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Pre-Schauder Bases in Topological Vector Spaces

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Abstract: A Schauder basis in a real or complex Banach space X is a sequence $(e_n)_{n\in\mathbb{N}}$ in X such that for every $x \in X$ there exists a unique sequence of scalars $(\lambda_n)_{n\in\mathbb{N}}$ satisfying that $x = \sum_{n=1}^{\infty} \lambda_n e_n$. Schauder bases were first introduced in the setting of real or complex Banach spaces but they have been transported to the scope of real or complex Hausdorff locally convex topological vector spaces. In this manuscript, we extend them to the setting of topological vector spaces over an absolutely valued division ring by redefining them as pre-Schauder bases. We first prove that, if a topological vector space admits a pre-Schauder basis, then the linear span of the basis is Hausdorff and the series linear span of the basis minus the linear span contains the intersection of all neighborhoods of 0. As a consequence, we conclude that the coefficient functionals are continuous if and only if the canonical projections are also continuous (this is a trivial fact in normed spaces but not in topological vector spaces). We also prove that, if a Hausdorff topological vector space admits a pre-Schauder basis and is w^*-strongly torsionless, then the biorthogonal system formed by the basis and its coefficient functionals is total. Finally, we focus on Schauder bases on Banach spaces proving that every Banach space with a normalized Schauder basis admits an equivalent norm closer to the original norm than the typical bimonotone renorming and that still makes the basis binormalized and monotone. We also construct an increasing family of left-comparable norms making the normalized Schauder basis binormalized and show that the limit of this family is a right-comparable norm that also makes the normalized Schauder basis binormalized.

Keywords: Schauder basis; topological vector space; monotone basis; Hausdorff topology

1. Introduction

Schauder bases were introduced for the first time in [1] in the setting of real or complex Banach spaces. However, they can be transported to a more general scope: the category of topological vector spaces over an absolutely valued division ring. We refer the reader to [2–5] for a more general perspective on this category. A topological vector space X over an absolutely valued division ring $\mathbb{K}$ is said to have a pre-Schauder basis $(e_n)_{n\in\mathbb{N}} \subset X$ provided that for every $x \in X$ there exists a unique sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{K}$ in such a way that $\sum_{n=1}^{\infty} \lambda_n e_n$ converges to x. For every $n \in \mathbb{N}$, the coefficient functionals and the canonical projections are, respectively, defined by

$$\begin{array}{cccc} e_n^* : & X & \to & \mathbb{K} \\ & x & \mapsto & \lambda_n \end{array} \tag{1}$$

and

$$\begin{array}{cccc} p_n : & X & \to & \operatorname{span}\{e_1,\ldots,e_n\} \\ & x & \mapsto & \sum_{i=1}^{n} \lambda_i e_i. \end{array} \tag{2}$$

For simplicity purposes, we convey that p_0 is the null projection. Notice that $p_n(x) = e_1^*(x)e_1 + \cdots + e_n^*(x)e_n$ and $p_n(x) - p_{n-1}(x) = e_n^*(x)e_n$ for all $n \in \mathbb{N}$ and all $x \in X$. As a consequence, if the coefficient functionals are all continuous, then the canonical projections are also continuous due to the vector character of the topology (see Lemma 1). In this situation, we say that $(e_n)_{n\in\mathbb{N}}$ is a Schauder basis. In normed spaces, it is a trivial fact that the coefficient functionals are continuous if and only if the canonical projections are also continuous. However, the same fact in the topological vector space setting is not trivial at all. In Section 2, we prove that the continuity of the canonical projections imply the continuity of the coefficient functionals in topological vector spaces over an absolutely valued division ring (Corollary 1). We also show in this section that the existence of a pre-Schauder basis on a topological vector space over an absolutely valued division ring implies the existence of a dense Hausdorff subspace, which is precisely the linear span of the basis (Theorem 1). We also construct explicitly non-Hausdorff topological vector spaces admitting Schauder bases (Theorem 2). In a topological vector space X, a biorthogonal system is a pair $\left(e_i, e_i^*\right)_{i\in I} \subseteq X \times X^*$ such that $e_i^*(e_j) = \delta_{ij}$ where δ_{ij} is the Kronecker δ. A biorthogonal system is said to be:

- expanding if $X = \operatorname{span}\{e_i : i \in I\}$;
- fundamental if $X = \overline{\operatorname{span}}\{e_i : i \in I\}$; and
- total if $X^* = \overline{\operatorname{span}}^{w^*}\{e_i^* : i \in I\}$.

We refer the reader to [6] for a complete perspective on biorthogonal systems in real or complex normed spaces. If $(e_n)_{n\in\mathbb{N}}$ is a pre-Schauder basis in a topological vector space X, then $(e_n, e_n^*)_{n\in\mathbb{N}}$ is clearly a fundamental biorthogonal system. In Section 2, we prove that, if X is Hausdorff and w^*-strongly torsionless, then $(e_n, e_n^*)_{n\in\mathbb{N}}$ is also total (Corollary 1). A topological space is Hausdorff provided that for every two different points there can be found disjoint neighborhoods of the points. Let us recall now the concept of w^*-strongly torsionless. Let M be a topological (left) module over a topological ring R. Then, M^* is a right R-module which can be endowed with the w^*-topology (the module topology on M^* inherited from the product topology of R^M). If A is a nonempty subset of M^*, then the preannihilator of A is defined by $A^\top := \bigcap_{m^*\in A} \ker(m^*)$. The topological module M is said to be w^*-strongly torsionless if for every proper w^*-closed submodule N of M^* and every $m^* \in M^* \setminus N$ there exists $m \in N^\top \setminus \{m^*\}^\top$. In virtue of the Hahn–Banach Theorem, real or complex locally convex Hausdorff topological vector spaces are w^*-strongly torsionless. If $A \subsetneq M$ is nonempty, then the annihilator of A is defined as $A^\perp := \{m^* \in M^* : A \subseteq \ker(m^*)\}$.

