a + l(a)]$

(where l is computable) is assigned to R and the definition of $A \cap [a, a+l(a)]$ and $B \cap [a, a+l(a)]$ is devoted to meet R. The R-strategy only becomes active at a stage $s+1$ such that

$$A_s \cap [a, a+l(a)] = \hat{\Phi}_s^{V_s} \cap [a, a+l(a)] \text{ and } B_s \cap [a, a+l(a)] = \hat{\Psi}_s^{V_s} \cap [a, a+l(a)]. \tag{3.4}$$

So changing either A on $[a, a+l(a)]$ or B on $[a, a+l(a)]$ at stage $s+1$ ensures that R is met unless $V \restriction a+l(a)+1$ changes later (recall that $\hat{\Phi}$ and $\hat{\Psi}$ are ibT-reductions hence a convergent computation $\hat{\Phi}_s^{V_s}(x)$ is not V-correct only if $V \restriction x+1 \neq V_s \restriction x+1$, and similarly for $\hat{\Psi}$). Then one argues that for sufficiently large $l(a)$ and for an appropiate order of the changes of A and B on $[a, a+l(a)]$, eventually V cannot repell one of the attacks whence R is met.

The most simple instance of this strategy gives a maximal pair in the c.e. sets (see the proof of Theorem 17 in [ASDFM13]). There it suffices to let $l(a) \geq a$, and the order in which the numbers in $[a, a+l(a)]$ are enumerated into A and B does not matter, as explained in Remark 2.5.5.

In the case of the construction of a maximal pair in the almost-c.e. sets, the strategy is somewhat more involved. Here it suffices to let $l(a) = 2^a$ and to use A and B alternatingly for the attacks where in each case the minimal change of the chosen set on the interval $[a, a+l(a)]$ is done (see the proof of Theorem 9.14.2 in Downey and Hirschfeldt [DH10]). In case of Fan's theorem where the pair (A, B) has to be maximal in the almost-c.e. degrees and B has to be c.e., the strategy becomes considerably more involved – using inductive calls of appropriate subprocedures – and the required bound $l(a)$ becomes much harder to describe. For the proof of Lemma 3.4.9, we will adapt Fan's strategy. (Note that the construction of sets $A_a^{\hat{\Phi},\hat{\Psi},V}$ and $B_a^{\hat{\Phi},\hat{\Psi},V}$ as in the lemma satisfying condition (3.3) corresponds to the definition of the parts $A \cap [a, a+l(a)]$ and $B \cap [a, a+l(a)]$ meeting requirement R above.) Due to the additional constraint that the c.e. set B has to be contained in the given infinite computable set D, the definition of the function l becomes dependent on D and the strategy has to be adjusted. In our proof – which we give now – we follow the presentation of the proof of Fan's theorem given in Brackmann [Bra13].

In order to show that the definition of the function l only depends on the given set D and not on the additional parameters of the requirements, i.e., the ibT-functionals $\hat{\Phi}$ and $\hat{\Psi}$ and the oracle set V, we first define l. The definition of l is based on some auxiliary functions to be used in the formal strategy for meeting (3.3). The intuition behind these functions will be explained when we discuss this strategy and prove its correctness.

Definition of the function l

We now define the function l using some auxiliary functions. First let d be the computable function defined by

$$d(b,k) = \mu z(z \geq k \ \& \ b+z \in D). \tag{3.5}$$

Next we define $l_A(b,k,n)$ and $l_B(b,k,n)$ by a simultaneous nested induction on $n \geq 1$ for all $b,k \in \omega$ where, for $n \geq 2$, an auxiliary function $h(b,k,n,i)$ is simultaneously defined for all $b,k,i \in \omega$, where the definition of $h(b,k,n,i)$ is by a side induction on $i \in \omega$.

Definition 3.4.11. *Let $b,k,i \in \omega$ be given. Let*

$$l_A(b,k,1) = 0, \ l_B(b,k,1) = 0, \ \textit{and } l_B(b,k,2) = d(b,k).$$

For $n \geq 2$ let

$$h(b,k,n,0) = d(b,k),$$

$$h(b,k,n,i+1) = l_B(b+i+1, h(b,k,n,i), n-1), \ \textit{and}$$

$$l_A(b,k,n) = \max(\{l_B(b,k,n)\} \cup \{i + \tilde{l}_{b,k,n,i} : \ 1 \leq i \leq d(b,k) + |n-1|\})$$

where

$$\tilde{l}_{b,k,n,i} = l_A(b+i, h(b,k,n,i-1), n-1).$$

For $n \geq 3$ let

$$l_B(b,k,n) = d(b,k) + |n-1| + h(b,k,n,d(b,k) + |n-1|)$$

where here and in the following, $|x|$ denotes the length of the binary representation of $x \in \omega$.

Then the function l is defined by

$$l(a) = l_A(a, 0, 2^{a+1}) \tag{3.6}$$

for all $a \in \omega$.

The intuition behind the functions in Definition 3.4.11 is described in Remarks 3.4.13 below.

By computability and infinity of D, d is computable, and so are the other auxiliary functions. So l is computable. Moreover, for later use note the following properties of the auxiliary functions which are immediate by definition (for $b, k, i \in \omega$).

$$d(b, k) \geq k \tag{3.7}$$

$$\forall\, n \geq 1\ (l_A(b, k, n) \geq l_B(b, k, n)) \tag{3.8}$$

$$\forall\, n \geq 2\ (l_B(b, k, n) \geq d(b, k)) \tag{3.9}$$

$$\forall\, n \geq 3\ (h(b, k, n, i + 1) \geq d(b + i + 1, h(b, k, n, i))) \tag{3.10}$$

$$\forall\, n \geq 3\ (h(b, k, n, i + 1) \geq h(b, k, n, i)) \tag{3.11}$$

Definition of the sets $A_a^{\hat{\Phi},\hat{\Psi},V}$ and $B_a^{\hat{\Phi},\hat{\Psi},V}$

Now fix a, ibT-functionals $\hat{\Phi}$ and $\hat{\Psi}$, an almost-c.e. set V, computable enumerations $\{\hat{\Phi}_s\}_{s\in\omega}$ and $\{\hat{\Psi}_s\}_{s\in\omega}$ of $\hat{\Phi}$ and $\hat{\Psi}$, respectively, and a computable almost-enumeration $\{V_s\}_{s\in\omega}$ of V. Uniformly in a, $\{\hat{\Phi}_s\}_{s\in\omega}$, $\{\hat{\Psi}_s\}_{s\in\omega}$ and $\{V_s\}_{s\in\omega}$, we have to give a computable almost-enumeration $\{A_{a,s}^{\hat{\Phi},\hat{\Psi},V}\}_{s\in\omega}$ and a computable enumeration $\{B_{a,s}^{\hat{\Phi},\hat{\Psi},V}\}_{s\in\omega}$ of an almost-c.e. set $A_a^{\hat{\Phi},\hat{\Psi},V}$ and a c.e. set $B_a^{\hat{\Phi},\hat{\Psi},V}$, respectively, such that $A_a^{\hat{\Phi},\hat{\Psi},V} \subseteq [a, a + l(a)]$, $B_a^{\hat{\Phi},\hat{\Psi},V} \subseteq [a, a + l(a)] \cap D$, and (3.3) hold.

In order to achieve this, by induction on $n \geq 1$, we define effective procedures $P_{b,k,n}$ ($b \geq a, k \in \omega$) which, started at a stage s_0 with given finite sets A_{s_0} and B_{s_0}, define a computable almost-enumeration $\{A_s\}_{s\geq s_0}$ of a set A and a computable enumeration $\{B_s\}_{s\geq s_0}$ of a set B. (The arbitrariness of the starting stage s_0 and the initial values A_{s_0} and B_{s_0} is used in the inductive step: when running procedure $P_{b,k,n}$ we may invoke procedures of the form $P_{b',k',n-1}$ which, called at stage $s_1 \geq s_0$, start with the initial parameters s_1, A_{s_1}, B_{s_1}.) We will argue that, for the procedure $P_{a,0,2^{a+1}}$ started at stage 0 with A_0 and B_0 being empty and for the sequences $\{A_s\}_{s\in\omega}$ and $\{B_s\}_{s\in\omega}$ defined in this way, $A^{\hat{\Phi},\hat{\Psi},V}_{a,s} = A_s$ and $B^{\hat{\Phi},\hat{\Psi},V}_{a,s} = B_s$ have the required properties.

Before we can define the procedures $P_{b,k,n}$ we need some more notation. Given a stage s and the current approximations A_s and B_s, we call s *critical (w.r.t. A_s and B_s)* if (3.4), i.e.,

$$A_s \cap [a, a+l(a)] = \hat{\Phi}^{V_s}_s \cap [a, a+l(a)] \ \text{ and } \ B_s \cap [a, a+l(a)] = \hat{\Psi}^{V_s}_s \cap [a, a+l(a)].$$

holds but with $l(a)$ replaced by $l_A(b,k,n)$ and a replaced by b. Procedure $P_{b,k,n}$ acts only at stages $s+1$ where s is critical. (If no $P_{b,k,n}$ acts at stage $s+1$ then $A_{s+1} = A_s$ and $B_{s+1} = B_s$.)

For an interval $[x,y]$ and a set X we identify $X \cap [x,y]$ with the binary string $X(x)X(x+1)\dots X(y)$ of length $y-x+1$. Finally, for a binary string σ of length n we let σ^+ be the lexicographical successor σ (of the same length), if it exists (i.e. if $\sigma \neq 1^n$), and we let $\sigma^+ = \sigma$ otherwise. More precisely,

$$\sigma^+ = \begin{cases} \sigma & \text{if } \sigma = 1^{|\sigma|} \\ \tau 1 & \text{if } \sigma \neq 1^{|\sigma|} \text{ and } \exists \tau \ (\sigma = \tau 0) \\ \tau^+ 0 & \text{if } \sigma \neq 1^{|\sigma|} \text{ and } \exists \tau \ (\sigma = \tau 1). \end{cases}$$

Definition 3.4.12. *Let $b \geq a$, $k \in \omega$, $n \geq 1$, $s_0 \in \omega$ and let A_{s_0}, B_{s_0} be any finite sets. The procedure $P_{b,k,n}$ started at stage s_0 with the initial sets A_{s_0} and B_{s_0} is defined by induction on n as follows.*

For $n = 1$, $P_{b,k,1}$ consists of the following two phases.

(*i*) *For the first critical stage $\tilde{s} \geq s_0$, set $A_{\tilde{s}+1} \upharpoonright b+1 = (A_{\tilde{s}} \upharpoonright b+1)^+$ and for all $x \geq b+1$, set $A_{\tilde{s}+1}(x) = A_{\tilde{s}}(x)$.*

(*ii*) *For the first critical stage $s_1 > \tilde{s}$, finish the procedure at stage s_1.*

For $n > 1$, by inductive hypothesis suppose that $P_{b',k',n-1}$ is already defined for all b' and k'. Let $d = d(b,k) + |n-1|$. Then $P_{b,k,n}$ consists of the following four phases.

(i)

1. *Run* $P_{b+1,h(b,k,n,0),n-1}$.
2. *Run* $P_{b+2,h(b,k,n,1),n-1}$.

$\vdots \qquad \vdots$

i. *Run* $P_{b+i,h(b,k,n,i-1),n-1}$.

$\vdots \qquad \vdots$

d. *Run* $P_{b+d,h(b,k,n,d-1),n-1}$.

We refer to the $P_{b+i,h(b,k,n,i-1),n-1}$-procedures of phase (i) as subprocedures *of $P_{b,k,n}$. We understand that the first subprocedure starts at stage s_0 and the $(i+1)$st subprocedure starts at the stage at which the ith one has finished, if this stage exists and does not start otherwise.*

(ii) *If all subprocedures of phase (i) finish, let $\tilde{s}$ be the first criticial stage after the stage that the last subprocedure finished. At stage $\tilde{s}+1$, set $B_{\tilde{s}+1} = B_{\tilde{s}} \cup \{b + d(b,k))\}$.*

(iii) *For the first critical stage $t > \tilde{s}$, set $A_{t+1} \upharpoonright b+1+d = (A_t \upharpoonright b+1+d)^+$ and for all $x \geq b+1+d$, set $A_{t+1}(x) = A_t(x)$.*

(iv) *For the first critical stage $s_1 > t$, finish the procedure at the stage s_1.*

Before we turn to the verification, in the following remark we give some intuition about the meaning of the variables b, k, n, i and how the auxiliary functions from Definition 3.4.11 are connected with the procedures $P_{b,k,n}$.

Remark 3.4.13. *Let $n \geq 1, b \geq a, k, i \in \omega$, let s_0 be a starting stage and, for simplicity, assume that A_{s_0} and B_{s_0} are empty (in Claim 2 below, we need a weaker assumption on A_{s_0} and B_{s_0}). Then $l_A(b,k,n), l_B(b,k,n)$ and $h(b,k,n,i)$ are defined such that the following hold.*

1. *b is the least and $b + l_A(b,k,n)$ is the largest number on which A may change during the run of $P_{b,k,n}$ (and they are changed if $P_{b,k,n}$ finishes).*

2. *If $n = 1$ then $P_{b,k,n}$ does not enumerate any number into B. If $n \geq 2$ then $b + d(b, k)$ is the least and $b + l_B(b, k, n)$ is the largest number which may be enumerated into B during the run of $P_{b,k,n}$ (and they are enumerated if $P_{b,k,n}$ finishes).*

3. *i is an auxiliary variable used in in the inductive step in the definition of $P_{b,k,n}$ (see phase (i) for $n > 1$ in Definition 3.4.12). The numbers $h(b, k, n, i)$ are the numbers "k'" used in phase (i) in the inductive step for the subprocedures of $P_{b,k,n}$. For consecutive i's, the values $h(b, k, n, i)$ code which numbers are enumerated into B by the ith subprocedure. More precisely, during the run of the ith subprocedure ($i \geq 1$), only numbers from the interval $[b + i + h(b, k, n, i-1), b + i + h(b, k, n, i)]$ may enter B. (We prove a slightly stronger statement in Claim 2.)*

Verification

Let $\{A_s\}_{s\in\omega}$ and $\{B_s\}_{s\in\omega}$ be the sequences defined by the procedure $P_{a,0,2^{a+1}}$ started at stage $s_0 = 0$ with $A_0 = B_0 = \emptyset$, and let $A^{\hat{\Phi},\hat{\Psi},V}_{a,s} = A_s$ and $B^{\hat{\Phi},\hat{\Psi},V}_{a,s} = B_s$. We will show that

$$A^{\hat{\Phi},\hat{\Psi},V}_{a} = \lim_{s\to\infty} A^{\hat{\Phi},\hat{\Psi},V}_{a} \text{ and } B^{\hat{\Phi},\hat{\Psi},V}_{a,s} = \lim_{s\to\infty} B^{\hat{\Phi},\hat{\Psi},V}_{a,s} \tag{3.12}$$

exist and have the required properties.

For this sake we prove some more general claims on the sequences $\{A_s\}_{s\geq s_0}$ and $\{B_s\}_{s\geq s_0}$ defined by the procedures $P_{b,k,n}$. We first observe that, in any case, these sequences are a computable almost-enumeration and a computable enumeration, respectively, and can be uniformly computed from a, $\{\hat{\Phi}_s\}_{s\in\omega}$, $\{\hat{\Psi}_s\}_{s\in\omega}$, $\{V_s\}_{s\in\omega}$, the parameters b, k, n and the initial values s_0, A_{s_0}, B_{s_0}.

Claim 1. Let $b, k \in \omega$ and $n \geq 1$ be given and suppose that $P_{b,k,n}$ starts at stage s_0 with the finite sets A_{s_0} and B_{s_0} where $B_{s_0} \subseteq D$. Then $\{A_s\}_{s\geq s_0}$ and $\{B_s\}_{s\geq s_0}$ are uniformly computable in a, $\{\hat{\Phi}_s\}_{s\in\omega}$, $\{\hat{\Psi}_s\}_{s\in\omega}$, $\{V_s\}_{s\in\omega}$, b, k, n, s_0, A_{s_0}, B_{s_0}. Furthermore, it holds that

1. $\forall x \forall s \geq s_0$ $(A_s \restriction x \leq_{lex} A_{s+1} \restriction x)$,

2. $\forall x \forall s \geq s_0$ $(B_s(x) \leq B_{s+1}(x))$,

3. $\forall s \geq s_0$ $(B_s \subseteq D)$.

Proof. Uniform computability of $\{A_s\}_{s \geq s_0}$ and $\{B_s\}_{s \geq s_0}$ follows from the (uniform) effectivity of the procedures $P_{b,k,n}$. The second part of the claim (i.e., *1.*,*2.* and *3.*) follows by an easy induction on n where in the inductive step, we use the fact that $b + d(b,k) \in D$ which is clear by definition of d. $\square$

Note that Claim 1 applied to the procedure $P_{a,0,2^{a+1}}$ with the initial values $s_0 = 0$ and $A_0 = B_0 = \emptyset$ shows that the sequences $\{A_{a,s}^{\hat{\Phi},\hat{\Psi},V}\}_{s\in\omega}$ and $\{B_{a,s}^{\hat{\Phi},\hat{\Psi},V}\}_{s\in\omega}$ are uniformly computable in a, $\{\hat{\Phi}_s\}_{s\in\omega}$, $\{\hat{\Psi}_s\}_{s\in\omega}$, and $\{V_s\}_{s\in\omega}$, that the former is a computable almost-enumeration while the latter is a computable enumeration whence (3.12) exist and the sets $A_a^{\hat{\Phi},\hat{\Psi},V}$ and $B_a^{\hat{\Phi},\hat{\Psi},V}$ are almost-c.e. and c.e., respectively. Moreover, $B_a^{\hat{\Phi},\hat{\Psi},V} \subseteq D$. So it only remains to show that

$$A_a^{\hat{\Phi},\hat{\Psi},V} \subseteq [a, a + l(a)] \text{ and } B_a^{\hat{\Phi},\hat{\Psi},V} \subseteq [a, a + l(a)] \tag{3.13}$$

and (3.3) hold. The former easily follows from the next claim which states that the functions $d(b,k), l_B(b,k,n)$ and $l_A(b,k,n)$ are chosen in such a way that $P_{b,k,n}$ changes A only inside the interval $[b, b + l_A(b,k,n)]$ and B only inside $[b+d(b,k), b+l_B(b,k,n)]$, provided that A_{s_0} and B_{s_0} are empty on the respective intervals.

Claim 2. Suppose that a procedure $P_{b,k,n}$ starts at stage s_0, finishes at stage s_1 and A_{s_0}, B_{s_0} are finite sets such that

$$\begin{aligned} B_{s_0} \cap [b + d(b,k), b + l_B(b,k,n)] &= \emptyset, \\ A_{s_0} \cap [b, b + l_A(b,k,n)] &= \emptyset. \end{aligned} \tag{3.14}$$

Then at stage s_1, it holds that

$$A_{s_1} \cap [b, b + l_A(b,k,n)] = \{b\}. \tag{3.15}$$

Furthermore, for all $s \geq s_0$,

$$\begin{aligned} B_s \upharpoonright b + d(b,k) &= B_{s+1} \upharpoonright b + d(b,k), \\ B_s \cap [b + l_B(b,k,n) + 1, \infty) &= B_{s+1} \cap [b + l_B(b,k,n) + 1, \infty), \\ A_s \upharpoonright b &= A_{s+1} \upharpoonright b, \\ A_s \cap [b + l_A(b,k,n) + 1, \infty) &= A_{s+1} \cap [b + l_A(b,k,n) + 1, \infty). \end{aligned} \tag{3.16}$$

Finally, if $P_{b,k,n}$ does not finish then (3.16) *holds for all stages $s \geq s_0$.*

Proof. The proof is by induction on $n \geq 1$. First suppose that $P_{b,k,n}$ finishes at stage $s_1 \geq s_0$. For $n = 1$, the claim holds because we only change $A(b)$ from 0 to 1 and do not change B. For $n > 1$, by inductive hypothesis assume that Claim 2 holds for $n-1$ and all b' and k'. Consider phase (i) of $P_{b,k,n}$. Let $d = d(b,k) + |n-1|$ and $\{t_i\}_{1 \leq i \leq d}$ be the sequence of stages at which the subprocedures $P_{b+i,h(b,k,n,i-1),n-1}$ finish, respectively and let $t_0 = s_0$.

In the following, we argue that the respective intervals $[b, b + l_A(b,k,n)]$ for A and $[b + d(b,k), b + l_B(b,k,n)]$ for B are big enough so that all subprocedures $P_{b+i,h(b,k,n,i-1),n-1}$ change A and B only inside these intervals. For $n = 2$, during phase (i), by inductive hypothesis, $P_{b+i,h(b,k,2,i-1),1}$ only changes $A(b+i)$ for $1 \leq i \leq d(b,k)+1$ from 0 to 1 during $[t_{i-1}, t_i)$ (note that by inductive hypothesis, $A(b+i)$ is not changed during $[s_0, t_{i-1})$). Moreover, $P_{b',k',1}$-procedures do not change B. Hence, in order to show that the subprocedures of $P_{b,k,2}$ change A and B only inside $[b, b + l_A(b,k,2)]$ and $[b, b + l_B(b,k,2)]$, respectively, it suffices to show that $l_A(b,k,2) \geq d(b,k) + 1$. But this holds since by Definition 3.4.11,

$$\begin{aligned} l_A(b,k,2) &\geq d(b,k) + 1 + l_A(b + d(b,k) + 1, h(b,k,2,d(b,k)), 1) \\ &= d(b,k) + 1 = d. \end{aligned}$$

Thus, at stage t_d, it holds that

$$A_{t_d} \cap [b, b + l_A(b,k,2)] = \{b+1, \dots, b+d\},$$

and for all $s \geq s_0$,

$$\begin{aligned}
B_s \upharpoonright b+1+d(b+1,d(b,k)) &= B_{s+1} \upharpoonright b+1+d(b+1,d(b,k)), \\
B_s \cap [b+l_B(b,k,2)+1,\infty) &= B_{s+1} \cap [b+l_B(b,k,2)+1,\infty), \\
A_s \upharpoonright b+1 &= A_{s+1} \upharpoonright b+1, \\
A_s \cap [b+l_A(b,k,2)+1,\infty) &= A_{s+1} \cap [b+l_A(b,k,2)+1,\infty).
\end{aligned}$$

For $n \geq 3$, we first note that by (3.11), $h(b,k,n,i)$ is nondecreasing in i. For notational convenience, let us abbreviate (compare to Definition 3.4.11)

$$\begin{aligned}
\tilde{l}_{b,k,n,i} &= l_A(b+i,h(b,k,n,i-1),n-1), \text{ and} \\
d_{b,k,n,i} &= d(b+i,h(b,k,n,i-1)).
\end{aligned}$$

Then by Definition 3.4.11 and (3.7) and (3.10), for all $1 \leq i \leq d = d(b,k)+|n-1|$, we have that

$$l_B(b,k,n) \geq (i+h(b,k,n,i)), \tag{3.17}$$

$$l_A(b,k,n) \geq i+\tilde{l}_{b,k,n,i}, \tag{3.18}$$

$$d_{b,k,n,i+1} \geq h(b,k,n,i) \geq d_{b,k,n,i}. \tag{3.19}$$

In particular, by (3.19), the intervals $[b+i+d_{b,k,n,i}, b+i+h(b,k,n,i)]$ ($i \leq d$) are pairwise disjoint. Hence, in order to show that during phase (i), all subprocedures change A and B only inside the respective intervals, it suffices to show that for all $1 \leq i \leq d+1$,

$$\begin{aligned}
B_{t_{i-1}} \cap [b+i+d_{b,k,n,i}, b+l_B(b,k,n)] &= \emptyset, \\
A_{t_{i-1}} \cap [b,b+l_A(b,k,n)] &= \{b+1,\dots,b+i-1\},
\end{aligned} \tag{3.20}$$

and for all $1 \leq i \leq d$ and $s \geq t_{i-1}$ (recall that $l_B(b+i,h(b,k,n,i-1),n-1) =$

$h(b,k,n,i))$,

$$\begin{aligned} B_s \restriction b+i+d_{b,k,n,i} &= B_{s+1} \restriction b+i+d_{b,k,n,i}, \\ B_s \cap [b+i+h(b,k,n,i)+1,\infty) &= B_{s+1} \cap [b+i+h(b,k,n,i)+1,\infty), \\ A_s \restriction b+i &= A_{s+1} \restriction b+i, \\ A_s \cap [b+i+\tilde{l}_{b,k,n,i}+1,\infty) &= A_{s+1} \cap [b+i+\tilde{l}_{b,k,n,i}+1,\infty). \end{aligned} \tag{3.21}$$

(Hence, during the run of the ith subprocedure, only numbers from $[b+i+d_{b,k,n,i}, b+i+h(b,k,n,i)]$ may enter B). For that matter, we show (3.20) and (3.21) by induction on i.

For $i=1$, (3.20) holds since $t_0 = s_0$ and by (3.17) (note that by (3.7), $d_{b,k,n,1} = d(b+1, d(b,k)) \geq d(b,k)$). Thus, by inductive hypothesis on $n-1$, we may apply the claim to the subprocedure $P_{b+1,h(b,k,n,0),n-1}$ starting at stage s_0 with A_{s_0} and B_{s_0}. Hence, (3.21) holds for $i=1$ and (3.20) holds for $i \leq 2$. Next, assume that $i \leq d$ is such that (3.20) holds for all $1 \leq j \leq i$ and (3.21) holds for all $j < i$. Then by (3.17), (3.18) and (3.19), it holds that

$$\begin{aligned} B_{t_{i-1}} \cap [b+i+d_{b,k,n,i}, b+i+h(b,k,n,i)] &= \emptyset \\ A_{t_{i-1}} \cap [b+i, b+i+\tilde{l}_{b,k,n,i}] &= \emptyset. \end{aligned}$$

Whence, again by inductive hypothesis on $n-1$, we may apply Claim 2 to the subprocedure $P_{b+i,h(b,k,n,i-1),n-1}$ starting at stage t_{i-1} with $A_{t_{i-1}}$ and $B_{t_{i-1}}$. But this implies (3.21) for i and (3.20) for $i+1$.

Thus, for all $n \geq 2$, at stage t_d, it holds that

$$A_{t_d} \cap [b, b+l_A(b,k,n)] = \{b+1, \ldots, b+d\}, \tag{3.22}$$

and for all $s \geq s_0$,

$$\begin{aligned} B_s \restriction b+1+d(b+1,d(b,k)) &= B_{s+1} \restriction b+1+d(b+1,d(b,k)), \\ B_s \cap [b+l_B(b,k,n)+1,\infty) &= B_{s+1} \cap [b+l_B(b,k,n)+1,\infty), \\ A_s \restriction b+1 &= A_{s+1} \restriction b+1, \\ A_s \cap [b+l_A(b,k,n)+1,\infty) &= A_{s+1} \cap [b+l_A(b,k,n)+1,\infty), \end{aligned} \tag{3.23}$$

holds. Since $P_{b,k,n}$ finishes, phase (ii) is reached and there exists a least critical stage $\tilde{s} \geq t_d$. At stage $\tilde{s}$, it holds that $B_{\tilde{s}+1} = B_{\tilde{s}} \cup \{b + d(b,k)\}$. By (3.7), $b+1+d(b+1, d(b,k)) > b + d(b,k)$. So by (3.23) and since we did not change B during $(t_d, \tilde{s}]$, we have $b + d(b,k) \notin B_{\tilde{s}}$. Hence, the enumeration of $b + d(b,k)$ into B at stage $\tilde{s}+1$ indeed changes $B(b + d(b,k))$ from 0 to 1 during phase (ii). Since phase (iii) is reached, there exists a least critical stage $t > \tilde{s}$. At stage $t+1$, we replace $A_t \upharpoonright b+1+d$ by $(A_t \upharpoonright b+1+d)^+$. By (3.22), (3.23) and since we do not change A during $(t_d, t]$, by definition of $^+$, we have that

$$A_{t+1} \cap [b, b + l_A(b,k,n)] = \{b\}.$$

(In particular, during phase (iii), we change A(b) from 0 to 1.) During phase (iv), we change neither A nor B. But this implies (3.15) and, for all $s \geq s_0$, (3.16).

Finally, if $P_{b,k,n}$ does not finish, then the above argument shows that (3.16) holds for all $s \geq s_0$ (the details are left to the reader). $\square$

Note that Claim 2 implies (3.13) as follows. For $P_{a,0,2^{a+1}}$, $s_0 = 0$ and $A_0 = B_0 = \emptyset$, the hypotheses of Claim 2 are satisfied. So (3.16) holds for all sufficiently large s. Since, by (3.8) and (3.6), $l_B(a, 0, 2^{a+1}) \leq l_A(a, 0, 2^{a+1}) = l(a)$ and since $A_a^{\hat{\Phi},\hat{\Psi},V}$ and $B_a^{\hat{\Phi},\hat{\Psi},V}$ are the limit of the A_s and B_s, respectively, this implies (3.13).

It remains to show that (3.3) holds. In order to do so we will argue that the failure of (3.3) implies that the procedure $P_{a,0,2^{a+1}}$ started at stage $s_0 = 0$ with the sets $A_0 = B_0 = \emptyset$ will finish and that this in turn will imply that, for $V \upharpoonright a+1$ viewed as a binary number, $V \upharpoonright a+1 \geq 2^{a+1}$ which is impossible. Again we show this by proving a more general claim. Before we state the claim we introduce some notation to be used in formulating and proving the claim.

We identify strings $\sigma \in 2^{<\omega}$ with the natural number

$$\ulcorner \sigma \urcorner = \sum_{i<|\sigma|} \sigma_i 2^{|\sigma|-i-1},$$

and likewise, we identify numbers with binary strings via their binary expansions. We shortly write $\sigma < \tau$ for $\ulcorner \sigma \urcorner < \ulcorner \tau \urcorner$. Note that we have to take care about how exactly we identify numbers with strings (and vice versa). Namely, $\sigma < \tau$ does not imply in general that $\sigma <_{lex} \tau$. But if σ and τ have the same length, then indeed $\sigma <_{lex} \tau \Leftrightarrow \ulcorner\sigma\urcorner < \ulcorner\tau\urcorner$, i.e., $<$ coincides with the lexicographical ordering. In particular, for every binary string σ, every number $y < 2^{|\sigma|}$ corresponds to a unique τ such that $|\tau| = |\sigma|$ and $\ulcorner\tau\urcorner = y$.

So unless otherwise stated, when we define operations that involve both strings and numbers, we tacitly assume that all strings involved have the same length and the lengths of binary representations of the numbers involved are at most as long as the strings.

In this way, if σ is a binary string and y is a number such that $\ulcorner\sigma\urcorner + y < 2^{|\sigma|}$ (or if $y \leq \ulcorner\sigma\urcorner$), we may define $\sigma + y$ ($\sigma - y$) as the unique binary string τ of length $|\sigma|$ such that $\ulcorner\tau\urcorner = \ulcorner\sigma\urcorner + y$ ($\ulcorner\tau\urcorner = \ulcorner\sigma\urcorner - y$). In particular, if $y = 1$, then $\tau = \sigma^+$ ($\tau^+ = \sigma$).

Similarly, if $y < 2^{|\sigma|}$ and $y \cdot \ulcorner\sigma\urcorner < 2^{|\sigma|}$, we define the product $y \cdot \sigma$ and finally, if σ and τ are binary strings such that $|\sigma| = |\tau|$ and $\sigma < \tau$, we write $\tau - \sigma$ for the unique number z such that $\sigma + z = \tau$. Note that, for any binary strings σ, τ, it holds that $\sigma\tau \geq 2^{|\tau|} \cdot 0^{|\tau|}\sigma$ which we shortly write as $\sigma\tau \geq 2^{|\tau|} \cdot \sigma$ (this is justified by that fact that $\ulcorner 0^{|\tau|}\sigma\urcorner = \ulcorner\sigma\urcorner$).

Now the claim is as follows.

Claim 3. Assume that procedure $P_{b,k,n}$ starts at stage s_0, finishes at stage s_1, and at stage s_0, it holds that

$$A_{s_0} \cap [b, b + l_A(b,k,n)] = B_{s_0} \cap [b + d(b,k), b + l_B(b,k,n)] = \emptyset.$$

Then it holds that

$$V_{s_1} \upharpoonright b + 1 - V_{s_0} \upharpoonright b + 1 \geq n.$$

Proof. The proof is by induction on $n \geq 1$.

Suppose $n = 1$. In this case, the procedure changes $A(b)$ at a critical stage from 0 to 1 once during $[s_0, s_1]$. By the fact that $\hat{\Phi}$ is an ibT-reduction and

$\{V_s\}_{s\in\omega}$ is a computable almost enumeration, this forces $V_{s_1} \upharpoonright b+1 >_{lex} V_{s_0} \upharpoonright b+1$. So $V_{s_1} \upharpoonright b+1 - V_{s_0} \upharpoonright b+1 \geq 1$.

