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"LEARNING AND CAPACITY EXPANSION  
IN A NEW MARKET UNDER UNCERTAINTY"

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# LEARNING AND CAPACITY EXPANSION IN A NEW MARKET UNDER UNCERTAINTY

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## Abstract

A competitive, dynamic model of entry into a new industry is set up and both its positive and normative properties are studied. The main assumptions are that uncertainty with respect to its eventual size prevails and that later waves of entrants are able to observe how profitable earlier entrants had been. The major result reported is that the equilibrium rate of entry lags behind the optimum one.

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## 1. Introduction

When a new market opens (as a result of a product being newly invented, for instance) or when an existing market starts to expand (due to the discovery of new uses for the product transacted in it), uncertainty regarding its eventual size is likely to prevail and to affect the pace of entry into it. Surely, firms will not enter into the market at once. Instead, some will enter initially, while others will wait and see what the consequences of that early entry had been. If the market - in retrospect - turns out to be "large", then early entrants seem to have entered "at the right time". On the other hand, if the reverse is true, then those who had waited seem to have been "wise to postpone entry". What determines success in such circumstances?

The purpose of the present paper is to consider a situation where success is random or, in other words, where excessive profits are due to luck. The basic tenet of the model below is that no market participant possesses superior decision-making capabilities or superior knowledge. Ex-ante, they all face uncertainty in making their entry decision. At that point in time, the timing of entry entails a tradeoff between gaining better information (which the "wait and see" strategy delivers) and realizing large profits (which the "immediate entry" strategy might deliver). Ex-post, some firms generate higher profits than others but that, again, is attributable to luck only and is the inevitable consequence of the uncertainty they have initially faced. The goal of this paper, then, is to formalize this uncertainty, analyse the time-patterns of entry under it and examine their welfare properties.

The problem of growth and capacity expansion in a new market had previously been examined by Spence [1979], but his analysis is more pertinent to the oligopoly case under certainty. Here we look, instead, at the evolution of a new industry under uncertainty and we do so under competitive (i.e. free-entry) conditions. The focus of the analysis is on the traditional comparison between the competitive evolution and the optimal one. This comparison acquires interest in

our setting because markets for future contingences which are written on the size of the new market are (by assumption) non-existent. Relatedly, our setting assumes away the possibility of contracts between present and future generations of entrants. The primary finding concerning this welfare comparison is that the competitive pace of expansion lags behind the socially optimal one, this result being closely related to what K.J. Arrow [1962] had pointed out more than 25 years ago in his seminal paper on the Economics of Learning by Doing.

Our second objective is to expand the literature on the "Economics of Learning" to the multi-agent setting. Previous studies on this subject [1], [2], [3], [4], had used a Bayesian setting (as we do) and had emphasized the tradeoff between learning via experimentation vs. maximizing present payoffs given the current state of knowledge. More importantly, perhaps, their focus had been on the implications of this tradeoff on the observed time-series of agents' actions. While valuable, these papers focus on the single agent case (monopoly, the household consumer, etc.) and, as a result, give no consideration to the consequences of agents interacting with each other (either strategically or via markets). In particular, in such settings it is not possible to explore the implications of one agent being able to costlessly learn from another agent's experience (by observing the outcome of the latter's action). In contrast, our formulation focuses on the multi-agent case and in doing so it brings into light an externality which arises under those conditions. Broadly speaking, this externality is due to the fact that existing firms are not rewarded for the informational benefits which their actions confer upon future generations of entrants and, hence, fail to take these benefits into account in making their own entry decisions. For this reason, the actual pace of capacity expansion is inefficiently slow.

The rest of the paper is organized as follows. The next section sets up the model. The section following it introduces the equilibrium concept, proves its existence and uniqueness and provides two examples illustrating the results. Section 4 sets up the planner's objective function, derives a set of first order conditions which are necessary for its optimum, compares them with the

equilibrium conditions as stated forth in section 3 and completes the parametric examples started there. Section 5 contains concluding comments.

## 2. Formulation

### 2.A. Consumers

The consumption side of the market is described by a stationary (inverse) demand function,

$$p(Q, \theta) = \begin{cases} 1, & Q < \theta \\ 0, & Q \geq \theta \end{cases} \quad (2.1)$$

where  $\theta \in \mathbb{R}_+$ . At  $t=0$ ,  $\theta$  is an unknown parameter. Its prior distribution is described by the c.d.f.  $F(\theta)$  with a corresponding p.d.f.  $f(\theta)$ . The following restrictions on  $F$  are maintained throughout the paper.

**Assumption A.1.**  $F(\cdot)$  is strictly increasing, continuously differentiable and has a finite first moment.

This specification of demand may seem restrictive, and admittedly, it is. We have borrowed it from the optimal inventory literature, this paper being a competitive, dynamic contribution to it. The major advantage of (2.1) is its tractability: not only with respect to the establishment of general existence and uniqueness results (see section 3.A below), but also in terms of computations and the derivation of closed-form expressions for both the equilibrium and the optimum (see the examples in sections 3.B and 4.C). On the other hand, the central idea which underlies this paper, namely, that potential entrants get to observe the outcomes of extant firms' actions and hence get to learn from their experience, does not seem to hinge upon this particular choice of a parametric demand function.

## 2.B. Producers

Time is discrete and is indexed by  $t=0,1,2,\dots$ . At  $t=0$ ,  $x_0$  firms are present in the market and at each subsequent time period,  $y_t$  new firms enter. The act of entry entails an irreversible investment, the per unit cost of which is  $c$ . Thus, the social cost -- in terms of period  $t$  dollars -- associated with the entry of  $y_t$  firms is  $cy_t$ . Firms are non-atomic and, once entered, are infinitely lived (i.e., each firm's investment is assumed not to depreciate). Each firm is capable of producing one unit per period -- at zero variable cost. Letting  $x_t$  denote the number of existing firms at time  $t$  (after entry in that period had taken place), we have:

$$x_t = x_0 + \sum_{\tau=1}^t y_\tau, \quad x_0 \geq 0 \text{ being initially given.} \quad (2.2)$$

By our foregoing assumptions,  $x_t$  is also the available (and perfectly inelastic) supply in the industry at time  $t$ . The product market opens in each and every time period and equilibrium in it is determined by the stationary demand, (2.1), and the supply,  $x_t$ . More explicitly,

$$p_t = \begin{cases} 1, & \theta < x_t \\ 0, & \theta \geq x_t \end{cases} \quad (2.3)$$

Below the entry sequence,  $(y_t)$ , will be alternatively considered either as market-determined -- assuming free-entry and imposing a zero-profit condition -- or as a control variable chosen by a "social planner". In either case, a common

discount factor,  $\beta = \frac{1}{1+r}$ , is assumed. The following joint restriction is imposed.

**Assumption A.2:**  $c < \frac{1}{1-\beta}$

(This assumption eliminates the uninteresting possibility of a degenerate outcome where entry does not occur at all).

**2.C. Information and Time Structure**

Potential firms (as discussed above there is an infinite supply of them at cost  $c$ ) and the planner are symmetrically informed. At the beginning of time  $t$ , it is publicly known whether there is still an excess demand in the market, i.e., whether  $\theta > x_{t-1}$  or whether the market had already been saturated,  $\theta \leq x_{t-1}$ . Within our parametric example this information is revealed by the equilibrium price in the product market at time  $t-1$ , i.e., by the relationship (2.3) for  $t \geq 2$ . For  $t=1$  it is assumed that  $\theta > x_0$  (and, thus, that  $F(\cdot)$  which is introduced in section 2.A is actually  $F(\cdot, \theta > x_0)$ . For simplicity's sake, however, we shall omit an explicit mention of the  $\theta > x_0$  condition).