Suppose now that X is a real or complex normed space admitting a pre-Schauder basis $(e_n)_{n\in\mathbb{N}}$. The set $V := \{x \in X : \forall n \in \mathbb{N}\ \|p_n(x)\| \leq 1\}$ is clearly absolutely convex. Since $(p_n(x))_{n\in\mathbb{N}}$ is a convergent sequence to x, it is not hard to check that V is also absorbing. Thus, $\mathrm{cl}(V)$ is a barrel of X. If X is a Banach space, then it can be proved that V is closed for convex series and therefore $\mathrm{int}(V) = \mathrm{int}(\mathrm{cl}(V))$. Since Banach spaces are barrelled, we conclude that $\mathrm{int}(V) = \mathrm{int}(\mathrm{cl}(V))$ is not empty. This implies that all the canonical projections are continuous and the set $\{p_n : n \in \mathbb{N}\}$ is bounded in $\mathcal{B}(X)$. In particular, the coefficient functionals are also continuous due to the norm properties. As a consequence, in Banach spaces, all pre-Schauder bases are Schauder bases and $\mathrm{bc}\left((e_n)_{n\in\mathbb{N}}\right) := \sup\{\|p_n\| : n \in \mathbb{N}\}$ is called the basis constant. This basis constant is key in order to renorm Banach spaces with Schauder bases to make those bases monotone. We refer the reader to [7–10] for recent developments on Schauder bases in Banach spaces.

In Section 3, we prove that every Banach space with a normalized Schauder basis admits an equivalent norm closer to the original norm than the typical bimonotone renorming and that still makes the basis binormalized and monotone (Theorem 3). We also construct an increasing family of left-comparable norms making the normalized Schauder basis binormalized (Theorem 4 and show that the limit of this family is a right-comparable norm that also makes the normalized Schauder basis binormalized (Corollary 2)).

2. Impact of Pre-Schauder Bases on the Vector Topology

First off, notice that a pre-Schauder basis on a topological vector space must be a linearly independent set. Indeed, if $\lambda_1, \ldots, \lambda_k$ are scalars such that $\lambda_1 e_1 + \cdots + \lambda_k e_k = 0$, then the sequence

$(\gamma_n)_{n\in\mathbb{N}} \subseteq \mathbb{K}$ defined by $\gamma_i = \lambda_i$ for $1 \leq i \leq k$ and $\gamma_i = 0$ for $i \geq k+1$ verifies that $\sum_{n\in\mathbb{N}} \gamma_n e_n$ converges to 0. Since the constant sequence equal to 0 also verifies the previous convergence, by uniqueness it must occur that $\gamma_i = 0$ for $1 \leq i \leq k$.

Recall that in a topological space X, the filter of neighborhoods of an element $x \in X$ is usually denoted by $\mathcal{N}_x^X$ (or simply by $\mathcal{N}_x$ if there is no confusion with X). Recall also that the following conditions are trivially equivalent:

- $y \in \cap\mathcal{N}_x := \bigcap_{V\in\mathcal{N}_x} V$.
- If $(y_i)_{i\in I}$ is a net in X converging to y, then it also converges to x.

If X is a topological vector space, then it is not hard to show that $y \in \cap\mathcal{N}_x$ if and only if $x \in \cap\mathcal{N}_y$. In fact, $x + \cap\mathcal{N}_y = \cap\mathcal{N}_{x+y}$ for all $x, y \in X$. On the other hand, in [11], it was shown that $W_X := \cap\mathcal{N}_0$ is a closed vector subspace of X whose inherited topology is the trivial topology and which is topologically complemented with any of its linear complements. Any subspace Y of X verifying that $Y \cap W_X = \{0\}$ is trivially Hausdorff. In particular, X is Hausdorff if and only if $W_X = \{0\}$.

Recall also that if M is a subset of a vector space X, then the series span of M is defined as

$$\operatorname{sspan}(M) := \left\{ x \in X : \exists\ (\lambda_n)_{n\in\mathbb{N}} \subseteq \mathbb{K} \text{ and } (m_n)_{n\in\mathbb{N}} \subseteq M \text{ such that } \sum_{n\in\mathbb{N}} \lambda_n m_n \text{ converges to } x \right\}.$$

Notice that if X is a topological vector space with a pre-Schauder basis $(e_n)_{n\in\mathbb{N}}$, then $X = \operatorname{sspan}\left((e_n)_{n\in\mathbb{N}}\right) = \overline{\operatorname{span}}\left((e_n)_{n\in\mathbb{N}}\right)$.

Remark 1. *In a topological vector space over an absolutely valued division ring, by the continuity of the addition, for every neighborhood V of 0 we can find another neighborhood U of 0 such that $U + U \subseteq V$. Observe that $-U$ is also a neighborhood of 0 since multiplying by -1 is a homeomorphism. By choosing $W := U \cap (-U)$, we obtain a symmetric neighborhood of 0 ($W = -W$) such that $W + W \subseteq V$.*

Theorem 1. *Let X be a topological vector space over an absolutely valued division ring $\mathbb{K}$ admitting a pre-Schauder basis $(e_n)_{n\in\mathbb{N}}$. If $\sum_{n=1}^{\infty} \lambda_n e_n$ converges to $x \in X$, then $\sum_{n=1}^{\infty} \lambda_n e_n$ converges to every $y \in \cap\mathcal{N}_x$. In other words,*

$$\cap\mathcal{N}_x = \{y \in X : e_n^*(y) = e_n^*(x) \text{ for all } n \in \mathbb{N}\}.$$

In particular,

$$W_X = \{x \in X : e_n^*(x) = 0 \text{ for all } n \in \mathbb{N}\},$$

$\operatorname{span}\{e_n : n \in \mathbb{N}\} \cap W_X = \{0\}$ *and* $W_X \setminus \{0\} \subseteq \operatorname{sspan}\{e_n : n \in \mathbb{N}\} \setminus \operatorname{span}\{e_n : n \in \mathbb{N}\}$.