Next, suppose $n+1 > 1$ and by inductive hypothesis, suppose the claim holds for all procedures $P_{b',k',n}$. Let $d = d(b,k) + |n|$ and let $\{t_i\}_{1\leq i\leq d}$ denote the stages at which the subprocedures $P_{b+i,h(b,k,n+1,i-1),n}$ finish, respectively and let $t_0 = s_0$ and $\sigma = V_{s_0} \upharpoonright b+1$. We show by induction on $0 \leq i \leq d$ that

$$V_{t_i} \upharpoonright b+1+i \geq 2^i\sigma + (2^i - 1)n. \tag{3.24}$$

For $i = 0$, this holds by definition of σ. Next, assume that (3.24) holds for $i < d$. Then we observe that

$$V_{t_i} \upharpoonright b+1+i+1 \geq 2\cdot(V_{t_i} \upharpoonright b+1+i)$$

which together with the inductive hypothesis on n

$$V_{t_{i+1}} \upharpoonright b+1+i+1 - V_{t_i} \upharpoonright b+1+i+1 \geq n$$

yields

$$\begin{aligned} V_{t_{i+1}} \upharpoonright b+1+i+1 &\geq 2(V_{t_i} \upharpoonright b+1+i) + n \\ &\geq 2^{i+1}\sigma + (2^{i+1} - 2)n + n \\ &= 2^{i+1}\sigma + (2^{i+1} - 1)n. \end{aligned}$$

Hence, at stage t_d, it holds that

$$V_{t_d} \upharpoonright b+1+d \geq 2^d\sigma + (2^d - 1)n = 2^d(\sigma + n) - n$$

By a calculation with carryovers, $2^d(\sigma+n) - n$ has the form

$$(\sigma + n - 1)1^{d-|n|}j_0 \dots j_{|n|-1}$$

where $j_l < 2$ $(l < |n|)$, since $d \geq |n|$. In particular, $\ulcorner\sigma\urcorner + n - 1 < 2^{b+1}$ (so $\sigma + n - 1$ is a well-defined binary string of length $b+1$), since otherwise

$V_{t_d} \upharpoonright b+1 \geq 2^{b+1}$ which is impossible. Since $P_{b,k,n}$ finishes, there exists a least critical stage $\tilde{s} \geq t_d$ and a least critical stage $t > \tilde{s}$. By definition of phase (ii) of $P_{b,k,n}$, we enumerate $b+d(b,k)$ into B at stage $\tilde{s}+1$. By choice of $d-|n| = d(b,k)$ and the fact that $\hat{\Psi}$ is an ibT-reduction, there must be a stage $s \in (\tilde{s}, t]$ such that $V_s \upharpoonright b+1+d(b,k) > (\sigma+n-1)1^{d(b,k)}$; hence,

$$V_{\tilde{s}} \upharpoonright b+1+d(b,k) \geq (\sigma+n)0^{d(b,k)}, \text{ i.e.,}$$
$$V_{\tilde{s}} \upharpoonright b+1 \geq \sigma+n.$$

Similarly, it must be that $\ulcorner\sigma\urcorner + n \not\geq 2^{b+1}$ (whence $|\sigma+n| = b+1$) since otherwise $V_t \upharpoonright b+1 \geq 2^{b+1}$. By the proof of Claim 2, A(b) changes from 0 to 1 at stage $t+1$. Hence, since $\hat{\Phi}$ is an ibT-reduction, this forces a change of V below $b+1$ until stage s_1, i.e., (recall that $\sigma = V_{s_0} \upharpoonright b+1$)

$$V_{s_1} \upharpoonright b+1 > (V_{s_0} \upharpoonright b+1) + n,$$

or, equivalently,

$$V_{s_1} \upharpoonright b+1 - V_{s_0} \upharpoonright b+1 \geq n+1.$$

So the claim holds for $n+1$ too. □

We conclude the proof of Lemma 3.4.9 by deducing (3.3) from Claim 3. Consider the procedure $P_{a,0,2^{a+1}}$ started at stage $s=0$ with $A_0 = B_0 = \emptyset$. Since $V \upharpoonright a+1 < 2^{a+1}$ it follows by Claim 3 that the procedure does not finish. So there are only finitely many critical stages s for the corresponding sets A_s and B_s. Since $A^{\hat{\Phi},\hat{\Psi},V}_{a,s} = A_s$ and $B^{\hat{\Phi},\hat{\Psi},V}_{a,s} = B_s$ this implies that there is a stage s_1 such that

$$\forall s \geq s_1 \; \exists\, x \in [a, a+l(a)] \; (A^{\hat{\Phi},\hat{\Psi},V}_{a,s}(x) \neq \hat{\Phi}^{V_s}_s(x) \text{ or } B^{\hat{\Phi},\hat{\Psi},V}_{a,s}(x) \neq \hat{\Psi}^{V_s}_s(x)).$$

However, this implies (3.3) by the use-principle. This completes the proof of Lemma 3.4.9.

3.4.3 Totally ω-C.E. Degrees and Complex Almost-C.E. Sets

We conclude the results of this chapter by looking at complex almost-c.e. sets and how they are related to sets of not totally ω-c.e. degree.

Definition 3.4.14 (Kanovich)**.** *Let h be a computable order. A set A is h-*complex *if $C(A \upharpoonright n) \geq h(n)$ for all n. A set A is* complex *if A is h-complex for some computable order.*

Kanovich (see Theorem 8.16.7 in [DH10]) shows that a c.e. set A is complex if and only if A is wtt-complete. Since any almost-c.e. set is wtt-equivalent to a c.e. set and since Kjos-Hanssen, Merkle and Stephan [KHMS11] observe that the class of complex sets is closed upwards under wtt-reducibility, the complex almost-c.e. sets coincide with the wtt-hard almost-c.e. sets.

Lemma 3.4.15. *An almost-c.e. set A is complex if and only if A is wtt-hard (for the class of c.e. sets).*

Note that, for any wtt-hard almost-c.e. set C and any number m, the m-bounded left-shift $C_{-m} = \{x : x + m \in C\}$ of C is wtt-hard (since $C \equiv_{cl} C_{-m}$) and almost-c.e. too and, for any set A such that $A \leq_{cl} C$ holds there exists m_0 such that $A \leq_{\mathrm{ibT}} C_{-m_0}$ holds. So in order to show that u.s.p. sets are not cl-reducible to any complex almost-c.e. set, it suffices to show the following theorem.

Theorem 3.4.16. *Let A be an almost-c.e. set with the universal similarity property and let C be a wtt-hard almost-c.e. set. Then $A \not\leq_{\mathrm{ibT}} C$.*

The theorem is immediate by Lemma 3.4.8 and the following equivalence. For the sake of completeness, we give the proof.

Lemma 3.4.17. *Let A be an almost-c.e. set. The following are equivalent.*

(i) A is not ibT-reducible to any wtt-hard almost-c.e. set.

(ii) For any infinite computable set D there is a computably enumerable subset B of D such that (A, B) is an ibT-maximal pair in the almost-c.e. sets.

Proof. $(i) \Rightarrow (ii)$. The proof is by contraposition. Assume that there is an infinite computable set D such that, for all c.e. subsets B of D, there is an almost-c.e. set C such that $A \leq_{\text{ibT}} C$ and $B \leq_{\text{ibT}} C$. Fix $B \subseteq D$ such that B is wtt-complete. Then, by $B \leq_{\text{ibT}} C$, the almost-c.e. set C is wtt-hard and $A \leq_{\text{ibT}} C$.

$(ii) \Rightarrow (i)$. Again, the proof is by contraposition. Fix a wtt-hard almost-c.e. set C such that $A \leq_{\text{ibT}} C$. First observe that there is a strictly increasing computable function f such that C is f-T-hard, i.e., such that any c.e. set B is reducible to C by an f-bounded Turing reduction. (Namely, there is a computable function g, such that any c.e. set is g-bounded Turing reducible to the universal c.e. set $K_u = \{\langle e, x\rangle : x \in W_e\}$, and, by choice of C, there is a computable function h such that K_u is h-bounded Turing reducible to C. So, since w.l.o.g. we may assume that the functions g and h are strictly increasing, the computable function $f = h \circ g$ will do.) Now let $D = \{f(x) : x \in \omega\}$ be the range of f. Then D is infinite and computable. So, in order to show that (ii) fails, it suffices to show that, for any c.e. subset B of D, B is ibT-reducible to C. So fix such a set B. Then there is a c.e. set $\hat{B}$ such that B is the f-shift $\hat{B}_f = \{f(x) : x \in \hat{B}\}$ of $\hat{B}$. Since, by choice of f, $\hat{B} \leq_{f\text{-T}} C$, this easily implies that $B \leq_{\text{ibT}} C$. □

Finally, the converse of Theorem 3.4.16 holds as well.

Theorem 3.4.18. *Let* $\mathbf{a}$ *be a c.e. Turing degree which is totally* ω*-c.e., and let* A *be any almost-c.e. set in* $\mathbf{a}$*. There is a wtt-hard almost-c.e. set* B *such that* $A \leq_{\text{ibT}} B$*.*

So, in total, we have the following result which proves a conjecture by Greenberg.

Theorem 3.4.19. *For a c.e. Turing degree* $\mathbf{a}$*, the following are equivalent.*

1. $\mathbf{a}$ *is not totally* ω*-c.e.*

2. *There exists an almost-c.e. set* $A \in \mathbf{a}$ *such that* $A \not\leq_{cl} B$ *holds for any complex almost-c.e. set* B*.*

3.5 Summary

The main results of this chapter can be summarized as follows. First of all, we proved that array noncomputability for almost-c.e. (wtt-)degrees coincides with that of c.e. degrees.

Theorem. *For a c.e. Turing degree (wtt-degree)* $\mathbf{a}$*, the following are equivalent.*

1. $\mathbf{a}$ *is a.n.c.*

2. $\mathbf{a}$ *is a.c.e.-a.n.c.*

3. *For any v.s.a.* $\mathcal{F}$ *there exists a (purely)* $\mathcal{F}$*-a.c.e.-a.n.c. set* $A \in \mathbf{a}$.

So, by [DJS90], a.c.e.-a.n.c. (wtt-) degrees are closed upwards.

Corollary. *For any two c.e. (wtt-) degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that* $\mathbf{a} \leq \mathbf{b}$ *holds and such that* $\mathbf{a}$ *is a.c.e.-a.n.c.,* $\mathbf{b}$ *is a.c.e.-a.n.c., too.*

Then in contrast to c.e. a.n.c. sets, there exist almost-c.e. sets which are $\mathcal{F}$-a.c.e.-a.n.c. for all very strong arrays $\mathcal{F}$. These are by definition the sets with the universal similarity porperty. It turns out that these sets exist precisely in the c.e. not totally ω-c.e. degrees.

Theorem. *For a c.e. Turing degree* $\mathbf{a}$ *the following are equivalent.*

(i) There is an almost-c.e. set A *with the universal similarity property in* $\mathbf{a}$.

(ii) $\mathbf{a}$ *is not totally* ω*-c.e.*

Then for certain properties $\mathcal{P}$, using u.s.p. sets facilitates to prove that c.e. not totally ω-c.e. degrees contain sets with a certain property $\mathcal{P}$ by showing that u.s.p. sets have property $\mathcal{P}$. The idea is to check whether $\mathcal{P}$ can be forced by a list of requirements $\mathcal{R}_e$, where, for each $\mathcal{R}_e$ the attempt is to prove that there exists a v.s.a. $\mathcal{F}_e$ and an $\mathcal{F}_e$-a.c.e. set B_e such that any set which is $\mathcal{F}_e$-similar to B_e meets $\mathcal{R}_e$; hence, any $\mathcal{F}_e$-a.c.e.-a.n.c. set meets $\mathcal{R}_e$ and any u.s.p. set A has property $\mathcal{P}$. Moreover, if we can show that the v.s.a. $\mathcal{F}_e$ can be chosen independently of the requirement $\mathcal{R}_e$, i.e., $\mathcal{F} = \mathcal{F}_e$ holds for all e (where $\mathcal{F} = \mathcal{F}_0$)

then it follows that any a.c.e.-a.n.c. set A has property $\mathcal{P}$; whence, any a.n.c. degree contains a set with property $\mathcal{P}$. This reaffirms the idea that not totally ω-c.e. degrees provide a uniform form of multiple permitting. In this chapter, we gave two instances of this methodology. First, we showed that any c.e. not totally ω-c.e. degree contains an almost-c.e. CB-random set, thereby strengthening the result from [BDN12]. In order to do that, we needed the following lemma.

Lemma. *Let f be a computable function. There is a c.v.s.a.i. $\mathcal{F} = \{F_n\}_{n\in\omega}$ such that any $\mathcal{F}$-a.c.e.-a.n.c. set A is f-ML-random.*

Then we may immediately deduce that we get the desired result.

Theorem. *Any almost-c.e. set with the universal similarity property is CB-random. Hence, any c.e. Turing degree* $\mathbf{a}$ *which is not totally ω-c.e. contains an almost-c.e. set $A \in \mathbf{a}$ which is CB-random.*

Secondly, we proved a conjecture by Greenberg that the c.e. not totally ω-c.e. degrees are characterized as those c.e. degrees which contain an almost-c.e. set which is not cl-reducible to any complex almost-c.e. set. For the if-direction, we needed a lemma on maximal pairs.

Lemma. *Let A be a an almost-c.e. set with the universal similarity property and let D be any infinite computable set. There is a c.e. set $B \subseteq D$ such that (A, B) is an* ibT*-maximal pair in the almost-c.e. sets.*

Finally, since complex almost-c.e. sets are just the almost-c.e. sets which are wtt-hard for the class of c.e. sets by Lemma 3.4.15 and since the question whether an almost-c.e. set A is cl-reducible to a wtt-hard almost-c.e. set can be converted in to into the question whether, for any infinite computable set D, A is half of a maximal pair, where the second half B is chosen to be c.e. and a subset D by Lemma 3.4.17, this shows that any u.s.p. set is not cl-reducible to any complex almost-c.e. set; hence, any c.e. not totally ω-c.e. set contains a set which is not cl-reducible to any complex almost-c.e. set. So, since Theorem 3.4.18 provides the converse, in summary, we have the following result.

Theorem. *For a c.e. Turing degree* $\mathbf{a}$*, the following are equivalent.*

1. $\mathbf{a}$ *is not totally* ω*-c.e.*

2. *There exists an almost-c.e. set* $A \in \mathbf{a}$ *such that* $A \not\leq_{cl} B$ *holds for any complex almost-c.e. set* B.

Chapter 4

Multiple Permitting and r-Maximality

4.1 Introduction

In 1944, Emil Post addressed in his paper [Pos44] the famous question whether there exists an incomplete and noncomputable c.e. Turing degree. Today, this is known as *Post's problem*. In his paper, he also introduced the notions of simple, hypersimple and hyperhypersimple sets. Although it is known that one cannot solve Post's problem using simple, hypersimple (h-simple) and hyperhypersimple (hh-simple) sets as projected by Post, these properties have later been investigated independently and other notions of simplicity have been proposed, e.g. maximal sets by Myhill [Myh56]. An overview of the best known simplicity notions and their relations among each other is given in Fig. 4.1 (see Soare [Soa87, p.211]).

$$\begin{tikzcd}
 & \text{r-maximal} \arrow[r] & \text{sh-simple} \arrow[r] & \text{fsh-simple} \arrow[dr] & & \\
\text{maximal} \arrow[ur] \arrow[dr] & & & & \text{h-simple} \arrow[r] & \text{simple} \\
 & \text{quasi-maximal} \arrow[r] & \text{hh-simple} \arrow[r] \arrow[uu] & \text{dense simple} \arrow[ur] & &
\end{tikzcd}$$

Figure 4.1: Most common simplicity properties and their relations among each other. An arrow $P \to Q$ indicates that property P implies property Q but not vice versa.

Now given a subclass $\mathcal{C}$ of the c.e. sets, it is an interesting question which of the simplicity notions may hold for sets in $\mathcal{C}$. An example of a class for which this question was investigated is the class of the *array noncomputable* (a.n.c.) sets.

In [DJS90], Downey, Jockusch and Stob describe exactly which of the properties in Fig. 4.1 may hold for a.n.c. sets and which not. On the negative side, they showed that a.n.c. sets can neither be dense simple nor strongly hypersimple (sh-simple). On the positive side, they showed that there is a finitely strongly hypersimple (fsh-simple) a.n.c. set.

In this chapter we address the question if the boundaries are the same if we replace a.n.c. sets by their weak truth table (wtt) degrees. On the negative side, recently Ambos-Spies [AS18, Theorem 3] showed that no a.n.c. wtt-degree contains a dense simple set, thereby extending the corresponding result in [DJS90] from sets to wtt-degrees. In contrast to [DJS90], however, we show here that the positive bound turns out to be stronger for a.n.c. wtt-degrees than for a.n.c. sets. More precisely, we prove the following.

Theorem 4.1.1. *There exists a c.e. wtt-degree $\boldsymbol{a}$ which is a.n.c. and* r*-maximal.*

By Theorem 4.1.1 and the result by Ambos-Spies [AS18], we completely describe which of the simplicity properties in Fig. 4.1 may hold for any a.n.c. wtt-degree.

The outline of this chapter is as follows. In Section 4.2, we give the basic definitions that are needed for the proof of Theorem 4.1.1. In Section 4.3, we give the basic idea of the proof and give the formal construction. Finally, we verify that the construction yields a set with the required properties.

Our results in this chapter are published in [Mon18].

4.2 Preliminaries

First of all, recall the definition of an r-maximal set.

Definition 4.2.1. *A c.e. set A is* r-maximal *if it is coinfinite and for every computable set R, either $R \cap \bar{A}$ or $\bar{R} \cap \bar{A}$ is finite.*

For the proof of Theorem 4.1.1, it is convenient to use a characterization of the a.n.c. wtt-degrees given by multiply permitting sets. Multiply permitting sets have been introduced by Ambos-Spies in [AS18]. For the definition of multiply permitting sets, recall from Definition 2.2.1 that an infinite sequence of finite sets $\mathcal{F} = \{F_n\}$ is a very strong array (v.s.a.) if the sets F_n are uniformly given by their canonical index such that they are mutually disjoint, nonempty and growing in size. Then multiply permitting c.e. sets are defined as follows.

Definition 4.2.2 ([AS18])**.** *Let $\mathcal{F} = \{F_n\}_{n\in\omega}$ be a v.s.a., let f be a computable function, let A be a c.e. set, and let $\{A_s\}_{s\in\omega}$ be a computable enumeration of A. Then A is $\mathcal{F}$-*permitting via f and $\{A_s\}_{s\in\omega}$ *if, for any partial computable function ψ,*

$$\exists^\infty n \; \forall x \in F_n \; (\psi(x)\downarrow \Rightarrow \; A \restriction f(x)+1 \neq A_{\psi(x)} \restriction f(x)+1) \tag{4.1}$$

*holds. A is $\mathcal{F}$-*permitting via f *if there is a computable enumeration $\{A_s\}_{s\in\omega}$ of A such that A is $\mathcal{F}$-permitting via f and $\{A_s\}_{s\in\omega}$; A is $\mathcal{F}$-*permitting *if A is $\mathcal{F}$-permitting via some computable f; and A is* multiply permitting *if A is $\mathcal{F}$-permitting for some v.s.a. $\mathcal{F}$. Finally, a c.e. wtt-degree $\boldsymbol{a}$ is* multiply permitting *if there is a multiply permitting set $A \in \boldsymbol{a}$.*

As shown in [AS18, Lemma 2 and Theorem 2], the c.e. multiply permitting wtt-degrees coincide with the a.n.c. wtt-degrees.

Theorem 4.2.3. *For a c.e. wtt-degree $\boldsymbol{a}$, the following are equivalent.*

1. *$\boldsymbol{a}$ is a.n.c.*
2. *$\boldsymbol{a}$ is multiply permitting.*
3. *Every c.e. set $A \in \boldsymbol{a}$ is multiply permitting.*

So for the sake of proving Theorem 4.1.1, it suffices to construct a c.e. set A which is both multiply permitting and r-maximal. Before we state the formal construction of such a set A, let us give some idea of the proof.

4.3 Proof of Theorem 4.1.1

The construction of A is divided into two parts. First, we construct a c.e. set B such that B is r-maximal and such that the complement $\bar{B}$ is "big enough" (which is made precise in (4.2) below). Then we define a v.s.a. $\mathcal{F}$ and construct A as a c.e. superset of B such that A is $\mathcal{F}$-permitting via the identity function. So first let us make precise what B looks like.

Lemma 4.3.1. *There exists a c.v.s.a.i. $\mathcal{G} = \{G_n\}_{n\in\omega}$ and an* r*-maximal set B such that*

$$\exists^\infty n \ (|G_n \cap \bar{B}| \geq (n+1)^2). \tag{4.2}$$

We claim that from Lemma 4.3.1, we can define a c.e. set A as required.

Proof of Theorem 4.1.1 using Lemma 4.3.1. Fix a c.v.s.a.i. $\mathcal{G} = \{G_n\}_{n\in\omega}$ and B as in Lemma 4.3.1 and fix a computable enumeration $\{B_s\}_{s\in\omega}$ of B. Let $\mathcal{F} = \{F_n\}_{n\in\omega}$ be the unique v.s.a. such that $F_n = \{x_0^n, \dots, x_n^n\}$, where $x_0^n, \dots, x_n^n$ are the first $n+1$ elements of G_{n+1} in order of magnitude (note that $F_n \subseteq G_{n+1}$ since $|G_n| \geq n+1$ for all n; hence, $\max(G_n) < \min(F_n)$ as $\mathcal{G}$ is a c.v.s.a.i.). Then we define a computable enumeration $\{A_s\}_{s\in\omega}$ of A in stages s as follows, where A_s denotes the finite set of numbers that are enumerated into A by stage s.

Construction of A.

Stage 0. $A_0 = \emptyset$.

Stage $s+1$. Given A_s, let N_s be the set of all $n \in \omega$ such that

(a) $G_n \not\subseteq A_s$ and

(b) $\exists\, e, x\ (e \leq n \ \&\ x \in F_n \ \&\ \varphi_{e,s}(x)\uparrow \ \&\ \varphi_{e,s+1}(x)\downarrow)$

hold (note that $N_s \subseteq \omega \restriction s$ and $\varphi_e(x) \leq s$ holds for any such e, x, s). Let $A_{s+1} = B_{s+1} \cup A_s \cup \{\min(G_n \setminus A_s) :\ n \in N_s\}$.

We claim that the so constructed set A is multiply permitting and r-maximal. Clearly, it suffices to show that A is multiply permitting (namely, every multiply permitting set is noncomputable; hence, $\bar{A}$ is infinite, and every coinfinite c.e. superset of an r-maximal set is r-maximal as well). To this end, let e be given. By (4.2), fix $n \geq e$ such that $|G_n \cap \bar{B}| \geq (n+1)^2$. We claim that any such n witnesses that the inner clause of (4.1) holds for φ_e in place of ψ with $f(x) = x$. Since by Lemma 4.3.1, there exist infinitely many such n, this proves the claim. By construction and by convention on converging computations, it suffices to show that $G_n \not\subseteq A_s$ holds for any stage s such that (b) holds. For a proof by contraposition, let s be a stage such that $G_n \subseteq A_s$. Now for any stage s', a number may enter $A \cap G_n$ only if (b) holds at stage $s'+1$ or if x enters B at stage $s'+1$. But on the one hand, there are at most $|F_n| \cdot (n+1) = (n+1)^2$ numbers that may enter $A \cap G_n$ via (b). On the other hand, $G_n \cap \bar{B} \geq (n+1)^2$ by choice of n. So $G_n \subseteq A_s$ can only hold if $\varphi_{e,s}(x)\downarrow$ holds for all $x \in F_n$; hence, (b) cannot hold for any stage $t \geq s$. □ □

Thus, it remains to show that Lemma 4.3.1 holds.

4.3.1 Proof of Lemma 4.3.1: Construction of B

For the proof of Lemma 4.3.1, we effectively construct a c.e. set B in stages s where B_s denotes the finite set of numbers which are enumerated into B by stage s. Before we give the formal construction, let us discuss some of the ideas behind it and introduce some of the concepts to be used in the construction.

We give the definition of the c.v.s.a.i. $\mathcal{G} = \{G_n\}_{n\in\omega}$ in advance. We define $\{G_n\}_{n\in\omega}$ as the unique c.v.s.a.i. such that $\min(G_0) = 0$, $\min(G_{n+1}) = \max(G_n)+1$ and

$$|G_n| = 2^{\frac{n(n+1)}{2}}(n+1)^2 \tag{4.3}$$

holds. Then it suffices to construct B such that (4.2) holds and such that B meets for all e the requirements

$$\mathcal{Q}_e : V_e^0 \cup V_e^1 = \omega \;\Rightarrow\; \exists i \leq 1\; \forall^\infty n\; (V_e^i \cap G_n \subseteq B). \tag{4.4}$$

where $\{(V_e^0, V_e^1)\}_{e\in\omega}$ is an effective enumeration of all pairs of disjoint c.e. sets. Such an enumeration can be easily obtained as follows. Given $e = \langle e_0, e_1\rangle$ and stage s, let $t_{e,s}$ be the largest stage $t \leq s$ such that $W_{e_0,t} \cap W_{e_1,t} = \emptyset$ and let

$V^i_{e,s} = W_{e_i,t_{e,s}}$ for $i \leq 1$. Then $V^0_{e,s} \cap V^1_{e,s} = \emptyset$ for all $e, s \in \omega$ and if W_{e_0} and W_{e_1} are disjoint then $V^i_e = W_{e_i}$ for all $i \leq 1$. We call a requirement $\mathcal{Q}_e$ *infinitary* if the hypothesis of $\mathcal{Q}_e$ holds.

Clearly, it is undecidable whether a requirement is infinitary (in fact, it is not hard to show that this question is Π_2-complete). So we have to effectively approximate this question in the course of the construction. For this we define T as the full binary tree as a priority tree. A node $\alpha \in T$ of length n codes a guess at which of the first n requirements are infinitary where, for $e < n$, $\alpha(e) = 0$ codes that $\mathcal{Q}_e$ is infinitary. Correspondingly we call e an *infinitary edge of* α in this case. Then the *true path* TP is the infinite path through T satisfying $TP(e) = 0$ iff $\mathcal{Q}_e$ is infinitary. In order to approximate TP at stage s of the construction we use the following *length of agreement function*

$$l(e,s) = \mu y(V^0_{e,s}(y) = V^1_{e,s}(y) = 0). \tag{4.5}$$

By choice of the sequence $\{(V^0_e, V^1_e)\}_{e\in\omega}$, $l(e,s)$ is nondecreasing in s for all e and, for fixed e, it is unbounded iff $\mathcal{Q}_e$ is infinitary. Based on $l(e,s)$, we define the set of α-stages by induction on $|\alpha|$ as follows. Every stage is a λ-stage. An α-stage s is called *α-expansionary* if $s = 0$ or $l(|\alpha|, s) > l(|\alpha|, t)$ for all α-stages $t < s$. Then a stage s is an $\alpha 0$-stage if it is α-expansionary and an $\alpha 1$-stage if it is an α-stage but not α-expansionary. The current approximation δ_s of $TP \upharpoonright s$ at the end of stage s is the unique node α of length s such that s is an α-stage, and we say that α is *accessible* at stage $s+1$ if α is an initial segment of δ_s, i.e., $\alpha \sqsubseteq \delta_s$. Note that $TP = \liminf_{s\to\infty} \delta_s$, i.e., $TP \upharpoonright n$ is the leftmost node of length n which is accessible infinitely often for every n. As usual, we say for two nodes α and β that α has *higher priority than* β and denote it by $\alpha < \beta$ iff $\alpha \sqsubset \beta$ (i.e., α is a proper initial segment of β) or α is to the left of β, denoted by $\alpha <_{left} \beta$, i.e., there exists $\gamma \in T$ such that $\gamma 0 \sqsubseteq \alpha$ and $\gamma 1 \sqsubseteq \beta$.

Now the strategy for meeting the requirements $\mathcal{Q}_e$ which at the same time satisfies (4.2) is based on a variant of the e-state definition used in the construction of a maximal set as e.g. given in [Soa87]. We assign the intervals G_n to the nodes $\alpha \in T$ where at each stage at most one interval is assigned to each α. An unused interval is assigned to α only at a stage where α is accessible, and the interval assigned to α is cancelled if α is to the right of δ_s. In this case, G_n is *deleted*, i.e., all elements of G_n are enumerated into B. So an interval may be permanently assigned to α only if α is on the true path or to the left of it (we also make sure that intervals that are never assigned to any node are deleted as well). Moreover, for any number e, there will be only finitely many nodes to the left of $TP \restriction e+1$ which get a permanent interval assigned since only finitely many such nodes are ever accessible. So almost all intervals which are never deleted are assigned to nodes extending $TP \restriction e+1$ and hence have the correct guess about the type of the first $e+1$ requirements.

Now, for any interval G_n, any node α and any stage s, the *α-state of G_n at stage s*, denoted by $\sigma(\alpha, n, s)$ is a binary string of length $\leq k$ where k is the number of infinitary edges of α. In the following, let $e_0 < e_1 < \cdots < e_{k-1}$ be the infinitary edges of α. Then $|\sigma(\alpha, n, s)|$ is the greatest $j \leq k$ such that for any $j' < j$, $l(e_{j'}, s) > \max(G_n)$; hence, $V^0_{e_{j'},s}$ and $V^1_{e_{j'},s}$ partition G_n (note that for α on the true path $|\sigma(\alpha, n, s)| = k$ for sufficiently large s). Moreover, for $j' < j$, we choose the values $i_{j'}$ of $\sigma(\alpha, n, s)(j')$ inductively in such a way that enumerating $V^{i_{j'}}_{e_{j'}} \cap G_n$ into B will keep $|G_n \cap \bar{B}|$ at least as big as when we would enumerate $V^{1-i_{j'}}_{e_{j'}} \cap G_n$ into B (for the precise definition of the inductive step of $\sigma(\alpha, n, s)$, see (4.7) below). As we will show, this ensures that the inner clause of (4.2) holds for any n such that G_n is never deleted.

Finally, in order to guarantee that requirement $\mathcal{Q}_e$ is met it suffices to ensure that (in the limit) almost all of the states $\sigma(\alpha, n, s)$ of intervals G_n which are permanently assigned to ndoes α extending $TP \restriction e+1$ agree on the first $k'+1$ arguments where $k'+1 = |\{e' \leq e : TP(e') = 0\}|$. Namely, for infinitary $\mathcal{Q}_e$ this ensures that there is $i \leq 1$ such that $\sigma(\alpha, n, s)(k') = i$ in almost all of the cases above whence $V^i_e \cap G_n \subseteq B$ for almost all undeleted intervals G_n; hence, $V^i_e \subseteq^* B$.

Now the states can be unified in the above way as follows. Whenever intervals G_n and $G_{n'}$ are assigned to α and β, respectively, where $\alpha < \beta$, $|\alpha| < |\beta|$ and $\sigma(\alpha, n', s) <_{left} \sigma(\alpha, n, s)$ holds then the interval $G_{n'}$ is assigned to α in place of G_n. Note that this replacement must be done even if α is to the left of δ_s. For the formal definition of the states we first introduce an auxiliary notion.

For finite subsets $E, F \subseteq \omega$ the *density of E inside F*, denoted by $\rho(E, F)$, is defined as

$$\rho(E,F) = \frac{|E \cap F|}{|F|}, \tag{4.6}$$

where we set $\rho(E,\emptyset) = 0$. Then given $\alpha \in T$, stage s and $n \in \omega$, we let $\sigma(\alpha,n,s)$ denote the *α-state of G_n at stage s* and define it to be the longest string $\sigma \in \{0,1\}^{<\omega}$ such that $|\sigma| \leq |\{e < |\alpha| : \ \alpha(e) = 0\}|$ and, for all $j < |\sigma|$ such that $\alpha(e_j) = 0$ (where e_j is the $(j+1)$st infinitary edge of α in order of magnitude), $l(e_j, s) > \max(G_n)$ holds and $\sigma(j)$ is the least $i \leq 1$ such that

$$\rho(V^i_{e_j,s}, \bar{B}_s \cap G_n \setminus \bigcup_{l<j} V^{\sigma(l)}_{e_l,s}) \leq \frac{1}{2}. \tag{4.7}$$

We let

$$V_{\sigma(\alpha,n,s)} = G_n \cap \bigcup_{j<|\sigma(\alpha,n,s)|} V^{\sigma(\alpha,n,s)(j)}_{e_j,s}. \tag{4.8}$$

Then the construction is as follows.

Construction of B.

Stage 0. $B_0 = \emptyset$.

Stage $s+1$. Let B_s be given. We say that a node α *requires attention* at stage $s+1$ if $|\alpha| \leq s$ and either

(i) $\alpha \sqsubseteq \delta_s$ and no interval is assigned to α, or

(ii) $\alpha \leq \delta_s$, G_n is assigned to α and $\sigma(\alpha,n,s) \sqsupset \sigma(\alpha,n,s-1)$, or

(iii) $\alpha \leq \delta_s$, G_n is assigned to α, (ii) does not hold and there exists $\beta > \alpha$ and n' such that $|\beta| > |\alpha|$, $G_{n'}$ is assigned to β and $\sigma(\alpha,n',s) <_{left} \sigma(\alpha,n,s)$.

Let α be the node of highest priority which requires attention at stage $s+1$. Say that α *receives attention* and *acts via* the clause via which α requires attention.

If (i) holds, assign G_s to α at stage $s+1$.

If (ii) holds, enumerate all of $V_{\sigma(\alpha,n,s)}$ into B at stage $s+1$.

If (iii) holds, let β be the highest priority node which makes (iii) true and let $G_{n'}$ be its assigned interval. Cancel the assignment of G_n to α, assign $G_{n'}$ to α and enumerate all of $V_{\sigma(\alpha,n',s)}$ into B at stage $s+1$.

At stage $s+1$, *initialize* all nodes $\beta > \alpha$, i.e., cancel their assigned interval (if any). After α has received attention and the assignment of intervals to nodes has been declared at stage $s+1$, for all $n \leq s$, do the following: if G_n is not assigned to any node at stage $s+1$, *delete* G_n at stage $s+1$, i.e., enumerate all of G_n into B at stage $s+1$.

This ends the formal construction.

Verification

We prove in a series of claims that the so constructed set B has the required properties. Before, let us give some general remarks about the construction which we will tacitly use in the proofs below. Unless otherwise stated, they can be easily shown by induction on the stage s.

The construction is effective and $\{B_s\}_{s\in\omega}$ is a computable enumeration of B; hence, B is a c.e. set. At any stage s, there is a unique node $\alpha \leq \delta_s$ which receives attention at stage $s+1$. For all nodes α and stages s, α is assigned at most one interval at stage s, if G_n is the interval that is assigned to α at stage s then $|\alpha| \leq n < s$ and if α gets G_n assigned via (i) at stage $s+1$ then $n = s$ and $G_n \cap B_s = \emptyset$ since all intervals that are assigned to nodes by stage s have index less than s.

Moreover, the assignment of intervals to nodes is nondecreasing in s and strictly increasing with respect to the priority ordering, i.e., if $\alpha < \beta$, α is assigned G_n and β is assigned $G_{n'}$ at stage s then $n < n'$. Furthermore, $\sigma(\alpha, n, s) \sqsubseteq \sigma(\beta, n, s)$ holds whenever $\alpha \sqsubseteq \beta$ holds and if G_n is assigned to α at stage $s+1$ then

$$\sigma(\alpha, n, s) \sqsubseteq \sigma(\alpha, n, s+1). \tag{4.9}$$

Hence, $V_{\sigma(\alpha,n,s)} \subseteq V_{\sigma(\alpha,n,s+1)}$ in this case. A proof of (4.9) is given in Claim 1.

Finally, and importantly, for any α and $n, s \in \omega$, if α gets G_n assigned at stage $s+1$ then G_n has not been deleted at any stage $t \leq s$. In particular, if G_n is never deleted then from stage $n+1$ on, it is always assigned to a node and it is eventually permanently assigned to a node which is on or to the left of the true path. Now as first claim states we prove that (4.9) holds.

Claim 1. For all nodes α and all $n, s \in \omega$, if G_n is assigned to α at stage $s+1$ then $\sigma(\alpha, n, s) \sqsubseteq \sigma(\alpha, n, s+1)$ holds.