Allowing the entry decision at time  $t$  to depend on the price at  $t-1$ , is based on the more general idea that prospective entrants are able to obtain information regarding the future profitability of their target industry by reading financial statements, employment figures and general business news pertaining to the firms which are already operating in that industry. This information, however, is available only after a certain time lag, and the length of the time period in the model (1) corresponds exactly to this "observation lag". No doubt,



operating firms will attempt to conceal such information. However, these attempts are neither costless nor are they perfectly successful [1] and to the extent that some information does leak out the features highlighted by this model will be of relevance.

### 3. Equilibrium

#### 3.A. Existence and Uniqueness

Heuristically speaking, the equilibrium concept considered here is one where the net (expected) profit available to each entrant at each point in time is zero. More precisely, any sequence of investments,  $y = (y_t)_{t=1}^{\infty}$ , induces a random time-path of prices,  $(p_t(\theta, y))_{t=1}^{\infty}$  [2]. Furthermore, given this time-path and the information available at time  $t$  (see 1.C above), a conditional distribution of prices starting as of that date is computable (specifically, the joint distribution of  $p_t, p_{t+1}, \dots$ ) is well-defined once  $p_1, p_2, \dots, p_{t-1}$  and  $y$  are known). From this conditional distribution, in turn, one can determine the expected discounted revenue (starting as of time  $t$ ) of a firm which sells one unit of the product in perpetuity. This revenue is exactly the return which a rational entrant (at  $t$ ) should anticipate on an investment of  $c$  dollars, and -- in equilibrium -- the two (i.e., revenue and cost) must be equal. Thus, an equilibrium investment path,  $(y_t)_{t=1}^{\infty}$  is one for which the property of cost=expected discounted revenue holds at each and every point in time.

Notationally, given  $y = (y_t)_{t=1}^{\infty}$  and the corresponding  $x = (x_t)_{t=0}^{\infty}$  (see (2.2)), we first define:

$$p_{t,k} = \Pr(\theta \leq x_{t+k}, \theta > x_{t+k-1} | \theta > x_{t-1}), \quad (3.1)$$

$$q_{t,n} = \sum_{k=n+1}^{\infty} p_{t,k} = \Pr(\theta > x_{t+n} | \theta > x_{t-1}) = \frac{\Pr(\theta > x_{t+n})}{\Pr(\theta > x_{t-1})}, \quad (3.2)$$

and

$$r_t = \sum_{k=0}^{\infty} p_{t,k} \left( \sum_{n=0}^{k-1} \beta^n \right), \quad t = 1, 2, \dots; k = 0, 1, 2, \dots \quad (3.3)$$

( $p_{t,k}$  is the probability that  $c$  dollars invested at time  $t$  will yield a positive return for exactly  $k$  periods;  $q_{t,n}$  is the probability that they will yield a positive return for at least  $n+1$  periods;  $r_t$  is the lifetime revenue which this investment is expected to generate).

$$\text{NOTE: } r_t = \sum_{k=0}^{\infty} p_{t,k} \left( \sum_{n=0}^{k-1} \beta^n \right) = \sum_{n=0}^{\infty} \beta^n \left( \sum_{k=n+1}^{\infty} p_{t,k} \right) = \sum_{n=0}^{\infty} \beta^n q_{t,n}. \quad (3.4)$$

Next we introduce:

DEFINITION: A perfect foresight, free entry equilibrium (PFFEE) is a sequence of investments,  $(y_t)$ , for which the induced probabilities,  $(p_{t,k})$ , are such that

$$c = r_t, \quad t = 1, 2, \dots \quad (3.5)$$

NOTE: Since the  $r_t$ 's depend on the vector of capacity levels,  $(x_t)$ , condition (3.4) defines an infinite system of (non-linear) equations in an infinite number of "unknowns",  $(x_t)_{t=1}^{\infty}$ . Theorem 1 below establishes that a simple and unique solution to it exists and that it can be determined recursively.

THEOREM 1: Under assumption (A.1), a unique PFFEE exists and it can be determined serially from following set of conditions:

$$c = (1 + \beta c) [1 - F(x_{t-1} + y_t | \theta > x_{t-1})]. \quad (3.6)$$

NOTE: The RHS of (3.6) is the expected discounted revenue which is available to a firm entering at the beginning of time  $t$ , and it depends on the endogenous data,

$(x_t)$ , via the term  $1 - F(x_{t-1} + y_t | \theta > x_{t-1})$ .

PROOF: We start out by proving a pair of claims.

CLAIM 1:  $q_{t,n} = q_{t,0} q_{t+1,n-1}$ . (3.7)

PROOF:  $q_{t+1,n-1} = \frac{\Pr(\theta > x_{t+1+n-1})}{\Pr(\theta > x_t)} = \frac{\Pr(\theta > x_{t+n}) / \Pr(\theta > x_{t-1})}{\Pr(\theta > x_t) / \Pr(\theta > x_{t-1})} =$

$$= \frac{q_{t,n}}{q_{t,0}} \quad (\text{see expression (3.2)}).$$

CLAIM 2:  $r_t = q_{t,0} (1 + \beta r_{t+1})$ . (3.8)

PROOF:  $r_t = \sum_{n=0}^{\infty} \beta^n q_{t,n} = q_{t,0} + \sum_{n=1}^{\infty} \beta^n q_{t,n}$

$$= q_{t,0} + \beta \sum_{n=1}^{\infty} \beta^{n-1} q_{t,0} q_{t+1,n-1} = q_{t,0} + q_{t,0} \beta \sum_{n=0}^{\infty} \beta^n q_{t+1,n}$$

$$= q_{t,0} (1 + \beta r_{t+1}),$$

where the first equality is based on expression (3.4) and the third one on claim 1 (that is, on (3.7)).

Returning now to the proof of the theorem, it is clear from the way the

model had been set up that  $q_{t,0} = 1 - F(x_t | \theta > x_{t-1})$  (see expression (3.2)). Thus, substituting  $c$  (instead of  $r_t$  and  $r_{t+1}$ ) into (3.6) we get

$$c = (1 + \beta c) [1 - F(x_{t-1} + y_t | \theta > x_{t-1})] \quad (3.6)$$

Obviously, given an  $x_{t-1}$  this equation has a unique solution,  $y_t$  (here the strict monotonicity of  $F$  is being used), and our proof is complete.

### 3.B. Illustrative Examples

1. (Uniform) Specializing the fundamental zero-profit condition (3.6) to the case where  $F(\theta) = \theta$ ,  $0 \leq \theta \leq 1$  and assuming  $x_0 = 0$  we have:

$$\frac{1 - (x_{t-1} + y_t)}{1 - x_{t-1}} = \frac{c}{1 + \beta c} \quad (3.9)$$

From this we readily obtain:

$$y_t = \frac{1 - (1 - \beta)c}{1 + \beta c} (1 - x_{t-1}) \equiv \alpha (1 - x_{t-1}) = \alpha (1 - \alpha)^{t-1}, \quad (3.10a)$$

$$1 - x_t = (1 - \alpha)(1 - x_{t-1}), \quad (3.10b)$$

and

$$x_t = 1 - (1 - \alpha)^t. \quad (3.10c)$$

That is, in the uniform case the equilibrium is such that the  $t^{\text{th}}$  period investment is equal to a constant fraction,  $\alpha$  (which by assumption A.2 is strictly

between zero and one), of the maximum residual market,  $1-x_{t-1}$ .