Proof. Fix an arbitrary $y \in \cap\mathcal{N}_x$. By the observation prior to this theorem, we have that $x \in \cap\mathcal{N}_y$, which automatically implies that $\sum_{n=1}^{\infty} \lambda_n e_n$ converges to y and hence $e_n^*(y) = e_n^*(x)$ for all $n \in \mathbb{N}$. Conversely, assume that $y \in X$ verifies that $e_n^*(y) = e_n^*(x)$ for all $n \in \mathbb{N}$. This means that $\sum_{n=1}^{\infty} \lambda_n e_n$ converges to y. Suppose to the contrary that $y \notin \cap\mathcal{N}_x$. There exists $V \in \mathcal{N}_0$ such that $y \notin x + V$. According to Remark 1, let W be a symmetric neighborhood of 0 such that $W + W \subseteq V$. We will show that $(x + W) \cap (y + W) = \emptyset$. If there are $w_1, w_2 \in W$ with $x + w_1 = y + w_2$, then

$$y = x + (w_1 - w_2) \in x + (W - W) = x + (W + W) \subseteq x + V.$$

This is a contradiction. Therefore, $(x + W) \cap (y + W) = \emptyset$. Finally, since $\sum_{n=1}^{\infty} \lambda_n e_n$ converges to both x and y, we reach a contradiction with the fact that $(x + W) \cap (y + W) = \emptyset$. We have proved that

$$\cap\mathcal{N}_x = \{y \in X : e_n^*(y) = e_n^*(x) \text{ for all } n \in \mathbb{N}\}.$$

As a consequence,

$$W_X = \cap \mathcal{N}_0 = \{x \in X : e_n^*(x) = 0 \text{ for all } n \in \mathbb{N}\}. \tag{3}$$

Let us show now that $\operatorname{span}\{e_n : n \in \mathbb{N}\} \cap W_X = \{0\}$. If $x \in \operatorname{span}\{e_n : n \in \mathbb{N}\} \cap W_X$, then there are $\lambda_1, \ldots, \lambda_k \in \mathbb{K}$ such that $x = \lambda_1 e_1 + \cdots + \lambda_k e_k$ and by Equation (3) we have that $0 = e_n^*(x) = \lambda_n$ for all $1 \leq n \leq k$. As a consequence, $x = 0$. □

The observation preceding the previous theorem allows us to conclude the following corollary. However, we first need a couple of technical lemmas.

Lemma 1. *Let M be a topological module over a topological ring R. If $r \in \mathcal{U}(R)$, then the map*

$$\begin{array}{ccc} M & \to & M \\ m & \mapsto & rm \end{array} \tag{4}$$

is an isomorphism. If, in addition, R is a division ring, then UV is a neighborhood of 0 in M whenever U is a neighborhood of 0 in R and V is a neighborhood of 0 in M.

Proof. Indeed, the continuity of the previous map follows from the module topology. On the other hand, its inverse, given by $m \mapsto r^{-1}m$, is again continuous for the same reason. Now, assume that R is a division ring. Then,

$$UV = \bigcup_{u \in U \setminus \{0\}} uV$$

is a union of open sets, because the isomorphism $m \mapsto um$ maps the neighborhood V onto the neighborhood uV. □

Remark 2. *In a topological vector space X over an absolutely valued division ring $\mathbb{K}$, if U is a neighborhood of 0 in X, then by the continuity of the action of $\mathbb{K}$ on X, there exists a closed ball $\mathsf{B}_{\mathbb{K}}(0,\varepsilon)$ of center 0 and radius $\varepsilon > 0$ and another neighborhood W of 0 in X such that $\mathsf{B}_{\mathbb{K}}(0,\varepsilon)W \subseteq U$. Notice that $V := \mathsf{B}_{\mathbb{K}}(0,\varepsilon)W$ is a neighborhood of 0 in X (Lemma 1), which is in fact balanced since $\mathsf{B}_{\mathbb{K}}V = \mathsf{B}_{\mathbb{K}}(\mathsf{B}_{\mathbb{K}}(0,\varepsilon)W) = \mathsf{B}_{\mathbb{K}}(0,\varepsilon)W = V$, where $\mathsf{B}_{\mathbb{K}}$ is the unit ball of $\mathbb{K}$. In particular, this implies that the balanced neighborhoods of 0 form a basis of neighborhoods of 0 in X. Therefore, if a vector $x \notin W_X$, then we can find a balanced neighborhood V of 0 such that $x \notin V$.*

Lemma 2. *Let $\mathbb{K}$ be an absolutely valued division ring and X a topological vector space over $\mathbb{K}$. Let $x \in X$. Then, the map*

$$\begin{array}{ccc} \mathbb{K} & \to & \mathbb{K}x \\ \lambda & \mapsto & \lambda x \end{array} \tag{5}$$

is an isomorphism if and only if $x \notin W_X$.

Proof. The above map is continuous due to the vector character of the topology of X. Now, we distinguish between two cases:

- $x \in W_X$. In this case, the range of the above map is $\mathbb{K}x$, which is contained in W_X and thus endowed with the trivial topology. Since $\mathbb{K}$ is Hausdorff, the above map is not an isomorphism.
- $x \notin W_X$. Observe that, in particular, $x \neq 0$ and thus the above map is a linear isomorphism. Let us check that its inverse is continuous. Let $\varepsilon > 0$ and consider the open ball $\mathsf{U}_{\mathbb{K}}(0,\varepsilon)$ of center 0 and radius ε. In accordance with Remark 2, let V be a balanced neighborhood of 0 in X such that $x \notin V$. Observe that since $\mathbb{K}$ is a division ring, in accordance with Lemma 1, $\mathsf{U}_{\mathbb{K}}(0,\varepsilon)(V \cap \mathbb{K}x)$ is a neighborhood of 0 in $\mathbb{K}x$. We prove next that the image of $\mathsf{U}_{\mathbb{K}}(0,\varepsilon)(V \cap \mathbb{K}x)$ under the inverse of the above map is contained in $\mathsf{U}_{\mathbb{K}}(0,\varepsilon)$. Indeed, an element of $\mathsf{U}_{\mathbb{K}}(0,\varepsilon)(V \cap \mathbb{K}x)$ has the form

$\lambda(\gamma x)$ where $\lambda \in \mathsf{U}_{\mathbb{K}}(0, \varepsilon)$ and $\gamma \in \mathbb{K}$. If $\gamma = 0$, then the image of $\lambda(\gamma x)$ under the inverse map is 0, which trivially is in $\mathsf{U}_{\mathbb{K}}(0, \varepsilon)$. Thus, assume that $\gamma \neq 0$. In this case, observe that it must be $|\gamma| < 1$, since otherwise we would have that $|\gamma| \geq 1$ so $\frac{1}{|\gamma|} \leq 1$ and then the balancedness of V brings up the contradiction that

$$x = \frac{1}{\gamma}(\gamma x) \in \frac{1}{\gamma} V \subseteq V.$$

Therefore, $|\gamma| < 1$ so $|\lambda\gamma| < \varepsilon$ and hence the image of $\lambda(\gamma x)$ under the inverse map verifies that $\lambda\gamma \in \mathsf{U}_K(0, \varepsilon)$.