Proof. First, note that by monotonicity of the length of agreement function $l(e, s)$ in s for all e, $|\sigma(\alpha, n, s)| \leq |\sigma(\alpha, n, s+1)|$ holds. So it suffices to show that $\sigma(\alpha, n, s+1)(j) = \sigma(\alpha, n, s)(j)$ holds for all $j < |\sigma(\alpha, n, s)|$. We show the latter by induction on j. Note that the claim trivially holds if $B_{s+1} \cap G_n = B_s \cap G_n$. So w.l.o.g., we may assume that $B_{s+1} \cap G_n \neq B_s \cap G_n$ holds. Then since α is the only node which may enumerate numbers into G_n, by construction, α requires attention at stage $s+1$ via (ii) or (iii). In both cases, $B_{s+1} = B_s \cup V_{\sigma(\alpha,n,s)}$. Now fix $j < |\sigma(\alpha, n, s)|$ and, by inductive hypothesis, assume that $\sigma(\alpha, n, s) \restriction j = \sigma(\alpha, n, s+1) \restriction j$ holds. We have to show that $\sigma(\alpha, n, s+1)(j) = \sigma(\alpha, n, s)(j)$ holds. Let e_j be the $(j+1)$st infinitary edge of α in order of magnitude. In particular, $\sigma(\alpha \restriction e_j, n, s) = \sigma(\alpha \restriction e_j, n, s+1)$ holds by inductive hypothesis since $\sigma(\alpha, n, t) \restriction j = \sigma(\alpha \restriction e_j, n, t)$ holds for all stages t. Then by definition of $\sigma(\alpha, n, s)$, $l(e_l, s) > \max(G_n)$ holds for all $l \leq j$. Hence, as this implies $V^i_{e_l,s} \cap G_n = V^i_{e_l,s+1} \cap G_n$ for all $i \leq 1$ and $l \leq j$, we may deduce that

$$
\begin{aligned}
&\rho(V^i_{e_j,s+1}, \bar{B}_{s+1} \cap G_n \setminus \bigcup_{l<j} V^{\sigma(l)}_{e_l,s+1}) \\
&= \rho(V^i_{e_j,s+1}, \bar{B}_{s+1} \cap G_n \setminus V_{\sigma(\alpha \restriction e_j,n,s+1)}) \\
&= \rho(V^i_{e_j,s}, \bar{B}_{s+1} \cap G_n \setminus V_{\sigma(\alpha \restriction e_j,n,s)}) \\
&= \rho(V^i_{e_j,s}, \bar{B}_s \cap \bar{V}_{\sigma(\alpha,n,s)} \cap G_n \cap \bar{V}_{\sigma(\alpha \restriction e_j,n,s)}) \\
&= \rho(V^i_{e_j,s}, \bar{B}_s \cap G_n \cap \bar{V}_{\sigma(\alpha \restriction e_j,n,s)}) \\
&= \rho(V^i_{e_j,s}, \bar{B}_s \cap G_n \setminus \bigcup_{l<j} V^{\sigma(l)}_{e_l,s}),
\end{aligned}
$$

where the second equality holds by inductive hypothesis and the fourth equality holds since $V_{\sigma(\alpha\restriction e_j,n,s)} \subseteq V_{\sigma(\alpha,n,s)}$. Hence, $\sigma(\alpha,n,s+1)(j) = \sigma(\alpha,n,s)(j)$ follows by definition of $\sigma(\alpha,n,s)$. □

Then we can prove that all nodes on or to the left of the true path act only finitely often and that the former eventually get a permanent interval assigned. Fix e in the following.

Claim 2. Every $\alpha \leq TP \restriction e$ requires attention only finitely often and if $\alpha = TP \restriction e$ then α eventually gets a permanent interval assigned.

Proof. Let $\alpha \leq TP \restriction e$ be given and, for any stage s let $P(\alpha, s_0)$ be the property

If α is not initialized after stage s_0

then

α requires attention only finitely often.

First, we show that $P(\alpha, s_0)$ holds for any stage s_0. Let s_0 be a stage such that α is not initialized after stage s_0. Then every time α requires attention after stage s_0 it receives attention and acts. Now α requires attention via (i) at most once after stage s_0 because once α gets an interval assigned after stage s_0 it is always assigned an interval since α is not initialized. Moreover, if α does not require attention via (i) after stage s_0 it does not require attention at all since requiring attention via (ii) or (iii) assumes that there is an interval assigned to α. So w.l.o.g. we may assume that there is an interval which is assigned to α at stage s_0. For $s \geq s_0$ let $n(\alpha, s)$ be the index of th currently assigned interval to α at stage s.

Then α may require attention after stage s_0 only via (ii) or (iii). Now by (4.9), we claim that α may require attention via (iii) at most $2^{|\alpha|}$ times. For that purpose, let $\sigma^*(\alpha, s)$ be the binary string of length $|\alpha|$ extending $\sigma(\alpha, n(\alpha, s), s)$ such that, for all $|\sigma(\alpha, n(\alpha, s), s)| \leq j < |\alpha|$, it holds that $\sigma^*(\alpha, s)(j) = 1$. By (4.9) and by Claim 1, for all stages $s \geq s_0$ such that $n(\alpha, s) = n(\alpha, s+1)$, $\sigma^*(\alpha, s+1) \leq \sigma^*(\alpha, s)$ holds. Otherwise, if $n(\alpha, s) \neq n(\alpha, s+1)$ then by construction, α requires attention via (iii) at stage $s+1$. In this case, $\sigma^*(\alpha, s+1) < \sigma^*(\alpha, s)$. Whence,

$\sigma^*(\alpha, s)$ is nondecreasing in s. But since it can decrease only $2^{|\alpha|}$ many times, we conclude that the number of times α may require attention via (iii) is also bounded by $2^{|\alpha|}$. So fix the last stage $s_1 \geq s_0$ such that α requires attention via (iii). Then since $|\sigma(\alpha, n, s)| \leq |\alpha|$, α may require attention via (ii) at most $|\alpha|$ many times after stage s_1. This shows that $P(\alpha, s_0)$ holds.

So for a proof that α requires attention only finitely often it suffices to show that α is initialized only finitely often. By construction, α may be initialized only by a node $\beta < \alpha$ and only if β is ever accessible in the course of the construction. Now since $\alpha \leq TP \upharpoonright e$, there are only finitely stages s such that $\delta_s <_{left} \alpha$. Hence, there are only finitely many nodes $\beta < \alpha$ that may initialize α in the course of the construction. Let n be the number of all these nodes and order them by priority, say $\beta_0 < \beta_1 < \cdots < \beta_n = \alpha$. But now the claim follows by an easy induction on $k \leq n$ using the fact that β_0 is never initialized.

Finally, if $\alpha = TP \upharpoonright e$ then α is accessible infinitely often. So if s_0 is a stage such that α is not initialized after stage s_0 then either α is already assigned an interval at stage s_0, or, for the least stage $s_1 \geq s_0$ where α is accessible, α is assigned an interval at stage s_1. As argued above, this implies that α is assigned an interval at any stage $s \geq s_1$. Hence, if $s_2 \geq s_1$ is the last stage where α requires attention then the interval that is assigned to α at stage s_2 is permanently assigned to α. □

In Claim 4, we show that $\bar{B}$ is infinite by proving that the inner clause of (4.2) holds for all intervals G_n that are never deleted during the course of the construction. Note that there are infinitely many such intervals. Namely, by Claim 2, any node $\alpha \sqsubset TP$ gets a permanent interval assigned which by construction is never deleted and different from intervals which are permanently assigned to proper initial segments of α. The proof of Claim 4 is based on a technical claim.

Claim 3. Let α and $n, s \in \omega$ be given. Then $\rho(V_{\sigma(\alpha,n,s)}, G_n \cap \bar{B}_s) \leq 1 - \left(\frac{1}{2}\right)^{|\sigma(\alpha,n,s)|}$. In particular, if G_n is assigned to α at stage $s+1$ then $|G_n \cap \bar{B}_{s+1}| \geq 2^{-|\sigma(\alpha,n,s)|}|G_n \cap \bar{B}_s|$.

Proof. Note that the second part of the claim follows from the first one. Namely, by (4.6), it suffices to show that $\rho(\bar{B}_{s+1}, G_n) \geq 2^{-|\sigma(\alpha,n,s)|}\rho(\bar{B}_s, G_n)$. For that, we distinguish between the following two cases. If $B_{s+1} \cap G_n = B_s \cap G_n$ then the second part of the claim holds trivially. Otherwise, $B_{s+1} = B_s \cup V_{\sigma(\alpha,n,s)}$ holds by construction. Hence, by the first part of the claim,

$$\begin{aligned}
\rho(\bar{B}_{s+1}, G_n) &= 1 - \rho(B_{s+1}, G_n) \\
&= 1 - \rho(B_s \cup V_{\sigma(\alpha,n,s)}, G_n) \\
&= 1 - \rho(B_s, G_n) - (1 - \rho(B_s, G_n)) \cdot \rho(V_{\sigma(\alpha,n,s)}, \bar{B}_s \cap G_n) \\
&= (1 - \rho(V_{\sigma(\alpha,n,s)}, \bar{B}_s \cap G_n)) \cdot \rho(\bar{B}_s, G_n) \\
&\geq 2^{-|\sigma(\alpha,n,s)|} \cdot \rho(\bar{B}_s, G_n),
\end{aligned}$$

where, for the third equality, we use that $\rho(A \cup B, C) = \rho(A, C) + (1 - \rho(A, C)) \cdot \rho(B, C \setminus A)$ holds for arbitrary finite sets A, B and C.

The proof of the first part is by induction on $m = |\alpha|$. The claim holds for $m = 0$ since $V_{\sigma(\lambda,n,s)} = \emptyset$. So by inductive hypothesis, let $m \in \omega$ be given and assume that Claim 3 holds for all α of length m. Fix a node α of length $m+1$ and let $\alpha' = \alpha \restriction m$. Then if $\alpha(m) = 1$ or $|\sigma(\alpha, n, s)| < |\{e < |\alpha| : \alpha(e) = 0\}|$, it follows that $\sigma(\alpha, n, s) = \sigma(\alpha', n, s)$. So the claim follows by inductive hypothesis.

Otherwise, $\alpha(m) = 0$ and $|\sigma(\alpha, n, s)| = |\{e < |\alpha| : \alpha(e) = 0\}|$. Let $i = \sigma(\alpha, n, s)(m)$ and e_m be the $(m+1)$st infinitary edge of α in order of magnitude. Thus, by definition σ, $l(m, s) > \max(G_n)$ and $\sigma(\alpha, n, s) = \sigma(\alpha', n, s)i$. Then we observe that

$$\begin{aligned}
&\rho(V_{\sigma(\alpha,n,s)}, \bar{B}_s \cap G_n) \\
&= \rho(V_{\sigma(\alpha',n,s)}, \bar{B}_s \cap G_n) + (1 - \rho(V_{\sigma(\alpha',n,s)}, \bar{B}_s \cap G_n)) \cdot \rho(V^i_{e_m,s}, \bar{B}_s \cap G_n \backslash V_{\sigma(\alpha',n,s)}) \\
&\leq 1 - \left(\frac{1}{2}\right)^{|\sigma(\alpha',n,s)|} + \left(\frac{1}{2}\right)^{|\sigma(\alpha',n,s)|} \cdot \frac{1}{2} \\
&= 1 - \left(\frac{1}{2}\right)^{|\sigma(\alpha,n,s)|},
\end{aligned}$$

where we argue that the above inequality holds as follows. Let $f : [0,1] \times [0,1] \to [0,1]$ be defined by $f(x, y) = x + (1 - x)y$, where $[0,1]$ denotes the unit interval of the reals numbers. Then f is nondecreasing in the sense that $f(x,y) \le f(x',y')$ holds for all pairs $(x,y),(x',y') \in [0,1]\times[0,1]$ such that $x \le x'$ and $y \le y'$. Hence, by setting $x = \rho(V_{\sigma(\alpha',n,s)}, \bar{B}_s \cap G_n)$, $x' = 1 - \left(\frac{1}{2}\right)^{|\sigma(\alpha',n,s)|}$, $y = \rho(V^i_{m,s}, \bar{B}_s \cap G_n \setminus V_{\sigma(\alpha',n,s)})$ and $y' = \frac{1}{2}$, we infer that $x \le x'$ holds by inductive hypothesis and $y \le y'$ holds by definition of $\sigma(\alpha,n,s)$. □

Now we are ready to prove that the inner clause of (4.2) holds for all intervals G_n that are never deleted in the course of the construction.

Claim 4. For all n, if G_n is never deleted then $|G_n \cap \bar{B}| \ge (n+1)^2$.

Proof. Let $n \in \omega$ be given such that G_n is never deleted. By construction, for any stage $s \ge n$, there is a node γ such that G_n is assigned to γ at stage $s+1$. Moreover, if G_n is assigned to γ at stage s and to γ' at stage $s+1$ then γ' requires attention via (iii) which implies that $\gamma' < \gamma$ and $|\gamma'| < |\gamma|$. So we can argue that there exists $k \in \omega$ and a sequence of stages $s_0 = n < s_1 < \cdots < s_k$ and nodes $\gamma_0 > \gamma_1 > \cdots > \gamma_k$ such that $G_n \cap B_{s_0} = \emptyset$, G_n is assigned to γ_k at any stage $s > s_k$ and, moreover, for any $i < k$, G_n is assigned to γ_i at stage $s+1$ for any $s \in [s_i, s_{i+1})$ and

$$|\gamma_i| > |\gamma_{i+1}|, \tag{4.10}$$

$$G_n \cap B_{s_{i+1}} \subseteq (G_n \cap B_{s_i}) \cup V_{\sigma(\gamma_i,n,s_{i+1})} \tag{4.11}$$

hold. Note that $k \le n$ since $k \le |\gamma_0|$ by (4.10) and $|\gamma_0| \le n$ since G_n is assigned to γ_0 at stage s_0+1. In particular, $|\gamma_i| \le n-i$ by (4.10) for all $i \le k$. Furthermore, γ_k must be on or to the left of the true path. Otherwise, γ_k would be initialized after stage s_k and G_n would be deleted. So by Claim 2, fix the least stage $s_{k+1} > s_k$ such that γ_k does not require attention after stage s_{k+1}. Then $G_n \cap B = G_n \cap B_{s_{k+1}}$ and $G_n \cap B_{s_{k+1}} \subseteq (G_n \cap B_{s_k}) \cup V_{\sigma(\beta,n,s_{k+1})}$. We claim that

$$|G_n \cap \bar{B}_{s_{i+1}}| \ge 2^{-|\gamma_i|}|G_n \cap \bar{B}_{s_i}| \tag{4.12}$$

holds for all $i \le k$. So let $i \le k$ be given. Then if $G_n \cap B_{s_{i+1}} = G_n \cap B_{s_i}$ holds then (4.12) holds trivially. Otherwise, by Claim 1 and since $\sigma(\alpha,n,s) \sqsubseteq \sigma(\alpha,n,s+1)$

implies that $V_{\sigma(\alpha,n,s)} \subseteq V_{\sigma(\alpha,n,s+1)}$ holds we may assume that, for all stages $s \in [s_i, s_{i+1} - 1)$, it holds that $G_n \cap B_s = G_n \cap B_{s+1}$ and that $G_n \cap B_{s_{i+1}} = (G_n \cap B_{s_i}) \cup V_{\sigma(\alpha,n,s_{i+1})}$ holds. So (4.12) holds by Claim 3 and by the fact that $|\sigma(\gamma_i, n, s_{i+1})| \leq |\gamma_i|$ holds.

However, since $G_n \cap B_{s_0} = \emptyset$, this yields

$$|G_n \cap \bar{B}| \geq 2^{-(|\gamma_k|+|\gamma_{k-1}|+\cdots+|\gamma_0|)}|G_n| \geq (n+1)^2.$$

This completes the proof. □ □

Finally, we show that all $\mathcal{Q}$-requirements are met.

Claim 5. $\mathcal{Q}_e$ is met.

Proof. If the hypothesis of $\mathcal{Q}_e$ does not hold, $\mathcal{Q}_e$ is trivially met. So we may assume that $\mathcal{Q}_e$ is infinitary, i.e., $TP(e) = 0$ holds. We have to show that there exists $i \leq 1$ and $m \in \omega$ such that for all $n \geq m$, $V_e^i \cap G_n \subseteq B$. For the proof of this statement, suppose that $e = e_k$ where e_k denotes the $(k+1)$st infinitary edge of TP in order of magnitude and let M be the set of all nodes $\beta \sqsupseteq TP \upharpoonright e+1$ which eventually get a permanent interval assigned. For $\beta \in M$, let n_β denote the index of the last interval assigned to β and let $\sigma(\beta, n_\beta) = \lim_{s\to\infty} \sigma(\beta, n_\beta, s)$. Note that $\sigma(TP \upharpoonright e+1, n_\beta, s) \sqsubseteq \sigma(\beta, n_\beta, s)$; hence, $\sigma(TP \upharpoonright e+1, n_\beta) = \lim_{s\to\infty} \sigma(TP \upharpoonright e+1, n_\beta, s)$ exists, too and $\sigma(TP \upharpoonright e+1, n_\beta) \sqsubseteq \sigma(\beta, n_\beta)$. Moreover, $|\sigma(TP \upharpoonright e+1, n_\beta)| = k+1$ since $TP \upharpoonright e+1$ lies on the true path; hence, $|\sigma(\beta, n_\beta)| \geq k+1$ for any $\beta \in M$. Now let τ be the leftmost binary string of length $k+1$ such that $\tau \sqsubseteq \sigma(\beta, n_\beta)$ for infinitely many $\beta \in M$. We claim that

$$\forall^\infty \beta \in M \ (\tau \sqsubseteq \sigma(\beta, n_\beta)) \tag{4.13}$$

holds and that (4.13) suffices to prove Claim 5. First, assume that (4.13) holds. Let $\beta_0 \in M$ be such that (4.13) holds for all $\beta \in M$ with $\beta \geq \beta_0$. Let $m = n_{\beta_0}$ and $i = \tau(k)$. We claim that, for all $n \geq m$, $V_e^i \cap G_n \subseteq B$ holds. Fix $n \geq m$. We distinguish between the following two cases. If G_n is eventually deleted then $G_n \subseteq B$. So the claim trivially holds for n. Otherwise, G_n is never deleted. Then by construction, there is a unique $\beta \in M$ such that $n = n_\beta$. Namely, β must be on or to the left of the true path. But by choice of m, β cannot be to the left of $TP \upharpoonright e+1$ nor can $\beta < \beta_0$ hold because, by construction, $n_\beta < m$ holds in both cases. Thus, by (4.13), $\sigma(\beta, n_\beta)(k) = i$. So, by (4.9) and since $\beta \sqsupseteq TP \upharpoonright e+1$, β eventually enumerates $V_{\sigma(\beta,n_\beta)} \supseteq V_e^i \cap G_n$ into B.

Thus, to complete the proof, we show that (4.13) holds. By definition of τ, let $\beta_0' \in M$ be such that, for all $\beta \in M$ with $|\beta| \geq |\beta_0'|$, $\tau \leq \sigma(TP \upharpoonright e+1, n_\beta)$ holds. Fix any such node β and, by Claim 2, fix s_0 such that β does not require attention after stage s_0. In particular, $\sigma(\beta, n_\beta) = \sigma(\beta, n_\beta, s)$ for all $s \geq s_0$. We claim that $\sigma(\beta, n_\beta) \sqsupseteq \tau$. Otherwise, $\tau <_{left} \sigma(\beta, n_\beta)$ by choice of β. By choice of τ, let $\beta' \in M$ with $|\beta'| > |\beta|$ and $\beta' > \beta$ be such that $\sigma(\beta', n_{\beta'}) \sqsupseteq \tau$ and let $s_1 > s_0$ be a stage such that $\sigma(\beta', n_{\beta'}, s_1) = \sigma(\beta', n_{\beta'})$. Such a β' exists because by construction, there are only finitely many nodes in M which have higher priority than β. But then β requires attention via (iii) at stage $s_1 + 1$, contrary to choice of s_0. $\square$

By Claim 5, this completes the proof of Lemma 4.3.1 and hence of Theorem 4.1.1.

Chapter 5

Eventually Uniformly Weak Truth-Table Array Computability

5.1 Introduction

By a result of [BDG10], the class of the c.e. not totally ω-c.e. degrees coincides with the class of degrees that contain a c.e. set which is not wtt-reducible to any hypersimple set. In view of the fact that the c.e. not totally ω-c.e. degrees are a (proper) subclass of the a.n.c. degrees, Ambos-Spies asked what kind of simplicity notion is needed to characterize the a.n.c. Turing degrees (wtt-degrees or even sets) in this way (for an overview of the the classical simplicity notions and the relations between them, see Fig. 4.1)? One result pointing in this direction is given by Ambos-Spies [AS18, Theorem 3] showing that no a.n.c. set can be wtt-below a dense simple. In particular, we may consider the question whether the a.n.c. sets can be characterized by being not wtt-reducible to any dense simple set. Here, we give a negative answer to this question which we provide in two steps. First, we give a characterization of the c.e. sets that are wtt-reducible to dense simple sets by introducing the notion of *eventually uniformly weak truth-table array computable sets* (*e.u.wtt-a.c. sets* for short) and by showing that the c.e. sets with this property are precisely those that are wtt-reducible to dense simple sets. In fact, we show that we get the same characterization when replacing wtt-reducibility by ibT-reducibility and dense simple sets by maximal sets (and hence the same holds by inserting any simplicity notion between maximal and dense simple and any reducibility between ibT- and wtt-reducibility). Second, we give strict lower and upper bounds for the class of the c.e. e.u.wtt-a.c. sets. For the lower bound we introduce the notion of *wtt-superlow sets*. They are analogously defined as the superlow sets – a notion which goes back to work of Mohrherr [Moh86] and Bickford and Mills [BM82] – but with the bounded jump in place of the Turing jump and we show that any wtt-superlow set is e.u.wtt-a.c. (but not vice versa). For the upper bound we show that any c.e. e.u.wtt-a.c. set has array computable wtt-degree (but not vice versa). Indeed, we can show that there is c.e. Turing degree that contains a set which is not e.u.wtt-a.c. Furthermore, we show that there is a hierarchy of within the c.e. wtt-superlow sets and that there are sets which lie in the intersection of this hierarchy (so called *strongly wtt-superlow sets*). In fact, we show that Turing complete strongly wtt-superlow sets exist.

In order to define e.u.wtt-a.c. and wtt-superlow sets, we need the notion of the bounded jump operator. It is defined as an operator which is akin to the classical Turing jump but using wtt-functionals in place of Turing ones. Analogs of the Turing jump operator which are defined using only strong reducibilities are already studied by Ershov [Ers70], by Gerla [Ger79] and by Coles, Downey and LaForte in [CDL98]. However, the bounded jump of a set as we consider it here was firstly deeply investigated by Anderson and Csima in [AC14]. On the one hand, Anderson and Csima demonstrate that the bounded jump shares many of the properties that the Turing jump operator has but with respect to wtt-reductions. For instance, it is order preserving, no set is wtt-reducible to its bounded jump and the bounded jump of the empty set is computably isomorphic to the halting problem. Moreover, transfinite iterations of the bounded jump and its close relationship to the Ershov hierarchy are studied by Anderson in Csima in [AC14], by Coles, Downey and LaForte in [CDL98] and by Downey and Greenberg in [DG19]. Note that, in the former two cases, the corresponding authors use Kleene's ordinal notation as a recursive approach to ordinals above ω, while, in the latter case, Downey and Greenberg introduce so called canonical ordinals, i.e., ordinals whose Cantor normal form is computable. On the other hand, Anderson, Csima and Lange consider the analog of the low/high hierarchy for the bounded jump operator (which they call the bounded low/high hierarchy, respectively) and demonstrate in [ACL17] that the bounded jump and the Turing jump differ in this respect by showing the existence of both a low set which is bounded high and high set which is bounded low.

All the results of this chapter are included in a paper that is in preparation by Ambos-Spies, Downey and Monath [ASDM19]. The outline of this chapter is as follows. In Section 5.2, we give the definition of the bounded jump w.r.t. to a computable enumeration of all wtt-functionals and show that the analog of the Recursion Theorem (with Parameters) holds for the enumeration of the wtt-functionals as we define it. In Section 5.3, we give the definition of e.u.wtt-a.c. sets and we state and prove the main result of this chapter, namely that e.u.wtt-a.c. sets are characterized by being wtt- (ibT-) reducible to dense simple (maximal) sets (Theorem 5.3.2). In Section 5.4, we show that the c.e. wtt-degrees containing e.u.wtt-a.c. sets form an ideal in the c.e. wtt-degrees. In Section 5.5, we introduce the notion of wtt-superlow sets. We show that they form a subclass of the e.u.wtt-a.c. sets which immediately follows from the result that the wtt-superlow coincide with the sets which are ω-computably approximable (ω-c.a.) (Theorem 5.5.2; this result also holds for not necessarily c.e. sets). Thereby, we show that the wtt-superlow sets also coincide with the bounded low sets as defined in [ACL17]. In Subsection 5.5.2, we extend Theorem 5.5.2 by introducing the notion of *wtt-jump traceable sets* (Theorem 5.5.6; however, in this case, the equivalence only holds for c.e. sets). In Subsection 5.5.3, we show that there is a hierarchy of wtt-superlow sets (Theorem 5.5.10). In Subsection 5.5.4, we look at strong variants of wtt-superlow sets and wtt-jump traceable sets. We show that they are equivalent and that such sets exist (Theorem 5.5.13); in fact, we construct a Turing complete set with this property. In Section 5.6, we show that every e.u.wtt-a.c. set has array computable wtt-degree (Theorem 5.6.2). Finally, in Section 5.7, we state (without proof) that the lower and the upper bound of the c.e. e.u.wtt-a.c. sets (given by the c.e. wtt-superlow sets and the c.e. sets having array computable wtt-degree, respectively) are strict (Theorems 5.7.1 and 5.7.3, respectively). In the latter case, we have a slightly stronger result.

5.2 Preliminaries

Let us start by giving the definition of the bounded jump. The underlying notation is mostly adapted from [DG19].

Definition 5.2.1 ([DG19]). *For every set $X \subseteq \omega$ for any numbers $e_0, e_1, y \in \omega$, we define*

$$\hat{\Phi}^X_{\langle e_0, e_1 \rangle}(y) = \begin{cases} \Phi^X_{e_0}(y) & \text{if } \Phi^X_{e_0}(y)\downarrow,\ \varphi_{e_1}(y)\downarrow \text{ and } \varphi^X_{e_0}(y) \leq \varphi_{e_1}(y), \\ \uparrow & \text{otherwise,} \end{cases} \tag{5.1}$$

$$\hat{\varphi}_{\langle e_0, e_1 \rangle} = \varphi_{e_1}. \tag{5.2}$$

Given a set A, the (diagonal) bounded jump *and the* bounded jump function *of A, denoted by $A^\dagger$ ($A^\dagger_d$) and $\hat{J}^A$, respectively, are defined as*

$$A^\dagger = \{\langle e, x \rangle : \hat{\Phi}^A_e(x)\downarrow\}, \tag{5.3}$$

$$A^\dagger_d = \{e : \hat{\Phi}^A_e(e)\downarrow\}, \text{ and} \tag{5.4}$$

$$\hat{J}^A(e) = \hat{\Phi}^A_e(e). \tag{5.5}$$

For notational conveniences, we define the bounded jump $A^\dagger$ of a set A such that $A^\dagger$ codes all computations of partial wtt-functionals instead of only the diagonal computations, the latter one being denoted by $A^\dagger_d$. However, it is easy to see that $A^\dagger$ and $A^\dagger_d$ are computably isomorphic (see 3. of Lemma 5.2.4). Before we start examining some of the properties of $A^\dagger$ and $\hat{J}^A$ for a (c.e.) set A, let us give some general remarks on the definition of $\hat{\Phi}_e$ and introduce some terminology to be used below which is also mostly taken from [DG19]. First of all, we say that a Turing functional Φ is a *wtt-functional* if there exists a number $e \in \omega$ such that $\Phi = \hat{\Phi}_e$. Note that, for any set A and any total function g, $g \leq_{wtt} A$ holds iff there exists $e \in \omega$ such that $g = \hat{\Phi}^A_e$. So $\{\hat{\Phi}_e\}_{e \in \omega}$ incorporates all wtt-reductions.

Using $\{\hat{\Phi}_e\}_{e \in \omega}$, we may extend the definition of being wtt-reducible to a set A to partial functions. We say that a partial function $\varphi : \omega \to \omega$ is *wtt-reducible* to a set A, and denote it by $\varphi \leq_{wtt} A$, if there exists $e \in \omega$ such that $\varphi = \hat{\Phi}^A_e$. Furthermore, we say for sets A and B that A is *bounded computably enumerable* *in* B, *bounded c.e. in* B or *bounded* B*-c.e.* for short, if there exists a partial function φ which is wtt-reducible to B and such that $A = \mathrm{dom}(\varphi)$. In particular, $A^\dagger$ is bounded c.e. in A for all sets A.

We fix computable approximations $\hat{\Phi}^X_{\langle e_0,e_1\rangle,s}(y)$ $(s \geq 0)$ of $\hat{\Phi}^X_{\langle e_0,e_1\rangle}(y)$ where $\hat{\Phi}^X_{\langle e_0,e_1\rangle,s}(y)$ is defined iff $\hat{\Phi}^X_{\langle e_0,e_1\rangle}(y)$, $\Phi^X_{e_0,s}(y)$ and $\varphi_{e_1,s}(y)$ are defined. Then, for any c.e. set A and any fixed computable enumeration $\{A_s\}_{s\in\omega}$ of A, we have a canonical approximation to $A^\dagger$, denoted by $\{A^\dagger_s\}_{s\in\omega}$, such that, for all numbers e, x, it holds that $\langle e, x\rangle \in A^\dagger_s$ iff $\hat{\Phi}^A_e(x)[s]\downarrow$. We tacitly assume that this approximation to $A^\dagger$ is clear from the context whenever a c.e. set A and a computable enumeration of A is given to or constructed by us. Note that if $\hat{\Phi}^A_e(x)[s]\downarrow$ holds for infinitely many stages s then $\hat{\Phi}^A_e(x)\downarrow$ holds as the use of $\hat{\Phi}_e$ is bounded (this does not hold for Turing functionals in general).

Moreover, we will often make use of the Recursion Theorem (with Parameters) with respect to $\{\hat{\Phi}_e\}_{e\in\omega}$. For that, we need the following definition.

Definition 5.2.2. *A sequence of wtt-functionals* $\{\Psi_e\}_{e\in\omega}$ *is* uniformly computable *if* $\{\Psi_e\}_{e\in\omega}$ *is uniformly computable in the sense of Turing functionals and there exists a uniformly computable sequence of partial computable functions* $\{\psi_e\}_{e\in\omega}$ *such that, for any* $e \in \omega$*, the use of* Ψ_e *is bounded by* ψ_e*.*

Then the following lemma says that $\{\hat{\Phi}_e\}_{e\in\omega}$ is a Gödel numbering of the wtt-functionals so we may argue as in the proof of the classical Recursion Theorem (with Parameters) that the Recursion Theorem also holds for uniformly computable sequences of wtt-functionals.

Lemma 5.2.3 (Recursion Theorem (with Parameters))**.** *Let* $\{\Psi_e\}_{e\in\omega}$ *be a sequence of wtt-functionals and* $g : \omega \to \omega$ *and* $H : \omega^2 \to \omega$ *be total computable functions. Then the following holds.*

1. $\{\Psi_e\}_{e\in\omega}$ *is uniformly computable iff there exists a computable one-one function* $f : \omega \to \omega$ *such that* $\Psi^A_e = \hat{\Phi}^A_{f(e)}$ *holds for any number* e *and any set* A*.*

2. *There exists* $e \in \omega$ *such that* $\hat{\Phi}_{g(e)} = \hat{\Phi}_e$*.*

3. *There exists a computable function* $h : \omega \to \omega$ *such that* $\hat{\Phi}_{h(e)} = \hat{\Phi}_{H(h(e),e)}$ *holds for any* $e \in \omega$*.*

Proof. For the "only if"-part of 1., note that a sequence $\{\hat{\Phi}_{f(e)}\}_{e\in\omega}$, where $f:\omega\to\omega$ is a computable function is a uniformly computable sequence of wtt-functionals since the use bound $\{\hat{\varphi}_{f(e)}\}_{e\in\omega}$ is a uniformly computable sequence of partial computable functions. For the "if"-direction, by Definition 5.2.2, we may fix computable one-one functions $f_i:\omega\to\omega$ ($i\leq 1$) such that, for any $e\in\omega$, it holds that $\Psi_e=\Phi_{f_0(e)}$ and $\psi_e=\varphi_{f_1(e)}$. Then, by (5.1) and by assumption on Ψ_e, it holds that $\Psi_e=\hat{\Phi}_{f(e)}$ for the computable one-one function $f(e)=\langle f_0(e),f_1(e)\rangle$.

For the proofs of 2. and 3., it is easy to see that the proofs of the Recursion Theorem and the Recursion Theorem with Parameters can be carried out in the setting of uniformly computable wtt-functionals. In the following, we give a sketch of the proofs by outlining the critical parts. For the former let, for any numbers $e,x\in\omega$ and any set A,

$$\Psi_e^A(x)=\begin{cases}\hat{\Phi}^A_{\varphi_e(e)}(x) & \text{if } \varphi_e(e)\downarrow,\\ \uparrow & \text{otherwise.}\end{cases}$$

Then the sequence $\{\Psi_e\}_{e\in\omega}$ is a uniformly computable sequence of Turing functionals whose use if uniformly bounded by $\psi_e(x)=\hat{\varphi}_{\varphi_e(e)}(x)$. So since $\{\psi_e\}_{e\in\omega}$ is a uniformly computable sequence of partial computable functions, by 1., we may fix a computable function $d:\omega\to\omega$ such that $\Psi_e^A=\hat{\Phi}^A_{d(e)}$ holds for any $e\in\omega$ and any set A and we may let $i\in\omega$ be such that $\varphi_i(x)=g(d(x))$. Then, by virtually the same argument as in the original Recursion Theorem, it follows that $e=d(i)$ is a fixed point for g.

For 3., we argue analogously. Let

$$\Psi^A_{\langle x,y\rangle}(z)=\begin{cases}\hat{\Phi}^A_{\varphi_x(\langle x,y\rangle)}(z) & \text{if } \varphi_x(\langle x,y\rangle)\downarrow,\\ \uparrow & \text{otherwise.}\end{cases}$$

Then since $\{\Psi_e\}_{e\in\omega}$ is clearly a uniformly computable sequence of Turing functionals and $\{\psi_e\}_{e\in\omega}$, where $\psi_{\langle x,y\rangle}(z)=\hat{\varphi}_{\varphi_x(\langle x,y\rangle)}(z)$ is a uniformly computable sequence of partial computable functions bounding the use of $\Psi^A_{\langle x,y\rangle}$ for any $x,y\in\omega$ and any set A, we may easily argue as in the proof of the Recursion Theorem with Parameters that $h(x) = d(i, x)$ is as desired, where, by 1., $d : \omega^2 \to \omega$ is chosen such that $\Psi^A_{\langle x,y\rangle} = \hat{\Phi}^A_{d(x,y)}$ holds and $i \in \omega$ is chosen such that $\varphi_i(\langle x, y\rangle) = H(d(x, y), y)$ holds for all $x, y \in \omega$. $\square$

It is natural to ask what properties bounded jump operator shares with the classical Turing operator if we replace Turing reductions by wtt-reduction. In the following lemma, we list some of the common properties which can be found in [DG19, p.30 f].