Thus, the sequence of investments forms a geometric series with a first term =  $\alpha$  and a decay rate =  $1-\alpha$ . Furthermore, from the definition of  $\alpha$  (see (3.10a)) we see - as one would have expected - that investments are monotonically increasing in  $\beta$  (the discount factor) and decreasing in  $c$ .

2. (Exponential) Specializing (3.6) again to the case where

$$F(\theta, \lambda) = 1 - e^{-\lambda \theta}, \quad 0 \leq \theta < \infty \quad (\lambda \text{ being a constant parameter}) \quad (3.11)$$

and assuming  $x_0 = 0$ , we have:

$$e^{-\lambda y_t} = \frac{c}{1 + \beta c} \quad (3.12)$$

Thus:

$$y_t = \frac{1}{\lambda} \log \left( \frac{1 + \beta c}{c} \right) = y^e \quad (3.13a)$$

(a constant),

and

$$x_t = y^e t = \frac{t}{\lambda} \log \left( \frac{1 + \beta c}{c} \right) \quad (3.13b)$$

(a constant investment path makes sense here since the condition  $\theta > z$  affects the exponential density by shifting it  $z$  units to the right, leaving its shape unchanged).

In terms of comparative statics, (3.12) shows that investments are decreasing in  $\lambda$  (the distributions  $F(\cdot|\lambda)$  are stochastically ordered by  $\lambda$ , a larger  $\lambda$  corresponding to a smaller distribution); likewise,  $y^0$  is decreasing in  $c$  and is increasing in  $\beta$ .

For both examples 1 and 2, the implied time-patterns of investments are non-increasing (i.e.,  $y_t \geq y_{t+1}$ ,  $t \geq 1$ ). This property holds, more generally, under the following (sufficient) condition.

PROPOSITION 1: Assume the hazard ratio  $\frac{1-F(\cdot)}{f(\cdot)}$  is (strictly) decreasing. Then

the equilibrium investment sequence,  $(y_t)_{t=1}^{\infty}$ , is (strictly) decreasing.

PROOF: Note that the family of distributions,  $\left\{ \frac{F(\theta+z)-F(z)}{1-F(z)} \right\}_{z \geq 0}$ , is

stochastically decreasing in  $z$ .

$$\begin{aligned} \frac{\delta}{\delta z} \left\{ \frac{F(\theta+z)-F(z)}{1-F(z)} \right\} &= \\ &= \frac{[f(\theta+z)-f(z)][1-F(z)] + f(z)[F(\theta+z)-F(z)]}{[1-F(z)]^2} \\ &= \frac{f(\theta+z)[1-F(z)] - f(z)[1-F(\theta+z)]}{[1-F(z)]^2} > 0, \end{aligned}$$

where the last inequality follows from the monotonicity of  $\frac{1-F(\cdot)}{f(\cdot)}$ .

The proposition now follows immediately, if we rewrite the equilibrium condition, (3.6), as:

$$\frac{F(x_{t-1} + y_t) - F(x_{t-1})}{1 - F(x_{t-1})} = \frac{1 - (1-\beta)c}{1 + \beta c} < 1.$$

A useful implication of the above proof is stated in the next corollary. We shall have an occasion to apply it in the next section.

COROLLARY 1: Under the condition stated in Proposition 1,

$$\int_0^{\infty} \frac{\theta f(\theta+z)}{1-F(z)} d\theta < \int_0^{\infty} \theta f(\theta) d\theta, \quad z \in \mathbb{R}_+.$$



## 4. Optimum

### 4.A. Deriving the Conditions of Optimality

Our approach here is to set-up the period one social objective, expressing it as a function of the stock variables  $(x_1, x_2, \dots)$  (rather than the flow variables,  $(y_1, y_2, \dots)$ ) and derive from it the first-order conditions which are necessary, for an optimum. In the appendix we outline the more indirect, dynamic programming approach. Obviously, the two approaches yield the same set of optimality conditions.

Consider, then, an investment plan,  $(y_n)_{n=1}^{\infty}$ , and its associated total capacity sequence,  $(x_n)_{n=1}^{\infty}$  (where, as before,  $x_n = x_0 + \sum_{i=1}^n y_i$ ). For  $\theta \in (x_n, x_{n+1}]$ ,

it follows from our informational assumptions (see section 1.C above) that investments take place for the first  $n+1$  periods and that they cease immediately after  $y_{n+1}$  had occurred. At that point in time it becomes known that productive capacity,  $x_{n+1}$ , exceeds the actual demand -  $\theta$  and, as capital does not depreciate, further investments (at that point) would be socially wasteful. Thus, the realized investment path which arises when  $\theta \in (x_n, x_{n+1}]$  is  $(y_1, \dots, y_{n+1}, 0, 0, \dots)$  and the total cost associated with it (expressed in terms of period 1 dollars) is:

$$c \sum_{i=1}^{n+1} \beta^{i-1} y_i \quad (4.1)$$

Turning to consumers' benefits we note that they are constrained by the available productive capacity during the "growth phase" (i.e., during periods  $1, 2, \dots, n$ ) and are fully satisfied thereafter [3] (during the "mature phase"). In other words, consumers' stream of benefits is of the form  $(x_1, x_2, \dots, x_n, \theta, \theta, \dots)$

which, conditional on  $\theta \in (x_n, x_{n+1}]$  has an expected discounted value of:

$$\sum_{i=1}^n \beta^{i-1} x_i + \beta^n \frac{1}{F(x_{n+1}) - F(x_n)} \int_{x_n}^{x_{n+1}} \frac{\theta}{1-\beta} f(\theta) d\theta. \quad (4.2)$$

Thus, combining (4.1) and (4.2) and taking the expected value under the prior,  $f(\cdot)$ , we get a net social benefit of

$$= \sum_{n=0}^{\infty} \left( \int_{x_n}^{x_{n+1}} [-c \sum_{i=1}^{n+1} \beta^{i-1} y_i + \sum_{i=1}^n \beta^{i-1} x_i + \beta^n \frac{\theta}{1-\beta}] f(\theta) d\theta \right) =$$

$$= \sum_{n=0}^{\infty} \left\{ (-c \sum_{i=1}^{n+1} \beta^{i-1} y_i + \sum_{i=1}^n \beta^{i-1} x_i) [F(x_{n+1}) - F(x_n)] + \beta^n \int_{x_n}^{x_{n+1}} \frac{\theta}{1-\beta} f(\theta) d\theta \right\}. \quad (4.3)$$

Next, setting  $y_i = x_i - x_{i-1}$  and manipulating the resulting expression (see the appendix for full details) we get:

$$W(x) = cx_0 + \sum_{n=0}^{\infty} \beta^n [x_{n+1} \{ (1+\beta c) [1 - F(x_{n+1})] - c [1 - F(x_n)] \}] + \int_{x_n}^{x_{n+1}} \frac{\theta}{1-\beta} f(\theta) d\theta \quad (4.4)$$

(4.4) is the social objective function, and we first prove that a maximum to it exists.

**THEOREM 2:** Assume the hazard ratio,  $\frac{1-F(\theta)}{f(\theta)}$ , is decreasing in  $\theta$ . Then

$W(x)$  attains a maximum,  $x^*$ , which satisfies the property  $x_n^* \leq nM$ , where

$$M = \frac{1}{(1-\beta)c} \int_0^{\infty} \theta f(\theta) d\theta.$$

PROOF: The proposition is proven in two steps.

CLAIM 1: An investment plan,  $(y_n)_{n=1}^{\infty}$ , is not optimal unless  $y_n \leq M$  for all  $n \geq 1$ .