□

Remark 3. *Let M be a topological module over a topological ring R. It is not hard to show that, if R is Hausdorff, then M is w^*-strongly torsionless if and only if $\overline{\operatorname{span}}^{w^*}(Q) = \left(Q^\top\right)^\perp$ for all nonempty subset $Q \subseteq M^*$. In this situation, Q is clearly w^*-dense in M^* if Q separates points of M, that is, $Q^\top = \{0\}$.*

As mentioned in the Introduction, the real or complex Hausdorff locally convex topological vector spaces are w^*-strongly torsionless in virtue of the Hahn–Banach Theorem.

Corollary 1. *Let X be a topological vector space over an absolutely valued division ring $\mathbb{K}$. If X admits a pre-Schauder basis $(e_n)_{n\in\mathbb{N}}$, then $\operatorname{span}\{e_n : n \in \mathbb{N}\}$ is Hausdorff, the maps*

$$\begin{array}{ccc} \mathbb{K} & \to & \mathbb{K}e_n \\ \lambda & \mapsto & \lambda e_n \end{array} \tag{6}$$

are isomorphisms for every $n \in \mathbb{N}$, and the coefficient functionals of $(e_n)_{n\in\mathbb{N}}$ are continuous if and only if the canonical projections are also continuous. If, in addition, X is Hausdorff and w^*-strongly torsionless, then $(e_n, e_n^*)_{n\in\mathbb{N}}$ is total.

Proof. In virtue of Theorem 1, $\operatorname{span}\{e_n : n \in \mathbb{N}\} \cap W_X = \{0\}$, therefore $\operatorname{span}\{e_n : n \in \mathbb{N}\}$ is Hausdorff. Since $e_n \notin W_X$ for each $n \in \mathbb{N}$, by applying Lemma 2 we conclude that the maps

$$\begin{array}{ccc} \mathbb{K} & \to & \mathbb{K}e_n \\ \lambda & \mapsto & \lambda e_n \end{array}$$

are isomorphisms. Next, if the canonical projections p_n are continuous, then the coefficient functionals can be obtain by composing $p_n - p_{n-1}$ with the inverse of the previous map (here $p_0 = 0$). Finally, assume that X is Hausdorff and w^*-strongly torsionless. To show that $X^* = \overline{\operatorname{span}}^{w^*}\{e_n^* : n \in \mathbb{N}\}$, it only suffices (see Remark 3) to prove that $\{e_n^* : n \in \mathbb{N}\}$ separates points of X, that is, $\{e_n^* : n \in \mathbb{N}\}^\top = \{0\}$. Indeed, suppose to the contrary that $\{e_n^* : n \in \mathbb{N}\}$ is not separating. If $x \in X \setminus \{0\}$ and $e_n^*(x) = 0$ for all $n \in \mathbb{N}$, then the null constant sequence converges to x and thus 0 belongs to all neighborhoods of x, which implies that x belongs to all neighborhoods of 0, and this is a contradiction since $W_X = \{0\}$ and $x \neq 0$. □

Finally, we can easily construct a non-Hausdorff topological vector space admitting a Schauder basis. We remind the reader that the trivial topology is a vector topology on any vector space.

Theorem 2. *Let Z be a Hausdorff topological vector space over an absolutely valued division ring $\mathbb{K}$ admitting a (pre-)Schauder basis $(e_n)_{n\in\mathbb{N}}$ and let Y be a vector space of dimension strictly greater than 0 endowed with the trivial topology. Then, $(e_n)_{n\in\mathbb{N}}$ is a (pre-)Schauder basis in $X := Z \oplus Y$ (endowed with the product topology).*

Proof. First off, notice that $\mathcal{N}_0^X = \{V + Y : V \in \mathcal{N}_0^Z\}$, therefore, since Z is Hausdorff,

$$W_X = \cap \mathcal{N}_0^X = \bigcap_{V \in \mathcal{N}_0^Z} (V + Y) = \left(\bigcap_{V \in \mathcal{N}_0^Z} V \right) + Y = \left(\cap \mathcal{N}_0^Z \right) + Y = W_Z + Y = \{0\} + Y = Y.$$

Now, let $x \in X$ and write $x = z + y$ with $z \in Z$ and $y \in Y$. Observe that $x = z + y \in z + \cap \mathcal{N}_0^X = \mathcal{N}_z^X$ and thus $z \in \cap \mathcal{N}_x^X$. Therefore, if $\sum_{n=1}^\infty \lambda_n e_n$ converges to z, then it also converges to x. On the other hand, if $\sum_{n=1}^\infty \gamma_n e_n$ converges to x, then it also converges to z for the same previous reason. Since $(e_n)_{n\in\mathbb{N}}$ is a pre-Schauder basis on Z, it must happen that $\lambda_n = \gamma_n$ for all $n \in \mathbb{N}$. This shows that $(e_n)_{n\in\mathbb{N}}$ is a pre-Schauder basis in X and $e_n^*(x) = e_n^*(p_Z(x))$ for all $n \in \mathbb{N}$ and all $x \in X$, where $p_Z : X \to Z$ is the projection on Z. Finally, if $(e_n)_{n\in\mathbb{N}}$ is a Schauder basis in Z, then the coefficient functionals on Z, $e_n^*|Z$, are all continuous for every $n \in \mathbb{N}$. To see that the coefficient functionals on X, e_n^*, are all continuous, it only suffices to prove that the projection p_Z is continuous. This continuity is a consequence of the facts that $p_Z = I - p_Y$, where I is the identity on X, and that $p_Y : X \to Y$ is continuous because Y is endowed with the trivial topology. □

An illustrative example of Theorem 2 follows to conclude this section.