Lemma 5.2.4 ([DG19])**.** *Let A and B be any (not necessarily c.e.) sets. Then the following holds.*

1. *If $A \leq_{wtt} B$ then there exists a strictly increasing computable function $f : \omega \to \omega$ such that, for any $e \in \omega$, it holds that $\hat{\Phi}^A_e = \hat{\Phi}^B_{f(e)}$.*
2. *$A^\dagger$ is 1-complete for the class of bounded A-c.e. sets. In particular, $\emptyset'$ is computably isomorphic to $\emptyset^\dagger$.*
3. *There exists a strictly increasing computable function $f : \omega \to \omega$ such that, for any e, x and any set A, it holds that $\hat{\Phi}^A_e(x) = \hat{J}^A(f(\langle e, x\rangle))$. Hence, $A^\dagger$ is computably isomorphic to $A^\dagger_d$.*
4. *$A <_{wtt} A^\dagger$.*
5. *$A \leq_{wtt} B$ implies $A^\dagger \leq_1 B^\dagger$.*

However, not every property of the Turing jump carries over to the bounded jump as the following lemma of [DG19] shows.

Lemma 5.2.5 ([DG19], Lemma 3.6)**.** *There is a c.e. set B and a set A such that $A^\dagger \leq_1 B^\dagger$ holds but $A \not\leq_{wtt} B$.*

The fact that the converse of 5. in Lemma 5.2.4 fails is due to the fact that the Complement Lemma does not carry over to bounded-c.e. sets as Downey and Greenberg also show in [DG19, Proposition 3.1(3)]. However, the proof of Lemma 5.2.5 (and similarly for [DG19, Proposition 3.1(3)]) builds on the fact that the set A constructed there may change its mind whether a given x is in A or not more than once. This leaves the question open whether the Complement

Lemma and hence the converse of 5. in Lemma 5.2.4 hold if A is chosen to be computably enumerable. We can affirmatively answer both questions.

Lemma 5.2.6. *For any sets A and B such that A is c.e. or co-c.e., if A and $\overline{A}$ are both bounded-c.e. in B then $A \leq_{wtt} B$. In particular, if A and B are both c.e. then $A^\dagger \leq_1 B^\dagger$ implies that $A \leq_{wtt} B$ holds.*

Proof. For a proof of the first part of the lemma, fix sets A and B such that A is c.e. or co-c.e. and A and $\overline{A}$ are bounded-c.e. in B. By $\overline{A} =_{wtt} A$ w.l.o.g. we may assume that A is computably enumerable. So fix a computable enumeration $\{A_s\}_{s\in\omega}$ of A and fix a number e such that $\overline{A} = \mathrm{dom}(\hat{\Phi}^B_e)$. Then we can compute A from B by a Turing reduction whose use is computably bounded as follows.

Let $f(x) = \mu s(x \in A_s$ or $\hat{\varphi}_{e,s}(x)\downarrow)$. Then f is a total computable function as $\hat{\varphi}_e(x)\downarrow$ holds for any number $x \notin A$. Given x, with oracle B compute the least stage $s \geq f(x)$ such that either $x \in A_s$ or $\hat{\Phi}^B_{e,s}(x)\downarrow$. Then, by assumptions on A, stage s exists and $x \in A$ iff $x \in A_s$. Moreover, since, by convention on converging computations, $\hat{\varphi}_e(x) < f(x)$ if $\hat{\varphi}_e(x)\downarrow$, $B \restriction f(x)$ can compute the stage s.

For the second part of Lemma 5.2.6, it suffices to note that $A^\dagger \leq_1 B^\dagger$ implies that A and $\overline{A}$ are bounded-c.e. in B. So the second part follows from the first part. □

Next, we formulate and prove the main result of this paper.

5.3 C.E. Sets Which Are Bounded Turing Reducible To Maximal Sets

For our main result, we make the following definition.

Definition 5.3.1. *A set A is called* eventually uniformly wtt-array computable (e.u.wtt-a.c. *for short) if there exist computable functions $g, k : \omega^2 \to \{0,1\}$ and* *a computable order $h : \omega \to \omega$ such that, for all e, x,*

$$A^\dagger(x) = \lim_{s\to\infty} g(x,s), \tag{5.6}$$

$$k(x,s) \leq k(x,s+1), \tag{5.7}$$

$$k(x,s) = 1 \Rightarrow \ |\{t \geq s : g(x,t+1) \neq g(x,t)\}| \leq h(x), \tag{5.8}$$

$$\forall e\ (\hat{\Phi}_e^A \text{ total } \Rightarrow\ \forall^\infty x\ \exists s\ (k(\langle e,x\rangle, s) = 1)). \tag{5.9}$$

For functions g, k and h as above, we say that A is eventually uniformly wtt-array computable via *g, k and h, and we let* EUwttAC *denote the class of all c.e. e.u.wtt-a.c. sets.*

Now the main result is as follows.

Theorem 5.3.2 (Characterization Theorem)**.** *For a c.e. set A the following are equivalent.*

(i) A is eventually uniformly wtt-array computable.

(ii) A is wtt-reducible to some maximal (quasi-maximal, hh-simple, dense simple) set.

(iii) A is ibT-reducible to some maximal (quasi-maximal, hh-simple, dense simple) set.

The proof consists of the following two parts. First, we prove that the implication $(i) \Rightarrow (iii)$ holds.

Theorem 5.3.3. *Let A be c.e. and eventually uniformly wtt-array computable. Then A is ibT-reducible (hence wtt-reducible) to some maximal set.*

Then we show that $(ii) \Rightarrow (i)$ holds.

Theorem 5.3.4. *Let A and D be c.e. sets such that $A \leq_{wtt} D$ and D is dense simple. Then A is eventually uniformly wtt-array computable.*

By these theorems we may prove Theorem 5.3.2 as follows.

Proof of Theorem 5.3.2 assuming Theorems 5.3.3 and 5.3.4. By the above Theorems 5.3.3 and 5.3.4, it remains to show that the implication $(iii) \Rightarrow (ii)$ holds. However, this follows from the fact that $A \leq_{ibT} B$ implies that $A \leq_{wtt} B$ holds for any sets A and B. □

Proof of Theorem 5.3.3. Let $\{A_s\}_{s\geq 0}$ be a computable enumeration of A and fix computable functions $\hat{g}$, $\hat{k}$ and $\hat{h}$ which witness that A is e.u.wtt-a.c. according to Definition 5.3.1. We construct a c.e. set M in stages s, where M_s denotes the finite set of numbers which are enumerated into M by stage s such that M is maximal and $A \leq_{ibT} M$. Clearly, any such M witnesses that Theorem 5.3.3 holds.

Before we give the formal construction, let us discuss some of the ideas behind it and introduce some of the concepts to be used in the construction. We start with the task of making M maximal.

In order to make M maximal, it suffices to ensure that the complement of M is infinite,

$$|\overline{M}| = \omega, \tag{5.10}$$

and that M meets the requirements

$$\mathcal{R}_e : \ \overline{M} \subseteq^* W_e \text{ or } \overline{M} \subseteq^* \overline{W_e}. \tag{5.11}$$

for $e \in \omega$.

In order to achieve these goals, just as in the classical maximal set construction (as for instance in Soare [Soa87]), we use n-states and "optimize" the states of almost all elements in $\overline{M}$. Since we use a priority tree here, however, in our definition of the states the infinitary outcome (corresponding to the case that $W_e \cap \overline{M}$ is infinite) is denoted by 0 (as common on priority trees) and not by 1 as in the classical definition of states. So here the *n-state of* a number *x at stage s* is the unique binary string $\sigma(n,x,s)$ of length n such that, for $e < n$,

$$\sigma(n,x,s)(e) = 0 \ \text{ iff } \ x \in W_{e,s},$$

and the *(true) n-state of* x is the unique binary string $\sigma(n,x)$ of length n such that, for $e < n$,

$$\sigma(n, x)(e) = 0 \text{ iff } x \in W_e.$$

Note that requirements $\mathcal{R}_0, \ldots, \mathcal{R}_n$ are met if almost all elements of $\overline{M}$ have the same $(n+1)$-state. So, in order to meet the maximal-set requirements, it suffices to guarantee that, for any $n \geq 0$, almost all numbers in $\overline{M}$ have the same n-state. In the construction of M we achieve this by attempting to minimize the n-states of the numbers in $\overline{M}$ (which corresponds to the classical strategy of maximizing the (classically defined) n-states).

For this sake we use the full binary tree $T = \{0,1\}^{<\omega}$ as priority tree. Elements of T are called *nodes*. As usual, we say for two nodes α and β that α has *higher priority than* β and denote it by $\alpha < \beta$ iff $\alpha \sqsubset \beta$ (i.e., α is a proper initial segment of β) or α is to the left of β, denoted by $\alpha <_{left} \beta$, i.e., there exists $\gamma \in T$ such that $\gamma 0 \sqsubseteq \alpha$ and $\gamma 1 \sqsubseteq \beta$. Nodes are viewed as states in the following sense. A node $\alpha \in T$ of length n codes the guess that there are infinitely many numbers in $\overline{M}$ with n-state α. Then, assuming that $\overline{M}$ is infinite, there is a leftmost path through T such that, for any node α on this path, there are infinitely many elements of $\overline{M}$ which have state α. So it suffices to guarantee that almost all elements of $\overline{M}$ have state α.

In order to approximate the true path, for any node α and any stage s, we let

$$\begin{aligned} V_{\alpha,s} &= \overline{M_s} \upharpoonright s \cap \{y : \sigma(|\alpha|, y, s) = \alpha\} \\ &= \overline{M_s} \upharpoonright s \cap \{y : \forall\, e < |\alpha|\ (y \in W_{e,s} \Leftrightarrow \alpha(e) = 0)\} \end{aligned}$$

and

$$V_\alpha = \overline{M} \cap \{y : \sigma(|\alpha|, y) = \alpha\} = \overline{M} \cap \{y : \forall\, e < |\alpha|\ (y \in W_e \Leftrightarrow \alpha(e) = 0)\},$$

and we use the following *length of agreement function*

$$l(\alpha, s) = |V_{\alpha,s}|. \tag{5.12}$$

Based on $l(\alpha, s)$, we define the set of α-stages by induction on $|\alpha|$ as follows. Every stage is a λ-stage. An α-stage s is called *α-expansionary* if $s = 0$ or $l(\alpha 0, s) > l(\alpha 0, t)$ holds for all α-stages $t < s$. Then a stage s is an $\alpha 0$-stage if it is α-expansionary and an $\alpha 1$-stage if it is an α-stage but not α-expansionary. At stage s, the current approximation δ_s of the true path is the unique node α of length s such that s is an α-stage, and we say that α is *accessible* at stage $s+1$ if α is an initial segment of δ_s, i.e., $\alpha \sqsubseteq \delta_s$. Then the *true path* TP through T is defined by $TP = \liminf_{s\to\infty} \delta_s$, i.e., $TP \upharpoonright n$ is the leftmost node of length n which is accessible infinitely often (for every n).

Next we explore under which assumptions on M the true path TP actually has the desired properties, i.e., satisfies that, for any n, $TP \upharpoonright n$ is the leftmost node α of length n such that V_α is infinite. We start with some observations. Note that

$$V_{\alpha 0,s} = V_{\alpha,s} \cap W_{|\alpha|,s} \text{ and } V_{\alpha 1,s} = V_{\alpha,s} \cap \overline{W_{|\alpha|,s}}. \tag{5.13}$$

So $V_{\alpha,s}$ is the disjoint union of $V_{\alpha 0,s}$ and $V_{\alpha 1,s}$,

$$V_{\alpha,s} = V_{\alpha 0,s} \dot{\cup} V_{\alpha 1,s}, \tag{5.14}$$

and

$$l(\alpha, s) = l(\alpha 0, s) + l(\alpha 1, s). \tag{5.15}$$

Note that the analog of (5.14) holds for V_α too and that the equation can be extended to

$$V_{\alpha,s} = \dot{\bigcup}_{|\beta|=n} V_{\alpha\beta,s} \text{ and } V_\alpha = \dot{\bigcup}_{|\beta|=n} V_{\alpha\beta} \tag{5.16}$$

for any $n \geq 0$. Next note that, for any node α, $\{V_{\alpha,s}\}_{s\geq 0}$ is a computable approximation of V_α, i.e., for any number y,

$$V_\alpha(y) = \lim_{s\to\omega} V_{\alpha,s}(y). \tag{5.17}$$

Moreover, a number $y \in V_{\alpha,s}$ is in V_α unless y is enumerated into M after stage s or the $|\alpha|$-state of y decreases after stage s. So, if we let

$$\hat{V}_\alpha = \bigcup_{\{\alpha' : |\alpha'|=|\alpha| \ \& \ \alpha' \leq_{left} \alpha\}} V_{\alpha'},$$

then

$$\hat{V}_\alpha = \overline{M} \cap \left(\bigcup_{s \geq 0} \bigcup_{\{\alpha' : |\alpha'| = |\alpha| \ \& \ \alpha' \leq_{left} \alpha\}} V_{\alpha', s} \right). \tag{5.18}$$

In fact, if we say that α' is *stronger* than α ($\alpha' \prec \alpha$) if $\alpha' <_{left} \alpha$ or $\alpha \sqsubset \alpha'$ (i.e., viewed as state, either α' is less than α or α' contains more information than α) then, by definition of $\hat{V}_\alpha$ and (5.16), $\hat{V}_{\alpha'} \subseteq \hat{V}_\alpha$ for any α' which is stronger than α whence

$$\hat{V}_\alpha = \bigcup_{\{\alpha' : \alpha' \preceq \alpha\}} V_{\alpha'} = \overline{M} \cap \left(\bigcup_{s \geq 0} \bigcup_{\{\alpha' : \alpha' \preceq \alpha\}} V_{\alpha', s} \right). \tag{5.19}$$

We can now state the two crucial facts on TP used in the proof.

Claim 1 (Infinity Lemma). Assume (5.10). *For any node $\alpha \sqsubset TP$, the set S_α of the α-stages is infinite and*

$$\lim_{s \to \omega, \ s \in S_\alpha} l(\alpha, s) = \omega. \tag{5.20}$$

Moreover, if α' is to the left of TP then $S_{\alpha'}$ and $\hat{V}_{\alpha'}$ are finite.

Proof. For a proof of the first part, fix $\alpha \sqsubset TP$. Infinity of S_α is immediate by definition of TP. The proof of (5.20) is by induction on $|\alpha|$. We distinguish the following three cases. First assume that $\alpha = \lambda$. Then $S_\alpha = \omega$ and $V_\lambda = \overline{M}$. So (5.20) holds by infinity of $\overline{M}$. Next assume that $\alpha = \hat{\alpha}0$ for some node $\hat{\alpha}$. Then, by $\alpha \sqsubset TP$ there are infinitely many $\hat{\alpha}$-expansionary stages. So S_α is infinite and (5.20) holds by definition. Finally assume that $\alpha = \hat{\alpha}1$ for some node $\hat{\alpha}$. Then, by $\hat{\alpha}1 \sqsubset TP$ there are only finitely many $\hat{\alpha}0$-stages whence $l(\hat{\alpha}0, s)$ is bounded. By the former, $S_\alpha =^* S_{\hat{\alpha}}$ while, by the latter and by (5.15), there is a constant c such that $l(\alpha, s) + c \geq l(\hat{\alpha}, s)$ for all stages s. So infinity of (5.20) follows by inductive hypothesis.

For a proof of the second part, fix α' to the left of TP, let $\alpha = TP \upharpoonright |\alpha'|$ and let $\hat{\alpha}$ be the longest common initial segment of α' and α. Then $\hat{\alpha}0 \sqsubseteq \alpha'$ and $\hat{\alpha}1 \sqsubseteq \alpha$ whence $\hat{\alpha}$ and $\hat{\alpha}1$ are on the true path. By definition of TP, it follows that $S_{\hat{\alpha}0}$ is finite. Since, by $\hat{\alpha}0 \sqsubseteq \alpha'$, $S_{\alpha'} \subseteq S_{\hat{\alpha}0}$, $S_{\alpha'}$ is finite too. Finally, in order to show that $\hat{V}_{\alpha'}$ is finite, for a contradiction assume that $\hat{V}_{\alpha'}$ is infinite. Since $\hat{V}_{\alpha'}$ is the finite union of the sets $V_{\alpha''}$ where $|\alpha''| = |\alpha'|$ and $\alpha'' \leq_{left} \alpha'$, for notational convenience, w.l.o.g. we may assume that $V_{\alpha'}$ is infinite. It follows that $V_{\hat{\alpha}0}$ is infinite since, by $\hat{\alpha}0 \sqsubseteq \alpha'$, $V_{\alpha'} \subseteq V_{\hat{\alpha}0}$. By (5.17) this implies that $lim_{s\to\omega} l(\hat{\alpha}0) = \omega$. Since, by $\hat{\alpha} \sqsubset TP$, $S_{\hat{\alpha}}$ is infinite, it follows that there are infinitely many $\hat{\alpha}$-expansionary stages, hence $S_{\hat{\alpha}0}$ is infinite cotradicting the above observation that $S_{\hat{\alpha}0}$ is finite. □

Claim 2 (Maximal-Set Lemma). Assume that M is c.e. and coinfinite. If, for any $\alpha \sqsubset TP$, $\overline{M} \subseteq^* \hat{V}_\alpha$ then M is maximal.

Proof. Since $\hat{V}_\alpha$ is the finite union of V_α and the sets $V_{\alpha'}$ such that $\alpha' <_{left} \alpha$ and $|\alpha'| = |\alpha|$, it follows by the second part of the Infinity Lemma that $\overline{M} \subseteq^* V_\alpha$ for all $\alpha \sqsubset TP$. So, for any $n \geq 0$, almost all numbers in $\overline{M}$ have $(n+1)$-state $TP \restriction n+1$. As pointed out before, this implies that all requirements $\mathcal{R}_n$ are met. Since, by assumption, M is c.e. and coinfinite this implies that M is maximal. □

The Infinity Lemma shows that (assuming $\overline{M}$ is infinite), for any α on the true path and for any numbers r and k, there are infinitely many stages at which α is accessible and where we can pick k numbers greater than r of current state α which have not yet been enumerated into M. (Note that, for meeting a finitary requirement we typically need such a set of numbers, where later in the construction some of this numbers may be put into M and some of the numbers may be kept out of M.) On the other hand, the Maximal-Set Lemma tells us that if we make sure that infinitely many of the numbers we pick in this way are kept out of M and that, for any $\alpha \sqsubset TP$, up to finitely many exceptions, only those numbers picked for α or a stronger node α' are kept out of M then M is maximal. These observations lead to the following strategy ensuring maximality. We pick the numbers which become associated with a given node α for ensuring any of the additional finitary tasks in such a way that one of these numbers is never needed for this task (this will ensure that $\overline{M}$ infinite). Moreover, if the task assigned to the numbers associated with a state α can be taken over by the numbers associated with one stronger state (or, as in the following, associated with a finite collection of stronger states) then the original attempt becomes superfluous and we may cancel it and enumerate the corresponding numbers into M.

Having introduced the basic technical notions needed for the maximal set strategy, we now turn to the second goal of the construction, namely to ensure that the given computably enumerable e.u.wtt-a.c. set A is ibT-reducible to the maximal set M that we construct. We first note that this part requires the construction of a uniformly computable sequence of auxiliary wtt-functionals $\{\Psi_\alpha\}_{\alpha\in\{0,1\}^*}$, where we denote the partial computable use bound of Ψ_α by ψ_α. By identifying $\{0,1\}^*$ with ω in the standard way, by Lemma 5.2.3 (Recursion Theorem), we may assume that *in advance* we are given a computable function $f : \{0,1\}^* \to \omega$ such that

$$\Psi_\alpha = \hat{\Phi}_{f(\alpha)} \tag{5.21}$$

holds for all $\alpha \in \{0,1\}^*$. So, by letting $g(\langle\alpha,x\rangle,s) = \hat{g}(\langle f(\alpha),x\rangle,s)$, $k(\langle\alpha,x\rangle,s) = \hat{k}(\langle f(\alpha),x\rangle,s)$ and $h(\langle\alpha,x\rangle) = \hat{h}(\langle f(\alpha),x\rangle)$, we obtain

$$\lim_{s\to\infty} g(\langle\alpha,n\rangle,s) = \begin{cases} 0 & \text{if } \Psi_\alpha^A(n)\uparrow, \\ 1 & \text{otherwise} \end{cases} \tag{5.22}$$

$$k(\langle\alpha,n\rangle,s) \le k(\langle\alpha,n\rangle,s+1) \tag{5.23}$$

$$k(\langle\alpha,n\rangle,s) = 1 \;\Rightarrow\; |\{t \ge s : g(\langle\alpha,n\rangle,t+1) \ne g(\langle\alpha,n\rangle,t)\}| \le h(\langle\alpha,n\rangle) \tag{5.24}$$

$$\Psi_\alpha^A \text{ is total } \;\Rightarrow\; \forall^\infty n\; \exists\, s\; (k(\langle\alpha,n\rangle,s) = 1) \tag{5.25}$$

and we may use these equations in the construction. In a more detail, we construct a uniformly computable sequence of wtt-functionals $\{\tilde{\Psi}_{\langle\langle i,e\rangle,j\rangle}\}_{i,e,j\ge 0}$ (with $\{\psi_{\langle\langle i,e\rangle,j\rangle}\}_{i,e,j\ge 0}$ as uniformly computable use bound) such that, for fixed $i \ge 0$, we have a version of the construction where we define $\tilde{\Psi}_{\langle\langle i,e\rangle,j\rangle}$ only for $j = \varphi_i(\langle i,e\rangle)$ (that is, i is a guess for an index of the desired function f). In particular, for each $i \ge 0$, we consider the functions $\tilde{g}_i(\langle e,x\rangle,s) = \hat{g}(\langle\varphi_i(\langle i,e\rangle),x\rangle,s)$, $\tilde{k}_i(\langle e,x\rangle,s) = \hat{k}(\langle\varphi_i(\langle i,e\rangle),x\rangle,s)$ and $h_i(\langle e,x\rangle) = \hat{h}(\langle\varphi_i(\langle i,e\rangle),x\rangle)$ and we perform the construction w.r.t. to these functions. So in the ith version of the construction, the reader may replace any occurence of g by $\tilde{g}_i$, k by $\tilde{k}_i$ and h by $\tilde{h}_i$. Now by uniform effectivity of the construction, we may argue by 1. and 3. of Lemma 5.2.3 that there exists a computable function $\tilde{f} : \omega \to \omega$ such that, for any $i,e \ge 0$, it holds that $\tilde{\Psi}^A_{\langle\langle i,e\rangle,\tilde{f}(\langle i,e\rangle)\rangle} = \hat{\Phi}^A_{\tilde{f}(\langle i,e\rangle)}$. So, for an index i_0 such that $\varphi_{i_0} = \tilde{f}$ holds, it follows (via the identification of ω with $\{0,1\}^*$)

that $f(\alpha) = \tilde{f}(\langle i_0, \alpha\rangle)$, $\Psi_\alpha = \tilde{\Psi}_{\langle\langle i_0,\alpha\rangle, \tilde{f}(\langle i_0,\alpha\rangle)\rangle}$, $\psi_\alpha = \tilde{\psi}_{\langle\langle i_0,\alpha\rangle, \tilde{f}(\langle i_0,\alpha\rangle)\rangle}$ and $g = \tilde{g}_{i_0}$ ($k = \tilde{k}_{i_0}$, $h = \tilde{h}_{i_0}$) are as desired.

Now, coming back to the second goal of the construction, in order to ensure that A is ibT-reducible to M we use a variant of straight permitting: if a number x enters A at a "late" stage s then, in order to indicate that x is in A we enumerate a number $y \leq x$ into M at stage s or at a later stage. Note that if we reserve a number y for such a permitting and x does not enter A then y will not enter M too. So, in order to be compatible with the maximal set strategy, we have to ensure that the states of the permitters y are sufficiently small. In order to show that there are sufficiently many permitters of small state, we exploit that A is eventually uniformly wtt-array computable. The basic idea of how to obtain permitters (for almost all numbers x) of a given m-state α (on or to the right of TP) is as follows. We attempt to define a strong array $\{B^\alpha_n\}_{n\geq 0}$ of finite sets B^α_n, in the following called (α-)*blocks*. The α-blocks are defined one after the other in increasing order and we ensure that the numbers in B^α_{n+1} are greater than the numbers in B^α_n. Moreover, when an α-block becomes defined, say at stage $s+1$ then all of its element are not in M_s and have m-state α or stronger than α at stage s. (Note that (assuming that $\overline{M}$ is infinite), by the Infinity Lemma, for α on or to the right of the true path we will find such numbers no matter how large we want to make the blocks. So, for such α, all the α-blocks will become defined.) Now the idea is that the numbers y in block B^α_n serve as permitters for the numbers x in the interval $I^\alpha_n = [\max B^\alpha_n, \max B^\alpha_{n+1}]$ (note that these intervals cover all numbers $x \geq \max B^\alpha_0$). In order to guarantee that the size (i.e., cardinality) of B^α_n is large enough to provide the required numbers of permitters, we appropriately define the corresponding auxiliary wtt-functional Ψ_α. We let $\psi_\alpha(n) = \max B^\alpha_{n+1}$ (if the latter block becomes defined) be the use of $\Psi^X_\alpha(n)$. Moreover, if $\psi_\alpha(n)$ is defined then we ensure that $\Psi^A_\alpha(n)$ is defined too where – exploiting that, by (5.22), $g(\langle\alpha, n\rangle, s)$ approximates the domain of Ψ^A_α – we make sure that any enumeration of a number $x \in I^\alpha_n$ in A is followed by a change of $g(\langle\alpha, n\rangle, s+1) \neq g(\langle\alpha, n\rangle, s)$ at a later stage s. Now, since Ψ^A_α is total, it follows by (5.25) that (for almost all n) there is a least stage s_n such that $k(\langle\alpha, n\rangle, s_n) = 1$, and, by (5.24), the function $s \mapsto g(\langle\alpha, n\rangle, s)$ will change after stage s_n at most $h(\langle\alpha, n\rangle)$ times. So if we say that a number $x \in I^\alpha_n$ enters

A "late" if it does so after stage s_n then $h(\langle\alpha, n\rangle)$ permitters suffice for dealing with all numbers in I_n^α. So it suffices to let B_n^α have size $h(\langle\alpha, n\rangle)$.

The above explains how, for a single α on or to the right of the true path, we can ensure that $A \leq_{ibT} M$ and at the same time only numbers of state α or stronger state are left in $\overline{M}$ (namely, it suffices to enumerate all numbers which are not in any α-block into M). Moreover, by adding one more element to each α-block we can guarantee that no α-block becomes completely enumerated into M whence $\overline{M}$ will be infinite.

For the actual construction, however, we have to ensure that, for *any* α on the true path almost all numbers left in $\overline{M}$ have state α or stronger state. We achieve this by (1) doing the above strategy for all α and by (2) suspending the permitting numbers in block B_n^α (in the actual construction we say that block B_n^α becomes *frozen*) and enumerating them into M once we see that, for any number $x \in I_n^\alpha$, there is a node $\alpha' \prec \alpha$ and a number n' such that x is in the interval $I_{n'}^{\alpha'}$ covered by the α'-block $B_{n'}^{\alpha'}$ and x is considered to be "late" relative to this block too (i.e., $k(\langle\alpha', n'\rangle, s) = 1$ if this happens at stage $s+1$). As we will show, this will provide the required improvements of states.

There is one technical problem left, however. We cannot achieve that, for $\alpha \neq \alpha'$, the α-blocks and α'-blocks are disjoint. So when determining the sizes of the blocks we have to consider possible overlaps. By allowing the α'-strategy to use a number in the intersection of the blocks B_n^α and $B_{n'}^{\alpha'}$ only if α' is stronger than α, we have to ensure that any block B_n^α contains a core $\hat{B}_n^\alpha$ of size $h(\langle\alpha, n\rangle) + 1$ which does not intersect any α'-block for all stronger α'. The sole purpose of the priority tree is to resolve this problem. The interval B_n^α will be defined by one of the nodes β which extend α and have length $\langle|\alpha|, n\rangle$. As long as B_n^α is not yet defined there will be (at most) one such β "eligible" to define B_n^α. The stage when this node becomes eligible gives a lower bound on $\min B_n^\alpha$ and by initializing a node its eligibility can be (temporarily) deleted. This will suffice to avoid overlaps between α-blocks and α'-blocks for comparable α and α' and will give an eligible node β a bound on the sizes of the potential overlaps in terms of the higher priority nodes currently admissible.

Having explained the ideas of the construction and some of its technical features we now turn to the construction. Any stage $s+1$ consists of 5 steps (Stage 0 is vacuous).

In Step 1 the blocks are defined. We let the nodes β with $\alpha \sqsubseteq \beta$ and $|\beta| = \langle |\alpha|, n\rangle$ define the block B_n^α. We call such a node β a B_n^α*-node* and call B_n^α the *block associated with* β. Moreover we call two nodes *equivalent* if they are associated with the same block. If B_n^α is defined by (activity of) node β then we say that B_n^α has *priority* β. As long as B_n^α is not yet defined, there will be at most one B_n^α-node β which is *eligible*. This node attempts to define B_n^α. Once B_n^α is defined, no B_n^α-node will be eligible. A node β can become eligible only at a stage $s+1$ such that $\beta \sqsubset \delta_s$ or $\delta_s <_{left} \beta$. Once β is eligible, β stays eligible unless β becomes *initialized*. The only effect of initialization of a node is to make it non-eligible. If initialized, a node may become eligible at a later stage again. We write $B_n^\alpha[s]\downarrow$ if B_n^α is defined by the end of step 1 of stage s and we write $B_n^\alpha[s]\uparrow$ otherwise. Moreover, $B_n^\alpha\downarrow$ ($B_n^\alpha\uparrow$) denotes that B_n^α is eventually defined (never defined). For any α and n such that B_n^α is defined, we let

$$\hat{B}_n^\alpha = \{y \in B_n^\alpha : \nexists \alpha' \prec \alpha \ \nexists n' \ (B_{n'}^{\alpha'}\downarrow \ \& \ y \in B_{n'}^{\alpha'}\}$$

be the *core* of B_n^α. Similarly, for s such that $B_n^\alpha[s]\downarrow$, we let

$$\hat{B}_n^\alpha[s] = \{y \in B_n^\alpha : \nexists \alpha' \prec \alpha \ \nexists n' \ (B_{n'}^{\alpha'}[s]\downarrow \ \& \ y \in B_{n'}^{\alpha'}\}$$

be the *core* of B_n^α at stage s.

In Steps 2 and 3 the partial use functions ψ_α respectively the wtt-functionals Ψ_α are defined. We write $\psi_\alpha(n)[s]\downarrow$ if $\psi_\alpha(n)$ has been defined by the end of step 2 of stage s and write $\psi_\alpha(n)[s]\uparrow$ otherwise, and we write $\Psi_\alpha^A(n)[s]\downarrow$ if $\Psi_\alpha^{A_s}(n)$ has been defined by the end of step 3 of stage s and $\Psi_\alpha^A(n)[s]\uparrow$ otherwise. We say that the α-block B_n^α is *realized* at stage s if $\psi_\alpha(n)[s]\downarrow$ and we say that B_n^α is *truly realized* at stage s if B_n^α is realized at stage s and $k(\langle\alpha, n\rangle, s) = 1$; and B_n^α is *realized* (*truly realized*) if it is realized (truly realized) at some stage. Finally, we say that x is *(truly) covered by B_n^α (at stage s)* – or (truly) $\langle\alpha, n\rangle$*-covered (at stage s)* for short – if $\langle\alpha, n\rangle$ is (truly) realized (at stage s) and $x \in [\max B_n^\alpha, \psi_\alpha(n)]$; and we say that x is α*-covered (at stage s)* if x is $\langle\alpha, n\rangle$-covered (at stage s) for some n.

In Step 4 blocks become frozen. We say that a block B_n^α is *admissible* at stage s, if it is truly realized at stage s and has not been frozen by the end of step 4 of stage s.

In Step 5 numbers are enumerated into M, i.e., M_{s+1} becomes defined.

Now, using the notation introduced above, the steps of stage $s+1$ are as follows.

Step 1 (Defining the blocks B_n^α). A B_n^α-node β requires attention at stage $s+1$ if one of the following holds.

(a) (i) $B_n^\alpha[s]\uparrow$

(ii) $\beta \sqsubset \delta_s$ or $\delta_s <_{left} \beta$ and $|\beta| < s$.

(iii) Neither β nor any equivalent node β' such that $\beta' <_{left} \beta$ is eligible at stage s.

(iv) For any node β' such that $\beta' \sqsubset \beta$, the block associated with β' is defined at stage s.

(v) For any node β' such that $\beta <_{left} \beta'$, $|\beta'| = |\beta|$ and β' is not equivalent to β, the block associated with β' is defined at stage s.

(b) β is eligible at stage s, and there is a block B which is suitable for the definition of B_n^α by β at stage $s+1$. Here a block B is *suitable* for the definition of B_n^α by the B_n^α-node β at stage $s+1$ if B has the following properties.

(i) $r(\beta, s) < \min B$,

(ii) $B \subseteq \bigcup_{\{\alpha' : |\alpha'| = |\alpha| \text{ and } \alpha' \leq_{left} \alpha\}} V_{\alpha', s}$,

(iii) The block B has cardinality $|B| = F(\beta, s)$ where $F(\gamma, s)$ is defined (by induction on γ) by

$$F(\gamma, s) = 2 + H(\gamma) + \sum_{\{\gamma' : \gamma' <_{left} \gamma \text{ and } \gamma' \text{ is eligible at stage } s\}} F(\gamma', s)$$

where, for a $B_{n'}^{\alpha'}$-node γ, $H(\gamma) = h(\langle \alpha', n' \rangle)$ (and where $\sum_\emptyset = 0$). (Note that, at any given stage s, there are only finitely many eligible nodes, hence $F(\gamma, s)$ is well defined.)

Fix β minimal such that β requires attention.

If (a) holds then declare that β becomes eligible, set $r(\beta', s+1) = s$ for all $\beta' \geq \beta$, and initialize all nodes β' with $\beta < \beta'$ (i.e., no such β' is eligible at stage $s+1$).

If (b) holds then let $B^\alpha_n = B$ for the least (w.r.t. the canonical index) block B which is suitable for the definition B^α_n by β at stage $s+1$, let β be the *priority* of B^α_n, set $r(\beta', s+1) = s$ for all $\beta' > \beta$, and initialize all nodes β' such that $\beta \leq \beta'$.

If no node requires attention then Step 1 of stage $s+1$ is vacuous..