PROOF: Starting from any  $x_{n-1}$  and assuming  $\theta > x_{n-1}$ , consumers' flow of benefits

cannot exceed  $\int_{x_{n-1}}^{\infty} \frac{\theta}{1-\beta} f(\theta | \theta > x_{n-1}) d\theta$  (which is what a fully informed planner would attain by choosing  $y_n = \theta - x_{n-1}$ ). But,

$$\begin{aligned} \int_0^{\infty} \frac{\theta}{1-\beta} f(\theta | \theta > x_{n-1}) d\theta &= \frac{x_{n-1}}{1-\beta} + \int_0^{\infty} \frac{\theta - x_{n-1}}{1-\beta} f(\theta | \theta > x_{n-1}) d\theta = \\ &= \frac{x_{n-1}}{1-\beta} + \int_0^{\infty} \frac{\theta f(\theta + x_{n-1})}{(1-\beta)[1-F(x_{n-1})]} d\theta < \frac{x_{n-1}}{1-\beta} + cM, \end{aligned}$$

where the above inequality follows from the corollary following proposition 1 (in section 3). Thus, if  $y_n > M$  the continuation value of (4.4) would be below

$$\frac{x_{n-1}}{1-\beta} \text{ and this is impossible since } \frac{x_{n-1}}{1-\beta} \text{ is attainable by choosing } y_t = 0, t \geq n.$$

The importance of claim 2 is that it allows us to seek an optimum to  $W(x)$  over the restricted set  $A$ , where

$$A \equiv \{(x_n)_{n=1}^{\infty} | 0 \leq x_n \leq nM\}.$$

By Tychonoff theorem  $A$  is compact in the product topology. Thus, it remain to show

CLAIM 2: The functional (4.4) is continuous on  $A$ , where  $A$  is endowed with the product topology.

PROOF: Rewrite (4.4) as  $W(x) = cx_0 + \sum_{n=0}^{\infty} \beta^n u(x_n, x_{n+1})$ ,

where

$$u(z, z') = z' \left\{ (1 + \beta c)[1 - F(z')] - c[1 - F(z)] \right\} + \int \frac{z' - \theta}{z - 1 - \beta} f(\theta) d\theta,$$

and note that, by assumption A.1,  $u$  is a continuous and bounded function on  $\mathbb{R}^2$ . Hence, given an  $\epsilon > 0$ , an  $N(\epsilon)$  can be found for which

$$\sum_{n=N(\epsilon)+1}^{\infty} \beta^n |u(x_n, x_{n+1}) - u(x'_n, x'_{n+1})| < \epsilon/2, \text{ for all } x, x' \in A.$$

Let  $x \in A$  be given and let  $x$  denote its projection on  $\mathbb{R}^{N(\epsilon)}$  (i.e.,  $x_t = x_t$ ,  $t = 1, \dots, N(\epsilon)$ ). Then by the continuity of  $u(\dots)$ , a neighbourhood,  $O(\epsilon, x) \subseteq \mathbb{R}^{N(\epsilon)}$ , of  $x$  exists for which

$$\text{Max}_{1 \leq n \leq N(\epsilon)} |u(x_n, x_{n+1}) - u(x'_n, x'_{n+1})| < \epsilon/2N(\epsilon), x' \in O(\epsilon, x).$$

Thus, for any  $x' \in O(\epsilon) \equiv \{x' \in A \mid x' \in O(\epsilon, x)\}$  we have

$$|W(x) - W(x')| \leq \sum_{n=1}^{N(\epsilon)} \beta^n |u(x_n, x_{n+1}) - u(x'_n, x'_{n+1})| + \sum_{n=N(\epsilon)+1}^{\infty} \beta^n |u(x_n, x_{n+1}) - u(x'_n, x'_{n+1})| < \epsilon.$$

And our claim is proven.

Theorem 2 validates a differential approach to the maximization of (4.4), and we proceed to the derivation of its first-order conditions. Differentiating (4.4) with respect to  $x_t$  we get:

$$\frac{\partial W}{\partial x_t} = \beta^{t-1} \{ (1 + \beta c) [1 - F(x_t)] - c [1 - F(x_{t-1})] + \beta c (x_{t+1} - x_t) f(x_t) \}. \quad (4.5)$$

Finally, from (4.5), after division by  $\beta^{t-1} [1 - F(x_{t-1})]$ , the following proposition is derived.

**THEOREM 3:** The planner's first-order optimality conditions are given by

$$c = (1 + \beta c) [1 - F(x_t | \theta > x_{t-1})] + \beta c y_{t+1} f(x_t | \theta > x_{t-1}), \quad t=1, 2, \dots \quad (4.6)$$

Let us pose here for a moment and give these conditions an economic interpretation as a set of marginal equivalences. On the LHS of (4.6) we clearly have the cost associated with the entry of one extra firm at time  $t$ . On the RHS, the benefit is two-fold. First, there is the direct benefit which is due to the fact that productive capacity - and along with it consumption - is increased as a result of this investment act. The value which the product market attaches to this benefit is exactly equal to the first term on the RHS - as the proof of theorem 1 above shows. Second, there is the informational benefit which is due to the potential avoidance of the wasteful investment,  $y_{t+1}$ . This investment is wasteful whenever  $\theta \in [x_t, x_t + \Delta)$  ( $\Delta$  being an "infinitesimal"), and it is avoided since the incipient firm lowers the period  $t$  price from 1 to 0, signalling thereby the "unwarrantedness" of  $y_{t+1}$  (recall that period  $t$  price becomes known at  $t+1$  and that  $p_t = 0$  signifies the end of the growth era). Observe that  $\theta \in [x_t, x_t + \Delta)$  occurs

with probability  $f(x_t | \theta > x_{t-1})$  and that the amount of money (in terms of period  $t$  dollars) saved in that event is  $\beta c y_{t+1}$ . Hence, the second term on the RHS of (4.6) is indeed the (expected) informational value of the incipient firm's entry in period  $t$ .

#### 4.B. Comparing the Optimum with the Equilibrium

The discrepancy between the first-order optimality conditions, (4.6), and the PFEE conditions, (3.6), is now revealed. Clearly, the informational value of the entry act, i.e., the second term on the RHS of (4.6), is not taken into account in a free-market setting. Hence, private investments are reduced and the productive capacity which is actually available in the industry at each point in time is below that which would have prevailed under the social optimum. More formally, we have

**THEOREM 4:** The optimum sequence of productive capacity levels,  $(x_t^0)_{t=0}^\infty$ , is larger, term by term, than the equilibrium sequence,  $(x_t^E)_{t=0}^\infty$ .

**PROOF:** We proceed by induction. For  $t=0$  we certainly have  $x_t^0 = x_t^E$ . Assume now that  $x_t^0 \geq x_t^E$  for some  $t > 0$  and suppose, per absurdum, that  $x_{t+1}^0 < x_{t+1}^E$ . Then we must have

$$1 - F(x_{t+1}^0 | \theta > x_t^0) > 1 - F(x_{t+1}^E | \theta > x_t^0) \geq 1 - F(x_{t+1}^E | \theta > x_t^E),$$

using the strict monotonicity of  $F$  and the inductive hypothesis.

But then, by (3.6) and (4.6), respectively:

$$c = (1+\beta c)[1-F(x_{t+1}^0 | \theta > x_t^0)] < (1+\beta c)[1-F(x_{t+1}^0 | \theta > x_t^0)] =$$

$$= c - \beta c y_{t+2}^0 f(x_{t+1}^0 | \theta > x_t^0) \leq c.$$

This establishes the sought after contradiction. Thus,  $x_{t+1}^0 \geq x_t^0$  and the theorem is proven.