Example 1. *Let Y be any real or complex vector space of dimension strictly greater than 0 and consider it endowed with the trivial topology. Consider $X := c_0 \oplus Y$ endowed with the product topology. Notice X is not Hausdorff. In fact, $W_X = Y$. Finally, the canonical basis $(e_n)_{n\in\mathbb{N}}$ of c_0 is indeed a Schauder basis for X in virtue of Theorem 2.*

3. Renormings Concerning Schauder Bases

Notice that, if $(e_n)_{n\in\mathbb{N}}$ is a pre-Schauder basis in a topological vector space and $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence of nonzero scalars, then $(\lambda_n e_n)_{n\in\mathbb{N}}$ is also a pre-Schauder basis in X. As a consequence, if X is a real or complex normed space, then $\left(\frac{e_n}{\|e_n\|}\right)_{n\in\mathbb{N}}$ is also a pre-Schauder basis.

From now on, we only consider Banach spaces over the real or complex numbers. A Schauder basis $(e_n)_{n\in\mathbb{N}}$ in a Banach space X is said to be:

- normalized if $\|e_n\| = 1$ for all $n \in \mathbb{N}$;
- binormalized if $\|e_n\| = \|e_n^*\| = 1$ for all $n \in \mathbb{N}$;
- monotone if $\|p_n\| = 1$ for all $n \in \mathbb{N}$;
- strictly monotone if $\|p_n(x)\| < \|p_{n+1}(x)\|$ for all $n \in \mathbb{N}$ and all $x \in X \setminus \ker(e_{n+1}^*)$; and
- bimonotone if $\|p_m - p_n\| = 1$ for all $m \in \mathbb{N}$ and all $n \in \mathbb{N} \cup \{0\}$ with $m > n$.

Since $p_n(x) - p_{n-1}(x) = e_n^*(x) e_n$ for all $n \in \mathbb{N}$ and all $x \in X$, we have that $\|p_n - p_{n-1}\| = \|e_n^*\|\|e_n\|$ for all $n \in \mathbb{N}$, therefore a normalized Schauder basis is binormalized if and only if $\|p_n - p_{n-1}\| = 1$ for all $n \in \mathbb{N}$. As a consequence, a normalized bimonotone Schauder basis is monotone and binormalized.

In (Theorem 4.1.14, [12]), it is proved that, if $(e_n)_{n\in\mathbb{N}}$ is a Schauder basis on a Banach space X, then

$$\|x\|_m := \sup\{\|p_n(x)\| : n \in \mathbb{N}\} \tag{7}$$

is an equivalent norm on X that makes the Schauder basis monotone. In fact, it is verified that

$$\|\cdot\| \le \|\cdot\|_m \le \mathrm{bc}\left((e_n)_{n\in\mathbb{N}}\right) \|\cdot\|. \tag{8}$$

Observe that $(e_n)_{n\in\mathbb{N}}$ is monotone with $\|\cdot\|$ if and only if $\|\cdot\|_m = \|\cdot\|$. Let us provide now an example where $\|\cdot\| < \|\cdot\|_m$. For this, it suffices to find a nonmonotone Schauder basis. We have to introduce first a bit of notation. If X is a Banach space, then $\mathsf{NA}(X)$ stands for the set of norm-attaining functionals on X, in other words,

$$\mathsf{NA}(X) := \{x^* \in X^* : \exists\, x \in \mathsf{S}_X \text{ with } x^*(x) = \|x^*\|\}. \tag{9}$$

Example 2. *In [13], an equivalent renorming on the Banach space c_0 of sequences converging to 0 is found in such a way that $\mathsf{NA}(c_0)$ does not contain any vector subspace of dimension strictly greater than 1. The canonical basis of c_0 under this equivalent renorming is still a Schauder basis, but it is not monotone, since in (Theorem 3.1(1), [14]), it was proved that, if a Banach space X has a monotone Schauder basis, then $\mathsf{NA}(X)$ contains an infinite dimensional vector subspace.*

The previous renorming (7) has a couple of inconveniences:

- The Banach space loses all kind of smooth properties after applying the previous renorming. We refer the reader to [9] where it is shown that a uniformly Frechet smooth Banach space with a Schauder basis can be equivalently renormed to remain uniformly Frechet smooth and to make the basis monotone.
- The renorming $\|\cdot\|_m$ does not assure that the Schauder basis become binormalized even if $(e_n)_{n\in\mathbb{N}}$ is normalized with the original norm.

To obtain a binormalized Schauder basis from an equivalent renorming, another equivalent norm is used. Let X be a Banach space with a normalized Schauder basis $(e_n)_{n\in\mathbb{N}}$. Then,

$$\|x\|_M := \sup\{\|p_m(x) - p_n(x)\| : m \in \mathbb{N}, n \in \mathbb{N}\cup\{0\}, m > n\} \tag{10}$$

verifies the following properties (see [10], Exercise 4.28):

1. It is an equivalent norm on X. In fact,

$$\|\cdot\| \le \|\cdot\|_m \le \|\cdot\|_M \le 2\mathrm{bc}\left((e_n)_{n\in\mathbb{N}}\right)\|\cdot\|. \tag{11}$$

2. $\|e_n\|_M = \|e_n^*\|_M^* = 1$ for all $n \in \mathbb{N}$.
3. $\|p_m - p_n\|_M = 1$ for all $m \in \mathbb{N}$ and all $n \in \mathbb{N}\cup\{0\}$ with $m > n$.