Step 2 (Defining the partial computable use functions ψ_α). For any α and any n such that either $n = 0$ or $\psi_\alpha(n-1)[s]\downarrow$, $\psi^\alpha(n)[s]\uparrow$ and $B^\alpha_{n+1}[s]\downarrow$, let $\psi_\alpha(n) = \max B^\alpha_{n+1}$.

Step 3 (Defining the wtt-functionals Ψ_α). For any α and any n such that $\psi_\alpha(n)[s]\downarrow$ let

$$\Psi^A_\alpha(n)[s+1]\downarrow \text{ if } \Psi^A_\alpha(n)[s]\uparrow \text{ and } g(\langle\alpha, n\rangle, s) = 0, \tag{5.26}$$

and let

$$\begin{gathered}\Psi^A_\alpha(n)[s+1]\uparrow \ \text{ if } \Psi^A_\alpha(n)[s]\downarrow,\ g(\langle\alpha, n\rangle, s) = 1 \text{ and} \\ A_{s+1} \upharpoonright \psi_\alpha(n)+1 \neq A_s \upharpoonright \psi_\alpha(n)+1.\end{gathered} \tag{5.27}$$

In any other case let $\Psi^A_\alpha(n)[s+1]\downarrow$ if and only if $\Psi^A_\alpha(n)[s]\downarrow$.

Step 4 (Freezing blocks). A block B^α_n is *freezable* at stage $s+1$ if the following hold.

(i) $\langle|\alpha|, n\rangle < s$.

(ii) B^α_n is not frozen at stage s.

(iii) For any x covered by B^α_n there is a block $B^{\alpha_x}_{n_x}$ such that $\alpha_x \prec \alpha$, $B^{\alpha_x}_{n_x}$ is admissible at stage s, and $B^{\alpha_x}_{n_x}$ covers x.

If there is a freezable block then choose $q = \langle m, n\rangle$ minimal such that there is a freezable block B^α_n with $|\alpha| = m$ and fix the rightmost α such that

$|\alpha| = m$ and B_n^α is freezable. Declare that B_n^α *becomes frozen* at stage $s+1$.

Step 5 (Enumerating M. A number $y \notin M_s$ is enumerated into M at stage $s+1$ if (at least) one of the following hold.

(i) *(Freezing)* There is a block B_n^α which becomes frozen in Step 4 of stage $s+1$ and y is in the core $\hat{B}_n^\alpha[s+1]$ of B_n^α at stage $s+1$.

(ii) *(Enumerating nonblock numbers)* y is not in any block defined at stage $s+1$ and y is less than the maximum of a block defined at stage $s+1$.

(iii) *(Coding A into M)* There is a node α and a number n such that the block B_n^α is admissible at stage s and

$$\Psi_\alpha^A(n)[s]\downarrow \text{ and } \Psi_\alpha^A(n)[s+1]\uparrow \tag{5.28}$$

or

$$g(\langle\alpha, n\rangle, s) = 1 \text{ and } g(\langle\alpha, n\rangle, s+1) = 0, \tag{5.29}$$

holds, and y is the least element of the core $\hat{B}_n^\alpha[s+1]$ of B_n^α at stage $s+1$ which is not in M_s. In this case, call y an $\langle\alpha, n\rangle$-*coding number.*

This completes the construction. In the remainder of the proof we show that M has the required properties.

We first summarize the properties of the blocks we will need.

Claim 3. The definition of the blocks satisfies the following conditions.

(B_0) If B_n^α becomes defined at stage $s+1$ (i.e., $B_n^\alpha[s+1]\downarrow$ and $B_n^\alpha[s]\uparrow$) then $B_n^\alpha \cap M_s = \emptyset$.

(B_1) If B_n^α is defined then

$$B_n^\alpha \cap \overline{M} \subseteq \bigcup_{\{\alpha' : |\alpha'| = |\alpha| \text{ and } \alpha' \leq_{left} \alpha\}} V_{\alpha'}.$$

(B_2) If B_n^α is defined then $\langle|\alpha|, n\rangle \leq \min B_n^\alpha$.

(B_3) If B^α_{n+1} is defined then B^α_n is defined and $\max B^\alpha_n < \min B^\alpha_{n+1}$.

(B_4) If B^α_n is defined then, for the core

$$\hat{B}^\alpha_n = B^\alpha_n \setminus \bigcup_{\{(\alpha',n')\colon\, \alpha' \prec \alpha,\ n' \geq 0 \text{ and } B^{\alpha'}_{n'}\downarrow\}} B^{\alpha'}_{n'}$$

of B^α_n, $|\hat{B}^\alpha_n| > h(\langle \alpha, n \rangle) + 1$.

(B_5) If B^α_n is defined and $\alpha \prec \alpha'$ and $|\alpha'| \leq |\alpha|$ then $B^{\alpha'}_n$ is defined too.

(B_6) Assume that $\overline{M}$ is infinite. Then, for any α on or to the right of the true path, the blocks B^α_n are defined for all n.

(B_7) There is an infinite path p through $T = \{0,1\}^*$ such that, for any α, all blocks B^α_n ($n \geq 0$) are defined if and only if α is on or to the right of p.

Proof. With the exception of property (B_7) the proof only depends on the definition of the blocks and not on the other parts of the construction. In case of (B_7) we use that at any stage $s+1$ any number y which is enumerated into in M at stage $s+1$ is bounded by the maximum of some block existing at this stage.

We tacitly use that the restraint function is nondecreasing in both arguments, i.e., $r(\beta, s) \leq r(\beta', s')$ for $\beta \leq \beta'$ and $s \leq s'$, and that for any pair $\langle \alpha, n \rangle$ and any stage s there is at most one eligible B^α_n-node at stage s and there is no such node if $B^\alpha_n[s]\downarrow$.

(B_0). This is immediate by clause (ii) in the definition of suitability since, for any node α and any stage s, $V_{\alpha,s} \subseteq \overline{M_s}$.

(B_1). This is immediate by clause (ii) in the definition of suitability since, for any node α and any stage s,

$$\overline{M} \cap \bigcup_{\{\alpha':|\alpha'|=|\alpha| \text{ and } \alpha' \leq_{left} \alpha\}} V_{\alpha',s} \subseteq \bigcup_{\{\alpha':|\alpha'|=|\alpha| \text{ and } \alpha' \leq_{left} \alpha\}} V_{\alpha'}.$$

(B_2). If β is the priority of the block B^α_n and B^α_n becomes defined at stage $s+1$ then $r(\beta, s) < \min B^\alpha_n$ by clause (i) in the definition of suitability. Moreover, there is a stage $s'+1 \leq s$ such that β receives attention via (a) and becomes eligible at stage $s'+1$. By clause (ii) in (a), this implies that $|\beta| < s' = r(\beta, s'+1) \leq r(\beta, s)$. Finally, since β is a B^α_n-node, $|\beta| = \langle |\alpha|, n \rangle$.

(B$_3$). Assume that B^α_{n+1} becomes defined by β at stage $s+1$. Then, for the greatest stage $t < s+1$ such that β is not eligible at stage t, $t+1 \leq s$ and β receives attention via clause (a) at stage $t+1$. So, since there is B^α_n-node β' such that $\beta' \sqsubset \beta$, $B^\alpha_n[t+1] \downarrow$ whence $\max B^\alpha_n \leq t$. Moreover, $r(\beta, t+1) = t$ hence, by $t+1 \leq s$, $r(\beta, s) \geq t$. By the latter and by clause (i) in the definition of suitability of a block B, it follows that $t < \min B^\alpha_{n+1}$ which completes the proof of (B$_3$).

(B$_4$). Assume that B^α_n is defined. Fix the node β and the stage $s+1$ such that B^α_n has priority β and B^α_n becomes defined by activity of β at stage $s+1$. Then, given any state α' and any number n' such that

$$\alpha' \prec \alpha,\ B^{\alpha'}_{n'} \text{ is defined and } B^{\alpha'}_{n'} \cap B^\alpha_n \neq \emptyset, \tag{5.30}$$

it suffices to show that, for the priority β' of $B^{\alpha'}_{n'}$,

$$\beta' <_{left} \beta, \tag{5.31}$$

$$\beta' \text{ is eligible at stage } s, \tag{5.32}$$

and

$$|B^{\alpha'}_{n'}| = F(\beta', s) \tag{5.33}$$

hold. Namely, since different blocks have different priorities, it follows that

$$\begin{aligned}
|\hat{B}^\alpha_n| &= |B^\alpha_n \setminus \bigcup_{\{(\alpha',n'):\ \alpha' \prec \alpha,\ n' \geq 0 \text{ and } B^{\alpha'}_{n'} \downarrow\}} B^{\alpha'}_{n'}| \\
&\geq |B^\alpha_n| - \textstyle\sum_{\{(\alpha',n'):\ \alpha' \prec \alpha,\ n' \geq 0,\ B^{\alpha'}_{n'} \downarrow \text{ and } B^{\alpha'}_{n'} \cap B^\alpha_n \neq \emptyset\}} |B^{\alpha'}_{n'}| \\
&\geq F(\beta, s) - \textstyle\sum_{\{\beta' : \beta' <_{left} \beta \text{ and } \beta' \text{ is eligible at stage } s\}} F(\beta', s) \\
&\quad \text{(by definition of } B^\alpha_n \text{ and by (5.30) implying (5.31) - (5.33))} \\
&\geq H(\beta) + 2 \\
&\quad \text{(by definition of } F(\beta, s)) \\
&= h(\langle \alpha, n \rangle) + 2 \\
&\quad \text{(by definition of } H(\beta))
\end{aligned}$$

So, assuming that (5.30) implies (5.31) - (5.33), (B_4) holds.

Hence, for the remainder of the proof of (B_4), fix α' and n' such that (5.30) holds, and let β' be the priority of $B^{\alpha'}_{n'}$. We have to show that (5.31) - (5.33) hold. Fix $t < s$ maximal such that β is not eligible at stage t and fix $t' + 1 < s' + 1$ such that $B^{\alpha'}_{n'}$ becomes defined via β' at stage $s' + 1$ and t' is maximal such that $t' < s'$ and β' is not eligible at stage t'. Note that β becomes eligible at stage $t + 1$, β is not initialized (hence eligible) at any stage u such that $t + 1 \leq u < s + 1$, and B^α_n is defined by β at stage $s + 1$. Hence

$$t = r(\beta, t + 1) = r(\beta, s) < \min B^\alpha_n \leq \max B^\alpha_n \leq s. \tag{5.34}$$

Similarly, β' becomes eligible at stage $t' + 1$, β' is not initialized (hence eligible) at any stage u' such that $t' + 1 \leq u' < s' + 1$, and $B^{\alpha'}_{n'}$ is defined by β' at stage $s' + 1$. Hence

$$t' = r(\beta', t' + 1) = r(\beta', s') < \min B^{\alpha'}_{n'} \leq \max B^{\alpha'}_{n'} \leq s'. \tag{5.35}$$

Moreover, any node γ with $\beta < \gamma$ is initialized at stages $t + 1$ and $s + 1$, and any node γ' with $\beta' < \gamma'$ is initialized at stages $t' + 1$ and $s' + 1$. Finally note that, by $(\alpha, n) \neq (\alpha', n')$, $\beta \neq \beta'$ and the stages t, s, t', s' are mutually different.

Now, the proof of (5.31) - (5.33) is in two steps: before we prove (5.31), we show that (5.31) implies (5.32) and (5.33). So assume (5.31). Now, if $s'+1 < s+1$ then (by (5.31)) β is initialized at stage $s'+1$ hence $s'+1 < t+1$. So, by (5.34) and (5.35), $\max B^{\alpha'}_{n'} < \min B^{\alpha}_{n}$ contradicting (5.30). Similarly, if $s+1 < t'+1$, then by (5.34) and (5.35), $\max B^{\alpha}_{n} < \min B^{\alpha'}_{n'}$, again contradicting (5.30). So $t'+1 < s+1 < s'+1$ must hold. Now (5.35) is immediate by choice of t'. Moreover, by choice of t' too, no node $\gamma \leq \beta'$ is initialized at any stage $u' \in [t'+1, s'+1)$ whence no node $\gamma' < \beta'$ becomes active at any such stage. So a node $\gamma' < \beta'$ is eligible at stage s if and only if it is eligible at stage s'. By definition of F this implies that $F(\beta', s) = F(\beta', s')$. Equation (5.33) follows since $|B^{\alpha'}_{n'}| = F(\beta', s')$.

It remains to establish (5.31). By assumption $\alpha' \prec \alpha$ hence $\alpha' <_{left} \alpha$ or $\alpha \sqsubset \alpha'$. In the former case, (5.31) is immediate since $\alpha' \sqsubseteq \beta'$ and $\alpha \sqsubseteq \beta$. So, for the remainder of the argument assume that $\alpha \sqsubset \alpha'$ and, for a contradiction, assume that (5.31) fails, i.e. that $\beta' \sqsubset \beta$ or $\beta \sqsubset \beta'$ or $\beta <_{left} \beta'$. If $\beta' \sqsubset \beta$ then, by construction, $B^{\alpha'}_{n'}$ has to be defined before β can become eligible, i.e., $s'+1 < t+1$ whence $\max B^{\alpha'}_{n'} < \min B^{\alpha}_{n}$ contrary to (5.30). Similarly, if $\beta \sqsubset \beta'$ then $s+1 < t'+1$ hence $\max B^{\alpha}_{n} < \min B^{\alpha'}_{n'}$ contrary to (5.30).

This leaves the case that $\beta <_{left} \beta'$. If $s'+1 < t+1$ or $s+1 < s'+1$ then, as above, we may conclude from (5.34) and (5.35) that (5.30) fails (note that in the latter case, $s+1 < t'+1$ by $\beta < \beta'$). So w.l.o.g. $t+1 < s'+1 < s+1$. But since $\alpha \sqsubset \alpha'$ and since $\beta < \beta'$, $V_{\alpha',s'} \subseteq V_{\alpha,s'}$, $F(\beta, s') \leq F(\beta', s')$ and $r(\beta, s') \leq r(\beta', s')$. So since the block $B^{\alpha'}_{n'}$ is suitable for β' at stage $s'+1$, $B^{\alpha'}_{n'}$ or a subblock B of it will be suitable for β at stage $s'+1$ too. So, since β is eligible at stage $s'+1$, β will require attention at stage $s'+1$. Since $\beta < \beta'$ this contradicts the fact that β' receives attention. This completes the proof of (5.31) and the proof of (B_4).

(B_5). Fix α, α' and n such that $B^{\alpha}_{n}\downarrow$ and either $\alpha' \sqsubset \alpha$ or $\alpha <_{left} \alpha'$ and $|\alpha| = |\alpha'|$. It suffices to show $B^{\alpha'}_{n}\downarrow$. (Then the claim follows by induction on $|\alpha|$.) Let β be the priority of B^{α}_{n} and fix the least stage $s+1$ at which β becomes eligible hence requires attention via clause (a). Then the subclauses (iv) and (v) of (a) guarantee that, for any node β' such that either $\beta' \sqsubset \beta$ or $\beta <_{left} \beta'$,

$|\beta| = |\beta'|$ and β and β' are not equivalent, the block associated with β' is defined at stage s. But if $\alpha' \sqsubset \alpha$ then $B_n^{\alpha'}$ is associated with the proper initial segment $\beta' = \beta \restriction \langle |\alpha'|, n\rangle$ of β and if $\alpha <_{left} \alpha'$ and $|\alpha| = |\alpha'|$ then $B_n^{\alpha'}$ is associated with the node $\beta' = \alpha' 1^{|\beta|-|\alpha|}$ and $\beta <_{left} \beta'$, $|\beta| = |\beta'|$ and β and β' are not equivalent. So in either case $B_n^{\alpha'} \downarrow$.

(B_6). The proof is indirect. Assume that $\overline{M}$ is infinite and that there is a node α and a number n such that $TP \restriction |\alpha| \leq_{left} \alpha$ and B_n^α is not defined. Fix $q = \langle m, n\rangle$ minimal such that there is a node α of length m such that $TP \restriction m \leq_{left} \alpha$ and B_n^α is not defined and fix the rightmost corresponding α. Moreover, let β be the rightmost B_n^α-node. (Note that $\beta = \alpha 1^{q-m}$. In particular, $\alpha \sqsubseteq \beta$, $|\beta| = q$ and, by $TP \restriction |\alpha| \leq_{left} \alpha$ and by definition of β, $TP \restriction q \leq_{left} \beta$.)

We claim that there is a stage s^* such that no node β' with $\beta' < \beta$ which is not equivalent to β requires attention after stage s^*. This is shown as follows. Note that any node β' with $\beta' < \beta$ which is not equivalent to β is element of one of the following sets.

$$
\begin{aligned}
N_0 &= \{\beta' : |\beta'| \leq |\beta| \;\&\; \beta' <_{left} TP \restriction |\beta'|\} \\
N_1 &= \{\beta' : |\beta'| < |\beta| \;\&\; TP \restriction |\beta'| \leq_{left} \beta'\} \\
N_2 &= \{\beta' : |\beta'| = |\beta| \;\&\; TP \restriction |\beta| \leq_{left} \beta' <_{left} \beta \;\&\; \beta' \text{ is not a } B_n^\alpha\text{-node}\} \\
N_3 &= \{\beta' : |\beta| < |\beta'| \;\&\; \beta' <_{left} \beta\}
\end{aligned}
$$

So it suffices to show that for $i \leq 4$ there is a stage s_i such that no node in N_i requires attention after stage s_i.

$i = 0$. Fix t_0 minimal such that $TP \restriction q < \delta_s$ for all stages $s \geq t_0$. Then no $\beta' \in N_0$ can become eligible after stage t_0. So whenever a node $\beta' \in N_0$ requires attention at a stage $s+1 > t_0$, either the block associated with β' becomes defined (namely if β' acts at stage $s+1$) or β' becomes initialized (namely if a higher priority node $\beta'' < \beta'$ acts at stage $s+1$). In either case β' will not require attention after stage $s+1$. Since N_0 is finite, this gives the existence of the desired stage s_0.

$i = 1$. Note that by minimality of q, any node $\beta' \in N_1$ is associated with a block which eventually becomes defined. Since N_1 is finite, this gives the existence of the desired stage s_1.

$i = 2$. Note that, for any $\beta' \in N_2$, $\beta' <_{left} \beta$, $|\beta'| = |\beta|$ and β' and β are not equivalent. Since the block B_n^α associated with β is never defined, it follows, by clause (v) in the definition of requiring attention via (a), that no node in N_1 will ever require attention via (a). So $s_2 = 0$ will do.

$i = 3$. If $\beta' \in N_3$ then, for the proper initial segment $\beta'' = \beta' \upharpoonright |\beta|$ of β' of length $|\beta|$, either $\beta'' \in N_2$ or β'' is a B_n^α-node. In either case the block associated with β'' is never defined. So, by clause (iv) in the definition of requiring attention via (a), β' does not require attention via (a), hence does not require attention. So $s_3 = 0$ will do.

Having established the existence of s^*, we next claim that there is a stage $t^* > s^*$ and a B_n^α node $\hat{\beta}$ such that $\hat{\beta}$ is eligible at all stages $s \geq t^*$. Since, by choice of s^*, a B_n^α-node β' can be initialized at a stage $s+1 > s^*$ only if a B_n^α-node β'' to the left of it becomes active at stage $s+1$, and since by $B_n^\alpha \uparrow$ this implies that β'' acts via clause (a) hence becomes eligible at stage $s+1$, it suffices to show that some B_n^α-node β' will be eligible at some stage $s+1 > s^*$. For a contradiction assume that such β' and $s+1$ do not exist. By minimality of q and maximality of α, we may fix a stage $s^{**} \geq s^*$ such that for any $q' = \langle m', n' \rangle < n$ the block $B_{n'}^{\beta \upharpoonright m'}$ is defined at stage s^{**} and, for any α' with $|\alpha'| = |\alpha|$ and $\alpha <_{left} \alpha'$, the block $B_n^{\alpha'}$ is defined at stage s^{**} too. Then, for the rightmost B_n^α-node β and any $s \geq s^{**}$, the subclauses (i) (by $B_n^\alpha \uparrow$), (iii) (by assumption) and (iv) and (v) (by choice of s^{**}) in the definition of requiring attention (a) hold at stage s. So if we let s be the least stage $\geq s^{**}$ such that $TP \upharpoonright q \sqsubset \delta_s$ then β requires attention via (a), at stage $s+1$ hence becomes eligible (since by assumption and by choice of s^* no higher priority node requires attention). Contradiction.

So, for the remainder of the argument we may fix the B_n^α-node $\hat{\beta}$ which is permanently eligible after stage t^*. In order to get the final contradiction, we show that, eventually, there is a stage $s+1 > t^*$ such that $\hat{\beta}$ requires attention via (b) at stage $s+1$. Since no higher priority node requires attention after stage t^*, it follows that the block B_n^α becomes defined at stage $s+1$ contrary to choice of α and n. Now, by choice of t^*, for any node $\beta' \leq \hat{\beta}$, eligibility of β' does not change after stage t^*. So $r(\hat{\beta}, s) = r(\hat{\beta}, t^*)$ and $F(\hat{\beta}, s) = F(\hat{\beta}, t^*)$.

So in order to show that $\hat{\beta}$ eventually requires attention via (b), it suffices to show that there is a stage $s \geq t^*$ such that

$$|\bigcup_{\{\alpha':|\alpha'|=|\alpha| \text{ and } \alpha' \leq_{left} \alpha\}} V_{\alpha',s}| > r(\hat{\beta}, t^*) + F(\hat{\beta}, t^*).$$

But, since $TP \upharpoonright m \leq_{left} \alpha$, this follows from the fact, that, by the assumption that $\overline{M}$ is infinite and by the True Path Lemma, $V_{TP \upharpoonright m}$ is infinite.

This completes the proof of (B_6).

(B_7). First note that infinitely many blocks become defined. (Namely, otherwise, it follows that M is finite since, any number y is enumerated into M at stage $s+1$ is less than or equal to the maximum of a block B^α_n defined at stage s. So, by (B_6), infinitely many blocks will be defined contrary to assumption.) Now, call a node β a *block node* if for all $\beta' \sqsubseteq \beta$ the block associated with β' is defined, and let B be the set of all block nodes. Note that any initial segment of a block node is a block node again, and any priority of a block which becomes defined is a block node. Moreover, for $\langle \alpha, n \rangle \neq \langle \alpha', n' \rangle$, the blocks B^α_n and $B^{\alpha'}_{n'}$ (if defined) have different priorities. Since infinitely many blocks become defined, we may conclude that the set B of block nodes is an infinite subtree of the priority tree $T = \{0, 1\}^*$.

Now, by König's Lemma, let p be the leftmost infinite path through B. To show that p has the required properties, first fix α on p and $n \geq 0$. Then $\beta = p \upharpoonright \langle |\alpha|, n \rangle$ is a block node and B^α_n is associated with β. So B^α_n is defined. By (B_4) we may conclude that, for α to the right of the path p, the blocks B^α_n ($n \geq 0$) are defined too. Finally, fix α to the left of p and, for a contradiction, assume that B^α_n is defined for all $n \geq 0$. Then, the set of priorities β_n of the blocks B^α_n, $n \geq 0$, is an infinite subset of nodes in B all extending the node α. By $\alpha <_{left} p$ and by König's Lemma this contradicts the fact that p is the leftmost infinite path through B.

This completes the proof of (B_7) and the proof of Claim 3. $\square$

Next we summarize relevant properties of the use functions ψ_α and the wtt-functionals Ψ_α.

Claim 4. The partial functions ψ_α, $\alpha \in \{0, 1\}^*$, are uniformly computable. Moreover, for any α, the domain of ψ_α (at stage s) is an initial segment of ω, *and ψ_α is strictly increasing on its domain. Finally, ψ_α is total if and only if the blocks B^α_n are defined for all $n \geq 0$.*

Proof. Uniform computability follows by effectivity of (part 1 of) the construction. The second part of the claim is immediate by definition and by (B_3). The third part is immediate by definition. □

Claim 5. The functionals Ψ_α are uniformly computable and, for any X, α and n such that $\Psi^X_\alpha(n)$ is defined, $\psi_\alpha(n)$ is defined and the use of $\Psi^X_\alpha(n)$ is bounded by $\psi_\alpha(n)$. Moreover, for any α and n such that $\psi_\alpha(n)$ is defined, $\Psi^A_\alpha(n)$ is defined too.

Proof. The proof of the first part is straightforward. For a proof of the second part, for a contradiction assume $\psi_\alpha(n)\downarrow$ and $\Psi^A_\alpha(n)\downarrow$. Since Ψ is a wtt-functional, it follows that $\Psi^A_\alpha(n)[s]\uparrow$ for almost all s. Moreover, by (5.22), $g(\langle\alpha, n\rangle, s) = 0$ for almost all s. So there is a least stage s_0 such that $\psi_\alpha(n)[s_0]\downarrow$ and $\Psi^A_\alpha(n)\uparrow$ and $g(\langle\alpha, n\rangle, s) = 0$ for all stages $s \geq s_0$. By clause (5.26) in the definition of Ψ, this implies $\Psi^A_\alpha(n)[s_0 + 1]\downarrow$. Contradiction. □

Note that the first part of Claim 5 justifies that in advance we have fixed a computable function f satisfying (5.21).

Claim 6. Assume that ψ_α is total. Then the following hold.

(i) Ψ^A_α *is total.*

(ii) There is a number n_α such that, for any $n \geq n_\alpha$, there is a stage s such that $k(\langle\alpha, n\rangle, s) = 1$.

Proof. Part (i) is immediate by the second part of Claim 5. Part (ii) follows from part (i) by (5.23) and (5.25). □

For the remaining claims we need some more notation. Let p be the unique path through T defined in (B_7). Then, for any node α such that $\alpha \sqsubset p$ or $p <_{left} \alpha$, all blocks B^α_n are defined. So, by Claims 4 and X6, ψ_α and Ψ^A_α are total and we may fix n_α such that $\lim_{s\to\infty} k(\langle\alpha, n\rangle, s) = 1$ for all $n \geq n_\alpha$. It follows that, for $n \geq n_\alpha$, the block B^α_n will eventually become truly realized. So, if we let $x_\alpha = \max B^\alpha_{n_\alpha}$, then all numbers $x \geq x_\alpha$ are eventually truly α-covered.

Hence, for such x we may fix n^α_x and s^α_x such that n^α_x is the unique n such that x becomes covered by B^α_n and s^α_x is the least stage s such that x is truly covered by $B^\alpha_{n^\alpha_x}$ at stage s.

Claim 7. Let $\alpha \sqsubset p$ and let $x \geq x_\alpha$. There is a node $\alpha' \preceq \alpha$, a number $n \geq 0$ and a stage t such that the block $B^{\alpha'}_n$ covers x and is admissible at all stages $s \geq t$ (hence is never frozen).

Proof. Note that, by (B_2), there are only finitely many blocks which may cover x. So there is a stage t_0 such that any block which covers x and becomes frozen is frozen by stage t_0. So it suffices to show that, for almost all stages s, there is a block $B^{\alpha'}_n$ such that $\alpha' \preceq \alpha$, $B^{\alpha'}_n$ covers x and $B^{\alpha'}_n$ is admissible at stage s. This is established by proving the following two facts. (a) There is a stage s such that x is covered by a block $B^{\alpha'}_n$ where $\alpha' \preceq \alpha$ and $B^{\alpha'}_n$ is admissible at stage s. (b) If x is covered by a block $B^{\alpha'}_n$ which is admissible at stage s then, at any stage $s' > s$, x is covered by a block $B^{\alpha''}_{n'}$ such that $\alpha'' \preceq \alpha'$ and $B^{\alpha''}_{n'}$ is admissible at stage s'.

For a proof of (a) recall that x will be truly α-covered eventually. So there is a stage s and a number n such that the block B^α_n truly covers x at stage s. If B^α_n is not frozen at stage s then B^α_n is admissible at stage s and we are done. Otherwise, there is a stage $\hat{s} \leq s$ such that B^α_n becomes frozen at stage $\hat{s}$. But, by construction, this implies that there is a block $B^{\alpha'}_{n'}$ such that $\alpha' \preceq \alpha$, $B^{\alpha'}_{n'}$ covers x and $B^{\alpha'}_{n'}$ is admissible at stage $\hat{s}$. So (a) holds in this case too.

For a proof of (b), it suffices to consider the case of $s' = s + 1$. (Then the general case follows by induction.) So assume that $B^{\alpha'}_n$ covers x and is admissible at stage s. If $B^{\alpha'}_n$ does not become frozen at stage $s + 1$ then we are done. Otherwise, it follows by construction that there is a block $B^{\alpha''}_{n'}$ such that $\alpha'' \preceq \alpha'$, $B^{\alpha''}_{n'}$ covers x and $B^{\alpha''}_{n'}$ is admissible at stage $s + 1$. This completes the proof of (b) and the proof of the claim. □

Claim 8. Assume that B^α_n becomes defined and is never frozen. Then, for the core $\hat{B}^\alpha_n$ of B^α_n, $\hat{B}^\alpha_n \cap \overline{M} \neq \emptyset$. Similarly, if B^α_n is defined but not frozen at stage s then $\hat{B}^\alpha_n[s+1] \cap \overline{M_s} \neq \emptyset$.

Proof. We prove the first part of the claim. The second part is obtained by straightforward modifications of the proof.

We first show that a number $y \in \hat{B}^\alpha_n$ can be enumerated into M only if it is an $\langle \alpha, n\rangle$-coding number. For a contradiction assume that $y \in \hat{B}^\alpha_n$ is enumerated into M at stage $s+1$ and y is not an $\langle \alpha, n\rangle$-coding number. Then y cannot be enumerated into M as a nonblock number according to clause (ii). (Namely, if so, $B^\alpha_n[s+1] \uparrow$. Hence B^α_n becomes defined at a stage $t+1 > s+1$. But, by (B_0) this implies that $B^\alpha_n \cap M_{s+1} = \emptyset$ hence $y \notin B^\alpha_n$. The claim follows since $\hat{B}^\alpha_n \subseteq B^\alpha_n$.) Since B^α_n is never frozen, this leaves the case that $y \in \hat{B}^{\alpha'}_{n'}[s+1]$ for some $\langle \alpha', n'\rangle \neq \langle \alpha, n\rangle$ and y becomes enumerated into M since $B^{\alpha'}$ becomes frozen at stage $s+1$ or y is an $\langle \alpha', n'\rangle$-coding number. Since, by (B_0), y can't be in a block which is not yet defined at stage $s+1$, it follows by $y \in \hat{B}^{\alpha'}_{n'}[s+1]$ and by definition of the core $\hat{B}^{\alpha'}_{n'}$ that $y \in \hat{B}^{\alpha'}_{n'}$. So it suffices to show that $\hat{B}^{\alpha'}_{n'} \cap \hat{B}^\alpha_n = \emptyset$. Since the core of a block is contained in the block this is done as follows. If $\alpha' = \alpha$ the claim is immediate. So, by symmetry, w.l.o.g. $\alpha' \prec \alpha$. But then, by definition of $\hat{B}^\alpha_n$, $B^{\alpha'}_{n'} \cap \hat{B}^\alpha_n = \emptyset$.

Now, by the above and by construction, a number $y \in \hat{B}^\alpha_n$ is enumerated into M at stage $s+1$ only if B^α_n is admissible at stage s (hence $k(\langle \alpha, n\rangle, s) = 1$) and (5.28) or (5.29) holds. Moreover, at any such stage $s+1$ at most one number $y \in \hat{B}^\alpha_n$ is enumerated into M. So, by (B_0) and (B_4), it suffices to show that

$$|\{s \geq s_0 :\ (5.28) \text{ or } (5.29) \text{ holds}\}| < h(\langle \alpha, n\rangle) + 2 \tag{5.36}$$

where s_0 is minimal such that $k(\langle \alpha, n\rangle, s_0) = 1$.

Since between any two stages $s < s'$ for which (5.29) holds there must be a stage t such that $g(\langle \alpha, n\rangle, t) = 0$ and $g(\langle \alpha, n\rangle, t+1) = 1$,

$$\begin{aligned} 2 \cdot |\{s \geq s_0 :\ (5.29) \text{ holds}\}| &\leq |\{s \geq s_0 : g(\langle \alpha, n\rangle, s+1) \neq g(\langle \alpha, n\rangle, s)\}| + 1 \\ &\leq h(\langle \alpha, n\rangle) + 1 \end{aligned} \tag{5.37}$$

where the second inequality holds by (5.24). Moreover, since $\Psi^A_\alpha(n) \downarrow$ by Claim 5, any stage s at which (5.28) holds has to be followed by a stage $t > s$ such that $\Psi^A_\alpha(n)[t] \uparrow$ and $\Psi^A_\alpha(n)[t+1] \downarrow$, where $t < s'$ for the least stage $s' > s$ such that (5.28) holds (if there is such a stage s'). Since, by construction, $g(\langle \alpha, n\rangle, t) = 0$

and $g(\langle\alpha, n\rangle, s') = 1$ for any such stage t, it follows that

$$|\{s \geq s_0 :\ (5.28) \text{ holds}\}| \leq |\{s \geq s_0 :\ (5.29) \text{ holds}\}|$$

holds. So, by (5.37), (5.36) holds. $\square$

Claim 9. $A \leq_{ibT} M$.

Proof. It suffices to give an effective procedure which computes $A(x)$ from $M \upharpoonright x + 1$ for all sufficiently large x.

Let $x \geq x_\lambda$ and let s be the least stage such that there is a node α and a number n such that

(I) B_n^α covers x at stage s,

(II) B_n^α is admissible at stage s,

(III) $\Psi_\alpha^A(n)[s]\downarrow$ and $g(\langle\alpha, n\rangle, s) = 1$, and

(IV) $M \upharpoonright x + 1 = M_s \upharpoonright x + 1$.

Note that such a stage s exists. (Namely, by Claim 7, there is a block B_n^α which covers x and which is admissible at all sufficiently large stages. So (I) and (II) hold for all sufficiently large s. Moreover, since $\Psi_\alpha^A(n)\downarrow$ (by the second part of Claim 5), it follows that (III) holds for all sufficiently large s too (by (5.22)). Finally, (IV) obviously holds for all sufficiently large s.) Moreover, for any stage s, we can effectively check whether, among the finitely many blocks defined at stage s, there is a block B_n^α satisfying (I) - (III). So we can find the above stage s by using $M \upharpoonright x + 1$ as an oracle.