#### 4.C. Examples

Unlike the equilibrium (see (3.6)), the optimality conditions, (4.6), are not serially solvable starting from some initial condition,  $x_0$ . The reason is that (4.6) defines a second-order difference equation and, thus, obtaining a specific solution to it requires that this solution's value - at two distinct points - be specified. When the support of  $F$  is bounded, a second boundary condition is  $x_{\infty} = \bar{\theta}$ , where  $\bar{\theta}$  is the supremum of this support. (The appendix furnishes a rigorous justification for this boundary condition) otherwise, one has to select (by "inspection") among the solutions to (4.6) one which is actually an optimum. These two approaches are (respectively) illustrated in the two examples below.

1)  $x_0 = 0$  and  $F$  is uniform on  $[0,1]$ . In that case the first-order conditions, (4.6), specialize to:

$$c = (1+\beta c) \frac{1-x_t}{1-x_{t-1}} + \beta c (x_{t+1} - x_t) \frac{1}{1-x_{t-1}} .$$

or, after multiplying by  $1-x_{t-1}$  and rearranging,

$$\beta c x_{t+1} - (1+2\beta c)x_t + c x_{t-1} = c(1-\beta) - 1 \quad (4.7)$$

Which is a second-order linear difference equation with constant coefficients (see Goldberg [ ]). A particular solution to it is:

$$x_t = 1 - (1-\gamma)^t \quad (4.8a)$$

$$\text{where } \gamma = \frac{-1 + \sqrt{1 + 4\beta c(1-c+\beta c)}}{2\beta c} \quad (4.8b)$$

This solution is obtained by postulating its form (which, in turn, is motivated by the equilibrium solution, (3.10)) and then solving the quadratic equation,

$$\beta c \gamma^2 + \gamma - [1-c+\beta c] = 0 \quad (4.8c)$$

which results when we substitute (4.8a) into (4.7). By our maintained assumption, (A.2), only the larger of the two roots of (4.8c) (i.e., (4.8b)) is economically sensible, that is,  $\gamma \in (0,1)$ . While other solutions to (4.7) do exist (and are of the form  $x_t + k_1 x_t^{(1)} + k_2 x_t^{(2)}$ , where  $k_1$  and  $k_2$  are arbitrary constants and  $x_t^{(i)} = (m_i)^t$  ( $i=1,2$ ),  $m_1$  and  $m_2$  being the roots of the characteristic equation  $\beta c m^2 - (1+2\beta c)m + c = 0$ ), they do not satisfy the pair of boundary conditions  $x_0=0, x_\infty=1$  - unless  $k_1=k_2=0$ . Thus, we conclude that (4.8a) is, in fact, the unique maximizer of expression (4.4). From (4.8a), it is easy to show now that

$$y_t = \gamma (1-x_{t-1}) \quad (4.8d)$$



and

$$1-x_t = (1-\gamma)(1-x_{t-1}). \quad (4.8e)$$

Therefore, both the investment policy function and the observed time series of capacity levels are of the same form as those which emerge in equilibrium. The only difference is that  $\gamma > \alpha$  (which follows from (A.2)) and, thus, as theorem 4 asserts  $x_t^0 \geq x_t^e$  for each and every  $t$ .

Using equation (4.8c), which determines  $\gamma$ , we can also establish comparative statics results analogous to those established in section 3.B.1. We have

$$\frac{\partial \gamma}{\partial \beta} = \frac{c(1-\gamma^2)}{1+2\beta c\gamma} > 0, \quad (4.9a)$$

and

$$\frac{\partial \gamma}{\partial c} = -\frac{1-\beta(1-\gamma^2)}{1+2\beta c\gamma} < 0 \quad (4.9b)$$

Finally, further calculations show that the Bellman equation (see the appendix) is solved by a linear function

$$V(x) = v_0 + v_1 x, \quad \text{with}$$

$$v_0 = \frac{1-(1-\gamma)^2-2(1-\beta)c\gamma}{2(1-\beta)[1-\beta(1-\gamma)^2]} \quad (4.10a)$$

and

$$y_1 = \frac{2(1-\beta)cy + 2(1-\beta)(1-\gamma)^2 + 2\gamma - \gamma^2}{2(1-\beta)(1-\beta(1-\gamma)^2)} \quad (4.10b)$$

The result of numerical calculations for the endogenous variables  $(\alpha, \gamma, y_0, y_1)$  given the values of the exogenous data over a grid in the  $(\beta, c)$  plane is reported in Table 1 below.

$\beta \backslash c$	0.5	1	2	4	8
0.98					
0.95					
0.90					
0.80					
0.50					

 $\alpha$  - Values

$\beta \backslash c$	0.5	1	2	4	8
0.98					
0.95					
0.90					
0.80					
0.50					

 $\gamma$  - Values

$\beta \backslash c$	0.5	1	2	4	8
0.98					
0.95					
0.90					
0.80					
0.50					

 $\gamma_0$  - Values

$\beta \backslash c$	0.5	1	2	4	8
0.98					
0.95					
0.90					
0.80					
0.50					

 $\gamma_1$  - Values

TABLE 1

2)  $x_0 = 0$  and  $F$  is exponential with parameter  $\lambda > 0$ , i.e.  $F(\theta) = 1 - e^{-\lambda\theta}$ .

In that case the first-order conditions reduce to:

$$\begin{aligned}
 c &= (1+\beta c)e^{-\lambda(x_t-x_{t-1})} + \beta c(x_{t+1}-x_t)\lambda e^{-\lambda(x_t-x_{t-1})} \\
 &= e^{-\lambda y_t} [1+\beta c + \beta c \lambda y_{t+1}], \quad t=1,2,\dots,
 \end{aligned} \tag{4.11}$$

where  $y_t = x_t - x_{t-1}$ .

Defining:

$$\psi(y, y') = e^{-\lambda y} [1 + \beta c + \beta c \lambda y'], \tag{4.12}$$

we can express (4.11) more concisely as

$$\psi(y_t, y_{t+1}) = c, \quad t=1,2,\dots \tag{4.13}$$

Implicitly differentiating the curve  $\psi(y, y') = c$ , we note that along it one must have:

$$\frac{dy'}{dy} = -\frac{\psi_y}{\psi_{y'}} = 1 + \frac{1}{\beta c} + \lambda y' \geq 1 + \frac{1}{\beta c} = g > 1 \tag{4.14}$$

Thus, three types of solutions to the dynamic-system (4.13) exist (consult Figure 1 below).

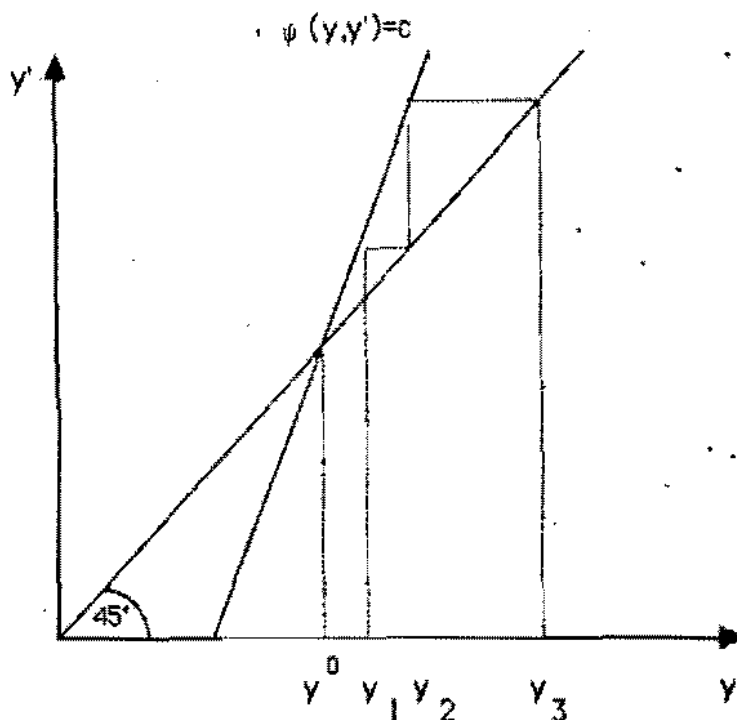


FIGURE 1

(i) A constant time-path  $y_t = y^0$ , where  $y^0$  is such that  $\psi(y^0, y^0) = c$ . Note that the function  $\zeta(y) \equiv \psi(y, y)$  is strictly diminishing in  $y$ , that  $\zeta(0) = 1 + \beta c > c$  (by assumption (A.2)) and that  $\lim_{y \rightarrow \infty} \zeta(y) = 0$ . Thus, a unique  $y^0$  as above exists.