The previous renorming in Equation (10) makes the normalized Schauder basis bimonotone and thus binormalized and monotone. Notice that $\|\cdot\| = \|\cdot\|_M$ if and only if the normalized Schauder basis $(e_n)_{n\in\mathbb{N}}$ is bimonotone with the original norm. The inconvenience presented by the renorming $\|\cdot\|_M$ is that it is further from the original norm than $\|\cdot\|_m$, as we can see in Equation (11). Being further from the original norm makes the renorming lose plenty of geometric properties.

We construct an equivalent norm on X closer to the original norm than the previous one and that still allows making the normalized Schauder basis binormalized and monotone. However, we first need to make several observations.

Notice that, if X is a Banach space with a normalized Schauder basis $(e_n)_{n\in\mathbb{N}}$, then

$$\|x\|_d := \sup\{|e_n^*(x)| : n \in \mathbb{N}\} = \sup\{\|p_n(x) - p_{n-1}(x)\| : n \in \mathbb{N}\} \tag{12}$$

is a norm on X left-comparable to its original norm verifying that $\|e_n\|_d = \|e_n^*\|_d^* = 1$ for all $n \in \mathbb{N}$. Indeed, for all $x \in X$ we have that

$$\begin{aligned}
\|x\|_d &= \sup\{\|p_n(x) - p_{n-1}(x)\| : n \in \mathbb{N}\} && (13)\\
&\le \sup\{\|p_m(x) - p_n(x)\| : m \in \mathbb{N}, n \in \mathbb{N}\cup\{0\}, m > n\} && (14)\\
&= \|x\|_M && (15)\\
&\le 2\mathrm{bc}\left((e_n)_{n\in\mathbb{N}}\right)\|x\|. && (16)
\end{aligned}$$

Since $(e_n)_{n\in\mathbb{N}}$ is normalized, $(e_n^*(x))_{n\in\mathbb{N}}$ converges to 0 for all $x \in X$, therefore X endowed with the norm $\|\cdot\|_d$ given in Equation (12) is isometric to a subspace of c_0 according to the embedding

$$\begin{array}{rcl} X & \to & c_0 \\ x & \mapsto & (e_n^*(x))_{n\in\mathbb{N}}. \end{array} \tag{17}$$

As a consequence, if $\|\cdot\|_d$ is equivalent to the original norm of X, then X is isomorphic to a subspace of c_0.

We are now at the right point to find an equivalent norm closer to the original norm than the norm given in Equation (10) and that still makes the normalized Schauder basis binormalized and monotone.

Theorem 3. *Let X be a Banach space with a normalized Schauder basis $(e_n)_{n\in\mathbb{N}}$. Then,*

$$\|x\|_{md} := \max\{\|x\|_m, \|x\|_d\} \tag{18}$$

verifies the following properties:

1. *It is an equivalent norm on X closer to the original norm that $\|\cdot\|_M$. In fact,*

$$\|\cdot\| \le \|\cdot\|_m \le \|\cdot\|_{md} \le \|\cdot\|_M \le 2\mathrm{bc}\left((e_n)_{n\in\mathbb{N}}\right)\|\cdot\|. \tag{19}$$

2. *$\|e_n\|_{md} = \|e_n^*\|_{md}^* = 1$ for all $n \in \mathbb{N}$.*
3. *$\|p_n\|_{md} = 1$ for every $n \in \mathbb{N}$.*

Proof.

1. It is trivial that $\|\cdot\|_{md}$ defines a norm on X. Notice in first place that $\|\cdot\|_m \le \|\cdot\|_M$ by Equation (11). By Equation (15), $\|\cdot\|_d \le \|\cdot\|_M$, thus $\|\cdot\|_{md} = \max\{\|\cdot\|_m, \|\cdot\|_d\} \le \|\cdot\|_M$. Finally, by Equation (11), we obtain the desired Equation (19).
2. Obviously, $\|e_n\|_{md} = \|e_n\|_d = \|e_n\|_m = \|e_n\| = 1$ for all $n \in \mathbb{N}$. Thus, it suffices to show that $\|e_n^*\|_{md}^* \le 1$ for all $n \in \mathbb{N}$. Indeed, if $x \in X$, then $|e_n^*(x)| \le \|x\|_d \le \|x\|_{md}$ for all $n \in \mathbb{N}$. This shows that $\|e_n^*\|_{md}^* \le 1$ for all $n \in \mathbb{N}$.
3. Again, since $\|e_n\|_{md} = 1$ for all $n \in \mathbb{N}$, it suffices to show that $\|p_n\|_{md} \le 1$ for every $n \in \mathbb{N}$. Indeed, fix arbitrary elements $n \in \mathbb{N}$ and $x \in X$ with $\|x\|_{md} \le 1$. We will follow two steps:

Step 1 Since $\|x\|_m \le 1$, we have that $\|p_k(x)\| \le 1$ for all $k \in \mathbb{N}$, therefore

$$\|p_k(p_n(x))\| = \|p_{\min\{k,n\}}(x)\| \le 1$$

for all $k \in \mathbb{N}$ and hence $\|p_n(x)\|_m \le 1$.

Step 2 Since $\|x\|_d \le 1$, we have that $|e_k^*(x)| \le 1$ for all $k \in \mathbb{N}$. Now,

$$e_k^*(p_n(x)) = \begin{cases} 0 & \text{if } k > m \\ e_k^*(x) & \text{if } k \le m \end{cases}$$

for all $k \in \mathbb{N}$, so $|e_k^*(p_n(x))| \le 1$ for all $k \in \mathbb{N}$ and hence $\|p_n(x)\|_d \le 1$.

As a consequence, $\|p_n(x)\|_{md} = \max\{\|p_n(x)\|_m, \|p_n(x)\|_d\} \le 1$.