We claim that $A(x) = A_s(x)$. For a proof, first note that B_n^α does not become frozen after stage s hence is admissible at all later stages. (To wit, if B_n^α becomes frozen at stage $s' + 1 > s$ then $\hat{B}_n^\alpha[s']$ is completely enumerated into M at stage $s' + 1$ whence, by the second part of Claim 8, there is a number $y \in \hat{B}_n^\alpha[s' + 1]$ such that $y \in M_{s'+1} \setminus M_{s'}$. Since $\hat{B}_n^\alpha[s' + 1]$ is contained in B_n^α and $\max B_n^\alpha \leq x$ it follows that $y \leq x$ hence $M \upharpoonright x + 1 \neq M_s \upharpoonright x + 1$ contrary to (iv).) Now, for a contradiction, assume that $A(x) \neq A_s(x)$. Fix $s' \geq s$ minimal such that a number $x' \leq x$ is enumerated into A at stage $s' + 1$. Then, assuming that

$\Psi^A_\alpha(n)[s'] \downarrow$ and $g(\langle\alpha, n\rangle, s') = 1$, $\Psi^A_\alpha(n)[s'+1] \uparrow$ by construction. So, in any case, there is a least stage s'' such that $s \leq s'' \leq s'$ and such that (5.28) or (5.29) holds for s' (in place of s). It follows, by construction and by Claim 8, that there is a number $y \in \hat{B}^\alpha_n[s'+1]$ which is newly enumerated into M at stage $s'+1$. But, as observed before, this contradicts (iv). □

Claim 10. $\overline{M}$ *is infinite.*

Proof. By (B_2) and by Claim 7 there are infinitely many blocks which are never frozen. So the claim follows by Claim 8. □

Claim 11. For any node $\alpha \sqsubset p$ *there are only finitely many blocks* $B^{\alpha'}_n$ *such that* $\alpha \prec \alpha'$, $n \geq 0$ *and* $B^{\alpha'}_n$ *is never frozen.*

Proof. Fix $\alpha \sqsubset p$. By (B_2) it suffices to show that any block $B^{\alpha'}_n$ such that $\alpha \prec \alpha'$ and $x_\alpha \leq \max B^{\alpha'}_n$ becomes frozen eventually. So fix such a block $B^{\alpha'}_n$ and, for a contradiction, assume that $B^{\alpha'}_n$ is never frozen. Note that, by $\alpha \sqsubset p$ and $\alpha \prec \alpha'$, α' is on p or to the right of p whence $B^{\alpha'}_n$ becomes defined, say at stage $s+1$. By Claim 7 we may fix a stage $t \geq s+1$ such that, for any of the finitely many numbers x covered by $B^{\alpha'}_n$ there is a block $B^{\alpha_x}_{n_x}$ such that $\alpha_x \preceq \alpha$ (hence $\alpha_x \prec \alpha'$), $B^{\alpha_x}_{n_x}$ covers x and $B^{\alpha_x}_{n_x}$ is admissible at all stages $t' \geq t$. So $B^{\alpha'}_n$ is freezable at all stages $t' \geq t$. Since there are only finitely many blocks $B^{\hat{\alpha}}_{\hat{n}}$ such that $\langle|\hat{\alpha}|, \hat{n}\rangle < \langle|\alpha'|, n\rangle$ or $\langle|\hat{\alpha}|, \hat{n}\rangle = \langle|\alpha'|, n\rangle$ and $\alpha' <_{left} \hat{\alpha}$, it follows that $B^{\alpha'}_n$ becomes frozen eventually. Contradiction. □

Claim 12. The true path TP *coincides with the path* p.

Proof. Claim 10 and (B_6) immediately imply that $p \leq_{left} TP$. For a proof of the converse, i.e., $TP \leq_{left} p$, it suffices to show that, for any given node $\alpha' <_{left} TP$, only finitely many α'-blocks become defined. Now, by Claim 10 and by the second part of the Infinity Lemma (Claim 1), there are only finitely many stages s such that $\delta_s <_{left} \alpha'$ or $\alpha' \sqsubset \delta_s$. So only finitely many α'-nodes can become eligible, hence only finitely many α'-blocks can be defined. □

Claim 13. For any α *on* TP, $\overline{M} \subseteq^* \hat{V}_\alpha$.

Proof. Fix $\alpha \sqsubset TP$. Since (by (B_1) and (5.19)) $\overline{M} \cap B^{\alpha'}_n \subseteq \hat{V}_\alpha$ for any block

$B_n^{\alpha'}$ such $\alpha' \preceq \alpha$, it suffices to show

$$\overline{M} \subseteq^* \bigcup_{\{(\alpha',n):\ \alpha' \preceq \alpha, n \geq 0 \text{ and } B_n^{\alpha'}\downarrow\}} B_n^{\alpha'}. \tag{5.38}$$

For a proof of (5.38), first recall that (by Claim 12) the true path TP coincides with the path p. So, by Claim 11, we may let B be the finite union of the blocks $B_n^{\alpha'}$ such that $\alpha \prec \alpha'$, $n \geq 0$ and $B_n^{\alpha'}$ is never frozen. Now, call a number y a *block number* if y is element of some block, and call a block number y an α'-*number* if α' is $\prec$-minimal such that y is in an α'-block. (Note that any number is element of at most finitely many blocks. So α' is well defined.) Then it suffices to show that any number $y \in \overline{M}$ which is not an α'-number for some $\alpha' \preceq \alpha$ is an element of B. So fix such y. We first observe that y is a block number. Namely, since there are infinitely many blocks, it follows by (B_2) that there is a stage s such that y is less than the maximum of a block defined at stage s. So if y is not a block number then y is enumerated into M at stage $s+1$ for the least such s contrary to choice of y. So we may fix α' and the corresponding unique n such that y is an α'-number and $y \in B_n^{\alpha'}$. It suffices to show that $B_n^{\alpha'}$ is contained in B. For a contradiction, assume that this is not the case. Since, by choice of y, $\alpha \preceq \alpha'$, this implies that there is a stage $s+1$ at which $B_n^{\alpha'}$ becomes frozen. So $\hat{B}_n^{\alpha}[s+1] \subseteq M_{s+1}$ by construction. But since y is an α'-number, y is in the core $\hat{B}_n^{\alpha'}$ of $B_n^{\alpha'}$. Since, obviously, $\hat{B}_n^{\alpha'} \subseteq \hat{B}_n^{\alpha'}[s+1]$ it follows that $y \in M$ contrary to assumption.

This completes the proof of Claim 13. □

Claim 14. M is maximal.

Proof. By effectivity of the construction and by Claims 10 and 13, the hypotheses of the Maximal-Set Lemma (Claim 2) are satisfied. □

By Claims 9 and 14, M has the required properties. This completes the proof the Theorem 5.3.3. □

In order to complete the proof of the Characterization Theorem it remains to give to the proof of Theorem 5.3.4.

Proof of Theorem 5.3.4. We use the following characterization of the dense simple sets in Robinson [Rob67]: a c.e. set D is dense simple if and only if D is coinfinite and, for every strong array $\{F_n\}_{n\geq 0}$ of mutually disjoint sets, there is a number m such that

$$\forall\, n \geq m\ (|F_n \cap \overline{D}| < n). \tag{5.39}$$

It suffices to define computable functions g, h and k witnessing that A is eventually uniformly wtt-array computable.

Fix a wtt-functional Γ such that $A = \Gamma^D$ and fix a computable function γ such that the use of Γ^D is bounded by γ where w.l.o.g. γ is strictly increasing. Moreover, fix computable enumerations $\{A_s\}_{s\geq 0}$, $\{D_s\}_{s\geq 0}$ and $\{\Gamma_s\}_{s\geq 0}$ of A, D and Γ, respectively, such that the length of agreement function

$$l(s) = \max\{y : A_s \upharpoonright y = \Gamma_s^{D_s} \upharpoonright y\}$$

is strictly increasing in s. (Such enumerations can be obtained by speeding up any given computable enumerations of A, D and Γ.) Note that this ensures

$$(x < l(s)\ \&\ A_{s+1}(x) \neq A_s(x)) \ \Rightarrow\ D_{s+1} \upharpoonright \gamma(x)+1 \neq D_s \upharpoonright \gamma(x)+1 \tag{5.40}$$

for all numbers x and stages s.

Now the computable functions $g, k : \omega^2 \to \{0,1\}$ and $h : \omega \to \omega$ are defined as follows. Define g by letting

$$g(\langle e, y\rangle, s) = \begin{cases} 1 & \text{if } \hat{\Phi}_{e,s}^{A_s}(y)\downarrow, \\ 0 & \text{otherwise,} \end{cases}$$

and let h be the order defined by

$$h(x) = (x+1)^2.$$

Finally, for the definition of k, define the auxiliary uniformly partial computable functions $\tilde{\varphi}_e$ by $\tilde{\varphi}_e(x) = \lim_{s\to\infty} \tilde{\varphi}_{e,s}(x)$ where

$$\tilde{\varphi}_{e,s}(x) = \begin{cases} x + \max\{\hat{\varphi}_{e,s}(y) : y \leq x\} & \text{if } \forall\, y \leq x\ (\hat{\varphi}_{e,s}(y)\downarrow), \\ \uparrow & \text{otherwise.} \end{cases}$$

Note that $\tilde{\varphi}_e$ is defined on an initial segment of ω, $\tilde{\varphi}_e$ is strictly increasing on its domain, $\tilde{\varphi}_e$ majorizes $\hat{\varphi}_e$ on its domain, and $\tilde{\varphi}_e$ is total if $\hat{\varphi}_e$ is total. So, for total $\hat{\Phi}^A_e$, $\tilde{\varphi}_e$ is total, strictly increasing and bounds the use of $\hat{\Phi}^A_e$. Now, the 0-1-valued function k is defined by letting $k(\langle e, x\rangle, s) = 1$ iff

$$\tilde{\varphi}_{e,s}(x)\downarrow\ \&\ l(s) > \tilde{\varphi}_e(x)\ \&\ |\overline{D_s} \upharpoonright \gamma(\tilde{\varphi}_{e,s}(x)) + 1| < \frac{(\langle e, x\rangle + 1)^2}{2}. \tag{5.41}$$

Obviously, the functions g, h and k are computable, and h is an order. Moreover, g is the canonical approximation of $A^\dagger$ whence (5.6) holds. So it only remains to show that the functions g, h and k satisfy conditions (5.7) - (5.9) in Definition 5.3.1 too.

For a proof of (5.7) it suffices to note that k is 0-1-valued and that the three clauses in equation (5.41) that characterize the stages s such that $k(\langle e, y\rangle, s) = 1$ persist if we replace s by a stage $t \geq s$. (For the second clause, recall that the length function $l(s)$ is nondecreasing in s.)

For a proof of (5.8) fix $x = \langle e, y\rangle$ and s such that $k(x, s) = 1$. By definition of g and h, it suffices to show that

$$|\{t \geq s : \hat{\Phi}^{A_t}_{e,t}(y)\downarrow\ \&\ \hat{\Phi}^{A_{t+1}}_{e,t+1}(y)\uparrow\}| < \frac{(\langle e, y\rangle + 1)^2}{2}. \tag{5.42}$$

(Namely, (5.42) guarantees that $g(x, t)$ switches from 1 to 0 less than $(x+1)^2 \cdot 2^{-1}$ times after stage s. So, since g is 0-1-valued, g may change on x after stage s at most $2((x + 1)^2 \cdot 2^{-1})$ $(= h(x))$ times.)

So fix t as in (5.42). Then $A_{t+1} \upharpoonright \hat{\varphi}_e(y) + 1 \neq A_t \upharpoonright \hat{\varphi}_e(y) + 1$. Note that, by $k(x, s) = 1$, (5.41) holds. So, by $\hat{\varphi}_e(y) \leq \tilde{\varphi}_e(y)$ (if defined), by (5.40) and by the first two clauses in (5.41), there is a number $\leq \gamma(\tilde{\varphi}_{e,s}(y))$ that is enumerated into D at stage $t + 1$. But, by the third clause in (5.41), the latter can happen for at most $\frac{(\langle e,y\rangle+1)^2}{2} - 1$ stages $t \geq s$. So (5.8) holds.

Finally, for a proof of (5.9), fix e such that $\hat{\Phi}_e^A$ is total. Then $\tilde{\varphi}_e$ is total, computable and strictly increasing (and so is γ). So we can define a computable partition of ω into nonempty intervals $\{F_n\}_{n\geq 0}$ by letting $F_0 = [0, \gamma(\tilde{\varphi}_e(0))]$ and $F_{n+1} = (\gamma(\tilde{\varphi}_e(n)), \gamma(\tilde{\varphi}_e(n+1))]$. Now, since D is dense simple, there is a number m such that (5.39) holds. So there is a constant c such that

$$\begin{aligned}|\overline{D} \upharpoonright \gamma(\tilde{\varphi}_e(n)) + 1| &= |\overline{D} \upharpoonright 1 + \max F_n| \\ &= \sum_{n' \leq n} |\overline{D} \cap F_{n'}| \leq (\sum_{n' \leq n} n') + c = \frac{n(n+1)}{2} + c\end{aligned}$$

for all $n \geq 0$. Since, by $x \leq \langle e, x\rangle$, $x(x+1) \cdot 2^{-1} + c < (\langle e, x\rangle + 1)^2 \cdot 2^{-1}$ for all sufficiently large x, it follows that, for almost all x, there is a stage s_x such that (5.41) holds for all $s \geq s_x$. Since $\lim_{s\to\infty} k(\langle e, x\rangle, s) = 1$ for any such x, this implies (5.9).

This completes the proof of Theorem 5.3.4. $\square$

5.4 Closure Properties of EUwttAC

In this section, we prove that EUwttAC is closed downwards under $\leq_{wtt}$ and closed under join. The former holds by the following slightly more general result where we do not require that the sets are computably enumerable.

Lemma 5.4.1. *Let A and B be any (not necessarily c.e.) sets such that $A \leq_{wtt} B$ and such that B is e.u.wtt-a.c. Then A is e.u.wtt-a.c., too.*

Proof. Fix computable functions g, k and h such that B is e.u.wtt-a.c. via g, k and h, and, by 1. of Lemma 5.2.4, fix a computable function f such that $\hat{\Phi}_e^A = \hat{\Phi}_{f(e)}^B$ for $e \geq 0$. Then A is e.u.wtt-a.c. via $\tilde{g}, \tilde{k}$ and $\tilde{h}$ where $\tilde{g}(\langle e, x\rangle, s) = g(\langle f(e), x\rangle, s)$, $\tilde{k}(\langle e, x\rangle, s) = k(\langle f(e), x\rangle, s)$ and $\tilde{h}(\langle e, x\rangle) = h(\langle f(e), x\rangle)$ (for $e, x, s \in \omega$). $\square$

For the closure under the join operation (and for some later applications), we need the following technical lemma.

Lemma 5.4.2. *Let A_0 and A_1 be c.e. sets. There exist strictly increasing computable functions $f_0, f_1 : \omega \to \omega$ such that, for all $e, x \in \omega$,*

$$\hat{\Phi}_e^{A_0 \oplus A_1}(x) \downarrow \Leftrightarrow (\hat{\Phi}_{f_0(e)}^{A_0}(x) \downarrow \ \& \ \hat{\Phi}_{f_1(e)}^{A_1}(x) \downarrow) \tag{5.43}$$

and

$$\hat{\Phi}_e^{A_0 \oplus A_1}(x) \downarrow \Rightarrow \ \exists\, i \leq 1 \ (\hat{\Phi}_{f_i(e)}^{A_i}(x) = \hat{\Phi}_e^{A_0 \oplus A_1}(x)). \tag{5.44}$$

Proof. Given computable enumerations $\{A_{i,s}\}_{s \geq 0}$ of A_i ($i \leq 1$), for each $i \leq 1$ and $e \geq 0$ define the functional $\Psi_{i,e}$ by letting, for any set Z and any number x,

$$\Psi_{i,e}^Z(x) \downarrow \Leftrightarrow \ \exists s \ (\hat{\Phi}_e^{A_0 \oplus A_1}(x)[s] \downarrow \ \& \ A_{i,s} \upharpoonright \hat{\varphi}_e(x) + 1 = Z \upharpoonright \hat{\varphi}_e(x) + 1)$$

and by setting

$$\Psi_{i,e}^Z(x) = \hat{\Phi}_e^{A_0 \oplus A_1}(x)[s]$$

for the least such s if $\Psi_{i,e}^Z(x)$ is defined. Note that the use of $\Psi_{i,e}^Z(x)$ is bounded by $\hat{\varphi}_e(x)$, and, for $i \leq 1$, $\{\Psi_{i,e}\}_{e \geq 0}$ is a uniformly computable sequence of wtt-functionals. So, by 1. of Lemma 5.2.3, there is a strictly increasing computable function f_i such that $\Psi_{i,e} = \hat{\Phi}_{f_i(e)}$. We claim that f_0 and f_1 are as desired.

Note that $\hat{\Phi}_e^{A_0 \oplus A_1}(x) \downarrow$ trivially implies that $\hat{\Phi}_{f_0(e)}^{A_0}(x)$ and $\hat{\Phi}_{f_1(e)}^{A_1}(x)$ are defined. So, assuming that $\hat{\Phi}_{f_0(e)}^{A_0}(x)$ and $\hat{\Phi}_{f_1(e)}^{A_1}(x)$ are defined, it suffices to show that $\hat{\Phi}_{f_i(e)}^{A_i}(x) = \hat{\Phi}_e^{A_0 \oplus A_1}(x)$ for some $i \leq 1$. By assumption, for $i \leq 1$ fix the least stage s_i such that $\hat{\Phi}_e^{A_0 \oplus A_1}(x)[s_i] \downarrow$ and $A_{i,s_i} \upharpoonright \hat{\varphi}_e(x) + 1 = A_i \upharpoonright \hat{\varphi}_e(x) + 1$ holds. Then, for $s = \max\{s_0, s_1\}$, it follows by the use-principle that $\hat{\Phi}_e^{A_0 \oplus A_1}(x) = \hat{\Phi}_e^{A_0 \oplus A_1}(x)[s]$. So, for the least $i \leq 1$ such that $s = s_i$, we may deduce that $\hat{\Phi}_{f_i(e)}^{A_i}(x) = \hat{\Phi}_e^{A_0 \oplus A_1}(x)$. □

By applying Lemma 5.4.2, now we can prove that EUwttAC is closed under join.

Lemma 5.4.3. *Let A_0 and A_1 be c.e. e.u.wtt-a.c. sets. Then $A_0 \oplus A_1$ is e.u.wtt-a.c. too.*

Proof. Fix computable functions g_i, k_i and h_i such that A_i is e.u.wtt-a.c. via g_i, k_i and h_i ($i \leq 1$). By Lemma 5.4.2, fix computable functions $f_i : \omega \to \omega$ ($i \leq 1$) such that (5.43) holds. Define the functions g, k and h by letting

$$g(\langle e, x\rangle, s) = g_0(\langle f_0(e), x\rangle, s) \cdot g_1(\langle f_1(e), x\rangle, s),$$
$$k(\langle e, x\rangle, s) = k_0(\langle f_0(e), x\rangle, s) \cdot k_1(\langle f_1(e), x\rangle, s), \text{ and}$$
$$h(\langle e, x\rangle) = h_0(\langle f_0(e), x\rangle) + h_1(\langle f_1(e), x\rangle)$$

(for all $e, x, s \in \omega$). We claim that $A_0 \oplus A_1$ is e.u.wtt-a.c. via g, k and h. Obviously, the functions g, k and h are computable. So it suffices to show (5.6) – (5.9) for $A = A_0 \oplus A_1$. Now, by choice of g_i and k_i, (5.6) is immediate by (5.43) and (5.7) is immediate. For a proof of (5.8), note that, for any $e, x, s \in \omega$, $g(\langle e, x\rangle, s+1) \neq g(\langle e, x\rangle, s)$ implies that $g_0(\langle f_0(e), x\rangle, s+1) \neq g_0(\langle f_0(e), x\rangle, s)$ or $g_1(\langle f_1(e), x\rangle, s+1) \neq g_1(\langle f_1(e), x\rangle, s)$ and $k(\langle e, x\rangle, s) = 1$ implies $k_0(\langle f_0(e), x\rangle, s) = k_1(\langle f_1(e), x\rangle, s) = 1$. So, by choice of g_i, k_i and h_i, $k(\langle e, x\rangle, s) = 1$ implies

$$\begin{aligned}
& |\{t \geq s : g(\langle e, x\rangle, t+1) \neq g(\langle e, x\rangle, t)\}| \\
\leq\ & |\{t \geq s : g_0(\langle f_0(e), x\rangle, t+1) \neq g_0(\langle f_0(e), x\rangle, t)\}| + \\
& |\{t \geq s : g_1(\langle f_1(e), x\rangle, t+1) \neq g_1(\langle f_1(e), x\rangle, t)\}| \\
\leq\ & h_0(\langle f_0(e), x\rangle) + h_1(\langle f_1(e), x\rangle) \\
=\ & h(\langle e, x\rangle).
\end{aligned}$$

Finally, for a proof of (5.9), fix e such that $\hat{\Phi}_e^{A_0 \oplus A_1}$ is total. Then, by (5.43), $\hat{\Phi}_{f_0(e)}^{A_0}$ and $\hat{\Phi}_{f_1(e)}^{A_1}$ are total too. So, by choice of k_0 and k_1, there is a number x_0 such that $\lim_{s\to\infty} k_0(\langle f_0(e), x\rangle, s) = 1$ and $\lim_{s\to\infty} k_1(\langle f_1(e), x\rangle, s) = 1$ for all $x \geq x_0$. By definition of k, this implies that $\lim_{s\to\infty} k(\langle e, x\rangle, s) = 1$ $x \geq x_0$. □

The above closure properties of EUwttAC show that the wtt-degrees of the c.e. e.u.wtt-a.c. sets are an ideal in the upper semilattice of the c.e. wtt-degrees. Moreover, by the Characterization Theorem, this ideal intersects all high c.e.

Turing degrees.

Theorem 5.4.4. *The class* $\mathrm{EUwttAC}_{wtt}$ *of the wtt-degrees of c.e. e.u.wtt-a.c. sets is an ideal in the upper semilattice of the c.e. wtt-degrees. Moreover, for any high c.e. Turing degree* $\mathbf{a}$*, there is a c.e. set* $A \in \mathbf{a}$ *such that* $deg_{wtt}(A) \in \mathrm{EUwttAC}_{wtt}$*.*

Proof. The first part of the theorem is immediate by Lemmas 5.4.1 and 5.4.3. For the second part of the theorem, note that, by Theorem 5.3.2, any maximal set is e.u.wtt-a.c. So the claim follows by Martin's Theorem [Mar66] which asserts that any high c.e. Turing degree contains a maximal set. □

In the remainder of the paper we relate the eventually uniformly wtt-array computable sets to the wtt-superlow sets and to the array computable sets. As we will show this provides strict lower respectively upper bounds. So let us first introduce wtt-superlow sets.

5.5 Weak Truth-Table Superlow Sets

5.5.1 Wtt-Superlow Sets and EUwttAC

The definition of wtt-superlow sets is as follows.

Definition 5.5.1. *A (not necessarily c.e.) set A is* wtt-superlow *if* $A^\dagger \leq_{tt} \emptyset'$*.*

In order to show that any (not necessarily c.e.) wtt-superlow set is eventually uniformly wtt-array computable, we characterize the wtt-superlow sets in terms of approximability of their bounded jumps. We first recall the relevant notions needed. A total function $f : \omega \to \omega$ is called *h-computably approximable via g* or *h-c.a. via g* for short, if $g : \omega^2 \to \omega$ is a computable function and $h : \omega \to \omega$ is a computable order such that $f(x) = \lim_{s\to\infty} g(x, s)$ and $|\{s : g(x, s+1) \neq g(x, s)\}| \leq h(x)$ (for any x), i.e., g is a computable approximation of f where the number of mind changes of g is computably bounded by h; f is called *h-computably approximable (h-c.a.)* if f is h-computably approximable (h-c-a.) via some computable function $g : \omega^2 \to \omega$; and f is *ω-computably approximable* or *ω-c.a.* for short if f is h-c.a. for some computable order h. (Note that if the range of f is bounded, say $f(x) \leq k$ for all x, then we may assume that the approximating function g is also bounded by k. So if A is an ω-c.a. set and g approximates A in the limit then in the following we tacitly assume that g is 0-1 valued.)

Lemma 5.5.2. *Let A be any (not necessarily c.e.) set. Then the following are equivalent.*

1. *A is wtt-superlow, i.e., $A^\dagger \leq_{tt} \emptyset'$.*
2. *$A^\dagger \leq_{wtt} \emptyset'$.*
3. *$A^\dagger$ is ω-c.a.*
4. *There exists an order h such that any set B which is bounded-c.e. in A is h-c.a.*

Proof. The equivalence of the first three properties 1., 2. and 3. is immediate by the general fact that, for *any* set B, $B \leq_{tt} \emptyset'$ iff $B \leq_{wtt} \emptyset'$ iff B is ω-c.a., see, e.g., [Odi99, III.8.14] and [DH10, Corollary 2.6.2]. Moreover, the implication "4. $\Rightarrow$ 3." is immediate too since $A^\dagger$ is bounded-c.e. in A. This leaves the implication "3. $\Rightarrow$ 4.".

So suppose that $A^\dagger$ is ω-c.a. Fix a computable function $g : \omega^2 \to \{0, 1\}$ and a computable order $\hat{h}$ such that $A^\dagger(x) = \lim_{s\to\infty} g(x, s)$ and $|\{s : g(x, s + 1) \neq g(x, s)\}| \leq \hat{h}(x)$ hold for all x. We claim that any bounded A-c.e. set is h-c.a. for the order $h(x) = \hat{h}(\langle x, x\rangle)$. So let B be a bounded A-c.e. set. Fix $e \in \omega$ such that $B = \mathrm{dom}(\hat{\Phi}^A_e)$. Then $x \in B$ iff $\langle e, x\rangle \in A^\dagger$. Define the computable function $\tilde{g} : \omega^2 \to \{0, 1\}$ by letting

$$\tilde{g}(x, s) = \begin{cases} B(x) & \text{if } x < e, \\ g(\langle e, x\rangle, s) & \text{otherwise.} \end{cases}$$

By definition, $B(x) = \lim_{s\to\infty} \tilde{g}(x, s)$ holds for all x. So it suffices to show that the number of mind changes of $s \mapsto \tilde{g}(x, s)$ is bounded by $h(x)$ for any x. The latter clearly holds if $x < e$. On the other hand, for $x \geq e$ we may argue that

$$\begin{aligned} & |\{s : \tilde{g}(x, s+1) \neq \tilde{g}(x, s)\}| \\ = & |\{s : g(\langle e, x\rangle, s+1) \neq g(\langle e, x\rangle, s)\}| \\ \leq & \hat{h}(\langle e, x\rangle) \\ \leq & h(x), \end{aligned}$$

where the latter inequality holds since $\hat{h}$ is a computable order. $\square$

Corollary 5.5.3. *Any (not necessarily c.e.) wtt-superlow set is e.u.wtt-a.c.*

Proof. Assume that A is wtt-superlow. Then, by Lemma 5.5.2, $A^\dagger$ is ω-c.a. So we may fix a computable order h and a computable function g such that $A^\dagger$ is h-c.a. via g. It follows that A is eventually uniformly wtt-array computable via g, k and h where we may let k be the constant function $k(x, s) = 1$. $\square$

From Lemma 5.5.2 we can further deduce that the class of the wtt-superlow sets is closed downwards under wtt-reducibility and that the class of the c.e. wtt-superlow sets is closed under join. So the class of the wtt-superlow c.e. wtt-degrees is an ideal in EUwttAC.

Corollary 5.5.4. *(a) Let A and B be any (not necessarily c.e.) sets such that $A \leq_{wtt} B$ and B is wtt-superlow. Then A is wtt-superlow too.*

(b) Let A_0 and A_1 be wtt-superlow c.e. sets. Then $A_0 \oplus A_1$ is wtt-superlow too.

Proof. (a). By wtt-superlowness of B, $B^\dagger \leq_{wtt} \emptyset'$ while, by $A \leq_{wtt} B$ and by part 5. of Lemma 5.2.4, $A^\dagger \leq_{wtt} B^\dagger$. Hence $A^\dagger \leq_{wtt} \emptyset'$. By Lemma 5.5.2 this implies that A is wtt-superlow.

(b). By Lemma 5.4.2 fix computable functions f_i $(i \leq 1)$ satisfying (5.43). Then, for all $e, x \in \omega$, it holds that

$$\langle e, x\rangle \in (A_0 \oplus A_1)^\dagger \;\Leftrightarrow\; \forall i \leq 1 \; (2\langle f_i(e), x\rangle + i \in A_0^\dagger \oplus A_1^\dagger).$$

Hence, $(A_0 \oplus A_1)^\dagger \leq_{tt} A_0^\dagger \oplus A_1^\dagger \leq_{tt} \emptyset'$. $\square$

5.5.2 Wtt-Superlowness and Wtt-Jump Traceability

For computably enumerable sets the equivalent characterizations of wtt-superlowness given in Lemma 5.5.2 can be expanded. In particular, a computably enumerable set A is wtt-superlow iff A is wtt-jump traceable where the latter is defined as follows.

Definition 5.5.5. *A set A is h-*wtt-jump traceable *via $\{V_e\}_{e\in\omega}$ if h is a computable order and $\{V_e\}_{e\in\omega}$ is a uniformly c.e. sequence of finite sets such that, for all $e \geq 0$, $|V_e| \leq h(e)$ and $\hat{J}^A(e)\downarrow$ implies $\hat{J}^A(e) \in V_e$; A is h-*wtt-jump traceable *if there exists a uniformly c.e. sequence $\{V_e\}_{e\in\omega}$ such that A is h-wtt-jump traceable via $\{V_e\}_{e\in\omega}$; and A is* wtt-jump traceable *if there exists a computable order h such that A is h-wtt-jump traceable. If A h-wtt-jump traceable via $\{V_e\}_{e\in\omega}$ then we say that $\{V_e\}_{e\in\omega}$ is an h-*trace *for $\hat{J}^A$.*

Theorem 5.5.6. *For a c.e. set A, A is wtt-superlow if and only if A is wtt-jump traceable.*

By Lemma 5.5.2, Theorem 5.5.6 is immediate by the following two lemmas. In these lemmas, in addition we analyze how the relevant orders are affected if we go from one notion to the other. (This analysis will be used below in the proof of Lemma 5.5.12).

Lemma 5.5.7. *Let A be a c.e. set, let h be a computable order, and suppose that $A^\dagger$ is h-c.a. Then A is $\hat{h}$-wtt-jump traceable for the computable order $\hat{h}(x) = \lceil \frac{h(\langle x,x\rangle)}{2} \rceil + 1$.*

Proof. We adapt some of the techniques from [Nie06, Theorem 4.1] where it is shown that the c.e. superlow sets coincide with the c.e. jump traceable sets.

Fix a computable function $g : \omega^2 \to \{0,1\}$ such that $A^\dagger$ is h-c.a. via g and fix a computable enumeration $\{A_s\}_{s\in\omega}$ of A. We show that there exists a number $d \in \omega$ and a uniformly c.e. sequence $\{V_e\}_{e\in\omega}$ such that A is h'-wtt-jump traceable via $\{V_e\}_{e\in\omega}$ for the computable order $h'(x) = \lceil \frac{h(\langle d,x\rangle)}{2} \rceil$. Then, obviously, A is

$\hat{h}$-wtt-jump traceable via $\{\hat{V}_e\}_{e\in\omega}$ via the uniformly c.e. sequence

$$\hat{V}_e = \begin{cases} \emptyset & \text{if } e < d \text{ and } \hat{J}^A(e)\uparrow \\ \{\hat{J}^A(e)\} & \text{if } e < d \text{ and } \hat{J}^A(e)\downarrow \\ V_e & \text{otherwise.} \end{cases}$$

Now, along with $\{V_e\}_{e\in\omega}$, we define an auxiliary wtt-functional Ψ in stages s where, by the Recursion Theorem, we may assume that in advance we know an index $d \in \omega$ such that $\Psi = \hat{\Phi}_d$ holds (the intuition behind $\Psi^A(x)$ is that its computation is a delayed version of the computation of $\hat{J}^A(x)$). In more detail, we define a uniformly computable sequence of wtt-functionals $\{\tilde{\Psi}_e\}_{e\in\omega}$ (intuitively, for any $e \in \omega$, we have a version for the definition of Ψ, where e is a guess for an index of Ψ). Then, in the construction, we make $\tilde{\Psi}_e^A(x)$ defined (undefined) at a certain stage $s+1$ only if $g(\langle e, x\rangle, s)$ correctly approximates the status of definedness of $\tilde{\Psi}_e^A(x)[s]$. Then, by the Recursion Theorem, there exists a number d such that $\tilde{\Psi}_d = \hat{\Phi}_d$. So $\Psi = \tilde{\Psi}_d$ is as desired. Now the definition of V_e and $\Psi^A(e)$ for given $e \in \omega$ is as follows.

Stage 0. Let $V_{e,0} = \emptyset$ and $\Psi^A(e)[0]\uparrow$.

Stage $s+1$. Let $V_{e,s}$ and $\Psi^A(e)[s]$ be given. If $\hat{\varphi}_e(e)[s]\uparrow$ or if $A_{s+1} \upharpoonright \hat{\varphi}_e(e)+1 \neq A_s \upharpoonright \hat{\varphi}_e(e)+1$ holds then let $\Psi^A(e)[s+1]\uparrow$ and $V_{e,s+1} = V_{e,s}$. Otherwise, distinguish between the following cases.

(i) If $\Psi^A(e)[s]\uparrow$, $\hat{J}^A(e)[s]\downarrow$ and $g(\langle d, e\rangle, s) = 0$ then let $\Psi^A(e)[s+1]\downarrow= \hat{J}^A(e)[s]$ with use $\hat{\varphi}_e(e)$ and let $V_{e,s+1} = V_{e,s}$.

(ii) If $\Psi^A(e)[s]\downarrow$ and $g(\langle d, e\rangle, s) = 1$ then let $\Psi^A(e)[s+1] = \Psi^A(e)[s]$ and $V_{e,s+1} = V_{e,s} \cup \{\Psi^A(e)[s]\}$.

If neither of the previous cases applies then let $\Psi^A(e)[s+1] = \Psi^A(e)[s]$ and $V_{e,s+1} = V_{e,s}$.

By effectivity of the construction, $\{V_e\}_{e\in\omega}$ is uniformly c.e. and Ψ is a wtt-functional. We claim that $\{V_e\}_{e\in\omega}$ and the number d obtained from the Recursion Theorem are as desired. We first prove that $\{V_e\}_{e\in\omega}$ is a trace for $\hat{J}^A$. So let $e \in \omega$ be given such that $\hat{J}^A(e)\downarrow$. Then we may fix the least stage s such that $\hat{J}^A(e)[s]\downarrow$ and $A \upharpoonright \hat{\varphi}_e(e)+1 = A_s \upharpoonright \hat{\varphi}_e(e)+1$. Since $\lim_{s\to\infty} g(\langle d,e\rangle, s) = \mathrm{dom}(\Psi^A)(e)$, it follows that there exists a stage s_0 such that (i) applies at stage s_0+1. So $\Psi^A(e)[s_0+1]\downarrow = \hat{J}^A(e)$ holds by construction; hence, for the least stage $s_1 > s_0$ such that (ii) applies at stage s_1+1, it follows that $\hat{J}^A(e) \in V_{e,s_1+1}$; hence, $\hat{J}^A(e) \in V_e$.