(ii) A time-path  $y_t$  which monotonically diverges to infinity at a supergeometric rate, i.e., one for which  $y_1 > y^0$  and  $y_{t+1} \geq g^t y_1$ , where  $g$  is defined in (4.14) (by (4.14) and figure 1,  $y_{t+1} - y_t \geq g(y_t - y_{t-1})$  from which this time pattern follows).

(iii) An investment path which monotonically declines to zero in finite time, i.e., one for which  $y_1 < y^0$  and  $y_t = 0$  for all  $t \geq$  some  $T$ .

In the appendix we show that neither (ii) nor (iii) constitutes an optimum, which leaves us with (i) as the unique maximizer of (4.4). Therefore, we have:

$$y_t = y^0, \quad (4.18a)$$

where  $y^0$  is the (unique) solution of  $e^{-\lambda y} [1 + \beta c + \beta c \lambda y] = c$  and,

$$x_t = y^0 t \quad (4.18b)$$

which, as in the uniform example, has the same form as the equilibrium sequence. Furthermore, one can show that  $y^0 \geq y^B$  which conforms, of course, with the general assertion contained in theorem 4.

Regarding comparative statics results we can implicitly differentiate the equation  $f(y, p, c, \lambda) = v(y, y; p, c, \lambda) = c$  which yields:

$$\frac{\partial y^0}{\partial \beta} = \frac{c(1 + \lambda y)}{1 + \beta c \lambda y} > 0, \quad (4.19a)$$

$$\frac{\partial y^0}{\partial c} = -\frac{\beta(1 + \lambda y)}{1 + \beta c \lambda y} < 0, \quad (4.19b)$$

and

$$\frac{\partial y^0}{\partial \lambda} = -y < 0. \quad (4.19c)$$

((4.19c) makes sense since a larger  $\lambda$  means a less favorable - in the first order stochastic dominance sense - distribution and thus smaller investments).

Finally, as in the uniform case, the value function for the exponential program turns out to be linear,  $V(x) = v_0 + v_1 x$ , with

$$v_0 = \frac{1 - e^{-\lambda y^0} - (1-\beta)c\lambda y^0}{(1-\beta)\lambda(1 - \beta e^{-\lambda y^0})} \quad (4.20a)$$

and

$$v_1 = \frac{1}{1-\beta} \quad (4.20b)$$

Table II below reports the values of the endogenous variables  $y^e, y^0, v_0$  and  $v_1$  for a grid of  $\beta, c$  values.

$\beta \backslash c$	0.5	1	2	4	8
0.98					
0.95					
0.90					
0.80					
0.50					

$y^e$ -Values

$\beta \backslash c$	0.5	1	2	4	8
0.98					
0.95					
0.90					
0.80					
0.50					

$y^o$ -Values

$\beta \backslash c$	0.5	1	2	4	8
0.98					
0.95					
0.90					
0.80					
0.50					

$v_0$ -Values

$\beta \backslash c$	0.5	1	2	4	8
0.98					
0.95					
0.90					
0.80					
0.50					

$v_1$ -Values

TABLE 2



## 5. Summary and Conclusions

This paper considers the competitive evolution of a new market under uncertainty. The effects of uncertainty on entry in a static model had previously been examined by [ ], whereas sequential entry under certainty had been considered by [ ], [ ]. Here the main premise is that entry occurs in waves, and that later entrants get to observe how successful earlier entrants had been and are able to act upon that information. Our primary conclusion under those conditions is that a divergence between the optimum and the equilibrium arises, which is similar in spirit to what the research and development literature had pointed out. In our setting, informational spillovers occur because the signalling content of entry (from which later entrants benefit) goes unrewarded. Hence, the actual rate of entry is inefficiently low. The viability of corrective measures (such as governmental subsidies or collaboration among prospective entrants) remains the subject of future research.

**NOTES**

- [1] If for no other reason they require cooperation among operating firms, and those are assumed to compete with each other.
- [2] Because of the special demand function, (2.1), we have postulated, the form of this time-path is very simple: prices are first equal to unity; then they drop to zero and remain there forever. From the perspective of time zero, then, the randomness which is embedded in this time-path is simply a matter of when will the drop in prices occur for the first time.
- [3] To elaborate: there is an excess demand in the industry during the growth phase, that is to say,  $\theta > x_i$  for  $i=1, \dots, n$  and an excess supply, i.e.,  $\theta \leq x_{n+1}$ , thereafter.

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## APPENDIX

1. The following lemma is needed both for the derivation of (4.4) in the text and for the boundary condition  $x_{\infty} = \bar{\theta}$ , imposed in section 4.C.1.

**LEMMA B.1:** Under assumptions A.1 and A.2, the optimal sequence of capacity levels,  $(x_n)_{n=1}^{\infty}$ , converges to  $\bar{\theta}$  ( $\bar{\theta}$  being the supremum of the distribution F).

**PROOF:**  $x_n$  is increasing and, thus convergent (possibly to infinity).

Assume, contrary to our claim, that  $x_n \rightarrow \bar{x} < \bar{\theta}$ . Let  $x' > \bar{x}$  be such that

$$\frac{1-F(x'|\theta > \bar{x})}{1-\beta} > c \quad \left( \frac{1}{1-\beta} > c, \text{ by A.2. Thus, by A.1, for } x' \text{ sufficiently} \right.$$

close to  $\bar{x}$  this inequality holds). Then as  $n \rightarrow \infty$ , we have

$$Z(n) = (x' - \bar{x}) \frac{1-F(x'|\theta > \bar{x})}{1-\beta} - c(x' - \bar{x}) \rightarrow c_0$$

where

$$c_0 = (x' - \bar{x}) \frac{1-F(x'|\theta > \bar{x})}{1-\beta} - c(x' - \bar{x}) > 0$$

(This exploits the continuity of  $F(\cdot, \cdot)$  in its second variable, which follows from assumption A.1)

Pick now an Integer  $N$  for which

$$c(\bar{x} - x_N) < \epsilon_0/4 \text{ and } Z(N) > \epsilon_0/2,$$

and consider the following alternative investment policy,  $(\tilde{Y}_t)_{t=1}^{\infty}$ :

$$\begin{cases} Y_t & t \leq N \\ x' - x_N & t = N+1 \\ 0 & t \geq N+2 \end{cases}$$