□

The norm $\|\cdot\|_d$ is in fact a member of a family of left-comparable norms on X verifying certain properties. We remind the reader that

$$\mathcal{P}_k^{\times}(A) := \{B \subseteq A : 0 < \mathrm{card}(B) \le k\}.$$

Theorem 4. *Let X be a Banach space with a normalized Schauder basis $(e_n)_{n\in\mathbb{N}}$. Fix an arbitrary $k \in \mathbb{N}$ and consider*

$$\|x\|_{d_k} := \sup\left\{\left\|\sum_{i\in A}\lambda_i e_i\right\| : x = \sum_{i=1}^{\infty}\lambda_i e_i,\ A \in \mathcal{P}_k^{\times}(\mathbb{N})\right\}. \tag{20}$$

Then,

1. $\|\cdot\|_{d_1} = \|\cdot\|_d$.
2. $\|p_j(x)\| \leq \|x\|_{d_k}$ *for all* $j \leq k$ *and all* $x \in X$.
3. $\|\cdot\|_{d_k} \leq 2k\mathrm{bc}\left((e_n)_{n\in\mathbb{N}}\right)\|\cdot\|$ *for all* $k \in \mathbb{N}$.
4. $\|\cdot\|_{d_{k_1}} \leq \|\cdot\|_{d_{k_2}}$ *if* $k_1 \leq k_2$.
5. $\|e_n\|_{d_k} = \|e_n^*\|_{d_k} = 1$ *for all* $n \in \mathbb{N}$.
6. $\|\sum_{i\in A} p_i - p_{i-1}\|_{d_k} = 1$ *for every* $A \in \mathcal{P}_k^{\times}(\mathbb{N})$.

Proof. First, notice that

$$\sum_{i\in A} \lambda_i e_i = \sum_{i\in A} p_i(x) - p_{i-1}(x)$$

for all $A \in \mathcal{P}_k^{\times}(\mathbb{N})$ and all $x \in X$. Then,

$$\|x\|_{d_k} = \sup\left\{\left\|\sum_{i\in A} p_i(x) - p_{i-1}(x)\right\| : x = \sum_{i=1}^{\infty} \lambda_i e_i,\ A \in \mathcal{P}_k^{\times}(\mathbb{N})\right\}$$

for all $x \in X$. The first five properties are easy to check. We only prove the last one. Indeed, fix an arbitrary $A \in \mathcal{P}_k^{\times}(\mathbb{N})$. If $i_0 := \max(A)$, then

$$\left(\sum_{i\in A} p_i - p_{i-1}\right)(e_{i_0}) = p_{i_0}(e_{i_0}) = e_{i_0},$$

therefore

$$\left\|\left(\sum_{i\in A} p_i - p_{i-1}\right)(e_{i_0})\right\|_{d_k} = \|e_{i_0}\|_{d_k} = 1,$$

so it suffices to show that

$$\left\|\sum_{i\in A} p_i - p_{i-1}\right\|_{d_k} \leq 1.$$

Fix an arbitrary $x = \sum_{i=1}^{\infty} \lambda_i e_i$ with $\|x\|_{d_k} \leq 1$. We show that

$$\left\|\left(\sum_{i\in A} p_i - p_{i-1}\right)(x)\right\|_{d_k} \leq 1.$$

Note that

$$\left(\sum_{i\in A} p_i - p_{i-1}\right)(x) = \sum_{i\in A} \lambda_i e_i.$$

Let $(\gamma_n)_{n\in\mathbb{N}}$ be defined by

$$\gamma_i := \begin{cases} \lambda_i & \text{if } i \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for every $B \in \mathcal{P}_k^{\times}(\mathbb{N})$ we have that

$$\sum_{i\in B} \gamma_i e_i = \begin{cases} \displaystyle\sum_{i\in A\cap B} \lambda_i e_i & \text{if } A \cap B \neq \emptyset, \\ 0 & \text{if } A \cap B = \emptyset. \end{cases}$$

Thus,

$$
\begin{aligned}
\left\| \left(\sum_{i \in A} p_i - p_{i-1} \right)(x) \right\|_{d_k} &= \left\| \sum_{i \in A} \lambda_i e_i \right\|_{d_k} && (21)\\
&= \left\| \sum_{i=1}^{\infty} \gamma_i e_i \right\|_{d_k} && (22)\\
&= \sup \left\{ \left\| \sum_{i \in B} \gamma_i e_i \right\| : B \in \mathcal{P}_k^{\times}(\mathbb{N}) \right\} && (23)\\
&= \sup \left\{ \left\| \sum_{i \in A \cap B} \lambda_i e_i \right\| : B \in \mathcal{P}_k^{\times}(\mathbb{N}) \right\} && (24)\\
&\leq \|x\|_{d_k} && (25)\\
&\leq 1. && (26)
\end{aligned}
$$

□

We remind the reader that a Schauder basis $(e_n)_{n\in\mathbb{N}}$ is said to be weakly unconditionally Cauchy (wuC) if the series $x = \sum_{n=1}^{\infty} \lambda_n e_n$ is weakly unconditionally Cauchy for every x. Unconditional Schauder bases are examples of wuC Schauder bases. We rely on the following well known result to prove our last corollary.

Theorem 5. *[Diestel, 1984; [15]] A series $\sum_{i=1}^{\infty} x_i$ in a normed space X is wuC if and only if there exists $H > 0$ such that*

$$
\begin{aligned}
H &= \sup \left\{ \left\| \sum_{i=1}^{n} a_i x_i \right\| : n \in \mathbb{N}, |a_i| \leq 1, i \in \{1, \ldots, n\} \right\} && (27)\\
&= \sup \left\{ \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| : n \in \mathbb{N}, \varepsilon_i \in \{-1, 1\}, i \in \{1, \ldots, n\} \right\} && (28)\\
&= \sup \left\{ \sum_{i=1}^{n} |f(x_i)| : f \in \mathsf{B}_{X^*} \right\}. && (29)
\end{aligned}
$$

We refer the reader to [16] for several interesting characterizations of completeness of normed spaces through weakly unconditionally Cauchy series.

Recall that $\mathcal{P}_{\mathtt{f}}^{\times}(A) := \{B \subseteq A : 0 < \mathrm{card}(B) < \infty\}$.