It remains to show that $\{V_e\}_{e\in\omega}$ is an h'-trace. For that, we observe that, by construction, a number x may be enumerated into V_e at stage $s+1$ only if $x = \Psi^A(e)[s]\downarrow$. So if $s_0 < s_1$ are stages such that $\Psi^A(e)[s_0]\downarrow \neq \Psi^A(e)[s_1]\downarrow$ and such that $\Psi^A(e)[s_i]$ enter V_e at stage s_i+1 $(i \leq 1)$ then, by construction, there must be a stage s such that $s \in (s_0, s_1)$ and such that $\Psi^A(e)[s+1]\uparrow$. Thus, by (i), there exists a stage $t \in (s, s_1)$ such that $\Psi^A(e)[t+1]\downarrow$. So, by (ii), we can argue that each new element that enters V_e corresponds to a change of $s \mapsto g(\langle d,e\rangle, s)$ from 1 to 0 and back to 1. Since there are at most $\lceil \frac{h(\langle d,e\rangle)}{2} \rceil$ many such stages, this completes the proof. $\square$

Lemma 5.5.8. *Let A be a c.e. set. There exists a strictly increasing computable function $f : \omega \to \omega$ such that, for any computable order h such that A is h-wtt-jump traceable, $A^\dagger$ is $\tilde{h}$-c.a. via the computable order $\tilde{h}(x) = 2h(f(x))+1$.*

Proof. Fix a computable enumeration $\{A_s\}_{s\geq 0}$ of A and consider the wtt-functional Ψ such that, for any oracle X and any input $e, x \in \omega$, it holds that

$$\Psi^X(\langle e,x\rangle) = \mu s(\hat{\Phi}_e^A(x)[s]\downarrow \;\&\; X \upharpoonright \hat{\varphi}_e(x)+1 = A_s \upharpoonright \hat{\varphi}_e(x)+1) \tag{5.45}$$

and, by 3. of Lemma 5.2.4, let $f : \omega \to \omega$ be a computable function such that $\Psi^X(n) = \hat{J}^X(f(n))$ holds for all oracles X and all numbers n.

Now fix a computable order h and suppose that A is h-wtt-jump traceable. By the latter, fix a uniformly c.e. sequence $\{V_e\}_{e\in\omega}$ which is an h-trace for $\hat{J}^A$.

Then, for all $n, e, x, s \in \omega$, let

$$t(n, s) = \max(V_{f(n),s}), \text{ and} \tag{5.46}$$

$$g(\langle e, x\rangle, s) = \begin{cases} 1 & \text{if } \hat{\Phi}_e^A(x)[t(\langle e, x\rangle, s)] \downarrow \text{ and} \\ & A_s \restriction \hat{\varphi}_e(x) + 1 = A_{t(\langle e,x\rangle,s)} \restriction \hat{\varphi}_e(x) + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.47}$$

We claim that $A^\dagger$ is $\tilde{h}$-c.a. via g for the computable order $\tilde{h}$ as given by the lemma. First of all, we show that $\lim_{s\to\infty} g(\langle e, x\rangle, s) = A^\dagger(\langle e, x\rangle)$ holds for all $e, x \in \omega$. First, suppose that $\hat{\Phi}_e^A(x) \uparrow$. Then $\hat{\Phi}_e^A(x)[s] \uparrow$ holds for almost all stages s; hence, $\lim_{s\to\infty} g(\langle e, x\rangle, s) = 0$, as desired. Otherwise, $\Psi^A(\langle e, x\rangle) \downarrow$; hence, $\hat{J}^A(f(\langle e, x\rangle)) \le t(\langle e, x\rangle, s)$ holds for almost all s by definition of f and by (5.47) which in turn implies that $\lim_{s\to\infty} g(\langle e, x\rangle, s) = 1$, as desired.

In order to show that the number of mind changes of $s \mapsto g(\langle e, x\rangle, s)$ is bounded by $2h(f(\langle e, x\rangle)) + 1$, by the fact that $g(\langle e, x\rangle, 0) = 0$, it suffices to show that the number of stages $s_0 < s_1$ such that $g(\langle e, x\rangle, s_0) = 1$, $g(\langle e, x\rangle, s_0 + 1) = 0$ and such that s_1 is the least stage greater than s_0 such that $g(\langle e, x\rangle, s_1) = 1$ is bounded by $h(f(\langle e, x\rangle))$. For the latter, let $e, x \in \omega$ be given and suppose that $s_0 < s_1$ are as above. We claim that $t(\langle e, x\rangle, s_0) < t(\langle e, x\rangle, s_1)$ holds. Otherwise, since $t(\langle e, x\rangle, s)$ is nondecreasing in s and by (5.47), it follows that

$$\hat{\Phi}_e^A(x)[t(\langle e, x\rangle, s_0)] \downarrow$$

and

$$A_{s_1} \restriction \hat{\varphi}_e(x) + 1 = A_{t(\langle e,x\rangle,s_0)} \restriction \hat{\varphi}_e(x) + 1.$$

Hence, $g(\langle e, x\rangle, s) = 1$ holds for all $s \in [s_0, s_1)$, contrary to choice of stage s_0. So for any such two stages $s_0 < s_1$ there exists a number which is enumerated into $V_{f\langle e,x\rangle}$. As $\{V_e\}_{e\in\omega}$ is an h-trace, this completes the proof. □

5.5.3 A Hierarchy of Wtt-Superlow Sets

We conclude the section by looking at strong variants of wtt-superlowness and by introducing a hierarchy of wtt-superlow sets. By Lemma 5.5.2 a set A is wtt-superlow if there is a computable order h such that $A^\dagger$ is h-c.a. So we may ask whether the function h depends on A or not. In this subsection we show that in general this is the case. In fact, we show that, for any computable order h_1, there is a (faster growing) computable order h_2 such that there is a c.e. set A such that the bounded jump $A^\dagger$ of A is h_2-c.a. but not h_1-c.a. and there is a (slower growing) computable order h_0 such that there is a c.e. set B such that the bounded jump $B^\dagger$ of B is h_1-c.a. but not h_0-c.a. On the other hand, in the next subsection we will show that there are noncomputable – in fact Turing complete – c.e. sets A such that $A^\dagger$ is h-c.a. for all computable orders.

The key to the hierarchy results in this subsection is the following technical lemma.

Lemma 5.5.9. *Let h, $\hat{h}$, H and $\hat{H}$ be computable orders such that, for $n \geq 0$,*

$$\hat{h}(n) = h(\langle n, n\rangle) \ \ \textit{and} \ \ H(n) = 2\hat{H}(n) + 1 \tag{5.48}$$

and such that there are a computable order $neg(n)$ and a strong array $\{F_n\}_{n\geq 0}$ of mutually disjoint finite sets satisfying

$$\forall\, n \ (|F_n| = neg(n) + 1) \tag{5.49}$$

and

$$\forall\, m \ (\sum_{\{n: neg(n)\leq m\}} (\hat{h}(\max F_n) + 1) \leq \hat{H}(m)). \tag{5.50}$$

There is a c.e. set A such that $A^\dagger$ is H-c.a. but not h-c.a.

Proof. By a finite injury argument, we give a computable enumeration $\{A_s\}_{s\geq 0}$ of a c.e. set A with the required properties. We make $A^\dagger$ H-c.a. via the canonical computable approximation $g : \omega^2 \to \{0, 1\}$ of $A^\dagger$ induced by $\{A_s\}_{s\geq 0}$ where (for $e, x \geq 0$)

$$g(\langle e, x\rangle, s) = 1 \ \Leftrightarrow\ \hat{\Phi}^A_e(x)[s] \downarrow. \tag{5.51}$$

For this sake it suffices to ensure that

$$m_g(\langle e, x\rangle) \leq H(\langle e, x\rangle)$$

for all $e, x \geq 0$ where

$$m_g(\langle e,x\rangle) = |\{s : g(\langle e,x\rangle, s+1) \neq g(\langle e,x\rangle, s)\}|$$

is the number of mind changes of g on $\langle e,x\rangle$. In order to achieve this, it suffices to meet the (negative) requirements

$$\mathcal{N}_{\langle e,x\rangle} : \ \hat{\varphi}_e(x) \downarrow \Rightarrow \ |(A \setminus A_{s_{\langle e,x\rangle}}) \upharpoonright \hat{\varphi}_e(x)+1| \leq \hat{H}(\langle e,x\rangle)$$

for $e, x \geq 0$ where $s_{\langle e,x\rangle}$ is the least stage s such that $\hat{\varphi}_{e,s}(x) \downarrow$. Namely, for any stage s such that $g(\langle e,x\rangle, s) = 1$ and $g(\langle e,x\rangle, s+1) = 0$, the definition of g implies that $s \geq s_{\langle e,x\rangle}$ and $A_{s+1} \upharpoonright \hat{\varphi}_e(x)+1 \neq A_s \upharpoonright \hat{\varphi}_e(x)+1$. Since $g(\langle e,x\rangle, s) = 1$ for any other stage s such that $g(\langle e,x\rangle, s+1) \neq g(\langle e,x\rangle, s)$, it follows that

$$\begin{aligned}
m_g(\langle e,x\rangle) &\leq 2 \cdot |\{s : g(\langle e,x\rangle, s) = 1 \ \& \ g(\langle e,x\rangle, s+1) = 0\}| + 1 \\
&\leq 2 \cdot |\{s \geq s_{\langle e,x\rangle} : A_{s+1} \upharpoonright \hat{\varphi}_e(x)+1 \neq A_s \upharpoonright \hat{\varphi}_e(x)+1|\}| + 1 \\
&\leq 2 \cdot |(A \setminus A_{s_{\langle e,x\rangle}}) \upharpoonright \hat{\varphi}_e(x)+1| + 1 \\
&\leq 2 \cdot \hat{H}(\langle e,x\rangle) + 1 \\
&= H(\langle e,x\rangle)
\end{aligned}$$

where the last inequality holds by $\mathcal{N}_{\langle e,x\rangle}$.

In order to guarantee that $A^\dagger$ is not h-c.a., we define an auxiliary wtt-functional Ψ together with a corresponding partial computable use bound ψ such that

$$dom(\Psi) \text{ is not } \hat{h}\text{-c.a.} \tag{5.52}$$

The proof that this guarantees that $A^\dagger$ is not h-c.a. is indirect. For a contradiction assume that $A^\dagger$ is h-c.a. Fix $\hat{g}$ such that $A^\dagger$ is h-c.a. via $\hat{g}$ and fix e such that $\Psi = \hat{\Phi}_e$. Then, for $x \geq 0$, the function $s \mapsto \hat{g}(\langle e,x\rangle, s)$ converges to $dom(\Psi)(x)$

with $\leq h(\langle e, x \rangle)$ mind changes. Since $h(\langle e, x \rangle) \leq \hat{h}(x)$ for all numbers $x \geq e$, this implies that $dom(\Psi)$ is $\hat{h}$-c.a. contrary to (5.52).

Since, for any order h, any h-c.a. set B is h-c.a. via a primitive recursive function, condition (5.52) can be broken up into the (positive) requirements

$$\mathcal{P}_n : \ dom(\Psi) \text{ is not } \hat{h}\text{-c.a. via } g_n.$$

$(n \geq 0)$ where $\{g_n\}_{n \geq 0}$ is a computable numbering of the primitive recursive functions of type $\omega^2 \to \{0, 1\}$.

The basic strategy for meeting requirement $\mathcal{P}_n$ is as follows. We pick a number y, called ($\mathcal{P}_n$-)*follower*, such that $\mathcal{P}_n$ may define Ψ and ψ on y. Then we ensure that the follower y witnesses that $\mathcal{P}_n$ is met by guaranteeing

$$dom(\Psi^A)(y) = \lim_{s \to \infty} g_n(y, s) \ \Rightarrow \ |\{s : g_n(y, s+1) \neq g_n(y, s)\}| > \hat{h}(y). \quad (5.53)$$

For this sake we pick $\hat{h}(y)+1$ numbers $z_0 < z_1 < \cdots < z_{\hat{h}(y)}$, called ($\mathcal{P}_n$-)*attackers*, which are not used as attackers by other strategies, let $\psi(y) = z_{\hat{h}(y)} + 1$ (note that this allows us to make a convergent computation $\Psi^A(y)[s] \downarrow$ divergent at stage $s+1$ by enumerating one of the attackers into A at this stage), and define Ψ on y as follows (where initially $\Psi^A(y)[0] \uparrow$). For any stage s such that $\Psi^A(y)[s] \uparrow$ and $g_n(y, s) = 0$ we let $\Psi^A(y)[s+1] \downarrow$ (note that this does not require to change the oracle A_s) and for any stage s such that $\Psi^A(y)[s] \downarrow$, $g_n(y, s) = 1$ and there is at least one attacker z_i left which is not yet in A, we put the least such z_i into A at stage $s+1$ and let $\Psi^A(y)[s+1] \uparrow$. Obviously, if the hypothesis of (5.53) holds, this guarantees that there are at least $\hat{h}(y) + 1$ stages s such that $g_n(y, s) = 1$ and $g_n(y, s+1) = 0$. So, in particular, (5.53) holds. Moreover, the functional Ψ defined in this way is a wtt-functional with partial computable bound ψ on the use.

Now, in order to make the $\mathcal{P}_n$-strategies compatible with the goal of meeting the negative requirements $\mathcal{N}_{\langle e, x \rangle}$, we have to adjust the basic strategy. In particular, it may happen that the $\mathcal{P}_n$-follower may be cancelled by a negative requirement, and the basic strategy for meeting $\mathcal{P}_n$ has to be started all over again with a new follower (and new attackers).

We say that $\mathcal{P}_n$ *injures* $\mathcal{N}_{\langle e,x\rangle}$ via follower y and corresponding attacker z at stage $s+1$ if $\hat{\varphi}_{e,s}(x)\downarrow$ (i.e., $s_{\langle e,x\rangle} \leq s$), $z \leq \hat{\varphi}_e(x)$ and $\mathcal{P}_n$ enumerates z into A at stage $s+1$. So, since attackers are the only numbers which may enter A, in order to ensure that $\mathcal{N}_{\langle e,x\rangle}$ is met it suffices to guarantee that there are at most $\hat{H}(\langle e,x\rangle)$ stages at which the requirement $\mathcal{N}_{\langle e,x\rangle}$ is injured. In order to achieve this, first we ensure that if a $\mathcal{P}_n$-follower y is appointed at stage $s+1$ then the corresponding attackers z_i are chosen to be $\geq s+1$ (in the actual construction we achieve this by letting $z_i = \langle y, s+1, i\rangle$ which, in addition, ensures that the sets of attackers associated with different followers are disjoint) whence, for any requirement $\mathcal{N}_{\langle e,x\rangle}$ such that $\hat{\varphi}_{e,s}(x)\downarrow$, $\mathcal{P}_n$ will not injure $\mathcal{N}_{\langle e,x\rangle}$ after stage s since $\hat{\varphi}_e(x) < s_{\langle e,x\rangle} \leq s \leq z_i$ for any attacker z_i associated with y (or with any $\mathcal{P}_n$-follower appointed later). Next we assign priorities to the requirements, and we ensure that a negative requirement $\mathcal{N}_{\langle e,x\rangle}$ cannot be injured by any lower priority positive requirement $\mathcal{P}_n$ as follows. If $\hat{\varphi}_e(x)$ becomes defined at stage s (i.e., if $s = s_{\langle e,x\rangle}$) then $\mathcal{N}_{\langle e,x\rangle}$ *initializes* the lower priority positive requirements $\mathcal{P}_n$ at stage s by cancelling the current follower y of $\mathcal{P}_n$ (if any) and the corresponding attackers. So the strategy for meeting $\mathcal{P}_n$ has to be restarted with a new follower and new attackers after stage s thereby guaranteeing that the new attackers are too large to injure $\mathcal{N}_{\langle e,x\rangle}$.

Note that $\mathcal{P}_n$ can be injured by any higher priority negative requirement at most once. So in order to guarantee that there will be a follower y of $\mathcal{P}_n$ left which is never cancelled (whence the basic strategy using follower y will succeed to meet $\mathcal{P}_n$) it suffices to assign a *reservoir* of followers to $\mathcal{P}_n$ which is greater than the number of the negative requriements that have higher priority than $\mathcal{P}_n$.

Here we achieve this by letting $\mathcal{N}_m$ have *higher priority* than $\mathcal{P}_n$ iff $m < neg(n)$ (and by letting $\mathcal{P}_n$ have higher priority than $\mathcal{N}_m$ otherwise) and by letting the finite set F_n be the reservoir of $\mathcal{P}_n$-followers. Then there are $neg(n)$ negative requirements of higher priority than $\mathcal{P}_n$ and, by (5.49) there are $neg(n)+1$ potential $\mathcal{P}_n$-followers. So the positive requirements $\mathcal{P}_n$ are met.

It remains to show that the negative requirements $\mathcal{N}_{\langle e,x\rangle}$ are met too. By initialization, $\mathcal{N}_{\langle e,x\rangle}$ can be injured only by the higher priority positive requirements, i.e., by the requirements $\mathcal{P}_n$ where $neg(n) \leq \langle e,x\rangle$. Moreover, for any such requirement $\mathcal{P}_n$, $\mathcal{N}_{\langle e,x\rangle}$ can be injured via one $\mathcal{P}_n$-follower only. Namely, if

$\mathcal{N}_{\langle e,x\rangle}$ becomes injured by $\mathcal{P}_n$ via y at stage $s+1$ then $s_{\langle e,x\rangle} \leq s$. So the attackers of any $\mathcal{P}_n$-followers which may be appointed later are greater than $s_{\langle e,x\rangle}$ hence cannot injure $\mathcal{N}_{\langle e,x\rangle}$. So $\mathcal{N}_{\langle e,x\rangle}$ can be injured by a single higher priority positive requirement $\mathcal{P}_n$ at most $\hat{h}(\max F_n) + 1$ times, since any $\mathcal{P}_n$-follower y is picked from the reservoir F_n and since y is associated with $\hat{h}(y) + 1$ attackers.

So, if we let $\mathcal{P}_n > \mathcal{N}_m$ denote that $\mathcal{P}_n$ has higher priority than $\mathcal{N}_m$, then, for any $\langle e, x\rangle$ such that $\hat{\varphi}_e(x) \downarrow$,

$$\begin{array}{rcl} |(A \setminus A_{s_{\langle e,x\rangle}}) \restriction \hat{\varphi}_e(x) + 1| & \leq & \sum_{\{n:\mathcal{P}_n > \mathcal{N}_{\langle e,x\rangle}\}} (\hat{h}(\max F_n) + 1) \\ & = & \sum_{\{n:neg(n) \leq \langle e,x\rangle\}} (\hat{h}(\max F_n) + 1) \\ & \leq & \hat{H}(\langle e, x\rangle) \end{array}$$

where the last inequality holds by assumption (5.50). So the negative requirements $\mathcal{N}_{\langle e,x\rangle}$ are met too.

Having outlined the construction, we conclude the proof by giving the formal construction. We start with some additional notation. Let $y_n[s]$ be the follower of $\mathcal{P}_n$ at stage s (if any); if $y_n[s] \downarrow$ let $z_{n,i}[s]$ $(i \leq \hat{h}(y_n[s]))$ be the attackers associated with $y_n[s]$; let $y^n_0 < y^n_1 < \cdots < y^n_{neg(n)}$ be the elements of F_n in order of magnitude; call a negative requirement *critical* at stage s if $\hat{\varphi}_{e,s}(x) \downarrow$ (i.e., $s_{\langle e,x\rangle} \leq s$); and let

$$l(n, s) = |\{\langle e, x\rangle < neg(n) : \hat{\varphi}_{e,s}(x) \downarrow\}| = |\{\langle e, x\rangle < neg(n) : s_{\langle e,x\rangle} \downarrow\leq s\}|$$

be the number of the negative requirements of higher priority than $\mathcal{P}_n$ which are critical at stage s. (Note that $s \mapsto l(n, s)$ is nondecreasing in s, $l(n, 0) = 0$ and $l(n, s) \leq neg(n)$ whence $y^n_{l(n,s)}$ is a well defined element of F_n.) In the construction all parameters persist unless explicitly stated otherwise.

Stage 0 is vacuous, i.e., $A_0 = \emptyset$, Ψ and ψ are nowhere defined, and no followers and attackers are defined.

Stage $s + 1$. Requirement $\mathcal{P}_n$ requires attention if

(a) either $n = s$ or $n < s$ and $l(n, s) < l(n, s + 1)$ or

(b) $n < s$ and $l(n, s + 1) = l(n, s)$ and (b1) $\Psi^A(y_n[s])[s] \uparrow$ & $g_n(y_n[s], s) = 0$ or

(b2) $\Psi^A(y_n[s])[s] \downarrow$ & $g_n(y_n[s], s) = 1$ and there is an attacker $z_{n,i}[s]$ which is not in A_s.

For any requirement $\mathcal{P}_n$ which requires attention act as follows according to the case via which the requirement requires attention.

(a) If $n < s$ and $l(n, s) < l(n, s+1)$ declare that $\mathcal{P}_n$ is *initialized* at stage $s+1$ and cancel the follower and attackers of $\mathcal{P}_n$ existing at stage s. In any case appoint $y_n[s+1] = y^n_{l(n,s+1)}$ as (new) $\mathcal{P}_n$-follower, assign $z_{n,i}[s+1] = \langle y_n[s+1], s+1, i\rangle$ as the corresponding attackers $(i \leq \hat{h}(y_n[s+1]))$, and let $\psi(y_n[s+1]) = z_{n,\hat{h}(y_n[s+1])}[s+1] + 1$.

(b) Distinguish the following subcases. If (b1) holds then let $\Psi^A(y_n[s])[s+1] \downarrow$. If (b2) holds then let $\Psi^A(y_n[s])[s+1] \uparrow$ and, for the least i such that $z_{n,i}[s] \notin A_s$, enumerate $z_{n,i}[s]$ into A.

This completes the construction. The correctness of the construction follows from the preceding discussion. A formal verification is left to the reader. $\square$

Theorem 5.5.10. *Let h_1 be any computable order. There are computable orders h_0 and h_2 such that the following hold.*

(a) There is a c.e. set A such that $A^\dagger$ is h_2-c.a. but not h_1-c.a.

(b) There is a c.e. set A such that $A^\dagger$ is h_1-c.a. but not h_0-c.a.

Proof. (a). Let h, $\hat{h}$, H, $\hat{H}$ be the computable orders defined by $h = h_1$, $\hat{h}(n) = \langle n, n\rangle$,

$$\hat{H}(n) = n \cdot (\hat{h}(\langle n, n\rangle) + 1), \tag{5.54}$$

and $H(n) = 2\hat{H}(n) + 1$ $(n \geq 0)$, let *neg* be the computable order $neg(n) = n + 1$, and let $\{F_n\}_{n\geq 0}$ be the strong array of mutually disjoint finite sets given by

$$F_n = |\{\langle n, k\rangle : k \leq n\}|.$$

We claim that (a) holds for $h_2 = H$. By Lemma 5.5.9 it suffices to show that (5.49) and (5.50) hold. The former is immediate. The latter holds by

$$\begin{array}{rcl}\sum_{\{n:neg(n)\leq m\}}(\hat{h}(\max F_n)+1) & = & \sum_{\{n:n+1\leq m\}}(\hat{h}(\langle n,n\rangle)+1) \\ & \leq & m\cdot(\hat{h}(\langle m,m\rangle)+1) \\ & = & \hat{H}(m)\end{array}$$

where the first equality holds by definition of $neg(n)$ and F_n while the last equality holds by definition of $\hat{H}$.

(b). Note that, for any computable orders $\tilde{h}$ and $\tilde{\tilde{h}}$ such that $\tilde{h}$ dominates $\tilde{\tilde{h}}$, any $\tilde{\tilde{h}}$-c.a. set is $\tilde{h}$-c.a. Moreover, for any computable order $\tilde{h}$ there is a computable order $\hat{H}$ such that $\tilde{h}$ dominates the computable order $H(n) = 2\hat{H}(n) + 1$. So w.l.o.g. we may assume that there is a computable order $\hat{H}$ such that h_1 is the corresponding computable order H, i.e., $h_1(n) = H(n) = 2\hat{H}(n) + 1$ for $n \geq 0$. It suffices to define computable orders h, $\hat{h}$, neg and a strong array $\{F_n\}_{n\geq 0}$ of disjoint finite sets such that h, $\hat{h}$, H, $\hat{H}$, neg and $\{F_n\}_{n\geq 0}$ satisfy the hypotheses of Lemma 5.5.9. Then (b) holds for $h_0 = h$.

Let neg be a strictly increasing computable function such that $\hat{H}(neg(n)) \geq s(n+1)$ where $s(n) = 0+1+\cdots+n$, and let $\{F_n\}_{n\geq 0}$ be the computable partition of ω into intervals such that $\max F_n + 1 = \min F_{n+1}$ and

$$|F_n| = neg(n) + 1.$$

Finally, let h be any computable order such that

$$h(\langle n,n\rangle) = m \text{ iff } n \in F_m$$

and let $\hat{h}(n) = h(\langle n,n\rangle)$.

It remains to show that (5.49) and (5.50) hold. The former is immediate by definition of F_n. For a proof of (5.50) fix m. W.l.o.g. we may assume that there is a number n such that $neg(n) \leq m$ (otherwise, (5.50) trivially holds since $\sum_{\emptyset}(\dots) = 0$). So, since neg is an order, there is a greatest such n, say n_0. It follows that

$$\begin{aligned}
\sum_{\{n:neg(n)\leq m\}}(\hat{h}(\max F_n)+1) &= \sum_{\{n:neg(n)\leq m\}}(n+1) \\
&\quad \text{(by definition of } h \text{ and } \hat{h}) \\
&\leq \sum_{\{n:n\leq n_0\}}(n+1) \\
&\quad \text{(by maximality of } n_0) \\
&= s(n_0+1) \\
&\leq \hat{H}(neg(n_0)) \\
&\quad \text{(by definition of } neg) \\
&\leq \hat{H}(m) \\
&\quad \text{(by } neg(n_0)\leq m)
\end{aligned}$$

which completes the proof of (5.50) and the proof of the theorem. □

5.5.4 Strongly Wtt-Superlow Sets

We show next that there is a c.e. set – in fact a Turing complete set – A such that $A^\dagger$ is h-c.a. for any order h.

Definition 5.5.11. *A set A is* strongly wtt-superlow *if $A^\dagger$ is h-computably approximable for any computable order h; and A is* strongly wtt-jump traceable *if A is h-wtt-jump traceable for any order h such that $h(0) > 0$.*

We first observe that the equivalence of wtt-superlowness and wtt-jump traceability for c.e. sets extends to strong wtt-superlowness and strong wtt-jump traceability.

Lemma 5.5.12. *Let A be a c.e. set. A is strongly wtt-superlow if and only if A is strongly wtt-jump traceable.*

Proof. First assume that A is strongly wtt-superlow. Then, given a computable order h such that $h(0) > 0$, we have to show that A is h-wtt-jump traceable. Let h' be a computable order such that $\lceil\frac{h'(\langle x,x\rangle)}{2}\rceil + 1 \leq h(x)$ for all $x \geq 0$. Then, by assumption, $A^\dagger$, is h'-c.a. But, by Lemma 5.5.7, this implies that A is h-wtt-jump traceable.

Now assume that A is strongly wtt-jump traceable. Then, given a computable order h, we have to show that $A^\dagger$ is h-c.a. Since any set which is h-c.a. is h'-c.a.

for any finite variant h' of h, w.l.o.g. we may assume that $h(0) \geq 3$. Fix a strictly increasing computable function f as in Lemma 5.5.8 and let h' be a computable order such that $h'(0) = 1$ and $2h'(f(x)) + 1 \leq h(x)$ for $x \geq 0$ (note that, by $h(0) \geq 3$ such h' exists). Then, by assumption, A is h'-wtt-jump traceable. But, by Lemma 5.5.8, this implies that $A^\dagger$ is h-c.a. $\square$

Theorem 5.5.13. *There exists a Turing complete set A which is strongly wtt-superlow.*

Proof. Fix a Turing complete set K and fix a computable enumeration $\{K_s\}_{s\geq 0}$ of K. Then we construct A in stages s, where A_s denotes the finite set of numbers enumerated into A by stage s and, for all e, we construct a uniformly c.e. sequence $\mathcal{V}^e = \{V_n^e\}_{n\geq 0}$ such that A and $\mathcal{V}^e$ meet for any e the requirement

$$\begin{aligned} \mathcal{R}_e : &\text{ If } \varphi_e \text{ is an order with } \varphi_e(0) > 0 \\ &\text{ then } \forall^\infty n \; [|V_n^e| \leq \varphi_e(n) \text{ and } (\hat{J}^A(n)\downarrow \Rightarrow \hat{J}^A(n) \in V_n^e)]. \end{aligned}$$

Note that meeting $\mathcal{R}_e$ for all numbers e ensures that we can find an h-trace for any given order h with $h(0) > 0$. For a verification, given h, let e be such that $h = \varphi_e$ and suppose $\mathcal{R}_e$ is met. Then take $\{V_n^e\}_{n\geq 0}$ as given by the construction and let n_0 be such that the conclusion of $\mathcal{R}_e$ holds for all $n \geq n_0$. Then the sequence $\tilde{\mathcal{V}}^e = \{\tilde{V}_n^e\}_{\geq 0}$, where

$$\tilde{V}_n^e = \begin{cases} V_n^e & \text{if } n \geq n_0, \\ \emptyset & \text{if } n < n_0 \text{ and } \hat{J}^A(n)\uparrow, \\ \{\hat{J}^A(n)\} & \text{otherwise.} \end{cases}$$

is uniformly c.e. as $\tilde{\mathcal{V}}^e$ is a finite variation of $\mathcal{V}^e$ and, by the fact that $h(0) > 0$, it is easy to see that $\tilde{\mathcal{V}}^e$ is an h-trace for $\hat{J}^A$.

In order to make A Turing complete, we define a total computable function

$\gamma : \omega^2 \to \omega$ (called a *marker function*) such that, for any $x, s \in \omega$,

$$\gamma(x, s) < \gamma(x + 1, s), \tag{5.55}$$

$$\gamma(x, s) \leq \gamma(x, s + 1), \tag{5.56}$$

$$\gamma^*(x) = \lim_{s \to \infty} \gamma(x, s) \text{ exists}, \tag{5.57}$$

$$x \in K_{s+1} \setminus K_s \ \Rightarrow \gamma(x, s + 1) \neq \gamma(x, s), \tag{5.58}$$

$$\gamma(x, s + 1) \neq \gamma(x, s) \ \Rightarrow A_{s+1} \restriction \gamma(x, s) + 1 \neq A_s \restriction \gamma(x, s) + 1, \tag{5.59}$$

where in the following, numbers of the form $\gamma(x, s)$ are called *markers*. Then from a marker function γ as above, we can compute K using A as an oracle as follows. Given x, compute with oracle A the least stage s such that $A \restriction \gamma(x, s) + 1 = A_s \restriction \gamma(x, s) + 1$ holds. Such a stage exists by (5.57). Then for any stage $t > s$, $\gamma(x, t) = \gamma(x, s)$ holds by (5.59); hence, $K(x) = K_s(x)$ holds by (5.58). Note that we did not use (5.55) and (5.56) for the definition of a Turing reduction from K to A; however, in the construction we make sure that only markers enter A (see the definition of A_s below). So by (5.55) and (5.56), we can argue that $\gamma(x, s) \notin A_s$ holds for all x, s.

For the definition of γ, we fix a computable enumeration $\{K_s\}_{s \in \omega}$ of K such that $|K_{s+1} \setminus K_s| = 1$ and let a_s be the unique element that enters K at stage $s + 1$ (note that such that a computable enumeration of K exists since K is noncomputable hence infinite). Then the idea is to define a computable sequence of numbers $\{x_s\}_{s \in \omega}$ such that $x_s \leq a_s$ holds for all s, to let, for all $x, s \in \omega$,

$$\begin{aligned} \gamma(x, 0) &= \langle x, 0 \rangle \\ \gamma(x, s + 1) &= \begin{cases} \gamma(x, s) & \text{if } x < x_s, \\ \langle x, s + 1 \rangle & \text{otherwise,} \end{cases} \end{aligned} \tag{5.60}$$

and to let $A_s = \{\gamma(x_t, t) : t < s\}$. Then $\{A_s\}_{s \geq 0}$ is a computable enumeration of A, (5.58) follows by the fact that $x_s \leq a_s$ holds and (5.55), (5.56) and (5.59) follow directly from the definition of γ and $\{A_s\}_{s \geq 0}$. So it remains to make sure that (5.57) holds, i.e., we have to define $\{x_s\}_{s \geq 0}$ in such a way that for any x there are only finitely many stages s such that $x_s \leq x$ holds. Since this will immediately follow from the strategy of how to meet the requirements $\mathcal{R}_e$ for all e, let us proceed in explaining how this is done. In the following, fix $e \geq 0$.

In order to measure at stage s of the construction whether we should believe that φ_e is an order or not, we check whether

$$\forall x < s\ (\varphi_{e,s}(x)\downarrow \Rightarrow \forall y \leq x\ (0 < \varphi_{e,s}(y)\downarrow \leq \varphi_{e,s}(x))) \tag{5.61}$$

holds, i.e., whether the domain of $\varphi_{e,s}$ is an initial segment of ω, whether $\varphi_e(0) > 0$ holds and whether $\varphi_{e,s}$ is nondecreasing on its domain. If a stage s appears such that (5.61) fails then we *cancel* $\mathcal{R}_e$ at stage $s+1$ and stop working on $\mathcal{R}_e$ immediately. Since $\mathcal{R}_e$ is trivially met if (5.61) fails at a stage s and since the question whether (5.61) holds for given e, s is decidable, w.l.o.g. we may assume in the following that (5.61) holds for all stages s.

Furthermore, at any stage s, we define a nondecreasing function $n(e,s)$ and a finite and strictly increasing sequence of numbers $\{y_k^e\}_{k<n(e,s)}$ by induction on k as follows. We let $y_0^e = 0$ and let y_{k+1}^e be the least $y > y_k^e$ such that $\varphi_e(y) > \varphi_e(y_k^e)$ holds, if such a y exists and we let y_{k+1}^e be undefined at stage s, otherwise; and we let $n(e,s)$ be the least k such that y_k^e is undefined at stage s (such a k exists as $\mathrm{dom}(\varphi_{e,s})$ is finite). In the following, we write $y_{k,s}^e\downarrow$ ($y_{k,s}^e\uparrow$) if y_k^e is defined (undefined) at stage s and we write $y_k^e\downarrow$ ($y_k^e\uparrow$) if there is (no) stage s such that $y_{k,s}^e\downarrow$ holds. So by definition, $y_{k,s}^e\downarrow$ ($y_k^e\downarrow$) holds iff (there exists a stage s such that) $k < n(e,s)$. Moreover, note the following properties about $n(e,s)$ and $\{y_{k,s}^e\}_{k<n(e,s)}$.