Denote the value of this policy at  $x_N$  (i.e., when  $x = x_N$  and it is known that  $\theta > x_N$ )

by  $u_A$  and denote the value of the original policy at the same state by  $u_0$ . Then we have:

$$u_0 \leq \int_{x_N}^{\bar{x}} \frac{\theta}{1-\beta} f(\theta|\theta > x_N) d\theta + \frac{\bar{x}}{1-\beta} [1 - F(\bar{x}|\theta > x_N)],$$

and

$$u_A = \int_{x_N}^{\bar{x}} \frac{\theta}{1-\beta} f(\theta|\theta > x_N) d\theta + \int_{\bar{x}}^{x'} \frac{\theta}{1-\beta} f(\theta|\theta > x_N) d\theta + \frac{x'}{1-\beta} [1 - F(x'|\theta > x_N)]$$

$$-c(x' - x_N) \geq \int_{x_N}^{\bar{x}} \frac{\theta}{1-\beta} f(\theta|\theta > x_N) d\theta + \frac{\bar{x}}{1-\beta} [F(x'|\theta > x_N) - F(\bar{x}|\theta > x_N)]$$

$$+ \frac{x'}{1-\beta} [1 - F(x'|\theta > x_N)] - c(x' - \bar{x}) - c(\bar{x} - x_N)$$

Thus,

$$u_A - u_0 \geq \frac{x' - \bar{x}}{1 - \beta} [1 - F(x' | \theta > x_N)] - c(x' - \bar{x}) - c(\bar{x} - x_N).$$

$$= Z(N) - c(\bar{x} - x_N) > e_0/2 - e_0/4 = e_0/4 > 0.$$

Therefore, at  $x = x_N$  the original policy is dominated by the alternative,  $(\tilde{y}_t)$ , which contradicts the former's presumed optimality. The proof is now complete.

## 2. Deriving expression (4.4) from (4.3).

The last term,  $\beta^n \int_{x_n}^{x_{n+1}} \frac{\theta}{1 - \beta} f(\theta) d\theta$ , inside the braces of (4.3) and (4.4) is

identical. Thus, it remains to deal with the terms preceding it.

Note, first, that

$$\sum_{i=1}^{n+1} \beta^{i-1} y_i = \sum_{i=1}^{n+1} \beta^{i-1} (x_i - x_{i-1}) = -x_0 + \beta^n x_{n+1} + (1 - \beta) \sum_{i=1}^n \beta^{i-1} x_i.$$

Thus,

$$-c \sum_{i=1}^{n+1} \beta^{i-1} y_i + \sum_{i=1}^{n+1} \beta^{i-1} x_i = (1 + \beta c - c) \left( \sum_{i=1}^n \beta^{i-1} x_i \right) + c x_0 - c \beta^n x_{n+1}.$$

Next,

$$\sum_{n=0}^{\infty} \left( \sum_{i=1}^n \beta^{i-1} x_i \right) [F(x_{n+1}) - F(x_n)] =$$

$$= \sum_{i=1}^{\infty} \beta^{i-1} x_i \left( \sum_{n=1}^{\infty} [F(x_{n+1}) - F(x_n)] \right) = \sum_{i=1}^{\infty} \beta^{i-1} x_i [1 - F(x_i)]$$

$$= \sum_{i=0}^{\infty} \beta^i x_{n+1} [1 - F(x_{n+1})],$$

where the first equality follows from a change in the order of summation and the second from Lemma B.1 (according to which  $F(x_{\infty})=1$ ).

Therefore,

$$\sum_{n=0}^{\infty} \left[ (1 + \beta c - c) \sum_{i=1}^n \beta^{i-1} x_i + c x_0 - c \beta^n x_{n+1} \right] [F(x_{n+1}) - F(x_n)]$$

$$= c x_0 + \sum_{n=0}^{\infty} \left\{ (1 + \beta c - c) \beta^n x_{n+1} [1 - F(x_{n+1})] - c \beta^n x_{n+1} [F(x_{n+1}) - F(x_n)] \right\}$$

$$= c x_0 + \sum_{n=0}^{\infty} \beta^n x_{n+1} \left\{ (1 + \beta c) [1 - F(x_{n+1})] - c [1 - F(x_n)] \right\},$$

which corresponds to the first two terms in expression (4.4). Our derivation is now complete.



### 3. The dynamic programming approach

The Bellman equation corresponding to the planner's program may be written as

$$V(x) = \text{Max}_{y \geq 0} \left\{ -cy + \int_0^{x+y} \frac{\theta}{x} f(\theta|\theta > x) d\theta + [x+y+\beta V(x+y)][1-F(x+y|\theta > x)] \right\} \quad (5.1)$$

**THEOREM 5:** Assume  $F$  has a compact support. Then there exists a unique and continuous solution to the Bellman equation.

**PROOF:** Consider the Banach space of continuous functions,  $C$ , on the support of  $F$ , together with the sup-norm on it. The RHS of (5.1) defines a mapping,  $T$ , on that space. We show that it is a contraction map and that the image of a continuous function under it is continuous as well. This, by the Banach fixed point theorem, is all that is needed.

Let then  $V \in C$ . By our maintained assumption, A.1, the RHS of (5.1) is continuous in the pair  $(x, y)$ . Thus, a maximum to it exists, and by the theorem of the maximum (see Berge [ ], p. ) the maximized value is continuous in  $x$ . This establishes that  $T$  is indeed into  $C$ .

Next, for any  $V \in C$  let  $H(y; x, V)$  be the RHS maximand and let  $h(x, V)$  be the corresponding maximizer (which, as we have just shown, does exist). Let  $V, \tilde{V} \in C$  be given. Then:

$$\begin{aligned} |TV(x) - T\tilde{V}(x)| &= |H(h(x, V); x, V) - H(h(x, \tilde{V}); x, \tilde{V})| = \\ &= H(h(x, \tilde{V}); x, \tilde{V}) - H(h(x, V); x, V), \end{aligned}$$

assuming without loss of generality that the left hand term is larger than the right hand one.

Continuing,

$$\begin{aligned} & \leq H(h(x, \tilde{Y}); x, \tilde{Y}) - H(h(x, \tilde{Y}); x, Y) \\ & = [x+y+\beta\tilde{V}(x+y)][1-F(x+y|\theta>x)] - [x+y+\beta V(x+y)][1-F(x+y|\theta>x)] \\ & = \beta[\tilde{V}(x+y)-V(x+y)][1-F(x+y|\theta>x)] \leq \beta\|\tilde{Y}-Y\|, \end{aligned}$$

where  $y = h(x, Y)$ . Thus,  $T$  is a contraction with modulus  $\beta$  and our claim is proven.

From this point on we shall proceed under the further assumption that  $V$  is, in fact, differentiable. This is not guaranteed by theorem 5, but holds, for instance, if one further restriction is imposed on the exogenous data. Namely, that the terms inside the braces of (4.4) are concave (see Benveniste and Scheinkman [ ], p. ). Differentiating the RHS of (5.1) the following first-order condition is obtained.

$$c = [1+\beta V'(x+y)][1-F(x+y|\theta>x)] - \beta f(x+y|\theta>x) \left[ V(x+y) - \frac{x+y}{1-\beta} \right]. \quad (5.2)$$

Furthermore, by the envelope theorem,

$$\begin{aligned} V'(x) &= c + \frac{f(x)}{1-F(x)} \left\{ \frac{x+y}{x} \frac{\theta}{1-\beta} f(\theta>x) dx - \frac{x}{1-\beta} + [x+y+\beta V(x+y)][1-F(x+y|\theta>x)] \right\} \\ &= c + \frac{f(x)}{1-F(x)} \left[ V(x) + cy - \frac{x}{1-\beta} \right]. \end{aligned} \quad (5.3)$$

using equation (5.1) to eliminate the term  $[x+y+\beta V(x+y)][1-F(x+y|\theta>x)]$ .