Corollary 2. *Let X be a Banach space with a normalized wuC Schauder basis $(e_n)_{n\in\mathbb{N}}$. Then,*

$$
\| \cdot \|_{d_\infty} = \sup_{k \in \mathbb{N}} \| \cdot \|_k \tag{30}
$$

is a right-comparable norm on X such that:

1. $\| \cdot \| \leq \| \cdot \|_m \leq \| \cdot \|_{d_\infty}$.
2. $\|e_n\|_{d_\infty} = \|e_n^*\|_{d_\infty} = 1$ *for all* $n \in \mathbb{N}$.
3. $\|\sum_{i \in A} p_i - p_{i-1}\|_{d_\infty} = 1$ *for every* $A \in \mathcal{P}_{\mathtt{f}}^{\times}(\mathbb{N})$.

Proof. According to Theorem 4 (2), we deduce that $\|\cdot\| \leq \|\cdot\|_m \leq \|\cdot\|_{d_\infty}$. We now show that $\|\cdot\|_{d_\infty}$ is well defined. Fix an arbitrary $x = \sum_{i=1}^{\infty} \lambda_i e_i \in X$. Since the previous series is weakly unconditionally Cauchy, Theorem 5 allows us to deduce that there exists a constant $H_x > 0$ such that

$$H_x = \sup\left\{\left\|\sum_{i=1}^{n} a_i(\lambda_i e_i)\right\| : n \in \mathbb{N}, |a_i| \leq 1, i \in \{1, \ldots, n\}\right\}.$$

Therefore,

$$\|x\|_{d_\infty} = \sup\left\{\left\|\sum_{i \in A} \lambda_i e_i\right\| : x = \sum_{i=1}^{\infty} \lambda_i e_i,\ A \in \mathcal{P}_{\mathfrak{f}}^{\times}(\mathbb{N})\right\} \leq H_x.$$

We leave the rest of the details of the proof to the reader. □

4. Conclusions

Schauder bases have never been profoundly studied in topological vector spaces, and whenever they have been, it was only in the Hausdorff setting over the real or complex numbers. In this sense, this manuscript is a further contribution on the study of Schauder bases in topological vector spaces over absolutely valued division rings.

We now proceed to enumerate the conclusions obtained from this work:

1. It is a trivial fact that in real or complex normed spaces the coefficient functionals of a pre-Schauder basis are continuous if and only if the canonical projections are also continuous. This is not trivial at all in real or complex topological vector spaces. We have accomplished this equivalence not only in real or complex topological vector spaces but also on topological vector spaces over an absolutely valued division ring (Corollary 1).
2. We have placed on the spotlight the strong impact of a pre-Schauder basis on the vector topology in the sense that the existence of a pre-Schauder basis in a topological vector space over an absolutely valued division ring forces the existence of a dense Hausdorff subspace, which is precisely the linear span of the basis (Theorem 1).
3. We have demonstrated the existence of Schauder bases in the non-Hausdorff setting (Theorem 2). Therefore, it makes sense to keep studying Schauder bases in the non-Hausdorff setting.
4. We construct (Theorem 3) an equivalent renorming that turns a normalized Schauder basis into a binormalized and monotone Schauder basis and this renorming is not far from the original norm, at least it is closer to the original norm than the typical bimonotone renorming given in Equation (10). This way, our renorming has a lower chance of losing geometrical properties than the renorming given in Equation (10).

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	linear dichroism

References

1. Schauder, J. Zur Theorie stetiger Abbildungen in Funktionalraumen. *Mathematische Zeitschrift* **1927**, *26*, 47–65. [CrossRef]
2. Kothe, G. *Topological Vector Spaces I*; Springer: New York, NY, USA , 1969.
3. Bourbaki, N. *Topological Vector Spaces*; Springer: Berlin, Germany, 1987; Chapters 1–5.
4. Warner, S. *Topological Fields*; Elsevier: Amsterdam, The Netherlands, 1989
5. Warner, S. Topological rings. In *North-Holland Mathematics Studies*; North-Holland Publishing Co.: Amsterdam, The Netherlands, 1993; p. 178.
6. Hájek, P.; Montesinos-Santalucía, V.; Vanderwerff, J.; Zizler, V. Biorthogonal systems in Banach spaces. In *CMS Books in Mathematics*; Springer: New York, NY, USA, 2008; p. 26.
7. Fabian, M.; Habala, P.; Hájek, P.; Montesinos-Santalucía, V.; Pelant, J.; Zizler, V. Schauder bases. In *Functional Analysis and Infinite-Dimensional Geometry. Canadian Mathematical Society/Société Mathématique du Canada*; Springer: New York, NY, USA, 2001; pp. 161–201.
8. McArthur, C.W. Developments in Schauder basis theory. *Bull. Am. Math. Soc.* **1972**, *78*, 877–908. [CrossRef]
9. Guirao, A.J.; Hájek, P. Schauder bases under uniform renormings. *Positivity* **2007**, *11*, 627–638. [CrossRef]
10. Fabian, M.; Habala, P.; Hájek, P.; Montesinos-Santalucía, V.; Zizler, V. Banach space theory: The basis for linear and nonlinear analysis. In *CMS Books in Mathematics*; Springer: New York, NY, USA, 2011.
11. Armario, R.; García-Pacheco, F.J.M.; Pérez-Férnandez, F.J. On the Krein-Milman Property and the Bade Property. *Linear Algebra Appl.* **2012**, *436*, 1489–1502. [CrossRef]
12. Megginson, R.E. An introduction to Banach space theory. In *Graduate Texts in Mathematics*; Springer: New York, NY, USA, 1998; p. 183.
13. Rmoutil, M. Norm-attaining functionals and proximinal subspaces. *J. Funct. Anal.* **2017**, *272*, 918–928. [CrossRef]
14. Acosta, M.D.; Aizpuru, A.; Aron, R.M.; García-Pacheco, F.J. Functionals that do not attain their norm. *Bull. Belg. Math. Soc. Simon Stevin* **2007**, *14*, 407–418.
15. Diestel, J. Sequences and series in Banach spaces. In *Graduate Texts in Mathematics*; Springer: New York, NY, USA, 1984.
16. Pérez-Fernández, F.J.; Benítez-Trujillo, F.; Aizpuru, A. Characterizations of completeness of normed spaces through weakly unconditionally Cauchy series. *Czechoslov. Math. J.* **2000**, *50*, 889–896. [CrossRef]