For $k > 0$, y_k^e is the kth number where φ_e takes a value which is greater than ever before. So by (5.61), $\varphi_e(y_k^e) > k$ holds if $y_{k,s}^e\downarrow$. For all $k, s \geq 0$, $y_{k,s}^e\downarrow$ implies that $y_{k',s}^e\downarrow$ holds for all $k' < k$; hence, $n(e,s)$ is nondecreasing in s. Besides, $y_k^e\uparrow$ holds if and only if $n(e,s) \leq k$ holds for all s and $n(e,s)$ has bounded range if and only if φ_e is bounded or not total. So we may assume in the following that $y_k^e\downarrow$ holds for all k.

Then we may define a computable enumeration $\{V_{n,s}^e\}_{s\geq 0}$ of V_n^e as follows. First of all, we keep $V_{n,s}^e = \emptyset$ for all stages s unless there exists k such that $y_{k+1,s}^e\downarrow$ and $n \in [y_k^e, y_{k+1}^e)$. Then if $k < e$, i.e., $n < y_e^e$ holds, we make sure that no element is ever enumerated in to V_n^e; hence, let $V_n^e = \emptyset$ holds for all $n < y_e^e$.

So assume that $k \geq e$ holds. Then we define $V^e_{n,s} = \emptyset$ for any stage s unless $\hat{J}^A(n)[s]\downarrow$ holds. Now for any such stage s, we distinguish between the following two cases. If $\hat{\varphi}_n(n)[s]\downarrow$ and $\gamma(k,s) \leq \hat{\varphi}_n(n)$ holds then we make sure that $\gamma(k,s)$ is moved at stage $s+1$ by defining $x_s \leq k$; otherwise, we enumerate $\hat{J}^A(n)[s]$ into V^e_n at stage $s+1$. Note that $\hat{J}^A(n)[s]$ is enumerated into V^e_n at stage $s+1$ only if $\hat{\varphi}_n(n) < \gamma(k,s)$ holds since $\gamma(k,s) \leq \hat{\varphi}_n(n)$ implies that $\gamma(k,t) \leq \hat{\varphi}_n(n)$ holds for all $t < s$ by (5.56). In the same way, we can argue that, once $\gamma(k,s)$ is moved at stage $s+1$, it follows that $\hat{\varphi}_n(n) < \gamma(k,t)$ holds for all $t > s$ by (5.56) and by convention on converging computations. So for each $n \in [y^e_k, y^e_{k+1})$ there exists at most one stage s such $\gamma(k,s)$ is moved at stage $s+1$.

Then for the definition of $\{x_s\}_{s\geq 0}$, by the above strategy, it suffices to let $x_s = a_s$ for a given stage $s+1$ unless there exists a requirement $\mathcal{R}_e$ and numbers $k,n \geq 0$ such that $k \geq e$, $y^e_{k+1,s}\downarrow$, $n \in [y^e_k, y^e_{k+1})$, $\hat{\varphi}_n(n)[s]\downarrow$ and $\gamma(k,s) \leq \hat{\varphi}_n(n)$ holds. In that case, we define x_s to be the minimum of a_s and all k for which there exist numbers e,n such that the just described situation holds. Note that, as $\{y^e_k\}_{k<n(e,s)}$ is strictly increasing and by convention on converging computations, $\max\{e,k,n\} \leq s$ holds; so since the question whether $y^e_{k,s}\downarrow$ and $\hat{\varphi}_n(n)[s]\downarrow$ at stage s is computable given e,k,n,s, it follows that the sequence $\{x_s\}_{s\geq 0}$ is computable.

So in total, for fixed k, the marker $\gamma(k,s)$ may be moved by the above strategy at most $y^e_{k+1} - y^e_k$ many times for each e and only for $e \leq k$ holds. Thus, we may argue that (5.57) holds. For fixed e,n such that the hypothesis of $\mathcal{R}_e$ holds, we may argue that at most $k+1$ many numbers may enter V^e_n in the course of the construction, where k is the unique number such that $n \in [y^e_k, y^e_{k+1})$ holds. Namely, if $k < e$ then $V^e_n = \emptyset$. Otherwise, by the above strategy, a number x may enter V^e_n at stage $s+1$ only if $x = \hat{J}^A(n)[s]\downarrow$ and only if $\hat{\varphi}_n(n) < \gamma(k,s)$. So, by definition of γ, by definition of $\{A_s\}_{s\geq 0}$ and by the use-principle, $\hat{J}^A(n)$ may take at most $k+1$ values in the course of the construction. So, since $k < \varphi_e(y^e_k) \leq \varphi_e(n)$ holds by (5.61), it follows that $|V^e_n| \leq \varphi_e(n)$. Finally, since $\hat{J}^A(n)[s] \in V^e_{n,s}$ holds for all sufficiently large stages s, $\hat{J}^A(n) \in V^e_n$ holds if $\hat{J}^A(n)\downarrow$ holds.

This gives the main idea of how to define $\{x_s\}_{s\geq 0}$ and $\mathcal{V}^e$ with the required properties. We now proceed to the formal construction.

Definition of $\mathcal{V}^e$ and $\{x_s\}_{s\geq 0}$.

Stage 0. Let $V^e_{n,0} = \emptyset$ for all e, n (x_0 is defined at stage 1).

Stage $s+1$. Let $V^e_{n,s}$ be given for all e, n and let x_{s-1} be given if $s > 0$. We say that a requirement $\mathcal{R}_e$ *requires attention* at stage $s+1$ if $e \leq s$, $\mathcal{R}_e$ is not cancelled at stage s and either

(1) (5.61) does not hold, or

(2) (1) does not hold and there exist $k \geq e$ and n such that $k+1 < n(e,s)$, $n \in [y^e_k, y^e_{k+1})$, $\hat{J}^A(n)[s]\downarrow$ and $\gamma(k,s) \leq \hat{\varphi}_n(n)$, or

(3) (1) does not hold, there exist $k \geq e$ and n such that $k+1 < n(e,s)$, $n \in [y^e_k, y^e_{k+1})$, $\hat{J}^A(n)[s]\downarrow$, (2) does not hold and $\hat{J}^A(n)[s] \notin V^e_{n,s}$.

If no requirement requires attention at stage $s+1$ let $x_s = a_s$ and $V^e_{n,s+1} = V^e_{n,s}$ for all e, n. Otherwise, for any e such that $\mathcal{R}_e$ requires attention at stage $s+1$, do the following. If (1) holds, cancel $\mathcal{R}_e$ at stage $s+1$ and let $x_s = a_s$. If (2) holds, let $k_{e,s}$ be the least k for which there exists n such that (2) hold for k, n accordingly, and let $x_s = \min(\{k_{e,s} : \mathcal{R}_e \text{ requires attention via (2)}\} \cup \{a_s\})$. If (3) holds then for any n for which there exists k such that (3) hold for k, n accordingly let $V^e_{n,s+1} = V^e_{n,s} \cup \{\hat{J}^A(n)[s]\}$.

This ends the formal construction.

Verification

We prove in a series of claims that A and $\mathcal{V}^e$ have the required properties. Before stating these claims let us first give some general remarks about the construction that will be tacitly used below. If not stated otherwise they can be shown by a straightforward induction on the stage number s.

First of all, the construction is effective so $\{x_s\}_{s\geq 0}$ is a computable sequence. So $\gamma(x,s)$ is a computable function, (5.55), (5.56) hold by (5.60), (5.58) holds by definition of $\{x_s\}_{s\geq 0}$ and (5.59) holds by definition of $\{A_s\}_{s\geq 0}$; hence, $\{A_s\}_{s\geq 0}$ is a computable enumeration of A, i.e., A is a c.e. set.

Moreover, $\mathcal{V}^e$ is uniformly c.e. (in fact, $\{V_n^e\}_{\langle e,n\rangle\geq 0}$ is uniformly c.e.). For any e, n, a number x may enter V_n^e at a stage $s+1$ only if $\mathcal{R}_e$ requires attention via (3). So $x = \hat{J}^A(n)[s]\downarrow$ and there exists k such that $e < k+1 < n(e,s)$, $n \in [y_k^e, y_{k+1}^e)$ and $\gamma(k,s) > \hat{\varphi}_n(n)$ by definition of (3).

Then for the first claim, we say that $\mathcal{R}_e$ requires attention via (2) respectively via (3) *with respect to* (k,n) at stage $s+1$ if k and n witness accordingly that (2) respectively (3) holds for $\mathcal{R}_e$ at stage $s+1$. Fix e, k, n in the following.

Claim 1. There are at most finitely many stages such that $\mathcal{R}_e$ requires attention via (2) or (3) with respect to (k,n), respectively.

Proof. We may assume that there is no stage s such that $\mathcal{R}_e$ requires attention via (1) since, otherwise, $\mathcal{R}_e$ is cancelled at stage $s+1$; hence, $\mathcal{R}_e$ does not require attention after stage s with respect to (k,n). Further, we may assume that there exists a stage s such that $k+1 < n(e,s)$, $n \in [y_k^e, y_{k+1}^e)$ and $\hat{J}^A(n)[s]\downarrow$ holds since otherwise, $\mathcal{R}_e$ will never require attention with respect to (k,n). In particular, $\hat{\varphi}_n(n)\downarrow$ holds. Now if $\mathcal{R}_e$ requires attention via (2) at stage $s+1$ with respect to (k,n) then $x_s \leq k$ holds by construction; hence, for all $t > s$, it holds that $\gamma(k,t) \geq \langle k, s+1\rangle > s > \hat{\varphi}_n(n)$; whence, $\mathcal{R}_e$ requires attention via (2) w.r.t. to (k,n) at most once in the course of the construction. Finally, since $\hat{\varphi}_n(n)\downarrow$ and since $\hat{J}^A(n)[s]$ may change only if a number $< \hat{\varphi}_n(n)$ enters A after stage s, $\mathcal{R}_e$ may require attention via (3) w.r.t. (k,n) at most $\hat{\varphi}_n(n)$ many times. $\square$

Then based on Claim 1, we can show that (5.57) holds.

Claim 2. For any k there exist at most finitely many stages s such that $\gamma(k,s) \neq \gamma(k,s+1)$.

Proof. Let $k \geq 0$ be given and, for a proof by induction on k suppose the claim to be true for all $k' < k$. By inductive hypothesis and since the infinite sequence $\{a_s\}_{s\geq 0}$ is injective (hence $\lim_{s\to\infty} a_s = \infty$), we may fix a stage s_0 such that $\gamma^*(k') = \gamma(k', s_0)$ holds for all $k' < k$ and $a_s > k$ holds for all $s \geq s_0$. Then by (5.60), for any stage $s \geq s_0$, $\gamma(k,s+1) \neq \gamma(k,s)$ may only hold if there exist

e, n such that $\mathcal{R}_e$ requires attention via (2) w.r.t. (k, n) at stage $s+1$; so in particular, $e \leq k$. So the claim follows by Claim 1 since for any k there can be at most $y^e_{k+1} - y^e_k$ many n such that $\mathcal{R}_e$ requires attention via (2) w.r.t. to (k, n). □

So, in particular, $\gamma^*(k) = \lim_{s\to\infty} \gamma(k, s)$ exists for all $k \geq 0$. Finally, we can show that all requirements are met.

Claim 3. $\mathcal{R}_e$ is met.

Proof. Since $\mathcal{R}_e$ is trivially met if the hypothesis of $\mathcal{R}_e$ does not hold, w.l.o.g., we may assume that φ_e is total, $\varphi_e(0) > 0$ holds and that φ_e is an order function. In particular, (5.61) holds for every stage s, $\lim_{s\to\infty} n(e, s) = \infty$ so $\{y^e_k\}_{k\geq 0}$ is an infinite, strictly increasing computable sequence and $\mathcal{R}_e$ never requires attention via (1) in the course of the construction.

Thus, in order to show that the conclusion of $\mathcal{R}_e$ holds, it suffices to prove that $|V^e_n| \leq \varphi_e(n)$ holds for all n and that $\hat{J}^A(n) \in V^e_n$ holds for all $n \geq y^e_e$ in case that $\hat{J}^A(n)\downarrow$ holds. For the former, note that $|V^e_n| \leq \varphi_e(n)$ holds for any $n < y^e_e$ since $\mathcal{R}_e$ may not require attention via (3) w.r.t. (k, n) for any $k < e$; hence, no number may enter V^e_n for any such n in the course of the construction. So it suffices to prove that $|V^e_n| \leq \varphi_e(n)$ holds for $n \geq y^e_e$. Fix $n \geq y^e_e$, fix the unique $k \geq e$ such that $n \in [y^e_k, y^e_{k+1})$ and fix the least stage s such that a number enters V^e_n at stage $s+1$. Then it must be that $\mathcal{R}_e$ requires attention via (3) w.r.t. (k, n) at stage $s+1$. So $\hat{J}^A(n)[s]\downarrow$, $\hat{J}^A(n)[s]$ enters V^e_n at stage $s+1$ and by definition of (3), it holds that $\gamma(k, s) > \hat{\varphi}_n(n)$. So by (5.55), the only markers which may be below $\hat{\varphi}_n(n)$ at stage s are of the form $\gamma(k', s)$ for $k' < k$. Furthermore, by (5.60), if $\gamma(k', s)$ is moved after stage s then $\gamma^*(k) > \hat{\varphi}_n(n)$ holds by convention on converging computations. So any $\gamma(k', s)$ with $k' < k$ may change the computation of $\hat{J}^A(n)$ at most once after stage s, i.e., $\hat{J}^A(n)$ may take at most $k+1$ different values in the course of the construction; whence, at most $k+1$ many numbers may enter V^e_n.

It remains to argue that $\hat{J}^A(n) \in V^e_n$ holds if $\hat{J}^A(n)\downarrow$. The latter implies that $\hat{J}^A(n)[s]\downarrow= \hat{J}^A(n)$ holds for almost all stages s. So, by construction, we are done if we show that there exists a stage s such that $\gamma(k, s) > \hat{\varphi}_n(n)$ holds, where k is the unique number such that $n \in [y^e_k, y^e_{k+1})$ holds since then it follows by (3) in the definition of requiring attention and by (5.56) that $\mathcal{R}_e$ eventually enumerates $\hat{J}^A(n)$. So suppose $\gamma(k,s) \leq \hat{\varphi}_n(n)$ holds for all stages $s \geq s_0$, where s_0 is least such that $\hat{\varphi}_n(n)[s_0] \downarrow$ (note that $\hat{J}^A(n) \uparrow$ if $\hat{\varphi}_n(n) \uparrow$). Since by (2) in the definition of requiring attention, $\gamma(k,s)$ is moved at the first stage $s > s_0$ such that $\hat{J}^A(n)[s] \downarrow$ holds, this implies that $\hat{J}^A(n)[s] \uparrow$ holds for all $s > s_0$, contrary to the assumption that $\hat{J}^A(n) \downarrow$. Hence, a stage as required exists. This completes the proof of Claim 3. $\square$

Since meeting all the requirements ensures that A is Turing complete and strongly-wtt jump traceable, this completes the proof of Theorem 5.5.13. $\square$

We conclude this section by showing that the class of the strongly wtt-superlow sets is downward closed under wtt-reducibility and that the class of the c.e. strongly wtt-superlow sets is closed under join. Compare with the corresponding results for the eventually uniformly wtt-array computable sets (Lemmas 5.4.1 and 5.4.3) and the wtt-superlow sets (Corollary 5.5.4).

Theorem 5.5.14. *(a) Let A and B be any (not necessarily c.e.) sets such that $A \leq_{wtt} B$ and B is strongly wtt-superlow. Then A is strongly wtt-superlow too.*

(b) Let A_0 and A_1 be strongly wtt-superlow c.e. sets. Then $A_0 \oplus A_1$ is strongly wtt-superlow too.

Proof. (a). Given a computable order h, it suffices to show that $A^\dagger$ is h-c.a. By 1. of Lemma 5.2.4 fix a strictly increasing computable function f such that, for $e \geq 0$, $\hat{\Phi}^A_e = \hat{\Phi}^B_{f(e)}$, hence

$$A^\dagger(\langle e, x\rangle) = B^\dagger(\langle f(e), x\rangle)$$

for $e, x \geq 0$. Now, since f is strictly increasing and so is $\langle \cdot, \cdot \rangle$ (in either argument), $\langle f(e), x\rangle \leq f(\langle e, x\rangle)$. So, for any order h' and any h'-bounded computable approximation g' of $B^\dagger$, g defined by $g(\langle e, x\rangle) = g'(\langle f(e), x\rangle)$ is a computable approximation of $A^\dagger$ and g is $h'(f(n))$-bounded. Since, for any computable order h there is a computable order h' such that $h'(f(n)) \leq h(n)$ for $n \geq 0$, and since, by assumption, $B^\dagger$ is h'-c.a. for any computable order h', this shows that $A^\dagger$ is h-c.a.

(b) Given a computable order h, it suffices to show that $(A_0 \oplus A_1)^\dagger$ is h-c.a. Fix strictly increasing computable functions $f_0, f_1 : \omega \to \omega$ as given by Lemma 5.4.2, let $f(n) = f_0(n) + f_1(n)$ and let h' be a computable order such that $h'(f(n)) \leq h(n)$ for $n \geq 0$. Then, since A_0 and A_1 are strongly wtt-superlow, we may fix h'-bounded computable approximations g_i of $A_i^\dagger$ $(i \leq 1)$. Now define g by

$$g(\langle e, x\rangle, s) = \min\{g_0(\langle f_0(e), x\rangle, s), g_1(\langle f_1(e), x\rangle, s)\}.$$

By (5.43), g is a computable approximation of $(A_0 \oplus A_1)^\dagger$. Moreover, by definition of g and by choice of g_0 and g_1, g is $\hat{h}$-bounded by for the computable order $\hat{h}$ defined by

$$\hat{h}(\langle e, x\rangle) = h'(\langle f_0(e), x\rangle) + h'(\langle f_1(e), x\rangle) \leq 2h'(f(\langle e, x\rangle)).$$

But, since f majorizes f_0 and f_1 and f and $\langle \cdot, \cdot \rangle$ are strictly increasing, it follows by choice of h' that $\hat{h}(n) \leq h'(f(n)) \leq n$ for $n \geq 0$. So $(A_0 \oplus A_1)^\dagger$ is h-c.a. $\square$

5.6 EUwttAC and Array Computable Sets

Having introduced a hierarchy of subclasses of EUwttAC, we continue to show that the class of c.e. sets having array computable (a.c.) wtt-degree are a superclass of EUwttAC, i.e., no e.u.wtt-a.c. c.e. set can be wtt-equivalent to an array noncomputable (a.n.c.) set. For this purpose, we use the fact that a.n.c. wtt-degrees, i.e., the wtt-degrees which contain an a.n.c. set can be characterized as those c.e. wtt-degrees whose c.e. members are all multiply permitting by [AS18, Lemma 2 and Theorem 2] (see also Theorem 4.2.3). Recall from Definition 4.2.2 that a c.e. set A is multiply permitting if there exists a very strong array (v.s.a.) $\mathcal{F} = \{F_n\}_{n\in\omega}$ (i.e., $\{F_n\}_{n\in\omega}$ is an infinite sequence such that the sets F_n are uniformly given by their canonical index, such that they are mutually disjoint, nonempty and growing in size), a computable function f and a computable enumeration of A such that, for any partial computable function ψ, (4.1) holds, i.e.,

$$\exists^\infty n \; \forall x \in F_n \; (\psi(x)\downarrow \Rightarrow \; A \restriction f(x) + 1 \neq A_{\psi(x)} \restriction f(x) + 1).$$

Then by [AS18, Lemma 1], it turns out that the property of being multiply permitting for a c.e. set does not depend on the choice of the very strong array.

Lemma 5.6.1 ([AS18])**.** *Let A be multiply permitting and let $\mathcal{F} = \{F_n\}_{n\in\omega}$ be a v.s.a. Then A is $\mathcal{F}$-permitting.*

Using Lemma 4.2.3, we can show that the following holds.

Theorem 5.6.2. *Let A be multiply permitting. Then A is not e.u.wtt-a.c.*

Proof. Suppose that A is multiply permitting. For a proof by contradiction, suppose that A is e.u.wtt-a.c. Fix computable functions $g, k : \omega^2 \to \{0, 1\}$ and a computable order h which witness that A is e.u.wtt-a.c. according to Definition 5.3.1. Let $\mathcal{F} = \{F_n\}_{n\in\omega}$ be the unique very strong array such that each F_n is an interval such that $|F_n| = \hat{h}(n)$, where $\hat{h}(n) = \lfloor \frac{h(\langle n,n\rangle)+1}{2} \rfloor$ (note that $\hat{h}$ is a computable order) and such that $\min(F_{n+1}) = \max(F_n) + 1$ holds for all n. By Lemma 5.6.1, we may fix a computable function f and a computable enumeration $\{A_s\}_{s\in\omega}$ such that A is $\mathcal{F}$-permitting via f and $\{A_s\}_{s\in\omega}$, where, w.l.o.g., we may assume that f is strictly increasing.

Then we define a wtt-functional Γ in stages s where, by Lemma 5.2.3, we may assume that in advance we know a number d such that $\Gamma = \hat{\Phi}_d$ holds. In particular, by (5.6), $\lim_{s\to\infty} g(\langle d, n\rangle, s) = 1$ holds iff $n \in \mathrm{dom}(\Gamma^A)$. In more detail, we define a uniformly computable sequence of wtt-functionals $\{\tilde{\Gamma}_e\}_{e\in\omega}$ and we declare $\tilde{\Gamma}^A_e(n)$ to be defined (undefined) at a stage $s+1$ only if $g(\langle e, n\rangle, s)$ correctly approximates (the status of definedness of) $\tilde{\Gamma}^A_e(n)[s]$ (so below the reader may replace Γ by $\tilde{\Gamma}_e$ and any occurence of d in any of the functions g and k by e). Then, by 1. and 2. of Lemma 5.2.3 there exists $d \in \omega$ such that $\tilde{\Gamma}_d = \hat{\Phi}_d$. So d and $\Gamma = \tilde{\Gamma}_d$ are as desired.

Then the definition of Γ is as follows, where we stick to the convention that $\Gamma^A(n)[s+1] = \Gamma^A(n)[s]$ holds for any n and any stage s unless otherwise stated. Fix n in the following.

Definition of $\Gamma^A(n)$.

Stage 0. Let $\Gamma^A(n)[0]\uparrow$.

Stage $s+1$. Let $\Gamma^A(n)[s]$ be given. Then we destinguish between the following two cases.

(1) If $\Gamma^A(n)[s]\uparrow$ and $g(\langle d,n\rangle, s)=0$ hold then declare $\Gamma^A(n)[s+1]\downarrow$.

(2) If $\Gamma^A(n)[s]\downarrow$, $g(\langle d,n\rangle, s)=1$, $k(\langle d,n\rangle, s)=1$ and it holds that $A_{s+1}\upharpoonright f(\max(F_n))+1\neq A_{s^-}\upharpoonright f(\max(F_n))+1$, where s^- is the largest stage $\leq s$ such that $\Gamma^A(n)[t]\downarrow$ holds for all $t\in[s^-,s]$, then declare $\Gamma^A(n)[s+1]\uparrow$.

By definition, Γ is a Turing functional and since, by clause (2), the use of Γ on input n is bounded by $f(\max(F_n))$, it follows that Γ is indeed a wtt-functional. Moreover, by (1), we may argue that Γ^A is total as we keep $\Gamma^A(n)[s]\uparrow$ for any stage s unless (1) holds. However, as $g(\langle d,n\rangle, s)$ correctly approximates the question whether or not $x\in\mathrm{dom}(\Gamma^A)$ holds, it follows that, for any stage s such that $\Gamma^A(n)[s]\uparrow$ there exists a least stage $t\geq s$ such that $\Gamma^A(n)[t]\uparrow$ and $g(\langle d,n\rangle, t)=0$. So for the least s such that $A_s\upharpoonright f(\max(F_n))+1=A\upharpoonright f(\max(F_n))+1$ and $\Gamma^A(n)[s]\downarrow$, it follows that $\Gamma^A(n)[t]\downarrow$ for all $t>s$. Hence, by (5.9), we may fix $n_0\in\omega$ such that $\lim_{s\to\infty}k(\langle d,n\rangle, s)=1$ holds for all $n\geq n_0$. Likewise, we can argue that for any stage s such that $\Gamma^A(n)[s]\downarrow$ there exists a least stage $t\geq s$ such that $\Gamma^A(n)[t]\downarrow$ and $g(\langle d,n\rangle, t)=1$. In particular, the clauses (1) and (2) always apply alternatingly to $\Gamma^A(n)$.

Now consider the partial computable function $\psi:\omega\to\omega$ which is defined as follows. Given n, let $x_0^n<\cdots<x_{\hat{h}(n)-1}^n$ be the elements of F_n. Then $\psi(x_i^n)$ is defined inductively such that, for all $i<\hat{h}(n)-1$, it holds that

$$\begin{aligned}\psi(x_0^n)&=\mu s(P(n,s)),\\ \psi(x_{i+1}^n)&=\mu s(s>\psi(x_i^n)\ \&\ P(n,s)\ \&\ \exists t\in(\psi(x_i^n),s)\ (\Gamma^A(n)[t]\uparrow)),\end{aligned}$$

where $P(n,s)$ holds iff $\Gamma^A(n)[s]\downarrow$, $g(\langle d,n\rangle, s)=1$ and $k(\langle d,n\rangle, s)=1$ holds. Note that, for all n, it holds that either $\mathrm{dom}(\psi)\cap F_n=\emptyset$ or $F_n\subset\mathrm{dom}(\psi)$. Namely, by definition, $\psi(x_i^n)\downarrow$ can only hold if $\psi(x_j^n)\downarrow$ holds for all $j<i$ and, if $\psi(x_i^n)\downarrow$ holds for some $i<\hat{h}(n)$ then, by (4.1) and since $\lim_{s\to\infty}g(\langle d,n\rangle, s)=1$ holds, there exists a stage $t>\psi(x_i^n)$ such that (2) applies at stage t in the definition of $\Gamma^A(n)$; hence, by (5.7), by definition of $P(n, s)$ and by totality of Γ^A, we may infer that $\psi(x_{i+1}^n)\downarrow$ holds. So since $\lim_{s\to\infty} k(\langle d, n\rangle, s) = 1$ holds for all $n \geq n_0$, it follows that there exist infinitely many n such that $\psi(x_0^n)\downarrow$; hence, $F_n \subset \mathrm{dom}(\psi)$ holds. However, for any such n, by definition of Γ, it follows that, for any $i \leq \hat{h}(n)$, there exists two stages $\psi(x_i^n) \leq s_0 < s_1$ such that $g(\langle d, n\rangle, s_i + 1) \neq g(\langle d, n\rangle, s_i)$ holds for all $i \leq 1$; and, if $i < \hat{h}(n)$ then $s_1 < \psi(x_{i+1}^n)$ holds. So for any $n \geq d$ such that $\psi(x_0^n)\downarrow$ holds the number of mind changes of $s \mapsto g(\langle d, n\rangle, s)$ after stage $\psi(x_0^n)$ is at least

$$2\hat{h}(n) > h(\langle n, n\rangle) > h(\langle d, n\rangle)$$

so (5.8) fails for any such n. However, as there are infinitely many $n \geq d$ such that $\psi(x_0^n)\downarrow$, we conclude that (5.8) fails, contrary to the assumption that A is e.u.wtt-a.c. This completes the proof. □

Corollary 5.6.3. *Let A be c.e. and e.u.wtt-a.c. Then any c.e. set B which is wtt-equivalent to A is array computable.*

Proof. By Lemma 4.2.3 and Theorem 5.6.2. □

5.7 Separations

In the preceding sections we have given lower and upper bounds for the class of the c.e. e.u.wtt-a.c. sets in terms of wtt-superlowness respectively array computability: any wtt-superlow set is e.u.wtt-a.c. (Corollary 5.5.3) and any c.e. set which is wtt-equivalent to a c.e. e.u.wtt-a.c. set is array computable (Corollary 5.6.3). We conclude our investigations of the e.u.wtt-a.c. sets by showing that these inclusions are proper. In fact, in case of the second inclusion we get a slightly stronger result by showing that there is an array computable c.e. Turing degree which contains a c.e. set which is not e.u.wtt-a.c. We start with the separation of wtt-superlowness and eventually uniform wtt-array computability on the c.e. sets. In order to separate the c.e. wtt-superlow sets from the c.e. e.u.wtt-a.c. sets, by the Characterization Theorem 5.3.2, it suffices to show the following.

Theorem 5.7.1 ([ASDM19])**.** *There is a maximal set M which is not wtt-superlow.*

Corollary 5.7.2. *There is an e.u.wtt-a.c. c.e. set which is not wtt-superlow.*

Proof. Immediate by the Characterization Theorem 5.3.2 and 5.7.1. □

And for the second separation we state (without proof) the result that there is an array computable Turing degree which contains a c.e. set which is not e.u.wtt-a.c. Again by the Characterization Theorem 5.3.2 it suffices to prove the following theorem.

Theorem 5.7.3 ([ASDM19])**.** *There is a c.e. set A such that the Turing degree of A is array computable and such that A is not wtt-reducible to any maximal set.*

Corollary 5.7.4. *There is an array computable c.e. Turing degree* **a** *which contains a computably enumerable set which is not eventually uniformly wtt-array computable.*

Proof. Immediate by the Characterization Theorem 5.3.2 and Theorem 5.7.3. □

Chapter 6

Conclusion

In this last chapter we would like to give an outlook for further research. We start with some questions related to the results in Chapter 2. There we have shown that there are c.a.n.c. c.e. degrees, i.e., c.e. (Turing) degrees **a** such that any c.e. set in **a** is wtt-equivalent to an array noncomputable set, and we have shown that there is a c.e. degree **b** such that any c.e. set in **b** is half of an ibT-maximal pair in the c.e. sets. Moreover, any degree with the latter property is c.a.n.c. since Ambos-Spies [AS16] has shown that the a.n.c. c.e. wtt-degrees coincide with the c.e. wtt-degrees which contain halves of ibT-maximal pairs in the c.e. sets. The question, however, whether the converse is true too remains open though in [AS16] it is shown that there is an a.n.c. set which is not half of any ibT-minimal pair in the c.e. sets.

Question 1. *Is there a c.a.n.c. degree which contains a c.e. set which is not half of any ibT-maximal pair in the c.e. sets?*

Moreover, related to these notions, we may take up the following question from [ASDFM13] and ask the corresponding question for the c.a.n.c. degrees.

Question 2. *Characterize the c.e. Turing degrees all of whose c.e. members are halves of ibT-maximal pairs in the c.e. sets.*

Question 3. *Characterize the c.a.n.c. degrees.*

More specifically we may ask about the possible jumps of the c.a.n.c. degrees. By Theorem 2.4.2, we know that c.a.n.c. degrees cannot be high. On the other hand, by Lemma 2.3.4 we know that the c.a.n.c. degrees are downward dense (whence there are low c.a.n.c. degrees). So the following question is of particular interest.

Question 4. *Is there a high$_2$ c.a.n.c. degree?*

Concerning Chapter 3 we know that, by Theorem 3.4.19, the wtt-degrees of sets with the universal similarity property cannot be complete (note that this also follows from a more direct argument as demonstrated in [ASLM18]). So in particular, the u.s.p. sets cannot be closed upwards in the wtt-degrees of (almost)-c.e. sets. Furthermore, Ambos-Spies and Losert have introduced in [ASL] variants of the universal similarity property in the setting of c.e. sets characterizing the c.e. not totally ω-c.e. degrees, e.g., the universally a.n.c. sets and the uniformly multiply permitting (u.m.p.) sets, amongst others. They show, using u.m.p. sets, that the c.e. not totally ω-c.e. degrees can be characterized as those c.e. degrees below which the 7-elemented lattice $\mathcal{S}_7$ can be embedded. Moreover, Losert used these variants in [Los18] in order to investigate the distribution of the wtt-degrees containing u.s.p. sets by showing, e.g., that every u.s.p. set is wtt-equivalent to a uniformly multiply permitting set and that any uniformly multiply permitting set computes a u.s.p. set. However, the following question is still unsolved.

Question 5. *Is there a c.e. set which is wtt-incomplete and which cannot be wtt-computed by any u.s.p. set?*

Finally, concerning Chapter 5, we showed that the e.u.wtt-a.c. c.e. sets can be characterized as the c.e. sets which are wtt-computable by maximal sets. Since the notion of e.u.wtt-a.c. sets is invariant under wtt-equivalence this gives rise to a characterization of the c.e. wtt-degrees that contain c.e. sets which are not wtt-reducible to any maximal sets. Moreover, if we define a c.e. Turing degree $\mathbf{a}$ to be *e.u.wtt-a.c. (wtt-superlow)* if all c.e. sets in $\mathbf{a}$ are e.u.wtt-a.c. (wtt-superlow), then the above characterization of the c.e. e.u.wtt-c.a. wtt-degrees trivially carries over to the c.e. Turing degrees. Since multiply permitting sets are not e.u.wtt-a.c., it follows that any a.n.c. Turing degree contains a c.e. set which is not wtt-reducible to any maximal set. However, since, by Theorem 5.7.3

there exists an array computable c.e. Turing degree which contains a c.e. set which is not wtt-reducible to any maximal set, it follows that neither the a.n.c. Turing degrees nor the c.e. not totally ω-c.e. Turing degrees capture the class of the c.e. e.u.wtt-a.c. Turing degrees. So we ask:

Question 6. *Which notion of (anti-)permitting corresponds to the class of e.u.wtt-a.c. Turing degrees?*

Finally, by Theorem 5.7.3, it follows that there exists a c.e. Turing degree which is array computable but not e.u.wtt-a.c. However, we do not have a separation of the e.u.wtt-a.c. and the wtt-superlow Turing degrees. So the following question remains an open problem.

Question 7. *Does there exist an e.u.wtt-a.c. Turing degree which is not wtt-superlow?*

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Errata

In the following, we provide a list of errata.

- **Page 11, line 10**: Replace "inside a given a.n.c. degree" by "inside a given a.n.c. Turing degree".
- **Page 15, line -3**: Replace "for all $m' < m$" by "for all $m' \leq m$".
- **Page 30, line 2**: Replace "for every computable order h" by "for every computable order h, where an order is a nondecreasing, unbounded function".
- **Page 67, line -3**: Replace "σ_i" by "$\sigma(i)$".
- **Page 94, line -8 and -9**: Replace "no set is wtt-reducible to its bounded jump" by "no set wtt-computes its bounded jump".
- **Page 105, line 9**: Replace "Assume (5.10)" by "Assume that M is coinfinite".
- **Page 118, line -12**: Replace "$i \leq 4$" by "$i < 4$".
- **Page 150, line -13**: Replace "and only for $e \leq k$ holds" by "such that $e \leq k$".