Evaluating now (5.3) at  $x+y$  (instead of  $x$ ) and using the resulting expression to

eliminate  $[V(x+y) - \frac{x+y}{1-\beta}]$  from (5.2) (the last term on its RHS), we obtain:

$$c = (1+\beta c)[1-F(x+y|\theta > x)] + \beta c y' f(x+y|\theta > x) \quad (5.4)$$

(where  $y'$  is the maximizer of the RHS of (5.1) for next period's state, i.e., for  $x' = x+y$ ). This coincides with (4.6) (setting  $x = x_{t-1}$ ,  $y = y_t$  and  $y' = y_{t+1}$ ) and our derivations are complete.

4. Proving the optimality of a constant investment path,  $y_t = y^0$  for the exponential case (section 4.C.2).

The following preliminary result is needed:

**LEMMA B.2:** Given an existing capacity level of  $x_0$  and the information that  $\theta > x_0$ , the continuation value of the planner's program under the exponential

distribution is  $\frac{x_0}{1-\beta} + k$ , where  $k \leq \infty$  is the maximal value of the planner's

objective when  $x=0$ .

**PROOF:** Denoting, more explicitly, the planner's objective by  $W(x; x_0)$  we have by (4.4):

$$W(x; x_0) = cx_0 + \sum_{n=0}^{\infty} \beta^n [x_{n+1} \{ (1+\beta c)[1-F(x_{n+1}|\theta > x_0)] - c[1-F(x_n|\theta > x_0)] \}]$$

$$+ \int_{x_n}^{x_{n+1}} \frac{\theta}{1-\beta} f(\theta|\theta > x_0) d\theta$$

$$= cx_0 + \sum_{n=0}^{\infty} \beta^n [x_{n+1} \{ (1+\beta c)[1-F(x_{n+1}-x_0|\theta>0)] - c[1-F(x_n-x_0|\theta>0)] \}$$

$$+ \int_{x_n}^{x_{n+1}} \frac{\theta}{1-\beta} f(\theta-x_0|\theta>0) d\theta,$$

where the latter equality follows from the fact that  $F$  is exponentially distributed. Introducing now the notation  $x'_n = x_n - x_0$ ,  $\theta' = \theta - x_0$  and writing  $F(z)$  instead of  $F(z|\theta > 0)$  we get:

$$W(x; x_0) = cx_0 + \sum_{n=0}^{\infty} \beta^n [x'_{n+1} + x_0] \{ (1+\beta c)[1-F(x'_{n+1})] - c[1-F(x'_n)] \}$$

$$+ \int_{x'_n}^{x'_{n+1}} \frac{\theta' + x_0}{1-\beta} f(\theta') d\theta'$$

$$= \sum_{n=0}^{\infty} \beta^n [x'_{n+1} \{ (1+\beta c)[1-F(x'_{n+1})] - c[1-F(x'_n)] \} + \int_{x'_n}^{x'_{n+1}} \frac{\theta'}{1-\beta} f(\theta') d\theta']$$

$$+ x_0 \{ c + \sum_{n=0}^{\infty} \beta^n [(1+\beta c)[1-F(x'_{n+1})] - c[1-F(x'_n)]] + \frac{F(x'_{n+1}) - F(x'_n)}{1-\beta} \}$$

$$= W(x;0) + x_0 \left[ c + \frac{1+\beta c - c}{1-\beta} + \left( \frac{1}{1-\beta} - c \right) \sum_{n=0}^{\infty} \beta^n [\beta F(x'_{n+1}) - F(x'_n)] \right]$$

$$= W(x;0) + x_0 \left[ \frac{1}{1-\beta} + \left( \frac{1}{1-\beta} - c \right) \left[ \sum_{n=0}^{\infty} \beta^{n+1} F(x'_{n+1}) - \sum_{n=0}^{\infty} \beta^n F(x'_n) \right] \right]$$

$$= W(x;0) + \frac{x_0}{1-\beta}$$

since  $F(x'_0) = 0$  and thus  $\sum_{n=1}^{\infty} \beta^n F(x'_n) = \sum_{n=0}^{\infty} \beta^n F(x'_n)$ .

Summarizing, we have shown that:

$$W(x; x_0) = \frac{x_0}{1-\beta} + W(x;0).$$

But  $W(x;0)$  is bounded by theorem 4 (note that the hazard ratio,  $\frac{1-F(\cdot)}{f(\cdot)}$ , is a

constant,  $\frac{1}{\lambda}$ ) and the lemma is proven.

**PROPOSITION 2:** The investment time paths outlined in (ii) and (iii) of section 4.C.2. above are not optimal.

**PROOF:** 1. We first rule out any investment path for which  $y_t \uparrow$ .

Specializing (5.1) to the exponential case we have:

$$V(x) = \text{Max}_{y \geq 0} \left\{ -cy + \frac{1}{1-\beta} [x(1-e^{-\lambda y}) + \frac{1-e^{-\lambda y}}{\lambda} - ye^{-\lambda y}] \right. \\ \left. + [x+y+\beta V(x+y)]e^{-\lambda y} \right\}. \quad (5.5)$$

We certainly have  $V(x) \geq \frac{x}{1-\beta}$  (by setting  $y = 0$ ). On the other hand, the term in

braces on the RHS of (5.5) equals to

$$-cy + \frac{1}{1-\beta} \left[ x + \frac{1-e^{-\lambda y}}{\lambda} - (x+y)e^{-\lambda y} \right] + [x+y+\beta V(x+y)]e^{-\lambda y} \\ = -cy + \frac{x}{1-\beta} + \frac{1-e^{-\lambda y}}{\lambda(1-\beta)} + \beta e^{-\lambda y} \left[ V(x+y) - \frac{x+y}{1-\beta} \right] \\ \leq -cy + \frac{x}{1-\beta} + \frac{1-e^{-\lambda y}}{\lambda(1-\beta)} + \beta e^{-\lambda y} k, \quad (5.6)$$

where the last step follows from lemma B.2.

But the expression on the RHS of (5.6), considered as a function of  $y$  only,

certainly tends to minus infinity as  $y$  tends to infinity. Thus, for sufficiently large  $y$  the RHS of (5.6) is below  $\frac{x}{1-\beta}$ . Furthermore, under the postulated policy (that is, one where  $y_t \uparrow \Rightarrow$ ), such  $y$ 's do eventually emerge and this contradicts the optimality of such policies and completes the proof of our first assertion.

2. We now show that  $y_t = 0$  cannot be optimal. For  $y > 0$ , the RHS of (5.5) exceeds

$$\begin{aligned} & -cy + \frac{1}{1-\beta} \left[ x + \frac{1-e^{-\lambda y}}{\lambda} - (x+y)e^{-\lambda y} \right] + \frac{x+y}{1-\beta} e^{-\lambda y} \\ & = -cy + \frac{x}{1-\beta} + \frac{1-e^{-\lambda y}}{\lambda(1-\beta)} \end{aligned} \quad (5.7)$$

where the last term on the RHS of (5.7) (i.e.,  $\frac{x+y}{1-\beta} e^{-\lambda y}$ ) is substituted in for

the larger term,  $[x+y+\beta V(x+y)]e^{-\lambda y}$ . Considering now the RHS of (5.7) as a function of  $y$ , we differentiate it with respect to that variable which yields

$$-c + \frac{1}{1-\beta} > 0 \text{ (by A.2) at } y = 0. \text{ Thus, a sufficiently small } y \text{ dominates } y=0 \text{ and}$$

our second assertion is established as well.

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